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RESPONSE OF AN EMPTY CYLINDRICAL GROUND SUPPORTED LIQUID STORAGE TANK TO BASE EXCITATION

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TABLE OF CONTENTS

ABSTRACT

The structure under consideration is an elastic cylindrical liquid storage tank attached to a rigid base slab. A finite element analysis is presented for the free vibrations of the empty tank permitting determination of natural frequencies and associated mode shapes. The response of the empty tank to (a) several simple deterministic base excitations as well as (b) artificial earthquake excitation is also determined by finite elements. Examples together with complete program listings are offered.

BACKGROUND

The problem of free and forced vibrations of thin elastic shells has been of interest to engineers for nearly a century. Most of the significant contributions to this area have been summarized in the 1973 monograph by A.W. Leissa [1]. In [1] the fundamentals of the various shell theories are presented and distinctions between popular theories examined, particularly with respect to curvature-displacement relations. Distinctions between the various theories are further exemplified through presentation of cylindrical shell natural frequencies as determined by each of approximately ten commonly employed theories.

A limited number of treatments of cylindrical shells subject to various dynamic loadings has been reported by P.M. Ogibalov [2], H. Kraus [3], G.B. Warburton [4], and P. Seide [5]. In general these investigations are limited to determination of responses to harmonic excitations applied normal to the shell surface or suddenly applied normal loads that remain constant with respect to time. Transient thermal effects are also investigated in [3]. The case of a shell subject to time-dependent edge loads is discussed in [3] and [5] and the principles outlined for determination of small responses governed by linearized shell equations. The particular case of a circular cylindrical shell subject to dynamically applied axial loading that increases linearly with respect to time has been considered on the basis of finite amplitude response by A.P. Coppa and W.A. Nash [6]. However, the techniques discussed in these works are not considered practical for application to the case of an applied edge loading that varies rapidly and in essentially a random fashion with

respect to time. From an analytical standpoint the case of time-dependent edge loads gives rise to significant complications due to the fact that the boundary conditions are inhomogenous. When a shell is subjected to time-dependent edge displacements rather than time-dependent edge forces the solution of the response problem becomes even more difficult.

The objective of the present investigation is to develop a computerized approach for determination of small amplitude elastic responses of an empty slab-supported cylindrical liquid storage tank subject to arbitrary base excitation. It is assumed that the base slab supporting the tank is rigid and that the tank does not separate from the slab during excitation. This study will form the basis for a subsequent investigation concerned with response determination of the same storage tank when filled to an arbitrary depth with liquid.

ANALYSIS

The finite element method is essentially a variant of the Rayleigh-Ritz approach and usually involves the principle of minimum potential energy. It has been shown to be effective in determination of shell behavior for a variety of both static and dynamic loadings. For the circular cylindrical tank with a vertical geometric axis under consideration here it is attractive to employ a series of ring-shaped elements extending from the base slab to the tank top, with each ring being bounded by a horizontal plane normal to the shell axis. Both in-plane as well as out-of-plane displacements and forces must be considered. These necessitate knowledge of the strain-displacement relations as indicated by an adequate shell theory. The theory employed here is that due to L. Sanders [7], and outlined in Appendix A.

The finite element method regards a complex structure as a finite assemblage of discrete elements, where each such element is a continuous structural member. This approach is thus, in essence, a discretization scheme since it expresses the displacement at any point of the continuous element in terms of a finite number of displacements at the boundaries of the element. This is accomplished through use of suitable interpolation functions. The points of intersection of these fictitious cuts in the complex structure are termed nodes or joints. The displacements of these nodes constitute the number of degrees of freedom of the system.

The equations of motion of an elastic system subject to external driving forces may be written in terms of elastic restoring forces, inertial forces, damping forces, and the applied forces. In the finite element approach

it is necessary to formulate the stiffness, mass, and damping matrices (denoted by K, M, and C respectively) for each element and then assemble these so as to characterize the complete structure. The element k_{ij} of the stiffness matrix represents the force required (or arising) at coordinate "i" due to a unit displacement of coordinate "j". The element m_{ij} of the mass matrix represents the inertial force corresponding to coordinate "i" due to a unit acceleration of coordinate "j". The elements of the damping matrix are defined in an analogous manner. However, damping is not treated in the present work.

The finite element method is particularly attractive for investigation of the response of structures subject to displacements introduced at the points of support of the structure. Details of this approach are given in Appendix B, but in general the imposition of support displacements is possible since one can partition the displacement vector U into certain components U_b associated with known support displacements with all other components being associated with the remaining (nonsupported) notes. Thus, the general equation of motion

$$M\ddot{x} + C\dot{x} + Kx = F(t) \tag{1}$$

may be written in the partitioned form [8]

$$\left[\begin{array}{c|c} M_{bb} & M_b^T \\ \hline M_b & M \end{array}\right]\left\{\begin{array}{c} \ddot{U}_{bt} \\ \hline \ddot{U}_t \end{array}\right\} + \left[\begin{array}{c|c} K_{bb} & K_b^T \\ \hline K_b & K \end{array}\right]\left\{\begin{array}{c} U_{bt} \\ \hline U_t \end{array}\right\} = \left\{\begin{array}{c} F_b \\ \hline 0 \end{array}\right\} \tag{2}$$

where damping in (1) has been neglected and U_{bt} and $\ddot{U}_{bt}$ in (2) are known support displacements and accelerations, respectively. All elements in the top line of Equation (2) pertain to base node parameters. Thus, K_{bb}

and M_{bb} denote forces at base nodes due to unit displacements at the base nodes and the superscript T of course denotes matrix transpose. K_b and M_b in the bottom row are coupling effects between the base nodes and the other (non-base) nodes. All other elements in the bottom row of Equation (2) pertain to non-base nodal parameters. Thus K and M represent stiffness and mass matrices of all non-base nodes.

At any time, the displacement vectors of the non-base nodes can be considered as a summation of two factors. The first vector U_s is a function of the instantaneous ground displacement, thus it can be called static. The second vector U_d is a function of the ground acceleration history, thus it is termed dynamic.

This approach furnishes a suitable method to reduce the equations of motion to the familiar form of forced vibrations:

$$[M]\{U_d\} + [K]\{U_d\} = F \tag{3}$$

where {F} is a defined driving forces vector to be derived hereafter. Thus:

$$\{U_t\} = \{U_s\} + \{U_d\} \tag{4}$$

The equations of motion are

$$\left[\begin{array}{c|c} M_{bb} & M_b^T \\ \hline M_b & M \end{array}\right]\left\{\begin{array}{c} \ddot{U}_{bt} \\ \hline \ddot{U}_s + \ddot{U}_d \end{array}\right\} + \left[\begin{array}{c|c} K_{bb} & K_b^T \\ \hline K_b & K \end{array}\right]\left\{\begin{array}{c} U_{bt} \\ \hline U_s + U_d \end{array}\right\} = \left\{\begin{array}{c} F_b \\ \hline 0 \end{array}\right\} \tag{5}$$

The equations of the off-base elements are

$$[M_b]\{\ddot{U}_{bt}\} + [M]\{\ddot{U}_s\} + [M]\{\ddot{U}_d\} + [K_b]\{U_{bt}\} + [K]\{U_s\} + [K]\{U_d\} = 0 \tag{6}$$

Now it is attractive to define U_s as a displacement vector so that when it is associated with the ground displacement vector U_{bt} the resulting motion of the structure corresponds to no internal strain energy. This

condition implies that

$$[K_b]\{U_{bt}\} + [K]\{U_s\} = 0 \tag{7}$$

In other words the vector $\{U_s\}$ is developed through rigid body displacements consistent with $\{U_{bt}\}$. Thus from (7)

$$U_s = -[K]^{-1}[K_b]\, U_{bt}$$

This phenomena has also been demonstrated numerically and the resulting static displacement U_s is nothing but a series of U_{bt} or

$$\{U_s\} = \begin{Bmatrix} U_{s1} \\ \hline U_{s2} \\ \hline U_{s3} \\ \hline \vdots \\ \hline U_{sN} \end{Bmatrix} = \begin{Bmatrix} U_{bt} \\ \hline U_{bt} \\ \hline U_{bt} \\ \hline \\ \hline U_{bt} \end{Bmatrix} \tag{8}$$

where N is the total number of elements and $\{U_{si}\}$ is the displacement vector of node i = $\{U_{bt}\}$ for all values of i and $\{U_{bt}\}$ is a (4 x 1) vector representing the axial, tangential, and radial displacements as well as the rotation of the generator at the base.

Thus, the off-base node equations yield

$$\begin{aligned} [M]\{\ddot{U}_d\} + [K]\{U_d\} &= -[M_b]\{\ddot{U}_{bt}\} - [M]\{\ddot{U}_s\} \\ [M]\{\ddot{U}_d\} + [K]\{U_d\} &= -[[M_b] + [M][K]^{-1}[K_b]]\{\ddot{U}_{bt}\} \\ &= [\text{effective mass matrix}]\ \{\ddot{U}_{bt}\} \\ &= [M_{eff}]\ \{\ddot{U}_{bt}\} \end{aligned} \tag{9}$$

It should be pointed out that for most practical tank dimensions the driving forces developed due to the mass $[M][K][K_b]$ are much larger than those developed by $[M_b]$. This has been demonstrated numerically.

The ground acceleration vector $\ddot{U}_{bt}$ will be proved to be equal to:

$$\ddot{U}_g(t)\begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix},$$

where $\ddot{U}_g(t)$ is the ground acceleration amplitude at time t.

Since the base of the tank is excited by a ground displacement and acceleration acting in its plane and in the constant direction $\theta = 0$, no axial acceleration component develops and the ground acceleration will be completely defined by its amplitude value $\ddot{U}_g(t)$:

$$\ddot{U}_g(t) = \text{Peak} \cdot f(t) \tag{10}$$

The peak is an acceleration value independent of time and f(t) is a non-dimensional function of time.

The associated base-node displacement vector U_{bt} is derived by use of Fig. 1, viz:

$$\begin{aligned} u(o,\theta,t) &= 0 \\ v(o,\theta,t) &= -\ddot{U}_g(t) \cdot \sin\theta = -\text{Peak} \cdot f(t) \cdot \sin\theta \\ w(o,\theta,t) &= +\ddot{U}_g(t) \cdot \cos\theta = +\text{Peak} \cdot f(t) \cdot \cos\theta \\ \frac{\partial w}{\partial z}(o,\theta,t) &= 0.0 \end{aligned} \tag{11}$$

Since the excitation function is described in the previous form (11) to be associated with m = 1, obviously only the first circumferential harmonic will be excited, and thus the vibration of the tank can be prescribed by super-position of certain contributions of different axial

modes corresponding to m = 1 only (see Appendix A, assumed form of loads and displacements).

$$\ddot{U}_b(t) = \text{Peak} \cdot f(t) \cdot \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} \tag{12}$$

Let

$$\{P_{eff}\} = \text{Peak} \cdot [M_{eff}] \cdot \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix}$$

$\therefore$ the equations of motion reduce to

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{P_{eff}\} \cdot f(t)$$

which is the desired form of forced vibration to which the modal analysis technique will be applied.

<u>Modal Analysis Solutions</u>

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{P_{eff}\} \cdot f(t)$$

Let

$$\{\ddot{U}_d\} = [X]\{\ddot{A}\}$$

$$\{U_d\} = [X]\{A\}$$

$\therefore$ [X] is the rectangular mode matrix formed as a set of mode vectors (n x m) where

n = number of degrees of freedom of the non-base elements

m = number of modes considered in the analysis

{A} = mode participation factor vector = m x 1

$$\therefore [M][X]\{\ddot{A}\} + [K][X]\{A\} = \{P_{eff}\} \cdot f(t)$$

$$\{A\} \equiv \{A(t)\} \; ; \quad \{\ddot{A}\} = \{\ddot{A}(t)\}$$

$$f(t) = \frac{\ddot{U}_g(t)}{\text{Peak}}$$

Premultiply by $[X]^T$; (m x n)

$$\therefore\ [X]^T[M][X]\{\ddot{A}\} + [X]^T[K][X]\{A\} = [X]^T\{P_{eff}\} \cdot f(t)$$
$$= \{GP\} \cdot f(t)$$

Now, use the orthogonality condition:

$$\{X_n\}^T[M]\{X_m\} = 0 \qquad m \neq n$$

Obviously the resulting matrix $[X]^T[M][X] = [GM]$ is a diagonal matrix since the (generalized m x n mass matrix) nonvanishing terms are only $[X_n]^T[M][X_n] = GM(n,n)$.

The same concept holds for $[X]^T[K][X] = [GS] \equiv$ diagonal matrix where Ω^2 is the squared eigenvalue diagonal matrix $= [\Omega^2][GM]$:

$$[\Omega^2] = \begin{bmatrix} \omega_1^2 & & & & \\ & \omega_2^2 & & & \\ & & \omega_3^2 & & \\ & & & \ddots & \\ & & & & \omega_n^2 \end{bmatrix}_{m \times m}$$

Thus GM, as well as GS can be considered as vectors,

$$\begin{bmatrix} GM(1,1) \\ GM(2,2) \\ GM(3,3) \\ GM(m,m) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} GM(1,1) \cdot \omega_1^2 \\ GM(2,2) \cdot \omega_2^2 \\ \vdots \\ GM(m,m) \cdot \omega_n^2 \end{bmatrix} \quad \text{respectively.}$$

Thus m independent equations result:

$$GM(I,I) \cdot \ddot{A}(I) + \omega(I) \cdot \omega(I) + A(I) \cdot GM(I,I) = GP(I) \cdot f(t)$$

where I refers to the mode number.

Thus:

$$\ddot{A}(I) + \omega^2(I) \cdot A(I) = \frac{GP(I)}{GM(I,I)} \cdot f(t)$$

which are the equations of m independent lumped masses each representing the participation of the corresponding I-th mode.

Now, A(I) can be found using Duhamel Integration to account for the initial conditions (just before the instant t), i.e. to consider the whole acceleration record imposed on the structure, viz:

$$A(I) = \frac{GP(I)}{GM(I,I) \cdot \omega(I)} \cdot \int_0^t f(\tau) \cdot \sin \omega(t-\tau) d\tau$$

$$= \frac{PIN(I)}{GM(I,I) \cdot \omega(I)}$$

$$\text{where } PIN(I) = \left(\int_0^t f(\tau) \sin \omega(t-\tau) d\tau\right) \cdot GP(I)$$

$$\therefore \quad A(I) = \frac{GP(I)}{GM(I,I)} f(t) - (\omega(I))^2 \cdot A(I)$$

Now from the original equations of motion the displacement and acceleration nodal vectors are determined:

$$\{U_d\} = [X]\{A\}$$
$$\{\ddot{U}_d\} = [X]\{\ddot{A}\}$$

The accuracy of the modal analysis approach depends on the number of modes involved in the superposition. The latter depends on how close or scarce the natural frequencies of the structure are spaced.

The accuracy of the method can be examined through the satisfaction of the original external equilibrium equation:

$$[M]\{\ddot{U}_d\} + [K]\{U_d\} = \{P_{eff}\} \cdot f(t)$$

For the structure considered it was found that the superposition of four modes offered only a crude approximation since the external equilibrium equation failed to be satisfied by as much as thirty percent. Use of eight modes reduced the maximum discrepancy to about ten percent. It was worth

the effort to cross-check the program by use of a coarse idealization of the tank involving a limited number of degrees of freedom, i.e. a limited total number of natural modes. In that case the external equilibrium equation was satisfied perfectly.

Reactions of the Base

From the equations of base vibrations:

$$[M_{bb} | M_b^T] \left\{ \begin{array}{c} \ddot{U}_{bt} \\ \hline \ddot{U}_s + \ddot{U}_d \end{array} \right\} + [K_{bb} | K_b^T] \left\{ \begin{array}{c} U_{bt} \\ \hline U_s + U_d \end{array} \right\} = \{F_b\}$$

Now, U_s and U_s were proved to be equal to

$$\{\ddot{U}_s\} = \begin{bmatrix} I \\ - \\ I \\ - \\ I \\ - \\ I \\ - \\ \vdots \end{bmatrix} \{\ddot{U}_{bt}\} \quad \text{and } \{U_s\} = \begin{bmatrix} I \\ - \\ I \\ - \\ I \\ - \\ I \\ - \\ I \\ \vdots \end{bmatrix} \{U_{bt}\}$$

where [I] is a(4 x 4) identity matrix, N/4 of which form the relating matrix between the resulting static non-base node displacements and the base node imposed displacements. Also N = number of non-base node degrees of freedom and since M_b^T contains nonzero elements only in the first four columns $M_b^T \cdot \ddot{u}_s$ can be expressed as

$$[M_b^T]'[I]\{\ddot{U}_{bt}\}$$

where $[M_b^T]'$ is the 4 x 4 matrix including the nonzero elements

$$[M_{bb} + M_b^{T'}I]\{\ddot{U}_{bt}\} + [M_b^T]'\{\ddot{U}_d\} + [K_{bb} + K_b^{T'}I]\{U_{bt}\} + K_b^T U_d = \{F_b\}$$

but $$[K_{bb} + K_b^{T'}I]\{U_{bt}\} = 0$$

$$\therefore \quad F_b = [M_{bb} + M_b^{T'}I]\{\ddot{U}_{bt}\} + [M_b^T]\{\ddot{U}_d\} + [K_b^T]\{U_d\}$$

Of course the most significant part of the base force is attributed to the displacements of the non-base nodes, i.e. $[K_b]^T\{U_d\}$.

COMPUTER IMPLEMENTATION

The expressions for displacements and forces arising because of base excitation (as given in the previous section) are well-suited to computer implementation. This is achieved through use of two main programs. The first main program, see Page D-4, is termed the free vibration program. Its main tasks are:

(a) To develop the element mass matrix for the typical ring element. This is done in subroutine MASS, see page D-9, using the final expression for the mass integral which is carried out by hand computation.

(b) To develop the ring element's stiffness matrix in subroutine STIFF, see page D-10. This is obtained by numerical integration using Gaussian integration numbers stored on a file under the designation GASN. Thus, it is clear that prior to running the first main program it is necessary to run the Gaussian integration program NIXW listed on page D-17.

(c) To assemble the structural mass and stiffness matrices in subroutine ASSTR, see page D-8.

(d) To apply the boundary conditions on the structural matrices according to the given boundary conditions of the physical problem. This is processed in subroutine BOUN, see page D-7.

(e) To extract the eigenvalues and natural modes. This is accomplished in subroutine EGN, see page D-13, using the Cholesky reduction method to obtain the first mode. Then, the orthogonality conditions are utilized to obtain the higher modes.

(f) To develop the stress-displacement matrices indicating the three orthogonal components of force per unit length of shell middle surface, i.e.

the axial, tangential, and radial forces, as well as the bending moments M_{zz} and $M_{\theta\theta}$ and the twisting moment $M_{\theta z}$. These are shown in Figures 15 and 16. These stress-displacement matrices are computed at six cross sections, namely at the top, bottom, and middle of the typical ring element at both $\theta = 0°$ and $\theta = \pi/2$. This is performed in subroutine STRMAT, see page D-6.

(g) To store on disc files the symmetric mass and stiffness matrices in triangular form. Also, to store the modes and the corresponding natural frequencies together with the stress-displacement matrices.

The second main program, page D-20, is the response program, the major tasks of which are:

(a) To retrieve the stored data from the disc through the use of control cards.

(b) To partition the original mass and stiffness matrices into the required matrices used in the analysis. This is achieved through use of subroutine EXTRACT, see page D-29.

(c) To develop the effective mass matrix M_{eff} discussed previously. This is done in subroutine EFFMASS, see page D-27.

(d) To apply modal analysis techniques by creating the generalized mass and stiffness matrices in subroutine MODAN, see page D-28. This is of course accomplished after retrieving the stored modes and associated frequencies.

(e) To solve for the participation factor of the modes employed by calculating the Duhamel integral for each. This is carried out using either of two subroutines. For the ground acceleration pulse in deterministic form of a rectangle, one sine wave, or a triangle, the closed form solution is

obtained for each prescribed value of time from subroutine INTE, see page D-30. For the ground motion in the form of the artificial earthquake accelerogram available through the National Information Service - Earthquake Engineering (PSEQGN) [11], numerical intergration for each mode is carried out in subroutine INT, see page D-26. The earthquake record is stored on a disc file.

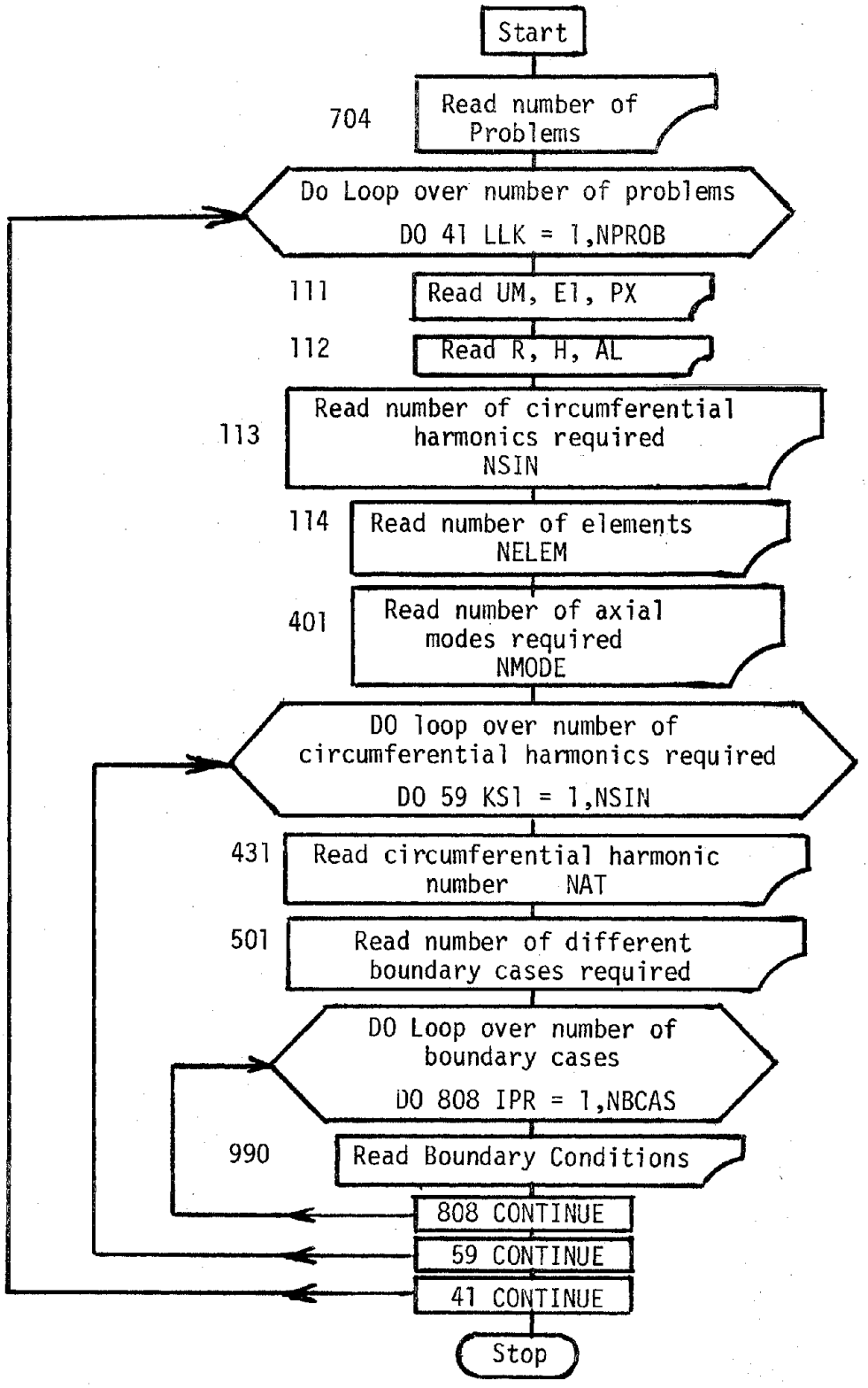
(f) To obtain the nodal displacements through superposition of the modes and their participation factors, as well as the external reaction developed at the base slab. Also, the stresses developed at the six cross sections mentioned previously are computed in the main program RESP, see page D-21, for each time station.

(g) To carry out an external equilibrium check.

It is to be noted that both of these main programs are written in Fortran IV language and implemented on a CDC Cyber computer. Data input to both main programs follow the scheme outlined in the following two charts. It is to be noted that several tanks can be investigated in the same computer run in which case the number of tanks being considered is entered into Line 704. However, this number is unity for the common case of considering only one tank at a time, hence in that case unity is entered in Line 704.

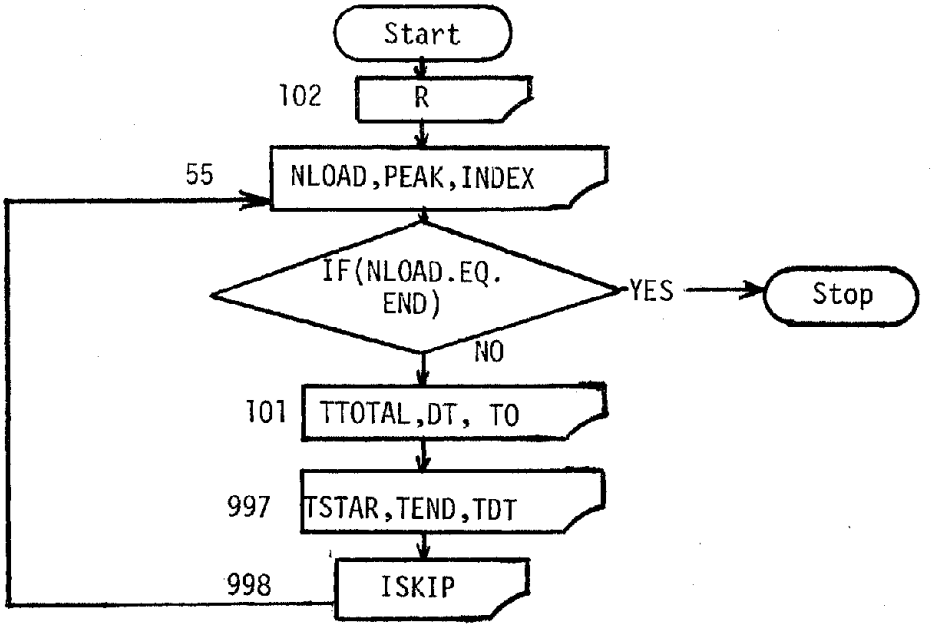
FIRST MAIN PROGRAM

DATA FEEDING



SECOND MAIN PROGRAM

DATA FEEDING



where

TTOTAL = Total time under consideration

DT = Time increment

TO = Total time duration of load application, used only for closed form solution

R = Radius of tank

M = Number of natural modes used for response superposition

NLOAD = Load indication. For example this is RECPU for rectangular pulse

or TRIPU for triangular pulse

or SINPU for sinusoidal pulse

or ARTEQ for artificial earthquake

PEAK = MAX AMPLITUDE of ground acceleration or the value used for normalization.

55	INDEX	Indicates the method of solution required can be CLOSD for closed form solution or NUMER for numerical integration.
997	TSTAR =	Initial time for which response is to be analyzed.
997	TEND =	Final time for which response is to be analyzed.
997	TDT =	Time interval between each two successive points under consideration as for response.
998	ISKIP =	Number of points to be skipped as for external equilibrium check, stress analysis and detailed printing between two successively considered time stations.

EXAMPLES

I. Free Vibrations of Empty Storage Tank

Let us consider the slab-supported tank discussed in [9]. This structure is 40 feet high and 60 feet in radius with a steel wall having thickness of one inch. First, let us determine the empty tank natural frequencies for the case of the tank clamped at the base slab and free at the top. The computer program of Appendix D is utilized for this frequency determination. To utilize the program one enters into Lines 111, 112, 113, 114, 401, 431, 501, and 990 of the main program on Page D-4 the following data:

Line 111: UM = ρ = density of tank material = 0.733×10^{-3} lb·sec^2/in^4

E1 = E = Young's modulus = 30×10^6 lb/in^2

PX = nu = Poisson's ratio = 0.3

Line 112: R = tank radius = 720 inches

H = tank wall thickness = 1 inch

AL = tank altitude = 480 inches

Line 113: NSIN = total number of circumferential wave patterns that analyst desires to investigate = 1

Line 114: NELEM = number of ring-shaped finite elements representing the tank = 9

Line 401: NMODE = number of axial waves under consideration = 36 (Printout indicates frequencies and displacements for modes 1, 2, ... 36)

Line 431: NAT = number of circumferential waves in pattern under consideration (i.e. "instantaneous" number of circumferential waves) = 1. This number specifies which one of those patterns listed under NSIN is

currently being investigated.

Line 501: NBCAS = total number of cases involving different sets of boundary conditions that analyst desires to investigate = 1.

Line 990: NBC = denotes boundary conditions at base and top of tank. First, enter CL if base is clamped or SM if base is simply supported. Next, enter CL if top is clamped, or SM if it is simply supported, or FR if it is free. Do not introduce a space between the designations of these two boundary conditions.

This completes all necessary input to the computer program.

The program output is displayed as follows: From Subroutine EGN on page D-13, line 108 INA represents the number of waves in the axial direction and omega in line 6990 denotes the corresponding natural frequency in Hz. These natural frequencies are as follows:

Mode	Frequency (Hz) Ref. [9]	Present work
1	34.04	34.08
2	43.87	43.91
3	44.58	44.64
4	45.10	45.19
5	45.80	45.92
6	47.97	47.13
7	-	49.03
8	-	51.86
9	-	55.79
10	-	61.91

11	-	69.28
12	-	79.00
13	-	84.33
14	-	91.45
15	-	107.07
16	-	111.14
17	-	126.17
18	-	147.94
19	-	168.62
20	-	205.18
21	-	210.55
22	-	325.09
23	-	351.53
24	-	486.49
25	-	572.91
26	-	649.62
27	-	823.44
28	-	826.61
29	-	1010.55
30	-	1097.67
31	-	1178.23
32	-	1284.02
33	-	1396.31
34	-	1707.09
35	-	1990.22
36	-	2169.27

For each of the above natural frequencies the relative (normalized) displacements u, v, and w together with the slope dw/dz are displayed in the computer printout in the form of columns (with these headings) immediately after printing of the natural frequency. In these displays of displacements and slope the first (top) line represents tank displacements and slope at the junction of the tank with the rigid base slab (base node) and the last (bottom) line represents the corresponding quantities at the tank top. As an example, the third (axial) mode values are found to be:

u	v	w	dw/dz
.00000000	0.00000000	0.00000000	0.00000000
-.00025816	-.00050568	.02571988	.00065707
.00114042	-.00070022	.04663320	.00005942
-.00208424	-.00034154	.03029001	-.00064278
-.00235623	.00037411	-.01117024	-.00084468
-.00174015	.00094758	-.04620948	-.00041509
-.00068429	.00095662	-.04882863	.00032133
.00003875	.00037045	-.01675913	.00083742
-.00011887	-.00042156	.02879214	.00082998
-.00118237	-.00036095	.06570091	.00061867

Plots of u, v, w, and dw/dz for the first four axial modes appear in Figures 2 through 5 inclusive.

II. Free Vibrations of Cylindrical Shells with Diverse Boundary Conditions

Let us compare the predictions of the present analysis with results due to D.F. Vronay and B. L. Smith [10]. These investigators considered a cylindrical shell of length 12 inches, radius 3 inches, thickness 0.010 inches, and having properties $E = 29.6 \times 10^6$ lb/in^2, $\nu = 0.29$, and $\rho = 0.733 \times 10^{-3}$ lb·sec^2/in^4. We seek to determine the natural frequencies of this shell for four sets of boundary conditions, namely both ends clamped

designated as C-C, one end clamped, the other simply supported designated as C-SS, both ends simply supported denoted as SS-SS, and one end clamped and the other free denoted by C-F. The computer program of Appendix D is utilized for this frequency determination. To employ this program one enters into Lines 111, 112, 113, 114, 401, 431, 501, and 990 of Page D-4 the following data:

Line 111: UM = ρ = density of shell material = 0.733×10^{-3} $lb \cdot sec^2/in^4$

E1 = E = Young's modulus = 29.6×10^6 lb/in^2

PX = nu = Poisson's ratio = 0.29

Line 112: R = shell radius = 3 inches

H = shell wall thickness = 0.010 inches

AL = shell altitude = 12 inches

Line 113: NSIN = total number of circumferential wave patterns that analyst desires to investigate = 5.

Line 114: NELEM = number of ring-shaped finite elements representing the shell = 9.

Line 401: NMODE = number of axial waves under consideration = 4.

Line 431: NAT = number of circumferential waves in pattern under consideration = 1, 2, 3, 4, and 5 successively.

Line 501: NBCAS = total number of cases involving different sets of boundary conditions that analyst desires to investigate = 4.

Line 990: NBC = denotes boundary conditions at both ends of shell. First, enter CL if first end is clamped or SM if it is simply supported. Next, enter CL if other

end is clamped, or SM if it is simply supported, or FR if it is free. Do not introduce a space between the designations of these two boundary conditions.

This completes all necessary input to the computer program.

The program output is displayed as follows: From Subroutine EGN on Page D-13, line 108, INA represents the number of waves in the axial direction and omega in line 6990 denotes the corresponding natural frequency in Hz. These natural frequencies are indicated in the following table in parentheses, preceded by values found in [10].

Axial Mode	Circumferential Mode	C-C	C-SS	SS-SS	C-F
1	1	3423 (3474)	3423 (3471)	3423 (3467)	1205 (1221)
	2	1917 (1976)	1916 (1968)	1915 (1960)	479 (490)
	3	1154 (1219)	1153 (1209)	1152 (1200)	255 (262)
	4	765 (831)	764 (820)	763 (810)	219 (219)
	5	581 (643)	580 (632)	579 (621)	280 (276)
2	1	6412 (6575)	6412 (6574)	6411 (6574)	3818 (3978)
	2	3902 (4065)	3901 (4060)	3900 (4055)	2136 (2211)
	3	2536 (2707)	2534 (2697)	2532 (2867)	1240 (1307)
	4	1752 (1931)	1750 (1917)	1748 (1903)	800 (863)
	5	1287 (1469)	1285 (1452)	1283 (1435)	597 (655)
3	1	8493 (8741)	8496 (8741)	8491 (8739)	6817 (7167)
	2	5832 (6152)	5839 (6150)	5839 (6146)	4406 (4671)
	3	4051 (4388)	4049 (4382)	4047 (4374)	2824 (3033)
	4	2918 (3269)	2916 (3258)	2913 (3246)	1906 (2102)
	5	2190 (2548)	2187 (2533)	2184 (2517)	1371 (1562)
4	1	9418 (9690)	9416 (9686)	9421 (9673)	9147 (8331)
	2	7299 (7736)	7298 (7734)	7296 (7732)	6228 (6725)
	3	5442 (5968)	5441 (5965)	5439 (5960)	4401 (4833)
	4	4100 (4670)	4096 (4664)	4094 (4656)	3158 (3565)
	5	3165 (3756)	3161 (3746)	3157 (3735)	2343 (2739)

III. Cylindrical Tank Whose Base Slab is Subject to a Single Rectangular Pulse Load

Let us consider again the tank of Ref. [9] where the tank wall is rigidly clamped to the base slab and the tank top is freely supported. The excitation is a rectangular pulse, as indicated in Figure 6. This acceleration is applied in the direction $\theta = 0°$.

It is necessary to run the first main program as shown in Example I prior to running the second main program which describes the input excitation. Note that after the first main program has been run once that it is filed, so it is not necessary to run it again for each separate base excitation being considered.

It is now necessary to enter into the second main program of Appendix D, page D-21, the following data characterizing base excitation:

Line 101:	TTOTAL = 1 second
	DT = 0.001 second
	TO = 0.1 second
Line 102:	R = 720 inches
Line 103:	M = 8 modes
Line 55:	NLOAD = RECPU
	PEAK = 120 inches/second^2
	INDEX = CLOSD

Also, an "end" card is required. This completes all necessary input to the computer program. The printed computer output for the closed form solution presents axial as well as radial components of shell middle surface displacement (u and w respectively) together with slope dw/dz at $\theta = 0°$. The tangential displacement v is also indicated at $\theta = \pi/2$. An example of

these responses is indicated below for t = 0.5 seconds after initiation of the pulse. Eight modes were considered. There, the figures in the top row correspond to the node at the base of the tank and the figures in the bottom row correspond to the node at the tank top.

U	V	W	DW/DZ
-.81852437E-04	-.46740950E-03	.18208493E-02	.30334876E-04
-.17984490E-03	-.92366075E-03	.31572421E-02	.26386298E-04
-.28671131E-03	-.13511467E-02	.45354341E-02	.20652330E-04
-.39758616E-03	-.17331961E-02	.52762242E-02	.12245377E-04
-.50799044E-03	-.2.595841E-02	.61866043E-02	.17011134E-04
-.61734933E-03	-.23205191E-02	.61154179E-02	-.22548246E-04
-.69542760E-03	-.25174721E-02	.48646779E-02	-.13219555E-04
-.75196334E-03	-.26635742E-02	.52660782E-02	.20202177E-04
-.82088053E-03	-.27557380E-02	.62569511E-02	.18612571E-04

At t = 0.5 seconds the external applied load is zero. The computer display of the sum of inertia and internal forces at each node at t = 0.5 seconds indicated extremely good agreement with this zero value. For this same pulse, the corresponding results found by direct numerical integration (replacing CLOSD in Line 55 by NUMER), (using eight modes) of the equations of motion are:

U	V	W	DW/DZ
-.93929171E-04	-.52839347E-03	.21513162E-02	.33915542E-04
-.20570788E-03	-.10436033E-02	.35190931E-02	.27507251E-04
-.32607525E-03	-.15262609E-02	.51559075E-02	.26867135E-04
-.45305417E-03	-.19574746E-02	.59719966E-02	.86783334E-05
-.57489212E-03	-.23262383E-02	.66951275E-02	.15334656E-04
-.69178481E-03	-.26244118E-02	.66587444E-02	-.20182781E-04
-.77856233E-03	-.28532298E-02	.56321753E-02	-.71536281E-05
-.84830809E-03	-.30230959E-02	.62032553E-02	.20950682E-04
-.93053598E-03	-.31284692E-02	.71307191E-02	.14981571E-04

Agreements of these two appraoches for other values of time are comparable but omitted for brevity.

Again, for the sake of brevity no other numerical results are tabulated here but instead Figure 7 indicates the time dependent behavior of radial displacement w at the tank top for $\theta = 0°$. Also, the bottom traces in Figure 7 indicate the behavior of radial displacement w at node 6 (top of the central element) as well as axial displacement at that same

node for $\theta = 0°$. These plots are for the first 1.00 seconds with time origin taken at the instant of pulse application.

Significant forces and moments per unit length of the shell middle surface are given by the second main program and as an example the values for t = 0.5 seconds are given below. Here the first three columns represent axial, tangential, and in-plane shear forces respectively at $\theta = 0°$. The fourth column represents the bending moment M_{zz}, the fifth column the tangential bending moment $M_{\theta\theta}$ and the sixth column represents the twisting moment M_{θ}.

STRESSES AT THETA = 0.0

AXIAL F.	TANGT. F.	IN-PLANE SH.	AXIAL MT.	TANGT. MT.	TORSION
-59.60	-15.18	0.	-7.427	-2.228	0.
-36.99	45.30	0.	2.765	.8360	0.
-32.64	83.27	0.	-.6162	-.1741	0.
-23.50	125.6	0.	1.123	.3524	0.
-19.72	141.7	0.	-.6732	-.1849	0.
-11.23	168.6	0.	1.058	.3374	0.
-5.803	156.4	0.	1.867	.5783	0.
-9.363	94.99	0.	-2.407	-.7109	0.
-3.849	104.8	0.	-.4119	-.1113	0.
5.493	147.5	0.	.1791	.7061E-01	0.

The corresponding values at $\theta = 0°$ are

STRESSES AT THETA = 90.0 DEGREES

AXIAL F.	TANGT. F.	IN-PLANE SH.	AXIAL MT.	TANGT. MT.	TORSION
-.6713E-04	-.2014E-04	-101.1	-.9854E-05	-.2956E-05	-.1756E-01
-.4908E-04	.6010E-04	-98.63	.3669E-05	.1109E-05	.6360E-01
-.4330E-04	.1105E-03	-92.71	-.8176E-06	-.2311E-06	.5371E-01
-.3125E-04	.1667E-03	-82.94	.1490E-05	.4675E-06	.3970E-01
-.2617E-04	.1880E-03	-70.26	-.8932E-06	-.2453E-06	.1903E-01
-.1490E-04	.2237E-03	-55.40	.1404E-05	.4477E-06	.3393E-01
-.7699E-05	.2075E-03	-39.64	.2477E-05	.7672E-06	-.6940E-01
-.1242E-04	.1260E-03	-25.96	-.3194E-05	-.9432E-06	-.4240E-01
-.5106E-05	.1390E-03	-13.73	-.5466E-06	-.1477E-06	.4879E-01
.7288E-05	.1957E-03	-6.795	.2376E-06	.9368E-07	.4549E-01

IV. Cylindrical Tank Whose Base Slab is Subject to Single Sinusoidal Pulse Load

The tank of Example III is reconsidered here, for excitation in the form of a single sinusoidal pulse, as indicated in Figure 8. This acceleration is also applied in the direction $\theta = 0°$.

Again, the first main program has already been run, so it is unnecessary to run it again. For the base excitation under consideration now, it is necessary to enter into the second main program of Appendix D the following data characterizing base excitation:

Line 55: NLOAD = SINPU

PEAK = 120 inches/second^2

INDEX = CLOSD

This completes all necessary input to the computer program.

The printed computer output of displacements for t = 0.5 seconds after initiation of the pulse is as follows:

TIME= .50000

U	V	W	DW/DZ
.41841928E-05	.21168633E-04	-.12699475E-03	-.14221379E-05
.90744454E-05	.41414549E-04	-.12790795E-03	.69186850E-07
.13451164E-04	.60441915E-04	-.20024378E-03	-.21875769E-05
.18821063E-04	.77469908E-04	-.25501342E-03	.27207151E-06
.23548499E-04	.91864731E-04	-.23701829E-03	-.71824744E-08
.27520883E-04	.10376167E-03	-.24440821E-03	-.27392337E-07
.31118018E-04	.11311563E-03	-.25751647E-03	-.63079009E-06
.34775136E-04	.11974903E-03	-.27936318E-03	.72220160E-06
.37363903E-04	.12361491E-03	-.18973398E-03	.18415839E-05

For this same pulse, the corresponding results found by direct numerical integration of the equations of motion (replacing CLOSD in Line 55 by NUMER) are:

TIME= .50000

U	V	W	(W/D)
.45110867E-05	.23269174E-04	-.12975100E-03	-.15470992E-05
.98161637E-05	.45619638E-04	-.14582139E-03	-.27894220E-06
.14793719E-04	.66589156E-04	-.22096339E-03	-.19043123E-05
.20479905E-04	.85369053E-04	-.26435806E-03	.18340200E-06
.25543938E-04	.10141784E-03	-.26658283E-03	-.41209183E-06
.30888886E-04	.11467257E-03	-.27171650E-03	.37250512E-06
.33952230E-04	.12511155E-03	-.27849401E-03	-.94230393E-06
.38050447E-04	.13258939E-03	-.30936957E-03	.64843471E-06
.41116894E-04	.13690073E-03	-.21487833E-03	.22512543E-05

Again, agreements for other values of time are comparable.

Figure 9 indicates the time dependent behavior of radial displacement w at the tank top for $\theta = 0°$. Also Figure 9 indicates the behavior of radial displacement w at node 6 as well as axial displacement at that same node.

V. Cylindrical Tank Whose Base Slab is Subject to Single Triangular Pulse Load

This example considers again the same tank of Example III, but now with the rigid base slab subject to excitation in the form of a single triangular pulse, as indicated in Figure 10. As in Example III, this is applied in the direction $\theta = 0°$.

The first main program need not be run again. For this particular base excitation, it is necessary to enter into the second main program of Appendix D the following data characterizing base excitation:

Line 55 NLOAD = TRIPU

PEAK = 120 inches/second^2

INDEX = CLOSD

This completes all necessary input to the computer program.

As in example IV, the printed computer output of displacements for t = 0.5 seconds after initiation of the pulse is as follows:

TIME= .50000

U	V	W	DW/DZ
.10785180E-04	.55378303E-04	-.25344254E-03	-.29001747E-05
.22597025E-04	.10928750E-03	-.28081243E-03	-.23115159E-06
.32916638E-04	.16092664E-03	-.40270811E-03	-.34942107E-05
.43985714E-04	.20896062E-03	-.51069075E-03	-.40785976E-06
.54011250E-04	.25210669E-03	-.54919748E-03	-.17185934E-05
.63946369E-04	.28973484E-03	-.67998620E-03	-.28018775E-05
.75454273E-04	.32029266E-03	-.83560242E-03	-.29818605E-05
.88595208E-04	.34188597E-03	-.90955793E-03	.95783405E-06
.10002630E-03	.35373068E-03	-.77810897E-03	.30627995E-05

For this same pulse, the corresponding results found by direct numerical integration of the equations of motion are:

TIME= .50000

U	V	W	DW/DZ
.11776005E-04	.61257601E-04	-.26807146E-03	-.32106263E-05
.24729702E-04	.12100055E-03	-.31859217E-03	-.71040037E-06
.36341184E-04	.17815728E-03	-.44763490E-03	-.32477642E-05
.48344197E-04	.23130870E-03	-.55189529E-03	-.82329672E-06
.59533570E-04	.27915377E-03	-.63427298E-03	-.25568314E-05
.71146923E-04	.32069488E-03	-.77023393E-03	-.22477814E-05
.83827430E-04	.35402631E-03	-.90912022E-03	-.31550583E-05
.98073568E-04	.37767895E-03	-.99692442E-03	.98177097E-06
.11046927E-03	.39068319E-03	-.84353252E-03	.36478079E-05

As before, agreements for other values of time are comparable.

Figure 11 indicates the time dependent behavior of radial displacement w at the tank top for $\theta = 0°$. Figure 11 indicates the behavior of radial displacement w at node 6 as well as axial displacement at that same node.

Comparison of the above results for Examples III, IV and V as obtained by the finite element approach with results of direct numerical integration lend confidence to use of the finite element program presented here for arbitrary base excitation.

VI. Cylindrical Tank Whose Base Slab is Subject to Artificial Earthquake Excitation

The tank of [9] is employed again in this example. The tank wall is rigidly clamped to the base slab and the top is freely supported. The artificial earthquake accelerogram available through the National Information Service - Earthquake Engineering - Computer Program Applications (PSEQGN) was considered to be the exciting mechanism acting on the rigid base slab in the horizontal direction along the line $\theta = 0°$.

The artificial earthquake record was imposed upon the base slab for five seconds and the tank response determined at 0.1 second intervals during the time period t = 0 to t = 5 seconds using time increments of 0.001 second. In using the artificial earthquake record the assigned maximum ground acceleration was taken to be g/2 although the record itself is normalized in terms of a unit value of g. The input to the rigid base was in terms of acceleration. The data cards employed and values assigned were as follows:

Line 55 NLOAD = ARTEQ

PEAK = 384 inches/second^2

INDEX (leave blank)

The time history of radial displacement at the tank top at $\theta = 0°$ during the time interval t = 4.0 to 4.5 seconds appears as shown in Figure 12. The axial, tangential, and in-plane shearing stresses as well as moments M_{zz}, $M_{\theta\theta}$, and $M_{z\theta}$ at $\theta = 0°$ are tabulated below at the time t = 4.3 seconds where the values in the top row correspond to base nodes and values in the bottom row correspond to nodes at the top of the tank.

STRESSES AT THETA=0.0

AXIAL F.	TANGT. F.	IN-PLANE SH.	AXIAL MT.	TANGT. MT.	TORSION
385.4	115.9	0.	45.82	13.75	0.
301.2	-175.1	0.	-21.19	-6.387	0.
285.9	-270.9	0.	14.34	4.261	0.
200.8	-738.1	0.	-4.988	-1.589	0.
138.0	-1070.	0.	-11.33	-3.529	0.
122.5	-1067.	0.	7.533	2.132	0.
63.05	-1219.	0.	-6.567	-2.113	0.
36.33	-1204.	0.	-3.825	-1.288	0.
33.53	-1100.	0.	8.183	2.326	0.
-19.63	-1258.	0.	-.6866	-.3509	0.

Corresponding values at $\theta = 90°$ at $t = 4.3$ seconds are tabulated below:

STRESSES AT THETA=90.0 DEGREES

AXIAL F.	TANGT. F.	IN-PLANE SH.	AXIAL MT.	TANGT. MT.	TORSION
.5127E-03	.1538E-03	776.1	.6080E-04	.1824E-04	.1347
.3997E-03	-.2323E-03	763.0	-.2811E-04	-.8475E-05	-.1733
.3807E-03	-.3595E-03	737.1	.1903E-04	.5653E-05	-.3343
.2664E-03	-.9793E-03	683.3	-.6618E-05	-.2108E-05	-.6816
.1831E-03	-.1420E-02	595.6	-.1504E-04	-.4682E-05	-.1062
.1826E-03	-.1416E-02	489.7	.9994E-05	.2829E-05	-.7725E-01
.8365E-04	-.1618E-02	373.4	-.8713E-05	-.2804E-05	-.1893
.4820E-04	-.1598E-02	249.0	-.5075E-05	-.1709E-05	.1926
.4449E-04	-.1459E-02	125.4	.1086E-04	.3087E-05	-.4624E-01
-.2505E-04	-.1669E-02	53.38	-.9110E-06	-.4656E-06	-.2263

ACKNOWLEDGMENT

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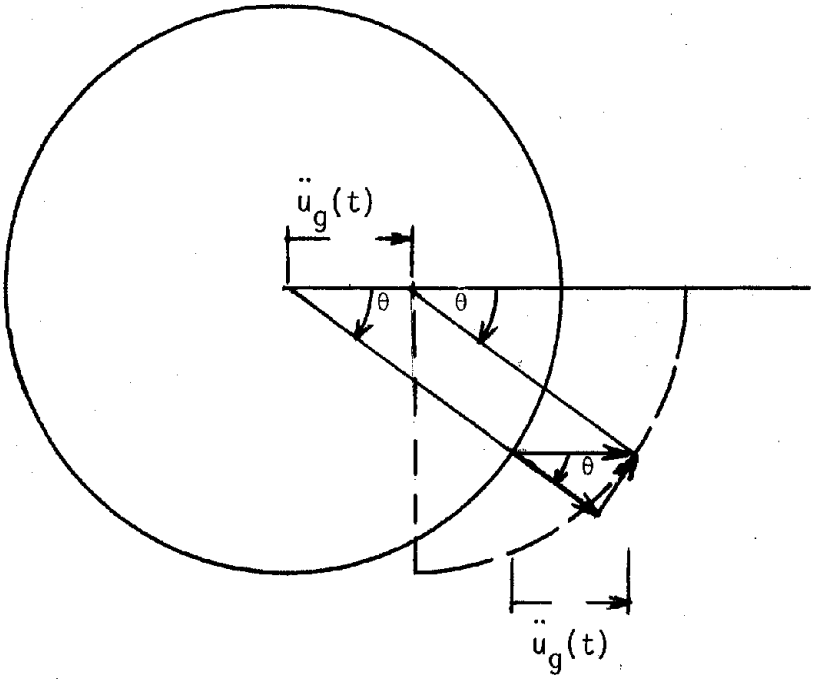


FIGURE 1

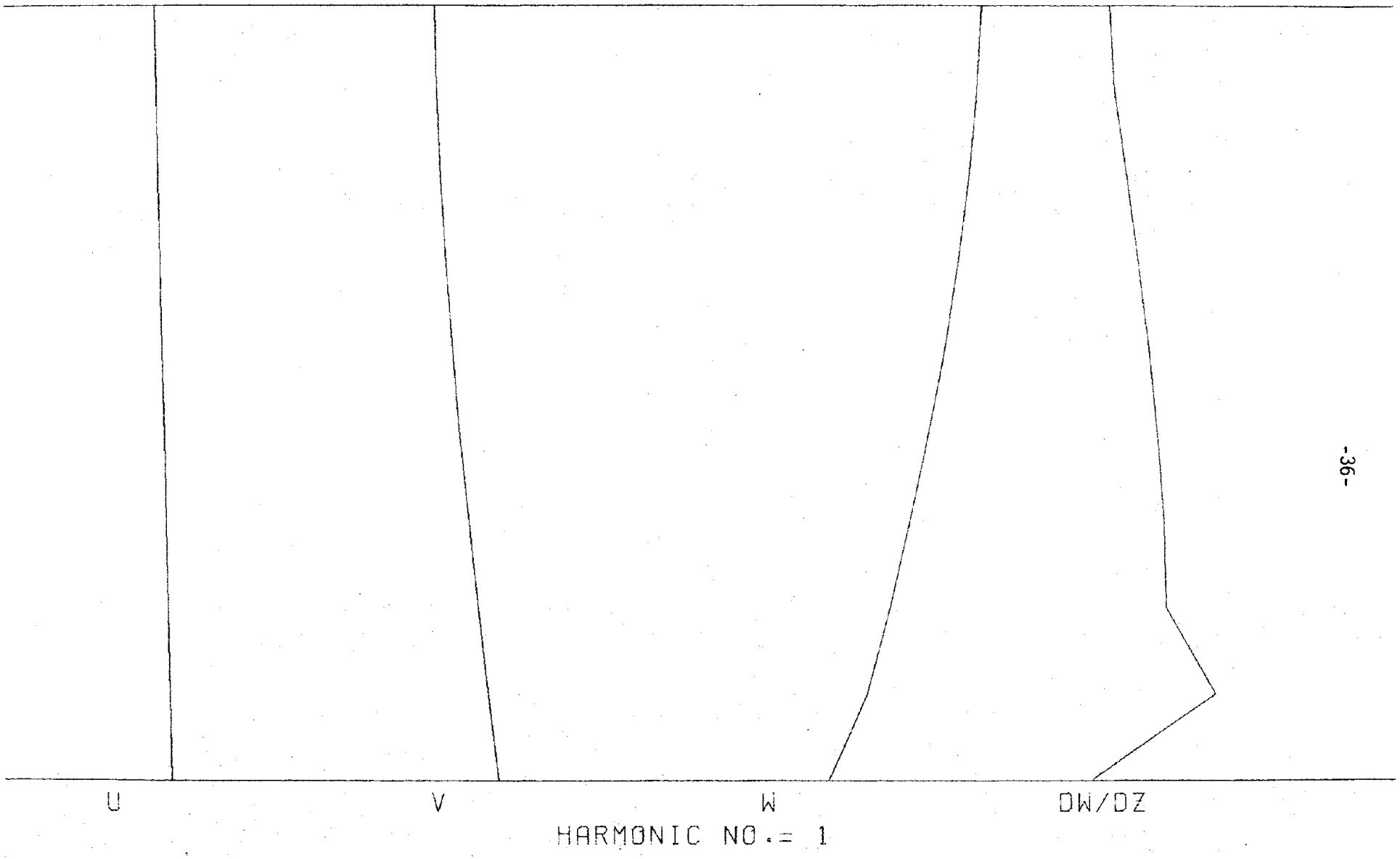


FIGURE 2

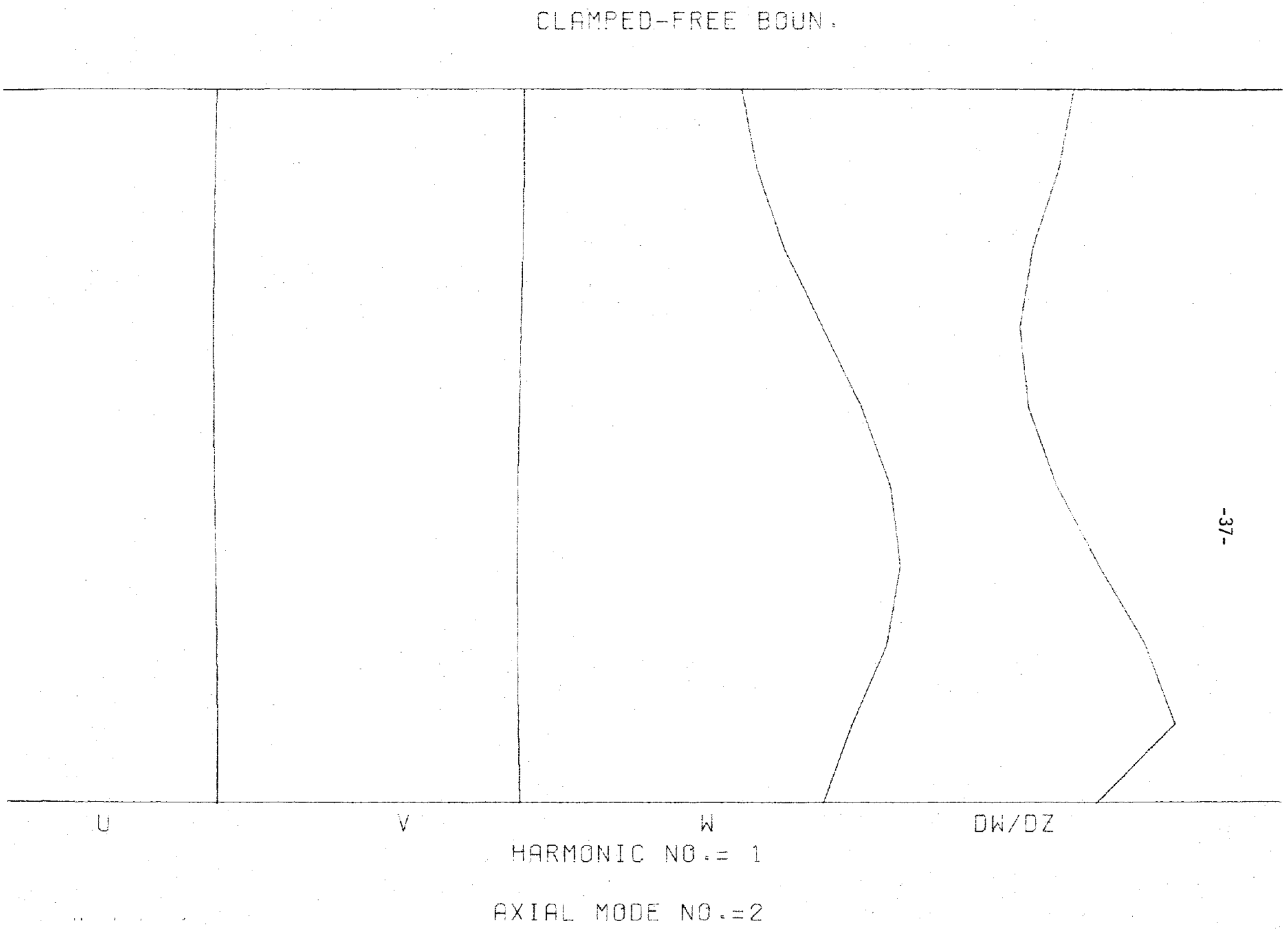


FIGURE 3

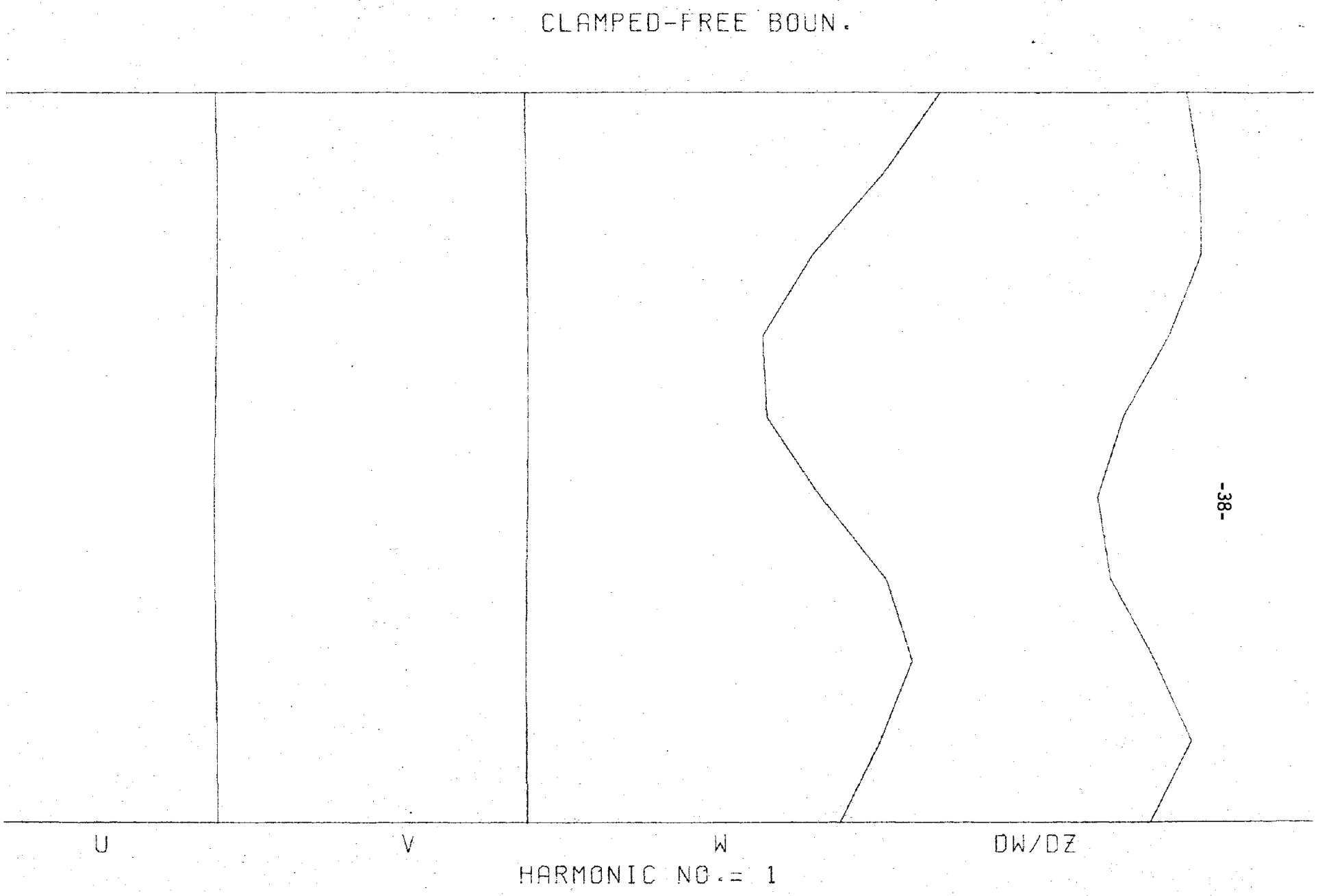
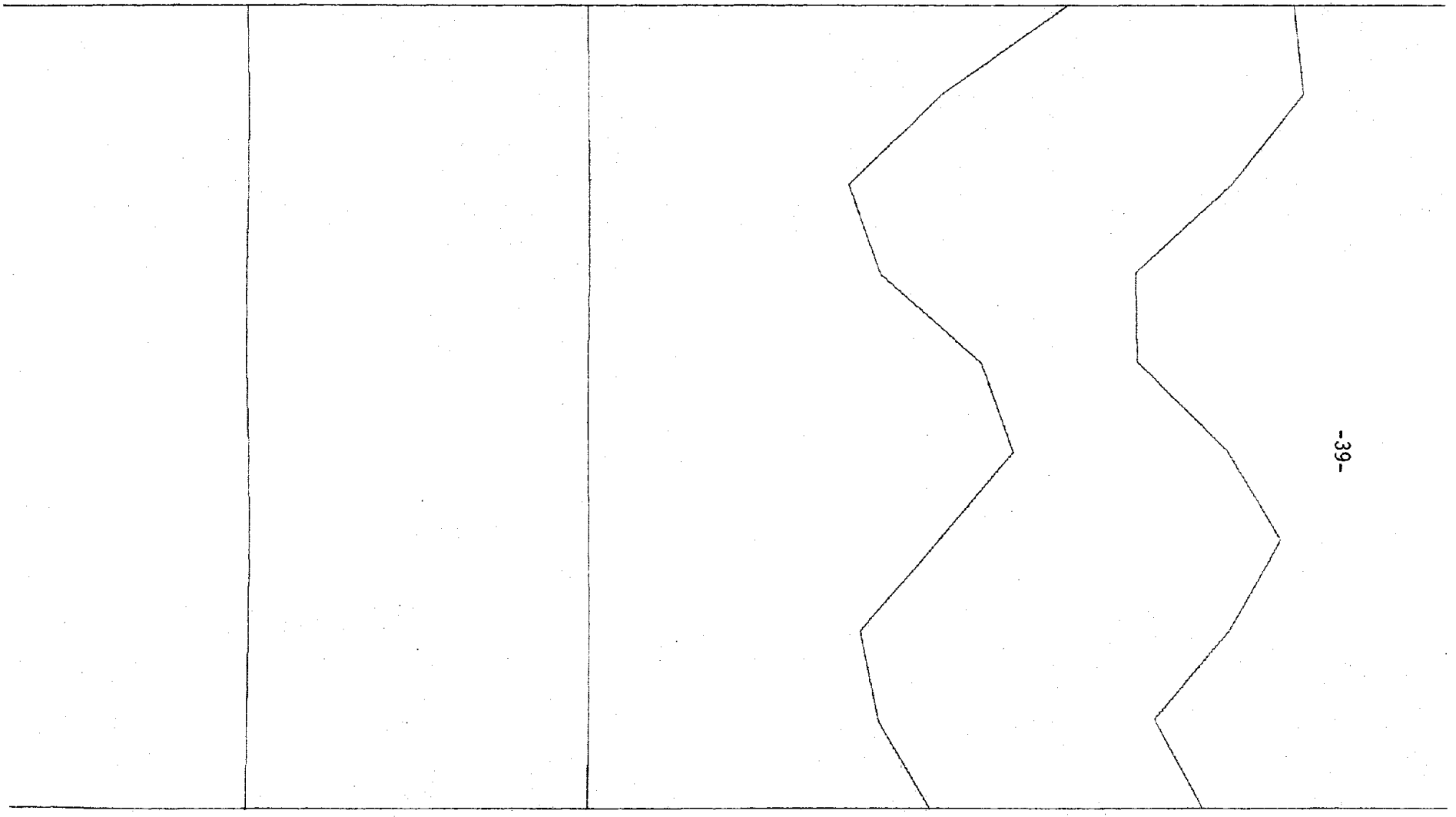


FIGURE 4

CLAMPED-FREE BOUN.



HARMONIC NO.= 1

AXIAL MODE NO.=4

FIGURE 5

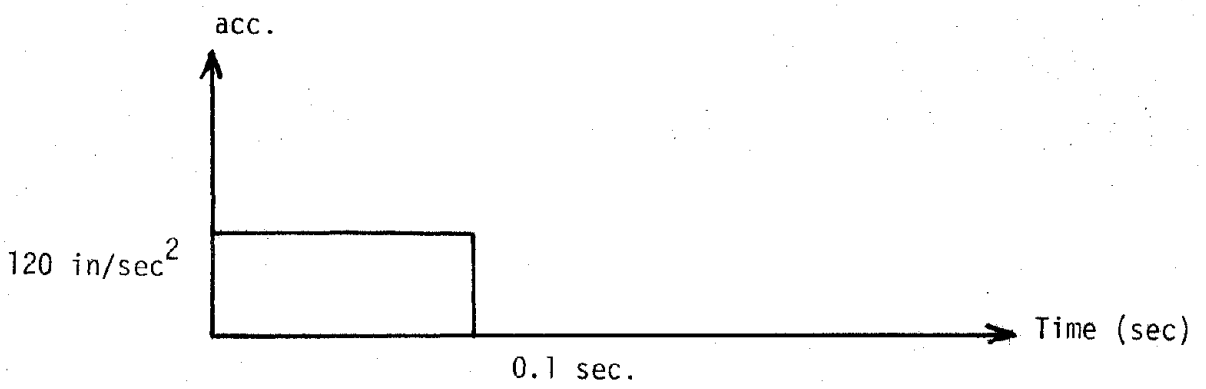


FIGURE 6

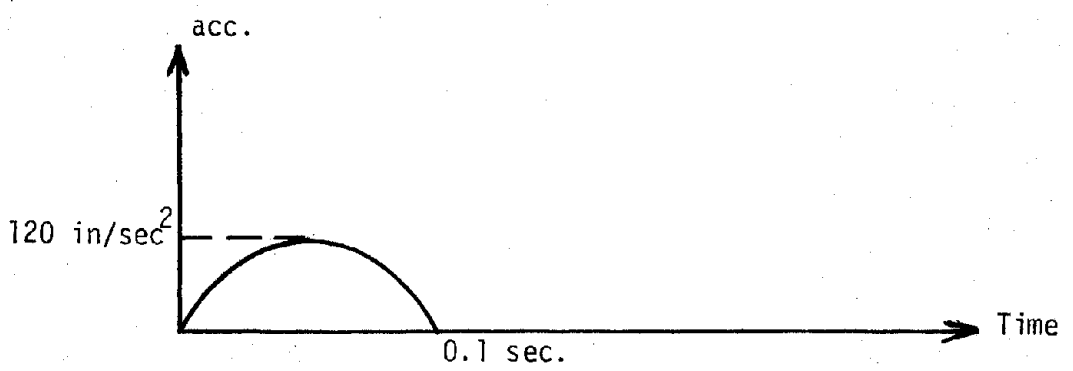


FIGURE 8

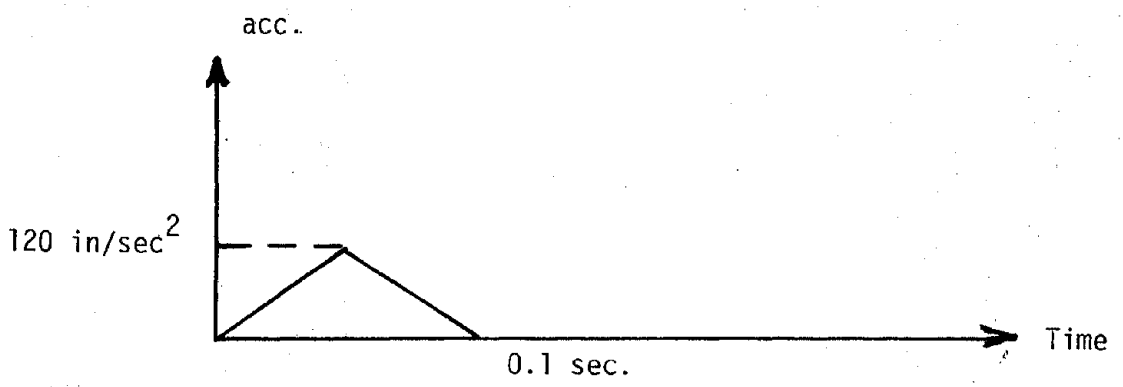


FIGURE 10

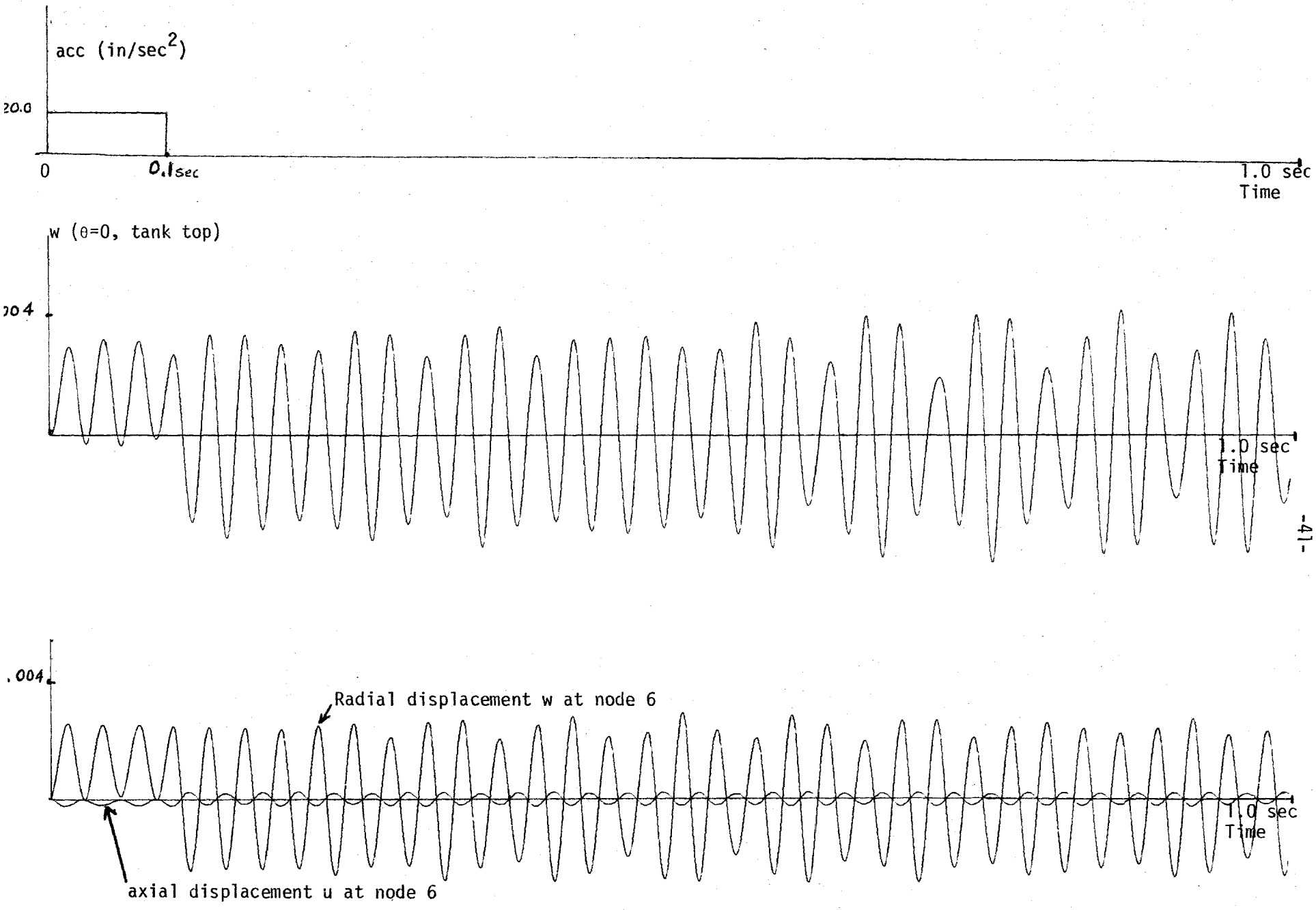

acc (in/sec^2)
20.0
0
0.1 sec
1.0 sec
Time
w (θ=0, tank top)
.004
1.0 sec
Time
.004
Radial displacement w at node 6
axial displacement u at node 6
1.0 sec
Time

FIGURE 7

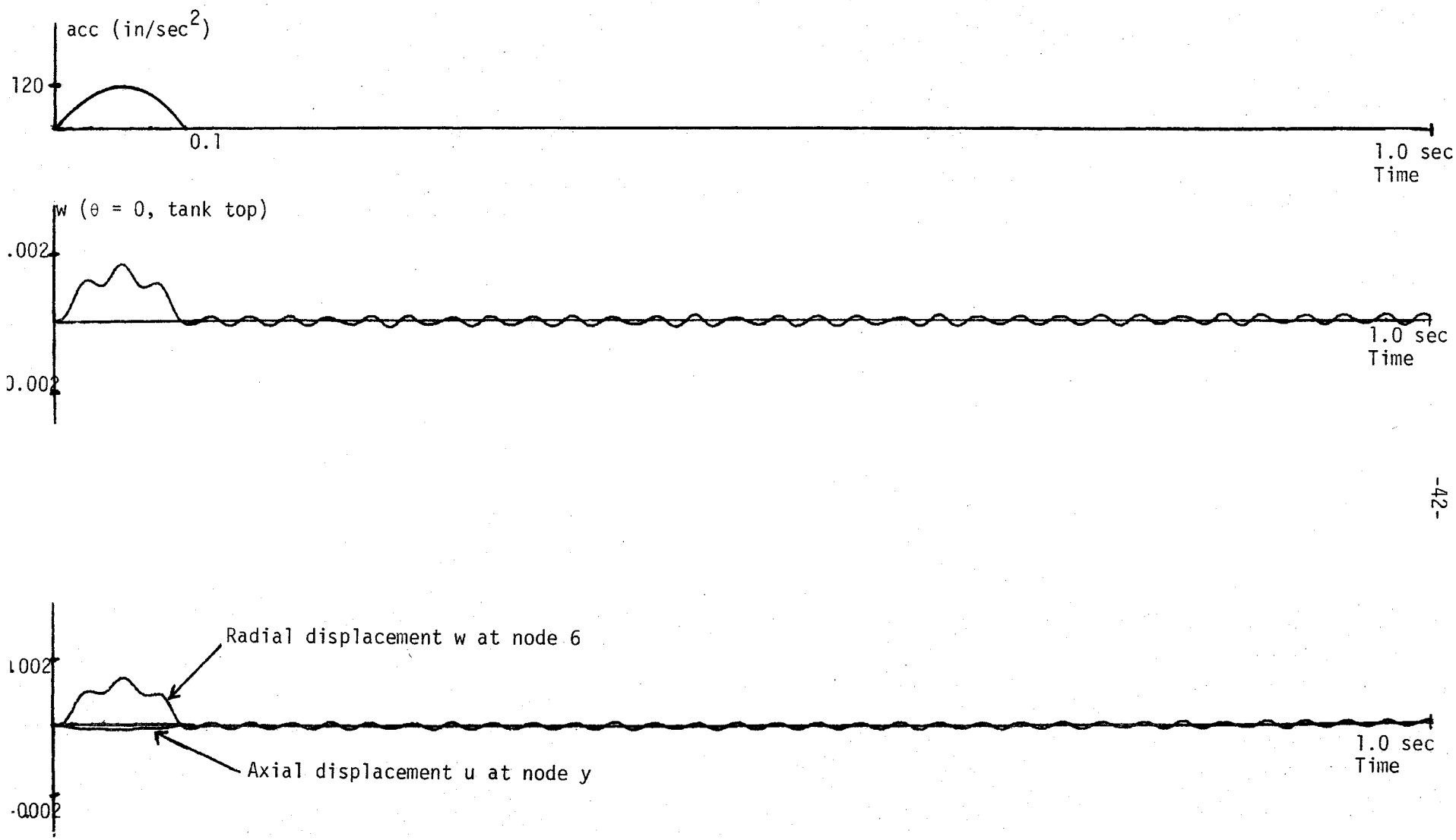

acc (in/sec^2)
120
0.1
1.0 sec
Time
w (θ = 0, tank top)
.002
0.002
1.0 sec
Time
Radial displacement w at node 6
.002
Axial displacement u at node y
-0.002
1.0 sec
Time

FIGURE 9

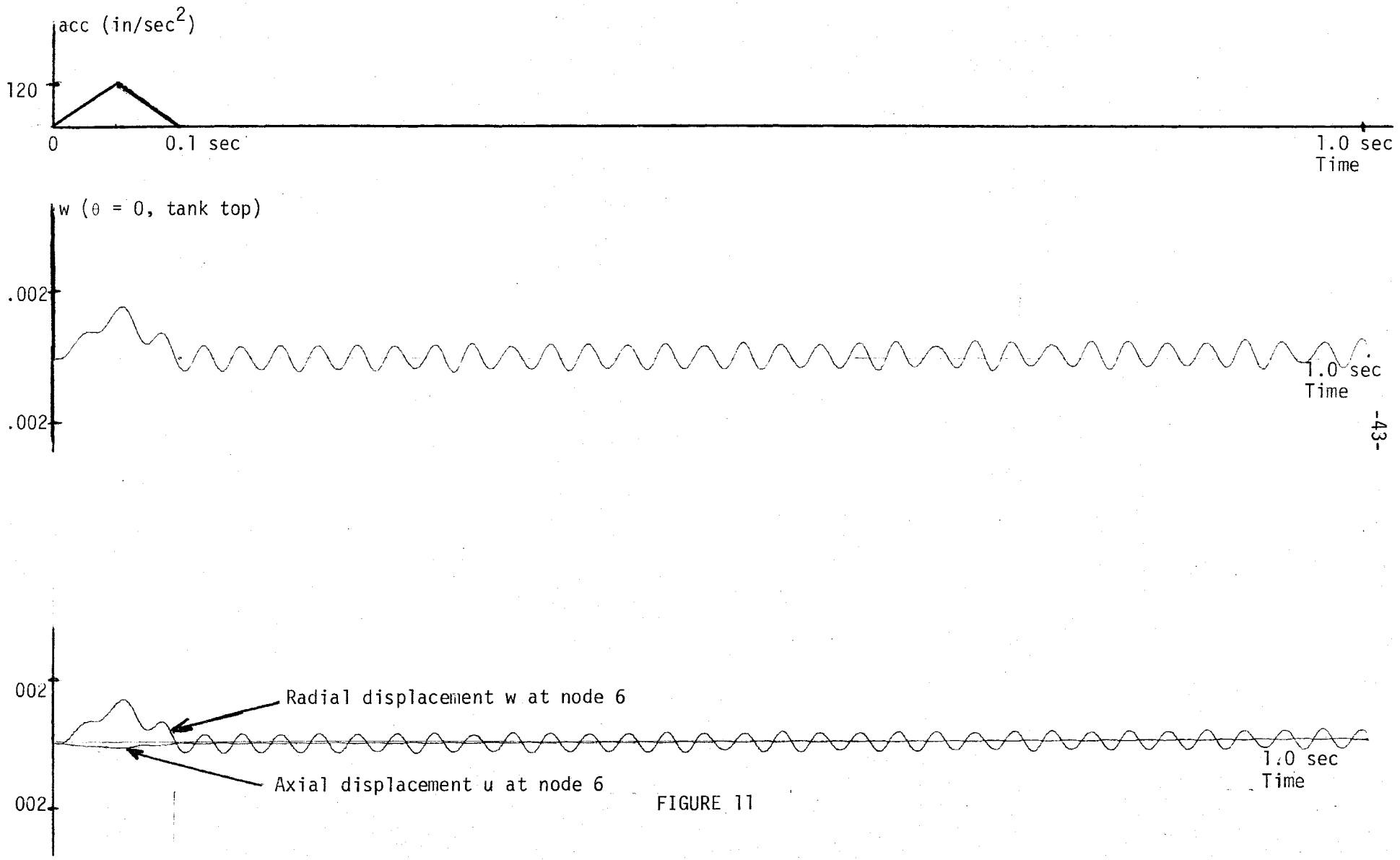

acc (in/sec^2)
120
0
0.1 sec
1.0 sec
Time
w (θ = 0, tank top)
.002
.002
1.0 sec
Time
002
002
Radial displacement w at node 6
Axial displacement u at node 6
1.0 sec
Time

FIGURE 11

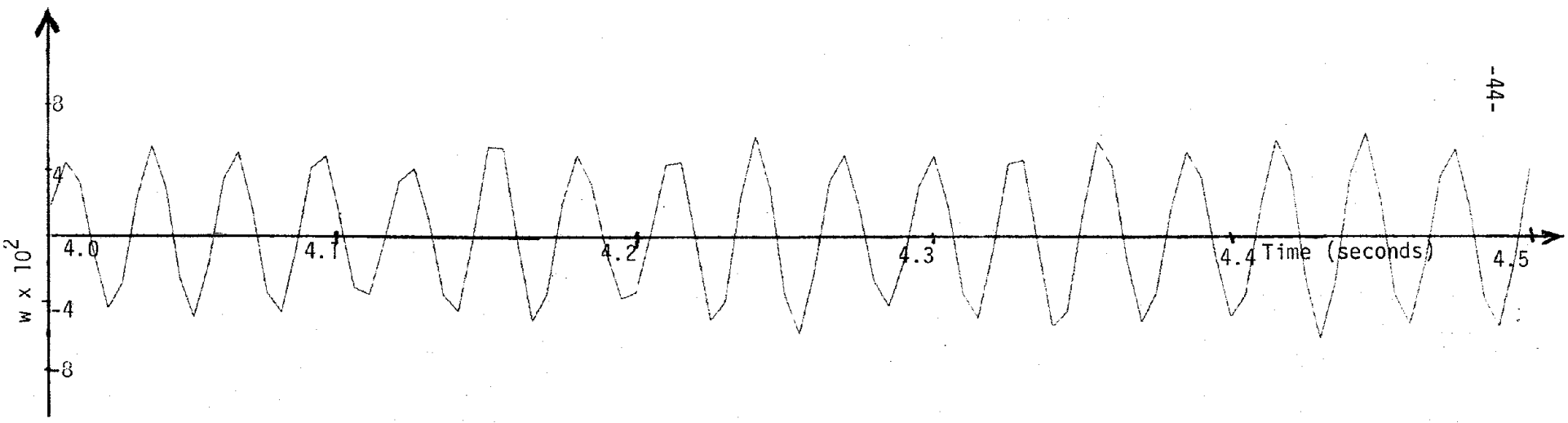


FIGURE 12

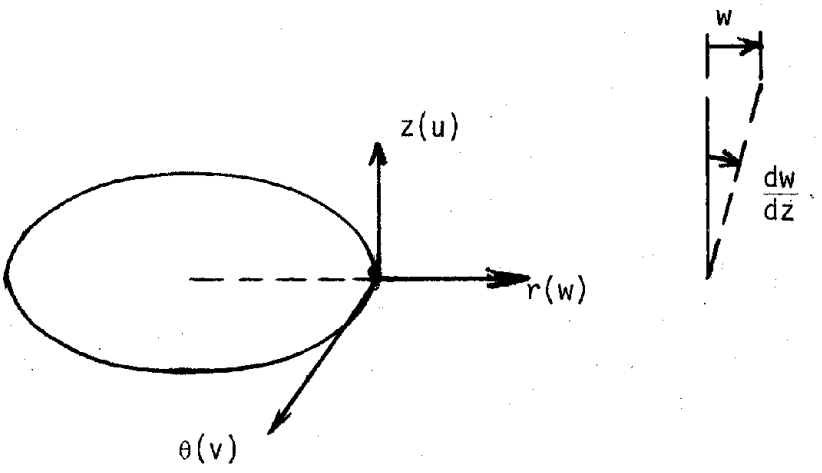
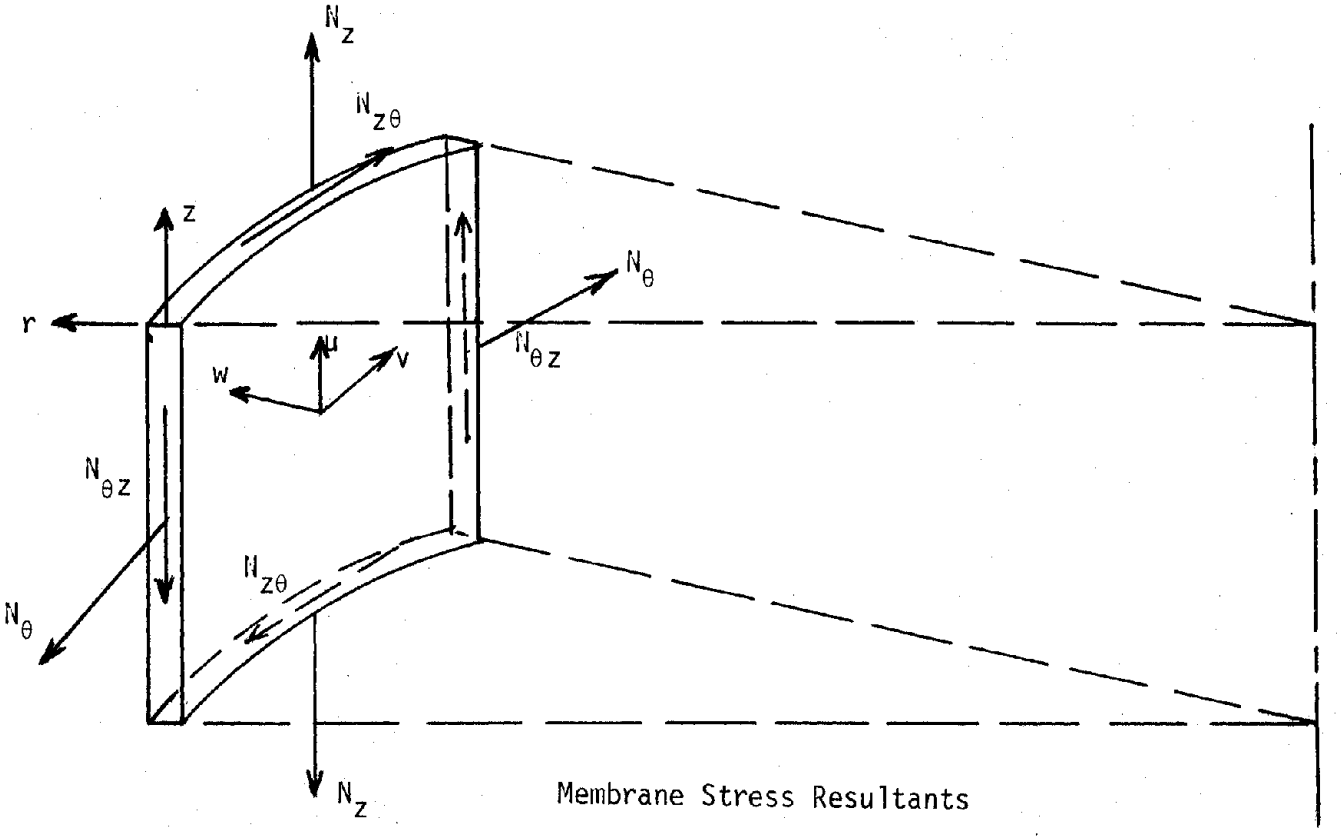
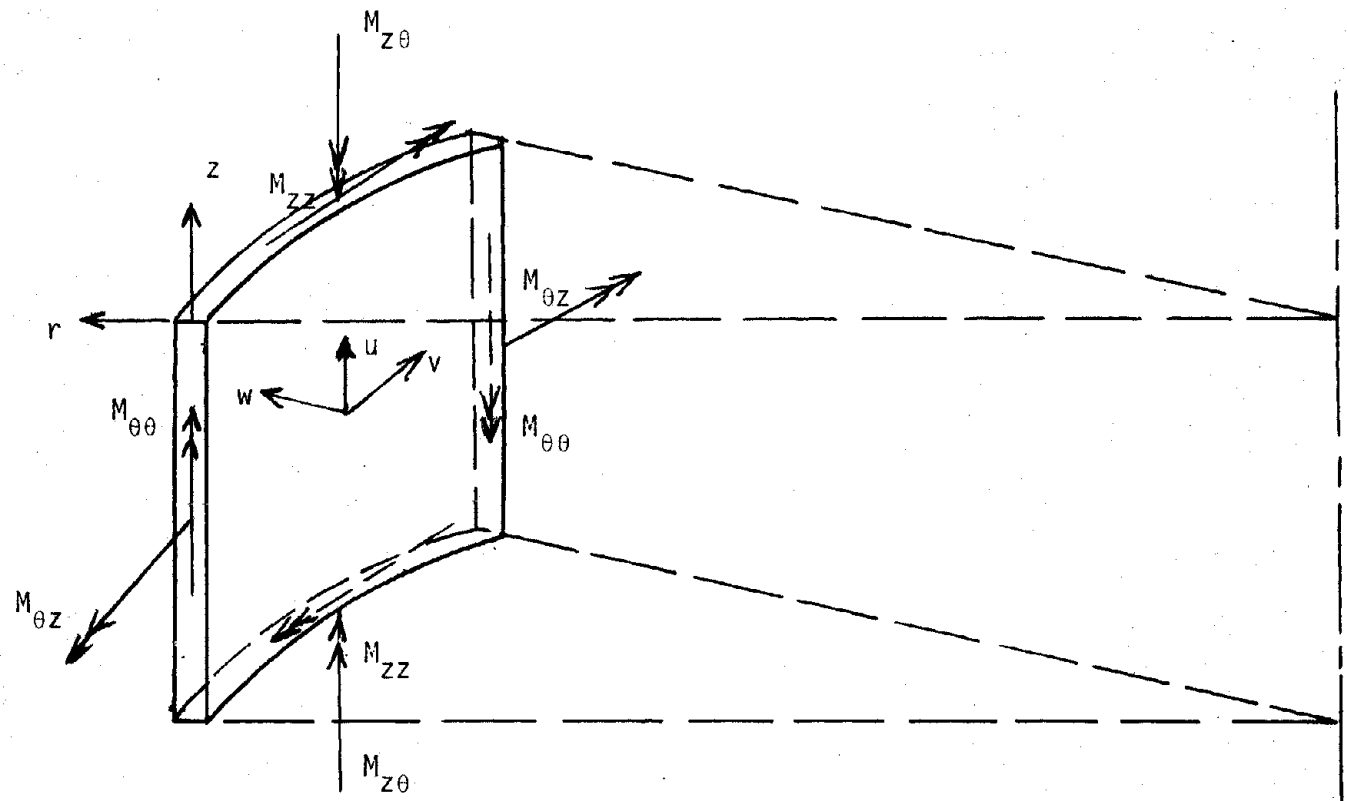


FIGURE 13



Membrane Stress Resultants

FIGURE 14



Bending and Torsional Stress Resultants

FIGURE 15

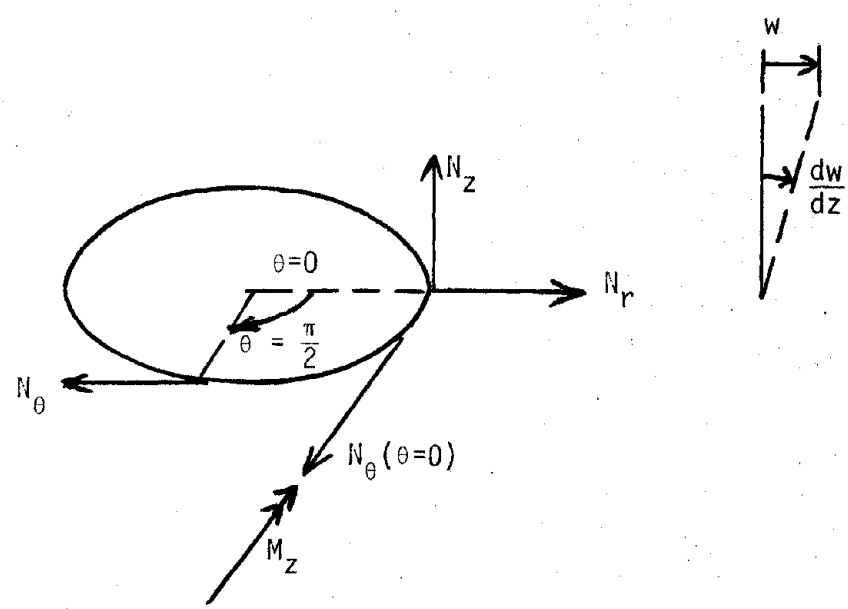


FIGURE 16

APPENDIX A

RESUME OF SANDERS' SHELL THEORY

The present investigation is based upon the shell theory due to J.L. Sanders, Jr. [7]. A brief resume of these equations for a thin, elastic, circular cylindrical shell follows.

Let us consider a right, circular cylindrical shell of radius a and thickness h. Let the quantities, r, θ, and z denote radial, circumferential, and axial coordinates respectively of a point on the shell middle surface. The corresponding displacement components are denoted by w, v, and u, as indicated in Figure 13. The force and moment resultants action on an element of the shell are depicted in Figures 14 and 15 respectively. The equilibrium equations in terms of these force resultants are:

$$
\begin{aligned}
a\frac{\partial N_z}{\partial z} + \frac{\partial \bar{N}}{\partial z} - \frac{1}{2a}\frac{\partial \bar{M}}{\partial \theta} &= -aq_z + a\rho_s h\ddot{u} \\
a\frac{\partial \bar{N}}{\partial z} + \frac{\partial N_\theta}{\partial \theta} + Q_\theta + \frac{1}{2}\frac{\partial \bar{M}}{\partial z} &= -aq_\theta + a\rho_s h\ddot{v} \\
N_\theta - a\frac{\partial Q_z}{\partial z} - \frac{\partial Q_\theta}{\partial \theta} &= aq_r - a\rho_s h\ddot{w} \qquad \text{(A-1)} \\
a\,Q_z &= a\frac{\partial M_z}{\partial z} + \frac{\partial \bar{M}}{\partial \theta} \\
a\,Q_\theta &= a\frac{\partial \bar{M}}{\partial z} + \frac{\partial M_\theta}{\partial \theta}
\end{aligned}
$$

where q_r, q_θ , and q_z are applied forces per unit area of the shell middle surface in the radial, tangential, and axial directions respectively; ρ_s denotes density of the shell material, and

$$
\begin{aligned}
\bar{N} &= \frac{1}{2}(N_{z\theta} + N_{\theta z}) + \frac{1}{4a}(M_{z\theta} - M_{\theta z}) \\
\bar{M} &= \frac{1}{2}(M_{z\theta} + M_{\theta z})
\end{aligned}
$$

By Sanders' theory, the curvatures are given by

$$k_z = -\frac{\partial^2 w}{\partial z^2}$$

$$k = -\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) \tag{A-2}$$

$$k_z = -\frac{2}{a}\frac{\partial^2 w}{\partial z \partial \theta} + \frac{3}{2a}\frac{\partial v}{\partial z} - \frac{1}{2a^2}\frac{\partial u}{\partial \theta}$$

and the strain-displacement relations by

$$\varepsilon_z = \frac{\partial u}{\partial z}$$

$$\varepsilon_\theta = \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + w\right) \tag{A-3}$$

$$\varepsilon_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{2}\frac{\partial u}{\partial \theta}$$

Accordingly, the force and moment resultants may be expressed in the forms

$$N_z = k[\varepsilon_z + \nu\varepsilon_\theta]$$

$$N_\theta = k[\varepsilon_\theta + \nu\varepsilon_z]$$

$$\overline{N} = k\left(\frac{1-\nu}{2}\right)\varepsilon_{z\theta} \tag{A-4}$$

$$M_z = D[k_z + \nu k_\theta]$$

$$M_\theta = D[k_\theta + \nu k_z]$$

$$\overline{M} = D\left(\frac{1-\nu}{2}\right)k_{z\theta}$$

where ν denotes Poisson's ratio and

$$k = \frac{Eh}{(1-\nu^2)} \qquad \text{(extensional rigidity)}$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad \text{(bending rigidity)}$$

Substitution of Equations (A-2), (A-3), and (A-4) in (A-1) yields the shell equations in terms of displacements:

$$\alpha^2 \frac{\partial^2 u}{\partial z^2} + \left(\frac{1-\nu}{2}\right)\left(1 + \frac{\alpha^2}{4}\right)\frac{\partial^2 u}{\partial \theta^2} + \frac{a}{2}\left[1 + \nu - \frac{3}{4}(1-\alpha)\alpha^2\right] \frac{\partial^2 v}{\partial\theta\partial z}$$

$$+ a\alpha \frac{\partial w}{\partial z} + \frac{a\alpha^2}{2}(1-\nu) \frac{\partial^2 w}{\partial\theta^2\partial z} = - \frac{a^2}{k} q_z + \frac{a^2}{k} \rho_s h\ddot{u}$$

$$\frac{a}{2}\left[1 + \nu - \frac{3}{4}(1-\nu)\alpha^2\right] \frac{\partial^2 u}{\partial\theta\partial z} + \frac{a^2(1-\nu)}{2}\left(1 + \frac{q}{4}\alpha^2\right) \frac{\partial^2 v}{\partial z^2}$$

$$+ (1+\alpha^2) \frac{\partial^2 v}{\partial\theta^2} + \frac{\partial w}{\partial\theta} - \alpha^2\left[\frac{\partial^3 w}{\partial\theta^3} + a^2\left(\frac{3}{2} - \frac{\nu}{2}\right)\right] \frac{\partial^3 w}{\partial z^2\partial\theta}$$

$$= - \frac{a^2}{k} q_\theta + \frac{a^2}{k} \rho_s h\ddot{v} \qquad \text{(A-5)}$$

$$a\nu \frac{\partial u}{\partial z} + \frac{a}{2}(1-\nu)\alpha^2 \frac{\partial^3 u}{\partial z\partial\theta^2} + \frac{\partial v}{\partial\theta} - \alpha^2\left[\frac{\partial^3 v}{\partial\theta^3} + a^2\left(\frac{3}{2} - \frac{\nu}{2}\right)\right] \frac{\partial^3 v}{\partial\theta\partial z^2}$$

$$+ w + \nabla^4 w = \frac{a^2}{k} q_r - \frac{a^2}{k} \rho_s h\ddot{w}$$

where

$$\alpha^2 = \frac{1}{12}\left(\frac{h}{a}\right)^2$$

$$\nabla^4 w = a^4 \frac{\partial^4 w}{\partial z^4} + 2a^2 \frac{\partial^4 w}{\partial z^2\partial\theta^2} + \frac{\partial^4 w}{\partial\theta^4}$$

In the finite element analysis carried out here, the significant forces and moments (per unit length of shell middle surface) that were investigated are indicated in Figure 16 in their positive directions. At the base of the cylindrical tank the external axial force must equal the internal force N_z and similarly for N_θ and M_z.

The equations (A-5) governing motion of the shell permit representation of the applied loads in the form

$$
\begin{aligned}
q_z &= \sum_m q_z^m (z,t) \cos m\theta \\
q_\theta &= \sum_m q_\theta^m (z,t) \sin m\theta \\
q_r &= \sum_m q_r^m (z,\theta) \cos m\theta
\end{aligned}
\qquad \text{(A-6)}
$$

while simultaneously the displacements may be expressed as

$$
\begin{aligned}
u &= \sum_m u_m(z,t) \cos m\theta \\
v &= \sum_m v_m (z,t) \sin m\theta \\
w &= \sum_m w_m (z,t) \cos m\theta
\end{aligned}
\qquad \text{(A-7)}
$$

It is to be noted the m-th harmonic of displacement is coupled only to the m-th harmonic of loading.

APPENDIX B

DEVELOPMENT OF MATRICES EMPLOYED IN FINITE ELEMENT ANALYSIS

For a specified harmonic m the displacement function vector $\{u\}$ assigned for the typical ring-shaped element is expressed, according to Sanders' theory for any point (z,θ) in terms of a generalized displacement vector $\{u_m\}$ (a function of z only) as follows

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \{u\} = [T'] \{u_m\} = \begin{bmatrix} \cos m\theta & 0 & 0 \\ 0 & \sin m\theta & 0 \\ 0 & 0 & \cos m\theta \end{bmatrix} \begin{Bmatrix} u_m \\ v_m \\ w_m \end{Bmatrix} \qquad \text{(B-1)}$$

where u_m and w_m are the axial and radial displacements at $\theta = 0°$ and v_m is evaluated at $\theta = \pi/2$.

The displacement function is also expressed in a polynomial form in terms of a generalized coordinate vector $\{A\}$ whose length equals the number of degrees of freedom per element:

$$\{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [P] \{A\} \qquad \text{(B-2)}$$

The nodal displacement vector $\{\delta\}$ which constitutes the number of degrees of freedom of the element can be employed to define the generalized coordinate vector $\{A\}$ by substituting in Equation (B-2) the nodal coordinates. Thus, a square matrix C involving the dimensions of the element results:

$$\{\delta\} = \left\{\begin{array}{c} \delta_u \\ \delta_v \\ \delta_w \\ \delta_{w'} \\ \hline \delta_u \\ \delta_v \\ \delta_w \\ \delta_{w'} \end{array}\right\} \begin{array}{c} \text{①} \\ \\ \\ \\ \\ \text{②} \\ \\ \\ \end{array} = [C]\{A\} \tag{B-3}$$

$$\therefore \qquad \{A\} = [C]^{-1}\{\delta\} \tag{B-4}$$

$$\{u\} = [P][C]^{-1}\{\delta\} = [N]\{\delta\} \tag{B-5}$$

where [N] is the shape function matrix relating the displacements at a general point on the element to the nodal displacements $\delta(\theta)$.

By Sanders' theory, the compatibility equations are introduced through the strain-displacement relations as follows:

$$\{\varepsilon\} = [\mathcal{L}]\{u\}$$

$$\left\{\begin{array}{c} \varepsilon_z \\ \varepsilon_\theta \\ \varepsilon_{z\theta} \\ K_z \\ K_\theta \\ K_{z\theta} \end{array}\right\} = [\mathcal{L}] \left\{\begin{array}{c} u \\ v \\ w \end{array}\right\} \tag{B-6}$$

where $\mathcal{L}$ is a differential operator matrix. Thus:

$$
\begin{aligned}
\{\varepsilon\} &= [\mathcal{L}]\{u\} \\
&= [\mathcal{L}][T]\{u_m\} \\
&= [\bar{\bar{T}}][\bar{\mathcal{L}}]\{u_m\} \\
&= [\bar{\bar{T}}][\bar{\mathcal{L}}][N]\{\delta_m\} \\
&= [\bar{\bar{T}}][B]\{\delta_m\}
\end{aligned}
\tag{B-7}
$$

where

$$[B] = [\bar{\mathcal{L}}][N] \tag{B-8}$$

The differential operator matrix involves differentiation with respect to both z and θ. Obviously the deformation matrix [T] involves functions of θ, i.e.

$$
[T] = \begin{bmatrix} \cos m\theta & & & \\ & \cos m\theta & & \\ & & \cos m\theta & \\ & & & \cos m\theta \end{bmatrix}
$$

The differentiations with respect to θ are performed on T and the result is a new differential operator matrix $[\bar{\mathcal{L}}]$ which involves differentiation with respect to z only.

Since [N], the shape function, is a function of z only it follows that B is in terms of z only and that was the purpose of the previous modification. These operations are shown in the following matrix notations: (see B matrix development in Appendix C):

$$[\mathcal{L}] = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{1}{a}\frac{\partial}{\partial \theta} & \frac{1}{a} \\ \frac{1}{a}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial z^2} \\ 0 & \frac{1}{a^2}\frac{\partial}{\partial \theta} & -\frac{1}{a^2}\frac{\partial^2}{\partial \theta^2} \\ -\frac{1}{2a^2}\frac{\partial}{\partial \theta} & \frac{3}{2a}\frac{\partial}{\partial z} & -\frac{2}{a}\frac{\partial^2}{\partial z \partial \theta} \end{bmatrix}$$

$$[T] = \left[\begin{array}{ccc|ccc} \cos m\theta & & & & & \\ & \sin m\theta & & & 0 & \\ & & \cos m\theta & & & \\ & & & \cos m\theta & & \\ \hline & 0 & & & \cos m\theta & \\ & & & & & \sin m\theta \\ & & & & & & \cos m\theta \\ & & & & & & & \cos m\theta \end{array}\right]$$

$$\therefore \qquad [\mathcal{L}][T] = [\bar{T}][\mathcal{L}]$$

$$
= \begin{bmatrix}
(\cos m\theta)\frac{\partial}{\partial z} & 0 & 0 \\
0 & (\cos m\theta)\frac{m}{a} & \frac{1}{a}(\cos m\theta) \\
\frac{m}{a}\sin m\theta & (\sin m\theta)\frac{\partial}{\partial z} & 0 \\
0 & 0 & -(\cos m\theta)\frac{\partial^2}{\partial z^2} \\
0 & \frac{m}{a^2}\cos m\theta & \frac{m^2}{a^2}\cos m\theta \\
\frac{m}{2a^2}\sin m\theta & \frac{1}{a}(\sin m\theta)\frac{\partial}{\partial z} & \frac{2m}{a}(\sin m\theta)\frac{\partial}{\partial z}
\end{bmatrix}
$$

$$
= \left[\begin{array}{ccc|ccc}
\cos m\theta & & & & & \\
 & \cos m\theta & & & & \\
 & & \sin m\theta & & & \\
\hline
 & & & \cos m\theta & & \\
 & & & & \cos m\theta & \\
 & & & & & \sin m\theta
\end{array}\right]
\begin{bmatrix}
\frac{\partial}{\partial z} & 0 & 0 \\
0 & \frac{m}{a} & \frac{1}{a} \\
-\frac{m}{a} & \frac{\partial}{\partial z} & 0 \\
0 & 0 & -\frac{\partial^2}{\partial z^2} \\
0 & \frac{m}{a^2} & \frac{m^2}{a^2} \\
-\frac{m}{2a^2} & \frac{3}{2a}\frac{\partial}{\partial z} & \frac{2m}{a}\frac{\partial}{\partial z}
\end{bmatrix}
$$

$$
= [\bar{\bar{T}}][\mathcal{L}]
$$

<u>Stress-Strain Relationships</u>

The stress vector is

$$\{\sigma\} = \begin{Bmatrix} N_z \\ N_\theta \\ N_{z\theta} \\ M_z \\ M_\theta \\ M_{z\theta} \end{Bmatrix} = [D]\{\varepsilon\} = \frac{Eh}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & & & & \\ \nu & 1 & & & & \\ & & \frac{1-\nu}{2} & & & \\ & & & \frac{h^2}{12} & \frac{\nu h^2}{12} & \\ & & & \frac{\nu h^2}{12} & \frac{h^2}{12} & \\ & & & & & \frac{h^2(1-\nu)}{24} \end{bmatrix} \begin{Bmatrix} \varepsilon_z \\ \varepsilon_\theta \\ \varepsilon_{z\theta} \\ K_z \\ K_\theta \\ K_{z\theta} \end{Bmatrix}$$

$$= [D][\bar{\bar{T}}][B]\{\delta_m\}$$

where D is the elastic coefficient matrix involving E, H, and ν, $\bar{\bar{T}}$ is a transformation matrix involving functions of θ only, and B is the linear strain-displacement relationship that involves functions of the lineal axial dimension z only. Thus, the stress-generalized nodal displacement (corresponding to a specific harmonic m) matrix is

$$[D][\bar{\bar{T}}][B]$$

<u>Derivation of the Element Stiffness Matrix K</u>

$$U = \frac{1}{2}\int_{vol} \{\varepsilon\}^T\{\sigma\}\, dv$$

$$= \frac{1}{2}\int_{vol} \{\varepsilon\}^T\{D\}\{\varepsilon\}\, dv$$

But since

$$\{\varepsilon\}^T = \{\delta_m\}^T[B]^T[\bar{\bar{T}}]^T$$

then

$$U = \frac{1}{2}\int_{vol} \{\delta_m\}^T[B]^T[\bar{\bar{T}}]^T[D][\bar{\bar{T}}][B]\{\delta_m\}\, dv$$

$$= \frac{1}{2} \{\delta_m\}^T \int_{vol} [B]^T [\bar{\bar{T}}][D][\bar{\bar{T}}][B] \, dv \{\delta_m\}$$

$$= \frac{1}{2} \{\delta_m\}^T [K] \{\delta_m\}$$

$$\therefore \quad [K] = \int_{vol} [B]^T [\bar{\bar{T}}][D][\bar{\bar{T}}][B] \, dv$$

$$= h \int_0^L \int_0^{2\pi} [B]^T [\bar{\bar{T}}]^T [D][\bar{\bar{T}}][B] a d\theta dz$$

$$= \pi h a \int_0^L [B]^T [D][B] dz$$

where L denotes shell length of the ring element.

In the computer program the element stiffness matrix was developed by using its elementary component matrices and through linear integration using the Gaussian integration method along the length of the element.

Derivation of the Element Mass Matrix M

The element mass matrix is developed by considering the expression for the element kinetic energy, viz:

$$V = \int_{vol} \{u\}^T \rho_s \frac{\partial^2}{\partial t^2} \{u\} \, dv$$

But

$$\{u\} = \{N\}\{\delta\} = [N][T']\{\delta_m\}$$

$$\therefore \quad V = \int_{vol} \rho_s \{\delta_m\}^T [T']^T [N]^T [N][T'] \frac{\partial^2}{\partial t^2} \{\delta_m\} \, dv$$

$$= \{\delta_m^T\} \rho_s \int [T']^T [N]^T [N][T'] \, dv \{\ddot{\delta}_m\}$$

$$= \{\delta_m\}^T [M] \{\ddot{\delta}_m\}$$

$$\therefore \quad [M] = \rho_s \int [T']^T [N]^T [N][T'] \, dv$$

$$= \rho_s h\pi \int_0^L [N^T][N]\, dz$$

The mass matrix element was evaluated by performing the matrix multiplication $[N]^T[N]$ and by carrying out the integration along the element length longhand. The final values are then substituded into the computer program in subroutine MASS. This is carried out in Appendix C.

APPENDIX C

DETAILED DEVELOPMENT OF MATRICES

Derivation of the B. Matrix

We have

$$\underset{6\times3}{[\bar{\mathcal{L}}]}\ \underset{3\times8}{[N]} = \underset{6\times8}{[B]}$$

$$\begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{m}{a} & \frac{1}{a} \\ -\frac{m}{a} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial z^2} \\ 0 & \frac{m}{a^2} & \frac{m^2}{2a} \\ \frac{m}{2a^2} & \frac{3}{2a}\frac{\partial}{\partial z} & \frac{2m}{a}\frac{\partial}{\partial z} \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1-\frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 & 0 \\ 0 & 1-\frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 \\ 0 & 0 & z-\frac{3z^2}{L^2}+\frac{2z^3}{L^3} & 2-\frac{2z^2}{L}+\frac{z^3}{L} & 0 & 0 & \frac{3z^2}{L^2}-\frac{2z^3}{L^3} & \frac{-z^2}{L}-\frac{z^3}{L^2} \end{bmatrix}$$

$$B_{11} = \frac{\partial}{\partial z}\left(1 - \frac{z}{L}\right) = \frac{-1}{L}$$

$$B_{12} = B_{13} = B_{14} = 0 \qquad B_{15} = \frac{1}{L}, \qquad B_{16} = B_{17} = B_{18} = 0$$

$$B_{21} = 0, \qquad B_{22} = \frac{m}{a}\left(1 - \frac{z}{L}\right), \qquad B_{23} = \frac{1}{a}\left(1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3}\right)$$

$$B_{24} = \frac{1}{a}\left(z - \frac{2z^2}{L} + \frac{z^3}{L^2}\right) \qquad B_{25} = 0$$

$$B_{26} = \frac{m}{a}\left(\frac{z}{L}\right) \qquad B_{27} = \frac{1}{a}\left(\frac{3z^2}{L^2} - \frac{2z^3}{L^3}\right)$$

$$B_{28} = \frac{1}{a}\left(\frac{-z^2}{L} + \frac{z^3}{L^2}\right)$$

$$B_{31} = -\frac{m}{a}\left(1 - \frac{z}{L}\right) \qquad B_{32} = -\frac{1}{L} \qquad B_{33} = B_{34} = 0$$

$$B_{34} = -\frac{m}{a}\frac{z}{L} \qquad B_{36} = \frac{1}{L} \qquad B_{37} = B_{38} = 0$$

$$B_{41} = 0, \qquad B_{42} = 0 \qquad B_{43} = -\frac{\partial^2}{\partial z^2}\left(1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3}\right) = \frac{6}{L^2} - \frac{12z}{L^3}$$

$$B_{44} = -\frac{\partial^2}{\partial z^2}\left(z - \frac{2z^2}{L} + \frac{z^3}{L^2}\right) = \frac{4}{L} - \frac{6z}{L^2}$$

$$B_{45} = B_{46} = 0 \qquad B_{47} = \frac{-\partial^2}{\partial z^2}\left(\frac{3z^2}{L^2} - \frac{2z^3}{L^3}\right) = -\frac{6}{L^2} + \frac{12z}{L^3}$$

$$B_{48} = \frac{-\partial^2}{\partial z^2}\left(\frac{-z^2}{L} + \frac{z^3}{L^2}\right) = \frac{z}{L} - \frac{6z}{L}$$

$$B_{51} = 0 \qquad B_{52} = \frac{m}{a^2}\left(1 - \frac{z}{L}\right) \qquad B_{53} = \frac{m^2}{a^2}\left(1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3}\right)$$

$$B_{54} = \frac{m^2}{a^2}\left(2 - \frac{2z^2}{L} + \frac{z^3}{L^2}\right) \qquad B_{55} = 0 \qquad B_{56} = \frac{m}{a^2}\frac{z}{L}$$

$$B_{57} = \frac{m^2}{a^2}\left(3\frac{z^2}{L^2} - \frac{2z^3}{L^3}\right), \qquad B_{58} = \frac{m^2}{a^2}\left(\frac{-z^2}{L} + \frac{z^3}{L^2}\right)$$

$$B_{61} = \frac{m}{2a^2}\left(1 - \frac{z}{L}\right) \qquad B_{62} = \frac{3}{2a}\left(-\frac{1}{L}\right)$$

$$B_{63} = \frac{2m}{a}\left(-\frac{6z}{L^2} + \frac{6z^2}{L^3}\right) \qquad B_{65} = \frac{m}{2a^2}\left(\frac{z}{L}\right) \qquad B_{66} = \frac{3}{2a}\left(\frac{1}{L}\right)$$

$$B_{64} = \frac{2m}{a}\left(1 - \frac{4z}{L} + \frac{3z^2}{L^2}\right) \qquad B_{67} = \frac{2m}{a}\left(\frac{6z}{L^2} - \frac{6z^2}{L^3}\right)$$

$$B_{68} = \frac{2m}{a}\left(\frac{-2z}{L} + \frac{3z^2}{L^2}\right)$$

<u>Derivation of the Element Shape Function Matrix $N = Pc^{-1}$</u>

$$[P] = \begin{bmatrix} 1 & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z & z^2 & z^3 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2L & 2L^2 \end{bmatrix}$$

$$[C]^{-1} = \frac{1}{L^3} \begin{bmatrix} L & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -L^2 & 0 & 0 & 0 & L^2 & 0 & 0 & 0 \\ 0 & L^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -L^2 & 0 & 0 & 0 & L^2 & 0 & 0 \\ 0 & 0 & L^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3L & -2L^2 & 0 & 0 & 3L & -L^2 \\ 0 & 0 & 2 & L & 0 & 0 & -2 & L \end{bmatrix}$$

$$[N] = [P][C]^{-1}$$

$$N_{11} = 1 - \frac{z}{L}$$

$$N_{12} = N_{13} = N_{14} = 0 \qquad N_{15} = \frac{z}{L}$$

$$N_{16} = 0 = N_{17} = N_{18}$$

$$N_{12} = 0 \qquad N_{13} = N_{14} = 0$$

$$N_{15} = \frac{z}{L} \qquad N_{16} = 0 \qquad N_{17} = N_{18} = 0$$

$$N_{21} = 0 \qquad N_{22} = 1 - \frac{z}{L}$$

$$N_{23} = 0 \qquad N_{24} = N_{25} = 0 \qquad N_{26} = \frac{z}{L}$$

$$N_{27} = 0 \qquad N_{28} = 0$$

$$N_{31} = 0 \qquad N_{32} = 0 \qquad N_{33} = 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3}$$

$$N_{34} = z - \frac{2z^2}{L} + \frac{z^3}{L^2}$$

$$N_{35} = N_{36} = 0$$

$$N_{37} = \frac{3z^2}{L^2} - \frac{2z^3}{L^3}$$

$$N_{38} = \frac{-z^2}{L} + \frac{z^3}{L^2}$$

$$[N] = \begin{bmatrix} 1 - \frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 & 0 \\ 0 & 1 - \frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 \\ 0 & 0 & 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3} & 2 - \frac{2z^2}{L} + \frac{z^3}{L^2} & 0 & 0 & \frac{3z^2}{L^2} - \frac{2z^3}{L^3} & \frac{-z^2}{L} + \frac{z^3}{L^2} \end{bmatrix}$$

Derivation of the Product Matrix $[N]^* = [N^T][N]$

$$[N]^* = \begin{bmatrix} 1 - \frac{z}{L} & 0 & 0 \\ 0 & 1 - \frac{z}{L} & 0 \\ 0 & 0 & 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3} \\ 0 & 0 & 2 - \frac{2z^2}{L} + \frac{z^3}{L^2} \\ \frac{z}{L} & 0 & 0 \\ 0 & \frac{z}{L} & 0 \\ 0 & 0 & \frac{3z^2}{L^2} - \frac{2z^3}{L^3} \\ 0 & 0 & \frac{-z^2}{L} - \frac{z^3}{L^2} \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 - \frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 & 0 \\ 0 & 1 - \frac{z}{L} & 0 & 0 & 0 & \frac{z}{L} & 0 & 0 \\ 0 & 0 & 1-\frac{3z^2}{L^2}+\frac{2z^3}{L^3} & 2-\frac{2z^2}{L^2}+\frac{z^3}{L^2} & 0 & 0 & \frac{3z^2}{L^2} - \frac{2z^3}{L^3} & \frac{-z^2}{L} + \frac{z^3}{L^2} \end{bmatrix}$$

$N^*_{11} = (1 - \frac{z}{L})^2$ $\quad N^*_{12} = N^*_{13} = N^*_{14} = 0$

$N^*_{15} = (1 - \frac{z}{L})(\frac{z}{L})$ $\quad N^*_{16} = N^*_{17} = N^*_{18} = 0$

$N^*_{21} = 0$ $\quad N^*_{22} = (1 - \frac{z}{L})^2$ $\quad N^*_{23} = 0$ $\quad N^*_{24} = 0$

$N^*_{25} = 0$ $\quad N^*_{26} = \frac{z}{L}(1 - \frac{z}{L})$ $\quad N^*_{27} = N_{28} = 0$

$N^*_{31} = N^*_{32} = 0$ $\quad N^*_{33} = 1 - \frac{6z^2}{L^2} + \frac{4z^3}{L^3} + \frac{9z^4}{L^4} - \frac{12z^5}{L^5} + \frac{4z^6}{L^6}$

$N^*_{34} = 2 - \frac{2z^2}{L} - \frac{2z^3}{L^2} + \frac{9z^4}{L^3} - \frac{7z^5}{L^4} + \frac{2z^6}{L^5}$

$N^*_{44} = z^2 - \frac{4z^3}{L} + \frac{6z^4}{L^2} - \frac{4z^5}{L^3} + \frac{z^6}{L^4}$

$$N^*_{45} = N_{46} = 0$$

$$N^*_{47} = \frac{3z^3}{L^2} - \frac{8z^4}{L^3} + \frac{7z^5}{L^4} - \frac{2z^6}{L^5}$$

$$N^*_{48} = \frac{-z^3}{L} + \frac{3z^4}{L^2} - \frac{3z^5}{L^3} + \frac{z^6}{L^4}$$

$$N^*_{51} = \frac{z}{L}(1 - \frac{z}{L}) \qquad N^*_{52} = N^*_{53} = N^*_{54} = 0$$

$$N^*_{55} = (\frac{z}{L})^2 \qquad N^*_{56} = N^*_{57} = N^*_{58} = 0$$

$$N^*_{61} = 0 \qquad N^*_{62} = \frac{z}{L}(1 - \frac{z}{L}) \qquad N^*_{63} = N_{64} = 0$$

$$N^*_{65} = 0 \qquad N^*_{66} = (\frac{z}{L})^2$$

$$N^*_{71} = N^*_{72} = 0 \qquad N^*_{73} = \frac{3z^2}{L^2} - \frac{2z^3}{L^3} - \frac{9z^4}{L^4} + \frac{12z^5}{L^5} - \frac{4z^6}{L^6}$$

$$N^*_{78} = \frac{-3z^4}{L^3} + \frac{5z^5}{L^4} - \frac{2z^6}{L^5}$$

$$N^*_{35} = N_{36} = 0$$

$$N^*_{37} = \frac{3z^2}{L^2} - \frac{2z^3}{L^3} - \frac{9z^4}{L^4} - \frac{12z^5}{L^5} - \frac{4z^6}{L^6}$$

$$N^*_{38} = \frac{-z^2}{L} + \frac{z^3}{L^2} + \frac{3z^4}{L^3} - \frac{5z^5}{L^4} + \frac{2z^6}{L^5}$$

$$N^*_{74} = \frac{3z^3}{L^2} = \frac{8z^4}{L^3} + \frac{7z^5}{L^4} - \frac{2z^6}{L^5}$$

$$N^*_{75} = N^*_{76} = 0$$

$$N^*_{77} = \frac{9z^4}{L^4} - \frac{12z^5}{L^5} + \frac{4z^6}{L^6}$$

$$N^*_{78} = \frac{-3z^4}{L^3} + \frac{5z^5}{L^4} - \frac{2z^6}{L^5}$$

$$N^*_{81} = N^*_{82} = 0$$

$$N^*_{83} = -\frac{z^2}{L} + \frac{z^3}{L^2} + \frac{3z^4}{L^3} - \frac{5z^5}{L^4} + \frac{2z^6}{L^5}$$

$$N^*_{84} = \frac{-z^3}{L} + \frac{3z^4}{L^2} - \frac{3z^5}{L^3} + \frac{z^6}{L^4}$$

$$N^*_{85} = N^*_{86} = 0$$

$$N^*_{88} = \frac{z^4}{L^2} - \frac{2z^5}{L^3} + \frac{z^6}{L^4}$$

<u>Development of the Integral Matrix $\int_0^L N^7\, dz$</u>

$$\int_0^L N^*_{11}\, dz = \int_0^L (1 - \frac{2z}{L} + \frac{z^2}{L^2})dz = (z - \frac{z^2}{L} + \frac{z^3}{3L^2}) = L(1 - 1 + \frac{1}{3}) = \frac{L}{3}$$

$$\int_0^L N^*_{15}\, dz = \int_0^L (\frac{z}{L} - \frac{z^2}{L^2})dz = (\frac{z^2}{2L} - \frac{z^3}{3L^2})_0^L = L(\frac{1}{2} - \frac{1}{3}) = \frac{L}{6}$$

$$\int_0^L N^*_{22}\, dz = \int_0^L (1 - \frac{z}{L})^2 = \frac{L}{3} = \int N^*_{11}\, dz$$

$$\int_0^L N^*_{26}\, dz = \frac{L}{6} = \int N^*_{15}\, dz$$

$$\int_0^L N^*_{33}\, dz = \int_0^L (1 - \frac{6z^2}{L^2} + \frac{4z^3}{L^3} + \frac{9z^4}{L^4} - \frac{12z^5}{L^5} + \frac{4z^6}{L^6})\, dz$$

$$= L(1 - \frac{6}{3} + 1 + \frac{9}{5} - \frac{12}{6} + \frac{4}{7})$$

$$\int_0^L N^*_{37}\, dz = \int_0^L (\frac{3z^2}{L^2} - \frac{2z^3}{L^3} - \frac{9z^4}{L^4} + \frac{12z^5}{L^5} - \frac{4z^6}{L^6})\, dz$$

$$= L(1 - \frac{2}{4} - \frac{9}{5} + \frac{12}{6} - \frac{4}{7})$$

$$\int_0^L N^*_{38}\, dz = \int_0^L (\frac{-z^2}{L} + \frac{z^3}{L^2} + \frac{3z^4}{L^3} - \frac{5z^5}{L^4} + \frac{2z^6}{L^5})\, dz$$

$$= L^2(-\frac{1}{3} - \frac{1}{4} + \frac{3}{5} - \frac{5}{6} + \frac{2}{7})$$

$$\int_0^L N_{43}^* \, dz = \int_0^L (z - \frac{2z^2}{L} - \frac{2z^3}{L^2} + \frac{8z^4}{L^3} - \frac{7z^5}{L^4} + \frac{2z^6}{L^6}) \, dz$$

$$= L^2(\frac{1}{2} - \frac{2}{3} - \frac{2}{4} + \frac{8}{5} - \frac{7}{6} + \frac{2}{7})$$

$$\int_0^L N_{44}^* \, dz = \int_0^L (z^2 - \frac{4z^3}{L} + \frac{6z^4}{L^2} = \frac{4z^5}{L^3} + \frac{z^6}{L^4})$$

$$= L^3(\frac{1}{3} - \frac{4}{4} + \frac{6}{5} - \frac{4}{6} + \frac{1}{7})$$

$$\int_0^L N_{47}^* \, dz = \int_0^L (\frac{3z^2}{L^2} - \frac{8z^4}{L^3} + \frac{7z^5}{L^4} = \frac{2z^6}{L^5}) \, dz$$

$$= L(\frac{3}{3} - \frac{8}{5} + \frac{7}{6} - \frac{2}{7})$$

$$\int_0^L N_{48}^* \, dz = \int_0^L (\frac{-z^3}{L} + \frac{3z^4}{L^2} - \frac{3z^5}{L^3} + \frac{z^6}{L^4})$$

$$= L^3(\frac{-1}{4} + \frac{3}{5} - \frac{3}{6} + \frac{1}{7})$$

$$\int_0^L N_{51}^* \, dz = \int_0^L (\frac{z}{L} - \frac{z^2}{L^2}) \, dz = L(\frac{1}{2} - \frac{1}{3}) = \frac{L}{6}$$

$$\int_0^L N_{55}^* = \int_0^L \frac{z^2}{L^2} \, dz = \frac{L}{3}$$

$$\int_0^L N_{62}^* = \int_0^L \frac{z}{L} - \frac{z^2}{L^2} = \frac{L}{6} \qquad \int_0^L N_{66}^* \, dz = \frac{L}{3}$$

$$\int_0^L N_{73}^* =$$

$$\int_0^L N_{77}^* \, dz = \int_0^L \left(\frac{9z^4}{L^4} - \frac{12z^5}{L^5} + \frac{4z^6}{L^6}\right) dz = L\left(\frac{9}{5} - \frac{12}{6} + \frac{4}{7}\right)$$

$$\int_0^L N_{78}^* \, dz = \int_0^L \left(\frac{-3z^4}{L^3} + \frac{5z^5}{L^4} - \frac{2z^6}{L^5}\right) dz = L^3\left(\frac{-3}{5} + \frac{5}{6} - \frac{2}{7}\right)$$

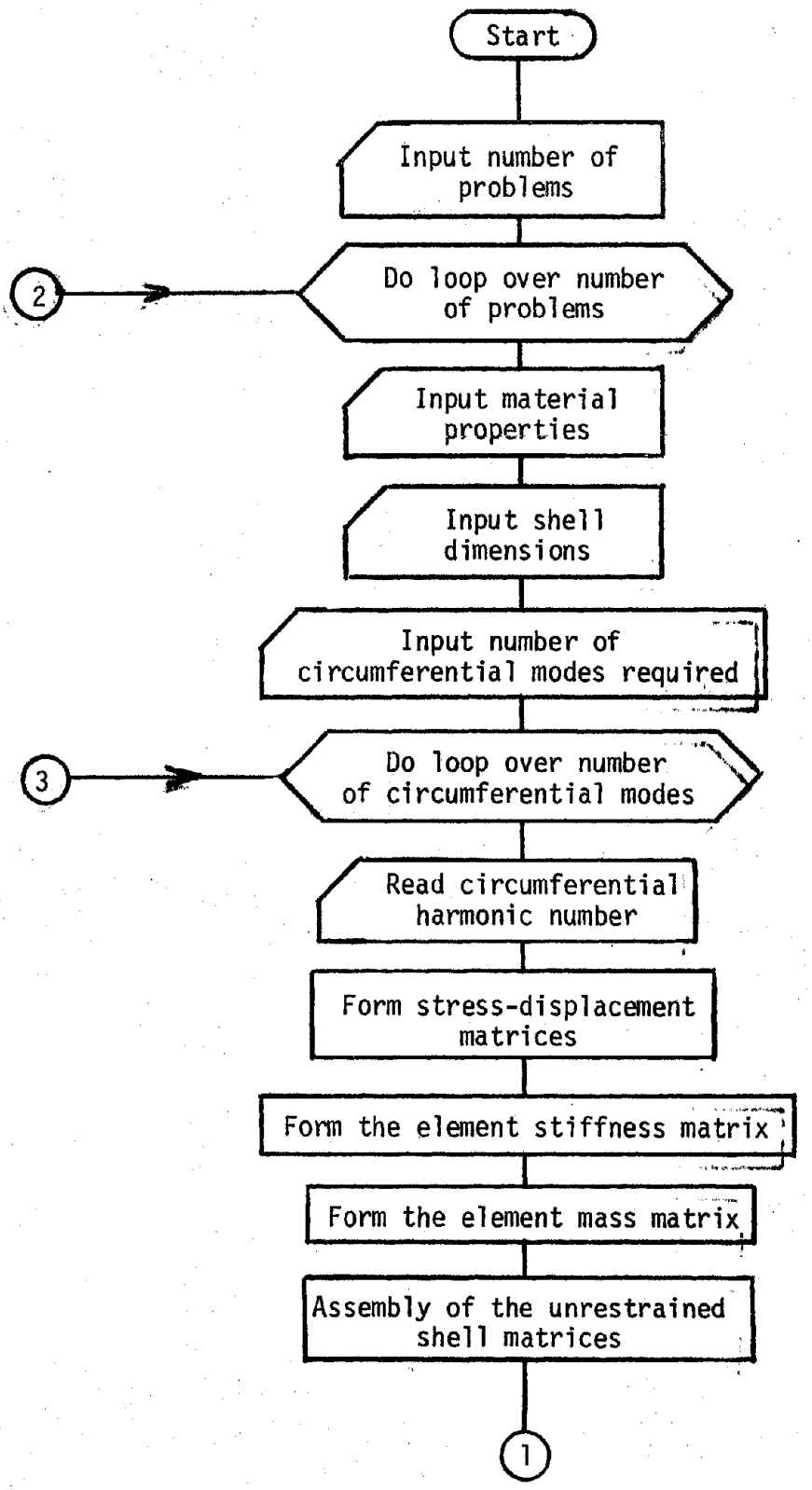
$$\int_0^L N_{88}^* = \int_0^L \left(\frac{z^4}{L^2} - \frac{2z^5}{L^3} + \frac{z^6}{L^4}\right) dz = L^3\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right)$$

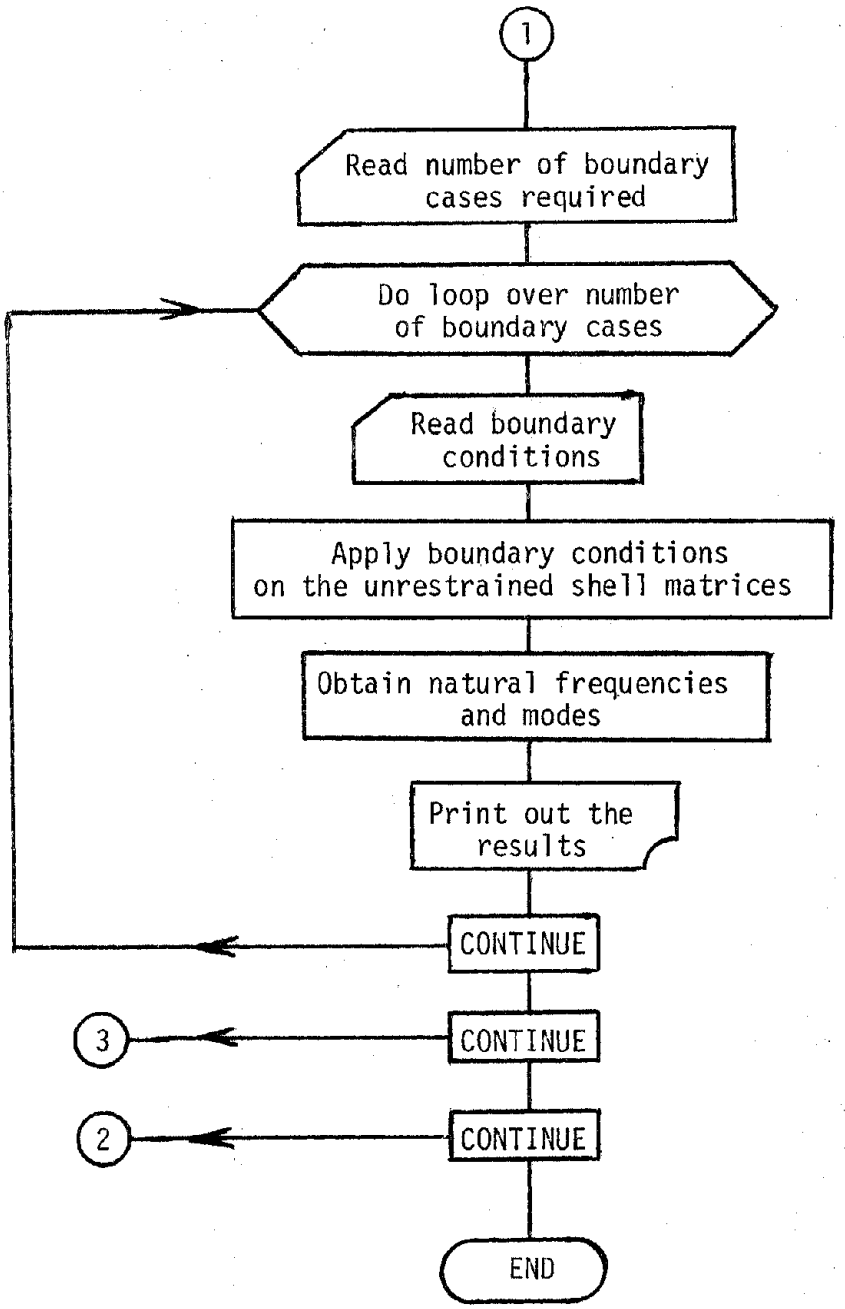
$$\int_0^L N^* dz = \begin{bmatrix} \frac{L}{3} & 0 & 0 & 0 & \frac{L}{6} & 0 & 0 & 0 \\ 0 & \frac{L}{3} & 0 & 0 & 0 & \frac{L}{6} & 0 & 0 \\ 0 & 0 & \frac{78L}{210} & \frac{11L^2}{210} & 0 & 0 & \frac{54L}{420} & \frac{-13L^2}{420} \\ 0 & 0 & \frac{11L^2}{210} & \frac{2L^3}{210} & 0 & 0 & \frac{13L^2}{420} & \frac{-3L^3}{420} \\ \frac{L}{6} & 0 & 0 & 0 & \frac{L}{3} & 0 & 0 & 0 \\ 0 & \frac{L}{6} & 0 & 0 & 0 & \frac{L}{3} & 0 & 0 \\ 0 & 0 & \frac{54L}{420} & \frac{13L^2}{420} & 0 & 0 & \frac{78L}{210} & \frac{-11L^2}{210} \\ 0 & 0 & \frac{-13L^2}{420} & \frac{-3L^2}{420} & 0 & 0 & \frac{-11L^2}{210} & \frac{2L^3}{210} \end{bmatrix}$$

APPENDIX D

FLOW CHART AND PROGRAM LISTINGS

FLOW CHART OF THE FIRST MAIN PROGRAM




1
Read number of boundary cases required
Do loop over number of boundary cases
Read boundary conditions
Apply boundary conditions on the unrestrained shell matrices
Obtain natural frequencies and modes
Print out the results
CONTINUE
3
CONTINUE
2
CONTINUE
END

```
 GET(TAPE5=GASM)
FTN.
LGO.
SAVE(TAPE10=MODES)
  SAVE(TAPE9=MODAL)
```

```
      PROGRAM MAIN(INPUT,OUTPUT,TAPE4,TAPE5,TAPE7,TAPE10,TAPE9,TAPE20)
   20 DIMENSION D(40,41)
   30 DIMENSION ST(8,8),AMAS(8,8)
   70 DIMENSION OMEG(5)
  701 FORMAT(10I8)
  702 FORMAT(8G10.4)
  704 READ 701,NPROB
  110    DO 41 LLK=1,NPROB
      PRINT 1
    1 FORMAT(1H1)
  111 READ 702,UM,E1,PX
      PRINT 772
  772 FORMAT(//,10X,* MATERIAL PROPERTIES*,/)
      PRINT 703,UM,E1,PX
  703 FORMAT(//,5X,* DENSITY OF SHELL MATERIAL=*,G10.4
     *,//,5X,* MODULUS OF ELASTICITY=*,G10.4
     *,//,5X,* POISSON RATIO=*,G10.4)
  182 PRINT 771
  112 READ 702,R,H,AL
  771 FORMAT(///,10X,*STRUCTURAL GEOMETRY*,/)
  170 PRINT 7,R,H,AL
    7 FORMAT(2X,*RADIUS=*,F9.3,5X,*THICKNESS=*,F6.3,5X,*HEIGHT=*,F9.3)
  113 READ 701,NSIN
  114 READ 701,NELEM
      NP=NELEM+1
      NDF=4
            NFREE=NDF*NP
  230   ELN=AL/FLOAT(NELEM)
C       NDF=NUMBER OF DEGREE OF FREEDOM PER NODE
C       NBAND=HALF BAND WIDTH
C       NFREE=NUMBER OF DEGREES OF FREEDOM
  401 READ 701,NMODE
      PRINT 43,NELEM
   43 FORMAT(//,5X,* NO. OF RING ELEMENTS=*,I3)
      PRINT 51,NMODE,NSIN
   51 FORMAT(//,5X,I2,* AXIAL MODES TO BE CONSIDERED FOR*,I2,
     G* CIRCUMFERENTIAL NO.S*,/)
  430 DO 59 KS1=1,NSIN
  431 READ 701,NAT
  450 ANT=NAT
      CALL STRMAT(ELN,R,H,ANT,PX,E1)
  501 READ 701,NBCAS
      PRINT 52,NBCAS
   52 FORMAT(//,5X,* NO. OF BOUNDARY CASES CONSIDERED=*,I2)
  502 DO 808 IPR=1,NBCAS
  504 IF(IPR .GT. 1) GO TO 801
  520 CALL ASSTR(NELEM,R,UM,ANT,PX,R,ELN,D,E1)
      NFREE1=NFREE+1
      WRITE(4)(NFREE)
  570 WRITE(4)((D(I,J),J=1,NFREE1 ),I=1,NFREE)
      REWIND 4
  801 CONTINUE
  990 READ 911,NBC
  911 FORMAT(A4)
  620 CALL BOUN(NFREE,NAT,D,NO,NBC)
      NO1=NO+1
  653 WRITE(7)((D(I,J),J=1,NO1 ),I=1,NO)
      REWIND 7
  691 DO 91 I=1,40
  692 DO 91 J=1,41
   91 D(I,J)=0.
  700 CALL EGN(D,NO,NMODE,E1,NBC)
```

```
808   CONTINUE
 59   CONTINUE
 41   CONTINUE
790   END
```

```
      SUBROUTINE STPMAT(AL,R,H,ANT,P,E)
      DIMENSION B(8,8),DM(8,8),DBT(6,8,6)
    4 FORMAT(///,10X,*STRESS-DISPLACEMENTS MATRICES*,/)
      PRINT 4
      HPI=1.570795
      HAL=AL/2.
      CALL DMATX(H,P,DM)
      DO 5 I=1,6
      DO 5 J=1,6
    5 DM(I,J)=DM(I,J)*E*H/(1.-P*P)
    3 FORMAT(6(5X,F10.4))
       DO 100 NN=1,6
       IF(NN .LT. 4) THETA=0.0
       IF(NN .GE. 4) THETA=HPI
      IF(NN .EQ. 1) X=0.0
      IF(NN .EQ. 2) X=AL
      IF(NN .EQ. 3) X=HAL
      IF(NN .EQ. 4) X=0.0
      IF(NN .EQ. 5) X=AL
      IF(NN .EQ. 6) X=HAL
      THETAM=THETA*ANT
      CALL BMATX(AL,R,ANT,X,B)
      COSIN=COS(THETAM)
      SINE=SIN(THETAM)
*     T- *B
      DO 10 J=1,8
      B(1,J)=B(1,J)*COSIN
      B(2,J)=B(2,J)*COSIN
      B(3,J)=B(3,J)*SINE
      B(4,J)=B(4,J)*COSIN
      B(5,J)=B(5,J)*COSIN
   10 B(6,J)=B(6,J)*SINE
*     D*T-*B
      DO 20 I=1,6
      DO 20 J=1,8
      DBT(I,J,NN)=0.0
      DO 20 K=1,6
   20 DBT(I,J,NN)=DBT(I,J,NN)+DM(I,K)*B(K,J)

      WRITE(20)(NN)
      PRINT 1,NN
      WRITE(20)((DBT(I,J,NN),J=1,8),I=1,6)
      PRINT 2,((DBT(I,J,NN),J=1,8),I=1,6)
    1 FORMAT(//,10X,*NN=*,I6,/)
    2 FORMAT(8(5X,F10.4))
  100 CONTINUE
      RETURN
      END
```

```
800   SUBROUTINE BOUN(NFREE,NAT,D,NO,NBC)
805   DIMENSION D(40,41)
      READ 911,NBC
911   FORMAT(A4)
815   IF(NBC .EQ. 4HCLFR ) GO TO 1
820   IF(NBC .EQ. 4HCLCL)  GO TO 2
825   IF(NBC .EQ. 4HCLSM ) GO TO 3
830   IF(NBC .EQ. 4HSMSM) GO TO 4
 55   FORMAT(1H1)
  1   PRINT 55
      PRINT 11
 11   FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-FREE CYL*)
845   PRINT 101,NAT
101   FORMAT(/, *  FOR CIRCUMFERENTIAL HARM. M=*,I3,/)
855   NO=NFREE-4
860   GO TO 33
  2   PRINT 55
      PRINT 12
 12   FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-CL CYL.*)
875   PRINT 101,NAT
880   NO=NFREE-8
 33   DO 777 I=1,NO
      NO1=NO+1
      DO 777 J=1,NO1
777   D(I,J)=D(I+4,J+4)
900   RETURN
  3   PRINT 55
      PRINT 13
 13   FORMAT(//,*NATURAL MODES AND FREQ. FOR A CL-SIMPLE CYL.*)
915   PRINT 101,NAT
920   NO=NFREE-7
      NO1=NO-1
      DO 111 I=1,NO1
930   DO 112 J=1,NO
112   D(I,J)=D(I+4,J+4)
111   D(I,NO+1)=D(I+4,NFREE+1)
945   I=NO
950   DO 113 J=1,NO
113   D(I,J)=D(NFREE,J+4)
960   D(NO,NO+1)=D(NFREE,NFREE+1)
      D(NO,NO)=D(NFREE,NFREE)
965   RETURN
  4   PRINT 55
      PRINT 14
 14   FORMAT(//,*NATURAL MODES AND FREQ. FOR A SIMPLE SIMPLE CYL.*)
980   PRINT 101,NAT
985   NO=NFREE-6
      NO2=NO-1
      DO 222 I=1,NO2
995   DO 221 J=1,NO
221   D(I,J)=D(I+3,J+3)
222   D(I,NO+1)=D(I+3,NFREE+1)
1010  I=NO
1015  DO 223 J=1,NO
223   D(I,J)=D(NFREE,J+3)
1025  D(NO,NO+1)=D(NFREE,NFREE+1)
      D(NO,NO)=D(NFREE,NFREE)
1030  RETURN
1035  END
```

```
1170   SUBROUTINE ASSTR(NELEM,H,UM,ANT,PX,R,XB,D,F)
1172   DIMENSION D(40,41)
1175   DIMENSION AMAS(8,8),ST(8,8)
1180   NBAND=8
  54   FORMAT(//,10X,*-----------------------*,/)
1310   CALL STIFF(H,XB,ANT,PX,R,ST)
       DO 40 I=1,8
       DO 40 J=1,8
  40   ST(I,J)=ST(I,J)*F
  10   FORMAT(8(5X,G10.4))
1320   CALL MASS(UM,R,XB,H,AMAS)
1324   DO 100 I=1,NELEM
1325   IN=(I-1)*4
1330   DO 20 II=1,NBAND
1340   DO 20 JJ=1,II
1350   K=IN+II
1360   L=IN+JJ
  20   D(K,L)=D(K,L)+AMAS(II,JJ)
1380   DO 30 JJ=1,NBAND
1390   DO 30 II=1,JJ
1400   K=IN+II
1410   L=IN+JJ+1
  30   D(K,L)=D(K,L)+ST(II,JJ)
 100   CONTINUE
1440   RETURN
1450   END
```

```
 2860  SUBROUTINE MASS(RHO,R,AL,H,A)
 2870   DIMENSION A(8,8)
C       INITIALIZE MASS MATRIX
 2890   DO 116 J=1,8
 2900   DO 116 I=1,8
 2910   A(I,J)=0.
  116   CONTINUE
C       CONSTRUCT MASS MATRIX
            PI=3.1415927
 2950  CONST=R*PI*RHO*H
            A(1,1)=A(2,2)=A(5,5)=A(6,6)=CONST*AL/3.
            A(5,1)=A(1,5)=A(6,2)=A(2,6)=CONST*AL/6.
            A(3,3)=A(7,7)=CONST*13.*AL/35.
 2990   A(4,3)=A(3,4)=CONST*11.*AL**2/210.
 3000   A(7,8)=A(8,7)=-A(4,3)
            A(4,4)=A(8,8)=CONST*AL**3/105.
            A(7,3)=A(3,7)=CONST*AL*9./70.
 3030  A(4,7)=A(7,4)=CONST*13.*AL*AL/420.
            A(8,3)=A(3,8)=-CONST*13.*AL**2/420.
            A(8,4)=A(4,8)=-CONST*AL**3/140.
   54  FORMAT(//,10X,*-----------------------------*,/)
 3068  RETURN
            END
```

```
3100   SUBROUTINE STIFF(H,AL,AM,P,R,SUM)
3110   DIMENSION X(20),W(20),DM(8,8),BB(8,8),DB(8,8),BD(8,8)
     X ,SUM(8,8)
3140   READ(5) NI
3150    DO 21 I=1,NI
  21   READ(5) X(I),W(I)
       REWIND 5
3170    A=0. $ B=AL
3180    DO 12 I=1,NI
3190    X(I)=(B-A)/2.*X(I)+(B+A)/2.
  12    W(I)=(B-A)/2.*W(I)
3210    DO 13 I=1,8
3220    DO 13 J=1,8
3230    SUM(I,J)=0.
  13    CONTINUE
3250   CALL DMATX(H,P,DM)
3260    DO 23 I=1,NI
3270    CALL BMATX(AL,P,AM,X(I),BB)
3280   CALL MBTM(DM,BB,DB,6,6,8)
3290    CALL MBTTM(BB,DB,BD,8,6,8)
3300    DO 22 J=1,8
3310    DO 22 K=1,8
  22    SUM(J,K)=SUM(J,K)+W(I)*BD(J,K)
  23    CONTINUE
3340   CONST=R*3.1415927*H/(1.-P*P)
3350    DO 1 I=1,8
3360    DO 1 J=1,8
   1    SUM(I,J)=SUM(I,J)*CONST
  54   FORMAT(/,20X,* ==========================*,//)
3400    RETURN
3410    END
```

```
3430    SUBROUTINE MBTM(D,B,DB,L,M,N)
3440    DIMENSION D(8,8),B(8,8),DB(8,8)
3450    DO 25 J=1,N
3460    DO 25 I=1,L
3470    DB(I,J)=0.
3480    DO 25 K=1,M
   25   DB(I,J)=DB(I,J)+D(I,K)*B(K,J)
3500    RETURN
3510    END
C
3530    SUBROUTINE MBTTM(D,B,DB,L,M,N)
3540    DIMENSION D(8,8),B(8,8),DB(8,8)
3550    DO 26 J=1,N
3560    DO 26 I=1,L
3570    DB(I,J)=0.
3580    DO 26 K=1,M
   26   DB(I,J)=DB(I,J)+D(K,I)*B(K,J)
3600    RETURN
3610    END
```

```
C
      SUBROUTINE DMATX(H,P,DM)
      DIMENSION DM(8,8)
       DO 27 I=1,6
       DO 27 J=1,6
   27 DM(I,J)=0.
       H2=H*H
      DM(1,1)=DM(2,2)=1.
      DM(1,2)=DM(2,1)=P
      DM(3,3)=(1.-P)/2.
      DM(4,4)=DM(5,5)=H2/12.
      DM(5,4)=DM(4,5)=P*H2/12.
      DM(6,6)=H2*(1.-P)/24.
       RETURN
       END
C
       SUBROUTINE BMATX(AL,R,AM,X,B)
       DIMENSION B(8,8)
       X2=X**2
       X3=X**3
       AL2=AL**2
       AL3=AL**3
       AM2=AM**2
       R2=R**2
       DO 29 I=1,6
       DO 29 J=1,8
       B(I,J)=0.
   29  CONTINUE
       B(1,1)=B(3,2)=-1./AL
       B(1,5)=B(3,6)=1./AL
       B(2,2)=AM*(1.-X/AL)/R
       B(2,3)=(1.-3.*X2/AL2+2.*X3/AL3)/R
       B(2,4)=X*(1.-2.*X/AL+X2/AL2)/R
       B(2,6)=AM*X/R/AL
       B(2,7)=X2*(3./AL2-2.*X/AL3)/R
       B(2,8)=X2*(-1./AL+X/AL2)/R
       B(3,1)=-B(2,2)
       B(3,5)=-B(2,6)
       B(4,3)=(6.-12.*X/AL)/AL2
       B(4,4)=(4.-6.*X/AL)/AL
       B(4,7)=-B(4,3)
       B(4,8)=(2.-6.*X/AL)/AL
       B(5,2)=B(2,2)/R
       B(5,3)=B(2,3)*AM2/R
       B(5,4)=B(2,4)*AM2/R
       B(5,6)=B(2,6)/R
      B(5,7)=B(2,7)*AM2/R
       B(5,8)=B(2,8)*AM2/R
       B(6,1)=AM*(1.-X/AL)/2./R2
       B(6,2)=-3./2./R/AL
       B(6,3)=-12.*AM*X*(1.-X/AL)/R/AL2
       B(6,4)=2.*AM*(1.-4.*X/AL+3.*X2/AL2)/R
       B(6,5)=AM*X/2./R2/AL
       B(6,6)=3./2./R/AL
       B(6,7)=-B(6,3)
       B(6,8)=-2.*AM*X*(2.-3.*X/AL)/R/AL
       RETURN
       END
```

```
 5500   SUBROUTINE EGN(D,ND,NMODE,E,NRC)
 5510   DIMENSION D(40,41)
 5520   DIMENSION V1(40),V2(40)
C       PRE-EIGENVALUE CHOLESKY REDUCTIONS
 6010   INA=1
            ND1=ND+1
 6032   READ(7)((D(I,J),J=1,ND1 ),I=1,ND)
       REWIND 7
 6040    DO 76 MA=1,ND
 6050    DO 76 MAS=MA,ND
            MA1=MA+1
            MAS1=MAS+1
            GASH=D(MA,MAS1)
            GISH=D(MAS,MA)
            MASH=1
   79    IF(MA-MASH) 77,77,78
   78    GASH=GASH-D(MASH,MA1)*D(MASH,MAS1)
            GISH=GISH-D(MA,MASH)*D(MAS,MASH)
            MASH=MASH+1
 6150    GO TO 79
   77    IF(MAS-MA) 81,81,119
   81    IF(GISH) 118,82,82
  118    GISH=0.
   82    IF(GASH) 83,84,84
   83    GASH=0.
   84    DIAG1=SQRT(GASH)
            DIAG2=SQRT(GISH)
 6230    IF(DIAG1.EQ.0.) GO TO 85
  119    D(MA,MAS1)=GASH/DIAG1
   85    IF(DIAG2.EQ.0.) GO TO 86
            D(MAS,MA)=GISH/DIAG2
   86     CONTINUE
   76    CONTINUE
C        FORM U/UL
 6300    DO 87 MA=1,ND
 6310    DO 87 MAS=MA,ND
            MAS1=MAS+1
            GASH=D(MAS,MA)
            MASH=MA
   91    MASH=MASH+1
 6360    IF(MAS-MASH) 88,89,89
   89    GASH=GASH-D(MA,MASH)*D(MASH-1,MAS1)
 6380    GO TO 91
   88    D(MA,MAS1)=GASH/D(MAS,MAS1)
   87     CONTINUE
C        MULTIPLICATION TO GET (U*ULE-1*ULTE-1*UT)
 6420    DO 92 MA=1,ND
 6430    DO 92 MAS=MA,ND
            MAS1=MAS+1
            GASH=0.
 6460    DO 93 MASH=MAS1,ND1
            GASH=GASH+D(MA,MASH)*D(MAS,MASH)
   93     CONTINUE
            D(MA,MAS1)=GASH
   92     CONTINUE
            MODE=NMODE
C        PU 1.0 IN V1 FROM 1 TO ND AND ITERATIVE
  115    DO 94 I=1,ND
   94    V1(I)=1.
            NUMIT=1
  121    ALAM2=0.
 6570    DO 95 I=1,ND
            I1=I+1
```

```
 6600   DO 96 J=1,I
            GASH=GASH+V1(J)*D(J,I1)
   96    CONTINUE
 6630   IF(I-ND) 97,98,98
   97   DO 99 J=I1,ND
            GASH=GASH+V1(J)*D(I,J+1)
   99    CONTINUE
   98   V2(I)=GASH
            ALAM2=ALAM2+GASH*GASH
   95   CONTINUE
            ALAMB=SQRT(ALAM2)
            SIGSQ=0.
 6720   DO 101 I=1,ND
            GASH=V2(I)/ALAMB
            GAS=V1(I)-GASH
            SIGSQ=SIGSQ+GAS*GAS
            V1(I)=GASH
  101    CONTINUE
            ZT=1./10.**12
            NUMIT=NUMIT+1
 6800   IF(SIGSQ-ZT) 102,102,103
  103   IF(NUMIT-150) 121,102,102
  102    CONTINUE
          PRINT 11
 6830   PRINT 104,NUMIT
  104  FORMAT(*   NO OF ITERATIONS=*,I3,/)
C        TO MULTIPLY (UE-1)*(U*X)
            I=ND
  109   GASH=V1(I)
            J=ND
  107   IF(J-I) 105,105,106
  106   GASH=GASH-V2(J)*D(J,I)
            J=J-1
 6920   GO TO 107
  105   V2(I)=GASH/D(I,I)
            I=I-1
 6950   IF(I) 108,108,109
  108  PRINT 995,INA
       WRITE(10,3)(INA)
    3 FORMAT(6I3)
C        OMEGA IN CYCLE/SEC
 6980   OMEGA=SQRT(1./ALAMB)/2./3.1415927
 6990   PRINT 112,OMEGA
       WRITE(10,1)(OMEGA)
       RES=0.0
        PRINT 12
    1 FORMAT(4E14.8)
       IF(NBC .NE. 4HCLFR) GO TO 500
       WRITE(9)(ND)
       WRITE(9)(OMEGA)
       WRITE(9)(V2(I),I=1,ND)
  500 IF(NBC .EQ. 4HSMSM) GO TO 40
       IF(NBC .EQ. 4HCLSM) GO TO 30
       PRINT 111,RES,RES,RES,RES
       WRITE(10,1)(RES,RES,RES,RES)
       PRINT 111,(V2(I),I=1,ND)
       WRITE(10,1)((V2(I),I=1,ND))
       IF(NBC .EQ. 4HCLCL) PRINT 111,RES,RES,RES,RES
       IF(NBC .EQ. 4HCLCL) WRITE(10,1)(RES,RES,RES,RES)
       GO TO 7040
   30 PRINT 111,RES,RES,RES,RES
       WRITE(10,1)(RES,RES,RES,RES)
       ND1=ND-1
```

```
      WRITE(10,1)((V2(I),I=1,ND1))
      PRINT 111,RES,RES,RES,V2(ND)
      WRITE(10,1)(RES,RES,RES,V2(ND))
      GO TO 7040
   40 PRINT 111,RES,RES,RES,V2(1)
      WRITE(10,1)(RES,RES,RES,V2(1))
       ND1=ND-1
      PRINT 111,(V2(I),I=2,ND1)
      WRITE(10,1)((V2(I),I=2,ND1))
      PRINT 111,RES,RES,RES,V2(ND)
      WRITE(10,1)(RES,RES,RES,V2(ND))
  995  FORMAT(//,10X,*AXIAL NO. =*,I3)
   11 FORMAT(//,20X,25(2H--))
   12 FORMAT (30X,* MODE SHAPE*,/,15X,*U*,20X,*V*,20X,*W*,20X,*DW/DZ*)
  111 FORMAT(4(5X,F16.8))
  112  FORMAT(//,10X,*NATURAL FREQUENCY=*,E20.10)
    2 FORMAT(8F16.8)
C        CHANGING TO NEXT MODE
 7040    DO 113 I=1,ND
 7050     DO 113 J=I,ND
             J1=J+1
  113    D(I,J1)=D(I,J1)-ALAMB*V1(I)*V1(J)
 6965   INA=INA+1
             MODE=MODE-1
 7090    IF(MODE) 114,114,115
  114    CONTINUE
 7110   RETURN
 7120   END
```

```
FTN.
LGO.
SAVE(TAPE1=GASN)
```

(COMPUTER CONTROL CARDS)

```
      PROGRAM NIXW(INPUT,OUTPUT,TAPE1)
      DIMENSION X(60),W(60)
    7 COMMON NI
      READ 701,NPR,NINT
  701 FORMAT(8I10)
   13 DO 12 I=1,NPR
      NI=NINT
      CALL  GDATA(NI,X,W)
      PRINT 701,NI
      WRITE(1)(NI)
      DO 12 J=1,NI
      PRINT 702,X(J),W(J)
  702 FORMAT(2(4X,E20.10))
   12 WRITE(1)(X(J),W(J))
      END
      SUBROUTINE GDATA (NI,X,W)
      DIMENSION X(60), W(60)
      NR=(NI+1)/2
      XI=0.
      DX=1./FLOAT(NI)
      CALL ROOTS (XI,DX,NR,W)
      DO 11 I=1,NR
      J=NR-I+1
      K=NI-I+1
      X(I)=-W(J)
   11 X(K)=W(J)
      DO 12 I=1,NR
      J=NI-I+1
      XX=X(J)
      CALL FNSNS(XX,F,FP,2)
      XX=(1.-XX*XX)*FP*FP
      W(J)=2./XX
   12 W(I)=W(J)
   17 RETURN
      END
      SUBROUTINE FNSNS(X,F,FP,IM)
  138 COMMON NI
      P1=0
      P2=1.
      I=0
   11 AI=I
      P3=((2.*AI+1.)*X*P2-AI*P1)/(AI+1.)
      P1=P2
      P2=P3
      I=I+1
      IF(I-NI) 11,12,12
   12 F=P2
      IF(IM.EQ.0)GO TO 17
      AI=NI
      FP=(AI*X*P2-AI*P1)/(X*X-1)
   17 RETURN
      END
      SUBROUTINE ROOTS(X1,DXX,NRT,RT)
      DIMENSION RT(60)
  230 LO=0 $ FO=0.
      DM=10.
      DD=0
  260 DX=DXX
      DO 20 NI=1,NRT
      K=0
      KP=0
   21 DO 56 JJ=1,10
      DO 30 J=1,11
      CALL FNSNS(X1,F1,FP,0)
```

```
      X=X1
      IF(ABS(F1).LE.1.E-09) GO TO 36
      IF(F1.LT.0.) L1=-1
      IF(F1.GT.0.) L1=1
      IF(J.EQ.1) GO TO 31
      F3=F1-F0
      IF(L1.NE.L0) GO TO 41
   31 X0=X1
      F0=F1
      L0=L1
      X1=X0+DX
   30 CONTINUE
      DX=DM*DX
   50 CONTINUE
   32 K=K+1
      IF(F3.EQ.0.) GO TO 49
      X=(F1*X0-F0*X1)/F3
      IF(X.LE.X0 .OR. X.GE.X1) GO TO 49
      CALL FNSNS(X,F,FP,0)
      IF(ABS(F) .LE.1.E-09) GO TO 36
      IF(F.LT.0.) LL=-1
      IF(F.GT.0.) LL=1
      IF(LL.NE.L1) GO TO 33
      IF(LL.NE.L0) GO TO 34
   33 X0=X
      F0=F
      L0=LL
      F3=F1-F0
      GO TO 35
   34 X1=X
      F1=F
      L1=LL
      F3=F1-F0
   35 XX=ABS(X1-X0)
      IF(XX.LE.1.E-07) GO TO 36
      IF(K.LE.100) GO TO 32
   49 K=0
      KP=KP+1
      IF(KP.GT.10) GO TO 36
      DX=(X1-X0)/10.
      X1=X0
      DM=1.
      GO TO 21
   41 XX=X1-X0
      IF(ABS(XX).LE.1.E-07) GO TO 36
      IF(F3.EQ.0.) GO TO 49
      X=X0-F0/F3
      DO 40 II=1,1000
      CALL FNSNS(X,F,FP,1)
      IF (FP.EQ.0.) GO TO 32
      XA=X-F/FP
      XX=ABS(XA-X)
      X=XA
      IF(XX.LE.1.E-07) GO TO 36
      IF(XA.LE.X0 .OR. XA.GE.X1) GO TO 32
   40 CONTINUE
   36 DX=(X-DD)*0.5
  900  IF(DX.EQ.0.) DX=DXX
      DM=1.
      X1=X+DX
      DD=X
      RT(NI)=X
   42 PRINT 45,NI,X
```

```
20 CONTINUE
      RETURN
      END
```

```
FRORES,CM200000,T200.
ACCOUNT,A0L7000,AASSDDA.
GET(TAPE7=ACC)
      GET(TAPE12=STRMAT)
GET(TAPE9=MODAL)
  GET(TAPE10=MAT)
FTN.
LGO.
```

```
      PROGRAM RESP(INPUT,OUTPUT,TAPE1,TAPE2,TAPE3,TAPE4,
     *TAPE7,TAPE8,TAPE9,TAPE10,TAPE11,TAPE12,TAPE19)
      DIMENSION DTB(6,8,6),STR1(12),STR2(12),STR(60),STRPI(60)
      DIMENSION STRCO(6),STRCP(6)
      DIMENSION UE(8)
      DIMENSION  BMASS(36,4),BSTF(36,4),SMASS(36,36),STIF(36,36)
      DIMENSION                          EFFM(36,4),RACC(4),PEF(36),U(36)
       DIMENSION FB3(4)
      DIMENSION A1(36,4),RDTS(4)
       DIMENSION GA(5001),GY(5001)
      DIMENSION STBB(4,4),BUF(4)
      DIMENSION BBM(4,4)
      DIMENSION OME(10),X(36,10),CM(10,10),CP(10),A(10),PIN(10),RCRD(10)
      DIMENSION OMED(10),ADD(10),GS(10,10),GMI(10,10),GSA(10)
      DIMENSION            UDD(36),FB1(4),FB2(4),FB(4) ,FB4(4)
      DIMENSION P1(36),P2(36),P(36),PEXT(36)
      PI=3.14159
  102 READ 4,R
   99 FORMAT(F10.4)
   88 FORMAT(10F10.4)
      DO 66 NREAD=1,6
      READ(12)(NN)
   66 READ(12)((DTB(I,J,NN),J=1,8),I=1,6)
    1 FORMAT(//,*   INDEX=*,I5)
    2 FORMAT(5(5X,G10.4))
      CALL EXTRCT
      READ(11)(NFREE)
      N=NFREE-4
      NELEM=N/4
      NTO=NELEM*6
       NST=NTO+6
*     MB
      READ(11)((BMASS(I,J),J=1,4),I=1,N)
*     M
      READ(11)((SMASS(I,J),J=1,N),I=1,N)
*     MBB
      READ(11)((BBM(I,J),J=1,4),I=1,4)
*     KB
      READ(11)(( BSTF(I,J),J=1,4),I=1,N)
*     K
      READ(11)(( STIF(I,J),J=1,N),I=1,N)
*     KBB
      READ(11)((STBB(I,J),J=1,4),I=1,4)
    3 FORMAT(9(3X,G12.6))
  103 READ 8,M
    8 FORMAT(I2)
      DO 10 J=1,M
      READ(9)(ND)
C     OMEGA IN CYCLE/SEC
      READ(9)(OME(J))
      READ(9)(X(I,J),I=1,ND)
   10 CONTINUE
      PI2=2.*3.14159
      DO 20 I=1,M
   20 OME(I)=OME(I)*PI2
C     OMEGA NOW IN RAD./SEC
      DO 15 I=1,4
      DO 15 J=1,4
      STBB(I,J)=STBB(I,J)+BSTF(J,I)
   15 BBM(I,J)=BBM(I,J)+BMASS(J,I)
      DO 77 J=1,M
      RCRD(J)=0.0
```

```
   77 OMED(J)=OME(J)*SQRT(1.-RCRD(J)*RCRD(J))
      CALL MODAN(SMASS,OME,X,GM,N,M,STIF,GS)
      NB=4
      CALL EFFMASS(STIF,SMASS,BSTF,BMASS,N,NB,EFFM,A1,STBB,BUF)
   55 READ 5,NLOAD,PEAK,INDEX
  101 READ 4,TTOTAL ,DT,TO
      NDUR=TO/DT+1.0001
      NTIME=TTOTAL/DT+1.0001
*     FOR ARTIFICIAL E.Q. RECORD PEAK=G
    5 FORMAT(A5,5X,F10.4,5X,A5)
      IF(NLOAD .EQ. 5HEND  ) GO TO 9999
      IF(INDEX .EQ. 5HCLOSD) GO TO 1004
      IF(NLOAD .NE. 5HARTEQ) GO TO 56
*     GROUND ACCELERATION VALUES SHOULD BE NORMALIZED
*     ACC.S ARE IN TERMS OF G
      READ(7,88)(GA(I),I=1,NTIME)
*      DISP.S ARE IN TERMS OF FEETS
      READ(8,88)(GY(I),I=1,NTIME)
*     BACC REPRESENT THE PEAK VALUES
   56 IF(NLOAD .EQ. 5HRECPU .A. INDEX .EQ. 5HNUMER) GO TO 1011
      IF(NLOAD .EQ. 5HTRIPU .A. INDEX .EQ. 5HNUMER) GO TO 1012
      IF(NLOAD .EQ. 5HSINPU .A. INDEX .EQ. 5HNUMER) GO TO 1013
 1011 DO 1014 I=1,NDUR
 1014 GA(I)=1.0
      GO TO 1004
 1012 NHDR=NDUR/2+1.0001
      DO 1015 I=1,NHDR
 1015 GA(I)=FLOAT(I-1)/FLOAT(NHDR-1)
      NHD1=NHDR+1
      DO 1016 I=NHD1,NDUR
 1016 GA(I)=1.0-FLOAT(I-NHDR)/FLOAT(NHDR-1)
      GO TO 1004
 1013 DO 1017 I=1,NDUR
 1017 GA(I)=SIN(PI*FLOAT(I-1)/FLOAT(NDUR-1))
 1004  PRINT 5555
      IF(NLOAD .EQ. 5HRECPU) PRINT 112,TO
      IF(NLOAD .EQ. 5HTRIPU) PRINT 113 ,TO
      IF(NLOAD .EQ. 5HSINPU) PRINT 114 ,TO
      IF(NLOAD .EQ. 5HARTEQ) PRINT 115,TTOTAL
  112 FORMAT(/,10X,*TANK IS EXCITED BY A RECTANGULAR PULSE GROUND ACC.AP
     QPLIED FOR*,G15.5,*SEC.*,/)
  113 FORMAT(/,10X,*TANK IS EXCITED BY A TRIANGULAR PULSE GROUND ACC. AP
     QPLIED FOR*,G15.5,*SEC.*,/)
  114 FORMAT(/,10X,*TANK IS EXCITED BY A SINOSOIDAL PULSE GROUND ACC. AP
     QPLIED FOR*,G15.5,*SEC.*,/)
  115 FORMAT(/,10X,*TANK IS EXCITED BY AN ARTIFICIAL E.Q. APPLIED FOR*,G
     Q15.5,*SEC.*,/)
      PRINT 70,PEAK
   70 FORMAT(//,10X,* MAX. GROUND ACCELERATION VALUE=*,G10.2,
     Q * IN./SEC/SEC*,/)
 5555 FORMAT(1H1)
      IF(INDEX .EQ. 5HARTEQ .OR. INDEX .EQ. 5HNUMER) PRINT 1005
 1005 FORMAT(//,10X,* NUMERICAL SOLUTION OF DAHMEL INTEGRAL *,/)
      IF(INDEX .EQ. 5HCLOSD) PRINT 1006
 1006 FORMAT(//,20X,*CLOSED FORM SOLUTION*,/)
      BACC(1)=0.0
      BACC(2)=-PEAK
      BACC(3)= PEAK
      BACC(4)=0.0
      DO 50 I=1,N
      REF(I)=0.0
      DO 50 J=1,NB
```

```
   50 PEF(I)=PEF(I)+FFFM(I,J)*BACC(J)
      DO 60 I=1,M
      BP(I)=0.0
      DO 60 J=1,N
   60 BP(I)=BP(I)+X(J,I)*PEF(J)
    4 FORMAT(4(5X,G10.4))
      READ 4,TSTAR,TEND,TDT
      READ 999,ISKIP
  999 FORMAT(I7)
      JUMP=1
      ITURN=0
      NSTAR=TSTAR/DT+1.001
      NEND=TEND/DT+1.001
      NDT=TDT/DT
      NPT=(NEND-NSTAR)/NDT+1
*     ***********************************************************************
*     ***********************************************************************
      HRDIS=0.0
      DO 1000 ITIM1=NSTAR,NEND,NDT
      ITURN=ITURN+1
      ITIME=ITIM1-1
      T=FLOAT(ITIME)*DT
*       CHANGE DISP. INTO IN.S
      BDIST=GY(ITIME) *12.0
      IF(NLOAD .EQ. 5HARTEQ .OR. INDEX .EQ. 5HNUMER)
     *CALL INT(GA            ,PIN,GP,T,ITIME,DT,M,OME,PEAK,OMED,RCRD)
      IF(NLOAD .NE. 5HARTEQ .A. INDEX .EQ. 5HCLOSD)
     * CALL INTE(T,TO,OME,NLOAD,M,GP,PIN)
      DO 110 I=1,M
  110 A(I)=PIN(I)/(GM(I,I)*OMED(I))
      DO 120 I=1,N
      U(I)=0.0
      DO 120 J=1,M
  120 U(I)=U(I)+X(I,J)*A(J)
      HRDIS=AMAX1(HRDIS,ABS(U(N-1)))
      IF(ITURN .NE. JUMP) GO TO 1000
       DO 111 I=1,M
  111 ADD(I)=GP(I)*GA(ITIME)/GM(I,I)-OMED(I)*OMED(I)*A(I)
*     CHANGE BACC TO REPRESENT THE ACTUAL VALUES
      BACCT=PEAK*GA(ITIME)
      BDIS(1)=BDIS(4)=0.0
      BDIS(2)=-BDIST    $    BDIS(3)=BDIST
      DO 121 I=1,N
       UDD(I)=0.0
      DO 121 J=1,M
  121 UDD(I)=UDD(I)+X(I,J)*ADD(J)
      BACC(1)=BACC(4)=0.0
      BACC(2)=-BACCT
      BACC(3)= BACCT
*    FB1=MBT*UDD
*    FB2=KBT*U
*    FB3=(MBB+MBT *I)* BACC
*    FB4=(KBB+KBT *I) *BDIS
       DO 125 I=1,4
      FB3(I)=0.0
      FB4(I)=0.0
      FB1(I)=FB2(I)=FB(I)=0.0
      DO 126 J=1,4
      FB1(I)=FB1(I)+BMASS(J,I)*UDD(J)
      FB3(I)=FB3(I)+BBM(I,J)*BACC(J)
      FB4(I)=FB4(I)+STBB(I,J)*BDIS(J)
  126 FB2(I)=FB2(I)+BSTF(J,I)*U(J)
```

```
  125 FR(I)=(FR1(I)+FR2(I)+FR3(I))/(3.14159*R)
*     ***************************************************************
*******************   CHECK   ********************
      DO 401 I=1,N
  401 PEXT(I)=PEF(I)*GA(ITIME)
*   M*UDD=P1
*   K*U=P2
*   CHECK  .... P1+P2=PEXT
      DO 402 I=1,N
      P1(I)=0.0
      P2(I)=0.0
      P(I)=0.0
      DO 403 J=1,N
      P1(I)=P1(I)+SMASS(I,J)*UDD(J)
  403 P2(I)=P2(I)+STIF(I,J)*U(J)
  402 P(I)=P1(I)+P2(I)
      DO 500 I=1,NST
      STRPI(I)=0.0
  500 STR(I)=0.0
      UE(1)=UE(2)=UE(3)=UE(4)=0.0
      DO 501 NEL=1,NELEM
      DO 502 J=5,8
      IF(NEL .GT. 1) UE(J-4)=UE(J)
  502 UE(J)=U((NEL-2)*4+J)
      DO 503 I=1,6
      STRCO(I)=STRCP(I)=0.0
      STR1(I)=STR1(I+6)=0.0
      STR2(I)=STR2(I+6)=0.0
      DO 503 J=1,8
      STRCO(I)=STRCO(I)+DTB(I,J,3)*UE(J)
      STRCP(I)=STRCP(I)+DTB(I,J,6)*UE(J)
      STR2(I)=STR2(I)+DTB(I,J,4)*UE(J)
      STR2(I+6)=STR2(I+6)+DTB(I,J,5)*UE(J)
      STR1(I)=STR1(I)+DTB(I,J,1)*UE(J)
  503 STR1(I+6)=STR1(I+6)+DTB(I,J,2)*UE(J)
      DO 504 J=1,12
      NN=(NEL-1)*6+J
      STRPI(NN)=STRPI(NN)+STR2(J)
  504 STR(NN)=STR(NN)+STR1(J)
  501 CONTINUE
      DO 506 I=7,NTO
      STRPI(I)=STRPI(I)/2.
  506 STR(I)=STR(I)/2.
            CALL PRIN(T,RACCT,BDIST,PIN,A,ADD,U,UDD,FB,P1,P2,P,PEXT,STR,
     * STRPI,NST,N,M)
      JUMP=JUMP+ISKIP
 1000 CONTINUE
*     ***************************************************************
      PRINT 507,HRDIS
  507 FORMAT(//,10X,*MAX. RADIAL DISPLACEMENT AT TANK TOP=*,G15.5,*IN.*)
      GO TO 55
 9999 CONTINUE
      END
```

```
      SUBROUTINE PRIN(T,RACCT,BDIST,PIN,A,ADD,U,UDD,FB,P1,P2,P,PEXT,STR,
     * STRPI,NST,N,M)
      DIMENSION STR(60),STRPI(60),A(10),ADD(10),PIN(10),FB(4)
      DIMENSION P1(36),P2(36),P(36),PEXT(36),U(36),UDD(36)
    7 FORMAT(20X,25(2H--),/,15X,*TIME=*,F10.5,/)
      PRINT 9
 1019 PRINT 7,T
*     PRINT 2,(PIN(I),I=1,M)
*     PRINT 2,(A(I),I=1,M)
*     PRINT 2,(ADD(I),I=1,M)
      PRINT 114,U(N-1)
  114 FORMAT(/,10X,*RADIAL DISPL. AT TANK TOP=*,G15.5,*IN.*)
  116 FORMAT(/,10X,*GROUND DISPLACEMENT        =*,G15.5,*IN.*)
  115 FORMAT(/,10X,*GROUND ACCELERATION        =*,G15.5,*IN./SEC/SEC*)
*     PRINT 9
      PRINT 12
      PRINT 111,(U(I),I=1,N)
      PRINT 9
    9 FORMAT(//)
  111 FORMAT(4(5X,G16.8))
   12 FORMAT(/,15X,*U*,20X,*V*,20X,*W*,20X,*DW/DZ*)
  112 FORMAT(/,25X,*EXTERNAL GENERALIZED FORCES AT BASE*,/)
      PRINT 112
  113 FORMAT(10X,*AXIAL FORCE*,7X,*TANGT. FORCE*,7X,*RADIAL FORCE*,7X,
     Q*AXIAL MT.*,/)
      PRINT 113
      PRINT 111,(FB(I),I=1,4)
  600 FORMAT(12(1X,G10.4))
      PRINT 9
  404 FORMAT(30X,*GENERALIZED INERTIAL FORCES P1 AT NON-BASE NODES*,/)
      PRINT 404
      PRINT 600,(P1(I),I=1,N)
      PRINT 9
  405 FORMAT(30X,*GENERALIZED INTERNAL FORCES P2 AT NON-BASE NODES*,/)
      PRINT 405
      PRINT 600,(P2(I),I=1,N)
  406 FORMAT(50X,* P1+P2*,/)
      PRINT 9
      PRINT 406
      PRINT 600,(P(I) ,I=1,M)
      PRINT 9
  407 FORMAT(30X,*GENERALIZED EXTERNAL FORCES AT NON-BASE NODES*,/)
      PRINT 407
      PRINT 600,( PEXT(I),I=1,N)
      PRINT 9
  509 FORMAT(4X,*AXIAL F.*,7X,*TANGT. F.*,5X, *IN-PLANE SH.*,5X,
     Q *AXIAL MT.*,5X,*TANGT. MT.*,7X,*TORSION*,/)
  507 FORMAT(//,40X,* STRESSES AT THETA=0.0 *,/)
      PRINT 507
      PRINT 509
      PRINT 505,(STR(I),I=1,NST)
  508 FORMAT(//,30X,* STRESSES AT THETA=90.0 DEGREES*,/)
      PRINT 508
      PRINT 509
      PRINT 505,(STRPI(I),I=1,NST)
  505 FORMAT(6(5X,G10.4))
      RETURN
      END
```

```
      SUBROUTINE INT(ACT ,          ,PIN,P,TIME,NT,DT,M,OME,PEAK,OMED,RCRD)
      DIMENSION ACT(5001),PIN(10),OME(10),OMED(10),RCRD(10),P(10)
      DIMENSION SINE(10),EXPO(10)
      DO 15 J=1,M
      BIN(J)=0.0
      DO 10 IT=1,NT
      TA=FLOAT(IT)*DT
      SINE(J)=SIN(OMED(J)*(TIME-TA))
      EXPO(J)=EXP(-RCRD(J)*OME(J)*(TIME-TA))
   10 PIN(J)=PIN(J)+ACT(IT)*EXPO(J)*SINE(J)
   15 PIN(J)=PIN(J)*P(J)*DT
      RETURN
      END
```

```
      SUBROUTINE EFFMASS(STIF,SM,BSTF,BMASS,N,NB,A2,A1,STBR,BUF)
      DIMENSION UA(4),UF(36)
      DIMENSION SM(36,36),BSTF(36,4),BMASS(36,4),A2(36,4),STIF(36,36)
      DIMENSION ST(36,36)
      DIMENSION A1(36,4)
      DIMENSION STBR(4,4),                    BUF(4)
      BUD(1)=BUD(4)=0.0
      BUD(2)=-1.0    $     BUD(3)=1.0
    3 FORMAT(12(1X,G10.4))
      DO 6 I=1,N
      DO 6 J=1,N
    6 ST(I,J)=STIF(I,J)
C           CALL INVERSION SUBROUTINE FOR STIF
      CALL INVMT(STIF,N,N,36,DET)
C         HERE STIF IS THE INVERSION OF THE ORIGINAL MATRIX
*A1=K-1 KB
      DO 10 I=1,N
      DO 10 J=1,NB
      A1(I,J)=0.0
      DO 10 K=1,N
   10 A1(I,J)=A1(I,J)+STIF(I,K)*BSTF(K,J)
      DO 20 I=1,N
*     A2=M K-1 KB
      DO 20 J=1,NB
      A2(I,J)=0.0
      DO 20 K=1,N
   20 A2(I,J)=A2(I,J)+SM(I,K)*A1(K,J)
      PRINT 5
    5 FORMAT(1H1)
      DO 30 I=1,N
      DO 30 J=1,NB
   30 A2(I,J)=A2(I,J)-BMASS(I,J)
      UA(1)=UA(4)=0.
       UA(3)=+1.0     $ UA(2)=-1.0
       DO 40 I=1,N
      UF(I)=0.0
      DO 40 J=1,ND
   40 UF(I)=UF(I)+A2(I,J)*UA(J)
      DO 50 I=1,N
      UF(I)=0.
      DO 50 J=1,NB
   50 UF(I)=UF(I)+BMASS(I,J)*UA(J)
* ST2=-KBT K-1 KB
      DO 90 I=1,N
      DO 90 J=1,N
   90 STIF(I,J)=ST(I,J)
      RETURN
      END
```

```
      SUBROUTINE MODAN(SMASS,OME,X,GM,N,M,STIF,GS)
      DIMENSION GS(10,10),GM(10,10),X(36,10),OME(10),XM(36,10)
      DIMENSION STIF(36,36),SMASS(36,36)
   15 FORMAT( 5X,I5)
      DO 10 I=1,N
      DO 10 J=1,M
      XM(I,J)=0.0
      DO 10 K=1,N
   10 XM(I,J)=XM(I,J)+SMASS(I,K)*X(K,J)
C  GM=XT*M*X
      DO 20 I=1,M
      DO 20 J=1,M
      GM(I,J)=0.0
      DO 20 K=1,N
   20 GM(I,J)=GM(I,J)+X(K,I)*XM(K,J)
      DO 30 I=1,N
      DO 30 J=1,M
       XM(I,J)=0.0
       DO 30 K=1,N
   30 XM(I,J)=XM(I,J)+STIF(I,K)*X(K,J)
      DO 40 I=1,M
      DO 40 J=1,M
      GS(I,J)=0.0
       DO 40 K=1,N
   40 GS(I,J)=GS(I,J)+X(K,I)*XM(K,J)
      RETURN
      END
```

```
      SUBROUTINE EXTRCT
      DIMENSION D(40,41)
      READ(10)(NFREF)
      NFREE1=NFREF+1
      READ(10)((D(I,J),J=1,NFREE1),I=1,NFREF)
      REWIND 10
       DO 10 I=1,NFREE
      DO 10 J=1,I
   10 D(J,I)=D(I,J)
      WRITE(11)(NFREE)
*     WRITE MR
      WRITE(11)((D(I,J),J=1,4),I=5,NFREE)
*     WRITE M
      WRITE(11)((D(I,J),J=5,NFREE),I=5,NFREE)
      WRITE(11)((D(I,J),J=1,4),I=1,4)
      READ(10)(NFREE)
      READ(10)((D(I,J),J=1,NFREE1),I=1,NFREE)
      DO 20 I=1,NFREE
      DO 20 J=2,NFREE1
   20 D(I,J-1)=D(I,J)
      DO 30 J=1,NFREE
      DO 30 I=1,J
   30 D(J,I)=D(I,J)
*     WRITE KR
      WRITE(11)((D(I,J),J=1,4),I=5,NFREE)
*     WRITE K
      WRITE(11)((D(I,J),J=5,NFREE),I=5,NFREE)
*      WRITE KBR
      WRITE (11)((D(I,J),J=1,4),I=1,4)
      REWIND 11
      RETURN
      END
```

```
      SUBROUTINE INTE(T,TO,OME,NLOAD,M,P,PIN)
      DIMENSION P(10),OME(10),PIN(10)
      TH=TO/2.
      IF(NLOAD .EQ. 5HREGPU) GO TO 10
      IF(NLOAD .EQ. 5HTRIPU) GO TO 20
      IF(NLOAD .EQ. 5HSINPU) GO TO 30
      PRINT 1
    1 FORMAT(* TYPE OF LOAD INVALIED*)
      STOP
   10 IF(T .LT. TO) GO TO 11
      DO 21 I=1,M
   21 PIN(I)=P(I)/OME(I)*(COS(OME(I)*(T-TO))-COS(OME(I)*T))
      RETURN
   11 DO 31 I=1,M
   31 PIN(I)=P(I)/OME(I)*(1.-COS(OME(I)*T))
      RETURN
   20 IF(T .LT. TH) GO TO 12
      IF(T .LT. TO ) GO TO 22
      DO 32 I=1,M
   32 PIN(I)=P(I)/OME(I)/OME(I)/TH*(2.*SIN(OME(I)*(T-TH))-SIN(OME(I)*
     C (T-TO ))-SIN(OME(I)*T))
      RETURN
   12 DO 42 I=1,M
   42 PIN(I)=P(I)/OME(I)/TH*(T-SIN(OME(I)*T)/OME(I))
      RETURN
   22 DO 52 I=1,M
   52 PIN(I)=P(I)/OME(I)/TH*(TO -T+2.*SIN(OME(I)*(T-TH))/OME(I)-SIN(OME
     Q(I)*T)/OME(I))
      RETURN
   30 PI=3.14159
      IF(T .LT. TO) GO TO 13
      DO 23 I=1,M
   23 PIN(I)=-P(I)*PI*TO/(OME(I)*TO*OME(I)*TO-PI*PI)*(SIN(OME(I)*(T-TO))+
     C +SIN(OME(I)*T))
      RETURN
   13 DO 33 I=1,M
   33 PIN(I)=P(I)*TO/(OME(I)*TO*OME(I)*TO-PI*PI)*(OME(I)*TO*SIN(PI*T/TO)
     Q -PI*SIN(OME(I)*T))
      RETURN
      END
```

```
      SUBROUTINE INVMT(A,NR,NC,IDIM,DELT)
8010  DIMENSION A(IDIM,IDIM),LABEL(50)
      DELT=1.0
8040  DO 21 J1=1,NR
   21 LABEL(J1)=J1
8060  DO 291 J1=1,NR
  101 TEMP=0.0
8080  DO 121 J2=J1,NR
      IF(ABS(A(J2,J1)).LT.TEMP) GO TO 121
      TEMP=ABS(A(J2,J1))
8110  IBIG=J2
  121 CONTINUE
8130  IF(IBIG.EQ.J1) GO TO 201
8140  DO 141 J2=1,NC
8150  TEMP=A(J1,J2)
8160  A(J1,J2)=A(IBIG,J2)
  141 A(IBIG,J2)=TEMP
8180  I=LABEL(J1)
8190  LABEL(J1)=LABEL(IBIG)
8200  LABEL(IBIG)=I
8210  DELT=-DELT
  201 TEMP=A(J1,J1)
8240  DELT=DELT*A(J1,J1)
8260  A(J1,J1)=1.0
8270  DO 221 J2=1,NC
  221 A(J1,J2)=A(J1,J2)/TEMP
8290  DO 281 J2=1,NR
8300  IF(J2.EQ.J1) GO TO 281
8310  TEMP=A(J2,J1)
8320  A(J2,J1)=0.0
8330  DO 241 J3=1,NC
  241 A(J2,J3)=A(J2,J3)-TEMP*A(J1,J3)
  281 CONTINUE
  291 CONTINUE
  301 N1=NR-1
8380  DO 391 J1=1,N1
8390  DO 321 J2=J1,NR
8400  IF(LABEL(J2).NE.J1) GO TO 321
8410  IF(J2.EQ.J1) GO TO 391
8420  GO TO 341
  321 CONTINUE
  341 DO 361 J3=1,NR
8450  TEMP=A(J3,J1)
8460  A(J3,J1)=A(J3,J2)
  361 A(J3,J2)=TEMP
8480  LABEL(J2)=LABEL(J1)
  391 CONTINUE
5001  RETURN
8510  END
```