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FINITE ELEMENT ANALYSIS OF A SEISMICALLY EXCITED CYLINDRICAL STORAGE TANK, GROUND SUPPORTED, AND PARTIALLY FILLED WITH LIQUID

Samia Hafiz Shaaban

William A. Nash

Department of Civil Engineering
University of Massachusetts
Amherst, Massachusetts 01002

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ABSTRACT

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BACKGROUND

A previous report [1] by these same investigators developed a finite element approach for determination of small amplitude elastic responses of an empty slab-supported cylindrical liquid storage tank subject to arbitrary base excitation. It was assumed that the base slab supporting the tank is rigid and that the tank does not separate from the slab during excitation. The present investigation continues the work presented in [1], but with the significant addition of an inviscid, incompressible liquid filling the tank to an arbitrary depth. Again, finite elements are employed to represent both the elastic tank as well as the liquid. Natural frequencies and associated mode shapes of the coupled liquid-elastic system are found through use of finite elements. Also, the special case of the natural frequencies and associated mode shapes of a liquid in a rigid container is investigated. Next, using modal superposition, a program is developed for determination of the response of the coupled liquid-elastic system to arbitrary base excitation.

ANALYSIS

Governing Equations

For the elastic circular cylindrical tank with a vertical geometric axis under consideration here, we shall employ a series of ring-shaped finite elements extending from the base slab to the tank top, with each ring being bounded by a horizontal plane normal to the shell axis. Both in-plane as well as out-of-plane displacements and forces in the shell must be considered. Again, as in [1], the shell theory due to J. L. Sander, Jr. [2] is employed to represent the small, elastic deformations of the cylindrical tank. Let the radius of the tank be R and its thickness be h. Further, let the quantities r, θ, and z denote radial, circumferential, and axial coordinates respectively of a point on the middle surface of the shell. The corresponding displacement components are denoted by w, v, and u. The equations of motion of the elastic tank in terms of w, v, and u are given in [1].

The liquid in the tank is assumed to be homogeneous, incompressible, and inviscid. Further, the flow is taken to be irrotational and only small amplitude liquid motions are considered. Lastly, it is assumed that there are no sources, sinks, or cavities anywhere in the liquid. Under these conditions the motion obeys the Laplace equation

$$\nabla^2 p\ (r,\ \theta,\ z) = 0 \tag{1}$$

where p represents total pressure at any point. The total pressure is the sum of the static and dynamic pressures, viz:

$$p = p_{st} + p_{dyn}$$

where p_{st} is the pressure that would exist if there were no motion and p_{dyn} arises because of motion of the liquid. Since the static pressure obeys Laplace's equation, obviously the dynamic pressure does also. Henceforth, the dynamic pressure will be denoted by p for brevity.

The Bernoulli equation may be expressed in the form:

$$gz + \frac{p}{\rho_f} + \frac{p_{st}}{\rho_f} + (1/2)v^2 + \frac{\partial \Phi}{\partial t} = 0 \qquad (2)$$

where z is as defined for the shell with origin at the liquid surface, g is the gravitational constant, ρ_f denotes liquid density, v the magnitude of velocity at any point in the liquid, t denotes time, and Φ is the velocity potential. Since the liquid is nonviscous, the motion is irrotational, and the oscillations are of small amplitude, the velocity squared term in (2) may be neglected in comparison with other terms. Also, for z measured positive upward from the liquid surface we have:

$$gz + \frac{p_{st}}{\rho_f} = 0 \qquad (3)$$

Thus, (2) becomes:

$$\frac{p}{\rho_f} + \frac{\partial \Phi}{\partial t} = 0 \qquad (4)$$

Boundary Conditions

At the liquid free surface, the vertical velocity component is given by:

$$v_z\ (r,\ \theta,\ 0) = \left[\frac{d\xi(r,\theta)}{dt}\right]_{z=0} = \left[\frac{\partial\Phi}{\partial z}\right]_{z=0} \tag{5}$$

$$\xi = \frac{-1}{g}\left(\frac{\partial\Phi}{\partial t}\right)$$

where ξ is the superelevation of the free surface over the undisturbed surface level. The linearized free surface condition may be expressed in the form:

$$\left[\frac{\partial^2\Phi}{\partial t^2} + g\,\frac{\partial\Phi}{\partial z}\right]_{z=0} = 0 \tag{6}$$

Using (4) together with the relation $\rho_f g\xi = p$, this may be expressed in the form:

$$\frac{1}{g}\ \frac{\partial^2 p}{\partial t^2} + \frac{\partial p}{\partial z} = 0 \tag{7}$$

For the liquid under consideration the velocity vector $\overline{V}$ may be written in the form:

$$\begin{aligned}\overline{V} &= \text{grad}\ \Phi \\ &= \nabla\Phi\end{aligned} \tag{8}$$

Consequently, the boundary conditions expressing liquid-solid interaction along the elastic wall of the cylindrical tank as well as at the rigid bottom of the tank may be written as:

$$\overline{V} \cdot \overline{n} = \begin{cases} \left[\dfrac{\partial w}{\partial t}\right]_{r=R} & \text{in the wetted part of the tank wall} \\ \left[0\right]_{z=-H} & \text{at the rigid tank bottom} \end{cases} \tag{9}$$

Here, $\overline{n}$ is a unit vector normal to the liquid-shell boundary and H denotes depth of liquid in the tank. Thus, along the wetted elastic tank wall denoted by Σ we have:

$$\left[\frac{\partial \Phi}{\partial r} - \frac{\partial w}{\partial t}\right]_{r=R} = 0$$

where w is the radial displacement of the tank wall at any point (R,z,θ). Again, using (4), this becomes:

$$\left[\frac{\partial^2 w}{\partial t^2} + \frac{1}{\rho_f}\frac{\partial p}{\partial r}\right]_{r=R} = 0 \tag{10}$$

Since the liquid velocity in the z-direction is zero at the tank bottom, it follows from that:

$$\left[\frac{\partial \Phi}{\partial z}\right]_{z=-H} = 0 \tag{11}$$

In summary, motion of the liquid is completely defined by the Laplace equation (1) together with boundary conditions.

In a finite element approach to the coupled liquid-elastic tank problem, the finite element matrix equation is obtained either from the governing differential equation by using Galerkin's method, or from the variational equation by using a minimization technique [3]. Use of the Galerkin procedure necessitates knowledge of the governing differential equations of motion together with selection of a weighting function which may be chosen to be the same as the element shape function. Setting the first variation of the resulting integral equal to zero yields the desired finite element matrix equation. Use of the Euler-Lagrange method necessitates formulation of the kinetic energy (found by integrating over the liquid volume), the potential energy (found by integrating over the free surface), and the work done on the liquid by external effects (such as solid-liquid interface forces). Minimization of energy then yields the governing equations. In [3], it is demonstrated that both approaches yield the same finite element matrix equation provided the same type of element and the same shape function are employed in both treatments.

In [3], it is shown that an appropriate variational functional for the liquid is

$$I = \int_{t_1}^{t_2} (T - \Pi - W)dt \qquad (12)$$

where T, Π, and W represent the kinetic energy, the potential energy of the liquid, and the work done on the liquid respectively. These are given by [3]

$$T = (1/2)\,\rho_f \int_V \nabla\Phi \cdot \nabla\Phi\, dv$$

$$\Pi = (1/2)\int_F \xi(\rho_f \quad g\xi)\ ds \qquad (13)$$

$$W = \int_Z \rho_f \left(\frac{\partial w}{\partial t}\right) \Phi ds$$

where ρ_f denotes liquid density and ξ is the deviation of the liquid elevation from the static configuration. The kinetic energy is evaluated by integration over the liquid volume V, the potential energy by integration over the free surface F, and the work by integration over the liquid-tank interface Σ.

In the present investigation, it is most convenient to investigate the dynamic problem in terms of the liquid dynamic pressure p. If damping is neglected, this leads to a matrix differential equation involving only the pressure together with its second derivative with respect to time. In [3], Eq. 3.9 it is shown that the functional pertinent to the governing equation (1) together with boundary conditions (7) and (10) may be written in the form:

$$I = (1/2)\int_V \nabla p \cdot \nabla p\ dv - \frac{1}{2g}\int_F \left(\frac{\partial p}{\partial t}\right)^2 ds - \rho_f \int_\Sigma p\,\frac{\partial^2 w}{\partial t^2}\, ds = I_1 - I_2 - I_3 \qquad (14)$$

where the definitions of I_1, I_2, and I_3 are evident from (14).

Finite Element Idealization

The liquid is discretized into annular elements of rectangular cross-section. These elements may by considered to be formed from the intersection of concentric annular cylindrical surfaces with a set of horizontal planes. The intersection of these surfaces with the planes gives rise to nodal circles, as shown in Figure 1.

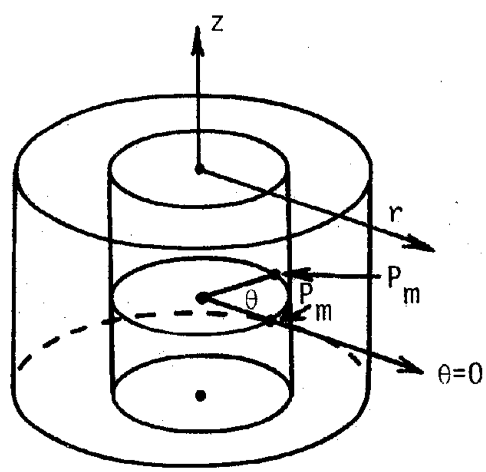


FIGURE 1

This three-dimensional problem can essentially be transformed into a two dimensional one by developing the pressure p in a Fourier series in the circumferential direction, viz:

$$p = \sum_m p_m \cos m\theta \qquad (15)$$

The problem of forced motion of the slab supported tank when excited by horizontal ground accelerations can be reasonably well described through consideration of only the first harmonic, m = 1 provided that one is concerned with obtaining the motions about the neutral equilibrium configuration. However, for the sake of generality, the following finite element matrices will be developed for an arbitrary number of harmonics m in the circumferential direction. Thus, let us set

$$p_m (r, z, \theta) = P_m (r, z, 0) \cos m\theta \tag{16}$$

Henceforth, the subscript m will be omitted for brevity.

Thus, the problem has been reduced to a two dimensional one in the plane indicated by r, z, $\theta = 0$ in Figure 1. Henceforth, we shall use (x,y) as local coordinates, which origin at the geometric center of the element, to denote the position of any point in this plane. The liquid pressure at any point in this plane is described using the nodal pressure parameters of the corresponding rectangular element surrounding it. Thus:

$$P(x,y) = [N] \{\delta_p\} \tag{17}$$

where [N] represents the element shape function and $\{\delta_p\}$ is the element nodal pressure vector. The shape function is obtained by assuming a suitable interpolation function which here is taken to be a linear variation of liquid pressure in both the x and y directions. Thus:

$$P(x,y) = \frac{1}{4ab} [(a-x)(b-y) \quad (a+x)(b-y) \quad (a+x)(b+y) \quad (a-x)(b+y)] \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ \delta p_3 \\ \delta p_4 \end{bmatrix} \tag{18}$$

Figure 2 indicates a typical element of length 2a in the radial (r) direction, 2b in the z (axial) direction, whose center (0,0) lies at a distance x_o from the geometric axis of the tank.

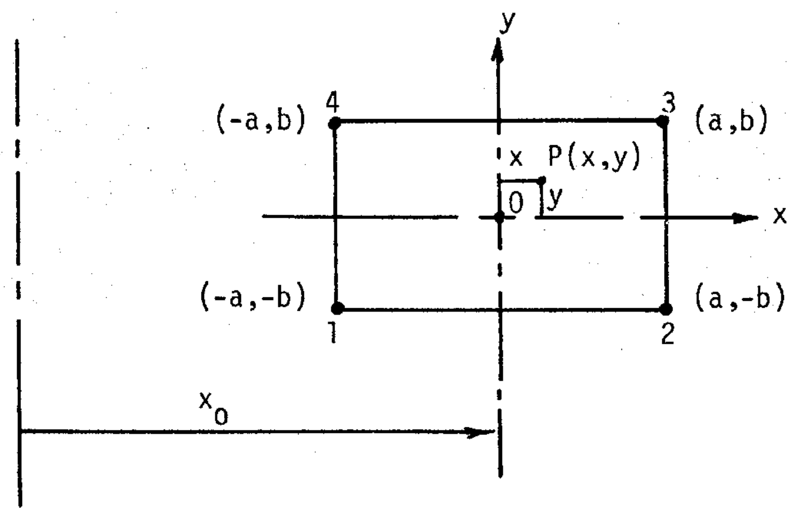


FIGURE 2
Liquid Element

From (16), we have:

$$\nabla p = \frac{\partial P}{\partial r} \cos(m\theta)\,\bar{i}_r + \frac{\partial P}{\partial z} \cos(m\theta)\,\bar{i}_z - \frac{m}{r} \sin(m\theta)\; P\,\bar{i}_\theta \tag{19}$$

$$\nabla p \cdot \nabla p = \left(\frac{\partial P}{\partial r}\right)^2 \cos^2(m\theta) + \left(\frac{\partial P}{\partial z}\right)^2 \cos^2(m\theta) + \frac{m^2}{r^2} \sin^2(m\theta)\, P^2 \tag{20}$$

It is now necessary to determine the functional (13). Substitution of the pressure (15) into the integral defining I_1 and integration over the liquid volume v yields:

$$I_1 = (1/2)\int_v \nabla p \cdot \nabla p \, dv$$

$$= (1/2)\int_r \int_z \int_\theta \left[\left(\frac{\partial P}{\partial r}\right)^2 \cos^2(m\theta) + \left(\frac{\partial P}{\partial z}\right)^2 \cos^2(m\theta) + \frac{m^2}{r^2} \sin^2(m\theta) P^2 \right] r d\theta dz\, dr$$

$$= \frac{\pi}{2} \int_r \int_z \left[\left(\frac{\partial P}{\partial r}\right)^2 + \left(\frac{\partial P}{\partial z}\right)^2 + \frac{m^2}{r^2} P^2 \right] r\, dr\, dz$$

$$= \frac{\pi}{2} \int_x \int_y \left[\left(\frac{\partial P}{\partial x}\right)^2 r\left(\frac{\partial P}{\partial y}\right)^2 + \frac{m^2 P^2}{(x_0 + x)^2} \right] (x_0 + x)\, dx\, dy \qquad (21)$$

$$I_1 = (1/2)\{\delta_p\}^T [K_e] \{\delta_p\} \qquad (22)$$

The element stiffness matrix $[K_e]$ is developed in detail in Appendix A. The integral defining I_2 is found by integrating over the liquid free surface F to be:

$$I_2 = \frac{1}{2g} \int_F \left(\frac{\partial p}{\partial t}\right)^2 ds$$

$$= \frac{1}{2g} \int_r \int_\theta \left(\frac{\partial P}{\partial t}\right)^2 \cos^2(m\theta)\, r d\theta\, dr$$

$$= \frac{\pi}{2g} \int_r \left(\frac{\partial P}{\partial t}\right)^2 r\, dr$$

$$= \frac{\pi}{2g} \int_x \left(\frac{\partial P}{\partial t}\right)^2 (x_0 + x)\, dx$$

$$= \frac{\pi}{2g} \int_x \{\dot{\delta}_p\}^T \{N\}^T [N] \{\dot{\delta}_p\} (x_0 + x)\, dx \qquad (23)$$

$$= 1/2 \{\dot{\delta}_p\}^T [M_e] \{\dot{\delta}_p\} \qquad (24)$$

The element mass matrix $[M_e]$ is found using (18) and is given in detail in Appendix A. The integral defining I_3 is found by integrating over the liquid-elastic shell interface Σ to be

$$I_3 = \rho_f \int_\Sigma p \frac{\delta^2 w}{\delta t^2}\, ds$$

$$= \rho_f \int \int_\theta P \frac{\partial^2 W}{\partial t^2} \cos^2(m\theta) R\, d\theta dz$$

$$= \rho_f \pi R \int_z P \frac{\partial^2 W}{\partial t^2}\, dz \qquad (25)$$

where R is the tank radius and

$$w(z, \theta) = W(z, 0) \cos(m\theta) \qquad (26)$$

The generalized radial displacement of the tank W may be represented in terms of the finite element generalized coordinates $\{\delta_u\}$ through the following:

$$W(z, 0) = [N_w] \{\delta_u\} \qquad (27)$$

Thus,

$$I_3 = \rho_f \pi R \int_z \{\delta_p\}^T [N]^T [N_w] \{\ddot{\delta}_u\} \, dz \qquad (28)$$

$$= \rho_f \{\delta_p\}^T [S_e] \{\ddot{\delta}_u\} \qquad (29)$$

From this the force matrix $[S_e]$ representing the coupling effect is determined. This is developed in detail in Appendix A. The assembled liquid mass and stiffness matrices are denoted by $[M_f]$ and $[K_f]$ respectively, and the coupling force matrix is assembled in [S].

The partial differential equations, in matrix form, governing liquid motion may be found by first realizing that the functional I (14) is of the form:

$$I = \int_{t_1}^{t_2} f(\delta_{p_1}, \delta_{p_2}, \dot{\delta}_{p_1}, \dot{\delta}_{p_2}, \ . \ . \ . \ t) \, dt \qquad (30)$$

Then, setting the first variation of this equal to zero, viz:

$$\delta I = 0 \qquad (31)$$

An Euler-Lagrange equation for each independent variable δ_{p_i} may be obtained from the expression:

$$\frac{\delta f}{\delta \delta_{p_i}} = \frac{\partial f}{\partial \delta_{p_i}} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{\delta}_{p_i}} \right) = 0 \qquad (32)$$

Substitution of (22), (24), and (29) into (14) yields:

$$I = 1/2\ \{\delta_p\}^T[K_f]\{\delta_p\} - 1/2\ \{\dot{\delta}_p\}^T[M_f]\{\dot{\delta}_p\} - \rho_f\{\delta_p\}^T[S]\{\ddot{\delta}_u\} \tag{33}$$

Thus, (32) leads to:

$$[K_f]\{\delta_p\} + [M_f]\{\ddot{\delta}_p\} - \rho[S]\{\ddot{\delta}_u\} = \{0\} \tag{34}$$

Also, the equation of motion of the elastic shell may be written in the form:

$$[M]\{\ddot{\delta}_u\} + [K]\{\delta_u\} = \{\delta_F\} \tag{35}$$

where $\{\delta_F\}$ denotes the generalized force vector at (z, 0) which may be expressed as

$$\{\delta_F\} = \{\delta_{F_e}\} + \{\delta_{F_p}\} \tag{36}$$

where $\{\delta_{F_e}\}$ represents external nodal forces including the static pressure of the liquid and $\{\delta_{F_p}\}$ represents nodal forces exerted on the shell arising from oscillations of the liquid. Also, [M] and [K] are the shell mass and stiffness matrices corresponding to a prescribed circumferential harmonic number m.

Free Vibrations of the Coupled System

Since we are interested in the free vibrations of the shell about the static equilibrium configuration (35) yields:

$$[M]\{\ddot{\delta}_u\} + [K]\{\delta_u\} = \{\delta_{F_p}\} \tag{37}$$

The generalized force vector corresponding to the dynamic pressure p_m on the inner surface of the shell is given by [5]:

$$\{\delta_{F_p}\} = -\pi R \int_z [N_w]^T [N]\{\delta_p\}\, dz$$

$$= -[S]^T\{\delta_p\}$$

$$\therefore \quad [M]\{\ddot{\delta}_u\} + [K]\{\delta_u\} + [S]^T\{\delta_p\} = 0 \qquad (38)$$

Thus, the free vibrations of the coupled liquid-elastic tank system may be expressed in the form:

$$\left[\begin{array}{c|c} M & 0 \\ \hline -\rho_f S & M_f \end{array}\right] \left\{\begin{array}{c} \ddot{\delta}_u \\ \hline \ddot{\delta}_p \end{array}\right\} + \left[\begin{array}{c|c} K & S^T \\ \hline 0 & k_f \end{array}\right] \left\{\begin{array}{c} \delta_u \\ \hline \delta_p \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \qquad (39)$$

Let us redefine the mass and stiffness matrices of the liquid as:

$$M_f = \frac{1}{\rho_f} M_f \qquad (40)$$

$$K_f = \frac{1}{\rho_f} K_f$$

Then, division of the second set of equations in (35) by ρ_f yields:

$$\left[\begin{array}{c|c} M & 0 \\ \hline -S & M_f \end{array}\right] \left\{\begin{array}{c} \ddot{\delta}_u \\ \hline \ddot{\delta}_p \end{array}\right\} + \left[\begin{array}{c|c} K & S^T \\ \hline 0 & K_f \end{array}\right] \left\{\begin{array}{c} \delta_u \\ \hline \delta_p \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \qquad (41)$$

These system matrices are nonsymmetric and extraction of eigenvalues and modes becomes extremely difficult, particularly when very large size matrices are concerned. In view of these serious numerical difficulties, let us adopt the approximation suggested by Hsiung and Weingarten [3] which consists of neglecting the free surface boundary condition (5). This simplification implies that (a) the liquid mass matrix $[M_f]$ corresponding to the free surface potential energy vanishes, and (b) the free surface pressure is zero. It is to be noted that in the present investigation the free surface condition was evaluated at the mean liquid level. Thus, the degrees of freedom corresponding to the free surface are constrained and can be omitted. Because of (a), we immediately have:

$$\begin{aligned} -[S]\{\ddot{\delta}_u\} + [K_f]\{\delta_p\} &= 0 \\ \{\delta_p\} &= [K_f]^{-1}[S]\{\ddot{\delta}_u\} \end{aligned} \qquad (42)$$

Thus:

$$\begin{aligned} [M]\{\ddot{\delta}_u\} + [K]\{\delta_u\} + [S]^T\{\delta_p\} &= \{0\} \\ \left[[M] + [S]^T[K_f]^{-1}[S]\right]\{\ddot{\delta}_u\} + [K]\{\delta_u\} &= \{0\} \end{aligned} \qquad (43)$$

This means that the shell mass matrix is augmented by an added mass matrix:

$$[ADM] = [S]^T[K_f]^{-1}[S] \qquad (44)$$

For the case of free vibrations of the system

$$\{\ddot{\delta}_u\} = -\omega^2 \{\delta_u\} \tag{45}$$

where ω is the natural frequency of the coupled system and the equation for eigenvalues is:

$$-\omega^2 [M + ADM] \{\delta_u\} + [K]\{\delta_u\} = \{0\} \tag{46}$$

The problem of the slab-supported partially-filled liquid storage container subject to seismic excitation of the base slab will thus lend itself to the response analysis detailed in [1] for the empty container provided that the shell mass matrix in [1] is replaced by the augmented mass matrix defined in (44) and (46). Details of this will be presented subsequently.

If one neglects the shell kinetic energy in comparison to the much larger kinetic energy of the liquid, the shell mass matrix [M] drops out and the problem reduces to:

$$-\omega^2 \left[[M_f] + [S][K]^{-1}[S]^T \right] \{\delta_p\} + [K_f]\{\delta_p\} = 0 \tag{47}$$

Numerical results obtained using this approach should agree quite closely with those found for a rigid tank. However, it is simpler to use a more direct analysis of liquid motion in a rigid tank, instead of employing (47).

In summary, the response of the coupled liquid-elastic tank system can be determined through superposition of the motions of the shell and the liquid found through neglect of free surface conditions, together with oscillation of the liquid in a rigid tank. For the range of geometries considered, results for natural frequencies of free vibration obtained on this basis agreed very well with those found through an entirely analytical (non-finite element) approach [6].

Response of the Coupled System to Base Excitation

The imposition of support displacements is solved for by partitioning the shell generalized displacement vector $\{\delta_u\}$ into components $\{\delta_{ub}\}$ associated with the known support displacements, with all other components being associated with the off-base nodes. Thus, the general equation of motion is written as:

$$[M + ADM]\{\ddot{\delta}_u\} + [K]\{\delta_u\} = \{\delta_{F_e}\} \tag{48}$$

where $\{\delta_{F_e}\}$ is the external generalized nodal force vector. It should be pointed out that the static liquid pressure forces are excluded from $\{\delta_{F_e}\}$ as mentioned in the discussion of (37). Also, the liquid dynamic pressure forces are excluded since the augmented mass matrix accounts for them. Thus, for the case of response under base excitations only, the governing equation (35) yields:

$$\left[\begin{array}{c|c} M_{bb} & M_b^T \\ \hline M_b & M \end{array}\right] \left\{\begin{array}{c} \ddot{\delta}_{ubt} \\ \hline \ddot{\delta}_{ut} \end{array}\right\} + \left[\begin{array}{c|c} K_{bb} & K_b^T \\ \hline K_b & K \end{array}\right] \left\{\begin{array}{c} \delta_{ubt} \\ \hline \delta_{ut} \end{array}\right\} = \left\{\begin{array}{c} \delta_{F_b} \\ \hline 0 \end{array}\right\} \tag{49}$$

which is identical with Equation (2) in [1]. Here, $\{\delta_{ubt}\}$ and $\{\ddot{\delta}_{ubt}\}$ are the known support displacements and accelerations, respectively, and

$\{\delta_{ut}\}$ and $\{\ddot{\delta}_{ut}\}$ are the total off-base displacements and accelerations corresponding to this response analysis.

All elements in the top line of Equation (49) pertain to base node parameters. Thus, K_{bb} and M_{bb} denote forces at base nodes due to unit displacements at the base nodes and the superscript T, of course, denotes matrix transpose. K_b and M_b in the bottom row are coupling effects between the base nodes and the other (non-base) nodes. All other elements in the bottom row of Equation (49) pertain to non-base nodal parameters. Thus, K and M are redefined to represent stiffness and mass matrices of all non-base nodes.

At any time, the displacement vectors of the non-base nodes can be considered as a summation of two vectors. The first vector $\{U_s\}$ is a function of the instantaneous ground displacement, thus it can be called static. The second vector $\{U_d\}$ is a function of the ground acceleration history, thus it is termed dynamic.

This approach furnishes a suitable method to reduce the equations of motion to the familiar form of forced vibrations:

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{F\} \quad (50)$$

Thus,

$$\{\delta_{u_t}\} = \{U_s\} + \{U_d\} \quad (51)$$

The equations of motion are:

$$\left[\begin{array}{c|c} M_{bb} & M_b^T \\ \hline M_b & M \end{array}\right] \left\{\begin{array}{c} \ddot{\delta}_{ubt} \\ \hline \ddot{U}_s + \ddot{U}_d \end{array}\right\} + \left[\begin{array}{c|c} K_{bb} & K_b^T \\ \hline K_b & K \end{array}\right] \left\{\begin{array}{c} \delta_{ubt} \\ \hline U_s + U_d \end{array}\right\} = \left\{\begin{array}{c} F_b \\ \hline 0 \end{array}\right\} \qquad (52)$$

The equations of the off-base elements are

$$[M_b]\{\ddot{\delta}_{ubt}\} + [M]\{\ddot{U}_s\} + [M]\{\ddot{U}_d\} + [K_b]\{\delta_{ubt}\} + [K]\{U_s\} + [K]\{U_d\} = 0 \qquad (53)$$

Now it is attractive to define U_s as a displacement vector so that when it is associated with the ground displacement vector U_{bt} the resulting motion of the structure corresponds to no internal strain energy. Hereafter, δ_{ubt} will be denoted by U_{bt} for brevity. This condition implies that:

$$[K_b]\{U_{bt}\} + [K]\{U_s\} = 0 \qquad (54)$$

In other words, the vector $\{U_s\}$ is developed through rigid body displacements consistent with $\{U_{bt}\}$. Thus, from (54)

$$\{U_s\} = -[K]^{-1}[K_b]\, U_{bt}$$

This phenomena has also been demonstrated numerically and the resulting static displacement U_s is nothing but a series of U_{bt} or

$$\{U_s\} = \begin{Bmatrix} U_{s1} \\ \hline U_{s2} \\ \hline U_{s3} \\ \hline \vdots \\ \hline U_{sN} \end{Bmatrix} = \begin{Bmatrix} U_{bt} \\ \hline U_{bt} \\ \hline U_{bt} \\ \hline \\ \hline U_{bt} \end{Bmatrix} \tag{55}$$

where N is the total number of elements and $\{U_{si}\}$ is the displacement vector of node i = $\{U_{bt}\}$ for all values of i and $\{U_{bt}\}$ is a (4 x 1) vector representing the axial, tangential, and radial displacements as well as the rotation of the generator at the base.

Thus, the off-base node equations yield

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = -[M_b]\{\ddot{U}_{bt}\} - [M]\{\ddot{U}_s\}$$

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = -[[M_b] - [M][K]^{-1}[K_b]]\{U_{bt}\}$$

$$= [\text{effective mass matrix}]\ \{\ddot{U}_{bt}\}$$

$$= [M_{eff}]\ \{\ddot{U}_{bt}\} \tag{56}$$

It should be pointed out that for most practical tank dimensions the driving forces developed due to the mass $[M][K][K_b]$ are much larger than those developed by $[M_b]$. This has been demonstrated numerically.

The ground acceleration vector $\ddot{U}_{bt}$ will be proved to be equal to:

$$\ddot{U}_g(t) \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix}$$

where $\ddot{U}_g(t)$ is the ground acceleration amplitude at time t.

Since the base of the tank is excited by a ground displacement and acceleration acting in its plane and in the constant direction $\theta = 0$, no axial acceleration component develops and the ground acceleration will be completely defined by its amplitude value $\ddot{U}_g(t)$:

$$\ddot{U}_g(t) = \text{Peak} \cdot f(t) \tag{57}$$

The peak is an acceleration value independent of time and f(t) is a non-dimensional function of time.

The associated base-node displacement vector U_{bt} is derived by use of Fig. 3, viz:

$$u(o,\theta,t) = 0$$

$$v(o,\theta,t) = -U_g(t) \cdot \sin\theta = -\text{Peak} \cdot f(t) \cdot \sin\theta$$

$$w(o,\theta,t) = -U_g(t) \cdot \cos\theta = +\text{Peak} \cdot f(t) \cdot \cos\theta$$

$$\frac{\partial w}{\partial z}(o,\theta,t) = 0.0 \tag{58}$$

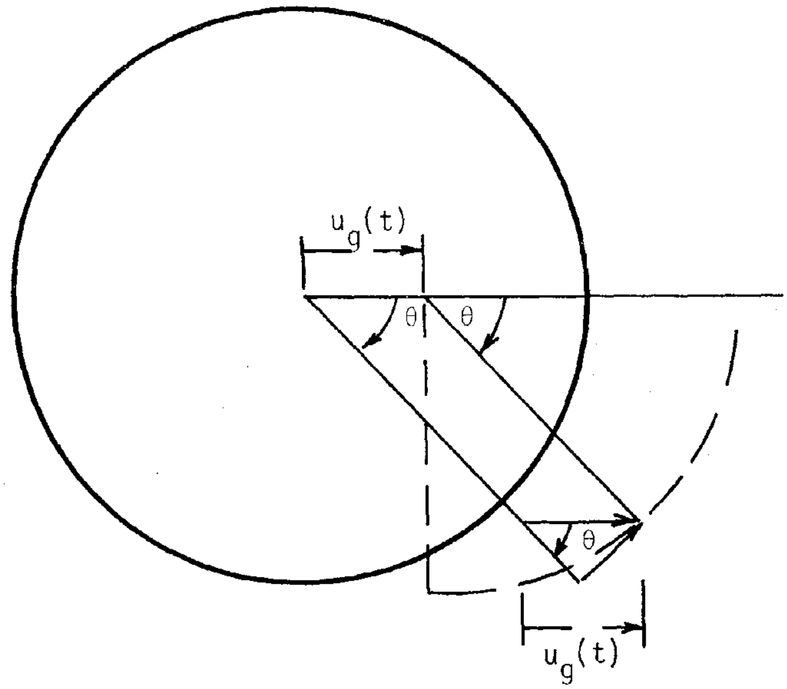


FIGURE 3

Since the excitation function is described in the previous form to be associated with m = 1, obviously only the first circumferential harmonic will be excited, and thus the vibration of the tank can be prescribed by super-position of certain contributions of different axial modes corresponding to m = 1 only (see Appendix A, in [1], for assumed form of loads and displacements).

$$\ddot{U}_b(t) = \text{Peak} \cdot f(t) \cdot \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix} \qquad (59)$$

Let

$$\{P_{eff}\} = \text{Peak} \cdot [M_{eff}] \cdot \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix}$$

$\therefore$ The equations of motion reduce to:

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{P_{eff}\} \cdot f(t)$$

which is the desired form of forced vibration to which the modal analysis technique will be applied.

Modal Analysis Solutions

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{P_{eff}\} \cdot f(t)$$

Let

$$\{\ddot{U}_d\} = [X]\{\ddot{A}\}$$

$$\{U_d\} = [X]\{A\}$$

$\therefore$ [X] is the rectangular mode matrix formed as a set of mode vectors (n x k) where

n = number of degrees of freedom of the non-base elements

k = number of modes considered in the analysis

{A}= mode participation factor vector = k x 1

$$\therefore [M][X]\{\ddot{A}\} + [K][X]\{A\} = \{P_{eff}\} \cdot f(t)$$

$$\{A\} \equiv \{A(t)\} \ ; \quad \{\ddot{A}\} \equiv \{\ddot{A}(t)\}$$

$$f(t) = \frac{\ddot{U}_g(t)}{\text{Peak}}$$

Premultiply by $[X]^T$; (k x n)

$$\therefore\ [X]^T[M][X]\{\ddot{A}\} + [X]^T[K][X]\{A\} = [X]^T\{P_{eff}\} \cdot f(t) = \{GP\} \cdot f(t)$$

Now, use the orthogonality condition:

$$\{X_n\}^T[M]\{X_k\} = 0 \qquad\qquad k \neq n$$

Obviously the resulting matrix $[X]^T[M][X] = [GM]$ is a diagonal matrix since the (generalized k x k mass matrix) nonvanishing terms are only $[X_n]^T[M][X_n] = GM(n,n)$.

The same concept holds for $[X]^T[K][X] = [GS] \equiv$ diagonal matrix where Ω^2 is the squared eigenvalue diagonal matrix $= [\Omega^2][GM]$:

$$[\Omega^2] = \begin{bmatrix} \omega_1^2 & & & & & & \\ & \omega_2^2 & & & & & \\ & & \omega_3^2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \omega_n^2 \end{bmatrix}_{k \times k}$$

Thus, GM, as well as GS can be considered as vectors,

$$\begin{bmatrix} GM(1,1) \\ GM(2,2) \\ GM(3,3) \\ GM(k,k) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} GM(1,1) & \omega_1^2 \\ GM(2,2) & \omega_2^2 \\ \vdots & \\ GM(k,k) & \omega_n^2 \end{bmatrix} \quad \text{respectively.}$$

Thus, k independent equations result:

$$GM(I,I) \cdot \ddot{A}(I) + \omega(I) \cdot \omega(I) \cdot A(I) \cdot GM(I,I) = GP(I) \cdot f(t)$$

where I refers to the mode number.

$$\ddot{A}(I) + \omega^2(I) \cdot A(I) = \frac{GP(I)}{GM(I,I)} \cdot f(t)$$

which are the equations of k independent lumped masses each representing the participation of the corresponding I-th mode.

Now, A(I) can be found using Duhamel integration to account for the initial conditions (just before the instant t), i.e. to consider the whole acceleration record imposed on the structure, viz:

$$A(I) = \frac{GP(I)}{GM(I,I) \cdot \omega(I)} \cdot \int_0^t f(\tau) \cdot \sin \omega(t-\tau)d\tau$$

$$= \frac{PIN(I)}{GM(I,I) \cdot \omega(I)}$$

$$\text{where } PIN(I) = (\int_0^t f(\tau) \sin \omega(t-\tau)d\tau) \cdot GP(I)$$

$$\therefore \ddot{A}(I) = \frac{GP(I)}{GM(I,I)} f(t) - (\omega(I))^2 \cdot A(I)$$

Now from the original equations of motion the displacement and acceleration nodal vectors are determined:

$$\{U_d\} = [X]\{A\}$$
$$\{\ddot{U}_d\} = [X]\{\ddot{A}\}$$

The accuracy of the modal analysis approach depends on the number of modes involved in the superposition. The latter depends on how close or scarce the natural frequencies of the structure are spaced.

The accuracy of the method can be examined through the satisfaction of the original external equilibrium equation:

$$[M]\ \{\ddot{U}_d\} + [K]\ \{U_d\} = \{P_{eff}\} \cdot f(t)$$

For the structure considered, it was found that the superposition of a few modes offered only a crude approximation since the external equilibrium equation failed to be satisfied by as much as thirty percent. Use of ten modes reduced the maximum discrepancy to about ten percent.

Reactions of the Base

From the equations of base vibrations:

$$[M_{bb}|M_b^T] \left\{ \frac{\ddot{U}_{bt}}{\ddot{U}_s + \ddot{U}_d} \right\} + [K_{bb}|K_b^T] \left\{ \frac{U_{bt}}{U_s + U_d} \right\} = \left\{ {}^{\delta}F_b \right\}$$

Now, $\{\ddot{U}_s\}$ and $\{U_s\}$ were proved to be equal to:

$$\{U_s\} = \begin{bmatrix} I \\ I \\ I \\ I \\ \vdots \end{bmatrix} \{U_{bt}\} \text{ and } \{\ddot{U}_s\} = \begin{bmatrix} I \\ I \\ I \\ I \\ I \\ \vdots \end{bmatrix} \{\ddot{U}_{bt}\}$$

where [I] is a (4x4) identity matrix, N/4 of which form the relating matrix between the resulting static non-base node displacements and the base node imposed displacements. Also N = number of non-base node degrees of freedom and since M_b^T contains nonzero elements only in the first four columns $M_b^T \cdot \ddot{U}_s$ can be expressed as:

$$[M_b^T]'[I]\{\ddot{U}_{bt}\}$$

where $[M_b^T]'$ is the 4 x 4 matrix including the nonzero elements

$$[M_{bb} + M_b^T{'}I]\{\ddot{U}_{bt}\} + [M_b^T]'\{\ddot{U}_d\} + [K_{bb} + K_b^T{'}I]\{U_{bt}\} + [K_b{}^T]\{U_d\} = \{F_b\}$$

but $[K_{bb} + K_b^T{'}I]\{U_{bt}\} = 0$

$$\therefore \quad \{\delta_{F_b}\} = [M_{bb} + M_b^T{'}I]\{\ddot{U}_{bt}\} + [M_b^T]\{\ddot{U}_d\} + [K_b^T]\{U_d\} \qquad (60)$$

Of course, the most significant part of the base force is attributed to the displacements of the non-base nodes, i.e. $[K_b]^T\{U_d\}$.

Liquid Oscillations in a Rigid Cylindrical Container

The fluid dynamic pressure arising due to liquid motion in a rigid cylindrical tank will be governed by a special case of (34). Since the rigid container degrees of freedom $\{\delta_u\}$ are restricted, $\{\delta_u\}$ can obviously be omitted and the governing equations yield:

$$[K_f]\{\delta_p\} + [M_f]\{\ddot{\delta}_p\} = \{0\} \qquad (61)$$

Although the fluid "mass" matrix $[M_f]$ is defined to be N_{DFF} x N_{DFF}, (where N_{DFF} is the total number of degrees of freedom of the liquid), the nonzero elements are those corresponding to the free surface generalized pressure vector only. A matrix condensation approach is employed to minimize the computer storage area as follows:

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{12}^T & K_{22} \end{array}\right] \left\{\begin{array}{c} \delta_{p_1} \\ \hline \delta_{p_2} \end{array}\right\} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{22} \end{array}\right] \left\{\begin{array}{c} \ddot{\delta}_{p_1} \\ \hline \ddot{\delta}_{p_2} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\} \qquad (62)$$

where the second set of equations corresponds to the free surface nodes (n_2 in number) and the first set corresponds to the remainder of the liquid nodes (n_1). This leads directly to:

$$K_{11}\,\delta_{p_1} + K_{12}\,\delta_{p_2} = 0$$

$$\delta_{p_1} = -[K_{11}]^{-1}\,[K_{12}]\{\delta_{p_2}\} \qquad (63)$$

Substituting this into the second set of equations (61) yields:

$$[-K_{12}^{T}\, K_{11}^{-1}\, K_{12} + K_{22}]\, \delta_{p_2} + M_{22}\ddot{\delta}_{p_2} = 0 \qquad (64)$$

$$[K_{cond}]\, \{\delta_{p_2}\} + [M_{cond}]\, \{\ddot{\delta}_{p_2}\} = 0 \qquad (65)$$

where

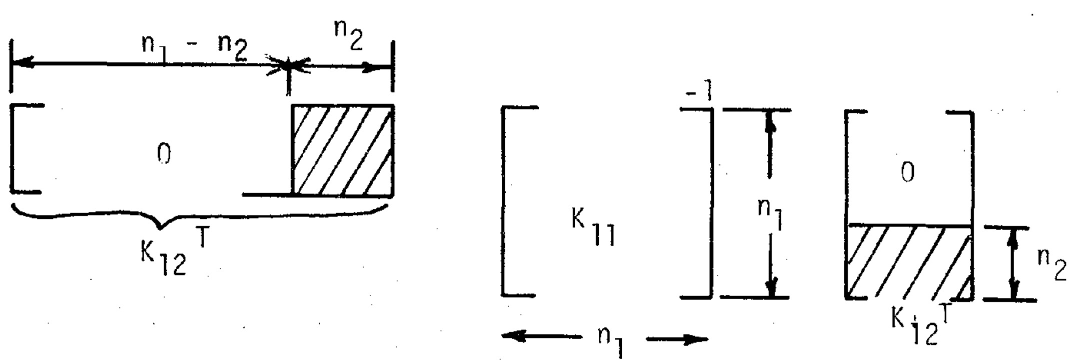
$$[K_{cond}] = [K_{22}] - [K_{12}]^{T}[K_{11}]^{-1}[K_{12}] \equiv (n_2 \times n_2) \qquad (66)$$

= The condensed stiffness matrix

$$[M_{cond}] = M_{22} \equiv (n_2 \times n_2) \qquad (67)$$

= The condensed mass matrix

The submatrix K_{12} $(n_1 \times n_2)$ also has a significantly smaller nonzero submatrix $\equiv n_2 \times n_2$ and the second matrix of (63) can be efficiently evaluated by use of this fact as follows:



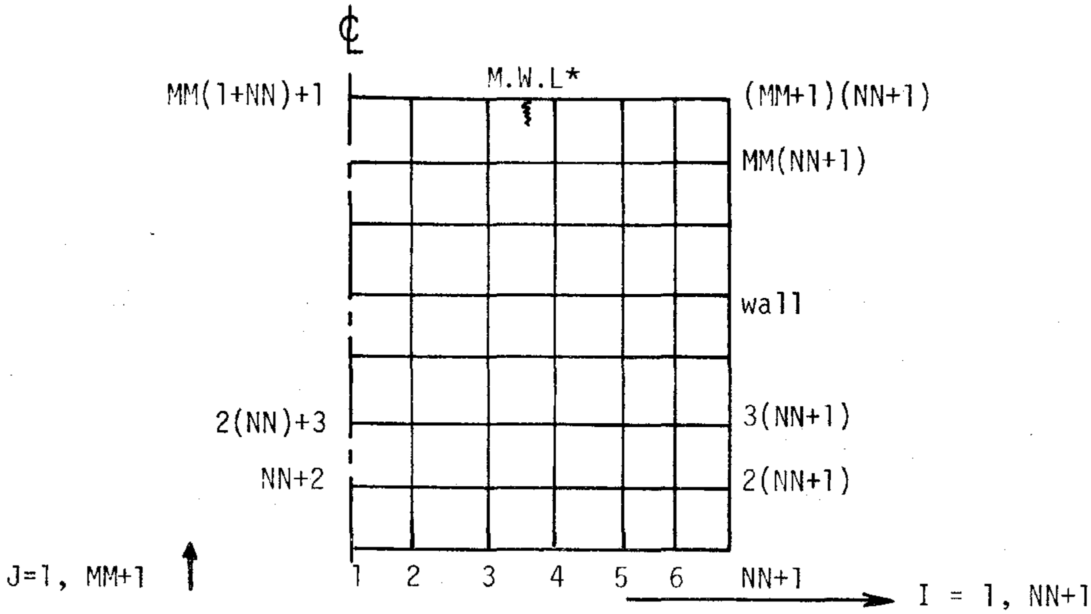
The direct inversion of $[K_{11}]$ is avoided and the last $(n_2 \times n_2)$ matrix resulting from the multiplication $K_{11}^{-1}K_{12}$ is the only portion treated, through the use of Gaussian elimination back substitution [4]. This, in fact, corresponds to the generalized nodal pressure vector $\{\delta p_3\}$ of the row immediately below the free surface.

Therefore, in the assembly of the stiffness matrix three submatrices are considered: $K_{11} = n_1 \times n_1$; $K_{12} = n_1 \times n_2$ (non-zero terms are $n_2 \times n_2$), and $K_{22} = n_2 \times n_2$. In the assembly of the mass matrix only the $M_{22} = n_2 \times n_2$ matrix is formulated.

The liquid matrices numbering schemes (for a rigid tank) are given in Figures 4a through 4d.

FIGURE 4a

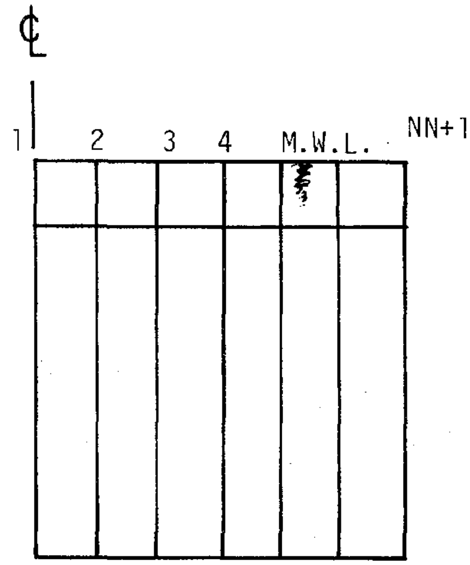
Liquid Degrees of Freedom numbering scheme pertinent to the stiffness matrix generated in program RIGID for symmetric harmonic modes.



* mean water level

FIGURE 4b

Liquid Degrees of Freedom numbering scheme pertinent to the condensed mass matrix generated in program RIGID for symmetric harmonic modes.



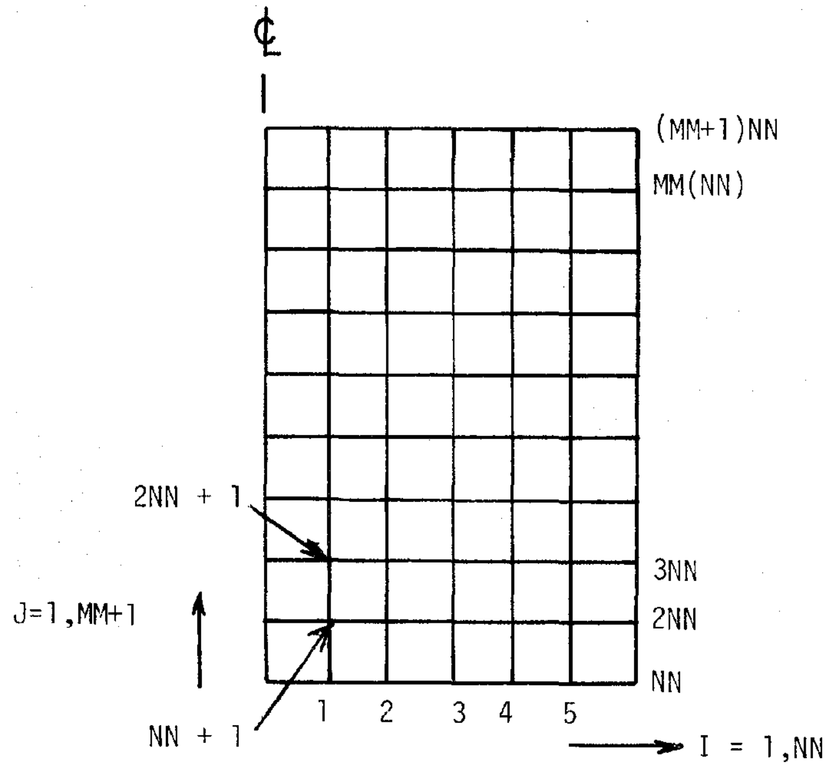


FIGURE 4c

Liquid Degrees of Freedom numbering scheme pertinent to the stiffness matrix generated in program RIGID for asymmetric harmonic modes.

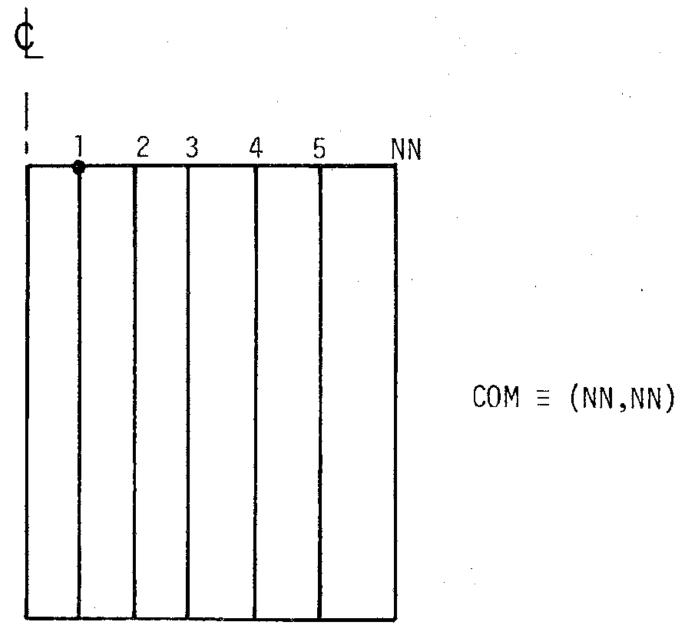


FIGURE 4d

Liquid Degrees of Freedom numbering scheme pertinent to the condensed mass matrix generated in program RIGID for asymmetric harmonic modes.

COMPUTER IMPLEMENTATION

Computer Programs

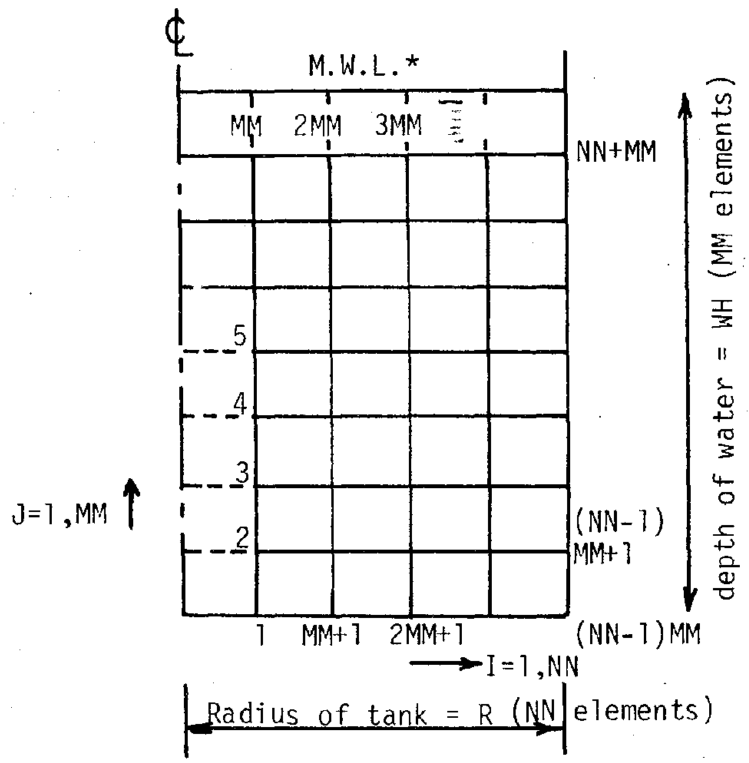
Three separate programs were developed in the present work. The first, program RIGID, determines liquid oscillation natural frequencies and associated mode shapes in a rigid circular cylindrical container fixed to a rigid base. In the early stages of this work this program served as a check on the formulation of the liquid "mass" and "stiffness" matrices and thus on the validity of the entire liquid idealization process. This is because in many cases, the data obtained were in good agreement with existing work involving rigid containers.

The second program, COUPLE, is employed to investigate natural frequencies and associated mode shapes of the coupled liquid-elastic tank system described by Equation (46). To this end the first main program described in [1] (MAIN) was modified slightly so as to correspond to two sets of ring-shaped finite elements representing the cylindrical tank. The first set of elements corresponds to the lower (wetted) surface of the tank and the second set to the portion of the tank above the liquid level (dry). The program corresponding to this representation is henceforth termed SHELL. A single run string was prepared of COUPLE and SHELL so as to be able to investigate the coupled liquid-elastic tank system. This also serves to retrieve the "added mass matrix" stored on a disc file by program COUPLE and to then add its terms to the corresponding shell mass matrix terms.

Program COUPLE carries out the following operations: a) It devises a numbering scheme for the liquid finite element mesh. This is accomplished in subroutine FLGEN which requires as input the number of liquid finite elements NN along the tank radius in a single row, the number of liquid finite elements MM in a single column, and the specified number of circumferential harmonics, m. This is illustrated in Figures 5a and 5b.

FIGURE 5a

Liquid Degrees of Freedom numbering scheme generated in program COUPLE for asymmetric harmonics pertinent to the liquid "stiffness" matrix. (m = 1, 3, 5, ...)



* mean water level

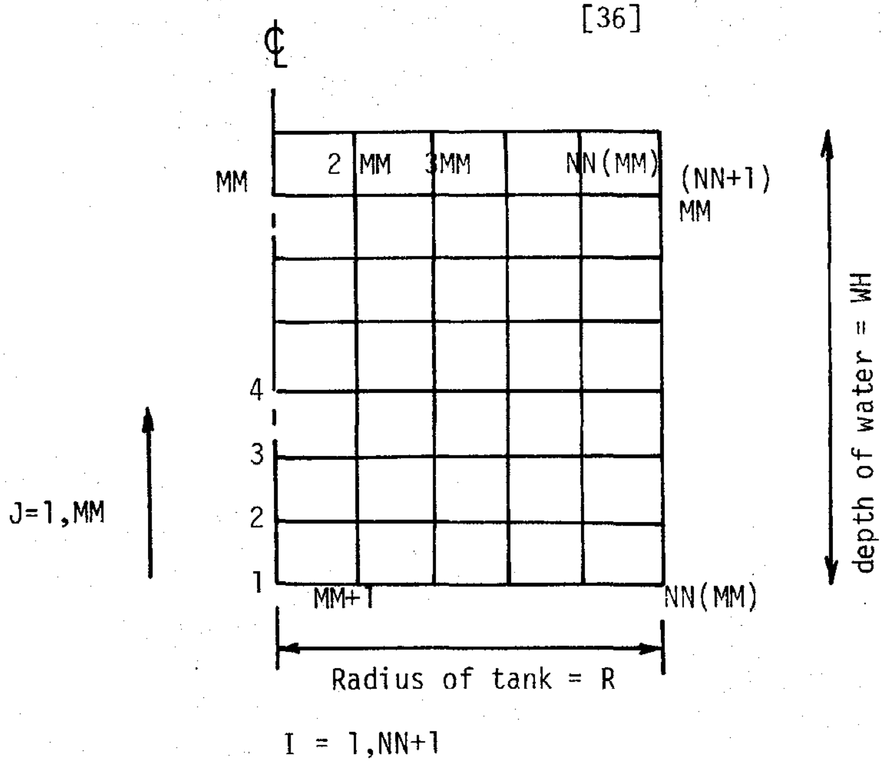


FIGURE 5b

Liquid Degrees of Freedom numbering scheme generated in program COUPLE for symmetric harmonics pertinent to the liquid "stiffness" matrix (m= 0, 2, 4, ...)

b) It evaluates a set of different liquid stiffness and coupling element matrices $[K_e]$ and $[S_e]$, each corresponding to a column of elements in the liquid idealization scheme. It is assumed that the liquid has been discretized into equal rectangular areas. This is accomplished in subroutines FSTIFF and FFORCE. c) It assembles the liquid stiffness matrix [K] in accordance with the numbering scheme mentioned in (a) above into a half-banded matrix stored in a linear array so as to minimize core allocation. The condensed coupling matrix is also assembled into an (MM, 2MM) matrix, $[\bar{S}]$. d) It evaluates the liquid added mass matrix defined in (44) and stores it on a disc file to be retrieved by SHELL.

The third program, RESPONSE, which follows after SHELL in the run string, accomplishes the following:

e) It evaluates the generalized forces developed at the tank wall nodes due to a unit ground acceleration in the horizontal direction.

f) It transforms the system properties into modal coordinates. That is, the generalized mass vector GM and the generalized force vector GP are evaluated. These operations are performed in the first section, PARTI. g) It retrieves in PARTII the ground acceleration record ACC previously generated utilizing program PSEQGN available through the National Information Service-Earthquake Engineering - Computer Program Applications, and which was stored on a disc file [7]. To improve the accuracy of the response computation the total time history under consideration is arbitrarily divided into smaller time intervals by "guiding" time stations, the modal velocities $\{\dot{A}\}$ and displacements $\{A\}$ of which are first determined independently in subroutine CONTROL. CONTROL calls subroutine RES at each time station to evaluate the Duhamel integral of the previous acceleration record. The vectors $\{A\}$ and $\{A\}$ are stored in the core array to be used as illustrated below:

h) It evaluates the specified nodes generalized displacements and prints the response history and stores it in disc files to be retrieved for automatic plotting purposes. The responses of the specified degrees-of-freedom designated as ND1, ND2, and ND3 are stored on tapes number 4, 5, and 6 respectively. These degrees-of-freedom are explained in detail on page (60) together with Figures 17 and 19.

Knowing the response history at any degree-of-freedom, the corresponding stresses can be found from the program RESP given in [1]. This is with regard to the internal forces developed, the reactions at the tank base, and the force equilibrium check if so desired.

Nature and Size of System Matrices

The original sizes of the system matrices are indicated in Equation (46) together with the numbering schemes shown in Figures 4 and 5 to be indicated below. For brevity, the following programming symbols were employed:

δ_u = NDFST x 1

δ_p = NDFF x 1

M and K = NDFST x NDFST

M_F and K_F = NDFF x NDFF

S = NDFF x NDFST

where

NN = number of liquid element in one row along the tank radius

MM = number of liquid elements in one column along the tank generator

MMT = total number of shell ring elements

NDFST = total number of shell degrees of freedom
= 4(MMT + 1)

NDFF = I.J

where

I = NN for asymmetric harmonic modes
= (NN + 1) symmetric harmonic modes

J = MM for the coupled case with zero pressure assumption at the free surface
= (MM + 1) for the fluid oscillation in a rigid cylindrical container

It is evident that these matrices can be *drastically* reduced in size if intelligently partitioned to separate the non-zero sub-matrices from the zero blocks. This approach is indeed essential to utilize the computer core storage area most efficiently. It also obviously validates the employment of finer system idealization schemes with the available core allocation.

The coupling matrix S originally denoted to be (NDFF x NDFST) contains non-zero terms corresponding to the fluid shell interface Σ only. Moreover, the fluid pressure is not directly affected by the shell nodal displacements above the water level. The axial and tangential displacements of the shell wetted surface also do not contribute to changes in the fluid pressure. Thus, the coupling is attributed only to the radial displacement w and the slope of the generater $\frac{\delta w}{\delta z}$ corresponding to the nodes at the wetted surface. This follows directly from the derivation of the coupling matrix as previously discussed.

Therefore, a condensed coupling matrix $[\bar{S}]$ that contains no zero blocks is employed, in which the number of rows diminishes from NDFF in [S] to MM, and the number of columns diminishes from NDFST to 2MM. $[\bar{S}]$ and $[\bar{S}]^T$ are shown by the shaded areas in Figure 6.

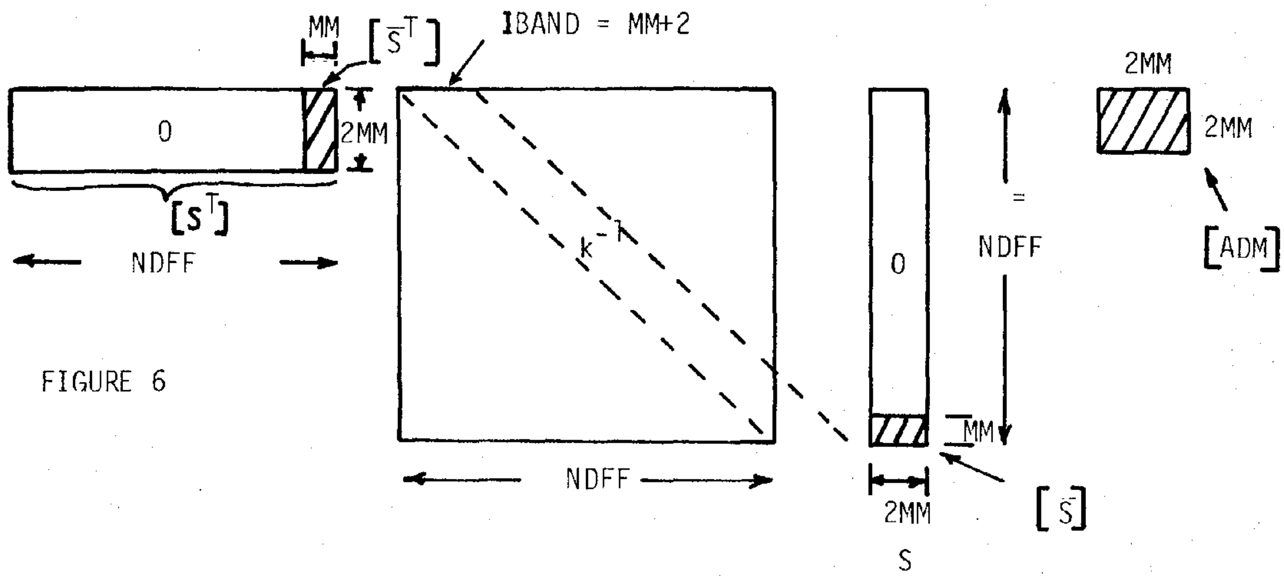


FIGURE 6

A numbering scheme is devised in program COUPLE to assemble the fluid matrices with special care paid to minimize the computation time as well as the computer storage area. The fluid interface degrees of freedom were numbered to lie in the end of δ_p as shown in Figures (5a) and (5b) so that an inversion Gaussian elimination back substitution technique would yield the desired multiplication $K^{-1}S$ into a (MM x 2MM) matrix only. The omitted upper portion of the resulting matrix contains non-zero terms, yet when premultiplied by S^T, it multiplies by a zero block and its contribution drops out.

Thus, the added mass matrix [ADM] developed by carrying out the previous operations is confined to a [2MM x 2MM] area. This is represented schematically in Figure 6.

The liquid and shell numbering schemes pertinent to $[\bar{S}]$ are given in Figures 7a and 7b respectively. It should be pointed out that the liquid stiffness matrix is half-banded and is assembled into a linear array to optimize the storage area implementation.

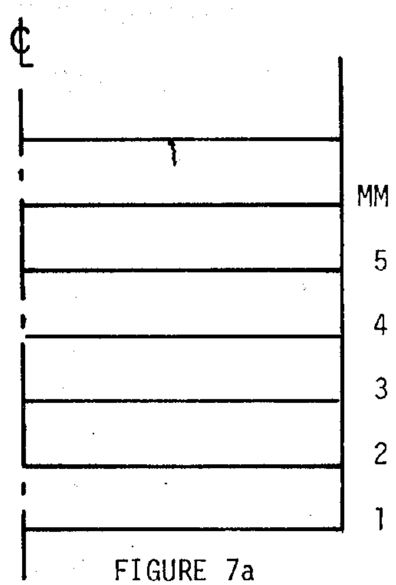


FIGURE 7a
Liquid Degrees of Freedom numbering pertinent to the condensed coupling matrix $[\bar{S}]$ for symmetric or asymmetric harmonics.

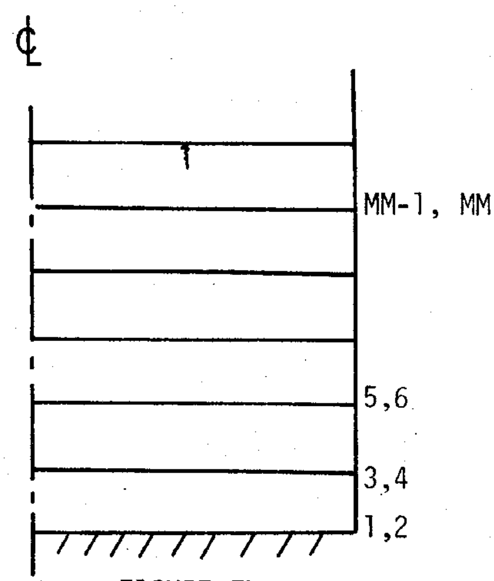


FIGURE 7b
Shell Degrees of Freedom numbering pertinent to the condensed coupling matrix $[\bar{S}]$ for symmetric or asymmetric harmonics.

EXAMPLES

1. Free Vibrations of Completely Filled Rigid Tank

Let us consider the slab-supported tank discussed in [1]. This tank is 40 feet high and 60 feet in radius, with rigid wall and slab. We seek to determine the natural frequencies and associated mode shapes when the tank is completely filled with water.

The computer program of Appendix B is utilized here. To use this program, one enters the following data:

CARD 1: DENF = liquid density = 0.9345×10^{-4} lb x sec^2/in^4 (though the result of the free vibration analysis is independent of ρ_f.)

R = tank radius = 720 inches

WH = depth of water = 480 inches

CARD 2: NN = number of liquid elements in one row along tank radius = 20

MM = number of liquid elements in one column along tank height = 20

CARD 3: NSIN = number of circumferential wave patterns that analyst desires to investigate. If this is greater than unity, the program indicates the response for each wave pattern from one wave through increasing integral values to the specified number. Here, NSIN = 1.

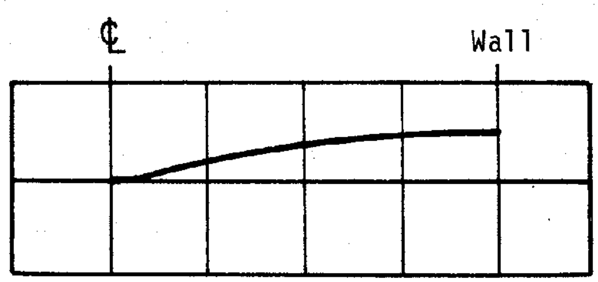
CARD 4: NMODE = number of axial waves under consideration = 5. (Printout indicates frequencies and free surface pressure vector for modes 1, 2, ... 5).

This completes all necessary input to the computer program.

The program output consists of liquid natural frequencies and free surface mode shapes. These natural frequencies are as follows:

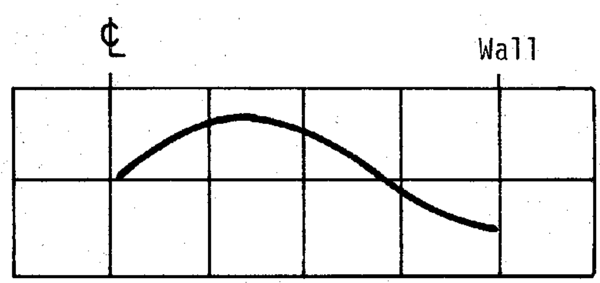
Axial Mode	Frequency (Hz)
1	0.15
2	0.27
3	0.34
4	0.40
5	0.46

Figures 8a through 8e show the liquid free surface corresponding to the plane $\theta = 0°$ for the first five axial modes whose frequencies are indicated above. The grid in these figures does not correspond to the finite element representation.



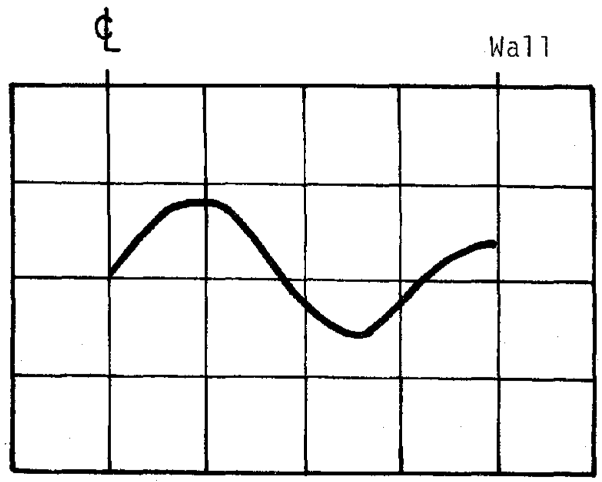
First Axial Mode

FIGURE 8a



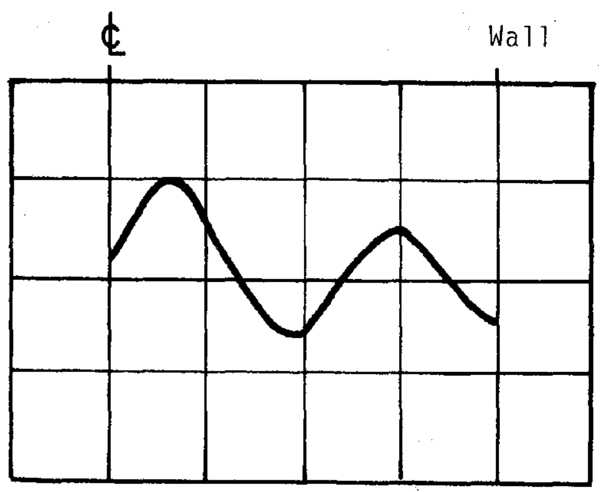
Second Axial Mode

FIGURE 8b



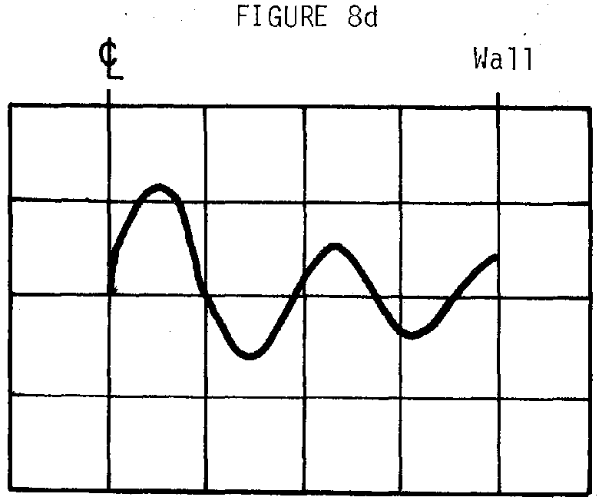
Third Axial Mode

FIGURE 8c



Fourth Axial Mode

FIGURE 8d



Fifth Axial Mode

FIGURE 8e

2. Free Vibrations of Completely Filled Elastic Tank

Let us consider the same tank discussed in the first example, but now with a steel wall one inch in thickness. We shall treat the elasticity of the tank wall. The tank contains water and we consider liquid depths of 25 percent, 40 percent, 60 percent, and 80 percent of the tank height, as well as the completely filled tank. The tank is clamped at the base and free at the top. We seek the natural frequencies and associated mode shapes of this system.

The computer program of Appendix C is utilized here. To employ this program for the case of the completely filled tank, one enters the following data pertinent to the liquid:

CARD 1: DENF = liquid density = 0.9345×10^{-4} lb. x sec^2/in^4

R = tank radius = 720 inches

WH = depth of water = 480 inches

CARD 2: NN = number of liquid elements in one row along tank radius = 20

MM = number of liquid elements in one column along tank height = 15

CARD 3: NHR = number of circumferential waves in pattern under consideration = 1.

Next, one enters the following data pertinent to the elastic tank:

CARD 4: UM = ρ = density of tank material = 0.733×10^{-3} lb x sec^2/in^4

EI = E = Young's modulus = 30×10^6 lb/in^2

PX = nu = Poisson's ratio = 0.3

CARD 5: R = tank radius = 720 inches

H = tank wall thickness = 1 inch

AL = tank altitude = 480 inches

CARD 6: NSIN = total number of circumferential wave patterns that analyst desires to investigate = 1 (Program C does not permit use of NSIN ≠ 1).

CARD 7: NELEM = number of ring-shaped finite elements representing the tank = 15

CARD 8: NELFS = number of shell finite elements corresponding to wetted surface = 15 (this must equal MM)

NELFR = number of shell finite elements corresponding to dry surface = 0 (obviously NELEM = NELFS + NELFR)

CARD 9: NMODE = number of axial waves under consideration = 10 (Printout indicates frequencies and displacements for modes 1, 2, ... 10).

CARD 10: NAT = number of circumferential waves in pattern under consideration (i.e. "instantaneous" number of circumferential waves) = 1. This number specifies which one of those patterns under NSIN is currently being investigated.

CARD 11: NBCAS = total number of cases involving different sets of boundary condidtions that analyst desires to investigate = 1 (The program listed in Appendix C does not permit use of NBCAS ≠ 1).

CARD 12: NBC = denotes boundary conditions at base and top of tank. First, enter CL if base is clamped, SM if base is simply supported. Next, enter CL if top is clamped, or SM if it is simply supported, FR if it is free. Do not introduce a space between the designations of these two boundary conditions.

This completes all necessary input to the computer program.

The program output consists of natural frequencies of the coupled liquid-elastic tank system together with mode shapes (along a generator). First, let us present results for the case of the tank *completely filled* with water. The first four natural frequencies are as follows:

Axial Mode	Frequency (Hz)
1	6.13
2	11.15
3	15.11
4	18.16

The program output also gives, for each of the above natural frequencies, the relative (normalized) displacements u, v, and w together with the slope dw/dz tabulated in the form of columns (with these headings) immediately after printing of the natural frequency. In these displays of displacements and slope, the first (top) line represents tank displacements and slope at the junction of the tank with the rigid base slab (base node) and the last (bottom) line represents the corresponding quantities at the tank top. As an example, the third (axial) mode values (for the tank completely filled with water) are found to be:

Natural Frequency = 0.1511308361E + 02

U	V	W	DW/DZ
0.00000000	0.00000000	0.00000000	0.00000000
-0.00009561	-0.00008448	0.01212372	0.00009423
-0.00029760	-0.00012035	0.01345022	-0.00032918
-0.00044688	-0.00009971	0.00600128	-0.00041985
-0.00047471	-0.00005514	-0.00390791	-0.00031737
-0.00037576	-0.00003085	-0.01262239	-0.00010702
-0.00018602	-0.00006547	-0.01732715	0.00014151
0.00002988	-0.00017956	-0.01646437	0.00034772
0.00019939	-0.00036897	-0.01029515	0.00044386
0.00026589	-0.00060617	-0.00083018	0.00039842
0.00020741	-0.00084932	0.00883707	0.00022673
0.00004390	-0.00105603	0.01553055	-0.00001645
-0.00016781	-0.00119749	0.01678216	-0.00026764
-0.00034528	-0.00126946	0.01121180	-0.00045008
-0.00041540	-0.00129727	0.00326332	-0.00018410
-0.00041504	-0.00131450	-0.00012743	-0.00007478

Plots of u, v, w, and dw/dz for the first five axial modes appear in Figures 9 through 13 inclusive. In the interest of brevity corresponding plots for water depths other than completely filled are not presented here. The natural frequencies of the coupled liquid-elastic tank system are, however, tabulated in Table 1 for various liquid depths ranging from empty to completely filled. Corresponding numbers of finite elements employed are also indicated. These natural frequencies are also plotted in Figure 14. An example of the use of the program of Appendix C for a half-filled tank is given as Example 3.

The effect of finite element mesh size on the coupled natural frequencies (for the case of the completely filled tank only) is indicated in Figures 15 and 16. Figure 15 shows the effect of varying the number of elements in the direction of the generator while holding the number of elements (NN) in the direction of the tank radius constant and equal to 30. Similarly, Figure 16 shows the effect of varying the number of elements in the direction of the tank radius while holding the number of elements (MM) in the direction of the tank generator constant and equal to 20.

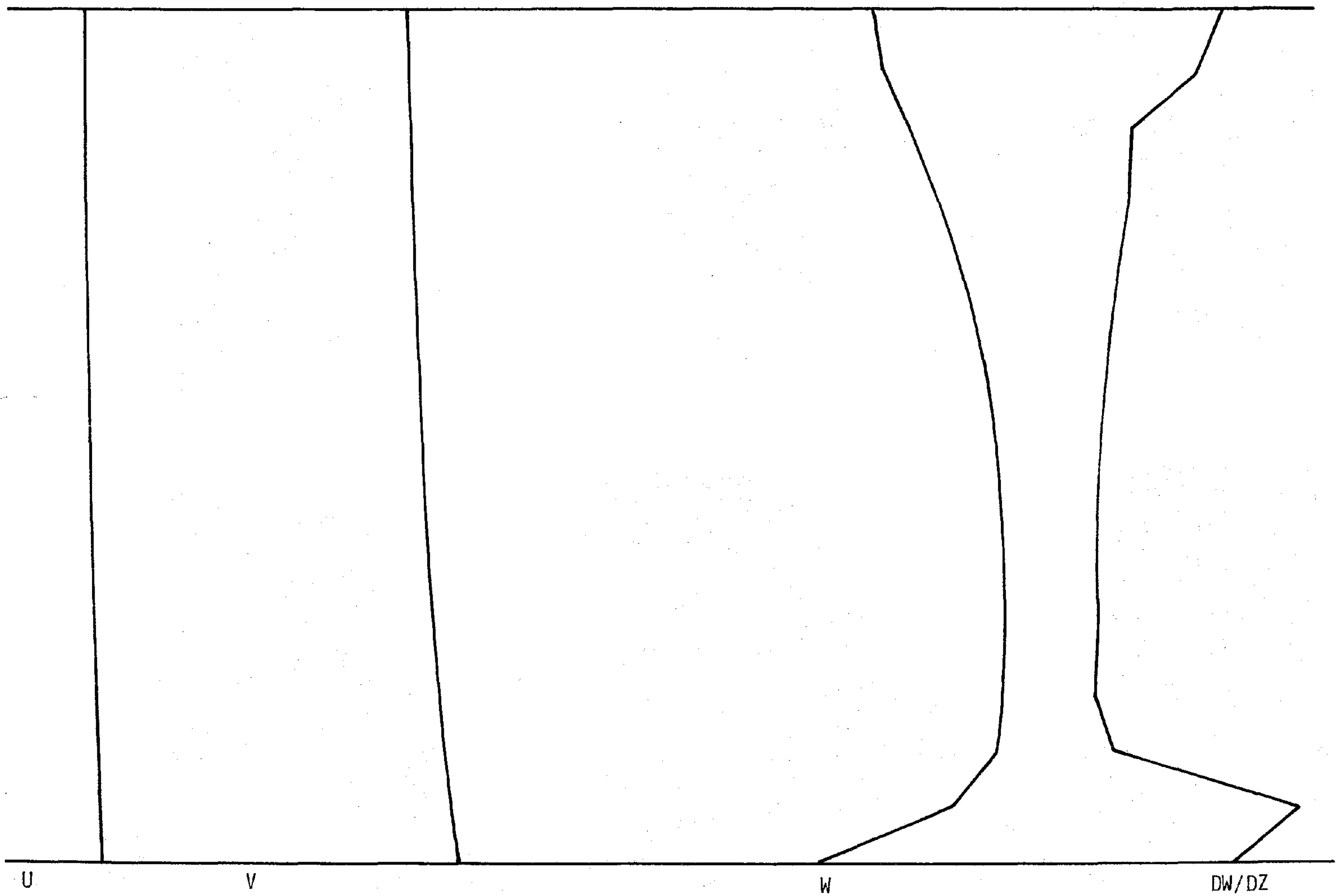
Percent Liquid in Tank	Freq. (Hz) First Axial Mode	Freq. (Hz) Second Axial Mode	Freq. (Hz) Third Axial Mode	Freq. (Hz) Fourth Axial Mode	Liquid Mesh Size (NN x MM)	Shell Mesh Size NELFS, NELFR
100 (Full)	6.13	11.15	15.11	18.16	20 x 15	15,0
80	7.16	12.93	17.38	21.22	20 x 12	12,3
60	8.85	15.79	21.33	27.22	22 x 10	10,7
50	10.14	17.85	24.34	32.17	30 x 10	10,10
40	11.18	20.78	29.64	35.04	30 x 10	10,15
25	16.47	31.85	34.62	43.98	30 x 8	8,22
0 (Empty)	34.06	43.87	44.53	44.98	----	0,15

TABLE 1

Coupled Natural Frequencies for Various Liquid Depths for Cylindrical Tank of 60 foot radius, 40 foot height, and 1 inch Wall Thickness

FIGURE 9

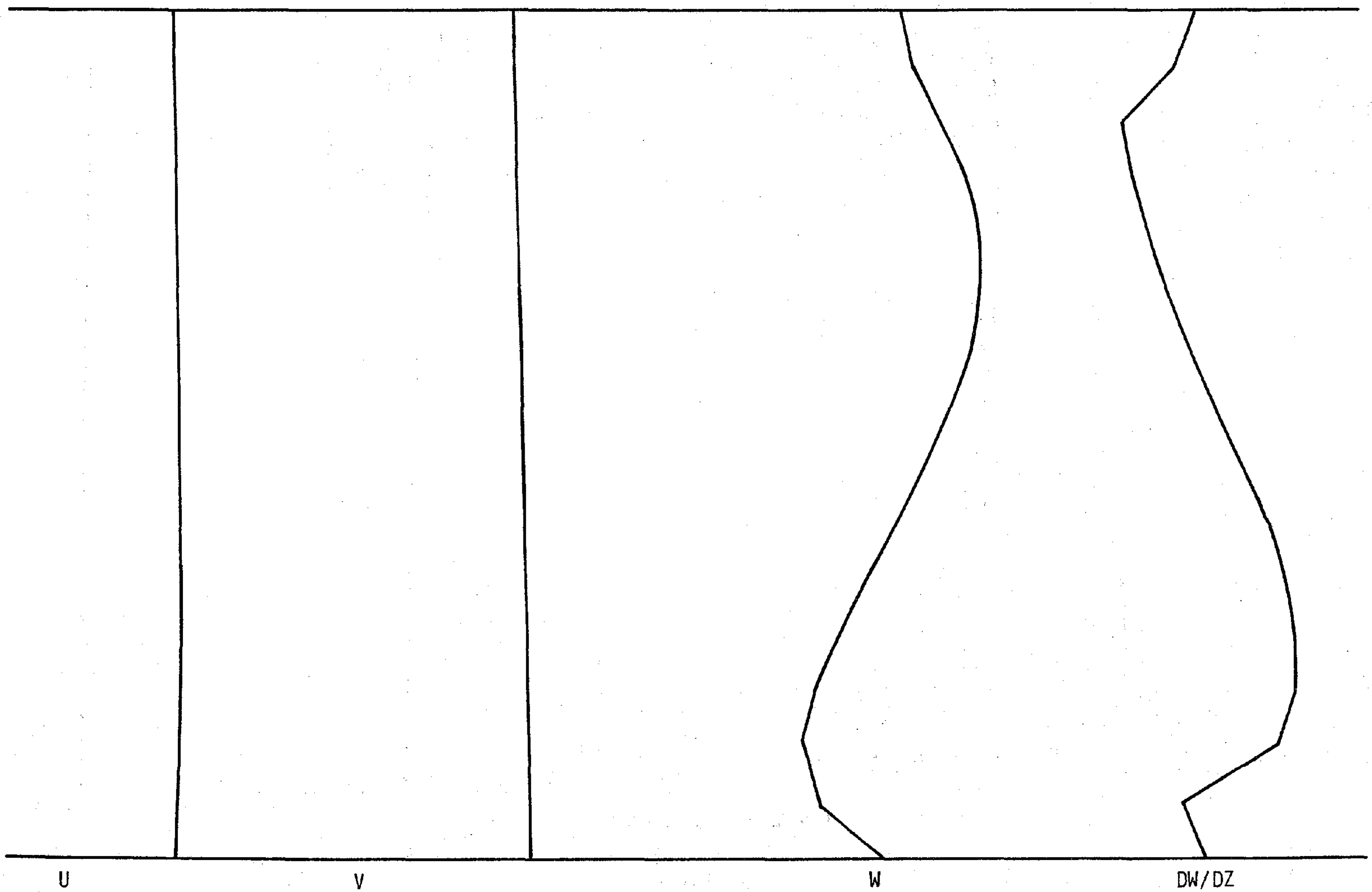
CLAMPED-FREE BOUNDARY



HARMONIC NO. = 1
AXIAL MODE NO. = 1

FIGURE 10

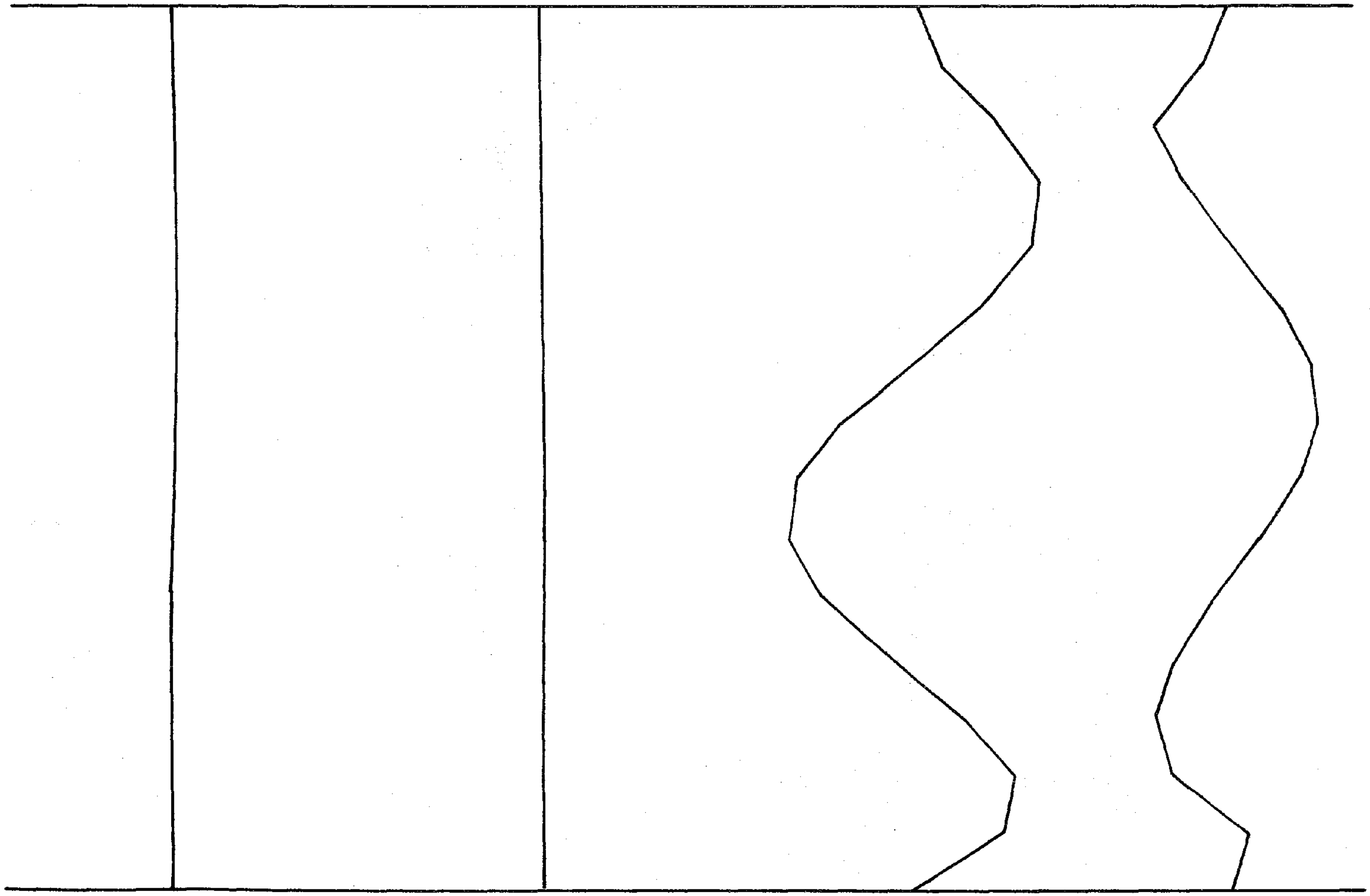
CLAMPED-FREE BOUNDARY



HARMONIC NO. = 1
AXIAL MODE NO. = 2

FIGURE 11

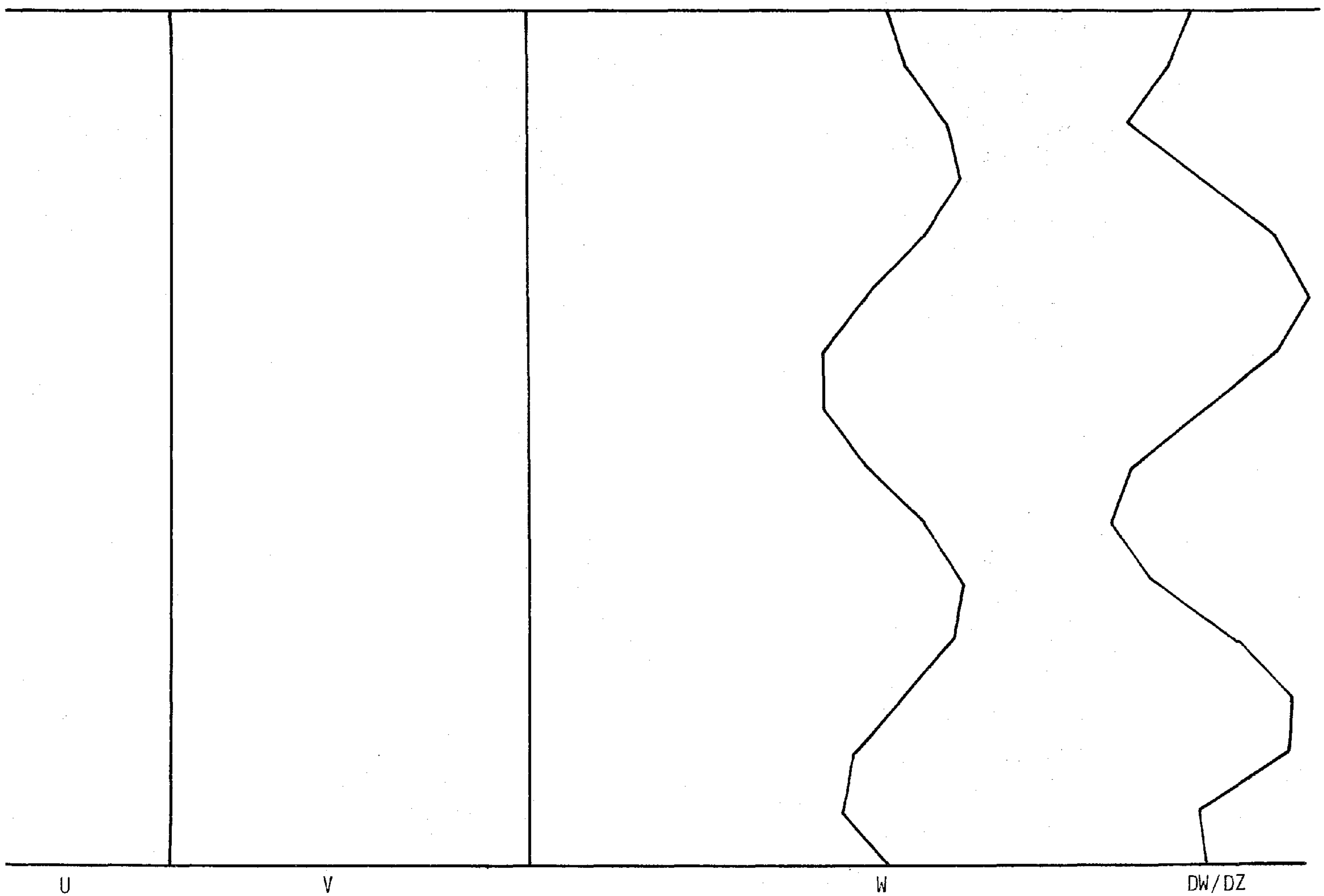
CLAMPED-FREE BOUNDARY



HARMONIC NO. = 1
AXIAL MODE NO. = 3

FIGURE 12

CLAMPED-FREE BOUNDARY

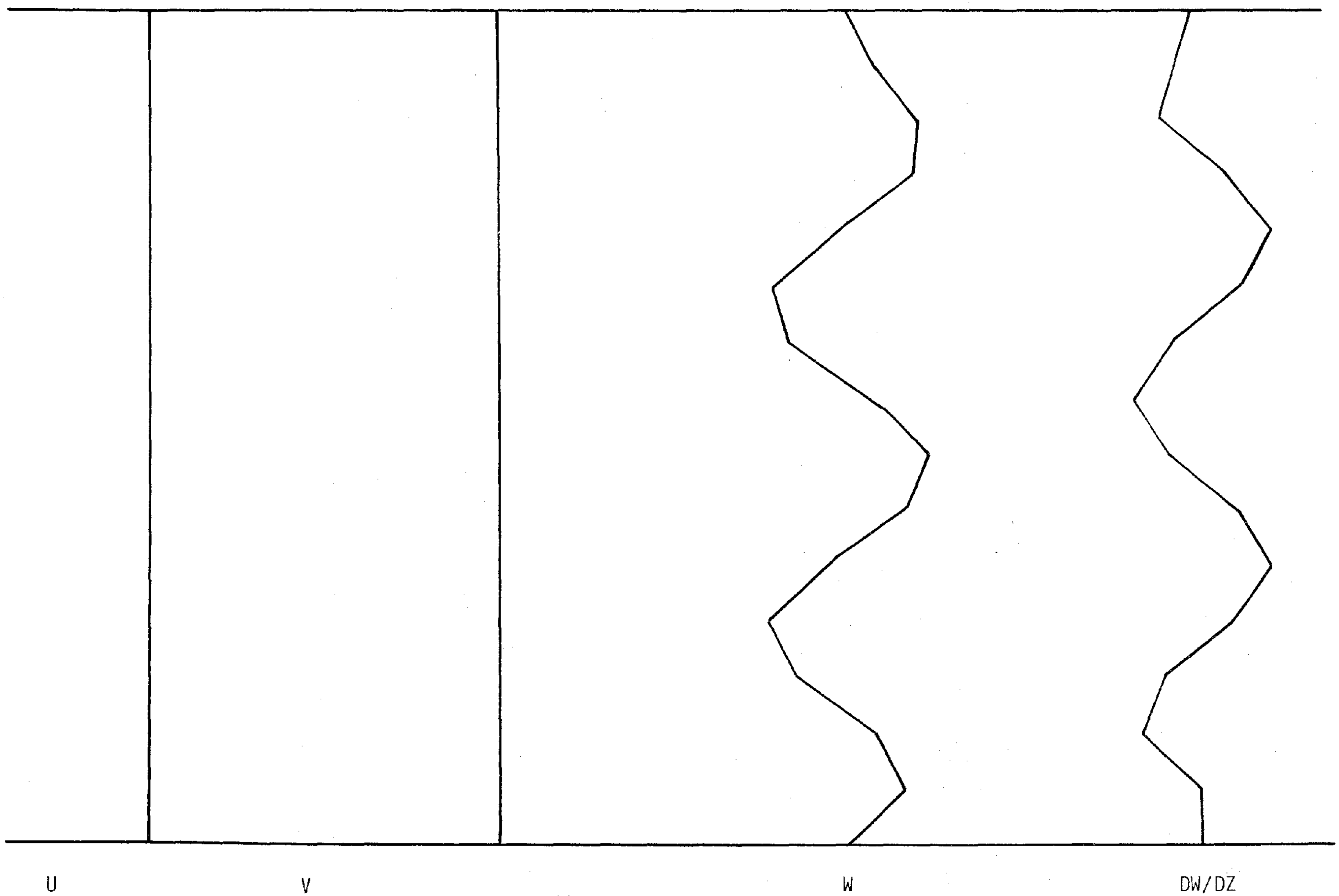


HARMONIC NO. = 1

AXIAL MODE NO. = 4

FIGURE 13

CLAMPED-FREE BOUNDARY



HARMONIC NO. = 1
AXIAL MODE NO. = 5

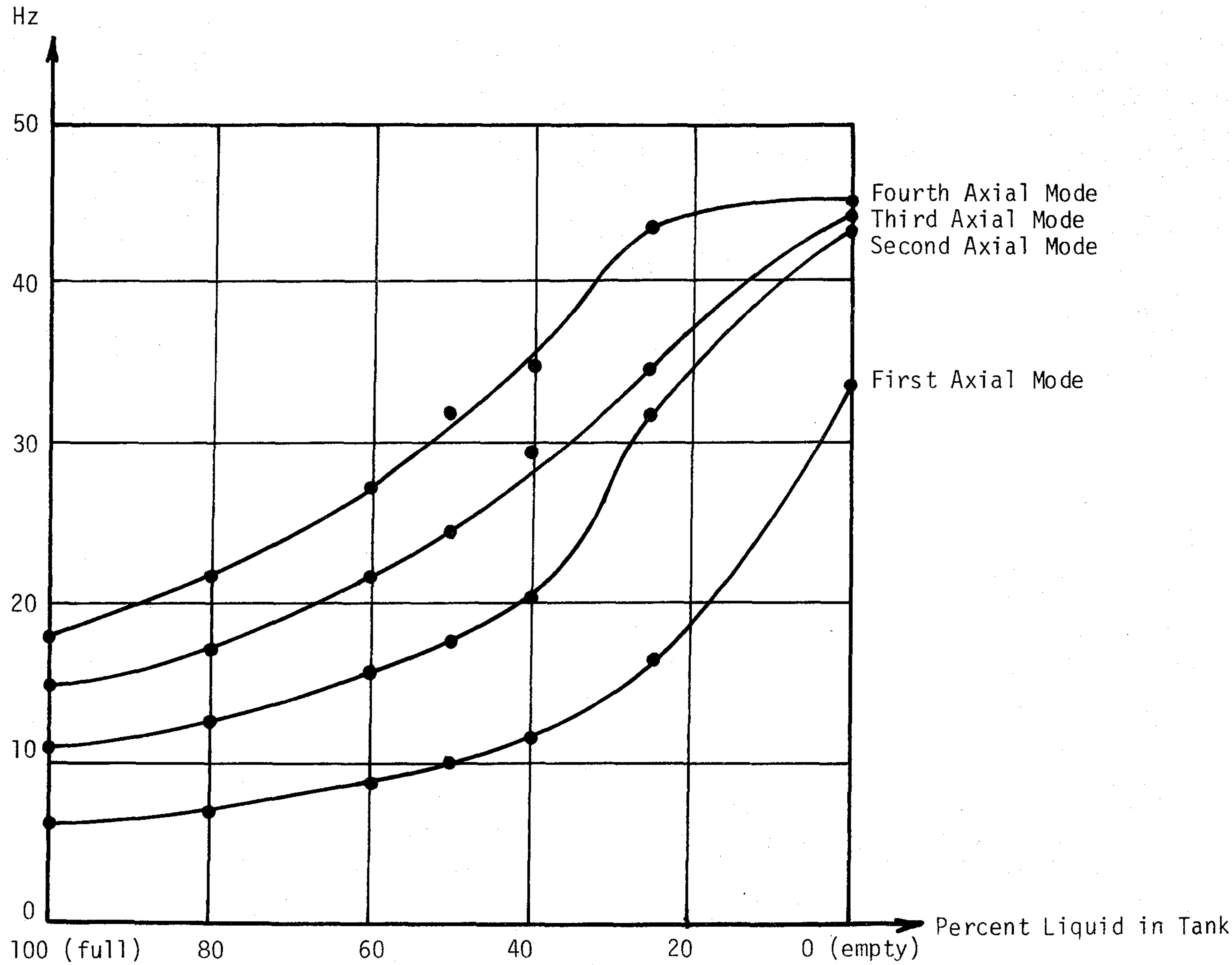


FIGURE 14
Effect of Liquid Depth on Coupled Natural Frequencies
(From data in Table 1)

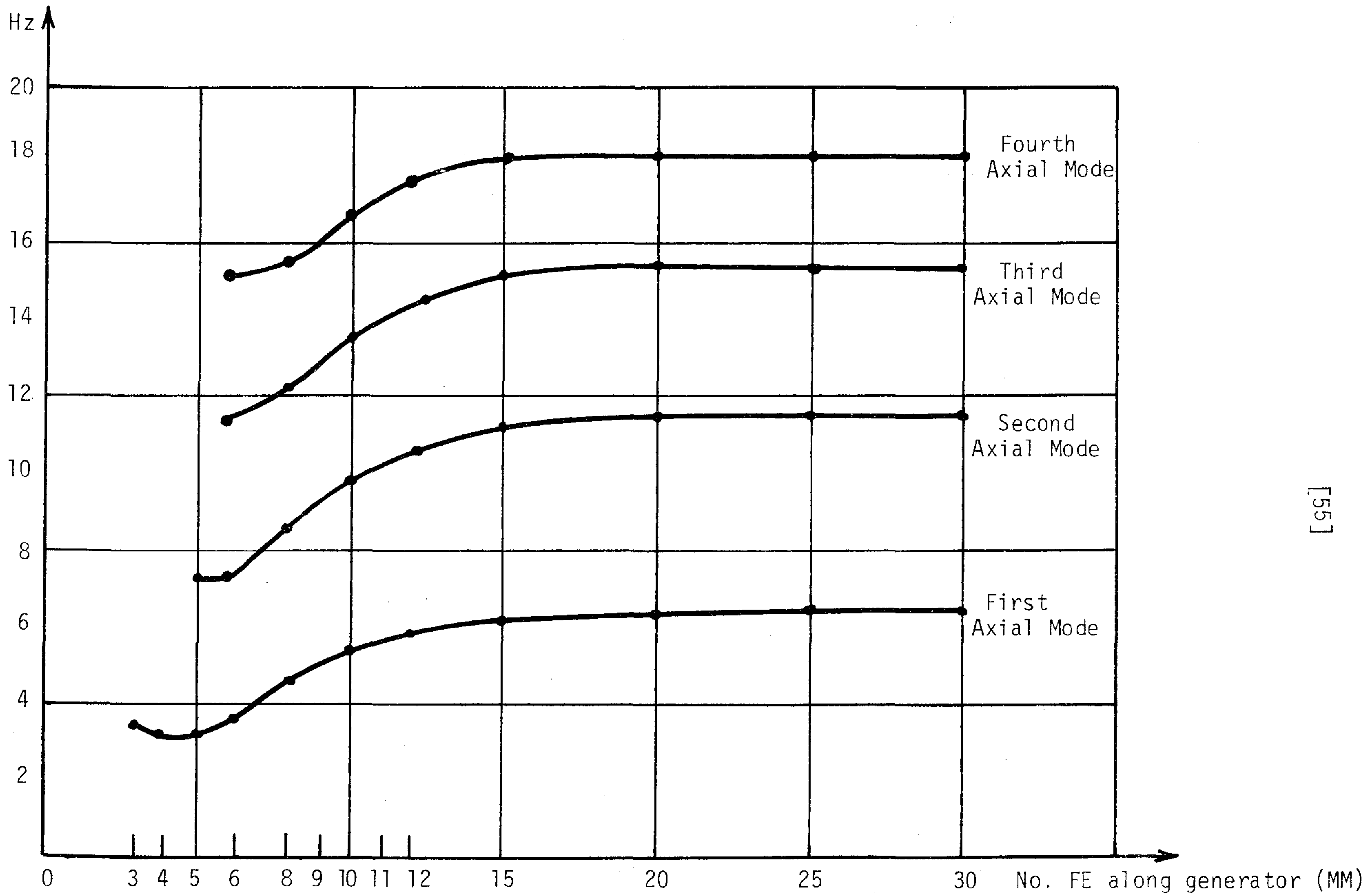


FIGURE 15

Effect on Frequency of Varying Number of Finite Elements in Direction of the Generator while Holding the Number of Elements in Radial Direction = 30. (Full Tank)

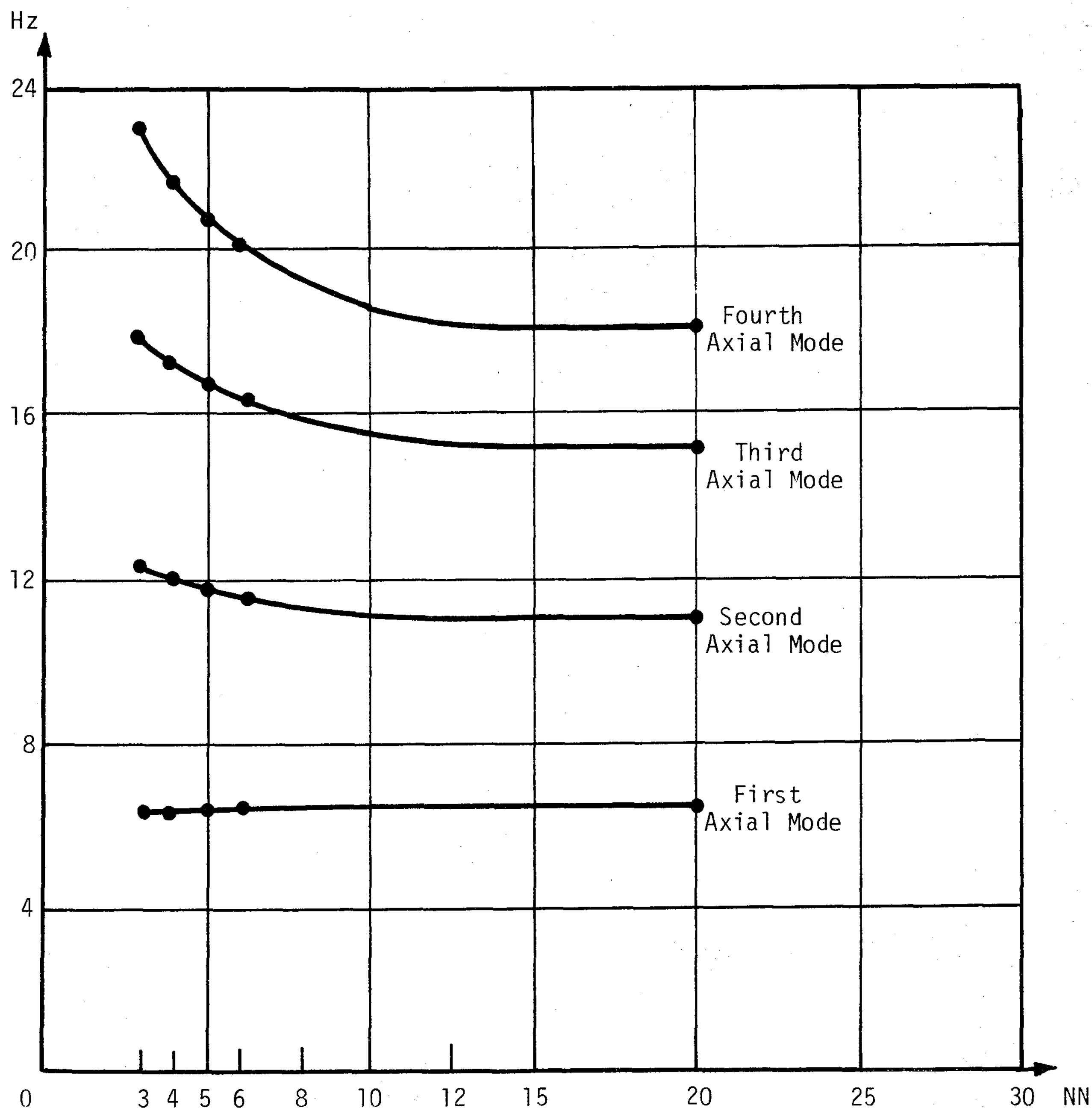


FIGURE 16

Effect on Frequency of Varying Number of Finite Elements in Radial Direction While Holding the Number of Elements in Direction of the Generator = 20. (Tank Full)

3. Free Vibrations of Partially Filled Elastic Tank

Let us consider the same tank discussed in Example 2, but now only *half-filled* with water.

Again, the computer program of Appendix C is used. One enters the following data pertinent to the liquid:

CARD 1: DENF = liquid density = 0.9345×10^{-4} lb x sec/in^4

R = tank radius = 720 inches

WH = depth of water = 240 inches

CARD 2: NN = number of liquid elements in one row along tank radius = 30

MM= number of liquid elements in one column along tank height = 10

CARD 3: NHR = number of circumferential waves in pattern under consideration = 1

Next, one enters the following data pertinent to the elastic tank:

CARD 4: UM = ρ = density of tank material = 0.733×10^{-3} lb x sec^2/in^4

E1 = E = Young's modulus = 30×10^6 lb/in^2

PX = nu = Poisson's ratio = 0.3

CARD 5: R = tank radius = 720 inches

H = tank wall thickness = 1 inch

AL = tank altitude = 480 inches

CARD 6: NSIN = total number of circumferential wave patterns that analyst desires to investigate = 1 (Program C does not permit use of NSIN $\neq$ 1)

CARD 7: NELEM = number of ring-shaped finite elements representing the tank = 15

CARD 8: NELFS = number of shell finite elements corresponding to wetted surface = 10. (This must equal MM)

NELFR = number of shell finite elements corresponding to dry surface = 5. (Obviously NELEM = NELFS + NELFR)

CARD 9: NMODE = number of axial waves under consideration = 10. (Printout indicates frequencies and displacements for modes 1, 2, ... 10)

CARD 10: NAT = number of circumferential waves in pattern under consideration (i.e. "Instantaneous" number of circumferential waves) = 1. This number specifies which one of those patterns under NSIN is currently being investigated.

CARD 11: NBCAS = total number of cases involving different sets of boundary conditions that analyst desires to investigate = 1. (The program of Appendix C does not permit use of NBCAS $\neq$ 1)

CARD 12: NBC = denotes boundary conditions at base and top of tank. First, enter CL if base is clamped or SM if base is simply supported. Next, enter CL if top is clamped, or SM if it is simply supported, or FR if it is free. Do not introduce a space between the designations of these two boundary conditions.

This completes all necessary input to the computer program.

The program output consists of natural frequencies of the coupled liquid-elastic tank system together with mode shapes (along a generator). For this half-filled tank the first four natural frequencies are:

Axial Mode	Frequency (Hz)	Frequency (Hz) [6]
1	10.15	9.39
2	17.85	15.90
3	24.35	20.40
4	32.18	---

In the interest of brevity, mode shapes are not presented here. It is of interest to compare the values 10.15, 17.85 Hz etc. obtained through the present finite element analysis with those found by an entirely analytical procedure due to T. Mouzakis [6] which are tabulated in the right hand column.

4. Cylindrical Tank Whose Base Slab is Subject to Artificial Earthquake Excitation.

Again, we consider the same tank discussed in the first example. Elasticity of the tank wall is considered and two cases are treated: a) the tank is completely filled with water, and, b) the tank is half-filled with water. The artificial earthquake accelerogram available through the National Information Service-Earthquake Engineering-Computer Program Applications (PSEQGN) [7] was considered to be the exciting mechanism acting on the rigid base slab in the horizontal direction along the line $\theta = 0°$. The response of the liquid-elastic tank system is desired. Specifically, for the completely filled tank (Case a), radial displacements are sought at the tank top, as well as at third points of the tank height. For the half-filled tank (Case b), radial displacements are desired at the tank top, at the surface of the liquid, and at half the liquid depth. All of these parameters are to be evaluated at $\theta = 0°$.

The program of Appendix D is utilized here. The artificial earthquake record was imposed upon the base slab for 10 seconds and the coupled liquid-elastic tank system response determined at 0.001 second intervals during the time period $t = 0$ to $t = 10$ seconds using time increments of 0.001 second. In using the artificial earthquake record the assigned maximum ground acceleration was taken to be g/2 although the record itself is normalized in terms of a unit value of g. The input to the rigid base was in terms of acceleration. Data cards employed and values assigned are as follows:

PART 1

CARD 1: M = number of modes used in superposition = 10 (obviously M cannot exceed NMODE.)

PART II

CARD 2: LREC = length of record = 100 points

NREC = number of intervals in record = 100

NRSTART = sequential number of the starting time "guide" station under consideration = 1

NREND = sequential number of the last time "guide" station = 99

CARD 3: DT = time increment between two successive time stations = 0.001 seconds

Case (a) - Completely filled tank

CARD 4: ND1 = first desired response according to numbering scheme shown in Figure 17 = 23-4 = 19

ND2 = second desired response according to numbering scheme shown in Figure 17 = 43-4 = 39

ND3 = third desired response according to numbering scheme shown in Figure 17 = 63-4 = 59

Case (b) - Half-filled tank

CARD 4: ND1 = first desired response (radial displacement at half liquid depth) according to numbering scheme shown in Figure 19 = 23-4 = 19

ND2 = second desired response (radial displacement at liquid surface) according to numbering scheme shown in Figure 19 = 43-4 = 39

ND3 = third desired response (radial displacement at tank top) according to numbering scheme shown in Figure 19 = 63-4 = 59

The time history of desired radial displacements during the time interval t = 0 to 10 seconds appears as indicated in Figure 18 for Case (a), i.e., the completely filled tank.

The time history of the specified radial displacements during the time interval t = 0 to 10 seconds appears as indicated in Figure 20 for Case (b), i.e., the half-filled tank. The radial response of the generator $\theta = 0°$ at time t = 7.15 seconds for the half-filled tank is indicated below where the value in the top row corresponds to the base mode and the value in the bottom row corresponds to the top of the tank. The intermediate values, of course, correspond to the radial displacements at the nodal points indicated in Figure 19. Responses at other values of time are also available from the computer output.

w	
0	
0.3046	
0.5261	
0.5567 (*)	(Node 4)
0.5133	
0.4585	(Node 6 - ND1)
0.3935	
0.2967	
0.1731	
0.0741	
0.0480	
0.0682	
0.0814	
0.0856	
0.0893	
0.1001	

It should be remembered that these radial displacements are all relative to the rigid slab and absolute motions could be obtained by superposing on the above the ground displacements. The displacement (*) of 0.5567 inches occurs at node number 4 (see Figure 19) and by inspection is the peak radial displacement of any point along the generator $\theta = 0°$ in the time interval from t = 0 to t = 10 seconds. The program of Appendix D displays the maximum response at ND1, ND2, and ND3 and the corresponding time when each peak occurs during the interval t = 0 to t = 10 seconds.

The axial, tangential, and in-plane shearing stresses as well as moments M_{zz}, $M_{\theta\theta}$, and $M_{z\theta}$ at $\theta = 0°$ are tabulated below at the time t = 7.15 seconds where the values in the top row correspond to base nodes and values in the bottom row correspond to nodes at the top of the tank.

STRESSES AT THETA =0.0

Axial F.	Tangt. F.	In-Plane Sh.	Axial Mt.	Tangt. Mt.	Torsion
-2466.	-739.8	0.0	-6078.	-1823.	0.0
-378.6	12180.	0.0	1370.	412.4	0.0
-107.3	21130.	0.0	964.5	291.8	0.0
-272.9	22040.	0.0	294.4	90.88	0.0
-315.3	19960.	0.0	55.22	18.89	0.0
-239.9	17490.	0.0	101.5	32.48	0.0
-129.8	14660.	0.0	266.5	81.64	0.0
-98.89	10510.	0.0	283.9	86.39	0.0
-182.9	5234.	0.0	96.12	29.45	0.0
-295.1	990.6	0.0	-313.5	-93.91	0.0
-78.69	472.2	0.0	-338.7	-101.6	0.0
-165.7	787.6	0.0	-230.6	-69.07	0.0
-464.0	897.8	0.0	-186.5	-55.83	0.0
-830.7	808.0	0.0	-156.9	-46.95	0.0
-1211.	709.3	0.0	-195.1	-58.40	0.0
-1304.	1021.	0.0	431.9	129.7	0.0

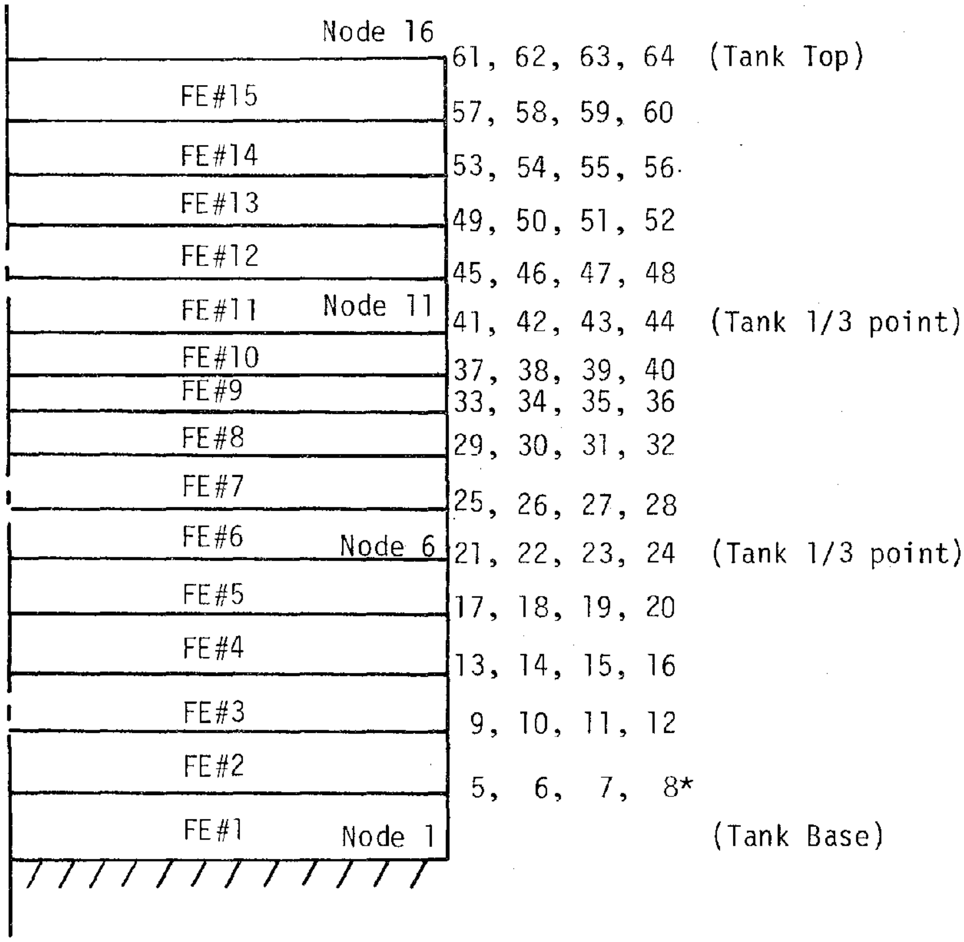


FIGURE 17

Shell Degree of Freedom Numbering System for Use in Response Determination (Program RESPONS)- Completely Filled Tank

*At each node, the numbered degrees of freedom correspond to u,v,w, and dw/dz respectively. For boundary conditions treated in this report the base nodes are not employed. Consequently, for use in Program RESPONS, correct designation of desired degree of freedom response is obtained by subtracting "4" from the number indicated in Figure 17.

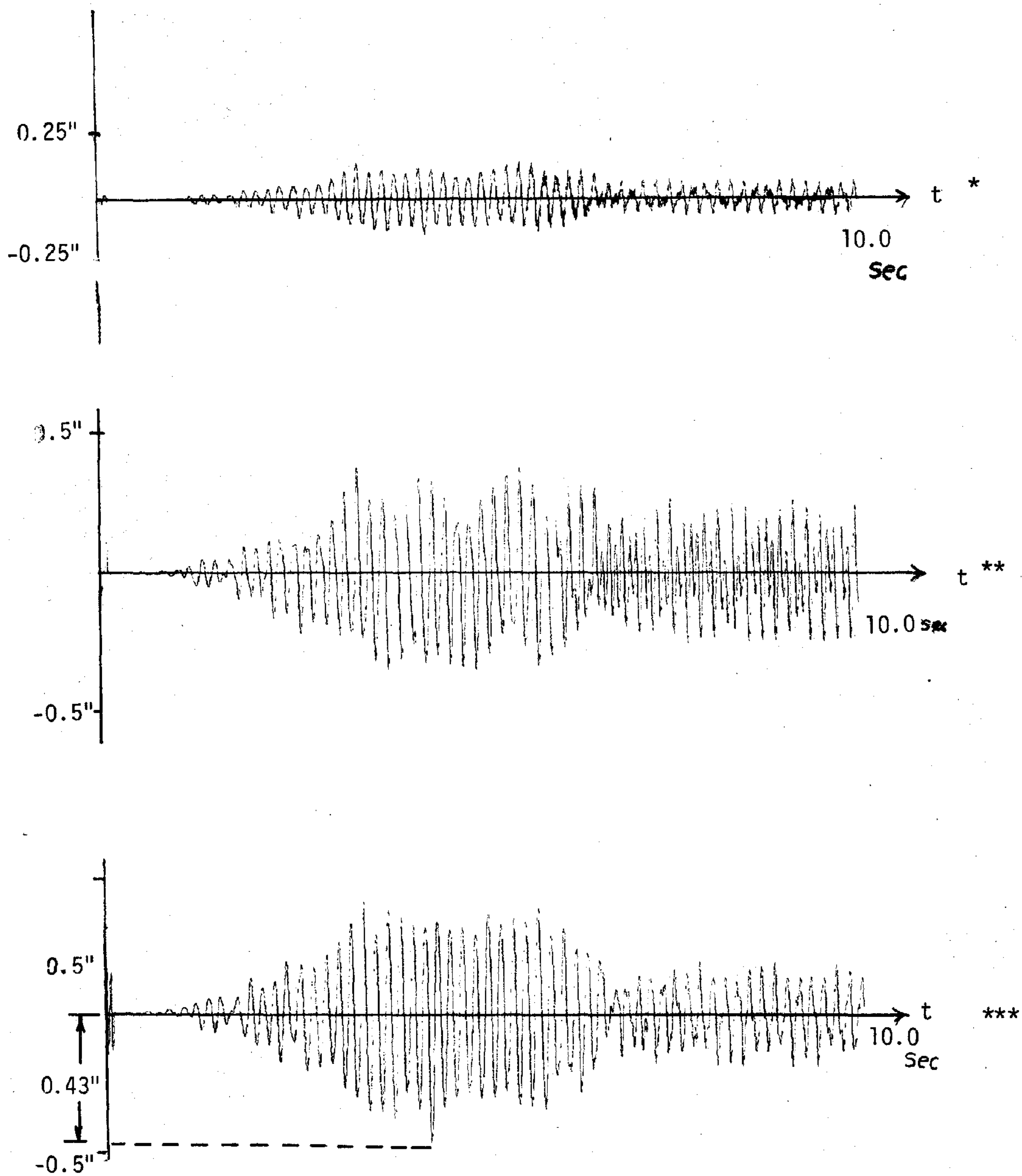


FIGURE 18

Time History of Radial Displacements at Third Points as well as at Tank Top for Completely Filled Tank (* indicates response at tank top, ** response at upper third point, and *** response at lower third point).

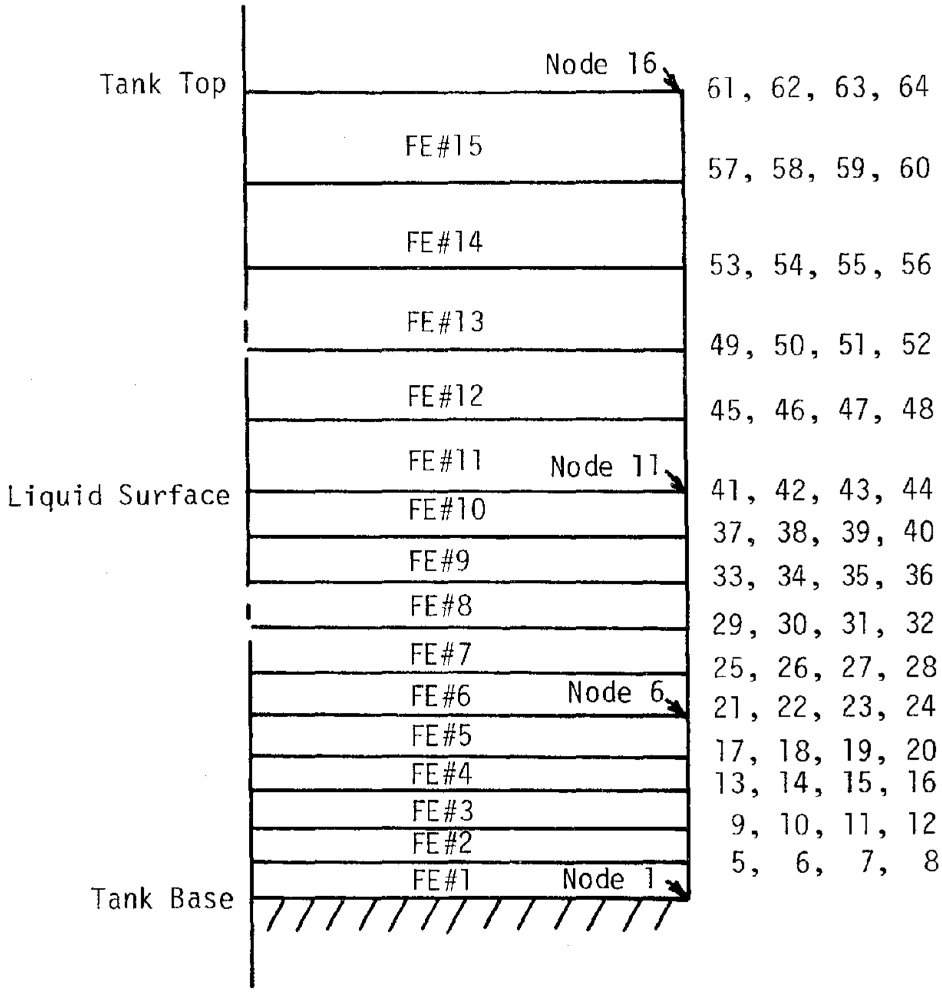


FIGURE 19

Shell Degree of Freedom Numbering System for Use in Response Determination (Program RESPONS)-Half-Filled Tank

*At each node, the numbered degrees of freedom correspond to u,v,w, and dw/dz respectively. For boundary conditions treated in this report, the base nodes are not employed. Consequently for use in Program RESPONS, correct designation of desired degree of freedom response is obtained by subtracting "4" from the number indicated in Figure 19.

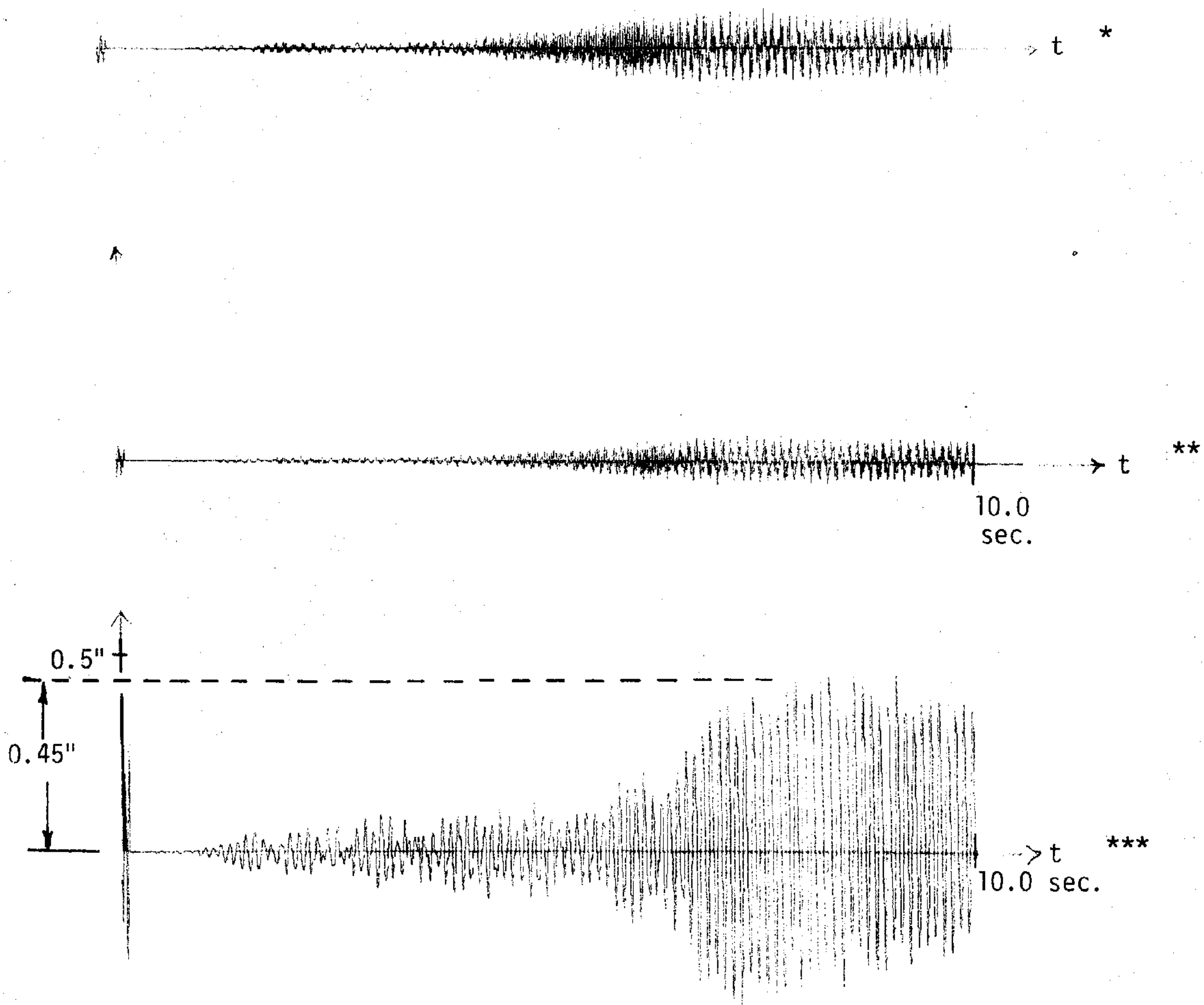


FIGURE 20

Time History of Radial Displacements in Half-Filled Tank
(* indicates response at tank top, ** response at surface of liquid, and *** response at mid-depth of liquid).

ABRIDGED METHOD OF COMPUTATION

In an effort to reduce the number of liquid degrees of freedom to a smaller value than has been employed till now in this investigation, yet maintain reasonable engineering accuracy, an investigation was made of the "active" volume of the liquid in the elastic tank. It was found that there exists a "liquid core" which is essentially stationary and thus the coupled system may be economically analyzed with acceptable accuracy by considering only an outer annular domain of liquid. The inner boundary of this domain is essentially a circular cylindrical surface and the dynamic pressure on it, as well as inside it, is presumably zero. This concept greatly reduces the liquid degrees of freedom from that previously presented.

Let us consider again the tank 40 feet high, 60 feet in radius, and with a one inch thick steel wall. The tank is clamped at the rigid base, free at the top, and completely filled with water. Various size "liquid cores" were postulated ranging from a zero radius (corresponding to the situation on page 44 of this report) to a radius equal to 5/6 of 60 feet. This "dead zone" radius appears as the abscissa in Figure 21. Natural frequencies of the coupled system having that size "dead zone" appear as the ordinates of this plot. Points at 1.00R on the abscissa correspond to the empty tank case discussed in [1] and those four points were plotted directly from results in [1].

These results indicate that, at least for this particular tank, the "dead zone" can be taken to be of the order of 80 percent of the tank radius and satisfactory values of coupled natural frequencies will be obtained through the use of about 20 percent of the original number of liquid degrees of freedom.

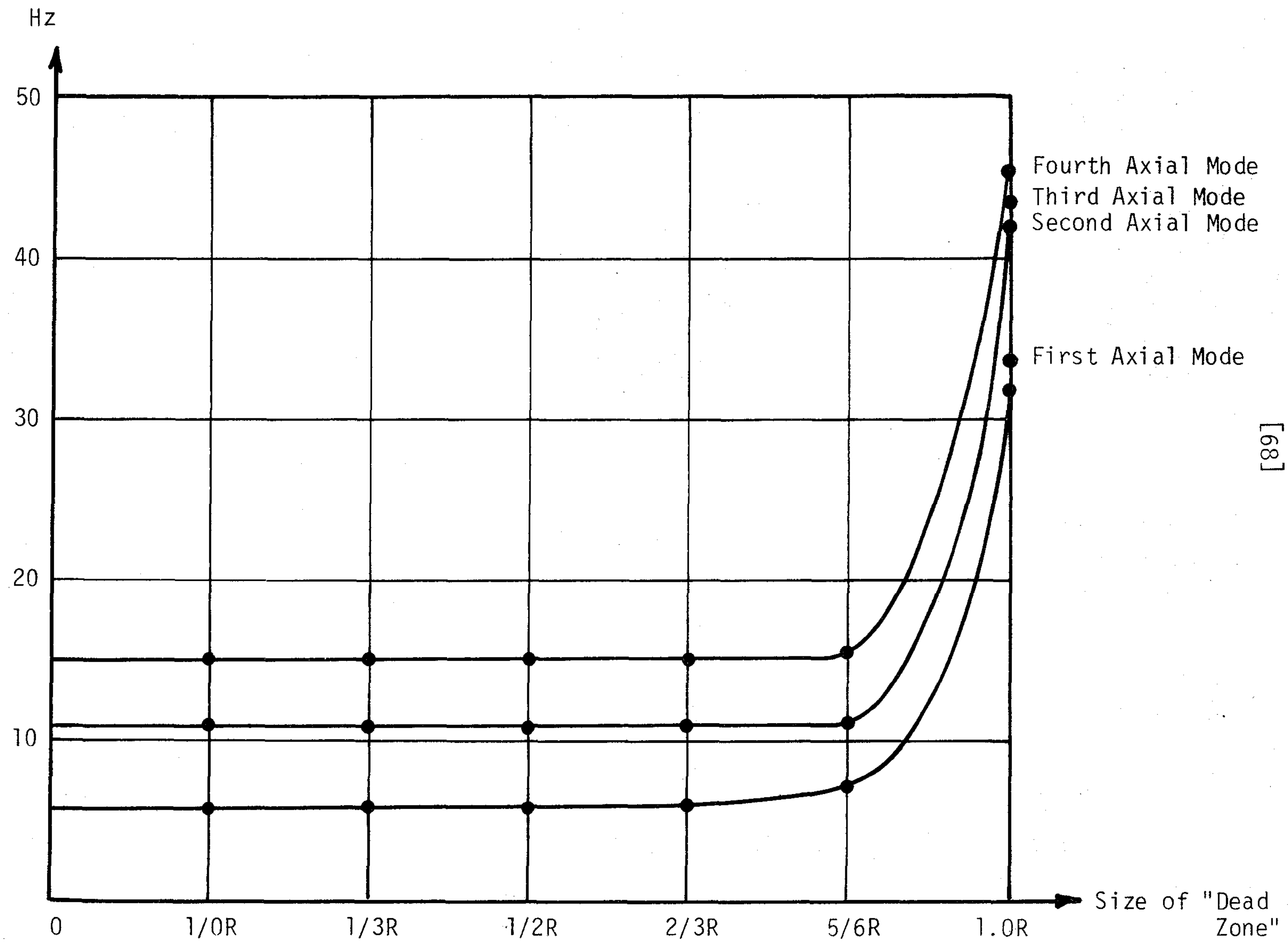


FIGURE 21

Predicted Natural Frequencies as a Function of Size of Central "Dead Zone"

ACKNOWLEDGMENT

The authors would like to express their thanks to Dr. C. I. Wu and Dr. J. M. Colonell for valuable discussions and comments offered during the course of this research.

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APPENDIX A

DEVELOPMENT OF MATRICES EMPLOYED IN FINITE ELEMENT ANALYSIS

Complete derivations of the element stiffness and mass matrices for the elastic tank are given in [1].

<u>Derivation of the Liquid Element "Stiffness" Matrix</u> $[K_e]$

This is defined in Equations (21) and (22) together with the numbering scheme shown in Figures 4 and 5. This may be written as:

$$\{\delta_p\}^T[K_e]\{\delta_p\} = \frac{\pi}{\rho}\int_x\int_y \left(\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 + \frac{m^2}{(x_o+x)^2}P^2\right)(x_o + x)\,dx\,dy$$

$$\therefore \quad P = [N]\{\delta_p\}$$

$$\frac{\partial P}{\partial x} = [\frac{\partial N}{\partial x}]\{\delta_p\}$$

$$\frac{\partial P}{\partial y} = [\frac{\partial N}{\partial y}]\{\delta_p\}$$

where δ_p is the generalized pressure of the element nodal circles

$[N]$ is the element shape function defined by

$$[N] = \frac{1}{4ab}[(a-x)(b-y)\ \ (a+x)(b-y)\ \ (a+x)(b+y)\ \ (a-x)(b+y)]$$

$$\therefore \quad [K_e] = \frac{\pi}{\rho}\int_{-a}^{a}\int_{-b}^{b}\left[[\frac{\partial N}{\partial x}]^T[\frac{\partial N}{\partial x}] + [\frac{\partial N}{\partial y}]^T[\frac{\partial N}{\partial y}] + \frac{m^2}{(x_o+x)^2}[N]^T[N]\right](x_o+x)dxdy$$

$$= \frac{\pi}{\rho}\int_{-a}^{a}\int_{-b}^{b}[[A_1] + [A_2] + m^2A_3]\,(x_o+x)\,dx\,dy$$

$$= \frac{\pi}{\rho}[[K_1] + [K_2] + [K_3]]$$

where $A_1 = [\frac{\partial N}{\partial x}]^T[\frac{\partial N}{\partial x}]$

$A_2 = [\frac{\partial N}{\partial y}]^T[\frac{\partial N}{\partial y}]$

$A_3 = [N]^T[N]/(x_o+x)^2$

$$[A_1] = \frac{1}{(4ab)^2}\begin{Bmatrix} -(b-y) \\ (b-y) \\ (b+y) \\ -(b+y) \end{Bmatrix} \quad [-(b-y) \quad (b-y) \quad (b+y) \quad -(b+y)]$$

$$= \frac{1}{(4ab)^2}\begin{bmatrix} (b-y)^2 & -(b-y)^2 & -(b^2-y^2) & (b^2-y^2) \\ -(b-y)^2 & (b-y)^2 & (b^2-y^2) & -(b^2-y^2) \\ -(b^2-y^2) & (b^2-y^2) & (b+y)^2 & -(b+y)^2 \\ +(b^2-y^2) & -(b^2-y^2) & -(b+y)^2 & (b+y)^2 \end{bmatrix} = \frac{1}{(4ab)^2}\begin{bmatrix} \ell_1 & -\ell_1 & -\ell_3 & \ell_3 \\ -\ell_1 & \ell_1 & \ell_3 & -\ell_3 \\ -\ell_3 & \ell_3 & \ell_2 & -\ell_2 \\ \ell_3 & -\ell_3 & -\ell_2 & \ell_2 \end{bmatrix}$$

Thus the determination of $[K_1]$ has been reduced to the evaluation of three double integrations as follows:

$$\int_{-b}^{b} \ell_1 \, dy = \frac{8}{3} b^3$$

$$\int_{-a}^{a} \int_{-b}^{b} \ell_1 \, dy \, (x+x_o)dx = \frac{8}{3} b^3 \, (2ax_o)$$

$$\int_{-b}^{b} \ell_2 \, dy = \frac{8}{3} b^3$$

$$\int_{-a}^{a} \int_{-b}^{b} \ell_2 \, dy \, (x+x_o)dx = \frac{8}{3} b^3 \, (2ax_o)$$

$$\int_{-b}^{b} \ell_3 \, dy = \frac{4}{3} b^3$$

$$\int_{-a}^{a} \int_{-b}^{b} \ell_3 \, dy \, (x+x_o)dx = \frac{4}{3} b^3 \, (2ax_o)$$

$$[K_1] = \frac{x_o b}{6a}\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

Similarly

$$[A_2] = \frac{1}{(4ab)^2} \begin{Bmatrix} -(a-x) \\ -(a+x) \\ (a+x) \\ (a-x) \end{Bmatrix} [-(a-x) \quad -(a+x) \quad (a+x) \quad (a-x)]$$

$$= \frac{1}{(4ab)^2} \begin{bmatrix} (a-x)^2 & (a^2-x^2) & -(a^2-x^2) & -(a-x)^2 \\ (a^2-x^2) & (a+x)^2 & -(a+x)^2 & -(a^2-x^2) \\ -(a^2-x^2) & -(a+x)^2 & (a+x)^2 & (a^2-x^2) \\ -(a-x)^2 & -(a^2-x^2) & (a^2-x^2) & (a-x)^2 \end{bmatrix}$$

$$= \frac{1}{(4ab)^2} \begin{bmatrix} t_1 & t_3 & -t_3 & -t_1 \\ t_3 & t_2 & -t_2 & -t_3 \\ -t_3 & -t_2 & t_2 & t_3 \\ -t_1 & -t_3 & t_3 & t_1 \end{bmatrix}$$

Here the evaluation of $[K_2]$ again reduces to evaluating three double integrals as follows:

$$\int_{-a}^{a} t_1 \ (x+x_o)dx = \frac{4a^3}{3} (2x_o-a)$$

$$\int_{-b}^{b} \int_{-a}^{a} t_1 \ (x+x_o)dxdy = \frac{8a^3b}{3} (2x_o-a)$$

$$\int_{-a}^{a} t_2 \ (x+x_o)dx = \frac{4a^3}{3} (2x_o+a)$$

$$\int_{-b}^{b} \int_{-a}^{a} t_2 \ (x+x_o)dxdy = \frac{8a^3b}{3} (2x_o+a)$$

$$\int_{-a}^{a} t_3 \ (x+x_o)dx = \frac{4a^3}{3} x_o$$

$$\int_{-b}^{b}\int_{-a}^{a} t_3 \ (x+x_o)dxdy = \frac{8a^3}{3} bx_o$$

$$\therefore [K_2] = \frac{ax_o}{6b} \begin{bmatrix} (2 - \frac{a}{x_o}) & 1 & -1 & -(2 - \frac{a}{x_o}) \\ 1 & (2 + \frac{a}{x_o}) & -(2 + \frac{a}{x_o}) & -1 \\ -1 & -(2 + \frac{a}{x_o}) & (2 + \frac{a}{x_o}) & 1 \\ -(2 - \frac{a}{x_o}) & -1 & 1 & (2 - \frac{a}{x_o}) \end{bmatrix}$$

The third additive matrix $[A_3]$ is given by

$$\frac{[N]^T[N]}{(x+x_o)^2} = \frac{1}{(x+x_o)^2} \frac{1}{(4ab)^2} \begin{Bmatrix} (a-x)(b-y) \\ (a+x)(b-y) \\ (a+x)(b+y) \\ (a-x)(b+y) \end{Bmatrix} [(a-x)(b-y)(a+x)(b-y)(a+x)(b+y)(a-x)(b+y)]$$

$$= \frac{1}{(4ab)^2(x+x_o)^2} \begin{bmatrix} (a-x)^2(b-y)^2 & (a^2-x^2)(b-y)^2 & (a^2-x^2)(b^2-y^2) & (a-x)^2(b^2-y^2) \\ (a^2-x^2)(b-y)^2 & (a+x)^2(b-y)^2 & (a+x)^2(b^2-y^2) & (a^2-x^2)(b^2-y^2) \\ (a^2-x^2)(b^2-y^2) & (a+x)^2(b^2-y^2) & (a+x)^2(b+y)^2 & (a^2-x^2)(b+y)^2 \\ (a-x^2)(b^2-y^2) & (a^2-x^2)(b^2-y^2) & (a^2-x^2)(b+y)^2 & (a-x)^2(b+y)^2 \end{bmatrix}$$

$$(x+x_o)\int_{-b}^{b} A_3 dy = \frac{1}{(4ab)^2(x+x_o)}(\frac{4b^3}{3}) \begin{bmatrix} 2(a-x)^2 & 2(a^2-x^2) & (a^2-x^2) & (a-x)^2 \\ 2(a^2-x^2) & 2(a+x)^2 & (a-x)^2 & (a^2-x^2) \\ (a^2-x^2) & (a+x)^2 & 2(a+x)^2 & 2(a^2-x^2) \\ (a-x)^2 & (a^2-x^2) & 2(a^2-x^2) & 2(a-x)^2 \end{bmatrix}$$

$$= \frac{b}{12a^2} \begin{bmatrix} 2e_1 & 2e_3 & e_3 & e_1 \\ 2e_3 & 2e_2 & e_2 & e_3 \\ e_3 & e_2 & 2e_2 & 2e_3 \\ e_1 & e_3 & 2e_3 & 2e_1 \end{bmatrix}$$

Here again only three different integrals are encountered:

$$\int_{-a}^{a} e_1 \, dx = (a+x_o)^2 \log_e\left(\frac{x_o+a}{x_o-a}\right) - 2a(2a+x_o) = E_1$$

$$\int_{-a}^{a} e_2 \, dx = (a-x_o)^2 \log_e\left(\frac{x_o+a}{x_o-a}\right) + 2a(2a-x_o) = E_2$$

$$\int_{-a}^{a} e_3 \, dx = (a^2-x_o^2) \log_e\left(\frac{x_o+a}{x_o-a}\right) + 2ax_o = E_3$$

Finally the matrix $[K_3]$ will be given by

$$[K_3] = \frac{m^2 b}{12a^2} \begin{bmatrix} 2E_1 & 2E_3 & E_3 & E_1 \\ 2E_3 & 2E_2 & E_2 & E_3 \\ E_3 & E_2 & 2E_2 & 2E_3 \\ E_1 & E_3 & 2E_3 & 2E_1 \end{bmatrix}$$

Derivation of the Liquid Element Mass Matrix $[M_e]$

The mass matrix corresponding to the free surface potential energy and defined by Equations (23) and (24) may be determined by performing the following integration about the free surface area, which, as an approximation, is taken to agree with the mean liquid level.

$$\{\ddot{\delta}_p\}^T [M_e] \{\ddot{\delta}_p\} = \frac{\pi}{g\rho} \int_{\substack{x \\ F.S.}} \left(\frac{\partial P}{\partial t}\right)^2 (x+x_o) dx$$

$$\therefore \frac{\partial P}{\partial t} = [\overset{*}{N}] \frac{\partial}{\partial t} \{\delta_p\}$$

$$\therefore [M_e] = \frac{\pi}{g\rho} \int_{-a}^{a} [\overset{*}{N}]^T [\overset{*}{N}] (x+x_o) dx$$

where $[\overset{*}{N}] = [N(x,b)] = \frac{1}{2a} [0 \quad 0 \quad (a+x) \quad (a-x)]$

$$[M_e] = \frac{\pi}{4a^2 \rho g} \int_{-a}^{a} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (a+x)^2 & (a^2-x^2) \\ 0 & 0 & (a^2-x^2) & (a-x)^2 \end{bmatrix} (x+x_o) dx$$

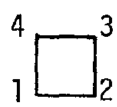
$$= \frac{\pi}{4a^2 \rho g} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4a^3}{3}(2x_o+a) & \frac{4a^3}{3} x_o \\ 0 & 0 & \frac{4a^3}{3} x_o & \frac{4a^3}{3}(2x_o+a) \end{bmatrix}$$

The mass matrix is (4x4) but the non-zero terms are (2x2) given by

$$[\overline{M}_e] = \frac{\pi a}{3\rho g} \begin{bmatrix} a + 2x_o & a \\ a & a + 2x_o \end{bmatrix}$$

where $[\overline{M}_e]$ is the non-zero element submatrix corresponding to the free surface generalized pressure vector $\begin{Bmatrix} \delta_{p_3} \\ \delta_{p_4} \end{Bmatrix}$ as illustrated below.

$$\begin{array}{c} \begin{array}{cc} \delta p_3 & \delta p_4 \end{array} \\ \left[\begin{array}{cc} & \\ & \\ & 2x2 \end{array} \right] \begin{array}{c} \delta p_3 \\ \\ \delta p_4 \end{array} \end{array}$$



The condensed assembled liquid mass matrix is thus L x L where L = number of elements along the radius of one row (for asymmetric modes) and L = unity plus the number of elements along the radius of any one row (for symmetric modes).

Derivation of the Shell-Liquid Coupling Force Matrix $[S_e]$

The coupling force matrix $[S_e]$ defined in Equations (28) and (29) is determined as follows:

$$\{\delta_p\}^T[\delta_e]\{\ddot{\delta}_u\} = \pi R \int_{z\Sigma} \{\delta_p\}^T[N]^T[N_w]\{\ddot{\delta}_u\}dz$$

$$[\delta_e] = \pi R \int_{-b}^{b} [\bar{\bar{N}}]^T[N_w]dy$$

where $\quad [\bar{\bar{N}}] = [N(a,y)] = \frac{1}{2b}[0 \quad (b-y) \quad (b+y) \quad 0]$

$$[N_w] = [0,\ 0,\ 1 - \frac{3y^2}{L^2} + \frac{2y^3}{L^3},\ y - \frac{2y^2}{L} + \frac{y^3}{L^2},\ 0,\ 0,\ \frac{3y^2}{L^2} - \frac{2y^3}{L^3},\ -\frac{y^2}{L} + \frac{y^3}{L^2}]$$

where L = the shell element height = 2b and $[N_w]$ corresponds to the shell element generalized nodal vector defined on page C-4 of [1]. The product $[\bar{\bar{N}}][N_w]$ will be denoted by B

$$B_{2,3} = (b-y)(1 - \frac{3y^2}{4b^2} + \frac{y^3}{4b^3})$$

$$B_{2,4} = (b-y)(y - \frac{y^2}{b} + \frac{y^3}{4b^2})$$

$$B_{2,7} = (b-y)(\frac{3}{4}\frac{y^2}{b^2} - \frac{y^3}{4b^3})$$

$$B_{2,8} = (b-y)(\frac{-y^2}{2b} + \frac{y^3}{4b^2})$$

$$B_{3,3} = (b+y)(1 - \frac{3y^2}{4b^2} + \frac{y^3}{4b^3})$$

$$B_{3,4} = (b+y)(y - \frac{y^2}{b} + \frac{y^3}{4b^2})$$

$$B_{3,7} = (b+y)(\frac{3y^2}{4b^2} - \frac{y^3}{4b^3})$$

$$B_{3,8} = (b+y)(\frac{-y^2}{2b} + \frac{y^3}{4b^2})$$

Performing the integration over the interface area yields

$$\int_{-b}^{b} B_{2,3}dy = \int_{-b}^{b} \left(\frac{-3y^2}{4b} - \frac{y^4}{4b^3}\right)dy = -0.6\ b^2$$

$$\int_{-b}^{b} B_{2,4}dy = \int_{-b}^{b} \left(-2y^2 - \frac{y^4}{4b^2}\right)dy = \frac{-43}{30}\ b^3$$

$$\int_{-b}^{b} B_{2,7}dy = \int_{-b}^{b} \left(\frac{3}{4}\frac{y^2}{b} + \frac{y^4}{4b^3}\right)dy = +0.6\ b^2$$

$$\int_{-b}^{b} B_{2,8}dy = \int_{-b}^{b} \left(\frac{-y^2}{2} - \frac{y^4}{4b^2}\right)dy = -\frac{13}{30}\ b^3$$

$$\int_{-b}^{b} B_{3,3}dy = \int_{-b}^{b} \left(-\frac{3y^2}{4b} + \frac{y^4}{4b^3}\right)dy = -0.4\ b^2$$

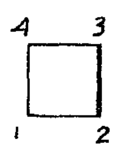
$$\int_{-b}^{b} B_{3,4}dy = \int_{-b}^{b} \frac{y^4}{4b^2}\ dy = 0.1\ b^3$$

$$\int_{-b}^{b} B_{3,7}dy = \int_{-b}^{b} \left(\frac{3}{4}\frac{y^2}{b} - \frac{y^4}{4b^3}\right)dy = 0.4\ b^2$$

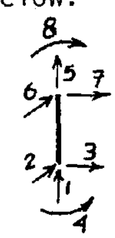
$$\int_{-b}^{b} B_{3,8}dy = \int_{-b}^{b} \left(\frac{-y^2}{2} + \frac{y^4}{4b^2}\right)dy = \frac{-7}{30}\ b^3$$

These non-zero terms are condensed into a [2x4] matrix relating the generalized nodal shell forces corresponding to $[w_i,\ w_i',\ w_{i+1},\ w_{i+1}']$ to the liquid generalized dynamic pressure at the nodes i, i+1. Here, primes denote differentiation with respect to z. These are shown below.

$$\begin{array}{c} \delta u_3 \quad \delta u_4 \quad \delta u_7 \quad \delta u_8 \\ \left[\begin{array}{c} \\ S_e \\ \\ \end{array} \right] \end{array} \begin{array}{c} \delta p_2 \\ \delta p_3 \end{array}$$



Liquid element DOF



shell element DOF

The condensed assembled liquid interaction force matrix is thus:

$$\left[\quad \bar{S} \quad \right]$$

Number of shell wetted surface elements

NELEFS = MM

2 * number of shell elements

2 * NELEFS = 2*MM

APPENDIX B


```
      PROGRAM RIGTD(INPUT,OUTPUT)
      DIMENSION FS11(9660),FS12(21,21),FS21(21,21),FS22(21,21)
      DIMENSION FSC(21,21),P(23)        ,COM(21,21),D(21,22)
      DATA  LINEAR/9660/ , NND /21/ , IBAND /23/, NDF1D /420/
    1 FORMAT(25(2H**))
   11 FORMAT ( 6(E10.4,3X))
   12 FORMAT ( 7(E10.4,3X))
      READ 701,NPROB
      DO 900 IPP=1,NPROB
      READ 702,DENF,R,WH
      PRINT 702, R,WH
      READ 701,NN,MM
      PRINT 701,NN,MM
      READ 701,NHARM
  701 FORMAT(10I8)
  702 FORMAT(8G10.4)
      IBAND=NN+2
      NDF1=(MM-1)*NN
      DO 800 NHR=1,NHARM
      XM=FLOAT(NHR)
      DO 5  I=1,NN
      DO 5  J=1,NN
       COM(I,J)=0.0
       FSC(I,J)=0.0
      FS12(I,J)=0.0
    5 FS22(I,J)=0.0
            CALL FLGEN(DENF,R,WH,XM,NN,MM,IBAND,LINEAR,FS11,FS12,FS22,
     0 COM, NND)
      PRINT 1
      DO 10  I=1,NN
      DO 10  J=1,NN
   10 FS21(J,I)=FS12(I,J)
      CALL BINV(FS11,FS12,P,NDF1,IBAND,NN,NN,NND,NND)
*     NOW FS12= FS11 INV *FS12
*     FS21 * FS11 INV *FS12
      DO 20  I=1,NN
      DO 20  J=1,NN
      FSC(I,J)=0.0
      DO 20 K=1,NN
   20 FSC(I,J)=FSC(I,J)+FS21(1,K)*FS12(K,J)
      DO 30  I=1,NN
      DO 30  J=1,NN
   30 FSC(I,J)=-FSC(I,J)+FS22(I,J)
      DO 40  J=1,NN
      DO 40  I=1,J
   40 D(I,J+1)=FSC(I,J)
      DO 50  I=1,NN
      DO 50  J=1,I
   50 D(I,J)=COM(I,J)
* CALL EGN
      NND1=NND+1
      PRINT 15, NHR
   15 FORMAT(1H1,//,10X,*CIRC. HARMONIC NO. M=*,I2,//)
      CALL EGN(D,NN,5,OMEGA,P,NND,NND1)
  800 CONTINUE
  900 CONTINUE
      END
 5500  SUBROUTINE EGN(D,ND,NMODE,OMEGA, V2,IDIM,IDIM1)
      DIMENSION D(IDIM,IDIM1),V2(IDIM)
      DIMENSION V1(124)
C      PRE-EIGENVALUE CHOLESKY REDUCTIONS
 6010  INA=1
            ND1=ND+1
   18 FORMAT(9(4X,E10.4),/)
```

```
 6040   DO 76 MA=1,ND
 6050   DO 76 MAS=MA,ND
            MA1=MA+1
            MAS1=MAS+1
            GASH=D(MA,MAS1)
            GISH=D(MAS,MA)
            MASH=1
   79   IF(MA-MASH) 77,77,78
   78   GASH=GASH-D(MASH,MA1)*D(MASH,MAS1)
            GISH=GISH-D(MA,MASH)*D(MAS,MASH)
            MASH=MASH+1
 6150   GO TO 79
   77   IF(MAS-MA) 81,81,119
   81   IF(GISH) 118,82,82
  118   GISH=0.
   82   IF(GASH) 83,84,84
   83   GASH=0.
   84   DIAG1=SQRT(GASH)
            DIAG2=SQRT(GISH)
 6230   IF(DIAG1.EQ.0.) GO TO 85
  119   D(MA,MAS1)=GASH/DIAG1
   85   IF(DIAG2.EQ.0.) GO TO 86
            D(MAS,MA)=GISH/DIAG2
   86    CONTINUE
   76   CONTINUE
C     FORM U/UL
 6300   DO 87 MA=1,ND
 6310   DO 87 MAS=MA,ND
            MAS1=MAS+1
            GASH=D(MAS,MA)
            MASH=MA
   91   MASH=MASH+1
 6360   IF(MAS-MASH) 88,89,89
   89   GASH=GASH-D(MA,MASH)*D(MASH-1,MAS1)
 6380   GO TO 91
   88   D(MA,MAS1)=GASH/D(MAS,MAS1)
   87    CONTINUE
C       MULTIPLICATION TO GET (U*ULE-1*ULTE-1*UT)
 6420   DO 92 MA=1,ND
 6430   DO 92 MAS=MA,ND
            MAS1=MAS+1
            GASH=0.
 6460   DO 93 MASH=MAS1,ND1
            GASH=GASH+D(MA,MASH)*D(MAS,MASH)
   93    CONTINUE
            D(MA,MAS1)=GASH
   92    CONTINUE
            MODE=NMODE
C       PU 1.0 IN V1 FROM 1 TO ND AND ITERATIVE
  115   DO 94 I=1,ND
   94   V1(I)=1.
            NUMIT=1
  121   ALAM2=0.
 6570   DO 95 I=1,ND
            I1=I+1
            GASH=0.
 6600   DO 96 J=1,I
            GASH=GASH+V1(J)*D(J,I1)
   96    CONTINUE
 6630   IF(I-ND) 97,98,98
   97   DO 99 J=I1,ND
            GASH=GASH+V1(J)*D(I,J+1)
   99    CONTINUE
   98   V2(I)=GASH
```

```
            ALAM2=ALAM2+GASH*GASH
   95   CONTINUE
            ALAMB=SQRT(ALAM2)
            SIGSQ=0.
 6720   DO 101 I=1,ND
            GASH=V2(I)/ALAMB
            GAS=V1(I)-GASH
            SIGSQ=SIGSQ+GAS*GAS
            V1(I)=GASH
  101    CONTINUE
            ZT=1./10.**12
            NUMIT=NUMIT+1
 6800   IF(SIGSQ-ZT) 102,102,103
  103   IF(NUMIT-150) 121,102,102
  102    CONTINUE
 6830   PRINT 104,NUMIT
  104  FORMAT(20X,15(2H--),/,*    NO. OF ITERATIONS=*,I3,/)
C       TO MULTIPLY (UE-1)*(U*X)
            I=ND
  109   GASH=V1(I)
            J=ND
  107   IF(J-I) 105,105,106
  106   GASH=GASH-V2(J)*D(J,I)
            J=J-1
 6920   GO TO 107
  105   V2(I)=GASH/D(I,I)
            I=I-1
 6950   IF(I) 108,108,109
  108  PRINT 995,INA
  995  FORMAT(/,20X,*AXIAL MODE NO.=*,I3,/)
 6960  PRINT 111,(V2(I),I=1,ND)
 6965  INA=INA+1
  111   FORMAT(4E16.8)
C      OMEGA IN RAD./SEC
 6980  OMEGA=SQRT(1./ALAMB)
      OMEGA=OMEGA/(2.*3.14159)
*     NOW OMEGA IS IN CYCL /SEC
 6990   PRINT 112,OMEGA
  112  FORMAT(//,10X,*NATURAL FREQUENCY=*,F16.8,/)
C       CHANGING TO NEXT MODE
 7040   DO 113 I=1,ND
 7050    DO 113 J=I,ND
            J1=J+1
  113   D(I,J1)=D(I,J1)-ALAMB*V1(I)*V1(J)
            MODE=MODE-1
 7090   IF(MODE) 114,114,115
  114   CONTINUE
 7110  RETURN
 7120  END
 1980  SUBROUTINE MASSF(A,XO,FM,DENF)
 1990  DIMENSION FM(4,4)
 2000  G=32.2*12.
 2010  DO 22 I=1,4
 2020  DO 22 J=1,4
   22  FM(I,J)=0.
      C=3.14159/DENF
 2040  FM(3,3)=  C *A*(2.*XO+A)/(3.*G)
 2060  FM(4,4)=  C *A*(2.*XO-A)/(3.*G)
 2070  FM(3,4)=FM(4,3)=  C *A*XO/(3.*G)
    1  FORMAT(/,10X,25(2H**),/)
 2090  RETURN
 2100  END
 2110  SUBROUTINE FSTIF(A,B,XO,FLAG,FK,XM,DENF)
 2120  DIMENSION A1(4,4),A2(4,4),A3(4,4),FK(4,4)
```

```
2122  DO 12 I=1,4
2123  DO 12 J=1,4
  12  A1(I,J)=A2(I,J)=A3(I,J)=0.
2140  V1=XO*B/A/6.
2150  A1(1,1)=A1(2,2)=A1(3,3)=A1(4,4)=2.*V1
2160  A1(1,2)=A1(2,1)=A1(3,4)=A1(4,3)=-2.*V1
2170  A1(1,3)=A1(3,1)=A1(2,4)=A1(4,2)=-1.*V1
2180  A1(2,3)=A1(3,2)=A1(1,4)=A1(4,1)=V1
   2  V2=XO*A/B/6.
2200  A2(1,1)=A2(4,4)=(2.-A/XO)*V2
2210  A2(1,3)=A2(3,1)=A2(2,4)=A2(4,2)=-V2
2220  A2(2,2)=A2(3,3)=(2.+A/XO)*V2
2230  A2(1,4)=A2(4,1)=-(2.-A/XO)*V2
2240  A2(2,3)=A2(3,2)=-(2.+A/XO)*V2
2250  A2(1,2)=A2(2,1)=A2(3,4)=A2(4,3)=V2
   3  V3=B/A/A/12.
2262  IF(A .EQ. XO) XO=XO+.001
2270  E1=((A+XO)*(A+XO)*ALOG((XO+A)/(XO-A))-2.*A*(2.*A+XO))*V3
2280  E2=((A-XO)*(A-XO)*ALOG((XO+A)/(XO-A))+2.*A*(2.*A-XO))*V3
2290  E3=((A-XO)*(A+XO)*ALOG((XO+A)/(XO-A))+2.*A*XO)*V3
2292  IF(A .EQ. XO) XO=XO-.001
2300  A3(1,1)=A3(4,4)=2.*E1
2310  A3(2,2)=A3(3,3)=2.*E2
2320  A3(1,2)=A3(2,1)=A3(3,4)=A3(4,3)=2.*E3
2330  A3(1,3)=A3(3,1)=A3(2,4)=A3(4,2)=E3
2340  A3(1,4)=A3(4,1)=E1
2350  A3(2,3)=A3(3,2)=E2
     DO 10 I=1,4
     DO 10 J=1,4
  10 FK(I,J)=3.14159*(A1(I,J)+A2(I,J)+A3(I,J)*XM*XM)/DENF
2490  RETURN
2495  END
     SUBROUTINE BINV(A,B,C,NN,NB,NEQ,MM,NEQD,MMD)
     DIMENSION A(2),B(MMD,NEQD),C(2)
     DIMENSION D(1000)
     PRINT 511
 511 FORMAT(10X,25(2H**))
     ND=NN-MM
4890  N=0
   5  N=N+1
     NL=(N-1)*NB
     IF(ABS(A(NL+1)).LT. 1.0E-10) A(NL+1)=1.0
 522 FORMAT(5X,E12.5)
      IF(N .LE. ND) GO TO 16
     NCON=N-ND
     DO 15 IB=1,NEQ
  15 B(NCON,IB)=B(NCON,IB)/A(NL+1)
  16 CONTINUE
4920  IF(N .EQ. NN) GO TO 45
4930  DO 10 K=2,NB
4940 C(K)=A(NL+K)
  10 A(NL+K)=A(NL+K)/A(NL+1)
4960  DO 30 L=2,NB
4970  I=N+L-1
4980  IF(NN .LT. I) GO TO 30
4990  J=0
     IL=(I-1)*NB
5000  DO 20 K=L,NB
5010  J=J+1
  20 A(IL+J)=A(IL+J)-C(L)*A(NL+K)
      IF(N .LE. ND) GO TO 26
     ICON=I-ND
     DO 25 IB=1,NEQ
  25 B(ICON,IB)=B(ICON,IB)-C(L)*B(NCON,IB)
```

```
   26 CONTINUE
   30  CONTINUE
 5050  GO TO 5
*      N= NO. OF EQU.
*      L= NO. OF UNKNOWN
*      K= SEQUENTIAL NO. OF UNKNOWN IN THE BAND
*      NL+K=LFS ... LINEAR SEQUENCE
   45 DO  100 IB=1,NEQ
      DO 70 II=1,MM
   70 D(II+ND)=B(II,IB)
      DO 75 II=1,ND
   75 D(II)=0.0
      N=NN
   40 N=N-1
      NL=(N-1)*NB
      IF( N .EQ. 0) GO TO 60
      DO 50 K=2,NB
      L=N+K-1
      IF( NN.LT. L) GO TO 50
      D(N)=D(N)-A(NL+K)*D(L)
   50 CONTINUE
      GO TO 40
   60 CONTINUE
      DO 80 II=1,MM
   80 B(II,IB)=D(II+ND)
  100 CONTINUE
      PRINT 511
      RETURN
 5150  END
      SUBROUTINE FLGEN(DENF,R,WH,XM,NN,MM,IBAND,LINEAR,FS11,FS12,FS22,
     0 COM, NND)
 1464  DIMENSION FM(4,4),FK(4,4)
 1469  DIMENSION N(4)
      DIMENSION FS11(LINEAR),FS12(NND,NND),FS22(NND,NND),COM(NND,NND)
      DX= R/ FLOAT(NN-1)
      DY=WH/FLOAT(MM-1)
      A=DX*0.5
      B=DY*0.5
      DO 5 I=1,LINEAR
    5 FS11(I)=0.0
      DO 10 I=1,NN
      DO 10 J=1,NN
      FS22(I,J)=0.0
      COM(I,J)=0.0
   10 FS12(I,J)=0.0
      NN1=NN-1
      DO 2000 I=1,NN1
 1740  XO=(FLOAT(I)-.5)*DX
 1745  CALL MASSF(A,XO,FM,DENF)
 1750  COM(I,I)=COM(I,I)+FM(4,4)
 1755  COM(I+1,I+1)=COM(I+1,I+1)+FM(3,3)
 1760  COM(I,I+1)=COM(I,I+1)+FM(4,3)
 1765  COM(I+1,I)=COM(I+1,I)+FM(3,4)
 2000  CONTINUE
      MM2=MM-2
*TRANSFORMATION FROM A SQUARE MATRIX TO BANDED  MATRIX
*    (K,L)=K,J)     ,J=L-K+1
*    TRANSFORMATION FROM A BAND TO LINEAR ARRAY
*     LFS=(K-1)*IBAND+J
      DO 1000 I=1,NN1
 1590  XO=(FLOAT(I)-.5)*DX
 1600  CALL FSTIF(A,B,XO,FLAG,FK,XM,DENF)
      DO 1000 J=1,MM2
      M=(J-1)*(NN-1)+I
```

```
1550  N(1)=(J-1)*NN+I
1560  N(2)=(J-1)*NN+I+1
1570  N(3)=J*NN+I+1
1580  N(4)=J*NN+I
      DO 55 II=1,4
      K=N(II)
      IPAST=K*IBAND-IBAND
      DO 51 JJ=1,4
      IF(N(JJ) . LT. N(II) ) GO TO 51
      L=N(JJ)-K+1
      LFS=IPAST+L
      FS11(LFS)=FS11(LFS)+FK(II,JJ)
   51 CONTINUE
   55 CONTINUE
 1000 CONTINUE
      J=MM-1
      DO 1010 I=1,NN1
      XO=(FLOAT(I)-0.5)*DX
      CALL FSTIF(A,B,XO,FLAG,FK,XM,DENF)
      N(1)=MM2*NN+I
      N(2)=MM2*NN+I+1
*     K22
      FS22(I,I)=FS22(I,I)+FK(4,4)
      FS22(I+1,I+1)=FS22(I+1,I+1)+FK(3,3)
      FS22(I,I+1)=FS22(I,I+1)+FK(4,3)
      FS22(I+1,I)=FS22(I+1,I)+FK(3,4)
      FS12(I,I)=FS12(I,I)+FK(1,4)
      FS12(I+1,I+1)=FS12(I+1,I+1)+FK(2,3)
      FS12(I,I+1)=FS12(I,I+1)+FK(1,3)
      FS12(I+1,I  )=FS12(I+1,I  )+FK(2,4)
      DO 56 II=1,2
      K=N(II)
      IPAST=K*IBAND-IBAND
      DO 57 JJ=1,2
      IF( N(JJ) .LT. N(II) ) GO TO 57
      L=N(JJ)-K+1
      LFS=IPAST+L
      FS11(LFS)=FS11(LFS)+FK(II,JJ)
   57 CONTINUE
   56 CONTINUE
 1010 CONTINUE
      RETURN
      END
```

```
      5
1.0       78.74      7.874
     21        21
      1
1.0       78.74      15.748
     21        21
      1
1.0       78.74      23.622
     21        21
      1
1.0       78.74      31.496
     21        21
      1
1.0       78.74      70.866
     21        21
      1
```

```
00.05.03.LIST.
00.05.03.ACCOUNT,A43Y000,.              B-8
00.05.03.COPYSBF(INPUT,OUTPUT)
00.05.04. COPY COMPLETE.
00.05.04.UEMS,        0.749KUNS.
00.05.04.UECP,        0.343SECS.
00.05.04.AESR,        1.0 OUNTS.
00.15.07.UCLP, 21,         2.112 B442EJX
```

APPENDIX C


```
TOTAL,CM200000,T700.
FTN(B=COUPLE)
COUPLE.
REWIND(TAPE1)
GET(TAPE5=GASN)
FTN(B=SHELL)
SHELL.
RETURN(TAPE5)
SAVE(TAPE4=MATHF)
SAVE(TAPE9=MODHF)
SAVE(TAPE20=STRMAT)
RETURN(TAPE4)
FTN(B=PART1)
GET(TAPE5=MATHF)
GET(TAPE3=MODHF)
PART1.
SAVE(TAPE2=DATAHF)
RETURN(TAPE5)
RETURN(TAPE4)
RETURN(TAPE6)
GET(TAPE1=ACC)
REWIND(TAPE2)
REWIND(TAPE3)
FTN(B=PART2)
PART2.
SAVE(TAPE4=W1HF)
SAVE(TAPE5=W2HF)
SAVE(TAPE6=W3HF)
```

```
      PROGRAM COUPLE (INPUT,OUTPUT,TAPE1,TAPE2)
      DIMENSION FS(31713),P(33),SC(31,62),SCT(62,31),ADM(62,62)
      DATA MMD/31/ , NDFSD/62/ , IBAND/33/ , LINEAR/31713/
   11 FORMAT(10(E10.4,3X))
* CARD 1
      READ 702,DENF,R,WH
* CARD 2
      READ 701 ,NN,MM
* CARD 3
      READ 701,NHR
      PRINT 701,NN,MM
      PRINT 702,DENF,R,WH
  701 FORMAT(10I8)
  702 FORMAT(8G10.4)
      INDEX=1
      RES=NHR-(NHR/2)*2
      IF(RES .EQ. 0.) INDEX=2
*    MM=NO. OF FLUID ELEMENTS ALONG THE GENERATOR
*     NN=NO. OF FLUID ELEMENTS ALONG THE RADIUS
      IBAND=MM+2
      NDFS=MM*2
      IF(INDEX .EQ. 1) NDFF=(NN)*MM
      IF(INDEX .EQ. 2) NDFF=(NN+1)*MM
      PRINT 701,IBAND,NDFS,NDFF
      PRINT 1111
 1111 FORMAT(1H1)
      XM=FLOAT(NHR)
              CALL FLGEN(DENF,R,WH,XM,NN,MM,NDFS,IBAND,MMD,NDFSD,LINEAR,
     Q FS,SC,INDEX)
      DO 10 I=1,MM
      DO 10 J=1,NDFS
   10 SCT(J,I)=SC(I,J)
      WRITE(2)(FS(I),I=1,LINEAR)
      REWIND 2
      CALL BINV(FS,SC,P,NDFF,IBAND,NDFS,MM,NDFSD,MMD)
*     NOW SC=FS INV *SC
      PRINT 1
      PRINT 11,((SC(I,J),J=1,NDFS),I=1,MM)
*     SCT * FS INV *SC =ADM
      DO 20 I=1,NDFS
      DO 20 J=1,NDFS
      ADM(I,J)=0.0
      DO 20 K=1,MM
   20 ADM(I,J)=ADM(I,J)+SCT(I,K)*SC(K,J)
      PRINT 1
    1 FORMAT(25(2H**))
      WRITE(1)((ADM(I,J),J=1,NDFS),I=1,NDFS)
      END
      SUBROUTINE BINV(A,B,C,NN,NB,NEQ,MM,NEQD,MMD)
      DIMENSION A(2),B(MMD,NEQD),C(2)
      DIMENSION D(1000)
      PRINT 511
  511 FORMAT(10X,25(2H**))
      ND=NN-MM
 4890  N=0
    5  N=N+1
      NL=(N-1)*NB
      IF(ABS(A(NL+1)).LT. 1.0E-10) A(NL+1)=1.0
  522 FORMAT(5X,E12.5)
       IF(N .LE. ND) GO TO 16
      NCON=N-ND
      DO 15 IB=1,NEQ
   15 B(NCON,IB)=B(NCON,IB)/A(NL+1)
   16 CONTINUE
```

```
      IF(N .EQ. NN) GO TO 45
      DO 10 K=2,NB
      C(K)=A(NL+K)
   10 A(NL+K)=A(NL+K)/A(NL+1)
      DO 30 L=2,NB
      I=N+L-1
      IF(NN .LT. I) GO TO 30
      J=0
      IL=(I-1)*NB
      DO 20 K=L,NB
      J=J+1
   20 A(IL+J)=A(IL+J)-C(L)*A(NL+K)
      IF(N .LE. ND) GO TO 26
      ICON=I-ND
      DO 25 IB=1,NEQ
   25 B(ICON,IB)=B(ICON,IB)-C(L)*B(NCON,IB)
   26 CONTINUE
   30 CONTINUE
      GO TO 5
*     N= NO. OF EQU.
*     L= NO. OF UNKNOWN
*     K= SEQUENTIAL NO. OF UNKNOWN IN THE BAND
*     NL+K=LFS ... LINEAR SEQUENCE
   45 DO  100 IB=1,NEQ
      DO 70 II=1,MM
   70 D(II+ND)=B(II,IB)
      DO 75 II=1,ND
   75 D(II)=0.0
      N=NN
   40 N=N-1
      NL=(N-1)*NB
      IF( N .EQ. 0) GO TO 60
      DO 50 K=2,NB
      L=N+K-1
      IF( NN.LT. L) GO TO 50
      D(N)=D(N)-A(NL+K)*D(L)
   50 CONTINUE
      GO TO 40
   60 CONTINUE
      DO 80 II=1,MM
   80 B(II,IB)=D(II+ND)
  100 CONTINUE
      PRINT 511
      RETURN
      END
      SUBROUTINE FLGEN(DENF,R,WH,XM,NN,MM,NDFS,IBAND,MMD,NDFSD,LINEAR,
     0 FS,SC,INDEX)
      DIMENSION FS(LINEAR),          SC(MMD,NDFSD)
      DIMENSION FM(4,4),FK(4,4),FF(2,4),N(4)
      DX=R/FLOAT(NN)
      DY=WH/FLOAT(MM)
      A=DX*0.5
      B=DY*0.5
      DO 10 I=1,LINEAR
   10 FS(I)=0.0
      DO 20 I=1,MMD
      DO 20 J=1,NDFSD
   20 SC(I,J)=0.0
*     TRANSFORMATION FROM A SQUARE MATRIX TO A BANDED MATRIX
*     (K,L) = (K,J)   , J=L-K+1
*     TRANSFORMATION FROM A BAND TO A LINEAR ARRAY
*     LFS=(K-1)*IBAND +J
      NN1=NN-1
      MM1=MM-1
```

```
      IF(INDEX .EQ. 1) NNX=NN-1
      IF(INDEX .EQ. 2) NNX=NN
      DO 1000 I=1,NNX
      IF(INDEX .EQ. 1) XO=FLOAT(I)*DX+A
      IF(INDEX .EQ. 2) XO=FLOAT(I-1)*DX+A
      CALL FSTIF(A,B,XO,FLAG,FK,XM,DENF)
      DO 1000 J=1,MM1
      N(1)=(I-1)*MM+J
      N(2)=I*MM+J
      N(3)=N(2)+1
      N(4)=N(1)+1
      DO 55 II=1,4
      K=N(II)
      IPAST=K*IBAND-IBAND
      DO 51 JJ=1,4
      IF(N(JJ) .LT. N(II) ) GO TO 51
      L=N(JJ)-K+1
      LFS=IPAST+L
      FS  (LFS)=FS  (LFS)+FK(II,JJ)
   51 CONTINUE
   55 CONTINUE
 1000  CONTINUE
      DO 1010 I=1,NNX
      IF(INDEX .EQ. 1) XO=FLOAT(I)*DX+A
      IF(INDEX .EQ. 2) XO=FLOAT(I-1)*DX+A
      CALL FSTIF(A,B,XO,FLAG,FK,XM,DENF)
      N(1)=I*MM
      N(2)=(I+1)*MM
      DO 65 II=1,2
      K=N(II)
      IPAST=K*IBAND-IBAND
      DO 61 JJ=1,2
       IF(N(JJ) .LT. N(II) ) GO TO 61
      L=N(JJ)-K+1
      LFS=IPAST+L
      FS(LFS)=FS(LFS)+FK(II,JJ)
   61 CONTINUE
   65 CONTINUE
 1010 CONTINUE
      IF(INDEX .EQ. 2) GO TO 76
      XO=A
      CALL FSTIF(A,B,XO,FLAG,FK,XM,DENF)
      DO 1020 J=1,MM1
      N(2)=J
      N(3)=J+1
      DO 75 II=2,3
      K=N(II)
      IPAST=IBAND*K-IBAND
      DO 71 JJ=2,3
       IF(N(JJ) .LT. N(II) ) GO TO 71
      L=N(JJ)-K+1
      LFS=IPAST+L
      FS(LFS)=FS(LFS)+FK(II,JJ)
   71 CONTINUE
   75 CONTINUE
 1020 CONTINUE
      J=MM
      IPAST=J*IBAND-IBAND
      LFS=IPAST+1
      FS(LFS)=FS(LFS)+FK(2,2)
   76 DO 40 J=1,NDFSD
      DO 40 I=1,MMD
   40 SC(I,J)=0.0
      XO=(FLOAT(NN)-0.5)*DX
```

```
      CALL FFORCE(      A,B,XO,FF)
      DO 3000 J=1,MM1
      NI=(J-1)*2
      DO 205 JJ=1,4
      L=NI+JJ
       SC(J,L)= SC(J,L)+FF(1,JJ)
  205  SC(J+1,L)= SC(J+1,L)+FF(2,JJ)
 3000 CONTINUE
      RETURN
      END
      SUBROUTINE FFORCE(      A,B,XO,FF)
      DIMENSION FF(2,4)
      PI=3.14159
      DO 10 I=1,2
      DO 10 J=1,4
   10  FF(I,J)=0.0
      V=PI*(XO+A)*B*0.5
      BV=V*B
      FF(1,1)=1.4*V
      FF(1,2)=-43.0/30.0*BV
      FF(1,3)=0.6*V
      FF(1,4)=-13.0/30.0*BV

      FF(2,1)=1.6*V
      FF(2,2)=0.1*BV
      FF(2,3)=0.4*V
      FF(2,4)=-7.0/30.0*BV
      RETURN
      END
 2110  SUBROUTINE FSTIF(A,B,XO,FLAG,FK,XM,DENF)
 2120  DIMENSION A1(4,4),A2(4,4),A3(4,4),FK(4,4)
 2122  DO 12 I=1,4
 2123  DO 12 J=1,4
   12  A1(I,J)=A2(I,J)=A3(I,J)=0.
 2140  V1=XO*B/A/6.
 2150  A1(1,1)=A1(2,2)=A1(3,3)=A1(4,4)=2.*V1
 2160  A1(1,2)=A1(2,1)=A1(3,4)=A1(4,3)=-2.*V1
 2170  A1(1,3)=A1(3,1)=A1(2,4)=A1(4,2)=-1.*V1
 2180  A1(2,3)=A1(3,2)=A1(1,4)=A1(4,1)=V1
    2  V2=XO*A/B/6.
 2200  A2(1,1)=A2(4,4)=(2.-A/XO)*V2
 2210  A2(1,3)=A2(3,1)=A2(2,4)=A2(4,2)=-V2
 2220  A2(2,2)=A2(3,3)=(2.+A/XO)*V2
 2230  A2(1,4)=A2(4,1)=-(2.-A/XO)*V2
 2240  A2(2,3)=A2(3,2)=-(2.+A/XO)*V2
 2250  A2(1,2)=A2(2,1)=A2(3,4)=A2(4,3)=V2
    3  V3=B/A/A/12.
 2262  IF(A .EQ. XO) XO=XO+.001
 2270  E1=((A+XO)*(A+XO)*ALOG((XO+A)/(XO-A))-2.*A*(2.*A+XO))*V3
 2280  E2=((A-XO)*(A-XO)*ALOG((XO+A)/(XO-A))+2.*A*(2.*A-XO))*V3
 2290  E3=((A-XO)*(A+XO)*ALOG((XO+A)/(XO-A))+2.*A*XO)*V3
 2292  IF(A .EQ. XO) XO=XO-.001
 2300  A3(1,1)=A3(4,4)=2.*E1
 2310  A3(2,2)=A3(3,3)=2.*E2
 2320  A3(1,2)=A3(2,1)=A3(3,4)=A3(4,3)=2.*E3
 2330  A3(1,3)=A3(3,1)=A3(2,4)=A3(4,2)=E3
 2340  A3(1,4)=A3(4,1)=E1
 2350  A3(2,3)=A3(3,2)=E2
      DO 10 I=1,4
      DO 10 J=1,4
   10 FK(I,J)=3.14159*(A1(I,J)+A2(I,J)+A3(I,J)*XM*XM)/DENF
 2490  RETURN
 2495  END
```

```
.000093   720.0     240.0
     20        10
      1
```

```
      PROGRAM SHELL(INPUT,OUTPUT,TAPE4,TAPE5,TAPE7,TAPE10,TAPE9,TAPE20,
     Q TAPE1)
      DIMENSION ADM(62,62)
      DIMENSION D(124,125)
  701 FORMAT(10I8)
  702 FORMAT(8G10.4)
      PRINT 1
    1 FORMAT(1H1)
* CARD 4
  111 READ 702,UM,E1,PX
      PRINT 772
  772 FORMAT(//,10X,* MATERIAL PROPERTIES*,/)
      PRINT 703,UM,E1,PX
  703 FORMAT(//,5X,* DENSITY OF SHELL MATERIAL=*,G10.4
     *,//,5X,* MODULUS OF ELASTICITY=*,G10.4
     *,//,5X,* POISSON RATIO=*,G10.4)
  162  PRINT 771
* CARD 5
      READ 702,R,H,AL,FL
  771  FORMAT(///,10X,*STRUCTURAL GEOMETRY*,/)
  170 PRINT 7,R,H,AL,FL
    7  FORMAT(2X,*RADIUS=*,F9.3,5X,*THICKNESS=*,F6.3,5X,*HEIGHT=*,F9.3
     Q ,10X,*FLUID HEIGHT*,F9.3 )
* CARD 6
  113 READ 701,NSIN
* CARD 7
  114 READ 701,NELEM
* CARD 8
      READ 701,NELFS,NELFR
      FRH=AL-FL
      NP=NELEM+1
      NDFS=(NELFS)*2
      NDF=4
            NFREE=NDF*NP
  230   ELN=AL/FLOAT(NELEM)
C        NDF=NUMBER OF DEGREE OF FREEDOM PER NODE
C        NBAND=HALF BAND WIDTH
C        NFREE=NUMBER OF DEGREES OF FFREEDOM
      ELN1=FL/FLOAT(NELFS)
      IF(NELFR .EQ. 0 ) GO TO 401
      ELN2=FRH/FLOAT(NELFR)
* CARD 9
  401 READ 701,NMODE
      PRINT 43,NELEM
   43 FORMAT(//,5X,* NO. OF RING ELEMENTS=*,I3)
      PRINT 51,NMODE,NSIN
   51 FORMAT(//,5X,I2,* AXIAL MODES TO BE CONSIDERED FOR*,I2,
     Q* CIRCUMFERENTIAL NO.S*,/)
  430  DO 59 KS1=1,NSIN
* CARD 10
  431 READ 701,NAT
  450  ANT=NAT
      CALL STRMAT(ELN1,R,H,ANT,PX,E1,NELFS,NELFR)
      IF(NELFR .EQ. 0 ) GO TO 501
      CALL STRMAT(ELN2,R,H,ANT,PX,E1,NELFS,NELFR)
* CARD 11
  501 READ 701,NBCAS
      PRINT 52,NBCAS
   52 FORMAT(//,5X,* NO. OF BOUNDARY CASES CONSIDERED=*,I2)
  502  DO 808 IPR=1,NBCAS
  504  IF(IPR .GT. 1) GO TO 801
  520 CALL ASSTR(NELEM,NELFS,NELFR,H,UM,ANT,PX,R,ELN1,ELN2,D,E1)
      NFREE1=NFREE+1
      READ(1)((ADM(I,J),J=1,NDFS),I=1,NDFS)
```

```
      DO 140 I=1,NDFS,2
      I2=2*I+1
      DO 140 J=1,I,2
      J2=2*J+1
       D( I2 , J2 )= D( I2 , J2 )+ ADM( I , J )
       D(I2+1,J2  )= D(I2+1,J2  )+ ADM(I+1,J  )
       D(I2+1,J2+1)= D(I2+1,J2+1)+ ADM(I+1,J+1)
      IF(I2 .EQ. J2) GO TO 140
       D( I2 ,J2+1)= D( I2 ,J2+1)+ ADM( I ,J+1)
  140 CONTINUE
      REWIND 1
      WRITE(4)(NFREE)
  570  WRITE(4)((D(I,J),J=1,NFREE1 ),I=1,NFREE)
      REWIND 4
  801 CONTINUE
* CARD 12
 990  READ 911,NBC
  911 FORMAT(A4)
  620  CALL BOUN(NFREE,NAT,D,NO,NBC)
      NO1=NO+1
  653  WRITE(7)((D(I,J),J=1,NO1 ),I=1,NO)
      REWIND 7
  691  DO 91 I=1,124
  692  DO 91 J=1,125
   91  D(I,J)=0.
  700  CALL EGN(D,NO,NMODE,E1,NBC)
  808  CONTINUE
   59  CONTINUE
   41  CONTINUE
  790  END
 5500  SUBROUTINE EGN(D,NO,NMODE,E,NBC)
      DIMENSION D(124,125),V1(124),V2(124)
C      PRE-EIGENVALUE CHOLESKY REDUCTIONS
 6010  INA=1
            NO1=NO+1
      READ(7)((D(I,J),J=1,NO1),I=1,NO)
 6040   DO 76 MA=1,NO
 6050   DO 76 MAS=MA,NO
            MA1=MA+1
            MAS1=MAS+1
            GASH=D(MA,MAS1)
            GISH=D(MAS,MA)
            MASH=1
   79   IF(MA-MASH) 77,77,78
   78   GASH=GASH-D(MASH,MA1)*D(MASH,MAS1)
            GISH=GISH-D(MA,MASH)*D(MAS,MASH)
            MASH=MASH+1
 6150   GO TO 79
   77   IF(MAS-MA) 81,81,119
   81   IF(GISH) 118,82,82
  118   GISH=0.
   82   IF(GASH) 83,84,84
   83   GASH=0.
   84   DIAG1=SQRT(GASH)
            DIAG2=SQRT(GISH)
 6230   IF(DIAG1.EQ.0.) GO TO 85
  119   D(MA,MAS1)=GASH/DIAG1
   85   IF(DIAG2.EQ.0.) GO TO 86
            D(MAS,MA)=GISH/DIAG2
   86    CONTINUE
   76   CONTINUE
C       FORM U/UL
 6300   DO 87 MA=1,NO
 6310   DO 87 MAS=MA,NO
```

```
          MAS1=MAS+1
          GASH=D(MAS,MA)
          MASH=MA
   91   MASH=MASH+1
 6360   IF(MAS-MASH) 88,89,89
   89   GASH=GASH-D(MA,MASH)*D(MASH-1,MAS1)
 6380   GO TO 91
   88   D(MA,MAS1)=GASH/D(MAS,MAS1)
   87    CONTINUE
C       MULTIPLICATION TO GET (U*ULE-1*ULTE-1*UT)
 6420   DO 92 MA=1,ND
 6430   DO 92 MAS=MA,ND
          MAS1=MAS+1
          GASH=0.
 6460   DO 93 MASH=MAS1,ND1
          GASH=GASH+D(MA,MASH)*D(MAS,MASH)
   93    CONTINUE
          D(MA,MAS1)=GASH
   92    CONTINUE
          MODE=NMODE
C       PU 1.0 IN V1 FROM 1 TO ND AND ITERATIVE
  115   DO 94 I=1,ND
   94   V1(I)=1.
          NUMIT=1
  121   ALAM2=0.
 6570   DO 95 I=1,ND
          I1=I+1
          GASH=0.
 6600   DO 96 J=1,I
          GASH=GASH+V1(J)*D(J,I1)
   96    CONTINUE
 6630   IF(I-ND) 97,98,98
   97   DO 99 J=I1,ND
          GASH=GASH+V1(J)*D(I,J+1)
   99    CONTINUE
   98   V2(I)=GASH
          ALAM2=ALAM2+GASH*GASH
   95   CONTINUE
          ALAMB=SQRT(ALAM2)
          SIGSQ=0.
 6720   DO 101 I=1,ND
          GASH=V2(I)/ALAMB
          GAS=V1(I)-GASH
          SIGSQ=SIGSQ+GAS*GAS
          V1(I)=GASH
  101    CONTINUE
          ZT=1./10.**12
          NUMIT=NUMIT+1
 6800   IF(SIGSQ-ZT) 102,102,103
  103   IF(NUMIT-150) 121,102,102
  102    CONTINUE
         PRINT 11
 6830   PRINT 104,NUMIT
  104  FORMAT(*    NO OF ITERATIONS=*,I3,/)
C       TO MULTIPLY (UE-1)*(U*X)
          I=ND
  109   GASH=V1(I)
          J=ND
  107   IF(J-I) 105,105,106
  106   GASH=GASH-V2(J)*D(J,I)
          J=J-1
 6920   GO TO 107
  105   V2(I)=GASH/D(I,I)
          I=I-1
```

```
 6950   IF(I) 108,108,109
  108   PRINT 995,INA
      WRITE(10,3)(INA)
    3 FORMAT(6I3)
C       OMEGA IN CYCLE/SEC
 6980   OMEGA=SQRT(1./ALAMB)/2./3.1415927
 6990   PRINT 112,OMEGA
      WRITE(10,1)(OMEGA)
      RES=0.0
       PRINT 12
    1 FORMAT(4E14.8)
      IF(NBC .NE. 4HCLFR) GO TO 500
      WRITE(9)(ND)
      WRITE(9)(OMEGA)
      WRITE(9)(V2(I),I=1,ND)
  500 IF(NBC .EQ. 4HSMSM) GO TO 40
      IF(NBC .EQ. 4HCLSM) GO TO 30
      PRINT 111,RES,RES,RES,RES
      WRITE(10,1)(RES,RES,RES,RES)
      PRINT 111,(V2(I),I=1,ND)
      WRITE(10,1)((V2(I),I=1,ND))
      IF(NBC .EQ. 4HCLCL) PRINT 111,RES,RES,RES,RES
      IF(NBC .EQ. 4HCLCL) WRITE(10,1)(RES,RES,RES,RES)
      GO TO 7040
   30 PRINT 111,RES,RES,RES,RES
      WRITE(10,1)(RES,RES,RES,RES)
      ND1=ND-1
      PRINT 111,(V2(I),I=1,ND1)
      WRITE(10,1)((V2(I),I=1,ND1))
      PRINT 111,RES,RES,RES,V2(ND)
      WRITE(10,1)(RES,RES,RES,V2(ND))
      GO TO 7040
   40 PRINT 111,RES,RES,RES,V2(1)
      WRITE(10,1)(RES,RES,RES,V2(1))
       ND1=ND-1
      PRINT 111,(V2(I),I=2,ND1)
      WRITE(10,1)((V2(I),I=2,ND1))
      PRINT 111,RES,RES,RES,V2(ND)
      WRITE(10,1)(RES,RES,RES,V2(ND))
  995  FORMAT(//,10X,*AXIAL NO. =*,I3)
   11 FORMAT(//,20X,25(2H--))
   12 FORMAT (30X,* MODE SHAPE*,/,15X,*U*,20X,*V*,20X,*W*,20X,*DW/DZ*)
  111 FORMAT(4(5X,F16.8))
  112  FORMAT(//,10X,*NATURAL FREQUENCY=*,E20.10)
    2 FORMAT(8E16.8)
C       CHANGING TO NEXT MODE
 7040   DO 113 I=1,ND
 7050     DO 113 J=I,ND
             J1=J+1
  113   D(I,J1)=D(I,J1)-ALAMB*V1(I)*V1(J)
 6965  INA=INA+1
             MODE=MODE-1
 7090   IF(MODE) 114,114,115
  114   CONTINUE
 7110  RETURN
 7120  END
      SUBROUTINE STRMAT(AL,R,H,ANT,P,E,NE1,NE2)
      DIMENSION B(8,8),DM(8,8),DBT(6,8,6)
    4 FORMAT(///,10X,*STRESS-DISPLACEMENTS MATRICES*,/)
      PRINT 4
      HPI=1.570795
      HAL=AL/2.
      CALL DMATX(H,P,DM)
      DO 5 I=1,6
```

```
      DO 5 J=1,6
    5 DM(I,J)=DM(I,J)*E*H/(1.-P*P)
    3 FORMAT(6(5X,G10.4))
       DO 100 NN=1,6
       IF(NN .LT. 4) THETA=0.0
       IF(NN .GE. 4) THETA=HPI
      IF(NN .EQ. 1) X=0.0
      IF(NN .EQ. 2) X=AL
      IF(NN .EQ. 3) X=HAL
      IF(NN .EQ. 4) X=0.0
      IF(NN .EQ. 5) X=AL
      IF(NN .EQ. 6) X=HAL
      THETAM=THETA*ANT
      CALL BMATX(AL,R,ANT,X,B)
      COSIN=COS(THETAM)
      SINE=SIN(THETAM)
*    T- *B
      DO 10 J=1,8
      B(1,J)=B(1,J)*COSIN
      B(2,J)=B(2,J)*COSIN
      B(3,J)=B(3,J)*SINE
      B(4,J)=B(4,J)*COSIN
      B(5,J)=B(5,J)*COSIN
   10 B(6,J)=B(6,J)*SINE
      WRITE(20)(NE1,NE2)
+     D*T-*B
      DO 20 I=1,6
      DO 20 J=1,8
      DBT(I,J,NN)=0.0
      DO 20 K=1,6
   20 DBT(I,J,NN)=DBT(I,J,NN)+DM(I,K)*B(K,J)
      WRITE(20)(NN)
      PRINT 1,NN
      WRITE(20)((DBT(I,J,NN),J=1,8),I=1,6)
      PRINT 2,((DBT(I,J,NN),J=1,8),I=1,6)
    1 FORMAT(//,10X,*NN=*,I6,/)
    2 FORMAT(8(5X,G10.4))
  100 CONTINUE
      RETURN
      END
  800  SUBROUTINE BOUN(NFREE,NAT,D,NO,NBC)
      DIMENSION D(124,125)
  815  IF(NBC .EQ. 4HCLFR ) GO TO 1
  820  IF(NBC .EQ. 4HCLCL)  GO TO 2
  825  IF(NBC .EQ. 4HCLSM ) GO TO 3
  830  IF(NBC .EQ. 4HSMSM) GO TO 4
   55 FORMAT(1H1)
    1 PRINT 55
      PRINT 11
   11  FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-FREE CYL*)
  845  PRINT 101,NAT
  101  FORMAT(/, *  FOR CIRCUMFERENTIAL HARM. M=*,I3,/)
  855  NO=NFREE-4
  860  GO TO 33
    2 PRINT 55
      PRINT 12
   12  FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-CL CYL.*)
  875  PRINT 101,NAT
  880  NO=NFREE-8
   33  DO 777 I=1,NO
      NO1=NO+1
      DO 777 J=1,NO1
  777  D(I,J)=D(I+4,J+4)
  900  RETURN
```

```
    3 PRINT 55
      PRINT 13
   13  FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-SIMPLE CYL.*)
  915  PRINT 101,NAT
  920  NO=NFREE-7
      NO1=NO-1
      DO 111 I=1,NO1
  930  DO 112 J=1,NO
  112  D(I,J)=D(I+4,J+4)
  111  D(I,NO+1)=D(I+4,NFREE+1)
  945  I=NO
  950  DO 113 J=1,NO
  113  D(I,J)=D(NFREE,J+4)
  960  D(NO,NO+1)=D(NFREE,NFREE+1)
      D(NO,NO)=D(NFREE,NFREE)
  965  RETURN
    4 PRINT 55
      PRINT 14
   14  FORMAT(//,*NATURAL MODES AND FREQ. FOR A SIMPLE SIMPLE CYL.*)
  980  PRINT 101,NAT
  985  NO=NFREE-6
      NO2=NO-1
      DO 222 I=1,NO2
  995  DO 221 J=1,NO
  221  D(I,J)=D(I+3,J+3)
  222  D(I,NO+1)=D(I+3,NFREE+1)
 1010  I=NO
 1015  DO 223 J=1,NO
  223  D(I,J)=D(NFREE,J+3)
 1025  D(NO,NO+1)=D(NFREE,NFREE+1)
      D(NO,NO)=D(NFREE,NFREE)
 1030  RETURN
 1035  END
      SUBROUTINE ASSTR(NELEM,NELFS,NELFR,H,UM,ANT,PX,R,ELN1,ELN2,D,E)
      DIMENSION D(124,125)
 1175  DIMENSION AMAS(8,8),ST(8,8)
      CALL STIFF(H,ELN1,ANT,PX,R,ST)
   10 FORMAT(8(5X,G10.4))
      DO 40 I=1,8
      DO 40 J=1,8
   40 ST(I,J)=ST(I,J)*E
      CALL MASS(UM,R,ELN1,H,AMAS)
      DO 100 I=1,NELFS
      IN=(I-1)*4
       DO 20 II=1,8
       DO 20 JJ=1,II
      K=IN+II
      L=IN+JJ
   20 D(K,L)=D(K,L)+AMAS(II,JJ)
      DO 30 JJ=1,8
      DO 30 II=1,JJ
      K=IN+II
      L=IN+JJ+1
   30 D(K,L)=D(K,L)+ST(II,JJ)
  100 CONTINUE
      IF(NELFR .EQ. 0 ) RETURN
      CALL MASS(UM,R,ELN2,H,AMAS)
      CALL STIFF(H,ELN2,ANT,PX,R,ST)
      DO 50 I=1,8
      DO 50 J=1,8
   50 ST(I,J)=ST(I,J)*E
      NELFS1=NELFS+1
      DO 200 I=NELFS1,NELEM
      IN=(I-1)*4
```

```
      DO 60 II=1,8
      DO 60 JJ=1,II
      K=IN+II
      L=IN+JJ
   60 D(K,L)=D(K,L)+AMAS(II,JJ)
      DO 70 JJ=1,8
      DO 70 II=1,JJ
      K=IN+II
      L=IN+JJ+1
   70 D(K,L)=D(K,L)+ST(II,JJ)
  200 CONTINUE
 1440  RETURN
 1450  END
 2860  SUBROUTINE MASS(RHO,R,AL,H,A)
 2870   DIMENSION A(8,8)
C       INITIALIZE MASS MATRIX
 2890   DO 116 J=1,8
 2900   DO 116 I=1,8
 2910   A(I,J)=0.
  116   CONTINUE
C       CONSTRUCT MASS MATRIX
            PI=3.1415927
 2950  CONST=R*PI*RHO*H
            A(1,1)=A(2,2)=A(5,5)=A(6,6)=CONST*AL/3.
            A(5,1)=A(1,5)=A(6,2)=A(2,6)=CONST*AL/6.
            A(3,3)=A(7,7)=CONST*13.*AL/35.
 2990   A(4,3)=A(3,4)=CONST*11.*AL**2/210.
 3000   A(7,8)=A(8,7)=-A(4,3)
            A(4,4)=A(8,8)=CONST*AL**3/105.
            A(7,3)=A(3,7)=CONST*AL*9./70.
 3030  A(4,7)=A(7,4)=CONST*13.*AL*AL/420.
            A(8,3)=A(3,8)=-CONST*13.*AL**2/420.
            A(8,4)=A(4,8)=-CONST*AL**3/140.
   54  FORMAT(//,10X,*-----------------------------*,/)
 3068  RETURN
            END
 3100  SUBROUTINE STIFF(H,AL,AM,P,R,SUM)
 3110  DIMENSION X(20),W(20),DM(8,8),BB(8,8),DB(8,8),BD(8,8)
     X ,SUM(8,8)
 3140  READ(5) NI
 3150   DO 21 I=1,NI
   21  READ(5) X(I),W(I)
      REWIND 5
 3170   A=0. $ B=AL
 3180   DO 12 I=1,NI
 3190   X(I)=(B-A)/2.*X(I)+(B+A)/2.
   12   W(I)=(B-A)/2.*W(I)
 3210   DO 13 I=1,8
 3220   DO 13 J=1,8
 3230   SUM(I,J)=0.
   13   CONTINUE
 3250  CALL DMATX(H,P,DM)
 3260   DO 23 I=1,NI
 3270   CALL BMATX(AL,R,AM,X(I),BB)
 3280  CALL MBTM(DM,BB,DB,6,6,8)
 3290   CALL MBTTM(BB,DB,BD,8,6,8)
 3300   DO 22 J=1,8
 3310   DO 22 K=1,8
   22   SUM(J,K)=SUM(J,K)+W(I)*BD(J,K)
   23   CONTINUE
 3340  CONST=R*3.1415927*H/(1.-P*P)
 3350   DO 1 I=1,8
 3360   DO 1 J=1,8
    1   SUM(I,J)=SUM(I,J)*CONST
```

```
54   FORMAT(/,20X,* ==========================*,//)
     RETURN
     END
     SUBROUTINE MBTM(D,B,DB,L,M,N)
     DIMENSION D(8,8),B(8,8),DB(8,8)
     DO 25 J=1,N
     DO 25 I=1,L
     DB(I,J)=0.
     DO 25 K=1,M
25   DB(I,J)=DB(I,J)+D(I,K)*B(K,J)
     RETURN
     END
C
     SUBROUTINE MBTTM(D,B,DB,L,M,N)
     DIMENSION D(8,8),B(8,8),DB(8,8)
     DO 26 J=1,N
     DO 26 I=1,L
     DB(I,J)=0.
     DO 26 K=1,M
26   DB(I,J)=DB(I,J)+D(K,I)*B(K,J)
     RETURN
     END
C
     SUBROUTINE DMATX(H,P,DM)
     DIMENSION DM(8,8)
     DO 27 I=1,6
     DO 27 J=1,6
27   DM(I,J)=0.
     H2=H*H
     DM(1,1)=DM(2,2)=1.
     DM(1,2)=DM(2,1)=P
     DM(3,3)=(1.-P)/2.
     DM(4,4)=DM(5,5)=H2/12.
     DM(5,4)=DM(4,5)=P*H2/12.
     DM(6,6)=H2*(1.-P)/24.
     RETURN
     END
C
     SUBROUTINE BMATX(AL,R,AM,X,B)
     DIMENSION B(8,8)
     X2=X**2
     X3=X**3
     AL2=AL**2
     AL3=AL**3
     AM2=AM**2
     R2=R**2
     DO 29 I=1,6
     DO 29 J=1,8
     B(I,J)=0.
29   CONTINUE
     B(1,1)=B(3,2)=-1./AL
     B(1,5)=B(3,6)=1./AL
     B(2,2)=AM*(1.-X/AL)/R
     B(2,3)=(1.-3.*X2/AL2+2.*X3/AL3)/R
     B(2,4)=X*(1.-2.*X/AL+X2/AL2)/R
     B(2,6)=AM*X/R/AL
     B(2,7)=X2*(3./AL2-2.*X/AL3)/R
     B(2,8)=X2*(-1./AL+X/AL2)/R
     B(3,1)=-B(2,2)
     B(3,5)=-B(2,6)
     B(4,3)=(6.-12.*X/AL)/AL2
     B(4,4)=(4.-6.*X/AL)/AL
     B(4,7)=-B(4,3)
     B(4,8)=(2.-6.*X/AL)/AL
```

```
      B(5,2)=B(2,2)/R
      B(5,3)=B(2,3)*AM2/R
      B(5,4)=B(2,4)*AM2/R
      B(5,6)=B(2,6)/R
      B(5,7)=B(2,7)*AM2/R
      B(5,8)=B(2,8)*AM2/R
      B(6,1)=AM*(1.-X/AL)/2./R2
      B(6,2)=-3./2./R/AL
      B(6,3)=-12.*AM*X*(1.-X/AL)/R/AL2
      B(6,4)=2.*AM*(1.-4.*X/AL+3.*X2/AL2)/R
      B(6,5)=AM*X/2./R2/AL
      B(6,6)=3./2./R/AL
      B(6,7)=-B(6,3)
      B(6,8)=-2.*AM*X*(2.-3.*X/AL)/R/AL
      RETURN
      END
```

```
0.000732  30000000.0      0.3
720.0     1.0       480.0      240.0
       1
      15
      10        5
      10
       1
       1
CLFR
```

APPENDIX D


```
      PROGRAM PARTI(INPUT,OUTPUT,TAPE2,TAPE3,TAPE5,TAPE6)
      DIMENSION D(64,65)
      DIMENSION SMASS(60,61),BMASS(60,4),BACC(4),UD(60),PEFF(60)
      DIMENSION X(60,10),GP(10),OME(10),XM(60,10)
      DIMENSION U(60)
      DIMENSION GM(10,10)
      EQUIVALENCE(D(1),SMASS(1))
      READ 100,M
  100 FORMAT(10I8)
      DATA M/10/
      READ(5)(NFREE)
      NFREE1=NFREE+1
      READ(5)((D(I,J),J=1,NFREE1),I=1,NFREE)
      REWIND 5
      DO 10 I=1,NFREE
      DO 10 J=1,I
   10 D(J,I)=D(I,J)
*     WRITE MB
      WRITE(6)((D(I,J),J=1,4),I=5,NFREE)
*    WRITE M
      WRITE(6)((D(I,J),J=5,NFREE),I=5,NFREE)
      BACC(1)=BACC(4)=0.0
      BACC(2)=-1.0
      BACC(3)=+1.0
      ND=NFREE-4
      PRINT 1,M,NFREE,ND
    1 FORMAT(//,6I8,//)
    2 FORMAT(/,10(E10.4,2X))
      REWIND 6
      READ(6)((BMASS(I,J),J=1,4),I=1,ND)
      READ(6)((SMASS(I,J),J=1,ND),I=1,ND)
      NN=ND/4
      DO 20 I=1,NN
      IS=(I-1)*4
      UD(IS+1)=BACC(1)
      UD(IS+2)=BACC(2)
      UD(IS+3)=BACC(3)
      UD(IS+4)=BACC(4)
   20 CONTINUE
*      M*UD=PEFF (ND,ND)*(ND,1)=(ND,1)
      DO 30 I=1,ND
      PEFF(I)=0.0
      DO 30 J=1,ND
   30 PEFF(I)=PEFF(I)+SMASS(I,J)*UD(J)
*    MB * BACC =U  (ND,4)*(4,1)=(ND,1)
      DO 40   I=1,ND
      U(I)=0.0
      DO 40 J=2,3
   40 U(I)=U(I)+BMASS(I,J)*BACC(J)
      DO 50 I=1,ND
   50 PEFF(I)=PEFF(I)-U(I)
      DO 55 J=1,M
      READ(3)(ND)
      READ(3)(OME(J))
      READ(3)(X(I,J),I=1,ND)
   55 CONTINUE
*      GP=XT *PEFF=(M,ND)*(ND,1)=(M,1)
      DO 60 I=1,M
      GP(I)=0.0
      DO 60 J=1,ND
   60 GP(I)=GP(I)+X(J,I)*PEFF(J)
      PEAK=384.0
      DO 70 I=1,M
   70 GP(I)=GP(I)*PEAK
```

```
*    GM=XT * M * X
      DO 75 I=1,ND
      DO 75 J=1,M
      XM(I,J)=0.0
      DO 75 K=1,ND
   75 XM(I,J)=XM(I,J)+SMASS(I,K)*X(K,J)
      DO 80 I=1,M
      DO 80 J=1,M
      GM(I,J)=0.0
      DO 80 K=1,ND
   80 GM(I,J)=GM(I,J)+X(K,I)*XM(K,J)
      WRITE(2)(M)
      WRITE(2)(GM(I,I),I=1,M)
      WRITE(2)(GP(I),I=1,M)
      WRITE(2)(OME(I),I=1,M)
      PRINT 2,(GM(I,I),I=1,M)
      PRINT 2,(GP(I),I=1,M)
      PRINT 2,(OME(I),I=1,M)
      END
```

```
      PROGRAM PARTII (INPUT,OUTPUT,TAPE1,TAPE2,TAPE3,TAPE4,TAPE5,TAPE6)
      DIMENSION ACC(100)
      DIMENSION X1(10),X2(10),X3(10)        D-4
      DIMENSION U(120)
      DIMENSION X(10),GM(10),GP(10),OME(10),A(10),Y(10),V(10)
      DIMENSION YO(100,10),VO(100,10)
      DIMENSION XARRAY(500),YARRAY(500)
      EQUIVALENCE(YO(1),XARRAY(1)),(VO(1),YARRAY(1))
* CARD 2
      READ 100,LREC,NREC,NRSTART,NREND
* CARD 3
      READ 200,DT
* CARD 4
      READ 100,ND1,ND2,ND3
* RESPONSE OF D.O.F NO. ND1 IS TO BE SAVED ON TAPE NO. 4
* RESPONSE OF D.O.F NO. ND2 IS TO BE SAVED ON TAPE NO. 5
* RESPONSE OF D.O.F NO. ND3 IS TO BE SAVED ON TAPE NO. 6
      DISM1=0.0
      DISM2=0.0
      DISM3=0.0
  100 FORMAT(10I8)
  200 FORMAT(F10.4)
      REWIND 1
      READ(2)(NMODES)
      M=NMODES
      PRINT 55,NMODES
   55 FORMAT(I10)
      READ(2)(GM(I),I=1,M)
      READ(2)(GP(I),I=1,M)
* GP HERE IS THE PEAK GENERALIZED FORCE VECTOR
      READ(2)(OME(I),I=1,M)
      REWIND 2
      DO 30 J=1,M
      READ(3)(ND)
      READ(3)(OME(J))
      READ(3)(U(I),I=1,ND)
      X1(J)=U(ND1)
      X2(J)=U(ND2)
      X3(J)=U(ND3)
   30 CONTINUE
      PRINT 55,ND
    2 FORMAT(1H1)
* OME IN CYC/SEC
      DO 35  J=1,M
   35 OME(J)=OME(J)*2.*3.14159
*OME IN RAD/SEC
      REWIND 3
             CALL  CONTROL(NREC,LREC,M,DT,YO,VO,ACC,OME,GP,GM,A,Y,V)
      DO 40 J=1,M
      Y(J)=0.0
   40 V(J)=0.0
      REWIND 1
      NR1=NRSTART-1
   88 FORMAT(10E10.4)
*     DO 5 IREAD=1,NR1
*   5 READ(1,88)(ACC(I),I=1,LREC)
*     DO 1000 IREC=NRSTART,NREND
      DO 1000  IREC=1,NREC
       DO 10 J=1,M
      Y(J)=YO(IREC,J)
   10 V(J)=VO(IREC,J)
      READ(1,88)(ACC(I),I=1,LREC)
      DO 50 ITIME=1,LREC
      TIME=FLOAT(ITIME)*DT
```

```
          CALL    INT(ACC,TIME,DT,LREC,M,OME,GP,GM,A,Y,V,ITIME)
*   INSERT HERE DISPLACEMENTS , STRESSES ,AND EXTERNAL EQUI.
      W1=W2=W3=0.0
      DO 20 J=1,M
      W1=W1+X1(J)*A(J)
      W2=W2+X2(J)*A(J)
      W3=W3+X3(J)*A(J)
   20 CONTINUE
      IF( DISM1 .GT. W1) GO TO 101
      DISM1=W1
      TIMM1=TIME
  101 IF( DISM2 .GT. W2) GO TO 202
      DISM2=W2
      TIMM2=TIME
  202 IF( DISM3 .GT. W3) GO TO 303
      DISM3=W3
      TIMM3=TIME
  303 CONTINUE
      WRITE(4,11)(W1)
      WRITE(5,11)(W2)
      WRITE(6,11)(W3)
   11 FORMAT(E10.4)
   50 CONTINUE
 1000 CONTINUE
      REWIND 4
      REWIND 5
      REWIND 6
         PRINT 2
 2001 FORMAT(//,25X,* RESPONSE OF D.O.F. NO. *,I2)
   12 FORMAT(/,(2X,10(E10.4,2X)))
      PRINT 2001,ND1
      DO 2000 III =1,NREC
      TREC=FLOAT((III-1)*LREC)*DT
      PRINT 199,TREC
      READ(4,11)(XARRAY(I),I=1,LREC)
      PRINT 12  ,(XARRAY(I),I=1,LREC)
 2000 CONTINUE
  199 FORMAT(/,15X,*TIME=*,F10.4,*SEC.*)
      PRINT 2002,DISM1  ,TIMM1
         PRINT 2
      PRINT 2001,ND2
      DO 3000 III=1,NREC
      TREC=FLOAT((III-1)*LREC)*DT
      PRINT 199,TREC
      READ(5,11)(XARRAY(I),I=1,LREC)
       PRINT 12,(XARRAY(I),I=1,LREC)
 3000 CONTINUE
      PRINT 2002,DISM2  ,TIMM2
      PRINT 2
      PRINT 2001,ND3
      DO 4000 III=1,NREC
      TREC=FLOAT((III-1)*LREC)*DT
      PRINT 199,TREC
      READ(6,11)(XARRAY(I),I=1,LREC)
       PRINT 12,(XARRAY(I),I=1,LREC)
 4000 CONTINUE
      PRINT 2002,DISM3  ,TIMM3
 2002 FORMAT(//,10X,* MAX. RESPONSE =*,F10.3, *AT TIME= *,F10.4)
      END
      SUBROUTINE CONTROL(NREC,LREC,M,DT,YO,VO,ACC,OME,GP,GM,A,Y,V)
      DIMENSION YOLD(10)
      DIMENSION ACC(LREC),YO(NREC,M),VO(NREC,M)
      DIMENSION Y(M),V(M),A(M),GP(M),GM(M)   ,OME(M)
      IREC=1
```

```
      DO 10 J=1,M
      YO(1,J)=VO(1,J)=0.0
   10 Y(J)=V(J)=0.0
   20  CONTINUE
      ITIME=LREC-1
      TIME=ITIME*DT
      READ(1,88)(ACC(I),I=1,LREC)
   88 FORMAT(10F10.4)
         CALL    INT(ACC,TIME,DT,LREC,M,OME,GP,GM,A,Y,V,ITIME)
      DO 30 J=1,M
   30 YOLD(J)=A(J)
      ITIME=LREC
      TIME=LREC*DT
         CALL    INT(ACC,TIME,DT,LREC,M,OME,GP,GM,A,Y,V,ITIME)
      IREC=IREC+1
      DO 40 J=1,M
      Y(J)=A(J)
      V(J)=(A(J)-YOLD(J))/DT
      YO(IREC,J)=Y(J)
   40 VO(IREC,J)=V(J)
      IF(IREC.LE. NREC) GO TO 20
      RETURN
      END
      SUBROUTINE INT(ACC,TIME,DT,LREC,M,OME,GP,GM,A,Y,V,ITIME)
      DIMENSION ACC(LREC)
      DIMENSION OME(M),GM(M)   ,GP(M),A(M),Y(M),V(M)
      DO 15 J=1,M
      PIN=0.0
      OMET=OME(J)*TIME
       DO 10 IT=1,ITIME
       TA=FLOAT(IT)*DT
       SINE=SIN(OME(J)*(TIME-TA))
   10  PIN=PIN+ACC(IT)*SINE
      PIN=PIN *GP(J)*DT
   15 A(J)=PIN/(GM(J)   *OME(J))+Y(J)*COS(OMET)+V(J)*SIN(OMET)/OME(J)
      RETURN
      END
```

100	100	1	99

0.001

19	39	59