CIVIL ENGINEERING STUDY 73-4
STRUCTURAL SERIES

DYNAMIC INSTABILITY AND ULTIMATE CAPACITY
OF INELASTIC SYSTEMS PARAMETRICALLY
EXCITED BY EARTHQUAKES--PART I

by
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ABSTRACT

A procedure of analysis is presented for determining the dynamic instability and response of framed structures subjected to pulsating axial loads, time-dependent lateral forces, or foundation movements. Included in the analytical work are the instability criterion of a structural system, the finite element technique of structural matrix formulation, and the computer solution methods.

Dynamic instability is defined by a region in relation to transverse natural frequency, longitudinal forcing frequency, and the magnitude of axial dynamic force. The axial pulsating load is expressed in terms of static buckling load for ensuring that the applied load is not greater than the buckling capacity of a structural system. Consequently, the natural frequency and static instability analyses are also included. For static instability analysis, both the concentrated and uniformly distributed axial loads have been investigated.

The displacement method has been used in this research for structural matrix formulation for which the elementary matrices of mass, stiffness, and stability have been developed by using the Lagrangian equation. The system matrices have been formulated by using the equilibrium and compatibility conditions of the constituent members of a system.

Two numerical integration techniques of the fourth-order
Runge-Kutta method and the linear acceleration method have been employed for the elastic and elasto-plastic response of continuous beams, shear buildings, and frameworks. The general considerations are the bending deformation, $p-\Delta$ effect, and the effect of girder shears on columns. For the elasto-plastic analysis, the effect of axial load on plastic moment is also included.

A number of selected examples are presented, and the results are illustrated in a series of charts, tables, and figures in which the significant effect of pulsating load on the amplitude of transverse vibration can be observed.
ACKNOWLEDGEMENTS

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I. INTRODUCTION

In recent years the theory of dynamic instability has become one of the newest branches of the structural dynamics and mechanics of deformable solids. The problems, which have been examined on the basis of the classical theory of vibrations and structural dynamics, emphasize the response history resulting from lateral, time-dependent excitations. It is known, when a rod is subjected to the action of longitudinal compressive force varying periodically with time, that for a definite frequency the transverse vibrations of the rod will have a rapidly increasing amplitude. Thus, the study of the formation of this type of vibration and the formulation of the methods for the prevention of their occurrence are necessary in the various areas of mechanics, transportation, industrial construction, and structures excited by earthquakes.

A. Purpose of Investigation

The purpose of this study has been to develop an analytical method for determining the behavior of dynamic instability and to study the response of structural systems subjected to longitudinal pulsating loads and lateral dynamic forces or foundation movements. The mathematical formulation, which is general enough for computer analysis of large structural systems, considers geometric and material nonlinearity.
B. Scope of Investigation

The scope of the study may be briefly stated as the derivation of instability criteria and the development of finite element formulation of structural matrices and the numerical methods of computer solution.

In Chapter III, the basic formulation of mass matrix, stiffness matrix, and stability matrix is presented by using the energy concept and finite element technique. The governing differential equation is expressed in terms of system matrix, which is formulated on structural geometric and equilibrium conditions.

In order to evaluate the dynamic instability regions, it is convenient to express the axial load in terms of static buckling load and the longitudinal forcing frequency in terms of natural frequency. Thus, in Chapter IV, the techniques for finding natural frequencies, buckling loads, and instability regions are presented. For the buckling load case, the uniform axial load is also investigated.

Two numerical integration techniques for dynamic response that use the fourth-order Runge-Kutta method and the linear acceleration method are presented in Chapter V in which a comparison of numerical solutions shows the accuracy of the presented methods.

Chapter VI contains examples of the dynamic response of various types of frameworks subjected to axial pulsating loads, lateral forces, or foundation movements.
The elasto-plastic case is given in Chapters VII and VIII for the formulation of member matrices and system matrix, plastic hinge rotations, and numerical solutions.

Two typical computer programs of elastic and elasto-plastic analyses of general types of rigid frames are given in the Appendix.
II. REVIEW OF LITERATURE

A. Structural Dynamics With Longitudinal Excitations

The behavior of structural systems subjected to both lateral and longitudinal excitations is little known. Most of the research work has been concentrated on the problem of an elastic column subjected to a periodically varying axial load for the purpose of searching for the stability criteria of double symmetric columns (1) as well as nonsymmetric columns (2).

Sevin E. (3), among other investigators, studied the effect of longitudinal impact on the lateral deformation of initially imperfect columns. Recently, Cheng and Tseng (4) investigated the effect of static axial load on the Timoshenko beam-column systems.

It seems that very little work has been done on either the criteria of dynamic instability or the response behavior of framed structures subjected to dynamic lateral and longitudinal excitations.

B. Structural Dynamics Without Longitudinal Excitations

The conventional structural dynamics problems have been generally solved by using three methods: lumped mass, distributed mass, and consistent mass. Before computer facilities were available, the lumped mass model with a finite degree of freedom had been extensively studied by a number
of investigators. With the advent of computers, research work on multistory structures was initiated by several investigators, namely N.M. Newmark, R.W. Clough, J.A. Blume, (6,7,9-12,17), and later by Cheng (13), E.L. Wilson, and I.P. King, (14,15,16).

For the distributed mass system, the early research work was limited to single members (18) or one-story frames (19). Later Levien and Hartz (20) used the dynamic flexibility matrix method to solve problems of one- and two-story rigid frames, and Cheng (4,13,29) solved free and forced vibrations of continuous beams and rigid frames by using the displacement method. The displacement and flexibility methods cited above may be considered exact in the sense that the members must be prismatic and that the structural joints are rigid.

In recent years, the finite element technique has been extensively used for solving structural dynamics problems. The method was initially proposed by Archer (21) for plane frameworks. Cheng (22) recently extended the technique to solve space frame problems. The model of the method is similar to the distributed mass system. The equation of motion, however, is expressed in an explicit form for which the solution effort is much less than that of the distributed mass model.

The fundamental behavior of the dynamic response of elasto-plastic systems can be found in standard texts (23,24). The elasto-plastic analysis method of beams and one-story frames
with distributed mass has appeared in references (25,26) in which the method is limited to simple structures.

For large structures, typical studies can be found in references (27,28). Berg and Dadeppo (27) investigated the response of a multistory elasto-plastic structures subjected to lateral dynamic forces, and Walpole and Sheperd (28) studied the behavior of reinforced concrete frames subjected to earthquake movements.
III. MATRIX FORMULATION OF ELASTIC STRUCTURAL SYSTEMS

The displacement matrix method has been used in the structural system formulation for static and dynamic instability analysis and dynamic response. The formulation involves deriving differential equations, element matrices of stiffness, mass, stability, and the matrix of general structural systems. The structures are plane frameworks in which the joints are rigid and the constituent members are prismatic. As shown in Fig. 3.1, the structure is subjected to time-dependent axial forces, \( N(t) \), and lateral dynamic loads, \( F(t) \), or foundation movement, \( G(t) \), and may have a superimposed uniform mass, \( m \), and a concentrated mass, \( M_i \), in addition to its own weight.

For the purpose of investigating large systems, the shears transmitted from girders to columns are taken into consideration, and the members are assumed to have bending deformation only.

A. Governing Differential Equation

Consider an arbitrary member of a structural system as shown in Fig. 3.2. The governing differential equations for such an element can be obtained by using the Lagrangian equation

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} - \frac{\partial V}{\partial q_i} = \frac{\partial W}{\partial q_i} = q_i
\]  

(3.1)
Fig. 3.1 General Problem

Fig. 3.2 Loading on a Typical Member
in which

\[ T = \text{kinetic energy} \]
\[ U = \text{strain energy of bending} \]
\[ V = \text{potential energy done by axial force} \]
\[ Q_i = \text{generalized forces} \]
\[ q_i = \text{generalized coordinates at node } i \text{ associated with } Q_i \]
\[ q_i = \text{generalized velocities} \]
\[ W = \text{work done by generalized external forces}. \]

Let \( \phi(x) \) be the shape function and \( q_i(t) \) be the time function of the beam motion, then the displacement of the beam can be expressed as

\[ y(x,t) = \sum_{i=1}^{n} q_i(t) \phi_i(x). \quad (3.2) \]

The kinetic energy for lateral displacement of the member is

\[ T = \frac{1}{2} \int_{0}^{L} m \left( \frac{\partial y}{\partial t} \right)^2 dx \quad (3.3) \]

where \( m \) is the mass per unit length.

The strain energy for bending of the member may be represented by

\[ U = \frac{1}{2} \int_{0}^{L} EI \left( \frac{\partial^2 y}{\partial x^2} \right)^2 dx \quad (3.4) \]
where $E$ and $I$ are Young's elastic modulus and moment of inertia, respectively.

The potential energy for the longitudinal force is

$$V = \frac{1}{2} \int_{0}^{L} N(t) \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)^2 \, dx.$$  \hfill (3.5)

By the substitution of Eq. (3.2), one may obtain

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{dq_i}{dt} \frac{dq_j}{dt} \int_{0}^{L} m \phi_i(x) \phi_j(x) \, dx$$  \hfill (3.6)

$$U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \int_{0}^{L} \frac{d^2 \phi_i(x)}{dx^2} \frac{d^2 \phi_j(x)}{dx^2} \, dx$$  \hfill (3.7)

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \int_{0}^{L} \frac{d \phi_i(x)}{dx} \frac{d \phi_j(x)}{dx} \, dx$$  \hfill (3.8)

or

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k} \sum_{l} \frac{d \dot{q}_i}{dt} \frac{d \dot{q}_j}{dt} \frac{m}{2} \mathbf{q}_i \mathbf{q}_j$$  \hfill (3.9)

$$U = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j$$  \hfill (3.10)

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}^l q_i q_j$$  \hfill (3.11)

where
\[ m_{ij} = \int_0^L m \phi_i(x) \phi_j(x) \, dx \]  
(3.12)

\[ k_{ij} = \int_0^L E I \phi_i''(x) \phi_j''(x) \, dx \]  
(3.13)

\[ s'_{ij} = \int_0^L N(t) \phi_i'(x) \phi_j'(x) \, dx. \]  
(3.14)

To include the concentrated masses in the formulation of \( m_{ij} \), let us consider masses \( M_k(x_k) \) acting at the positions \( x_k, \ k=1,2,\ldots,r \), then Eq. (3.12) should be expressed as

\[ m_{ij} = \int_0^L m \phi_i(x) \phi_j(x) \, dx + \sum_{k=1}^r M_k(x_k) \phi_i(x_k) \phi_j(x_k). \]  
(3.15)

The work done by external forces acting at the generalized coordinate \( q_i \) is

\[ W = \sum_{i=1}^n \sum_{j=1}^p \left\{ F_j(x_j) \phi_i(x_j) \right\} + \int_0^L f(x,t) \phi_i(x) \, dx \right) q_i \]  
(3.16)

where \( F_j(x_j) \) is the concentrated forces acting at positions \( x_j, \ j=1,2,\ldots,p \).

Let \( N(t) = (\alpha + \delta \cos \theta t)N_0 \), then Eq. (3.14) becomes

\[ s'_{ij} = (\alpha + \delta \cos \theta t) s_{ij}. \]  
(3.17)
where

\[ L_s = \int_0^L N_i \phi_j (x) \phi'_j (x) dx. \]

By substituting Eqs. (3.9), (3.10), (3.11), and (3.17) into Eq. (3.1) and by performing the operation shown in Eq. (3.1), the following governing differential equations of motion can be obtained:

\[
[m_{ij}] \ddot{q} + [k_{ij}] q - (\alpha + \beta \cos \theta) [s_{ij}] q = \{ f \} \tag{3.18}
\]

in which the matrices \([m_{ij}], [k_{ij}], \) and \([s_{ij}]\) are the matrices of mass, stiffness, and stability defined in Eqs. (3.12), (3.13), and (3.17), respectively. The term \([f]\) is the vector of equivalent generalized external forces. All the elements in \([m_{ij}], [k_{ij}], \) and \([s_{ij}]\) are derived in the next section.

For a structural system, the member matrices are assembled together by using the equilibrium and continuity conditions at nodal points and are discussed in Section C. Similar to Eq. (3.18), the system matrix may be written as

\[
[M] \ddot{X} + [K] X - (\alpha + \beta \cos \theta) [S] X = \{ F \} \tag{3.19}
\]

in which \([X]\) represents global coordinates; \([M], [K], \) and \([S]\) are the matrices of total structural mass, stiffness, and
stability, respectively, and may be formulated through the procedure of displacement method. Eq. (3.19) is the governing differential equation of motion to be used in this study of the dynamic instability and dynamic response.

B. Derivation of Members Mass, Stiffness, Stability Matrices

For the displacement method, it is generally preferable to formulate the mass matrix, stiffness matrix, and stability matrix of a typical member on the basis of a set of defined local coordinates; then, the system matrices can be formulated by transferring local coordinates to global coordinates by using equilibrium and compatibility conditions.

Let us consider a typical bar shown in Fig. 3.3 in which \( q_i \) \( (i=1,2,3,4) \) are local coordinates in the positive direction, and \( Q_i \) \( (i=1,2,3,4) \) are positive local generalized forces corresponding to \( q_i \). The compressive axial force, \( N(t) \), is considered to be positive. The displacements, \( q_i \), are due to the application of the generalized forces \( Q_i \). The displacement \( y(x,t) \) of the beam section at point, \( x \), and time, \( t \), may be written as

\[
y(x,t) = \sum_{i=1}^{4} q_i(t) \phi_i(x). \tag{3.20}
\]

If bending deformation is considered only, then the differential equation of beam deflection is \( \phi''(x)=0 \) for which
Fig. 3.3 Generalized Local Coordinates and Generalized Forces for a Typical Beam
the solution may be expressed in cubic polynomials

\[ \phi(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3. \]

This is the shape function in Eq. (3.20). Let the coordinates, \( q_i \), in Fig. 3.3 be displaced, one at each time, for a unit displacement; then \( \phi(x) \) becomes

\[ \begin{align*}
\phi_1(x) &= (x-2x^2/L+x^3/L^2) \\
\phi_2(x) &= (x^3/L^2-x^2/L) \\
\phi_3(x) &= (-1+3x^2/L^2-2x^3/L^3) \\
\phi_4(x) &= (3x^2/L^2-2x^3/L^3).
\end{align*} \]

(3.21) - (3.24)

Substituting Eqs. (3.21 to 3.24) into Eqs. (3.12 to 3.14) and performing the integration over the bar length, we can obtain \([m_{ij}], [k_{ij}], \) and \([s_{ij}]\) as follows:

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4
\end{bmatrix}
\begin{bmatrix}
4mL^3 \\
4mL^3 \\
-22mL^2 \\
13mL^2
\end{bmatrix}
\begin{bmatrix}
420 \\
420 \\
420 \\
420
\end{bmatrix}
\begin{bmatrix}
-3mL^3 \\
4mL^3 \\
13mL^2 \\
-22mL^2
\end{bmatrix}
\begin{bmatrix}
420 \\
420 \\
420 \\
420
\end{bmatrix}
\begin{bmatrix}
-22mL^2 \\
13mL^2 \\
156mL \\
-54mL
\end{bmatrix}
\begin{bmatrix}
420 \\
420 \\
420 \\
420
\end{bmatrix}
\begin{bmatrix}
13mL^2 \\
-22mL^2 \\
-54mL \\
156mL
\end{bmatrix}
\begin{bmatrix}
420 \\
420 \\
420 \\
420
\end{bmatrix}
\]

(3.25)
\[
\begin{align*}
\mathbf{Q}_1 & = \begin{bmatrix} 4EI/L & 2EI/L & -6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & -6EI/L^2 & -6EI/L^2 \\ -6EI/L^2 & -6EI/L^2 & 12EI/L^3 & 12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & 12EI/L^3 & 12EI/L^3 \\ k & & & \end{bmatrix} \mathbf{q}_1 \\
\mathbf{Q}_2 & = \begin{bmatrix} 2L/15 & -L/30 & -1/10 & -1/10 \\ -L/30 & 2L/15 & -1/10 & -1/10 \\ -1/10 & -1/10 & 6/5L & 6/5L \\ -1/10 & -1/10 & 6/5L & 6/5L \\ p & & & \end{bmatrix} \mathbf{q}_2 \\
\mathbf{Q}_3 & = \begin{bmatrix} 2L/15 & -L/30 & -1/10 & -1/10 \\ -L/30 & 2L/15 & -1/10 & -1/10 \\ -1/10 & -1/10 & 6/5L & 6/5L \\ -1/10 & -1/10 & 6/5L & 6/5L \\ k & & & \end{bmatrix} \mathbf{q}_3 \\
\mathbf{Q}_4 & = \begin{bmatrix} 2L/15 & -L/30 & -1/10 & -1/10 \\ -L/30 & 2L/15 & -1/10 & -1/10 \\ -1/10 & -1/10 & 6/5L & 6/5L \\ -1/10 & -1/10 & 6/5L & 6/5L \\ k & & & \end{bmatrix} \mathbf{q}_4 \\
\mathbf{k}_{ij} & = \begin{bmatrix} 2EI/L & 4EI/L & -6EI/L^2 & -6EI/L^2 \\ 2EI/L & 4EI/L & -6EI/L^2 & -6EI/L^2 \\ -6EI/L^2 & -6EI/L^2 & 12EI/L^3 & 12EI/L^3 \\ -6EI/L^2 & -6EI/L^2 & 12EI/L^3 & 12EI/L^3 \\ k & & & \end{bmatrix} \\
\mathbf{q}_1 & = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \end{bmatrix} \\
\mathbf{q}_2 & = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \end{bmatrix} \\
\mathbf{q}_3 & = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \end{bmatrix} \\
\mathbf{q}_4 & = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \end{bmatrix}
\end{align*}
\]

Note that \( \mathbf{Q}_1, \mathbf{Q}_2 \) and \( \mathbf{Q}_3, \mathbf{Q}_4 \) correspond to moments and shears, respectively; \( q_1, q_2 \) and \( q_3, q_4 \) correspond to
rotations and displacements, respectively; and $\ddot{q}_1$, $\ddot{q}_2$ and $\ddot{q}_3$, $\ddot{q}_4$ are accelerations due to rotations and displacements, respectively. For convenience, let us rewrite Eqs. (3.25, 3.26, 3.27) in the following condensed forms:

$$
\begin{align*}
\begin{bmatrix}
Q_m \\
Q_v
\end{bmatrix}_{m} &=
\begin{bmatrix}
[M_M] & [M_Y] \\
[M_V] & [M_Y]
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_r \\
\ddot{q}_s
\end{bmatrix} \\
\begin{bmatrix}
Q_m \\
Q_v
\end{bmatrix}_{k} &=
\begin{bmatrix}
[K_M] & [K_Y] \\
[K_V] & [K_Y]
\end{bmatrix}
\begin{bmatrix}
q_r \\
q_s
\end{bmatrix} \\
\begin{bmatrix}
Q_m \\
Q_v
\end{bmatrix}_{p} &=
\begin{bmatrix}
[S_M] & [S_Y] \\
[S_V] & [S_Y]
\end{bmatrix}
\begin{bmatrix}
q_r \\
q_s
\end{bmatrix}
\end{align*}
$$

(3.28) 

(3.29) 

(3.30)

in which the subscripts m, k, and p signify that the moments \{Q_m\} and shears \{Q_v\} are associated with \([m_{ij}]\), \([k_{ij}]\), and \([s_{ij}]\), respectively. The subscripts r and s signify the joint rotations and displacements, respectively.

C. System Matrices of Mass, Stiffness, and Stability

The displacement method of formulating structural
system matrix has been well documented (31,32). Following Cheng's recent work (13), one can rewrite the relationship between the generalized external forces, \( \{F\} \), and generalized external displacement, \( \{X\} \), as

\[
\begin{bmatrix}
\{F_r\} \\
\{F_s\}
\end{bmatrix} = \begin{bmatrix}
[A_m] [MMR] [A_m]^T & [A_m] [MMY] [A_V]^T \\
[A_V] [MVR] [A_m]^T & [A_V] [MVY] [A_V]^T
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
0 & [A_{mS}]
\end{bmatrix} \begin{bmatrix}
\ddot{x}_r \\
\ddot{x}_s
\end{bmatrix}
+ \begin{bmatrix}
[A_m] [KMR] [A_m]^T & [A_m] [KMY] [A_V]^T \\
[A_V] [KVR] [A_m]^T & [A_V] [KVY] [A_V]^T
\end{bmatrix} \begin{bmatrix}
\dot{x}_r \\
\dot{x}_s
\end{bmatrix}
\]

(3.31)

Knowing \( \{F_r\} \) and \( \{F_s\} \), one can find \( \{x_r\} \), \( \{x_s\} \), \( \{\dot{x}_r\} \), and \( \{\ddot{x}_s\} \) from Eq. (3.31) by using the numerical integration presented in Chapter V. Consequently, the member end moments and end shears can be obtained as follows:
\[
\begin{align*}
\begin{pmatrix} Q_m \\ Q_v \end{pmatrix} &= \begin{bmatrix} [MMR][A_m]^T & [MMY][A_v]^T \\ [MVR][A_m]^T & [MVR][A_v]^T \end{bmatrix} \begin{pmatrix} \ddot{X}_r \\ \ddot{X}_s \end{pmatrix} \\
&+ \begin{bmatrix} [KVR][A_m]^T & [KVR][A_v]^T \\ [KVY][A_m]^T & [KVY][A_v]^T \end{bmatrix} \begin{pmatrix} X_r \\ X_s \end{pmatrix} \\
&- \begin{bmatrix} [SMR][A_m]^T & [SMY][A_v]^T \\ [SVR][A_m]^T & [SVY][A_v]^T \end{bmatrix} \begin{pmatrix} X_r \\ X_s \end{pmatrix}
\end{align*}
\]

(3.32)

in which

\([A_m]\) = equilibrium matrix relating internal moments to external nodal moments;

\([A_v]\) = equilibrium matrix relating internal shears to external nodal forces;

\([F_r]\) = external nodal moments;

\([F_s]\) = external nodal forces;

\([X_r]\) = global rotations;

\([X_s]\) = global displacements;

\([\ddot{X}_r]\) = acceleration due to global rotations;

\([\ddot{X}_s]\) = acceleration due to global displacement;

\([A_{ms}]\) = diagonal matrix involves the inertial forces due to joint displacements; and

\(T\) = transpose of matrix.
Eqs. (3.31, 3.32) have been explained in detail in SUBROUTINE ASATA, ASATB, SATMV shown in the Appendix.

D. Shear Building Subjected to Lateral Forces

In many practical cases, the girder stiffnesses compared with those of columns are sufficiently large. Consequently, the structural joint rotations are very small and only the sway displacements are significant. By neglecting the global coordinates corresponding to the structural joint rotations, one can rewrite Eq. (3.31) as

$$\begin{align*}
[M]\ddot{x}_s + [K]x_s - [S]x_s &= \{F_s\} \\
\text{(3.33a)}
\end{align*}$$

where

$$\begin{align*}
[M] &= [A_v][MVY][A_v]^T + [A_{ms}] \\
[K] &= [A_v][KVY][A_v]^T \\
[S] &= [A_v][SVY][A_v]^T.
\end{align*}$$

When the axial load is $N(t) = (a + b\cos\theta t)N_0$, then Eq. (3.33a) becomes

$$\begin{align*}
[M]\dddot{x}_s + [K]\dot{x}_s - (a + b\cos\theta t)[S]x_s &= \{F_s\}. \\
\text{(3.33b)}
\end{align*}$$
IV. STATIC AND DYNAMIC STABILITY

A. Boundary of Dynamic Instability

When a structural framework is subjected to a transverse pulsating load, the framework will generally experience forced vibration with a certain frequency of the excitation. The amplitude of the vibration becomes larger and larger when the forcing frequency approaches the natural frequency of the vibrating system. The behavior is called resonance. However, when the frame is subjected to a pulsating axial load as shown in Eq. (3.19), an entirely different type of resonance is observed. The resonance occurs when a certain relationship exists between the natural frequency, the frequency of longitudinal forces and their magnitude. This resonance is called parametric resonance. The behavior of parametric resonance may be studied by using the governing differential equations of motion, Eq. (3.19).

Let us consider the time dependent axial forces only, then Eq. (3.19) becomes

\[ [M] \ddot{X} + \left( [K] - (a + \beta \cos \theta) [S] \right) X = 0 \]  \hspace{1cm} (4.1)

which represents a system of second-order differential equations with the periodic coefficient of the known Mathieu-Hill type. It has been observed that the Mathieu-Hill equation similar to Eq. (4.1) has periodic solutions with periods
T and 2T (T=2π/θ) at the boundaries of the instability region (2). The regions of instability may be determined by finding the periodic solutions of Eq. (4.1) in the form of a trigonometric series. The instability regions are bounded by two solutions with the same period. The stability regions are bounded by two solutions with different periods. The critical values of parameters α, β, and θ contained in Eq. (4.1) are obtained from the condition that Eq. (4.1) has periodic solutions. The stability or instability solutions of Eq. (4.1) correspond to the stability or instability of the structural system. The above-mentioned statement can be illustrated by the following derivation.

For the solution with period 2T, let the trial solution be in the form of a series

\[ \{X\} = \sum_{k=1,3,5,\ldots}^{\infty} (A_k \sin \frac{k\theta t}{2} + B_k \cos \frac{k\theta t}{2}) \tag{4.2} \]

in which \(A_k\) and \(B_k\) are vectors independent of time. By substituting Eq. (4.2) into Eq. (4.1), the following system of matrix equations can be obtained by comparing the coefficients of \(\sin \frac{k\theta t}{2}\) and \(\cos \frac{k\theta t}{2}\):

\[
\begin{align*}
([K] - (\alpha - \frac{1}{2}\beta)[S] - \frac{1}{4}\theta^2 [M])A_1 - \frac{1}{2}\beta [S]A_3 &= 0 \\
([K] - \alpha [S] - \frac{1}{4}k^2 \theta^2 [M])A_k - \frac{1}{2}\beta [S](A_{k-2} + A_{k+2}) &= 0 \\
(k &= 3, 5, 7, \ldots).
\end{align*}
\]
Solutions having the period $2T=4\pi/\theta$ can occur if the following conditions are satisfied:

\[
\begin{bmatrix}
[K] - (\alpha + \beta) [S] - \frac{\theta^2}{4} [M] & -\beta [S] & 0 & \cdots \\
-\beta [S] & [K] - \alpha [S] - \frac{\theta^2}{4} [M] & -\beta [S] & \cdots \\
0 & -\beta [S] & [K] - \alpha [S] - \frac{2\pi^2}{4} [M] & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix} = 0
\]

(4.3)

Similarly, for the solution with period $T$, let the trial solution be represented by

\[
\{x\} = \beta_0 + \sum_{k=2,4,6,\cdots}^{\infty} (A_k \sin k\theta t + B_k \cos k\theta t).
\]

(4.4)

Substituting Eq. (4.4) into Eq. (4.1) yields Eqs. (4.5) and (4.6) for the solution having the period $T=2\pi/\theta$.

For finding the regions of instability as sketched in Fig. 4.1, one may solve Eqs. (4.3), (4.5), and (4.6) for the
critical values of the parameters \((\alpha, \beta, N_0, \theta)\). The first region of instability (Region A) is determined from Eq. (4.3). Similarly, the second region of instability (Region C) is determined from Eqs. (4.5) and (4.6). The stability region (Region B) is confined by Region A and Region C.

\[\begin{bmatrix}
[K] - \alpha[S] - \theta^2[M] & -\gamma \beta [S] & 0 & \cdots \\
-\gamma \beta [S] & [K] - \alpha[S] - 4\theta^2[M] & -\gamma \beta [S] & \cdots \\
0 & -\gamma \beta [S] & [K] - \alpha[S] - 6\theta^2[M] & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix} = 0
\]

(4.5)
In practice, only the finite number of terms in the determinant is used for studying the principal instability regions. Thus when the first term of the series of Eq. (4.2) is considered (i.e., \( k=1, \{X\}=A_1\sin(\theta t/2)+B_1\cos(\theta t/2) \)), one may have

\[
| [K] - \alpha[S] - \beta[S] - \frac{\theta^2}{4}[M] | = 0 \tag{4.7}
\]

which corresponds to the first matrix element along the diagonal of Eq. (4.3). The solution of Eq. (4.7) gives the principal regions of dynamic instability. Similarly, from Eqs. (4.5) and (4.6), we may have

\[
| [K] - \alpha[S] - \theta^2[M] | = 0
\]
which give the secondary region (Region C of Fig. 4.1) of
dynamic instability. Note that Eq. (4.7) is an eigenvalue
equation which can be solved by the conventional method of
expanding the determinant equation, Eq. (4.7), into a
polynomial equation for the eigenvalue and its associated
eigenvector. For this research of studying large structural
systems, a different technique of matrix iteration has been
used by utilizing computer facilities (32).

B. Static Buckling Load and Natural Frequency

It may be observed from Eq. (4.7) that an instability
region is confined by the axial load and the ratio of axial
forcing frequency to the natural frequency. In order to en­sure
that the amount of axial load to be applied is not greater
than the elastic buckling capacity of the system, it is
essential to express the applied load in terms of buckling
load \( N_0 \), as \( \alpha N_0 \) and \( \beta N_0 \). \( \alpha \) and \( \beta \) are fractional numbers
less than one. In this section, the techniques of finding static
buckling load and natural frequency are discussed.

By observing Eq. (4.1), one may obtain the three groups
of eigenvalue problems classified as (a), (b), and (c) that
are shown below:
(a). For static buckling case when $\ddot{X}=0, \beta \cos \theta t=0$, then Eq. (4.1) becomes

$$([K] - \alpha [S])\{X\} = 0 \quad \text{or} \quad |[K] - \alpha [S]| = 0. \quad (4.8)$$

(b). For free vibration of harmonic motions without external axial loads, Eq. (4.1) may be written as

$$[M]\dddot{X} + [K]\{X\} = 0. \quad (4.9)$$

Let $\{X\} = \{A e^{i\omega t}\}$, then Eq. (4.9) becomes

$$|[K] - \omega^2 [M]| = 0 \quad (4.10)$$

which gives the natural frequency $\omega$.

(c). For the influence of static axial loads on the natural frequency, one can rewrite Eq. (4.1) as

$$|[K] - \alpha [S] - \omega^2 [M]| = 0 \quad (4.11)$$

from which one may observe that the compressive load will decrease the natural frequency, and the tensile force will increase the natural frequency.

Let Eqs. (4.7), (4.8), (4.10), and (4.11) be expressed in a standard eigenvalue form as
\[
\frac{1}{\lambda} \{X\} = [DM] \{X\} \tag{4.12}
\]

where \([DM]\) and \(\lambda\) in Eq. (4.7) signify either

\[
[DM] = [[K] - (\alpha + \frac{1}{2} \beta)[S]]^{-1}[M], \text{ and } \lambda = \theta^2/4 \tag{4.13}
\]

or

\[
[DM] = [[K] - (\alpha - \frac{1}{2} \beta)[S]]^{-1}[M], \text{ and } \lambda = \theta^2/4. \tag{4.14}
\]

\([DM]\) and \(\lambda\) in Eq. (4.8) represent

\[
[DM] = [K]^{-1}[S], \text{ and } \lambda = \alpha.
\]

For Eq. (4.10)

\[
[DM] = [K]^{-1}[S], \text{ and } \lambda = \omega^2,
\]

and for Eq. (4.11)

\[
[DM] = [[K] - \alpha[S]]^{-1}[M], \text{ and } \lambda = \omega^2.
\]

The matrix iteration method developed by Cheng (30) has been employed to obtain the eigenvalue \(\lambda\) and its associated eigenvector \(\{X\}\).

Example 4.1. Consider that the step beam given in Fig. 42a is subjected to an axial force \(N(t) = \alpha N_0 + \beta N_0 \cos \theta t\). The cross
section of segments AB, BC are 8.375"x3.465" and 6.925"x 3.465", respectively. Let \( E = 30 \times 10^6 \text{psi} \), \( \gamma = 490 \text{ lbs/ft}^3 \), \( L_{AB} = 144" \), \( L_{BC} = 96" \). Find the dynamic instability region.

Solution: Using the local coordinates, \( \{q\} \), and global coordinates, \( \{X\} \), shown in Fig. 4.2b and 4.2c, respectively, one can find the equilibrium matrices \([A_m]\), \([A_v]\) tabulated in Fig. 4.2d and then manipulate Eq. (3.19) for

\[
\begin{bmatrix}
\frac{4E I_{AB} L_{AB}^3}{420} + \frac{4E I_{BC} L_{BC}^3}{420} & -\frac{22m_{AB} L_{AB}^2}{420} + \frac{22m_{BC} L_{BC}^2}{420} \\
\frac{-22m_{AB} L_{AB}^2}{420} + \frac{22m_{BC} L_{BC}^2}{420} & \frac{156m_{AB} L_{AB}}{420} + \frac{156m_{BC} L_{BC}}{420}
\end{bmatrix}
\]  
(4.18)

\[
\begin{bmatrix}
\frac{4E I_{AB}}{L_{AB}} + \frac{4E I_{BC}}{L_{BC}} & -\frac{6E I_{AB}}{L_{AB}^2} - \frac{6E I_{BC}}{L_{BC}^2} \\
-\frac{6E I_{AB}}{L_{AB}^2} - \frac{6E I_{BC}}{L_{BC}^2} & \frac{12E I_{AB}}{L_{AB}^3} - \frac{12E I_{BC}}{L_{BC}^3}
\end{bmatrix}
\]  
(4.19)
(a) Given Problem

(b) Global Coordinates

(c) Local Coordinates

(d) Equilibrium Matrices

Fig. 4.2 Example 4.1
Thus substituting Eqs. (4.19) and (4.20) into Eq. (4.8) gives the static buckling load $N_0 = 2975$ kips. Using Eqs. (4.10) and (4.12) yields the natural frequency $\omega = 28.95$ cps. Let $\alpha = 0., 0.1, 0.2, 0.3, 0.4, 0.5,$ and $\beta = 0., 0.1, 0.2, 0.3, 0.4, 0.5,$ then one can find various values of $\theta$ from Eqs. (4.12), (4.13), (4.14). Expressing $\theta$ in terms of $\theta/\omega$ and then using parameters $\alpha$ and $\beta$, one can draw the instability regions shown in Fig. 4.3.(12).

C. Static Buckling Resulting From a Combined Action of Distributed and Concentrated Axial Forces

In the previous section, the static buckling load was assumed to be acting at the structural joints as a concentrated force. However, there are many cases where the longitudinal forces are distributed along the members. Typical examples may be the self-weight of chimneys, the self-weight of slender tall buildings, and the weight of walls attached to columns. The stability matrix for the above mentioned type of structures is different from that in Eq. (3.27).
It is well known that if a longitudinal compressive force is continuously distributed along a bar, the classical mathematical formulation becomes very sophisticated, because the differential equation of the deflection curve of the buckled bar will no longer be an equation with constant coefficients. Consequently, the direct integration of the equation can only be applied to simple bars, such as cantilever columns. It is the purpose of this section to present the stability matrix due to a combined action of distributed and concentrated axial forces.

1. Formulation of Stability Matrix

Consider the beam of Fig. 4.4a subjected to a concentrated axial force, N, and a uniformly distributed axial load, q. The generalized coordinates, \( q_i \), and generalized forces, \( Q_i \), are shown in Fig. 4.4b and c, respectively. Let \( N, q, Q_i, q_i \) are positive as shown, the displacement \( y(x) \) of the beam at point \( x \) due to \( q_i \) and \( Q_i \) may be expressed as

\[
y(x) = \sum_{i=1}^{4} q_i \phi_i(x).
\]  

(4.21)

For bending deformation only, the shape functions \( \phi(x) \) of Eq. (4.21) are the same as Eqs. (3.21, 3.22, 3.23, 3.24) shown below:
\[ \phi_1(x) = (x - 2x^2/L + x^3/L^2) \]
\[ \phi_2(x) = (x^3/L^2 - x^2/L) \]
\[ \phi_3(x) = (-1 + 3x^2/L^2 - 2x^3/L^3) \]
\[ \phi_4(x) = (3x^2/L^2 - 2x^3/L^3) \]

The total potential energy, \( V \), due to \( N \) and \( q \) is given by

\[ V = V_N + V_q \]

where \( V_N \) is the virtual work done by the axial force \( N \) on displacement \( \Delta \), and \( V_q \) is the virtual work done by the uniformly distributed axial load, \( q \), on displacement \( \Delta \); where \( \Delta \) is the displacement resulting from the displacements \( q_i \). For an element \( dx \) shown in Fig. 4.4d one may have

\[ d\Delta = ds - dx \quad (4.23) \]
\[ ds = dx\sqrt{1 + (dy/dx)^2} \quad (4.24) \]

for small deflection, Eq. (4.24) becomes

\[ ds = dx\sqrt{1 + \frac{1}{2}(dy/dx)^2} \quad (4.25) \]

Substituting Eq. (4.25) into Eq. (4.23) and then integrating over the length, one can obtain
\[
\Delta = \frac{1}{2} \int_0^L (\frac{dy}{dx})^2 \, dx .
\]

We can now write the work \( V_N \) as

\[
V_N = N\Delta = \frac{1}{2}N \int_0^L (\frac{dy}{dx})^2 \, dx
\]

(4.26)

Fig. 4.4 Typical Bar Subjected to Concentrated Axial Load \( N \) and Uniformly Distributed Load \( q \)
From Fig. 4.4d, \( d\Delta = ds - dx = \frac{1}{2}(dy/dx)^2 dx \), the work done by the load acting on the right side of \( x \) on \( d\Delta \) is

\[
dV_q = (L-x)d\Delta = q(L-x)\left\{ \frac{1}{2}(dy/dx)^2 \right\} dx
\]

Therefore, the total work produced by the distributed load over the length is

\[
V_q = \int_0^L dV_q = \frac{1}{2} \int_0^L q(L-x)(dy/dx)^2 dx.
\]  \hspace{1cm} (4.27)

The strain energy is

\[
U = \frac{1}{2} \int_0^L EI[y''(x)]^2 dx.
\]  \hspace{1cm} (4.28)

The virtual work done by forces \( Q_i \) on \( q_i \) may be written as

\[
W = \sum_{i} Q_i q_i.
\]  \hspace{1cm} (4.29)

By Lagrange's equation,

\[
\frac{\partial U}{\partial q_i} - \frac{\partial V}{\partial q_i} = \frac{\partial W}{\partial q_i}
\]  \hspace{1cm} (4.30)

upon which the substitution of Eqs. (4.26), (4.27), (4.28), (4.29) leads

\[
\{\nabla U\} - \{\nabla V_N\} - \{\nabla V_q\} = \{\nabla W\}.
\]  \hspace{1cm} (4.31)
From Eq. (4.21),

\[ y'(x) = \sum_{i} q_i \phi'_i(x), \quad (4.32) \]

\[ y''(x) = \sum_{i} q_i \phi''_i(x), \quad (4.33) \]

Thus substituting Eqs (4.32), (4.33) into Eqs (4.26), (4.27) and (4.28), respectively, gives

\[ u = \frac{1}{2} \sum_{i,j} k_{ij} q_i q_j = \frac{1}{2} \{ q \}^T [k_{ij}] \{ q \} \quad (4.34) \]

\[ v_N = \frac{1}{2} \sum_{i,j} s_{ij} q_i q_j = \frac{1}{2} \{ q \}^T [s_{ij}] \{ q \} \quad (4.35) \]

\[ v_q = \frac{1}{2} \sum_{i,j} g_{ij} q_i q_j = \frac{1}{2} \{ q \}^T [g_{ij}] \{ q \} \quad (4.36) \]

in which

\[ k_{ij} = \int_{0}^{L} EI \phi''_i(x) \phi''_j(x) \, dx \]

\[ s_{ij} = \int_{0}^{L} N \phi'_i(x) \phi'_j(x) \, dx \]

\[ g_{ij} = \int_{0}^{L} q(L-x) \phi'_i(x) \phi'_j(x) \, dx. \]

The substitution of Eqs. (4.34), (4.35), and (4.36) into Eq. (4.30) yields the results of Eq. (4.31) as
\{\nu\} = [k_{ij}]\{q\}

\{\nu_N\} = [s_{ij}]\{q\}

\{\nu_g\} = [g_{ij}]\{q\}

\{\nu_w\} = \{Q\}.

Therefore Eq. (4.31) may be rewritten as

\[ [k_{ij}]\{q\} - [s_{ij}]\{q\} - [g_{ij}]\{q\} = \{Q\} \tag{4.38} \]

in which \([k_{ij}]\) and \([s_{ij}]\) are exactly the same as Eqs. (3.26) and (3.27). The term \([g_{ij}]\) is the stability matrix due to a uniformly distributed axial load, and can be expressed as follows

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4
\end{bmatrix}
= \begin{bmatrix}
\frac{6gL^2}{60} & -gL^2 & 0 & 0 \\
-gL^2 & \frac{2gL^2}{60} & -gL & -gL \\
0 & -gL & 3g & 3g \\
0 & -gL & 3g & 3g
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix}
\tag{4.39}
\]

Through the displacement method discussed in Section C
of Chapter III, one can calculate the buckling load of a structure subjected to a simultaneous action of concentrated axial force $N$ and distributed axial load, $q$. The following examples are selected for the comparison of the numerical solution obtained by the present method with Timoshenko's rigorous mathematical approach (34).

2. Numerical Examples

Example 4.2. Consider the uniform cantilever column shown in Fig. 4.5a with a concentrated axial force, $N$, acting at end, $B$, and a uniform load, $q$, acting along the axis. Find either the critical load, $q_{cr}$, or the critical load, $N_{cr}$. Let the member length $L=240$ in., the uniform cross section $A=24$ in.$^2$, $I=96$ in.$^4$, and $E=30\times10^6$ psi.

Solution: Let the column be divided into five segments as shown in Fig. 4.5a. The global coordinates and local coordinates are shown in Fig. 4.5b and 4.5c, respectively, from which the equilibrium matrices $[A_m]$ and $[A_v]$ are established as follows:

$$[A_m] = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Fig. 4.5 Example 4.2
The eigenvalue equation of this problem is similar to Eq. (4.8) with the inclusion of \([g_{ij}]\). Using the digital computer program based on the matrix iteration method (32) yields the solutions shown in Tables I and II in which the comparison of the present solution with Timoshenko's solution is very satisfactory.

Example 4.3. Consider the simply supported uniform beam shown in Fig. 4.6a with a concentrated axial force, \(N\), acting at both ends, A and B, and a uniform load, \(q\), acting along the axis. Find the critical load, \(q_{cr}\), for given \(N\) and critical load, \(N_{cr}\), for given \(q\). Let \(L=240\) in., \(A=30.2376\) in.\(^2\), \(I=192\) in.\(^4\), and \(E=30\times10^6\) psi.

Solution: Let the beam be divided into five segments as shown in Fig. 4.6a. The generalized global coordinates and generalized local coordinates are shown in Figs. 4.6b and 4.6c, respectively, from which the equilibrium matrices \([A_m]\) and \([A_v]\) are established as follows:

\[
[A_v] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
(a) Loading

(b) Global Coordinates

(c) Local Coordinates

Fig. 4.6 Example 4.3
Similar to Example 4.2, the solutions obtained by using the computer program are shown in Tables III and IV in which a very good comparison between the present solution with Timoshenko's solution is shown.
Table I  Buckling Load $q_{cr}$ with $N$ Given of Example 4.2

<table>
<thead>
<tr>
<th>$b$</th>
<th>$N=bEI/L^2$ (lbs)</th>
<th>$a$</th>
<th>$q_{cr}=a\pi^2EI/L^3$ (lbs/in.)</th>
<th>$q_{cr}$ (lbs/in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^2/4.$</td>
<td>123,370.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2.28</td>
<td>114,000.00</td>
<td>0.25</td>
<td>128.51</td>
<td>130.99</td>
</tr>
<tr>
<td>2.08</td>
<td>104,000.00</td>
<td>0.50</td>
<td>257.02</td>
<td>269.06</td>
</tr>
<tr>
<td>1.91</td>
<td>95,500.00</td>
<td>0.75</td>
<td>385.53</td>
<td>385.42</td>
</tr>
<tr>
<td>1.72</td>
<td>86,000.00</td>
<td>1.00</td>
<td>514.04</td>
<td>514.70</td>
</tr>
<tr>
<td>0.96</td>
<td>48,000.00</td>
<td>2.00</td>
<td>1,028.08</td>
<td>1,019.80</td>
</tr>
<tr>
<td>0.15</td>
<td>7,500.00</td>
<td>3.00</td>
<td>1,542.13</td>
<td>1,538.97</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>3.18</td>
<td>1,634.66</td>
<td>1,632.92</td>
</tr>
</tbody>
</table>
Table II  Buckling Load $N_{cr}$ with $q$ Given of Example 4.2

<table>
<thead>
<tr>
<th>$a$</th>
<th>$q=a \pi^2 EI/L^3$ (lbs/in.)</th>
<th>$b$</th>
<th>$N_{cr}=bEI/L^2$ (lbs)</th>
<th>$N_{cr}$ (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>$\pi^2/4.$</td>
<td>123,370.00</td>
<td>123,372.92</td>
</tr>
<tr>
<td>0.25</td>
<td>128.51</td>
<td>2.28</td>
<td>114,000.00</td>
<td>114,184.90</td>
</tr>
<tr>
<td>0.50</td>
<td>257.02</td>
<td>2.08</td>
<td>104,000.00</td>
<td>104,898.80</td>
</tr>
<tr>
<td>0.75</td>
<td>385.53</td>
<td>1.91</td>
<td>95,500.00</td>
<td>95,499.12</td>
</tr>
<tr>
<td>1.00</td>
<td>514.04</td>
<td>1.72</td>
<td>86,000.00</td>
<td>86,064.87</td>
</tr>
<tr>
<td>2.00</td>
<td>1,028.08</td>
<td>0.96</td>
<td>48,000.00</td>
<td>47,358.83</td>
</tr>
<tr>
<td>3.00</td>
<td>1,542.13</td>
<td>0.15</td>
<td>7,500.00</td>
<td>7,276.96</td>
</tr>
<tr>
<td>3.18</td>
<td>1,634.66</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>b</td>
<td>( N = bEI/L^2 ) (lbs)</td>
<td>a</td>
<td>( q_{cr} = a\pi^2EI/L^3 ) (lbs/in.)</td>
<td>( q_{cr} ) (lbs/in.)</td>
</tr>
<tr>
<td>-----</td>
<td>--------------------------</td>
<td>--------</td>
<td>--------------------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>( \pi^2 )</td>
<td>986,965.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>8.63</td>
<td>836,000.00</td>
<td>0.25</td>
<td>1,028.09</td>
<td>1,025.64</td>
</tr>
<tr>
<td>7.36</td>
<td>736,000.00</td>
<td>0.50</td>
<td>2,056.18</td>
<td>2,057.84</td>
</tr>
<tr>
<td>6.08</td>
<td>608,000.00</td>
<td>0.75</td>
<td>3,084.26</td>
<td>3,084.26</td>
</tr>
<tr>
<td>4.77</td>
<td>477,000.00</td>
<td>1.00</td>
<td>4,112.35</td>
<td>4,112.62</td>
</tr>
</tbody>
</table>
Table IV: Buckling Load $N_{cr}$ with $q$ Given of Example 4.3

<table>
<thead>
<tr>
<th>$a$</th>
<th>$q=\pi^2 EI/L^3$ (lbs/in.)</th>
<th>$b$</th>
<th>$N_{cr}=bEI/L^2$ (lbs)</th>
<th>$N_{cr}$ (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>$\pi^2$</td>
<td>986,965.00</td>
<td>987,110.30</td>
</tr>
<tr>
<td>0.25</td>
<td>1,028.09</td>
<td>8.63</td>
<td>863,000.00</td>
<td>862,559.50</td>
</tr>
<tr>
<td>0.50</td>
<td>2,056.18</td>
<td>7.36</td>
<td>736,000.00</td>
<td>725,177.20</td>
</tr>
<tr>
<td>0.75</td>
<td>3,084.27</td>
<td>6.08</td>
<td>608,000.00</td>
<td>607,814.00</td>
</tr>
<tr>
<td>1.00</td>
<td>4,112.35</td>
<td>4.77</td>
<td>477,000.00</td>
<td>476,929.00</td>
</tr>
</tbody>
</table>
V. NUMERICAL INTEGRATION METHODS AND THEIR APPLICATION TO DYNAMIC RESPONSE

In the analysis of dynamic response, an exact or rigorous mathematical approach may be possible for a very simple structure subjected to a force expressable in a mathematical function. For practical problems of complicated structures and loadings, the direct mathematical integration becomes tedious, or, perhaps impossible. Therefore, it is often desirable and sometimes imperative to solve the equations of motion by step-by-step numerical integration procedures which are designed to utilize modern computational techniques.

Two well-known methods, the Runge-Kutta fourth-order method and the linear acceleration method, have been employed in this research for the general dynamic excitation of elastic as well as inelastic structures.

A. Fourth-Order Runge-Kutta Method

Consider the following second-order simultaneous differential equation

\[
\frac{d^2 x}{dt^2} = F(t, x, dx/dt)
\]  \hspace{1cm} (5.1)

of which the numerical integration by the fourth-order Runge-Kutta method may be expressed as (33)
\begin{align}
\{x\}_{i+1} &= \{x\}_i - (dt)\{x\}_i + \frac{(dt)}{6}([K_1]+[K_2]+[K_3]) \quad (5.2) \\
\{\dot{x}\}_{i+1} &= \{\dot{x}\}_i - \frac{1}{6}([K_1]+2[K_2]+2[K_3]+[K_4]) \quad (5.3)
\end{align}

where

\begin{align*}
[K_1] &= (dt)F(t_i,\{x\}_i,\{\dot{x}\}_i) \\
[K_2] &= (dt)F(t_i+\frac{dt}{2},\{x\}_i+\frac{dt}{2}\{\dot{x}\}_i,\{\dot{x}\}_i+\frac{dt}{2}[K_1]) \\
[K_3] &= (dt)F(t_i+\frac{dt}{2},\{x\}_i+\frac{dt}{2}\{\dot{x}\}_i+\frac{dt}{4}[K_1],\{\dot{x}\}_i+\frac{dt}{2}[K_2]) \\
[K_4] &= (dt)F(t_i+dt,\{x\}_i+dt\{\dot{x}\}_i+\frac{dt}{2}[K_2],\{\dot{x}\}_i+[K_3]).
\end{align*}

From either Eq. (3.19) or Eq. (3.31), one can write the acceleration equations as

\[\{\ddot{x}\} = [M]^{-1}(\{F\} - ([K] - (\alpha+\beta cosa)t)[S])\{x\}). \quad (5.4)\]

Because of the similarity between Eq. (5.1) and Eq. (5.4), the solution of Eq. (5.4) can be obtained by applying the fourth-order Runge-Kutta method.

The SUBROUTINE GFMKP in the appended computer programs is based on Eqs. (5.2 and 5.3) for which two examples are selected for the comparison of the numerical solution with the exact solution by direct integration.

Example 5.1. Find x and y of the following simultaneous second-order differential equations by using (a) direct
integration and (b) the fourth-order Runge-Kutta method.

\[ \begin{align*}
\frac{d^2x}{dt^2} + \frac{dy}{dt} + x - y &= \sin t \\
\frac{d^2y}{dt^2} + \frac{dx}{dt} + x - y &= 2t^2
\end{align*} \] 

of which the initial conditions are:

\[ x = 2., \ y = -4.5, \ \frac{dx}{dt} = -1., \ \text{and} \ \frac{dy}{dt} = -3.5 \ \text{at} \ t = 0. \]

Solution: (a) Using the given initial conditions one can find the following solution to Eq. (5.5) by the direct integration technique:

\[ x = 1 + t - 2t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + e^{-t} - \sin t \]

\[ y = -6 - 3t - 4t^2 + \frac{1}{6}t^3 + e^t - e^{-t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t \]

in which \( x \) and \( y \) are function of \( t \). Let \( t \) be varied in an interval of 0.1 sec., then the values of \( x \) and \( y \) are tabulated in Table V.

(b) Let Eq. (5.5) be rewritten in the following matrix form:
Using the computer program GFMKP the solution of $x$ and $y$ in Eq. (5.6) has been found for the interval of time $dt=0.004$ sec. The result is shown in Table VI. Comparing Table V with Table VI reveals that the difference is negligible. $x$ and $y$ obtained in (a) and (b) are plotted in Fig. 5.1.

Example 5.2. Find $x$, $y$, $z$ of the following simultaneous second-order differential equations by using (a) the direct integration method and (b) the fourth-order Runge-Kutta method.

\[
\begin{align*}
\frac{d^2x}{dt^2} + \frac{d^2z}{dt^2} - x &= 0 \\
\frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} - y &= 0 \\
\frac{d^2x}{dt^2} + y &= 2\cos t
\end{align*}
\]

of which the initial conditions are: $x=0$, $y=0$, $z=3$, $\frac{dx}{dy}=0$, $\frac{dy}{dt}=0$ and $\frac{dz}{dt}=1.5$ at $t=0$.

Solution: (a) The solutions to Eq. (5.7) are obtained by the direct integration method as

\[
\begin{align*}
x &= t \sin t \\
y &= t \sin t
\end{align*}
\]
\[ z = 1.5t - 2tsint - 2(1-cost) - 3. \]

The numerical values of \( x, y, z \) are tabulated in Table VII.

(b) Let Eq. (5.7) be rewritten in matrix form as

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{z} \\
\ddot{y} \\
\ddot{x}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z \\
y \\
x
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 2cost \end{bmatrix} \quad (5.8)
\]

The computer solution of Eq. (5.8) for \( dt=0.002 \) sec. is shown in Table VIII. The comparison between the results obtained by these two methods is very satisfactory. Fig. 5.2 shows the function of \( x, y, z \) vs time.

B. Linear Acceleration Method

The general expression of numerical integration of a second-order differential equation may be rewritten as (17)

\[
\begin{align*}
\{X\}_t &= \{X\}_{t-dt} + \{\dot{X}\}_{t-dt} (dt) + (\frac{1}{2} - B') \{\ddot{X}\}_{t-dt} (dt)^2 + B' \{\dddot{X}\}_t (dt)^2 \\
\{\dot{X}\}_t &= \{\dot{X}\}_{t-dt} + \frac{1}{2} \{\dddot{X}\}_{t-dt} + \{\dddot{X}\}_t (dt)
\end{align*}
\quad (5.9)
\]

\[
\{\dddot{X}\}_t = \{\dddot{X}\}_{t-dt} + \frac{1}{2} \{\dddot{X}\}_{t-dt} + \{\dddot{X}\}_t (dt)
\quad (5.10)
\]

in which the parameter \( B' \) is chosen to change the form
of the variation of acceleration in the time interval \( dt \).

When \( B' = 1/6 \), the motion solution corresponds to a linear variation of acceleration in the time interval \( dt \), and Eqs. (5.9) and (5.10) become

\[
\begin{align*}
\{x\}_t &= \{x\}_{t-dt} + (dt)\{\dot{x}\}_{t-dt} + \frac{1}{3}\{\ddot{x}\}_{t-dt} (dt)^2 + \frac{1}{6}\{\dot{x}\}_t (dt)^2 \\
\{\dot{x}\}_t &= \{\dot{x}\}_{t-dt} + \frac{1}{2}(dt)\{\ddot{x}\}_{t-dt} + \frac{1}{2}(dt)\{\dddot{x}\}_t
\end{align*}
\]

(5.11)

(5.12)

in which the subscripts \( t \) and \( t-dt \) denote the response at time, \( t \), and the previous, \( t-dt \), respectively. Thus the solution method is called a linear acceleration method.

Let the governing differential equation of motion of Eq. (3.19) be rewritten as

\[
[M]\{\dddot{x}\} + ([K] - (\alpha + \beta \cos \theta)[S])\{x\} = \{F\}
\]

(5.13)

which is actually a nonlinear differential equation, because the stability matrix \((\alpha + \beta \cos \theta)\ [S]\) is time dependent. The motion equation may be considered to be linear during a very short time duration, \( dt \), for which Eq. (5.13) can be expressed in an incremental form as

\[
[M]\{\Delta \dddot{x}\} + ([K] - (\alpha + \beta \cos \theta)[S])\{\Delta x\} = \{\Delta F\}
\]

(5.14)
in which

\( \{\Delta \ddot{X}\} = \text{incremental acceleration}; \)

\( \{\Delta X\} = \text{incremental displacement}; \) and

\( \{\Delta F\} = \text{incremental force}. \)

From Eqs. (5.11) and (5.12) we have

\[
\{\Delta \ddot{X}\} = \{\ddot{X}\}_t - \{\ddot{X}\}_t - dt = 3/dt(\{\Delta X\}) + \{B\} \tag{5.15}
\]

and

\[
\{\Delta \ddot{X}\} = \{\ddot{X}\}_t - \{\ddot{X}\}_t - dt = 6/dt^2\{\Delta X\} + \{A\} \tag{5.16}
\]

in which

\[
\{\Delta X\} = \{X\}_t - \{X\}_t - dt \tag{5.17}
\]

\[
\{A\} = -6/dt(\dot{X})_t - dt - 3(\ddot{X})_t - dt \tag{5.18}
\]

\[
\{B\} = -3(\dot{X})_t - dt/2(\ddot{X})_t - dt. \tag{5.19}
\]

Substituting Eqs. (5.15 to 5.19) into Eq. (5.14) yields the following symbolic form:

\[
[K']\{\Delta X\} = \{\Delta R\} \tag{5.20}
\]

in which

\[
[K'] = 6/dt^2[M] + [K] - [S'] \tag{5.21}
\]

\[
\{\Delta R\} = \{\Delta F\} - [M]\{A\} \tag{5.22}
\]

\[
[S'] = (a + \beta \cos \theta)[S]. \tag{5.23}
\]
Thus Eq. (5.14) is reduced to the pseudo-static form of Eq. (5.20) from which \( \Delta X \) can be solved as

\[
\{ \Delta X \} = [K']^{-1}\{ \Delta R \}. \tag{5.24}
\]

Using the pseudo-static form to find the dynamic response of a structure, one must repeatedly perform the following procedures:

\[
\{ A \} = -6/\text{dt}\{ \ddot{X} \} + 3\{ \dot{X} \} \text{dt}
\]

\[
\{ B \} = -3\{ \dot{X} \} \text{dt} + 2\{ \ddot{X} \} \text{dt}
\]

\[
\{ \Delta R \} = \{ \Delta F \} - [M]\{ A \}
\]

\[
[K'] = ([K] - [S'] + 6/\text{dt}^2[M])
\]

\[
\{ \Delta X \} = [K']^{-1}\{ \Delta R \}
\]

\[
\{ X \}_t = \{ X \}_t-\text{dt} + \{ \Delta X \}_t\text{dt}
\]

\[
\{ \dot{X} \}_t = \{ \dot{X} \}_t-\text{dt} + \{ \Delta \dot{X} \}_t\text{dt} = \{ \dot{X} \}_t-\text{dt} + 3/\text{dt}\{ \Delta X \}_t\text{dt} + \{ B \}
\]

\[
\{ \ddot{X} \}_t = \{ \ddot{X} \}_t-\text{dt} + \{ \Delta \ddot{X} \}_t\text{dt} = \{ \ddot{X} \}_t-\text{dt} + 6/\text{dt}^2\{ \Delta X \}_t\text{dt} + \{ A \}
\]

in which \([S']\) is different from time to time. Consequently, the structure is assumed to behave in a linear manner during each time increment, and the nonlinear response is obtained as a sequence of successive increments.
C. Modal Analysis

In analyzing the response of a structural system subjected to dynamic excitation, the governing differential equations of motion are usually composed of a set of coupled differential equations of second order. One of the approaches of solving these coupled equations is to uncouple the equations by using a technique of linear coordinate transformation. The linear transformation is obtained by assuming that the response is a superposition of the normal modes of a system multiplied by corresponding time-dependent generalized coordinates. The solutions to the uncoupled equations can be obtained by using Duhamel's integral. This analysis is called modal analysis (23,24).

D. Application of Numerical Integration Methods to a Structure Subjected to a Ground Acceleration

When a structure is excited by a ground acceleration, the motion equations of Eq. (3.19) may be expressed in terms of the following relative coordinates:

\[
\{x_s\}_{\text{relative}} = \{x_s\} - \{x_g\} \\
\{x_r\}_{\text{relative}} = \{x_r\} \\
\{\ddot{x}_s\}_{\text{relative}} = \{\ddot{x}_s\} - \{\ddot{x}_g\} \\
\{\ddot{x}_r\}_{\text{relative}} = \{\ddot{x}_r\}
\]

\[
\text{r relative}
\]

(5.25)
Table V  Values of x and y of Example 5.1
by Direct Integration Method

<table>
<thead>
<tr>
<th>Time (sec.)</th>
<th>x (inch)</th>
<th>y (inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.2000000E 01</td>
<td>-0.4500000E 01</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1885653E 01</td>
<td>-0.4877422E 01</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1745129E 01</td>
<td>-0.5309491E 01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1581950E 01</td>
<td>-0.5796082E 01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1399307E 01</td>
<td>-0.6337336E 01</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1200031E 01</td>
<td>-0.6933632E 01</td>
</tr>
<tr>
<td>0.6</td>
<td>0.09865822E 00</td>
<td>-0.7585610E 01</td>
</tr>
<tr>
<td>0.7</td>
<td>0.07610353E 00</td>
<td>-0.8294145E 01</td>
</tr>
<tr>
<td>0.8</td>
<td>0.05250612E 00</td>
<td>-0.9060350E 01</td>
</tr>
<tr>
<td>0.9</td>
<td>0.02799199E 00</td>
<td>-0.9885552E 01</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0264967E-01</td>
<td>-0.1077128E 02</td>
</tr>
<tr>
<td>1.1</td>
<td>-0.2349665E 00</td>
<td>-0.1171918E 02</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.5043706E 00</td>
<td>-0.1273120E 02</td>
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<tr>
<td>1.3</td>
<td>-0.7822802E 00</td>
<td>-0.1380931E 02</td>
</tr>
<tr>
<td>1.4</td>
<td>-0.1069665E 01</td>
<td>-0.1495567E 02</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.1367968E 01</td>
<td>-0.1617241E 02</td>
</tr>
<tr>
<td>1.6</td>
<td>-0.1679098E 01</td>
<td>-0.1746175E 02</td>
</tr>
<tr>
<td>1.7</td>
<td>-0.2005452E 01</td>
<td>-0.1882587E 02</td>
</tr>
<tr>
<td>1.8</td>
<td>-0.2349898E 01</td>
<td>-0.2026689E 02</td>
</tr>
<tr>
<td>1.9</td>
<td>-0.2715786E 01</td>
<td>-0.2178680E 02</td>
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<tr>
<td>2.0</td>
<td>-0.3106950E 01</td>
<td>-0.2338741E 02</td>
</tr>
</tbody>
</table>
Table VI Values of $x$ and $y$ of Example 5.1 by Runge-Kutta Method

<table>
<thead>
<tr>
<th>Time (sec.)</th>
<th>Runge-Kutta Method</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$x$ (inch)</td>
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<td>0.2000000E 01</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1885633E 01</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1745090E 01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1581895E 01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1399232E 01</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1199939E 01</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9864780E 00</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7609386E 00</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5249753E 00</td>
</tr>
<tr>
<td>0.9</td>
<td>0.279458E 00</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2638184E-01</td>
</tr>
<tr>
<td>1.1</td>
<td>-0.2350211E 00</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.5044181E 00</td>
</tr>
<tr>
<td>1.3</td>
<td>-0.7823184E 00</td>
</tr>
<tr>
<td>1.4</td>
<td>-0.1069688E 01</td>
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<tr>
<td>1.5</td>
<td>-0.1367956E 01</td>
</tr>
<tr>
<td>1.6</td>
<td>-0.1679055E 01</td>
</tr>
<tr>
<td>1.7</td>
<td>-0.2005371E 01</td>
</tr>
<tr>
<td>1.8</td>
<td>-0.2349773E 01</td>
</tr>
<tr>
<td>1.9</td>
<td>-0.2715609E 01</td>
</tr>
<tr>
<td>2.0</td>
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</table>
Fig. 5.1 Solutions of x and y of Example 5.1
Table VII  Value of x, y, z of Example 5.2 by Direct Integration Method

<table>
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<tr>
<th>Time (sec.)</th>
<th>Direct Integration Method</th>
</tr>
</thead>
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<td>x or y (inch)</td>
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<tr>
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<td>0.9983279E-02</td>
</tr>
<tr>
<td>0.2</td>
<td>0.397359E-01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8865470E-01</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1557643E 00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2397080E 00</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3387790E 00</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4509431E 00</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5738719E 00</td>
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<tr>
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Table VIII Values of $x$, $y$, $z$ of Example 5.2 by Runge-Kutta Method

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<th>Time (sec.)</th>
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<tr>
<td>0.4</td>
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<tr>
<td>0.5</td>
<td>0.2397076E 00</td>
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<td>0.1818587E 01</td>
</tr>
</tbody>
</table>
Fig. 5.2 Solutions of x, y and z of Example 5.2
in which
\[
\{x_g\} = \text{ground displacement}; \quad \text{and} \quad \{\dot{x}_g\} = \text{ground acceleration.}
\]

Upon substitution of Eq. (5.25) into Eq. (3.19), the motion equations become

\[
[M]\begin{bmatrix}
\ddot{x}_r \\
\ddot{x}_s
\end{bmatrix}
+ ([K] - (a + \beta \cos \theta t) [S]) \begin{bmatrix}
x_r \\
x_s
\end{bmatrix}_{\text{rel.}}
= -\ddot{x}_g [M] \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

(5.26)

If the joint rotations are neglected, then Eq. (5.13) becomes

\[
[M]\begin{bmatrix}
\dddot{x}_s
\end{bmatrix}_{\text{rel.}} + ([K] - (a + \beta \cos \theta t) [S]) \begin{bmatrix}
x_s
\end{bmatrix}_{\text{rel.}} = -\ddot{x}_g [M] \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

(5.27)

Example 5.3. Consider the shear building shown in Fig. 5.3 subjected to a ground acceleration \(\dddot{x}_g = (-8.\pi^2 \sin 4\pi t)\) in./sec\(^2\). The structure is assumed to be stationary at \(t=0\). Find the relative displacements \(y_1\) and \(y_2\).

Solution: Without considering the joint rotations, the diagrams of relative displacements and internal shears are shown in Figs. 5.4a and 5.4b, respectively. The governing differential equations of motion can be established as
Fig. 5.3 Example 5.3
(a) Relative displacements

(b) Internal Shears

Fig. 5.4 Diagrams for Example 5.3
The solutions to Eq. (5.28) by modal matrix method are

\[
\begin{pmatrix}
0.294 & 0 \\
0 & 0.177
\end{pmatrix}
\begin{pmatrix}
\ddot{Y}_1 \\
\ddot{Y}_2
\end{pmatrix} +
\begin{pmatrix}
113.4258 & -57.8703 \\
57.8703 & 57.8703
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}
= (-8\pi^2 \sin 4\pi t)
\begin{pmatrix}
-0.294 \\
-0.177
\end{pmatrix}
\]

(5.28)

in which \( \omega_1 = 10.0493 \text{ rad./sec.} \) and \( \omega_2 = 24.7338 \text{ rad./sec.} \).

Eq. (5.28) is also solved by the Runge-Kutta method and the linear acceleration method. The results obtained by using these three methods are shown in Tables IX, X, and XI. The values of \( Y_1 \) and \( Y_2 \) obtained by the Runge-Kutta method are plotted in Fig. 5.5.
Fig. 5.5 Dynamic Response of $Y_1$ and $Y_2$ of Example 5.3
Table IX Modal Matrix Solution of Example 5.3

<table>
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<th>Time (sec.)</th>
<th>$y_1$(inch)</th>
<th>$y_2$(inch)</th>
</tr>
</thead>
<tbody>
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<td>0.00000000E 00</td>
<td>0.00000000E 00</td>
</tr>
<tr>
<td>0.25</td>
<td>0.8464944E 00</td>
<td>0.1220889E 01</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.1363268E 01</td>
<td>-0.1980392E 01</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1368720E 01</td>
<td>0.1968805E 01</td>
</tr>
<tr>
<td>1.00</td>
<td>-0.8336927E 00</td>
<td>-0.1221648E 01</td>
</tr>
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<td>-0.1810213E-01</td>
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<td>0.1220779E 01</td>
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<td>-0.3398962E-01</td>
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<td>2.75</td>
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<tr>
<td>Time (sec.)</td>
<td>Runge-Kutta Method</td>
<td>Runge-Kutta Method</td>
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Table XI  Linear Acceleration Solution of Example 5.3

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VI. DYNAMIC RESPONSE OF ELASTIC STRUCTURAL SYSTEMS

The numerical integration techniques described in the preceding chapter will be used herein to study the instability behavior and displacement response of a structure subjected to time dependent axial forces as well as lateral forces or foundation movements. A number of selected examples given below have been studied by using digital computer programs based on the numerical integration techniques described in Chapter V.

A. Numerical Examples

Example 6.1. Consider a beam column shown in Fig. 6.1a subjected to \( N_t \) at both ends and periodic lateral force, \( F_t \), at point B. The periodic force, \( F_t \), is shown in Fig. 6.1b, and the axial force is \( N_t = (a + b \cos \omega t)N_0 \). The member properties are:

- Cross sectional area: \( A_{AB} = 30.24 \text{ in.}^2, A_{BC} = 24. \text{ in.}^2 \)
- Member length: \( L_{AB} = 144. \text{ in.}, L_{BC} = 96. \text{ in.} \)
- Moment inertia: \( I_{AB} = 192. \text{ in.}^4, I_{BC} = 96. \text{ in.}^4 \)

The static buckling load and natural frequency are found to be 2974.80 kips and 181.9423 rad./sec., respectively. The principal dynamic instability region for \( N_0 = 2974.80 \text{ kips} \), \( \omega = 181.9423 \text{ rad./sec.} \), \( a = 0 \), and \( \beta = 0.2 \) is shown in Fig. 6.2. Two cases of dynamic response are investigated by using the Runge-Kutta method with time interval \( dt = 0.004 \text{ sec.} \). As
indicated in Fig. 6.2, case A is for $\theta = 251.7872$ rad./sec. in the stability region, and case B is for $\theta = 364.00$ rad./sec. in the principal instability region. The lateral deflections at point B corresponding to case A and case B are shown in Fig. 6.3.

Example 6.2. Consider a two-story steel framework shown in Fig. 6.4a in which the masses lumped at the floor and the length and moment inertia of the constituent members are given. The columns of the frame are subjected to a time dependent axial force, $N_t = (a + \beta \cos \theta t)N_0$, and the base of the frame is excited by a ground acceleration, $\ddot{X}_g = (-8\pi^2 \sin 4\pi t)$ in./sec$^2$. After the static buckling load, $N_0$, and natural frequency, $\omega$, of the structural system have been found, the principal instability regions for $N_0 = 1001.626$ kips, $\omega = 10.0494$ rad./sec. are investigated. The results are shown in Fig. 6.5 for various axial loads corresponding to $\alpha = 0.1, 0.2, 0.4$, and $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$. Two cases of dynamic response sketched in Fig. 6.5 have been studied in which case A is for $\alpha = 0.1, \beta = 0.3$, and $\theta = 15.0$ rad./sec. in the stability region, and case B is for $\alpha = 0.1, \beta = 0.3$, and $\theta = 20.1$ rad./sec. in the instability region. The Runge-Kutta method with a time interval $dt = 0.025$ sec. has been employed for studying the relative displacements $y_1$ and $y_2$. The results associated with case A and case B are shown in Figs. 6.6 and 6.7. These two cases have also been investigated by the linear acceleration method with a time
interval \( dt=0.0125 \) sec. The relative displacements \( y_1 \) and \( y_2 \) are shown in Figs. 6.8 and 6.9.

B. Discussion of Results

For the cases in the instability region, the deflection response grows exponentially with time. The deflection response associated with the cases in the stability region, however, is quite stable. The results obtained by the Runge-Kutta method agree satisfactorily with those obtained by the linear acceleration method.
Fig. 6.1 Example 6.1

(a) Given Structure

(b) Forcing Function

(c) Global Coordinates

(d) Local Coordinates
Fig. 6.2 Dynamic Instability Region of Example 6.1
Fig. 6.3 Dynamic Response of Example 6.1

Case A
Case B

Displacement at B (in.)
Fig. 6.4 Example 6.2
Fig. 6.5 Dynamic Instability Region of Example 6.2
Fig. 6.6 \( y_1 \) of Example 6.2 by Runge-Kutta Method
Fig. 6.7 $y_2$ of Example 6.2 by Runge-Kutta Method
Fig. 6.8 $y_1$ of Example 6.2 by Linear Acceleration Method
Fig. 6.9 $y_2$ of Example 6.2 by Linear Acceleration Method
VII. MATRIX FORMULATION FOR ELASTO-PLASTIC STRUCTURAL SYSTEMS

When the deflection of a structural framework becomes sufficiently large, the internal moments of the constituent members may exceed the elastic limit. In such a case, the elastic analysis is no longer applicable, and the structure must be analyzed to include the inelastic deformation. Therefore, the elementary mass, stiffness, and stability matrices of a typical member must be derived to account for the deformation beyond the elastic limit.

A. Idealized Elasto-Plastic Moment-Rotation Characteristics

Let us assume that the constituent members of a frame have ideal elasto-plastic moment-rotation characteristics as shown in Fig. 7.1. The typical moment-rotation diagram has a linear relationship called elastic branch which varies from zero moment to the reduced plastic moment $M_{pc}$. The reduced plastic moment is evaluated to account for the effect of axial load on the plastic moment. For any further deformation, the member will have a plastic hinge at which the applied moment is $M_{pc}$. When the member has reverse deformation, the moment-rotation relationship becomes linear and parallel to the original elastic branch. The elastic behavior remains unchanged until the internal moment reaches $M_{pc}$. Consequently, a plastic hinge is assumed, and a constant moment is applied at the hinge.
The cyclic process is sketched in Fig. 7.1.

B. Reduced Plastic Moment

The influence of axial force on plastic moment is calculated according to ASCE manuals (35) as

(a) for wide-flange sections

when \( 0 \leq P \leq 0.15P_Y \)

\[ M_{pc} = M_p = F_Y Z \] (7.1)

when \( 0.15P_Y \leq P \leq P_Y \)

\[ M_{pc} = 1.18[1-(P/P_Y)]M_p \] (7.2)

(b) for rectangular section

\[ M_{pc} = [1 - (P/P_Y)^2]M_p \] (7.3)

where

- \( Z \) = plastic section modulus;
- \( P_Y = F_Y A \);
- \( F_Y \) = yielding stress of steel;
- \( A \) = cross sectional area;
- \( P \) = axial force;
- \( M_p = F_Y Z \) = plastic moment; and
- \( M_{pc} \) = reduced plastic moment.
Fig. 7.1 Idealized Moment-Rotation Relationships
C. Modified Elementary Mass, Stiffness, and Stability Matrices

An elastic analysis for dynamic response can only be carried out to the loading stage at which none of the internal moment reaches plastic moment. When an internal moment reaches plastic moment, the frame is then modified by inserting a real hinge at the location with a plastic moment applied at the hinge. Thus the mass, stiffness, and stability matrices of that member must be modified according to the hinge location.

Let the typical member shown in Fig. 7.2 have a hinge at j, then the shape functions of the member are

\[
\begin{align*}
\phi_1(x) &= (x^3/2L^2 - 3x^2/2L + x) \\
\phi_2(x) &= 0 \\
\phi_3(x) &= (-x^3/2L^3 + 3x^2/2L^2 - 1) \\
\phi_4(x) &= (3x^2/2L^2 - x^3/2L^3).
\end{align*}
\]

(7.4)

Fig. 7.2 Generalized Local Coordinates and Generalized Forces of a Beam with j End Hinged
Following the same procedure used in Chapter III, one can derive the mass, stiffness, and stability matrices as

\[
\begin{align*}
Q_1 &= \begin{pmatrix}
\frac{8mL^3}{420} & 0 & -\frac{36mL^2}{420} & \frac{11mL^2}{280} \\
0 & 420 & 0 & 0 \\
-\frac{36mL^2}{420} & 204mL & -39mL & 280 \\
\frac{11mL^2}{280} & 0 & -39mL & 99mL
\end{pmatrix} \\
\dot{q}_1 &= \begin{pmatrix}
\ddot{q}_1 \\
\ddot{q}_2 \\
\ddot{q}_3 \\
\ddot{q}_4
\end{pmatrix}
\end{align*}
\]

\[(7.5)\]

\[
\begin{align*}
Q_1 &= \begin{pmatrix}
\frac{3EI}{L} & 0 & -\frac{3EI}{L^2} & -\frac{3EI}{L^2} \\
0 & L^2 & 0 & 0 \\
-\frac{3EI}{L^2} & 3EI & \frac{3EI}{L^3} & \frac{3EI}{L^3} \\
-\frac{3EI}{L^2} & 0 & \frac{3EI}{L^3} & \frac{3EI}{L^3}
\end{pmatrix} \\
q_1 &= \begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix}
\end{align*}
\]

\[(7.6)\]
Similarly, let the typical member shown in Fig. 7.3 have a hinge at end i, then the shape functions for the boundary conditions of the member may be derived as

\[ \phi_1(x) = 0 \]

\[ \phi_2(x) = \left(\frac{x^3}{2L^2} - \frac{x}{2}\right) \]

\[ \phi_3(x) = \left(-\frac{x^3}{2L^3} + \frac{3x}{2L} - 1\right) \]

\[ \phi_4(x) = \left(-\frac{x^3}{2L^3} + \frac{3x}{2L}\right). \]
Consequently, the mass, stiffness, and stability matrices become

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4 
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{8mL^3}{420} & \frac{11mL^2}{280} & -\frac{36mL^2}{420} \\
0 & \frac{11mL^2}{280} & \frac{99mL}{420} & -\frac{39mL}{280} \\
0 & -\frac{36mL^2}{420} & -\frac{39mL}{280} & \frac{204mL}{420} 
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3 \\
\dot{q}_4 
\end{bmatrix}
\]

(7.9)
If a member has both ends hinged, then the stiffness and stability matrices become null, and the mass matrix is
D. System Matrices of Mass, Stiffness, and Stability

Because the mass, stiffness, and stability matrices of a member with one end or both ends hinged have been modified to account for the boundary conditions, the formulation of system mass, stiffness, and stability matrices can be made by following the same procedure described in Section C of Chapter III.
VIII. DYNAMIC RESPONSE OF ELASTO-PLASTIC STRUCTURES

A. Transfer Matrix for Plastic Moments and Their Associated Shears

When an internal moment at the nodal point of a member is equal to or greater than the reduced plastic moment, then a plastic hinge is assumed to be at the node with a constant moment, $M_{pc}$, applied at the hinge. Thus the plastic hinge is treated as a real hinge, and the member mass, stiffness, and stability matrices must be modified to satisfy the boundary conditions. If a plastic hinge forms at end $i$ of member $ij$, the moment $M_{pc}$ at that end must be carried over to end $j$ with the magnitude of $M_{pc}f_{co}$ ($f_{co}$ is the carry-over factor including the effect of axial force). Consequently, $M_{pc}f_{co}$ is treated as the external moment at joint $j$. The shears, which result from $M_{pc}$ and $M_{pc}f_{co}$ on the member $ij$, are then transferred to the structural nodes and become the external forces.

Let $\{FEM\}$, $\{FEV\}$ represent the plastic moments $M_{pc}$, $M_{pc}f_{co}$ and the shears due to $M_{pc}$, $M_{pc}f_{co}$, respectively; then the transfer matrix can be expressed as

$$\{TF\} = \begin{bmatrix} T_{F_r} \\ T_{F_s} \end{bmatrix} = \begin{bmatrix} [A_m] \{FEM\} \\ [A_v] \{FEV\} \end{bmatrix}$$

(8.1)

where $\{TF\} =$ external load matrix transferred from plastic moments $\{TF_r\} = [A_m]\{FEM\}$, and shears $\{TF_s\} = [A_v]\{FEV\}$. 


{TF} should be combined with the load matrix $\begin{bmatrix} F_r \\ F_s \end{bmatrix}$ in Eq. (3.31) for dynamic response of the elasto-plastic case.

The internal moments and shears can be evaluated from Eq. (3.32) and should be combined with moments \{FEM\} and shears \{FEV\} for the final solution.

B. Calculation of Plastic Hinge Rotation

As discussed previously, when an internal nodal moment reaches the plastic moment capacity, a real hinge is inserted at that node with a constant moment applied at the hinge which is allowed to rotate according to the material behavior shown in Fig. 7.1. When the hinge rotates in the direction of the plastic moment, the moment is assumed to be constant, and the rotation can increase indefinitely. When the hinge rotation is in the opposite direction of the moment, however, the plastic moment is removed, and the member becomes elastic. Thus the plastic hinge rotation must be calculated at each step of the numerical integrations and compared with the previous one, if any, in order to check the change of the sign of rotation. For an entire structural system, the hinge rotations may be obtained as follows:

$$\{H_r\} = ([FS]\{Q_m\} - [FY][A_v]^T\{X_s\}) - [A_m]^T\{X_r\} \tag{8.2}$$

in which the first term of the right side of Eq. (8.2) is
composed of the force-deformation relationship of the constituent members given in Eqs. (3.29, 3.30), and the second term is due to external nodal rotations. The typical element in the first term can be derived from Eqs. (3.29) and (3.30) as

\[
\{q_r\} = [K_{MR}+S_{MR}]^{-1}\{Q_m\} + [K_{MY}+S_{MY}][A_v]^T\{X_s\}. \tag{8.3}
\]

Thus the elements in Eq. (8.2) are

\[
[FS] = \begin{pmatrix}
[FM]_1 \\
[FM]_2 \\
& \ddots \\
& & [FM]_i \\
& & & \ddots \\
& & & & [FM]_n
\end{pmatrix}
\]

\[
\begin{aligned}
[FM]_i = \frac{1}{\text{DET}_i} & \begin{pmatrix}
\frac{4E_i}{L_i} - \frac{2N_i L_i}{15} & -\frac{2E_i}{L_i} + \frac{N_i L_i}{30} \\
-\frac{2E_i}{L_i} + \frac{N_i L_i}{30} & \frac{4E_i}{L_i} - \frac{2N_i L_i}{15}
\end{pmatrix} \\
\text{DET}_i &= \left(\frac{6E_i}{L_i} - \frac{N_i L_i}{10}\right)\left(\frac{2E_i}{L_i} - \frac{N_i L_i}{6}\right) \\
\end{aligned}
\]
Note that \( \{Q_m\} \) is the vector of internal nodal moments due to nodal displacements. The subscript \( i \) denotes the number of members. The element \( i \) of the vector \( \{H_i\} \) has value only if a plastic hinge exists at node \( i \).

### C. Numerical Examples

**Example 8.1** Let Example 6.1 be used to investigate the elasto-plastic dynamic response for \( \alpha = 0.0, \beta = 0.2 \) and \( \theta = 364. \) rad./sec. The deflections of point B for elastic and elasto-plastic cases are shown in Fig. 8.1.

**Example 8.2** Example 6.2 is used to investigate the elasto-plastic dynamic response for \( \alpha = 0.0, \beta = 0.3, \) and
Fig. 8.1 Dynamic Response of Example 8.1

Elastic Case

Elasto-Plastic Case

Displacement at B (in.)
\( \theta = 20.1 \text{ rad./sec.} \) The lateral deflections of \( y_1 \) and \( y_2 \) for both elastic and elasto-plastic cases are shown in Figs. 8.2 and 8.3.

D. Discussion of Results

From these two examples, it can be observed that the behavior of elastic case is different from that of the elasto-plastic case. The parametric resonance shows up clearly for the elastic case and cannot be observed for the elasto-plastic case. The reason is that the dynamic instability region is based on the assumption that the structure is elastic.
Fig. 8.2 Dynamic Response of $y_1$ of Example 8.2

--- Elastic Case

--- Elasto-Plastic

Displacement $y_1$ (in.)
Fig. 8.3 Dynamic Response of \( y_2 \) of Example 8.2
IX. SUMMARY AND CONCLUSIONS

An analytical method is presented for determining the behavior of dynamic instability and response of frameworks subjected to longitudinal pulsating loads and lateral dynamic forces or foundation movements. Some of the features of this work are summarized as follows:

1. Dynamic instability criteria are discussed and formulated in relation to the magnitude of axial force, the longitudinal forcing frequency, and the transverse frequency.

2. The displacement method is employed for structural matrix formulation for which the typical member matrices of mass, stiffness, and stability are derived.

3. Eigenvalues of free vibrations and static instability are investigated in this work. The static instability analysis includes both concentrated and uniformly distributed loads.

4. The elastic and elasto-plastic frameworks are analyzed for the response of displacements, internal moments, and shears resulting from dynamic lateral forces or ground accelerations. General considerations include bending deformation, geometric nonlinearity, the effect of girder shears on columns, and the effect of axial loads on plastic moments.
5. Two numerical methods, the fourth-order Runge-Kutta method and the linear acceleration method, are used to solve the nonlinear differential equations of motion. The solutions obtained by these two methods compare very satisfactorily.

6. A number of selected examples are presented from which it can be observed that the deflection response corresponding to the instability region grows exponentially with time.

7. The dynamic instability analysis yields the stability and instability regions from which one may design a structure to avoid the occurrence of parametric resonance.


BIBLIOGRAPHY (continued)


APPENDIX

Computer Programs
**LIST OF SYMBOLS USED IN COMPUTER PROGRAMS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>APV</td>
<td>= matrix relating girder shears to columns</td>
</tr>
<tr>
<td>AFP</td>
<td>= matrix relating vertical forces to axial force in columns</td>
</tr>
<tr>
<td>AF</td>
<td>= axial force in column</td>
</tr>
<tr>
<td>AM</td>
<td>= matrix ([A_m])</td>
</tr>
<tr>
<td>AV</td>
<td>= matrix ([A_v])</td>
</tr>
<tr>
<td>AMS</td>
<td>= matrix ([A_{ms}])</td>
</tr>
<tr>
<td>AREA</td>
<td>= cross sectional area</td>
</tr>
<tr>
<td>A, B, C, D</td>
<td>= constants of (K_1, K_2, K_3, K_4) for Runge-Kutta formula</td>
</tr>
<tr>
<td>ALPHA</td>
<td>= coefficient of axial load</td>
</tr>
<tr>
<td>BETA</td>
<td>= coefficient of axial load</td>
</tr>
<tr>
<td>DT</td>
<td>= small increment of time</td>
</tr>
<tr>
<td>FY</td>
<td>= yielding stress (F_y)</td>
</tr>
<tr>
<td>PSB</td>
<td>= static buckling load</td>
</tr>
<tr>
<td>PT</td>
<td>= time-dependent axial force</td>
</tr>
<tr>
<td>PM</td>
<td>= plastic moment (ZF_y)</td>
</tr>
<tr>
<td>PY</td>
<td>= cross sectional area times (F_y)</td>
</tr>
<tr>
<td>NM</td>
<td>= number of members</td>
</tr>
<tr>
<td>NP</td>
<td>= number of degrees of freedom</td>
</tr>
<tr>
<td>NPR</td>
<td>= number of degrees of freedom in joint rotation</td>
</tr>
<tr>
<td>NPS</td>
<td>= number of degrees of freedom in side sway</td>
</tr>
<tr>
<td>NVP</td>
<td>= number of vertical forces acting on columns</td>
</tr>
<tr>
<td>VA</td>
<td>= (\alpha) value</td>
</tr>
</tbody>
</table>
VB = \( \beta \) value
NPTS = number of time steps
XL = member length
XM = mass per unit length
XI = moment of inertia of cross section
XE = elastic Young's modulus
X = displacement
XT = velocity
XTT = acceleration
XEM = internal end moments
XEV = internal end shears
XXM = system mass matrix \([M]\)
XXK = system stiffness matrix \([K]\)
XXP = system stability matrix \([S]\)
T = time
ZP = plastic modulus
ZETA = \( \theta \) value
NPH = new plastic hinge
LPH = old plastic hinge
NRH = relieved plastic hinge
HR = plastic hinge rotation \(\{H_r\}\)
FEM = internal moment due to plastic moment
FEV = internal shear due to plastic moment
COFR = carry-over factor \(f_{CO}\)
FRY₁, FRY₂, FRY₃, FRY₄ = element of matrix [FY]
FMV₁, FMV₂, FMV₃, FMV₄ = element of matrix [L]
FRM₁, FRM₂, FRM₃, FRM₄ = element of matrix [FS]
PMR₁, PMR₂, PMR₃, PMR₄ = element of submatrix [SMR]
PMY₁, PMY₂, PMY₃, PMY₄ = element of submatrix [SMY]
PVR₁, PVR₂, PVR₃, PVR₄ = element of submatrix [SVR]
PYY₁, PYY₂, PYY₃, PYY₄ = element of submatrix [SVY]
SMR₁, SMR₂, SMR₃, SMR₄ = element of submatrix [KMR]
SMY₁, SMY₂, SMY₃, SMY₄ = element of submatrix [KMY]
SVR₁, SVR₂, SVR₃, SVR₄ = element of submatrix [KVR]
SVY₁, SVY₂, SVY₃, SVY₄ = element of submatrix [KVY]
XMR₁, XMR₂, XMR₃, XMR₄ = element of submatrix [MMR]
XMY₁, XMY₂, XMY₃, XMY₄ = element of submatrix [MMY]
XVR₁, XVR₂, XVR₃, XVR₄ = element of submatrix [MVR]
XYY₁, XYY₂, XYY₃, XYY₄ = element of submatrix [MVY]
FLOW CHART OF ELASTIC DYNAMIC RESPONSE PROGRAM

START

READ AND WRITE INPUT

SET UP MEMBER MASS MATRIX \([m_{ij}]\)
AND MEMBER STIFFNESS MATRIX \([k_{ij}]\)

SET UP STRUCTURAL MASS MATRIX \([M]\)
AND STRUCTURAL STIFFNESS MATRIX \([K]\)

INVERT MASS MATRIX \([M]\)

CALCULATE \(S^T M\) FOR \([M]\) AND \([K]\)

DEFINE AND WRITE INITIAL CONDITIONS

DO 9999 KK = 2, NPTS

FIND AXIAL FORCE IN COLUMNS

SET UP MEMBER STABILITY MATRIX \([s_{ij}]\)

SET UP STRUCTURAL STABILITY MATRIX \([S]\)

CALCULATE \(S^T S\)

CALL SUBROUTINE GFMKP

CALCULATE \(\{x\}, \{\dot{x}\}, \{\ddot{x}\}\) BY RUNGE-KUTTA METHOD

CALCULATE INTERNAL FORCES

\[ T = T + DT \]

9999 CONTINUE

STOP

END
FLOW CHART OF ELASTO-PLASTIC DYNAMIC RESPONSE PROGRAM

START

READ AND WRITE INPUT
CALCULATE REDUCED PLASTIC MOMENT

DEFINE AND WRITE INITIAL CONDITIONS

DO 9999 KK = 2, NPTS

FIND AXIAL FORCE IN COLUMNS

CALCULATE \{FEM\} AND \{FEV\}

SET UP MEMBER STABILITY MATRIX \[ S_{ij} \] AND STRUCTURAL STABILITY MATRIX \[ S \]

CALCULATE \( S^T \) FOR \[ S \]

SET MEMBER MASS MATRIX \[ m_{ij} \] AND MEMBER STIFFNESS MATRIX \[ k_{ij} \]
SET UP STRUCTURAL MASS MATRIX AND STRUCTURAL STIFFNESS MATRIX \[ K \]

INVERT MASS MATRIX \[ M \]

CALCULATE \( S^T \) FOR \[ M \] AND \[ K \]

IS THERE ANY MEMBER WITH BOTH ENDS HINGED?

yes

CALCULATE SECONDARY SHEAR \( SECDV \)
TRANSFER TO EXTERNAL JOINT LOAD

CALL SUBROUTINE GEXTP

CALL SUBROUTINE GFMKP
CALCULATE \( \{X\} \), \( \{X\} \), \( \{X\} \) BY RUNGE-KUTTA METHOD

CALCULATE INTERNAL FORCES

yes

CALCULATE HINGE ROTATION \( \{H_r\} \)

no
IS THERE ANY NEW PLASTIC HINGE FORMED?

- ADJUST \{FEM\} AND \{FEV\}

   yes
   no

IS THERE ANY OLD PLASTIC HINGE RELIEVED?

- ADJUST \{FEM\} AND \{FEV\}

   yes
   yes
   no

   \( T = T + DT \)

   9999 CONTINUE

   STOP

   END
**ELASTIC DYNAMIC RESPONSE**

1. DIMENSION ALPHA(10), PAR(10), BETA(10), RATIO(10)
2. DIMENSION AM(12,18), AMI(3,3), AV(12,18)
3. DIMENSION XL(10), XI(10), XM(10)
4. DIMENSION NPH(10)
5. DIMENSION ASAT(10,10), INDEX(50)
6. DIMENSION XXK(10,10), XMX(10,10), XXP(10,10)
7. DIMENSION SMR1(10), SMR2(10), SMY1(10), SMY2(10)
8. DIMENSION SMR3(10), SMR4(10), SMY3(10), SMY4(10)
9. DIMENSION XM1(10), XM2(10), XMY1(10), XMY2(10)
10. DIMENSION XM3(10), XM4(10), XMY3(10), XMY4(10)
11. DIMENSION PMR1(10), PMR2(10), PMY1(10), PMY2(10)
12. DIMENSION PMR3(10), PMR4(10), PMY3(10), PMY4(10)
13. DIMENSION SVR1(10), SVR2(10), SVY1(10), SVY2(10)
14. DIMENSION SVR3(10), SVR4(10), SVY3(10), SVY4(10)
15. DIMENSION PVR1(10), PVR2(10), PYY1(10), PYY2(10)
16. DIMENSION PVR3(10), PVR4(10), PYY3(10), PYY4(10)
17. DIMENSION XV1(10), XV2(10), XYY1(10), XYY2(10)
18. DIMENSION XV3(10), XV4(10), XYY3(10), XYY4(10)
19. DIMENSION XMP(18,10), XVP(18,10), XEM(18)
20. DIMENSION XMK(18,10), XMM(18,10), XVM(18,10)
21. DIMENSION A(10), B(10), C(10), D(10), XMI(10,10)
22. DIMENSION RX(10), RXT(10), RXT(10)
23. DIMENSION X(10), XT(10), XT(10)
24. DIMENSION XA(10), XB(10), XC(10), XD(10)
25. DIMENSION XA(10), XB(10), XC(10), XD(10)
26. DIMENSION XAC(10), XAC(10)
27. DIMENSION XE(18), XEM(18,10), XEM(18)
28. DIMENSION XEMPK(18), XEMPK(18), XEM(18)
29. DIMENSION AF(10), AFV(10,18), AFP(10,18), P(10)
30. DIMENSION AFV(10), AFV(10)
31. DIMENSION FMV1(10), FMV2(10), FMV3(10), FMV4(10)
32. DIMENSION AREA(10), FY(10), PY(10), RDP(18)
33. DIMENSION EDP(18), D(18), ZP(10)
34. DIMENSION AXIAL(10), AXIAL(10), PLIMIT(18), PREDUC(18)
35. 1 READ(1,2) NO
36. IF(NO) 52, 52, 3
37. 3 WRITE(3,1001) NO
38. READ(1,401) NM, NP, NPR, NPS, NVP
39. 40 READ(1,1009) NPH(I), I=1, NEM
40. READ(1,400) XL(I), I=1, NM
41. READ(1,400) (AYA(I), I=1, NM)
42. READ(1,400) (AXY(I), I=1, NM)
43. READ(1,400) (NYA(I), I=1, NM)
44. READ(1,400) (NYA(I), I=1, NM)
45. READ(1,400) (ZIP(I), I=1, NM)
46. READ(1,400) (ZIP(I), I=1, NM)
47. READ(1,400) (ZIP(I), I=1, NM)
48. READ(1,400) (ZIP(I), I=1, NM)
49. READ(1,400) (ZIP(I), I=1, NM)
50. READ(1,601) PSK, XE
51. READ(1,400) VA, VA, ZETA
52 WRITE(3,700)
53 WRITE(3,701)(XL(I),I=1,NM)
54 WRITE(3,702)
55 WRITE(3,701)(XI(I),I=1,NM)
56 WRITE(3,703)
57 WRITE(3,701)(XM(I),I=1,NM)
58 WRITE(3,704)
59 WRITE(3,701)(ALPHA(I),I=1,NM)
60 WRITE(3,705)
61 WRITE(3,701)(BETA(I),I=1,NM)
62 WRITE(3,706)
63 WRITE(3,701) PSI,XE
64 WRITE(3,4321) VA,V8,ZETA
65 WRITE(3,3348)
66 DO 3346 I=1,NM
67 PY(I)=AREA(I)*FY(I)
68 PM(I)=ZP(I)*FY(I)
69 WRITE(3,3347) I,AREA(I),FY(I),PY(I),ZP(I),PM(I)
70 CONTINUE
71 DO 402 I=1,NPR
72 DO 402 J=1,NEM
73 AM(I,J)=0.
74 DO 407 I=1,NPS
75 DO 407 J=1,NEM
76 AV(I,J)=0.
77 DO 414 I=1,NM
78 DO 414 J=1,NEM
79 AFV(I,J)=0.
80 DO 415 I=1,NM
81 DO 415 J=1,NVP
82 AFP(I,J)=0.
83 406 READ(1,403) I,J,AMIJ
84 IF(I) 404,404,405
85 AM(I,J)=AMIJ
86 GO TO 406
87 404 READ(1,403) I,J,AVIJ
88 IF(I) 408,408,409
89 AV(I,J)=AVIJ
90 GO TO 404
91 408 DO 410 I=1,NPS
92 DO 410 J=1,NPS
93 AMS(I,J)=0.
94 413 READ(1,403) I,J,AMSIJ
95 IF(I) 411,411,412
96 AMS(I,J)=AMSIJ
97 GO TO 413
98 411 READ(1,403) I,J,AFVII
99 IF(I) 417,417,416
100 AFVII,J)=AFVII
101 GO TO 411
102 417 READ(1,403) I,J,AFPIJ
103 IF(I) 418,419,419
104 419 AF(I,J)=AFPIJ
105 GO TO 417
106 418 WRITE(3,650)
107 WRITE(3,603) ((AM(I,J),J=1,NEM),I=1,NPR)
108 WRITE(3,651)
109 WRITE(3,603) ((AV(I,J),J=1,NEM),I=1,NPS)
110 WRITE(3,652)
111 WRITE(3,633) ((AMS(I,J),J=1,NPS),I=1,NM)
112 WRITE(3,653)
113 WRITE(3,603) ((AFV(I,J),J=1,NEM),I=1,NM)
114 WRITE(3,654)
115 WRITE(3,603) ((AFP(I,J),J=1,NVP),I=1,NM)
116 C FORMULATE MASS & STIFF. MATRIX
117 DO 1000 I=1,NM
118 CALL STIFFA(SMR1,SMR2,SMR3,SMR4,
119 & SMY1,SMY2,SMY3,SMY4,SVR1,SVR2,SVR3,SVR4,
120 & SVY1,SVY2,SVY3,SVY4,XMLR1,XLRZ,XMLR3,XMLR4,
121 & XMY1,XMY2,XMY3,XMY4,XVRL,XVRL,XVRL,XVRL,
122 & XYY1,XYY2,XYY3,XYY4,MN,XE,XI,XL,XM)
123 1000 CONTINUE
C SET UP XXM,XXK
124 CALL ASATA(NPR,NPS,NM,AM,AV,
125 & SMR1,SMR2,SMR3,SMR4,SMY1,SMY2,SMY3,SMY4,
126 & SVR1,SVR2,SVR3,SVR4,SVY1,SVY2,SVY3,SVY4,XXK)
127 CALL ASATB(NPR,NPS,NM,AM,AV,
128 & XMR1,XMR2,XMR3,XMR4,XMY1,XMY2,XMY3,XMY4,
129 & XVR1,XVR2,XVR3,XVR4,XVY1,XVY2,XVY3,XVY4,
130 & AMS,XXM)
131 CALL ASATM(NP,XXM,XMI)
C FORMULATE S*AT
132 CALL SATVM(NPR,NPS,NM,SMR1,SMR2,SMR3,
133 & SMR4,SMY1,SMY2,SMY3,SMY4,AM,AV,XXM)
134 CALL SATVM(NPR,NPS,NM,SMR1,SMR2,SMR3,
135 & SMR4,SMY1,SMY2,SMY3,SMY4,AM,AV,XXM)
136 CALL SATVM(NPR,NPS,NM,SMR1,SMR2,SMR3,
137 & SMR4,SMY1,SMY2,SMY3,SMY4,AM,AV,XXM)
C DEFINE THE INITIAL CONDITION
138 READ(1,900)(X(I),I=1,NP)
139 READ(1,900)(XT(I),I=1,NP)
140 READ(1,900)(XTT(I),I=1,NP)
141 DO 671 I=1,NEM
142 XEV(I)=0.
143 671 CONTINUE
144 T=0.
145 NPST=201
146 WRITE(3,9011T
147 DO 9000 I=1,NP
148 WRITE(3,9003) X(I),XT(I),XTT(I)
9000 CONTINUE
DO 9999 KK=2,NPTS
DO 930 I=1, NP
RX(I)=X(I)
RT(I)=XT(I)
RXTT(I)=XTT(I)
930 CONTINUE
RT=T
C FIND AXIAL FORCE
ZT=ZETA*T
CZT=COS(ZT)
DO 655 I=1, NVP
PT(I)=VA*PSB+VP*PSB*CZT
655 CONTINUE
DO 666 I=1, NEM
AFVV(I)=0.
DO 667 J=1, NVP
AFVV(I)=AFVV(I)+AFV(I,J)*XEV(J)
667 CONTINUE
DO 668 I=1, NEM
AFPP(I)=AFVV(I)+AFPP(I)
668 CONTINUE
DO 1100 I=1, NM
MN=I
CALL STIFPA(PMR1,PMR2,PMR3,PMR4,
&PMY1,PMY2,PMY3,PMY4,PVR1,PVR2,PVR3,PVR4,
&MN,XL,AF)
1100 CONTINUE
CALL ASATA(NPR,NPS,NM,AM,AV,
&PMR1,PMR2,PMR3,PMR4,PMY1,PMY2,PMY3,PMY4,
&PVR1,PVR2,PVR3,PVR4,PMV1,PMV2,PMV3,PMV4,XXP)
C CALCULATE S*AT FOR P
CALL SATMV(NPR,NPS,NM,PMR1,PMR2,PMR3,
&PMR4,PMY1,PMY2,PMY3,PMY4,AM,AV,XXP)
C CALCULATE A,B,C,D VECTOR
DO 3001 I=1, NP
XA(I)=PX(I)
XTA(I)=RXT(I)
3001 CONTINUE
TA=RT
CALL GFMKPTA(TA,DT,NP,NPR,VA,VB,ZETA,PSB,XA,XXP,
&XXK,XM1,A)
TR=RT+DT/2.
DO 931 I=1, NP
180 $x_b(i) = x(i) + (dt/2.)*rxt(i)$
181 $x_t(i) = rxt(i) + 0.5*a(i)$
182 CONTINUE
183 CALL GFMKP(TB, DT, NP, NPR, VA, VB, ZETA, PSB, XB, XXP, 
& XKK, XM1, B)
184 TC = RT + DT/2.
185 DO 934 I = 1, NP
186 XCI(I) = RX(I) + (DT/2.)*(RXT(I) + (DT/4.)*(A(I))
187 XTC(I) = RXT(I) + 0.5*B(I)
188 CONTINUE
189 CALL GFMKP(TC, DT, NP, NPR, VA, VB, ZETA, PSB, XC, XXP, 
& XKK, XM1, C)
190 TD = RT + DT
191 DO 936 I = 1, NP
192 XDI(I) = PX(I) + DT*RXT(I) + (DT/2.)*A(I)
193 XTD(I) = RXT(I) + C(I)
194 CONTINUE
195 CALL GFMKP(TD, DT, NP, NPR, VA, VB, ZETA, PSB, XD, XXP, 
& XKK, XM1, D)
196 DO 938 I = 1, NP
197 XI(I) = RX(I) + DT*RXT(I) + (DT/6.)*(A(I) + B(I) + C(I))
198 XT(I) = RXT(I) + (1./6.)*(A(I) + 2.*B(I) + 2.*C(I) + 
& D(I))
199 CONTINUE
200 TAC = RT + DT
201 DO 939 I = 1, NP
202 XAC(I) = X(I)
203 XTAC(I) = XT(I)
204 CONTINUE
205 CALL GFMKP(TAC, DT, NP, NPR, VA, VB, ZETA, PSB, XAC, 
& XXP, XKK, XM1, XTX)
206 DO 941 I = 1, NP
207 XTT(I) = XTT(I)/DT
208 CONTINUE
209 T = RT + DT
210 WRITE(3, 901) T
211 DO 9100 I = 1, NP
212 WRITE(3, 903) X(I), XT(I), XTT(I)
213 CONTINUE
214 C CALCULATE END FORCES
215 DO 890 I = 1, NEM
216 XEM(I) = 0.
217 XEV(I) = 0.
218 DO 890 J = 1, NP
219 XEM(I) = XEM(I) + (XMK(I, J) - XMP(I, J))*X(J) + 
& XMM(I, J)*XTT(J)
220 XEV(I) = XEV(I) + (XVK(I, J) - XVP(I, J))*X(J) + 
& XVM(I, J)*XTT(J)
221 CONTINUE
222 WRITE(3, 891) T
223 WRITE(3, 892) I, XEM(I), XEV(I)
9001 CONTINUE
T=RT+DT
9999 CONTINUE
2 FORMAT(15)
400 FORMAT(6F10.4)
401 FORMAT(5I5)
403 FORMAT(2I5,F10.4)
500 FORMAT(/10X,'NO. OF PROGRAMS=',I5)
601 FORMAT(2F10.2)
603 FORMAT(12F10.4)
633 FORMAT(2F10.4)
650 FORMAT(/10X,'AM MATRIX')
651 FORMAT(/10X,'AV MATRIX')
652 FORMAT(/10X,'AMS MATRIX')
653 FORMAT(/10X,'AFV MATRIX')
654 FORMAT(/10X,'AFP MATRIX')
700 FORMAT(/10X,'MEMBER LENGTH')
701 FORMAT(3E16.7)
702 FORMAT(/10X,'MEMBER MOMENT INERTIA')
703 FORMAT(/10X,'MEMBER MASS')
704 FORMAT(/10X,'ALPHA VALUE')
705 FORMAT(/10X,'BETA VALUE')
706 FORMAT(/10X,'LOAD P AND ELASTIC MODULUS')
E,4X,E16.7,4X,E16.7)
891 FORMAT(/10X,'END MOMENT,END SHEAR AT TIME=',
7F10.7)
900 FORMAT(6F10.4)
903 FORMAT(/10X,E16.7,10X,E16.7,10X,E16.7)
1001 FORMAT(1H1)
901 FORMAT(/10X,'X,XT,XTT AT TIME T=',F10.7)
1009 FORMAT(6I5)
3347 FORMAT(/10X,I5,E16.7)
3348 FORMAT(/10X,'MEMBER NO.',10X,'AREA',10X,'FY',
10X,‘PY',10X,'ZP',10X,'PM')
4321 FORMAT(/10X,'VA=',F10.4,5X,'VB=',F10.4,5X,
&'ZETA=',F10.4)
52 STOP
END
READ(1,400) (XL(I), I=1, NM)
READ(1,400) (ZETA(I), I=1, NM)
READ(1,400) (XI(I), I=1, NM)
READ(1,400) (ZP(I), I=1, NM)
READ(1,400) (FY(I), I=1, NM)
READ(1,400) (ALPHA(I), I=1, NM)
READ(1,400) (META(I), I=1, NM)
READ(1,601) PS9,XE
READ(1,400) VA, VB, ZETA
WRITE(3,700)
WRITE(3,701) (XL(I), I=1, NM)
WRITE(3,702)
WRITE(3,701) (XI(I), I=1, NM)
WRITE(3,703)
WRITE(3,703)
WRITE(3,701) (XL(I), I=1, NM)
WRITE(3,704)
WRITE(3,701) (ALPHA(I), I=1, NM)
WRITE(3,705)
WRITE(3,701) (BETA(I), I=1, NM)
WRITE(3,706)
WRITE(3,701) PS9, XE
WRITE(3,4321) VA, VB, ZETA
WRITE(3,3348)
DO 3346 I=1, NM
PY(I)=AREA(I)*FY(I)
PM(I)=ZP(I)*FY(I)
WRITE(3,3347) I, AREA(I), FY(I), PY(I), ZP(I), PM(I)
3346 CONTINUE
C CALCULATE REDUCED PLASTIC MOMENT
DO 1007 I=1, NM
AXIALF(I)=(ALPHA(I)*VA+ZETA(I)*VB)*PS9
PLIMIT(I)=0.15*PY(I)
REDUC(I)=((ALPHA(I)*VA+BETA(I)*VP)*PS9)/PY(I)
RDPM(I)=PM(I)*(1,-REDUC(I)*REDUC(I))
IL=2*I-1
JR=2*I
EDPM(IL)=RDPM(I)
EDPM(JR)=RDPM(I)
1007 CONTINUE
WRITE(3,3349)
DO 3447 I=1, NM
WRITE(3,3347) I, AXIALF(I), PLIMIT(I), RDPM(I)
3447 CONTINUE
DO 402 I=1, NPS
DO 402 J=1, NEM
AM(I,J)=0.
DO 407 I=1, NPS
DO 407 J=1, NEM
407 AV(I,J)=0.
DO 414 I=1,NM
DO 414 J=1,NEM
AFV(I,J)=0.
DO 415 I=1,NM
DO 415 J=1,NVP
AFP(I,J)=0.
READ(1,403) I,J,AMM
IF(I) 404,404,405
AMM(I,J)=AMM
GO TO 406
READ(1,403) I,J,AVJ
IF(I) 403,408,409
AVJ(I,J)=AVJ
GO TO 404
DO 408 DO 410 I=1,NPS
DO 410 J=1,NPS
AMS(I,J)=0.
READ(1,403) I,J,AMSI
IF(I) 414,414,414
AMSI(I,J)=AMSI
GO TO 412
READ(1,403) I,J,AFVI
IF(I) 411,411,412
AFVI(I,J)=AFVI
GO TO 411
READ(1,403) I,J,AFPI
IF(I) 417,417,418
AFPI(I,J)=AFPI
GO TO 417
WRITE(3,650)
WRITE(3,603) (AM(I,J),J=1,NM),I=1,NPR
WRITE(3,651)
WRITE(3,603) (AV(I,J),J=1,NEM),I=1,NPS
WRITE(3,652)
WRITE(3,633) (AMSI(I,J),J=1,NPS),I=1,NPR
WRITE(3,653)
WRITE(3,633) (AFVI(I,J),J=1,NEM),I=1,NM
WRITE(3,654)
WRITE(3,603) (AFPI(I,J),J=1,NVP),I=1,NM
DO 561 I=1,NM
FMV1(I)=-1./XL(I)
FMV2(I)=-1./XL(I)
FMV3(I)=-1./XL(I)
FMV4(I)=-1./XL(I)
561 CONTINUE
C DEFINE THE INITIAL CONDITION
DO 567 I=1,NEM
NPH(I)=0
MNPH(I)=0
NRH(I)=0
MN2R(I)=0
LPH(I)=0
PRHR(I) = 0.
MRAT(I) = 0.
LPHRD(I) = 0
CONTINUE
DO 4446 I = 1,NPS
XS(I) = 0.
CONTINUE
DO 5001 I = 1,NEM
SECDV(I) = 0.
CONTINUE
DO 9449 I = 1,NEP
PSE(I) = 0.
CONTINUE
DO 560 I = 1,NEM
FEV(I) = 0.
FEM(I) = 0.
CONTINUE
READ(1,900)(X(I),I = 1,NP)
READ(1,900)(XT(I),I = 1,NP)
READ(1,900)(XTT(I),I = 1,NP)
DO 671 I = 1,NEM
XEV(I) = 0.
CONTINUE
WRITE(3,674)
WRITE(3,900)(XEV(I),I = 1,NEM)
T = 0.
DT = 0.004
KZERO = 0
WRITE(3,901)T
DO 9000 I = 1,NP
WRITE(3,903)X(I),XT(I),XTT(I)
CONTINUE
WRITE(3,1920)(LPH(I),I = 1,NEM)
NPTS = 51
DO 9999 KK = 2,NPTS
WRITE(3,1001)
WRITE(3,1919)(LPH(I),I = 1,NEM)
DO 930 I = 1,NP
RX(I) = X(I)
RXT(I) = XT(I)
RXTT(I) = XTT(I)
CONTINUE
DO 5682 J = 1,NEM
RXEV(J) = XEV(J)
CONTINUE
RX = T
PRT = RT
WRITE(3,901) PRT
DO 1972 I = 1,NP
WRITE(3,903) RX(I),RXT(I),XTT(I)
CONTINUE
C FIND AXIAL FORCE
204 ZT=ZETA*T
205 CZT=COS(ZT)
206 DO 655 I=1,NVP
207 PT(I)=VA*PSR*V*A*PSR*CZT
208 CONTINUE
209 DO 666 I=1,NVP
210 AFVV(I)=0.
211 DO 666 J=1,NVP
212 AFVV(I)=AFVV(I)+AFV(I,J)*XEV(J)
213 CONTINUE
214 DO 667 I=1,NVP
215 AFPP(I)=0.
216 DO 667 J=1,NVP
217 AFPP(I)=AFPP(I)+AFP(I,J)*PT(J)
218 CONTINUE
219 DO 668 I=1,NVP
220 AF(I)=AFVV(I)+AFPP(I)
221 CONTINUE
C CALCULATE CARRY OVER FACTOR
222 DO 1601 I=1,NM
223 COFR(I)=(2.*XE*XIII)/XL(I)+AF(I)*XL(I)/30.)/
224 &((4.*XE*XII)/XL(I)-2.*AF(I)*XL(I)/15.)
225 CONTINUE
226 IL=2*I-1
227 JR=2*I
228 MIL=LPH(IL)
229 MJR=LPH(JR)
231 2011 IF(MJR) 2013,2013,2014
232 2013 FEM(IL)=0.
233 FEM(JR)=0.
234 GO TO 1020
235 2014 FEM(IL)=FEM(JR)*COFR(I)
236 FEM(JR)=FEM(JR)
237 GO TO 1020
239 2015 FEM(IL)=FEM(IL)
240 FEM(JR)=FEM(IL)*COFR(I)
241 GO TO 1020
242 2016 FEM(IL)=FEM(IL)
243 FEM(JR)=FEM(JR)
244 1020 CONTINUE
C FINDED END SHEAR
245 DO 562 K=1,NM
246 L=2*K-1
247 M=2*K
248 FEVIL)=FMV1(K)*FEM(L)+FMV2(K)*FEM(M)
249 FEVLM)=FMV3(K)*FEM(L)+FMV4(K)*FEM(M)
250 CONTINUE
251 DO 1000 I=1,NM
252 MN=1
IL = 2 * I - 1
JR = 2 * I
NIL = LPH(IL)
NJR = LPH(JR)
IF (NIL) 1011, 1011, 1002
1011 IF (NJR) 1003, 1003, 1004
1003 CALL STIFPA (PMR1, PMR2, PMR3, PMR4,
& PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
GO TO 1000
1004 CALL STIFPB (PMR1, PMR2, PMR3, PMR4,
& PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
GO TO 1000
1005 CALL STIFPC (PMR1, PMR2, PMR3, PMR4,
& PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
GO TO 1000
1006 CALL STIFPD (PMR1, PMR2, PMR3, PMR4,
& PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
CONTINUE
CALL ASATA (NPR, NPS, NM, AN, AV,
& PMR1, PMR2, PMR3, PMR4, PMY1, PMY2, PMY3, PMY4,
& PVR1, PVR2, PVR3, PVR4, PVY1, PVR2, PVY3, PVY4, X)
CALL SATMV (NPR, NPS, NM, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
CALL SATMV (NPR, NPS, NM, PVR1, PVR2, PVR3, PVR4,
& PVY1, PVY2, PVY3, PVY4, MN, XL, AF)
GO TO 1000
3102 IF (LPHR(JR) - LPH(JR)) 8101, 8102, 8101
3101 NIL = LPH(IL)
NJR = LPH(JR)
IF (NIL) 8011, 8011, 8002
3011 IF (NJR) 8003, 8003, 8004
9003 CALL STIFFA (SMR1, SMR2, SMR3, SMR4,
& SMY1, SMY2, SMY3, SMY4, SVR1, SVR2, SVR3, SVR4,
& SVY1, SVY2, SVY3, SVY4, XM1, XM2, XM3, XM4,
& XY1, XY2, XY3, XY4, XV1, XV2, XV3, XV4,
& XYV1, XYV2, XYV3, XYV4, MN, XE, XI, XL, XM)
GO TO 9000
3004 CALL STIFFB (SMR1, SMR2, SMR3, SMR4,
& SMY1, SMY2, SMY3, SMY4, SVP1, SVP2, SVP3, SVP4,
& SVY1, SVY2, SVY3, SVY4, XMP1, XMP2, XMP3, XMP4,
& XY1, XY2, XY3, XY4, XV1, XV2, XV3, XV4,
& XYV1, XYV2, XYV3, XYV4, MN, XE, XI, XL, XM)
GO TO 8000

9002 IF(INJ) 8005,8005,8006

3005 CALL STIFFC(SMR1,SMR2,SMR3,SMR4, &SMY1,SMY2,SMY3,SMY4,SVR1,SVR2,SVR3,SVR4, &SVY1,SVY2,SVY3,SVY4,XMR1,XMR2,XMR3,XMR4, &XMY1,XMY2,XMY3,XMY4,XVR1,XVR2,XVR3,XVR4, &XVY1,XVY2,XVY3,XVY4,MY,N,E,X1,XL,XM)

GO TO 8000

3006 CALL STIFFD(SMP1,SMR2,SMR3,SMR4, &SMY1,SMY2,SMY3,SMY4,SVP1,SVP2,SVP3,SVP4, &SVY1,SVY2,SVY3,SVY4,XMR1,XMR2,XMR3,XMR4, &XMY1,XMY2,XMY3,XMY4,XVP1,XVP2,XVP3,XVP4, &XVY1,XVY2,XVY3,XVY4,MY,N,XF,XI,XL,XM)

8000 CONTINUE

IF(KZERO.EQ.0) GO TO 8204

DO 8201 I=1,NEM

IF(LPHR(I).LE.LPH(I)) 3204,8201,3204

GO TO 8203

3204 CALL ASATA(NPR,NPS,NM,AM,AV, &SMR1,SMR2,SMR3,SMR4,SMY1,SMY2,SMY3,SMY4, &SVR1,SVR2,SVR3,SVR4,SVY1,SVY2,SVY3,SVY4,XXK)

CALL ASATD(NPR,NPS,NM,AM,AV, &SMR1,SMR2,SMR3,SMR4,SMY1,SMY2,SMY3,SMY4, &SVR1,SVR2,SVR3,SVR4,SVY1,SVY2,SVY3,SVY4,XXK)

CALL ASATM(NP,XXM,XM)

C RECORD LPH(I)

8203 DO 1081 I=1,NEM

LPHR(I)=LPH(I)

1081 CONTINUE

C CHECK IF THERE IS ANY MEMBER BOTH ENDS HINGED

4444 DO 5011 I=1,NM

IL=2*I-1

JR=2*I

NMIL=LPH(IL)

NMJR=LPH(JR)

IF(NMIL) 5011,5011,5012

5012 IF(NMJR) 5011,5011,5013

5011 CONTINUE

GO TO 4445

C CALCULATE SECONDARY SHEAR

5013 DO 4020 I=1,NM

IL=2*I-1
JR = 2*I
MJR = LPH(JR)
MMJL = LPH(IL)

IF(MMJL) 4012, 4012, 4011
JL = J(I)
ML = 2*K

SAVT(M,J) = SPVY3(K)*AV(J,L) + SPVY4(K)*AV(J,M)
SAVL(L,J) = SPVY1(K)*AV(J,L) + SPVY2(K)*AV(J,M)

GO TO 4020
4012 SPVY1(I) = AF(I)*(-1./XL(I))
SPVY2(I) = AF(I)*(-1./XL(I))
SPVY3(I) = AF(I)*(-1./XL(I))
SPVY4(I) = AF(I)*(-1./XL(I))
4013 CONTINUE
4020 CONTINUE
DO 4470 J = 1, NPS
DO 4470 K = 1, NM
L = 2*K-1
M = 2*K
SAVT(M,J) = SPVY3(K)*AV(J,L) + SPVY4(K)*AV(J,M)
SAVL(L,J) = SPVY1(K)*AV(J,L) + SPVY2(K)*AV(J,M)

4470 CONTINUE
4470 CONTINUE
DO 4480 I = 1, NEM
SECDV(I) = 0.
DO 4480 J = 1, NPS
SECDV(I) = SECDV(I) + SAVT(I,J)*XS(J)
SECDV(I) = SECDV(I) + SAVL(I,J)*XS(J)

4480 CONTINUE
C TRANSFER SECONDARY SHEAR TO EXTERNAL Joint
DO 4564 I = 1, NPS
II = I+NPR
PSF(II) = 0.
DO 4564 J = 1, NEM
PSF(II) = PSF(II) + AV(I,J)*SECDV(J)

4564 CONTINUE
WRITE(3,9031)(PSF(LL),LL = 1,NP)
4445 CALL GEXTP(AN,AV,NP,NPR,NM,PM,PM,PSF,PSF)
C CALCULATE A,B,C,D VECTOR
DO 3001 I = 1, NP
XA(I) = PX(I)
XTA(I) = RXT(I)
XTA(I) = RX(I)

3001 CONTINUE
TA = RT
CALL GMKP(TA,DT,NP,NPR,VA,VR,ZETA,PB,XA,XXP,
EKKH,XM1,PSF,TA)
TB = RT + DT/2.
DO 931 I = 1, NP
XB(I) = RX(I) + (DT/2.)*PXT(I)
XTB(I) = RXT(I) + 0.5*A(I)

931 CONTINUE
CALL GMKP(TB,DT,NP,NPR,VA,VR,ZETA,PB,XB,XXP,
EKKH,XM1,PSF,TA)
TC = RT + DT/2.
127

DO 934 I=1,NP

X(I)=RX(I)+DT/2.*(XT(I)+XT(I+DT/2.))*(A(I))

XT(I)=XT(I)+0.5*B(I)

934 CONTINUE

CALL GFMKP(TC,DT,NP,NPR,VA,VB,ZETA,PSB,XC,XXP,
&XXK,XMI,RSFT,C)

TD=RT+DT

DO 936 I=1,NP

X(I)=PX(I)+DT*XT(I)+(DT/2.)*(B(I)+C(I))

XTO(I)=PX(I)+C(I)

936 CONTINUE

CALL GFMKP(TD,DT,NP,NPR,VA,VB,ZETA,PSB,XD,XXP,
&XXK,XMI,RSFT,D)

DO 938 I=1,NP

X(I)=RX(I)+DT*XT(I)+(DT/6.)*(A(I)+B(I)+C(I))

XT(I)=XT(I)+(1./6.)*(A(I)+2.*B(I)+2.*C(I)+
&ED(I))

939 CONTINUE

TAC=RT+DT

DO 939 I=1,NP

XAC(I)=X(I)

XTAC(I)=XT(I)

939 CONTINUE

CALL GFMKP(TAC,DT,NP,NPR,VA,VB,ZETA,PSB,XAC,
&XXP,XXK,XMI,RSFT,XTT)

DO 941 I=1,NP

XTT(I)=XTT(I)/DT

941 CONTINUE

DO 7446 I=1,NPS

II=I+NPR

XSI(I)=X(II)

7446 CONTINUE

T=RT+DT

WRITE(3,901)T

DO 9100 I=1,NP

WRITE(3,903)X(I),XT(I),XTT(I)

9100 CONTINUE

C CALCULATE END FORCES

DO 890 I=1,NE

XEMPK(I)=0.

XEVPK(I)=0.

XEMM(I)=0.

XEV(I)=0.

DO 890 J=1,NP

XEMPK(I)=XEMPK(I)+XMK(I,J)-XM(I,J)*X(J)

XEVPK(I)=XEVPK(I)+XVK(I,J)-XV(I,J)*X(J)

XEMM(I)=XEMM(I)+XMM(I,J)*XT(I)

XEV(I)=XEV(I)+XVM(I,J)*XTT(J)

890 CONTINUE

DO 1890 I=1,NE

XEMPK(I)=XEMPK(I)+FEM(I)

XEVPK(I)=XEVPK(I)+FEV(I)+SEC(I)
1:50 CONTINUE
C
DO 450 I=1,NM
413
414 DFT(I)=((6.*XE*XI(I)/XL(I)*AF(I)*XL(I)/10.*)*
415
1(2.*XE*XI(I)/XL(I)*AF(I)*XL(I)/6.1)
416 FRM(I)=((4.*XE*XI(I)/XL(I))-2.*AF(I)*XL(I)/
417
6.1)))/DFT(I)
418 FRM3(I)=((4.*XE*XI(I)/XL(I))-2.*AF(I)*XL(I)/
419
6.1))/DFT(I)
420 FRM2(I)=-(2.*XE*XI(I)/XL(I))+(1.*AF(I)*XL(I)/
421
30.1))/DFT(I)
422 FRM3(I)=-(2.*XE*XI(I)/XL(I))+(1.*AF(I)*XL(I)/
423
6.1))/DFT(I)
424
450 CONTINUE
425
426
427 DP(L)=FRM1(K)*XEMP(L)+FRM2(K)*XEMP(M)
428 DR(M)=FRM3(K)*XEMP(L)+FRM4(K)*XEMP(M)
429
460 CONTINUE
430
431
432
433
434 FAVT(L,J)=FRY1(K)*AV(J,L)+FRY2(K)*AV(J,M)
435 FAVT(M,J)=FRY1(K)*AV(J,L)+FRY2(K)*AV(J,M)
436
470 CONTINUE
437
438
439
440 JJ=J+NPR
441 DO 490 I=1,NEM
442
443
444
445
446 DO 491 I=1,NEM
447 AMTX(I)=0.
448 DO 499 J=1,NPR
449 AMTX(I)=AMTX(I)+AM(J,I)*X(J)
450
499 CONTINUE
451 HR(I)=ENDR(I)-AMTX(I)
452
453
454 IF(ABS(HR(I))>.LE.0.000001) HP(I)=0.
455
456 WRITE(7,550)
457 DO 551 I=1,NEM
WRITE(3,552) I,HR(I),EMDR(I),AMTX(I),PRHP(I)
CONTINUE
DO 893 I=1,NEM
XEM(I)=XEMPK(I)+XEMM(I)
XEV(I)=XEVPK(I)+XEVM(I)
CONTINUE
WRITE(3,891) I
DO 9001 I=1,NEM
WRITE(3,892) I,XEM(I),XEV(I)
CONTINUE
C CHECK : IS THERE ANY NEW PLASTIC HINGE FORMED
DO 569 I=1,NEM
IF(MNRH(I) .EQ.0) 7856,7857,7857
7856 NPH(I)=1
GO TO 569
7857 IF(wp(I).EQ.1) GO TO 569
IF(wp(I).EQ.0) NPH(I)=1
CONTINUE
DO 590 I=1,NEM
LPH(I)=LPH(I)+NPH(I)
MNRH(I)=NPH(I)
IF(XEM(I).LT.0) GO TO 596
KC=1
GO TO 581
590 CONTINUE
KC=-1
FEM(I)=KC*EDPM(I)
NOM=(I+1)*2
IF(I.EQ.2*NOM) GO TO 582
IF(LPH(I+1).EQ.0) FEM(I+1)=FEM(I)
GO TO 590
582 IF(LPH(I-1).EQ.0) FEM(I-1)=FEM(I)
CONTINUE
DO 599 I=1,NEM
IF(NPH(I).EQ.0) GO TO 599
DO 2599 K=1,NEM
NPH(K)=0
CONTINUE
DO 5678 J=1,NP
X(J)=RX(J)
XT(J)=RXT(J)
XTT(J)=RXTT(J)
CONTINUE
DO 5680 J=1,NEM
XEV(J)=RXEV(J)
CONTINUE
DO 5683 J=1,NPS
JJ=J+NPR
XS(J)=RX(JJ)
CONTINUE
GO TO 1599
509 CONTINUE
 510      C      CHECK IS THERE ANY OLD PLASTIC HINGE PLEIVED
511      DO 2005 I=1,NEM
512      IF(MNPH(I).EQ.1) GO TO 2005
513      IF(LPH(I).EQ.0) GO TO 2005
514      IF(HR(I)) 2000,2001,2001
516      PRHR(I)).EQ.2003,2004,2004
517      IF(HRATIO(I).GT.ALOWP) GO TO 2003
518      GO TO 2005
519      2003 NRH(I)=-1
520      MNPH(I)=NRH(I)
521      2005 CONTINUE
522      DO 571 I=1,NEM
523      IF(NRH(I).EQ.0) GO TO 571
524      WRITE(3,595) I,NRH(I)
525      LPH(I)=LPH(I)+NRH(I)
526      NOM=(I+1)/2
527      IF(I.EQ.2*NOM) GO TO 572
528      IF(LPH(I+1).EQ.1) GO TO 573
529      FEM(I+1)=0.
530      FEM(I)=0.
531      GO TO 571
532      573 FEM(I)=FEM(I+1)*COFR(NOM)
533      GO TO 571
534      572 IF(LPH(I-1).EQ.1) GO TO 574
535      FEM(I-1)=0.
536      FEM(I)=0.
537      GO TO 571
538      574 FEM(I)=FEM(I-1)*COFR(NOM)
539      571 CONTINUE
540      DO 594 I=1,NEM
541      IF(NRH(I).EQ.0) GO TO 594
542      DO 3599 J=1,NEM
543      NRH(J)=0
544      3599 CONTINUE
545      T=RT
546      DO 5679 K=1,NP
547      X(K)=RX(K)
548      XT(K)=RXT(K)
549      XTT(K)=RXTT(K)
550      5679 CONTINUE
551      DO 5681 J=1,NEM
552      XEV(J)=PXEVI(J)
553      5681 CONTINUE
554      DO 5684 J=1,NPS
555      JJ=J+NP
556      XS(J)=RX(JJ)
557      5684 CONTINUE
558      DO 5686 L=1,NP
559      PSF(L)=0
CONTINUE
GO TO 1599
CONTINUE
C     CHOOSE THE MAX HINGE ROTATION
DO 583 I=1,NEM
IF(LPH(I).EQ.0) GO TO 583
IF(HR(I)) 585,586,586
585 IF(PRHR(I)) 587,588,588
586 IF(PRHR(I)) 588,587,587
587 IF(ABS(HR(I)) .GT. ABS(PRHR(I))) PRHR(I)=HR(I)
GO TO 583
588 PRHR(I)=HR(I)
589 583 CONTINUE
DO 580 I=1,NEM
LPHR(I)=LPH(I)
NPH(I)=0
MNPH(I)=0
NRH(I)=0
MNHR(I)=0
580 CONTINUE
DO 5685 J=1,NPS
JJ=J+NPR
XS(J)=X(JJ)
5685 CONTINUE
KZERO=KK
T=RT+DT
9999 CONTINUE
2 FORMAT(15)
400 FORMAT(6F10.4)
401 FORMAT(5I5)
402 FORMAT(2I5,F10.4)
500 FORMAT(//10X,'NO. OF PROGRAMS=',I5)
550 FORMAT(//5X,'HINGE ROTATION',18X,'HR',18X,'DF'
&,'10X,'AMTX'*,'13X,'MAX.* PRHR')
552 FORMAT(//10X,'POINT(',I5,''),10X,4F16.7)
593 595 FORMAT(//10X,'PLASTIC HINGE RELIEVED AT POINT',
&I5,2X,I5)
601 FORMAT(2F10.2)
603 FORMAT(12F10.4)
633 FORMAT(2F10.4)
650 FORMAT(//10X,'AM MATRIX')
651 FORMAT(//10X,'AV MATRIX')
652 FORMAT(//10X,'AMS MATRIX')
653 FORMAT(//10X,'AFV MATRIX')
654 FORMAT(//10X,'AFP MATRIX')
674 FORMAT(//10X,'INITIAL XEV')
700 FORMAT(//10X,'MEMBER LENGTH')
701 FORMAT(3E16.7)
702 FORMAT(//10X,'MEMBER MOMENT INERTIA')
703 FORMAT(//10X,'MEMBER MASS')
704 FORMAT(//10X,'ALPHA VALUE')
705 FORMAT(//10X,'BETA VALUE')
SUBROUTINE ASATA(NPR,NPS,NM,AM,AV,
&EAR1,EAR2,AMR3,AMR4,AMY1,AMY2,AMY3,AMY4,
&AVR1,AVR2,AVR3,AVR4,AVY1,AVY2,AVY3,AVY4,XXA)

DIMENSION AMR1(10),AMR2(10),AMR3(10),AMR4(10)
DIMENSION AMY1(10),AMY2(10),AMY3(10),AMY4(10)
DIMENSION AVR1(10),AVR2(10),AVR3(10),AVR4(10)
DIMENSION AVY1(10),AVY2(10),AVY3(10),AVY4(10)
DIMENSION AM(12,12),AV(12,12),XXA(12,12)

FORMULATE FRAME STIFFNESS & STABILITY MATRIX
SUBROUTINE ASATB(NPR,NPS,NM,AM,AV,AMR1,AMR2,AMR3,AMR4,AMY1,AMY2,AMY3,AMY4,AVR1,AVR2,AVR3,AVR4,AVY1,AVY2,AVY3,AVY4,AMS,XXA)

FORMULATE FRAME MASS MATRIX

DIMENSION AMR1(10),AMR2(10),AMR3(10),AMR4(10)
DIMENSION AMY1(10),AMY2(10),AMY3(10),AMY4(10)
DIMENSION AVR1(10),AVR2(10),AVR3(10),AVR4(10)
DIMENSION AVY1(10),AVY2(10),AVY3(10),AVY4(10)
DIMENSION AMS(4,4)

DO 419 I=1,NPR
    XXA(I,J)=0.
    L=2*K-1
    M=2*K
    XXA(I,J) = XXA(I,J) + AM(I,L) * AMR1(K) * AM(J,L) + AMR2(K) * AM(J,M) + AM(I,M) * AMR3(K) * AM(J,L) + AMR4(K) * AM(J,M)
CONTINUE

DO 420 I=1,NPS
    DO 421 J=1,NPS
        JJ=J+NPR
        XXA(I,JJ) = XXA(I,JJ) + AM(I,L) * AV1(K) * AV(J,L) + AVY1(K) * AV(J,M) + AM(I,M) * AVY3(K) * AV(J,L) + AVY4(K) * AV(J,M)
    CONTINUE
    RETURN
END
M = 2*K

XXA(I, J) = XXA(I, J) + AV(I, L)*(AVR1(K) * AM(J, L) + 
AVR2(K) * AM(J, M)) + AV(I, M)*(AVR3(K) * AM(J, L) + 
AVR4(K) * AM(J, M))

CONTINUE

DO 421 I = 1, NPS
DO 421 J = 1, NPS
JJ = J + NPS
XXA(I, JJ) = 0
DO 421 K = 1, NM
L = 2*K - 1
M = 2*K
XXA(I, JJ) = XXA(I, JJ) + AM(I, L)*(AVY1(K) * AV(J, L) + 
AVY2(K) * AV(J, M)) + AM(I, M)*(AVY3(K) * AV(J, L) + 
AVY4(K) * AV(J, M))

CONTINUE

RETURN
END

SUBROUTINE ASATM(NP, SAT, ASAT, ASATT)
DIMENSION ASAT(10, 10), ASAT1(10, 10), INDEX100)
DO 16 I = 1, NP
INDEX(I) = 0
16 AMAX = -1.
17 DO 18 I = 1, NP
IF (INDEX(I)) 18, 19, 19
18 TEMP = ABS(SAT(I, I))
19 IF (TEMP = AMAX) 18, 18, 20
20 ICOL = I
21 AMAX = TEMP
22 CONTINUE
23 IF (AMAX) 21, 29, 22
24 INDEX(ICOL) = 1
25 PIVOT = ASAT(ICOL, ICOL)
26 ASAT(ICOL, ICOL) = 1.0
27 PIVOT = 1./PIVOT
28 DO 23 J = 1, NP
29 ASAT(ICOL, J) = ASAT(ICOL, J) * PIVOT
30 DO 24 I = 1, NP
31 IF (I - ICOL) 25, 24, 75
32
25 TEMP = ASAT(I,icol)
26 ASAT(I,icol) = 0.0
27 ASAT(I,J) = ASAT(I,J) - ASAT(icol,j) * TEMP
28 CONTINUE
29 WRITE(3,100)
100 FORMAT('IOX, 'SINGULAR MATRIX OCCURS!')
28 RETURN
27 ASAT(I,J) = ASAT(I,J)
26 DC 26 J = 1,NP
25 DC 26 J = 1,NP
24 CONTINUE
23 GO TO 17
22 GO TO 29
21 DO 27 I = 1,NP
20 DO 27 J = 1,NP
19 SUBROUTINE SAT(I1,icol)
18 FORMULATE SAT
17 DIMENSION AM(12,18),AV(12,18),AMK(18,10)
16 DIMENSION AMR1(10),AMR2(10),AMR3(10),AMR4(10)
15 DIMENSION AMY1(10),AMY2(10),AMY3(10),AMY4(10)
14 DO 1500 J = 1,NPR
13 DO 1500 K = 1,M
12 L = 2*K-1
11 M = 2*K
10 AMK(L,J) = AMR1(K)*AM(J,L)*AMR2(K)*AM(J,M)
9 AMK(M,J) = AMR3(K)*AM(J,L)*AMR4(K)*AM(J,M)
8 CONTINUE
7 DO 600 J = 1,NPS
6 DO 600 K = 1, NM
5 L = 2*K-1
4 M = 2*K
3 JJ = J + NPP
2 AMK(L, JJ) = AMY1(K)*AV(J,L)*AMY2(K)*AV(J,M)
1 AMK(M, JJ) = AMY3(K)*AV(J,L)*AMY4(K)*AV(J,M)
0 CONTINUE
-1 RETURN
0 SUBROUTINE STIFFA(SMR1,SMR2,SMR3,SMR4,
-2 DIMENSION SMR1(10),SMR2(10),SMR3(10),SMR4(10),
-3 SMY1,SMY2,SMY3,SMY4,SVK1,SVK2,SVK3,SVK4,
-4 SVY1,SVY2,SVY3,SVY4,XML1,XML2,XML3,XML4,
-5 XMY1,XMY2,XMY3,XMY4,XVP1,XVR2,XVR3,XVR4,
-6 XVY1,XVY2,XVY3,XVY4,IX,IX1,IXL,XML)
-1 DIMENSION XI(10),XL(10),XM(10)
-2 DIMENSION SMR1(10),SMR2(10),SMR3(10),SMR4(10)
-3 DIMENSION SMR3(10),SMR4(10),SMY1(10),SMY2(10)
-4 DIMENSION SMR1(10),SMR2(10),SMR3(10),SMR4(10)
-5 DIMENSION SMR3(10),SMR4(10),SMR1(10),SMR2(10)
-6 DIMENSION SMR1(10),SMR2(10),SMR3(10),SMR4(10)
-7 DIMENSION SMR3(10),SMR4(10),SMR1(10),SMR2(10)
-8 DIMENSION SMR3(10),SMR4(10),SMR1(10),SMR2(10)
-9 DIMENSION SMR3(10),SMR4(10),SMR1(10),SMR2(10)
-10 DIMENSION SMR3(10),SMR4(10),SMR1(10),SMR2(10)
DIMENSION XVY3(10), XVY4(10)
SMR1(I) = (4.*XE*XI(I))/XL(I)
SMR2(I) = (2.*XE*XI(I))/XL(I)
SMR3(I) = (2.*XE*XI(I))/XL(I)
SMR4(I) = (4.*XE*XI(I))/XL(I)
SMY1(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY2(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY3(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY4(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR1(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR2(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR3(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR4(I) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMR1(lO) = XVY1(lO), XVY2(lO), XVY3(lO), XVY4(lO)
SMR2(lO) = XVY1(lO), XVY2(lO), XVY3(lO), XVY4(lO)
SMR3(lO) = XVY1(lO), XVY2(lO), XVY3(lO), XVY4(lO)
SMR4(lO) = XVY1(lO), XVY2(lO), XVY3(lO), XVY4(lO)
SMR1(10) = (4.*XE*XI(I))/XL(I)
SMR2(10) = (2.*XE*XI(I))/XL(I)
SMR3(10) = (2.*XE*XI(I))/XL(I)
SMR4(10) = (4.*XE*XI(I))/XL(I)
SMY1(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY2(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY3(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMY4(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR1(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR2(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR3(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SVR4(10) = (-6.*XE*XI(I))/(XL(I)*XL(I))
SMR1(1) = (3.*XE*XI(I))/XL(I)
SMR2(1) = 0.
SUBROUTINE STIFFC(SMR1,SMR2,SMR3,SMR4,
  SMY1,SMY2,SMY3,SMY4,SVR1,SVR2,SVR3,SVR4,
  ESVY1,ESVY2,ESVY3,ESVY4,XMR1,XMR2,XMR3,XMR4,
  EXMY1,EXMY2,EXMY3,EXMY4,EXVR1,EXVR2,EXVR3,EXVR4,
  EXXY1,EXXY2,EXXY3,EXXY4,EXI,XE,XI,XL,XM)
DIMENSION XI(10),XL(10),XM(10)
DIMENSION SMR1(10),SMR2(10),SMY1(10),SMY2(10)
DIMENSION SMR3(10),SMR4(10),SMY3(10),SMY4(10)
DIMENSION XMR1(10),XMR2(10),XMR3(10),XMR4(10)
DIMENSION SVR1(10),SVR2(10),SVR3(10),SVR4(10)
DIMENSION SVY1(10),SVY2(10),SVY3(10),SVY4(10)
DIMENSION XVY1(10),XVY2(10),XVY3(10),XVY4(10)
DIMENSION SMR4(10)=(-3.*XE*XI(1))/XL(I)*XL(I))
SMR2(I)=0.
SMR3(I)=0.
SMR4(I)=0.
SMY1(I)=(-3.*XE*XI(1))/XL(I)*XL(I))
SMY2(I)=(-3.*XI(1))/XL(I)*XL(I))
SMY3(I)=0.
SMY4(I)=0.
SVR1(I)=(-3.*XE*XI(1))/XL(I)*XL(I))
SVR2(I)=0.
SVR3(I)=(-3.*XE*XI(1))/XL(I)*XL(I))
SVR4(I)=0.
SVY1(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY2(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY3(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY4(I)=(3.*XE*XI(1))/XL(I)*XL(I))
XMR1(I)=(8.*XM(I))*XL(I)*XL(I)))/420.
XMR2(I)=0.
XMR3(I)=0.
XMR4(I)=0.
XMY1(I)=(-36.*XM(I))*XL(I)*XL(I))/420.
XMY2(I)=(+11.*XM(I))*XL(I)*XL(I))/280.
XMY3(I)=0.
XMY4(I)=0.
XVR1(I)=(-36.*XM(I))*XL(I)*XL(I))/420.
XVR2(I)=0.
XVR3(I)=(+11.*XM(I))*XL(I)*XL(I))/280.
XVR4(I)=0.
XVY1(I)=(+204.*XM(I))*XL(I))/420.
XVY2(I)=(-39.*XM(I))*XL(I))/230.
XVY3(I)=(-39.*XM(I))*XL(I))/280.
XVY4(I)=(+99.*XM(I))*XL(I))/420.
RETURN
END
SUBROUTINE STIFFC(SMR1,SMR2,SMR3,SMR4,
  SMY1,SMY2,SMY3,SMY4,SVR1,SVR2,SVR3,SVR4,
  ESVY1,ESVY2,ESVY3,ESVY4,XMR1,XMR2,XMR3,XMR4,
  EXMY1,EXMY2,EXMY3,EXMY4,EXVR1,EXVR2,EXVR3,EXVR4,
  EXXY1,EXXY2,EXXY3,EXXY4,EXI,XE,XI,XL,XM)
DIMENSION XI(10),XL(10),XM(10)
DIMENSION SMR1(10),SMR2(10),SMY1(10),SMY2(10)
DIMENSION SMR3(10),SMR4(10),SMY3(10),SMY4(10)
DIMENSION XMR1(10),XMR2(10),XMR3(10),XMR4(10)
DIMENSION SVR1(10),SVR2(10),SVR3(10),SVR4(10)
DIMENSION SVY1(10),SVY2(10),SVY3(10),SVY4(10)
DIMENSION XVY1(10),XVY2(10),XVY3(10),XVY4(10)
DIMENSION SMR4(10)=(-3.*XE*XI(1))/XL(I)*XL(I))
SMR2(I)=0.
SMR3(I)=0.
SMR4(I)=0.
SMY2(I)=(-3.*XI(1))/XL(I)*XL(I))
SMY3(I)=0.
SMY4(I)=0.
SVR1(I)=(-3.*XE*XI(1))/XL(I)*XL(I))
SVR2(I)=0.
SVR3(I)=(-3.*XE*XI(1))/XL(I)*XL(I))
SVR4(I)=0.
SVY1(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY2(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY3(I)=(3.*XE*XI(1))/XL(I)*XL(I))
SVY4(I)=(3.*XE*XI(1))/XL(I)*XL(I))
XMR1(I)=(8.*XM(I))*XL(I)*XL(I)))/420.
XMR2(I)=0.
XMR3(I)=0.
XMR4(I)=0.
XMY1(I)=(-36.*XM(I))*XL(I)*XL(I))/420.
XMY2(I)=(+11.*XM(I))*XL(I)*XL(I))/280.
XMY3(I)=0.
XMY4(I)=0.
XVR1(I)=(-36.*XM(I))*XL(I)*XL(I))/420.
XVR2(I)=0.
XVR3(I)=(+11.*XM(I))*XL(I)*XL(I))/280.
XVR4(I)=0.
XVY1(I)=(+204.*XM(I))*XL(I))/420.
XVY2(I)=(-39.*XM(I))*XL(I))/230.
XVY3(I)=(-39.*XM(I))*XL(I))/280.
XVY4(I)=(+99.*XM(I))*XL(I))/420.
RETURN
END
\begin{verbatim}
872 \texttt{SVM3(I)} = (-3.*XF*XI(I))/(XL(I)*XL(I))
373 \texttt{SVM2(I)} = 0.
374 \texttt{SVM1(I)} = 0.
875 \texttt{SVR4(I)} = (-3.*XF*XI(I))/(XL(I)*XL(I))
376 \texttt{SVR3(I)} = 0.
377 \texttt{SVR2(I)} = (-3.*XE*XI(I))/(XL(I)*XL(I))
378 \texttt{SVR1(I)} = 0.
379 \texttt{SVY1(I)} = (3.*XE*XI(I))/(XL(I)*XL(I))
380 \texttt{SVY2(I)} = (3.*XE*XI(I))/(XL(I)*XL(I))
381 \texttt{SVY3(I)} = (3.*XE*XI(I))/(XL(I)*XL(I))
382 \texttt{SVY4(I)} = (3.*XE*XI(I))/(XL(I)*XL(I))
383 \texttt{XMR4(I)} = (8.*XM(I)*XL(I)/XL(I))/420.
634 \texttt{XMR1(I)} = 0.
635 \texttt{XMR2(I)} = 0.
636 \texttt{XMR3(I)} = 0.
637 \texttt{XVY4(I)} = (-36.*XM(I)*XL(I)/XL(I))/420.
638 \texttt{XVY3(I)} = (1.1.*XM(I)*XL(I))/280.
639 \texttt{XVY2(I)} = 0.
640 \texttt{XVY1(I)} = 0.
641 \texttt{XVR4(I)} = (-36.*XM(I)*XL(I)/XL(I))/420.
642 \texttt{XVR3(I)} = 0.
643 \texttt{XVR2(I)} = (1.1.*XM(I)*XL(I)/XL(I))/280.
644 \texttt{XVR1(I)} = 0.
645 \texttt{XVY4(I)} = (+204.*XM(I)*XL(I))/420.
646 \texttt{XVY2(I)} = (-39.*XM(I)*XL(I))/280.
647 \texttt{XVY3(I)} = (-39.*XM(I)*XL(I))/280.
648 \texttt{XVY1(I)} = (+99.*XM(I)*XL(I))/420.
899 \texttt{RETURN}
900 \texttt{END}
901 \texttt{SUBROUTINE STIFFO(SMR1,SMR2,SMR3,SMR4,}
902 \texttt{SMY1,SMY2,SMY3,SMY4,SVR1,SVR2,SVR3,SVR4,}
903 \texttt{SVY1,SVY2,SVY3,SVY4,XMR1,XMR2,XMR3,XMR4,}
904 \texttt{XMY1,XMY2,XMY3,XMY4,XVP1,XVR2,XVR3,XVR4,}
905 \texttt{(XVY1,XVY2,XVY3,XVY4,I,XE,XI,ML,XM))}
906 \texttt{DIMENSION XI(10),XL(10),XM(10))
907 \texttt{DIMENSION SMR1(10),SMR2(10),SMY1(10),SMY2(10)}
908 \texttt{DIMENSION SMR3(10),SMR4(10),SMY3(10),SMY4(10)}
909 \texttt{DIMENSION XMR1(10),XMR2(10),XMY1(10),XMY2(10)}
910 \texttt{DIMENSION XMR3(10),XMR4(10),XMY3(10),XMY4(10)}
911 \texttt{DIMENSION SVR1(10),SVR2(10),SVY1(10),SVY2(10)}
912 \texttt{DIMENSION SVR3(10),SVR4(10),SVY3(10),SVY4(10)}
913 \texttt{DIMENSION XVR1(10),XVR2(10),XVY1(10),XVY2(10)}
914 \texttt{DIMENSION XVR3(10),XVR4(10),XVY3(10),XVY4(10)}
915 \texttt{DIMENSION XMR2(10)}
916 \texttt{DIMENSION SMR1(10)}
917 \texttt{DIMENSION SMR3(10)}
918 \texttt{DIMENSION SMR4(10)}
919 \texttt{DIMENSION SMY1(10)}
920 \texttt{DIMENSION SMY2(10)}
921 \texttt{DIMENSION SMY3(10)}
922 \texttt{DIMENSION SMY4(10)}
\end{verbatim}
SUBROUTINE STIFPA(PMR1, PMR2, PMR3, PMR4, PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4, PVY1, PVY2, PVY3, PVY4, I, XL, AF)

DIMENSION XI(10), XL(10), AF(10)
DIMENSION PMR1(10), PMR2(10), PMY1(10), PMY2(10)
DIMENSION PMR3(10), PMR4(10), PMY3(10), PMY4(10)
DIMENSION PVR1(10), PVR2(10), PVY1(10), PVY2(10)
DIMENSION PVR3(10), PVR4(10), PVY3(10), PVY4(10)

PMR1(I)=AF(I)*(-XL(I)/30.)
PMR2(I)=AF(I)*(-XL(I)/30.)
PMR3(I)=AF(I)*(2.*XL(I)/15.)
PMR4(I)=AF(I)*(2.*XL(I)/15.)
PMY1(I)=AF(I)*(-1./10.)
PMY2(I)=AF(I)*(-1./10.)
PMY3(I)=AF(I)*(-1./10.)
PMY4(I)=AF(I)*(-1./10.)
PVR1(I)=AF(I)*(-1./10.)
PVR2(I)=AF(I)*(-1./10.)
PVR3(I)=AF(I)*(-1./10.)
PVR4(I)=AF(I)*(-1./10.)
PVY1(I)=AF(I)*(6./(5.*XL(I)))
PVY2(I)=AF(I)*(6./(5.*XL(I)))
PVY3(I)=AF(I)*(6./(5.*XL(I)))
PVY4(I)=AF(I)*(6./(5.*XL(I)))
RETURN
SUBROUTINE STIFFP(I,PMR1,PMR2,PMR3,PMR4,
&PMY1,PMY2,PMY3,PMY4,PVR1,PVR2,PVR3,PVR4,
&PVY1,PVY2,PVY3,PVY4,I,XL,AF)
970 DIMENSION XL(10),AF(10)
971 DIMENSION PMR1(10),PVR2(10),PMY1(10),PMY2(10)
972 DIMENSION PMR3(10),PVR4(10),PMY3(10),PMY4(10)
973 DIMENSION PVR1(10),PVR2(10),PVY1(10),PVY2(10)
974 DIMENSION PVR3(10),PVR4(10),PVY3(10),PVY4(10)
975 PMR1(I)=AF(I)*(-1./5.)
976 PMR2(I)=AF(I)*0.
977 PMR3(I)=AF(I)*0.
978 PMR4(I)=AF(I)*0.
979 PMY1(I)=AF(I)*(-1./5.)
980 PMY2(I)=AF(I)*(-1./5.)
981 PMY3(I)=AF(I)*0.
982 PMY4(I)=AF(I)*0.
983 PVR1(I)=AF(I)*(-1./5.)
984 PVR2(I)=AF(I)*0.
985 PVR3(I)=AF(I)*(-1./5.)
986 PVR4(I)=AF(I)*0.
987 PVY1(I)=AF(I)*6./5.*XL(I))
988 PVY2(I)=AF(I)*6./5.*XL(I))
989 PVY3(I)=AF(I)*6./5.*XL(I))
990 PVY4(I)=AF(I)*6./5.*XL(I))
991 RETURN
END

SUBROUTINE STIFPC(I,PMR1,PMR2,PMR3,PMR4,
&PMY1,PMY2,PMY3,PMY4,PVR1,PVR2,PVR3,PVR4,
&PVY1,PVY2,PVY3,PVY4,I,XL,AF)
994 DIMENSION XL(10),AF(10)
995 DIMENSION PMR1(10),PVR2(10),PMY1(10),PMY2(10)
996 DIMENSION PMR3(10),PVR4(10),PMY3(10),PMY4(10)
997 DIMENSION PVR1(10),PVR2(10),PVY1(10),PVY2(10)
998 DIMENSION PVR3(10),PVR4(10),PVY3(10),PVY4(10)
999 PMR1(I)=AF(I)*0.
1000 PMR2(I)=AF(I)*0.
1001 PMR3(I)=AF(I)*0.
1002 PMR4(I)=AF(I)*(-1.*XL(I)/5.)
1003 PMY1(I)=AF(I)*0.
1004 PMY2(I)=AF(I)*0.
1005 PMY3(I)=AF(I)*(-1./5.)
1006 PMY4(I)=AF(I)*(-1./5.)
1007 PVR1(I)=AF(I)*0.
1008 PVR2(I)=AF(I)*(-1./5.)
1009 PVR3(I)=AF(I)*0.
1010 PVR4(I)=AF(I)*(-1./5.)
1011 PVY1(I)=AF(I)*6./5.*XL(I))
1012 PVY2(I)=AF(I)*6./5.*XL(I))
1013 PVY3(I)=AF(I)*6./5.*XL(I))
SUBROUTINE STIFPD(PMY1, PMY2, PMY3, PMY4,
PMY1, PMY2, PMY3, PMY4, PVR1, PVR2, PVR3, PVR4,
PVY1, PVY2, PVY3, PVY4, XI, AF)

DIMENSION XL(10), AF(10)

PMR1(I)=AF(I)*0.
PMR2(I)=AF(I)*0.
PMR3(I)=AF(I)*0.
PMR4(I)=AF(I)*0.
PMY1(I)=AF(I)*0.
PMY2(I)=AF(I)*0.
PMY3(I)=AF(I)*0.
PMY4(I)=AF(I)*0.
PVR1(I)=AF(I)*0.
PVR2(I)=AF(I)*0.
PVR3(I)=AF(I)*0.
PVR4(I)=AF(I)*0.
PVY1(I)=AF(I)*0.
PVY2(I)=AF(I)*0.
PVY3(I)=AF(I)*0.
PVY4(I)=AF(I)*0.

GEXTP(AM, AV, NP, NPP, NM, FEM, FEV, PSE,
PSFT)

DIMENSION FEV(12), FEM(12), PE(I,10), PSFT(10)

DIMENSION AM(12,12), AV(12,12), PSE(10)

NEP=NM*2
NPS=NP-NPR

DO 563 I=1,NPP
PE(I)=0.

DO 563 J=1,NEM
PE(I)=PE(I)+AM(I,J)*FEM(J)
CONTINUE

563 DO 564 I=1,NPS
II=I+NPR
PE(II)=0.

DO 564 J=1,NEM
PE(II)=PE(II)+AV(I,J)*FEV(J)
CONTINUE

564 DO 565 I=1,NP
RSFT(I)=PE(I)-PSE(I)
CONTINUE

565 RETURN

RETURN
1001    END
1002    SUBROUTINE GEMXP(T,DT,NP,NPR,VA,VS,ZETA,PAR,X,
   1 SXP,XXK,XXM(,SSET,G)
1003    DIMENSION XP(10,10),XXM(10),XXMPX(10),
   1 SXP,XXK(10,10),XXM(10,10)
1004    DIMENSION FT(10),XI(10),XXP(1,10),XXM(10,10),
   1 CG(10),DG(10)
1005    DIMENSION RSET(10),SFT(10)
1006    NPS=NP-NPR
1007    TS1=0.02
1008    TS2=0.02
1009    TD1=0.04
1010    TD2=0.08
1011    TD3=0.12
1012    TD4=0.16
1013    TD5=0.20
1014    TD6=0.24
1015    FO1=10000.
1016    FO2=10000.
1017    SLOP1=FO1/TS1
1018    SLOP2=FO2/TS2
1019    IF(T-TD1) 9006,9006,9007
1020  9006   SFT(1)=FO1-SLPO1*T
1021     GO TO 9008
1022  9007   IF(T-TD2) 9009,9009,9010
1023  9009   SFT(1)=-FO2+SLOP2*(T-TD1)
1024     GO TO 9003
1025  9010   IF(T-TD3) 9011,9011,9012
1026  9011   SFT(1)=-FO2+SLOP2*(T-TD1)
1027     GO TO 9003
1028  9012   IF(T-TD4) 9013,9013,9014
1029  9013   SFT(1)=-FO2+SLOP2*(T-TD3)
1030     GO TO 9003
1031  9014   IF(T-TD5) 9015,9015,9016
1032  9015   SFT(1)=-FO2+SLOP2*(T-TD3)
1033     GO TO 9003
1034  9016   IF(T-TD6) 9017,9017,9018
1035  9017   SFT(1)=-FO2+SLOP2*(T-TD5)
1036     GO TO 9003
1037  9018   SFT(1)=0.
1038  9008   DO 9004 I=1,NPR
1039     FT(I)=RSFT(I)
1040  9004  CONTINUE
1041     DO 9005 I=1,NPS
1042     IN=NPR+I
1043     FT(IN)=SFT(I)+RSFT(IN)
1044  9005  CONTINUE
1045     DO 9007 I=1,NP
1046     XMF(I)=0.
1047  9006  CONTINUE
1048     DO 9007 J=1,NP
1049     XXMPX(J)=0.
1109 \[ X_{M_{PK}(I,J)} = 0. \]
1110 DO 904 K = 1, N_P
1111 904 \[ X_{M_{PK}(I,J)} = X_{M_{PK}(I,J)} + X_{M_{I}(I,K)} = (X_{M_{PK}(K,J)} - X_{M_{K}(K,J)}) \]
1112 \[ X_{M_{PK}(I)} = X_{M_{PK}(I)} + X_{M_{PK}(I,J)} * X(J) \]
1113 \[ X_{M_{F}(I)} = X_{M_{F}(I)} + X_{M_{I}(I,J)} * F(T(J)) \]
1114 \[ D_{G}(I) = X_{M_{F}(I)} + X_{M_{PK}(I)} \]
1115 \[ G(I) = D_{G}(I) * D_T \]
1116 CONTINUE
1117 RETURN
1118 END