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# DYNAMIC STIFFNESS OF FOUNDATIONS: 2-D vs. 3-D SOLUTIONS

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Approximate formulae for the dynamic stiffness of embedded strip footings are developed through parametric studies. A comparison is then established between the static value of these stiffnesses and their frequency variation for two- and three-dimensional solutions. A plane strain model is demonstrated to be incapable of reproducing all the stiffness terms of a circular or square foundation.

As a preliminary method of analysis, construction of similar formulae and graphs provides estimates of the natural frequency and effective modal damping provided by two- and three-dimensional finite element solutions. The magnitude of error in adopting a plane strain model may then be evaluated.

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## ABSTRACT

Most computer programs used at present for soil structure interaction analyses are based on a two-dimensional (or pseudo three-dimensional) model of the soil. In this work the results of parametric studies are presented, leading to approximate formulae for the dynamic stiffnesses of embedded strip footings similar to those already available for circular foundations. A comparison is then established between the static values of these stiffnesses and their frequency variation for two- and three-dimensional solutions. It is shown that a plane strain model cannot reproduce all the stiffness terms of a circular or square foundation over the complete range of frequencies. Recognizing, on the other hand, that not all the terms may contribute significantly to the structural response, it is recommended that for each particular problem, before embarking on the use of finite element programs, preliminary analyses be conducted with formulae and graphs similar to those presented here, in order to estimate the natural frequencies and effective modal damping provided by two-dimensional and three-dimensional solutions. These estimates will allow assessment of the order of magnitude of the error involved in a plane strain model and will indicate whether more expensive studies with such a model are justified.

# PREFACE

The work described in this report was conducted by Moshe Jakub, a graduate research assistant in the Civil Engineering Department at M.I.T., under the supervision of Professor José M. Roesset. It was made possible through Grant AEN 7417835 from the National Science Foundation, Division of Advanced Environmental Research and Technology.

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1. Research Report R76-8 by Mohammed M. Ettouney, "Transmitting Boundaries: A Comparison," January 1976.

2. Research Report R76-9 by Mohammed M. Ettouney, "Nonlinear Soil Behavior in Soil Structure Interaction Analysis," February 1976.

3. Research Report R77-30, by José J. Gonzalez, "Dynamic Interaction between Adjacent Structures," September 1977.

4. Research Report R77-33 by F. Elsabee and J.P. Morray, "Dynamic Behavior of Embedded Foundations," September 1977.

5. Research Report R77-35, by Moshe Jakub, "Nonlinear Stiffness of Foundations," September 1977.

# DYNAMIC STIFFNESS OF FOUNDATIONS: 2-D vs. 3-D SOLUTIONS

## INTRODUCTION

A considerable amount of work has been done in recent years to determine the dynamic stiffnesses of foundations of various shapes, a problem of interest on its own (as for the design of machine foundations), but of particular importance in relation to the seismic analysis of nuclear power plants. Analytical solutions have been presented and tabulated for circular foundations on the surface of an elastic (15) or viscoelastic (16) halfspace, and algorithms have been developed to study circular and rectangular foundations (or for that purpose foundations of arbitrary shape) on the surface of a horizontally stratified soil deposit (4,9,18). These analytical or semi-analytical solutions have been limited until now to surface foundations and linear soil behavior, but approximate procedures to account for the effect of embedment (3,12,13,14) and for the nonlinear characteristics of soils (7) have also been suggested. Finally, finite element solutions are also available to study circular foundations embedded or on the surface (6) and surface footings of arbitrary shape (5).

In spite of all this work, most of the computer programs used in practice, such as LUSH (10), are based on a two-dimensional (plane strain) model of the soil and the structure. Even a more recent version, FLUSH (11), which claims to solve the three-dimensional problem has in fact a plane strain model with viscous dashpots added to the lateral faces in order to increase the radiation damping (this is at best a pseudo three-dimensional solution).

The main reason invoked for the use of two-dimensional models is one of economy: the solution of a completely three-dimensional model for an embedded foundation of arbitrary shape and a soil profile with variable properties and nonlinear behavior is today prohibitively expensive (although the only problem in the formulation of such a model is the derivation of realistic constitutive equations for the soil). This argument would imply that these two effects are accurately reproduced in present programs and that the increased accuracy over the approximate solutions mentioned above more than compensates for the errors introduced by the two-dimensional idealization. The first point is not entirely correct. Present finite element solutions assume perfect welding between the sidewalls of the foundation and the surrounding soil, but the actual conditions of the backfill, often uncertain, may significantly influence the effective depth of embedment; nonlinear soil behavior is normally simulated through an equivalent linearization technique, using an iterative scheme which requires several arbitrary assumptions; thus whether there is in fact an important increase in accuracy in these finite element models may be questionable. The second point requires more extensive studies than those conducted to date in order to assess the magnitude of the potential errors introduced by plane strain models (or the modified versions) for typical situations.

## PREVIOUS WORK

In 1974 Kausel (16) conducted a comparative study for a typical containment building using both a cylindrical finite element formulation, reproducing the exact three-dimensional condition for a horizontal seismic excitation, and an equivalent two-dimensional model. The building had a circular foundation with a radius of 57.5 ft. and was embedded 55 ft. in a soil stratum with a total depth of 170 ft. The shear wave velocity of the soil varied from 428 ft/sec at the surface to 934 ft/sec at the bottom, and the stratum was underlain by competent rock which was simulated as a rigid base. In the 2-D model the structure was represented by two coaxial columns of lumped masses and springs, attached to a massless, rigid foundation with flexible sidewalls. Masses and stiffnesses were normalized with respect to an equivalent width 2B, obtained by equating the area of a square foundation to that of the actual circular mat. The properties of the flexible sidewalls (deforming mainly in flexure) were selected so as to yield the same horizontal displacement at grade level as the cylindrical wall (deforming mainly in shear) under a uniform lateral load.

The results of this study indicated that for the case considered the agreement between the two models was reasonable and probably good enough for practical purposes. Differences in the amplified response spectra at various levels of the structure were of the order of 15%.

Luco and Hadjian (8), on the other hand, considered the case of a structure resting on the surface of an elastic, homogeneous halfspace. For this case they presented and discussed various alternatives for the selection of the width and thickness (out of the plane) of a strip footing so as to match the average values of the stiffness and damping terms of a circular foundation over a certain frequency range (it should be noticed that the static stiffness of a strip footing on a halfspace is zero). They decided that the best approach was to select the halfwidth of the footing B = 0.816R and the total thickness 2C = 1.25 x (2R) in order to match the stiffnesses; the natural frequencies of the soil structure system would remain then unchanged, a factor they considered of primary importance for the determination of equipment response. In this way, however, the damping terms for the strip footing were larger than for the circular foundation (32% in swaying and 18% in rocking over the high frequency range, and substantially more for low values of the dimensionless frequency $a_o$). Using this approach, they determined amplified response spectra at the base, top of the internal structure, and top of the containment building for a structure with a fundamental frequency of 3 cps on a rigid base, and 0.74 and 1.29 cps on the elastic foundation (with shear wave velocities for the soil of 600 and 1150 ft/sec respectively). The spectra at the top of the structures had peaks at the natural frequencies which were approximately 50% higher for the three-dimensional solution than for the equivalent two-dimensional model.

Several points of this study deserve careful consideration. In the first place, the selection of the equivalent width and thickness does not seem to be the most appropriate; while it is important to reproduce reasonably well the natural frequencies of the soil structure system, it should be noticed that these frequencies will vary at most with the square root of the foundation stiffnesses. Thus an error of even 25 or 30% on these stiffnesses will represent only an error of at most 10 to 15% on the frequencies (the error being larger as the soil becomes softer and the stiffnesses decrease). Considering the uncertainties in the actual values of the soil properties (both their values in situ for low levels of strain and their variation with shear strain), and even in the structural characteristics, this error does not seem unreasonable (one should consider in any case variations in the soil parameters and not rely on a single analysis). Reproduction of the radiation damping, particularly in rocking and in the neighborhood of the natural frequency of the combined system, would appear to be more important in order to obtain sensible results.

In the second place the unusually small natural frequencies of the structure on elastic foundation make the error in the radiation damping considerably larger than the 32 or 18% predicted by the formulae for the high frequency range (the effective values of damping in the first mode were 2.3 and 1.8% for the three-dimensional solution and 6.36 and 5% for the two-dimensional one). The fact that no internal damping was assumed for the soil aggravates the situation; clearly the difference in the results from 2% damping to a value of 5% is much larger than from, say, 7 to 11% or 12 to 16%.

Because of these facts one should expect that the differences between the two solutions would not be as large as the reported 50% in practical situations. It is important to realize, however, that differences will exist and that, as pointed out by the authors, it is not possible to select an equivalent two-dimensional model which will match all the stiffness terms over the complete frequency range. Furthermore, the fact that the radiation damping will be generally overestimated by the equivalent strip footing would indicate that increasing the dissipation of energy by adding viscous dashpots may make the situation worse instead of improving it.

In 1975 Berger, Lysmer and Seed (1) presented results of another comparative study. They concluded that the differences between their three-dimensional and their two-dimensional solutions, which were again significant, were due to deficiencies in the structural model, which would be an obvious source of errors. The horizontal component of motion at the base of the structure was, however, very similar for the two cases, and the study recommended the use of a complete two-dimensional soil structure model to obtain the base motion and then the use of an independent structural analysis program with a three-dimensional model of the building and the computed motion at the base to determine the structural response. A proper justification of this approach, beyond the comparison of the translational components of the base motion, was, however, lacking, and the rotational component which would be an important part of the total motion was not presented (and seemed to be ignored).

Finally Lysmer et al. (11) showed results for a nuclear reactor and two adjoining buildings using a purely two-dimensional model and the pseudo three-dimensional solution. Since the only difference between the two models is an increase in damping for the latter, the response in this case was significantly smaller (about 50%) when simulating three-dimensional effects, contrary to most of the other comparisons. The fact that a true three-dimensional solution was not presented and that there were two adjoining structures which were very deeply embedded make this example inappropriate for an adequate evaluation of the errors introduced by the two-dimensional approximation.

## SCOPE

It is clear that the use of a plane strain finite element model for the analysis of a structure resting on the surface of an elastic, homogeneous, halfspace is entirely unjustified. Not only are the available analytical solutions (15, 16) much more accurate than those that could be obtained with the discrete formulation, but the use of these solutions with the three-step approach (or substructure method) would be considerably more economical than the direct (or one-step) solution of the complete soil structure system with any of the available computer programs (LUSH or FLUSH).

In most practical cases, however, the soil properties will not be uniform with depth (the shear modulus will normally increase) or a deposit of finite depth will be underlain by much stiffer, rock-like material. The elastic, uniform halfspace is therefore a mathematical idealization rather than a true physical condition. In addition, most structural foundations, and those of nuclear power plants especially, will have some degree of embedment.

The purpose of this work is to conduct further comparisons between two-dimensional and three-dimensional solutions by obtaining approximate expressions for the dynamic stiffnesses of embedded strip footings and comparing them to those presented by Elsabee(3) for circular foundations.

The dynamic behavior of strip footings had already been extensively studied by Chang Liang (2), who presented a series of curves relating the static stiffnesses (and particularly the flexibilities) to layer depth and embedment. Here, following the work of Kausel (6) and Elsabee (3) for circular foundations, the static stiffnesses are determined first for surface foundations and approximate formulae are fitted to the numerical results. Parametric studies are conducted next for embedded foundations, within the range of embedment ratios of practical interest, and corrective factors are obtained by fitting straight lines. Finally, the variation with frequency of the stiffness coefficients is studied.

The importance of the error introduced by a two-dimensional idealization will depend in a practical situation on the characteristics of the soil profile and the structure considered. (It will be obviously negligible if the soil structure interaction effect is small). Using the results presented in this work and those already available for circular foundations one can, in each specific situation, carry out some simple preliminary computations to estimate the natural frequencies of the soil structure system and the effective modal damping for both types of models before embarking on expensive computer runs. This will allow the assessment of the validity of a plane strain analysis.

# FORMULATION

The parametric studies for surface foundations were carried out with an analytical-type formulation developed by Gazetas (4) and with a finite element model implemented by Chang Liang (2). The former is only applicable to a dynamic situation, i.e. to values of the frequency of vibration different from zero: the static stiffnesses were then obtained by extrapolating the values in the low frequency range. The latter discretizes the soil under the footing by a finite element mesh and places the consistent transmitting boundary developed by Waas (17) directly at the edge of the foundation. This boundary has been shown to reproduce with great accuracy the radiation of waves from the core region into the far field. By using these two approaches, with almost identical results, the validity of the extrapolation procedure to determine the static stiffnesses in the analytical solution and the adequacy of the finite element meshes used could be checked (two meshes were actually used, and a linear extrapolation was applied to obtain improved estimates).

The results for embedded foundations were obtained in all cases with the finite element model, assuming a massless, rigid foundation with the sidewalls (also rigid) welded to the backfill. In order to isolate the effect of embedment, the static stiffnesses were divided by those of an identical footing on the surface. To compute these ratios the values used were the ones corresponding to the same finite element meshes.

The variation of the dynamic stiffnesses with frequency was studied for surface foundations using again the analytical-type solution of Gazetas (4). For this purpose the stiffnesses were written in the form

$$K_x = K_{xo}(k_1 + i\ a_o c_1)(1 + 2iD)$$
$$K_\phi = K_{\phi o}(k_2 + i\ a_o c_2)(1 + 2iD)$$

where $K_{xo}$ and $K_{\phi o}$ are the static values.

$k_1$ $k_2$ $c_1$ $c_2$ are the stiffness coefficients, function of frequency.

D is the internal damping in the soil, of a hysteretic nature (frequency independent).

$a_0 = \Omega B/c_s$ is a dimensionless frequency; $\Omega$ is the frequency of excitation in rad/sec; B is the halfwidth of the footing, and $c_s$ is the shear wave velocity of the soil.

## STATIC STIFFNESSES

Surface Foundations. Figure 1 shows the variation of the horizontal static stiffness with layer depth for different values of Poisson's ratio $\nu$(0, 0.15, 0.3 and 0.45). It can be seen that for values of the ratio B/H (B is the halfwidth of the footing and H the thickness of the soil stratum) larger than 1/4, the computed values fall almost exactly on a straight line. For deeper strata the stiffness decreases much faster and tends, obviously, to zero as H increases.

Figure 2 shows the corresponding results for the rocking stiffness. The values for a halfspace are now finite (nonzero). The variation over the complete range studied, up to values of B/H = 1, is no longer linear but more closely parabolic. It should be noticed, however, that H = 2B represents already a rather shallow stratum (thickness equal to total foundation width). The range of practical interest may then be reduced to values of H/B from 1/8 to 1/2. Within this range linear approximations can furnish reasonable results.

By comparison, figure 3 shows the varation of the static stiffnesses versus layer depth for a circular foundation, as reported by Kausel (6), and figure 4 the same results for a square foundation with sides = 2B as computed by Gonzalez (5). Notice that in both cases the studies were limited to the range of R/H or B/H from 1/8 to 1/2.

For this reduced range straight lines were fitted to the computed stiffnesses. The resulting formulae are shown in tables 1 and 2.

Table 1 - Horizontal Static Stiffness

| Poisson's ratio $\nu$ | $K_{xo}/G$ |
|---|---|
| 0 | 0.904 (1 + 2.5 B/H) |
| 0.15 | 1.017 (1 + 2.37 B/H) |
| 0.30 | 1.175 (1 + 2.15 B/H) |
| 0.45 | 1.419 (1 + 1.95 B/H) |

Table 2 - Rocking Static Stiffness

| Poisson's ratio $\nu$ | $K_{\phi o}/GB^2$ |
|---|---|
| 0 | 1.891 (1 + 0.17 B/H) |
| 0.15 | 2.094 (1 + 0.17 B/H) |
| 0.30 | 2.394 (1 + 0.17 B/H) |
| 0.45 | 2.907 (1 + 0.24 B/H) |

Writing the static stiffnesses in the form

$$K_{xo} = aG(1 + b\ B/H)$$
$$K_{\phi o} = cGB^2(1 + d\ B/H)$$

Figure 5 shows the variation of the coefficients a, c with Poisson's ratio $\nu$. The available analytical expressions for both strip footings and circular foundations indicate that a should be inversely proportional to $2-\nu$, whereas c should vary inversely to $1-\nu$. These results are based, however, on the assumption of a smooth footing (the horizontal stiffness being associated only with horizontal shear forces and the rocking stiffness resulting exclusively from applied normal stresses). Expressions of these forms do not fit exactly the results obtained here (it was found that the coefficients a, c, tended to vary more closely with the inverse of $4-3\nu$). Over the range of practical interest for soils, however, with values of $\nu$ from 0.3 to 0.5, a good approximation is provided by

$$a = \frac{2.1}{2-\nu} \qquad c = \frac{1.62}{1-\nu}$$

As indicated by the expressions in tables 1 and 2, the coefficients b and d are also functions of Poisson's ratio. Within the range $\nu = 0.3$ to 0.5 one can again take average values of these coefficients without too much error. With these approximations the expressions for the static stiffnesses become

$$K_{xo} = \frac{2.1}{2-\nu} G(1 + 2\ B/H)$$

$$K_{\phi o} = \frac{1.62}{1-\nu} GB^2(1 + 0.2\ B/H)$$

The expressions used by Luco and Hadjian (8) in their study for a halfspace were

$$K_x = \frac{0.94\pi}{2-\nu} G \simeq \frac{2.95}{2-\nu} G$$

$$K_\phi = \frac{\pi}{2(1-\nu)} GB^2 \simeq \frac{1.57}{1-\nu} GB^2$$

The agreement for the rocking stiffness (1.62 vs. 1.57) is very good. The values of the horizontal stiffnesses differ, however, by almost 50%. This discrepancy is due to the fact that $K_{xo}$ for a halfspace is actually zero and the expression used by Luco and Hadjian corresponds to an average over the high frequency range. For a halfspace or a deep soil stratum, $K_x$ will increase with frequency initially and then remain essentially constant (with some oscillations) as will be shown later.

By comparison the formulae suggested by Kausel (6) for a circular foundation are

$$K_{xo} = \frac{8GR}{2-\nu} (1 + \frac{1}{2} R/H)$$

$$K_{\phi o} = \frac{8GR^3}{3(1-\nu)} (1 + \frac{1}{6} R/H)$$

The expressions for a square foundation proposed by Gonzalez (5) are

$$K_{xo} = \frac{4.2GB}{2-\nu}(1 + 2\ B/H)$$

$$K_{\phi o} = \frac{3.24GB^3}{1-\nu}(1 + 0.2\ B/H)$$

The agreement between the static stiffnesses computed from a two-dimensional, plane strain model and the true three-dimensional solution will depend therefore on the ratio B/H, particularly for the horizontal case. For very shallow layers the two-dimensional approximation may be reasonable, but as the depth of the stratum increases, it deteriorates considerably. Within the range of values of B/H of practical interest, 1/8 to 1/2, the horizontal stiffness of the strip footing is from 50 to 70% of that of a square footing and the rocking stiffness from 80 to 85%.

Embedded Foundations. The static stiffnesses of embedded foundations were obtained for embedment ratios E/B of 1/3, 2/3 and 1, and layer depths H = 8B, 4B and 2B. E is the depth of embedment and B is again the half-width of the footing. These solutions were obtained with the finite element computer program developed by Chang-Liang (2).

Following the work of Elsabee (3), these stiffnesses were divided by those of a surface foundation with the same width and total layer depth. These ratios are shown in figure 6. The stiffnesses were then written in the form

$$K_{xo} = \frac{2.1}{2-\nu}G(1 + 2\ B/H)(1 + \frac{\alpha}{\gamma}\ B/H)\gamma$$

$$K_{\phi o} = \frac{1.62}{1-\nu}GB^2(1 + 0.2\ B/H)(1 + \frac{\delta}{\beta}\ B/H)\beta$$

The coefficients $\alpha/\gamma$, $\gamma$, $\delta/\beta$ and $\beta$ were then computed and plotted versus E/B as shown in figure 7. From these figures the approximate expressions are obtained:

$$\gamma = 1 + 0.3\frac{E}{B} \qquad \frac{\alpha}{\gamma} = \frac{4}{3}\ \frac{E}{B}$$

$$\beta = 1 + \frac{E}{B} \qquad \frac{\delta}{\beta} = \frac{2}{3}\ \frac{E}{B}$$

valid for ratios $B/H \leq 1/2$ and $E/B \leq 2/3$.

The final formulae are then

$$K_{xo} = \frac{2.1}{2-\nu} G(1 + 2 \frac{B}{H})(1 + \frac{1}{3} \frac{B}{H})(1 + \frac{4}{3} \frac{B}{H})$$

$$K_{\phi o} = \frac{1.62}{1-\nu} GB^2(1 + 0.2 \frac{B}{H})(1 + \frac{E}{B})(1 + \frac{2}{3} \frac{E}{H})$$

The corresponding formulae for embedded circular foundations (3) are

$$K_{xo} = \frac{8GR}{2-\nu} GB^2(1 + \frac{1}{2} \frac{R}{H})(1 + \frac{2}{3} \frac{E}{R})(1 + \frac{5}{4} \frac{E}{H})$$

$$K_{\phi o} = \frac{8GR^3}{3(1-\nu)} (1 + \frac{1}{6} \frac{R}{H})(1 + 2 \frac{E}{R})(1 + 0.7 \frac{E}{H})$$

It is interesting to notice that the coefficients of the term E/H, ratio of embedment depth to total layer depth, are very similar for both cases. On the other hand the coefficients of the term E/B or E/R, ratio of embedment depth to footing halfwidth or foundation radius, are approximately half for the strip footing.

DYNAMIC STIFFNESSES

The variation of the dynamic stiffnesses with frequency was studied, using the analytical-type solution, for the case of a surface foundation, layer depths H = 8B and H = 4B and a soil deposit with a Poisson's ratio $\nu = 0.3$. Three values of internal soil damping D (of a hysteretic nature) were used: D = 0.05, 0.10, and 0.20.

The stiffnesses were written in the form

$$K_x = K_{xo}(k_1 + i\, a_o c_1)(1 + 2iD)$$

$$K_\phi = K_{\phi o}(k_2 + i\, a_o c_2)(1 + 2iD)$$

with $a_o = \Omega B/c_s$, $\Omega$ the frequency of excitation in radians/second, and $c_s$ the shear wave velocity of the soil.

Figures 8 to 10 show the variation of the horizontal stiffnesses $\frac{K_{xo}}{G} k_1$ and $\frac{K_{xo}}{G} c_1$, versus $a_o$ for the three values of D considered. It can be noticed that while the stiffnesses are strongly dependent on the ratio B/H in the low frequency range (including the static case) the values in the higher frequency range oscillate around those of the half-space (these oscillations correspond to the natural frequencies of the soil stratum). As the internal damping increases, these oscillations become less pronounced, and the effect of finite depth of the layer tends to disappear. The values of $\frac{K_{xo}}{G} k_1$ increase slightly with increasing D, but this increase is relatively small for typical values of damping in soils (of the order of 5 to 10%). The complex term of the dynamic stiffness, $\frac{K_{xo}}{G} c_1$, would be zero below the fundamental shear frequency of an elastic stratum ($a_o = \frac{\pi}{2} \frac{B}{H}$), since in this range of frequencies there cannot be any radiation of waves into the far field. Above this frequency, the values of this term approach again those corresponding to a halfspace with some oscillations. The effect of the internal soil damping D is to decrease these oscillations and to provide for a smoother continuous transition from the zero value for the static case to the halfspace values.

Figures 11 to 13 show the corresponding results for the rocking stiffness. The effect of layer depth on the term $\frac{K_{\phi o}}{GB^2} k_2$ is not very marked, and for H = 8B the solution is almost identical to that of a halfspace. The effect of increasing the internal damping D is again to reduce the oscillations and to increase the values of $k_2$ in the high frequency range (this increase is only significant for values of D of the order of 0.20 or larger). If the soil had no internal damping, the complex term $\frac{K_{\phi o}}{GB^2} c_2$ would be zero below the fundamental dilatational frequency of the stratum ($a_o = \frac{\pi}{2} \frac{B}{H} \frac{c_p}{c_s}$ with $c_p$ the P wave velocity of the soil and $c_s$ the shear wave velocity). Above this value of $a_o$ the results are again very similar to those of a halfspace. The existence of internal soil damping provides for nonzero values of $c_2$ below the stratum's natural frequency and a smooth transition from zero at $a_o = 0$ to the halfspace curve. Notice, however, that in the low frequency range the damping term $c_2$ will always be substantially smaller for a finite soil layer than for a halfspace.

Figures 14 to 16 and 17 to 19 show the variation with frequency of the terms $k_1c_1$ and $k_2c_2$. This is the form in which the dynamic stiffness coefficients are normally presented for circular foundations. The same basic observations can be made from these figures with respect to the effects of layer depth and of internal soil damping. In this case for higher frequencies the values of $k_1$ increase with layer depth since the product $K_{xo}\ k_1$ would tend to be constant (independent of B/H) and $K_{xo}$ increases with increasing B/H. For comparison, figures 20 and 21 show the dynamic stiffness coefficients for circular foundations presented by Kausel (6). These results correspond to the case of a soil profile with a Poisson's ratio $\nu = 0.33$ and an internal hysteretic damping $D = 0.10$. Results obtained by Gonzalez (5) indicate that the frequency variation of the dynamic stiffnesses for square foundations is very similar to that of circular foundations using an equivalent B (or R) in the expression for $a_o$.

These results indicate that for values of the dimensionless frequency $a_o$ larger than 1.5 to 2 times the fundamental frequency of the soil deposit, the horizontal stiffness term $K_{xo}k_1$ can be approximated (for a surface foundation) by the static value corresponding to H = 4B or

$$K_{xo}k_1 = \frac{3.15G}{2-\nu} = \frac{\pi G}{2-\nu}$$

almost independently of the layer depth (particularly if there is some internal damping in the soil, as would be normally the case). Luco and Hadjian (8) used this same expression for the halfspace multiplied by an averaging factor of 0.94.

It should be noticed that the decrease in the effect of layer depth with increasing frequency $a_o$ also takes place in a circular foundation. It is less dramatic in this case because the effect is smaller, since the corresponding factor is $1 + \frac{1}{2}\ \frac{R}{H}$ instead of $1 + 2\ \frac{B}{H}$. Thus for the same range the values of $a_o$, the dynamic horizontal stiffness of a circular foundation should be written simply as

$$K_{xo}k_1 = \frac{8GR}{2-\nu}$$

For the same range of frequencies the term $K_{xo}c_1$ is approximately 1.1 to 1.15 times $K_{xo}k_1$ or

$$K_{xo}c_1 \approx 2 \text{ to } 2.1G \quad \text{for } \nu = 0.3$$

Luco and Hadjian (8) used a value of $K_{xo}c_1 = 2G$ which is again in very good agreement with these results. Veletsos and Wei (15) have shown that for circular foundations the value of $c_1$ is not independent of Poisson's ratio and therefore the expression for $K_{xo}c_1$ in terms of $\nu$ is not identical to that of $K_{xo}k_1$. Within the range of values of practical interest ($\nu$ = 0.3 to 0.5), the variation is not, however, large. One can thus write approximately

$$K_{xo}c_1 \approx \frac{3.6G}{2-\nu}$$

whereas for a circular foundation

$$K_{xo}c_1 \simeq \frac{4.8GR}{2-\nu}$$

The rocking stiffness term $K_{\phi o}k_2$ is only slightly affected by the layer depth since the factor is 1 + 0.2 B/H. The disappearance of this factor with increasing values of $a_o$ is therefore harder to ascertain from the numerical results. Since the effect is, however, small for typical values of B/H, this question is not of great importance. The variation of this term with frequency is very similar for the strip footing and for the circular foundation. Over the range of frequencies studied it can be approximated as a straight line of the form

$$k_2 = 1 - 0.2a_o$$

The studies by Veletsos and Wei (15) over a more complete range of frequencies indicate that for circular foundations $k_2$ can be approximated by $1 - 0.2a_o$ for values of $\nu$ close to 0.5; for values of $\nu$ of the order of 0.3, this expression is valid (approximately) for $a_o \leq 2.5$. For $a_o \geq 2.5$, $k_2 \simeq 0.5$. A similar type of behavior can be expected for strip footings.

The term $c_2$ is still increasing slowly at a frequency $a_o = 0.5\pi$. It has then a value of about 0.4 for the strip footing and just about 0.2 for the circular foundation. For higher frequencies Luco and Hadjian (8) used values of $c_2 = 0.56$ for the strip footing and 0.4 for the circular foundation (at $\nu = 0.3$), which seem reasonable. Thus while the variation of $c_2$ with frequency looks similar for both cases, the values are somewhat higher for the strip footing, and the increase takes place faster. For a halfspace the results of Veletsos and Wei (15) for a circular foundation can be approximated (for typical values of $\nu$ between 0.3 and 0.5) by

$$c_2 \simeq \frac{0.35a_o^2}{1 + a_o^2}$$

For a strip footing on a halfspace, the corresponding approximation is more nearly

$$c_2 \simeq \frac{a_o^2}{1 + 2a_o^2}$$

## SELECTION OF AN EQUIVALENT FOOTING

In order to reproduce a three-dimensional foundation (circular or nearly square) by a two-dimensional model, it is necessary to select some equivalent dimensions for the strip footing. The width of the footing 2B will affect the various stiffness terms in different form, as well as the dimensionless frequency $a_o$. The thickness in the direction perpendicular to the plane 2C will just be a scaling factor applied to all the terms.

For the purpose of estimating the values of the stiffnesses provided by the models, the results of the previous sections can be summarized in the form of the following formulae:

<u>strip footing</u>

$$K_x = K_{xo}\,(k_1 + i\,\frac{\Omega B}{c_s}\,c_1)(1 + 2iD)$$

with

$$K_{xo} = \frac{2\pi GC}{2-\nu}\,(1 + \frac{1}{3}\,\frac{E}{B})(1 + \frac{4}{3}\,\frac{E}{H})$$

$k_1 = \frac{2}{\pi}\,(1 + 2\,\frac{B}{H})$ for $\Omega = 0$ (static case)

$= 1$ for $\Omega$ larger than 1.5 times the natural shear frequency of the layer. Between these two values of $\Omega$ a transition must be used as shown in figures 8 to 11 or 14 to 16.

$c_1 = 1.1$ for values of $\Omega$ larger than the natural frequency of the layer; a transition curve must be used for lower values of $\Omega$ , depending on the internal damping D.

$$K_\phi = K_{\phi o}\,(k_2 + i\,\frac{\Omega B}{c_s}\,c_2)(1 + 2iD)$$

with

$$K_{\phi o} = \frac{\pi GCB^2}{1-\nu}\,(1 + 0.2\,\frac{B}{H})(1 + \frac{E}{B})(1 + \frac{2}{3}\,\frac{E}{H})$$

$k_2 = 1 - 0.2\,\frac{\Omega B}{c_s}$ for $\frac{\Omega B}{c_s} \leq 2.5$ all $\nu$

$= 1 - 0.2\,\frac{\Omega B}{c_s}$ for $\frac{\Omega B}{c_s} \geq 2.5$ $\nu = 0.45$ to $0.5$

$= 0.5$ for $\frac{\Omega B}{c_s} \geq 2.5$ $\nu = 0.3$ to $0.4$

$c_2 = \frac{a_o^2}{1 + 2a_o^2}$ with $a_o = \frac{\Omega B}{c_s}$ for $\Omega$ larger than the fundamental dilatational frequency of the layer. For smaller values of $\Omega$ use a transition curve depending on the internal damping D.

<u>circular foundations</u>

$$K_x = K_{xo}\left(k_1 + i\frac{\Omega R}{c_s}c_1\right)(1 + 2iD)$$

with $$K_{xo} = \frac{8GR}{2-\nu}\left(1 + \frac{2}{3}\frac{E}{R}\right)\left(1 + \frac{5}{4}\frac{E}{H}\right)$$

$$k_1 = 1 + \frac{1}{2}\frac{R}{H} \quad \text{for } \Omega = 0 \text{ (static case)}$$

$= 1$ for $\Omega$ larger than 1.5 times the natural shear frequency of the layer. Between these two values of $\Omega$, use a transition curve as shown in figure 20.

$$c_1 \approx 0.6$$

$$K_\phi = K_{\phi o}\left(k_2 + i\frac{\Omega R}{c_s}c_2\right)(1 + 2iD)$$

with $$K_{\phi o} = \frac{8GR^3}{3(1-\nu)}\left(1 + \frac{1}{6}\frac{R}{H}\right)\left(1 + 2\frac{E}{R}\right)\left(1 + 0.7\frac{E}{H}\right)$$

$$k_2 = 1 - 0.2\frac{\Omega R}{c_s} \quad \text{for } \frac{R}{c_s} \leq 2.5 \quad \text{all } \nu$$

$$= 1 - 0.2\frac{\Omega R}{c_s} \quad \text{for } \frac{\Omega R}{c_s} \geq 2.5 \quad \nu = 0.45 \text{ to } 0.5$$

$$= 0.5 \quad \text{for } \frac{\Omega R}{c_s} \geq 2.5 \quad \nu = 0.3 \text{ to } 0.4$$

$$c_2 = \frac{0.35\,a_o^2}{1 + a_o^2} \quad \text{with } a_o = \frac{\Omega R}{c_s}$$ for $\Omega$ larger than the fundamental dilatational frequency of the layer. Transition depending on D for smaller values of $\Omega$.

Elsabee (3) has recommended for the transition curves in the case of circular foundations

$$c_1 = 0.65D\,\frac{\alpha}{1 - (1-2D)\alpha^2} \qquad \text{for } \alpha = \frac{a_o}{a_{o1}} \leq 1$$

$$c_2 = 0.50D\,\frac{\alpha}{1 - (1-2D)\alpha^2} \qquad \text{but } \frac{0.35a_o^2}{1 + a_o^2} \quad \text{for } \alpha = \frac{a_o}{a_{o1}} \leq \frac{c_p}{c_s}$$

Similar expressions could be used as a first approximation for the strip footing, although their applicability was not checked in this study.

As Luco and Hadjian (8) had already pointed out, it is not possible to select values of footing width and thickness so as to match all the stiffness terms over the complete frequency range. Accounting for a layer of finite depth and embedment complicates the problem further. Even so, it must be recognized that not all the terms have the same effect or importance on the structural response and that some degree of error may be tolerated in some of them. In order to assess the validity of a two-dimensional model it would be advisable, for each particular problem, to estimate the natural frequency of the soil structure system and the effective amount of damping. These quantities can be computed with various degrees of accuracy (and complexity). In the simplest case, if the structure is modelled as an equivalent single-degree-of-freedom system with a mass M (or weight w) concentrated at a height h above the base of the foundation, and a spring k such that $\omega_o = \sqrt{k/M}$ is the fundamental frequency of the structure on a rigid foundation (without any soil structure interaction effect), the natural frequency of the combined soil structure system can be estimated by

$$\omega = \frac{\omega_o}{\sqrt{1 + \frac{k}{K_{xo}k_1} + \frac{kh^2}{K_{\phi o}k_2}}}$$

A simple iterative procedure can be used to account for the frequency variation of $K_{xo}k_1$ and $K_{\phi o}k_2$.

Once the natural frequency $\omega$ is known, the effective damping can be estimated from

$$\beta = \beta_o \left(\frac{\omega}{\omega_o}\right)^2 + D\left[1 - \left(\frac{\omega}{\omega_o}\right)^2\right] + \frac{1}{2}\frac{\omega B}{c_s}\left(\frac{\omega}{\omega_o}\right)^2 \left[\frac{k}{K_{xo}k_1}\frac{c_1}{k_1} + \frac{kh^2}{K_{\phi o}k_2}\frac{c_2}{k_2}\right]$$

where $\beta_o$ is the structural damping (assumed to be of a hysteretic nature), D is the internal soil damping (also hysteretic) and β is the effective damping of the combined system. For circular foundations the term $\omega B/c_s$ should be replaced by $\omega R/c_s$.

By computing ω and β using the formulae for both strip footings and circular foundations, one can determine whether the error introduced by the two-dimensional approximation will be acceptable for the case considered.

## CONCLUSIONS AND RECOMMENDATIONS

A series of parametric studies were conducted to determine the dynamic stiffnesses of strip footings. From the results of these studies approximate formulae were derived which were then compared to those already available for circular and square foundations. The differences between these formulae are such that important errors may result in some cases when attempting to reproduce a three-dimensional problem by a two-dimensional model. Furthermore, it appears that increasing the amount of radiation damping for the strip footing, as done in some pseudo three-dimensional solutions, may not improve in general the agreement between the two models. It is therefore recommended that before embarking on the use of computer programs based on a plane strain model, some preliminary analyses be conducted to assess the magnitude of the errors involved in this idealization for the particular case under consideration.

## REFERENCES


1. Berger, E., Lysmer, J. and Seed, H.B., "Comparison of Plane Strain and Axisymmetric Soil Structure Interaction Analysis," 2nd ASCE Specialty Conference on Structural Design of Nuclear Plant Facilities, New Orleans, 1975.

2. Chang Liang, V., "Dynamic Response of Structures in Layered Soils," Research Report R74-10, Dept. of Civil Engineering, M.I.T., January 1974.

3. Elsabee, F. and Morray, J.P., "Dynamic Behavior of Embedded Foundations," Research Report R77-33. Dept. of Civil Engineering, M.I.T., September 1977.

4. Gazetas, G. and Roesset, J.M., "Forced Vibrations of Strip Footings on Layered Soils," Proc. National Structures Engineering Conference, ASCE, Madison, Wisconsin, August 1976.

5. Gonzalez, J.J., "Dynamic Interaction between Adjacent Structures," Research Report R77-30, Dept. of Civil Engineering, M.I.T., September 1977.

6. Kausel, E., "Forced Vibrations of Circular Foundations on Layered Media," Research Report R74-11, Dept. of Civil Engineering, M.I.T., January 1974.

7. Kausel, E., Roesset, J.M. and Christian, J.T., "Nonlinear Behavior in Soil Structure Interaction," Journal of the Geotechnical Engineering Division, ASCE, November 1976.

8. Luco, J.E. and Hadjian, A.H., "Two-Dimensional Approximations to the Three-Dimensional Soil Structure Interaction Problem," Nuclear Engineering and Design, 1974.

9. Luco, J.E., "Impedance Functions for a Rigid Foundation on a Layered Medium," Nuclear Engineering and Design, 1974.

10. Lysmer, J., Udaka, T., Seed, H.B. and Hwang, R.N., "LUSH, A Computer Program for Complex Response Analysis of Soil Structure Systems," Report EERC 74-4, University of California, Berkeley, 1974.

11. Lysmer, J., Seed, H.B., Udaka, T., Hwang, R.N. and Tsai, C.F., "Efficient Finite Element Analysis of Seismic Soil Structure Interaction," Report EERC 75-34, University of California, Berkeley, November 1975.

12. Novak, M., "Vibrations of Embedded Footings and Structures," Preprint 2029, ASCE National Structures Engineering Meeting, San Francisco, California, April 1973.

13. Novak, M. and Beredugo, Y.O., "The Effect of Embedment on Footing Vibrations," Proc. First Canadian Conference on Earthquake Engineering Research, Vancouver, May 1971.

14. Urlich, C.M. and Kuhlemeyer, R.L., "Coupled Rocking and Lateral Vibrations of Embedded Footings," Canadian Geotechnical Journal, Vol. 10, No. 2, May 1973.

15. Veletsos, A.S. And Wei, Y.T., "Lateral and Rocking Vibration of Footings," Journal of the Soil Mechanics and Foundations Division, ASCE, Vol. 97, September 1971.

16. Veletsos, A.S. and Verbic, B., "Vibration of Viscoelastic Foundations," International Journal of Earthquake Engineering and Structural Dynamics, Vol. 2, 1973.

17. Waas, G., "Earth Vibration Effects and Abatement for Military Facilities," Technical Report S71-14, U.S.A.E.W.E.S., September 1972.

18. Wong, H.L., "Dynamic Soil Structure Interaction," Report EERL 75-01, California Institute of Technology, May 1975.

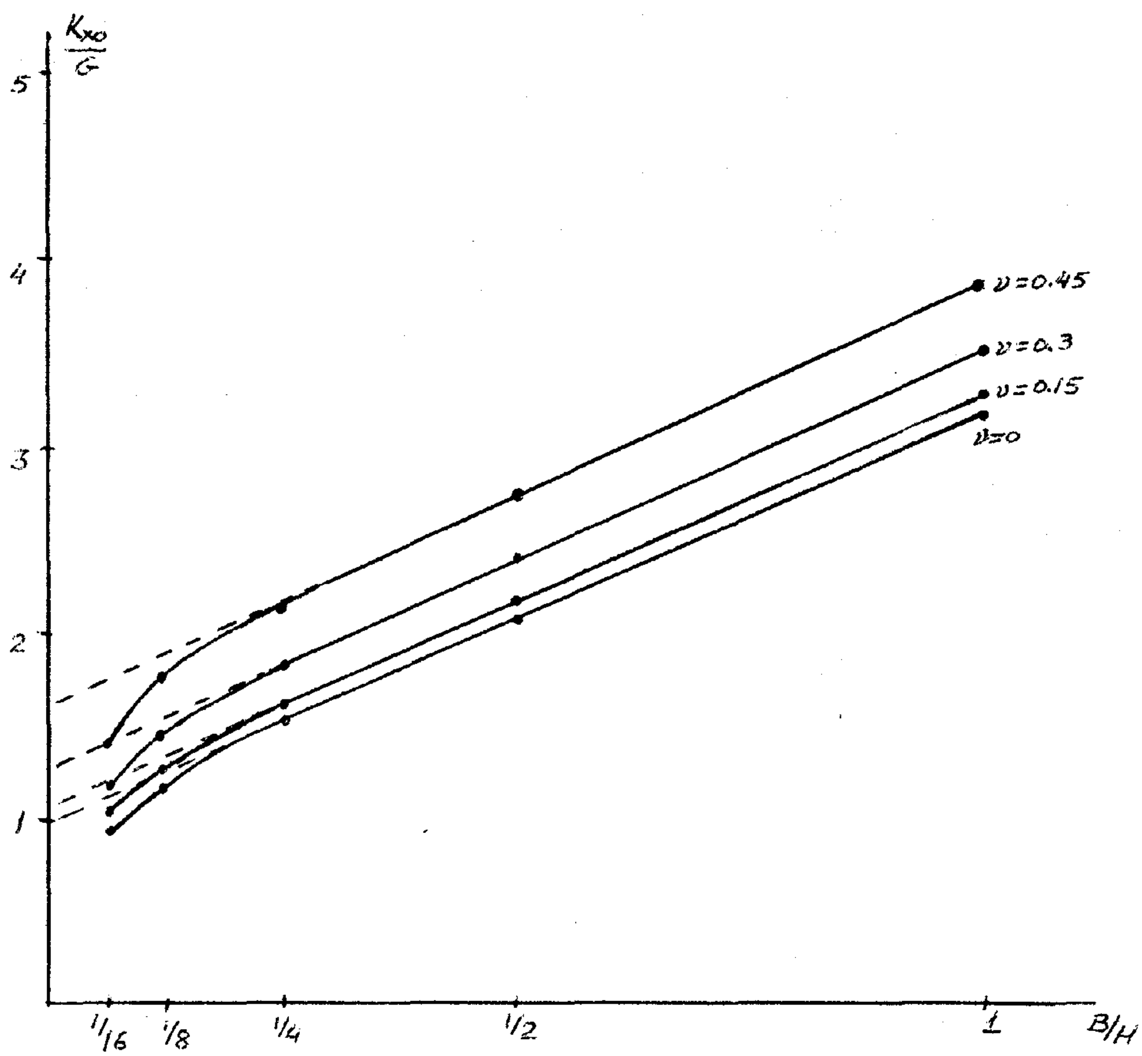



FIGURE 1 - VARIATION OF HORIZONTAL STATIC STIFFNESS WITH LAYER DEPTH - STRIP FOOTING

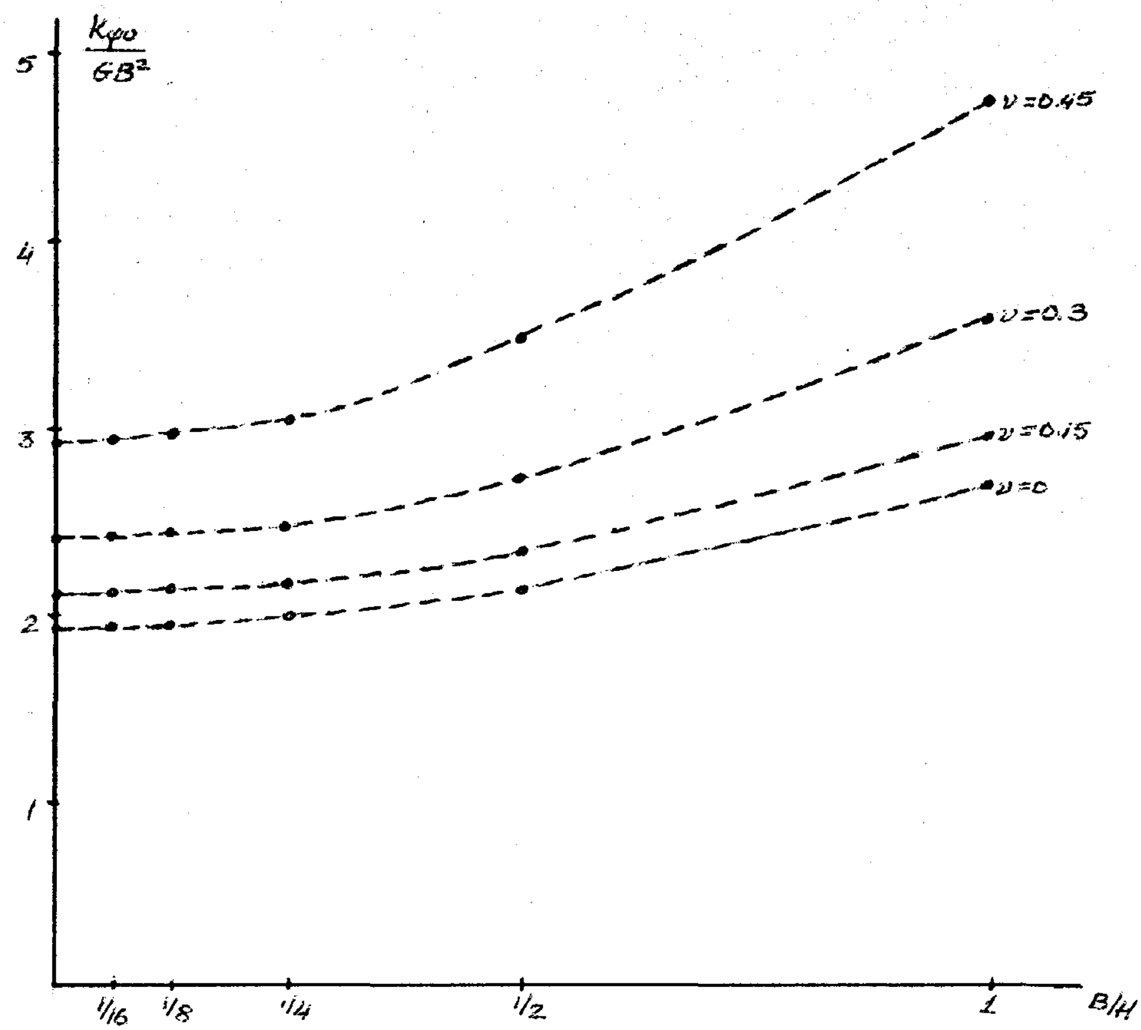



FIGURE 2 VARIATION OF ROCKING STATIC STIFFNESS WITH LAYER DEPTH – STRIP FOOTING

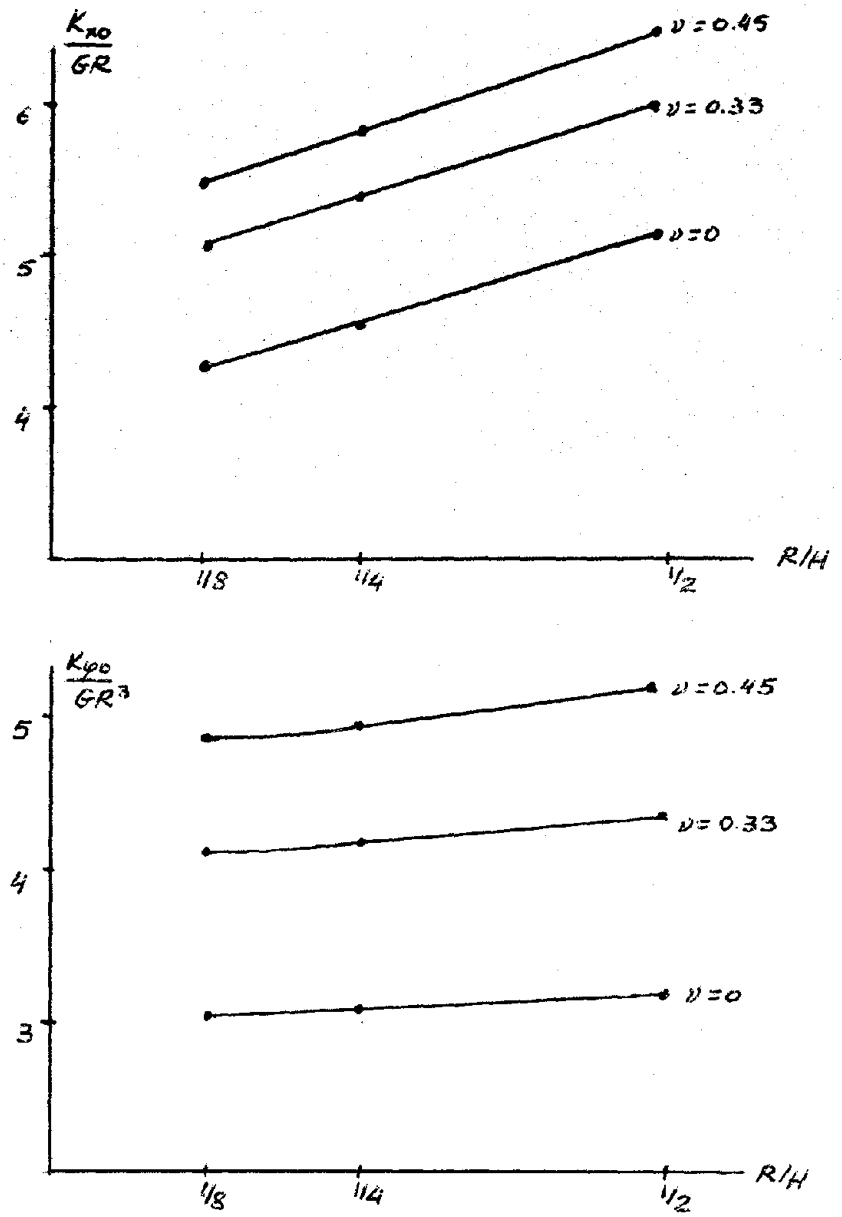



FIGURE 3 - VARIATION OF STATIC STIFFNESSES WITH LAYER DEPTH - CIRCULAR FOUNDATION (after Kausel, 6)

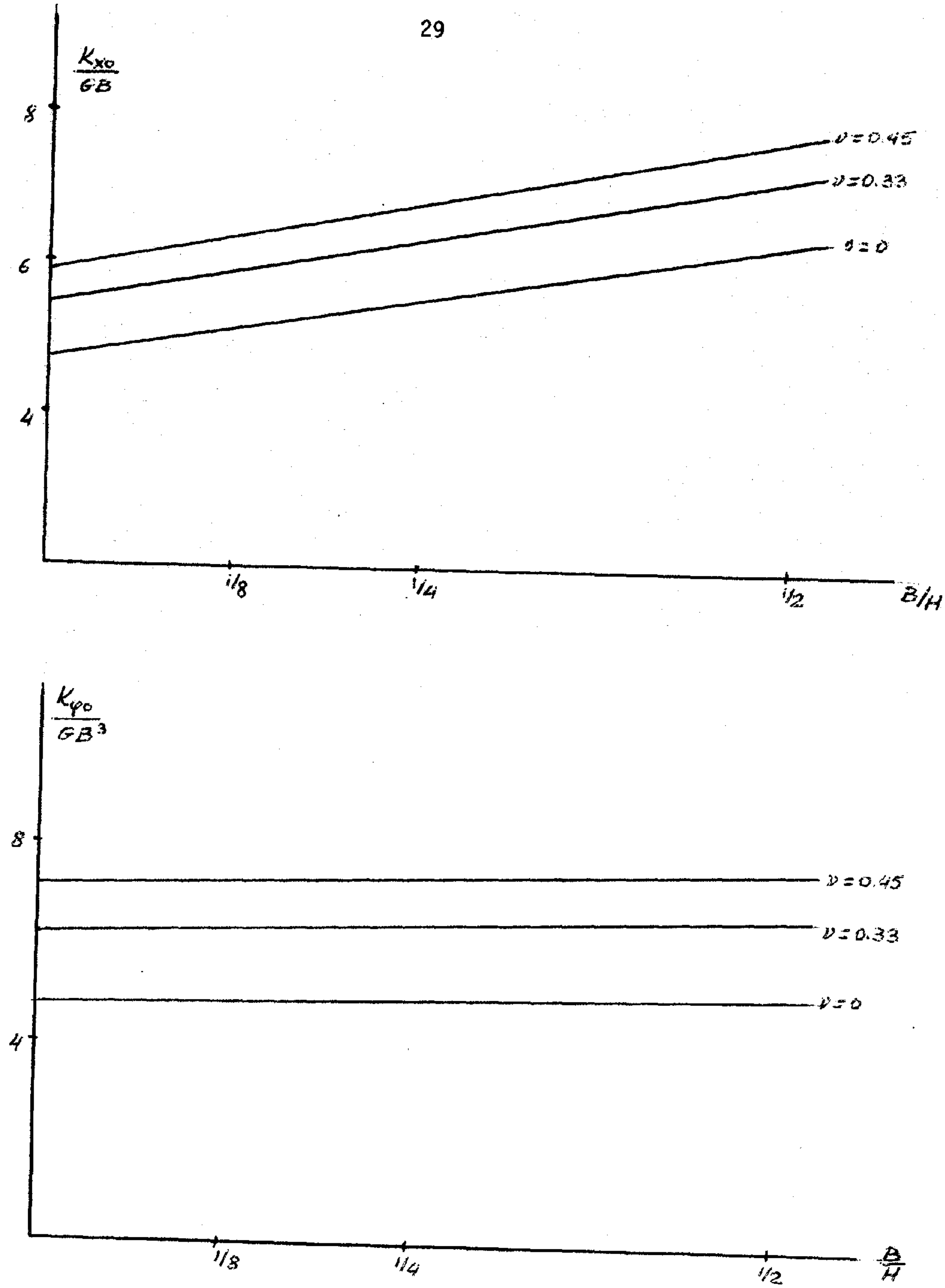



FIGURE 4. VARIATION OF STATIC STIFFNESSES WITH LAYER DEPTH. SQUARE FOUNDATION (after Gonzalez, 5)

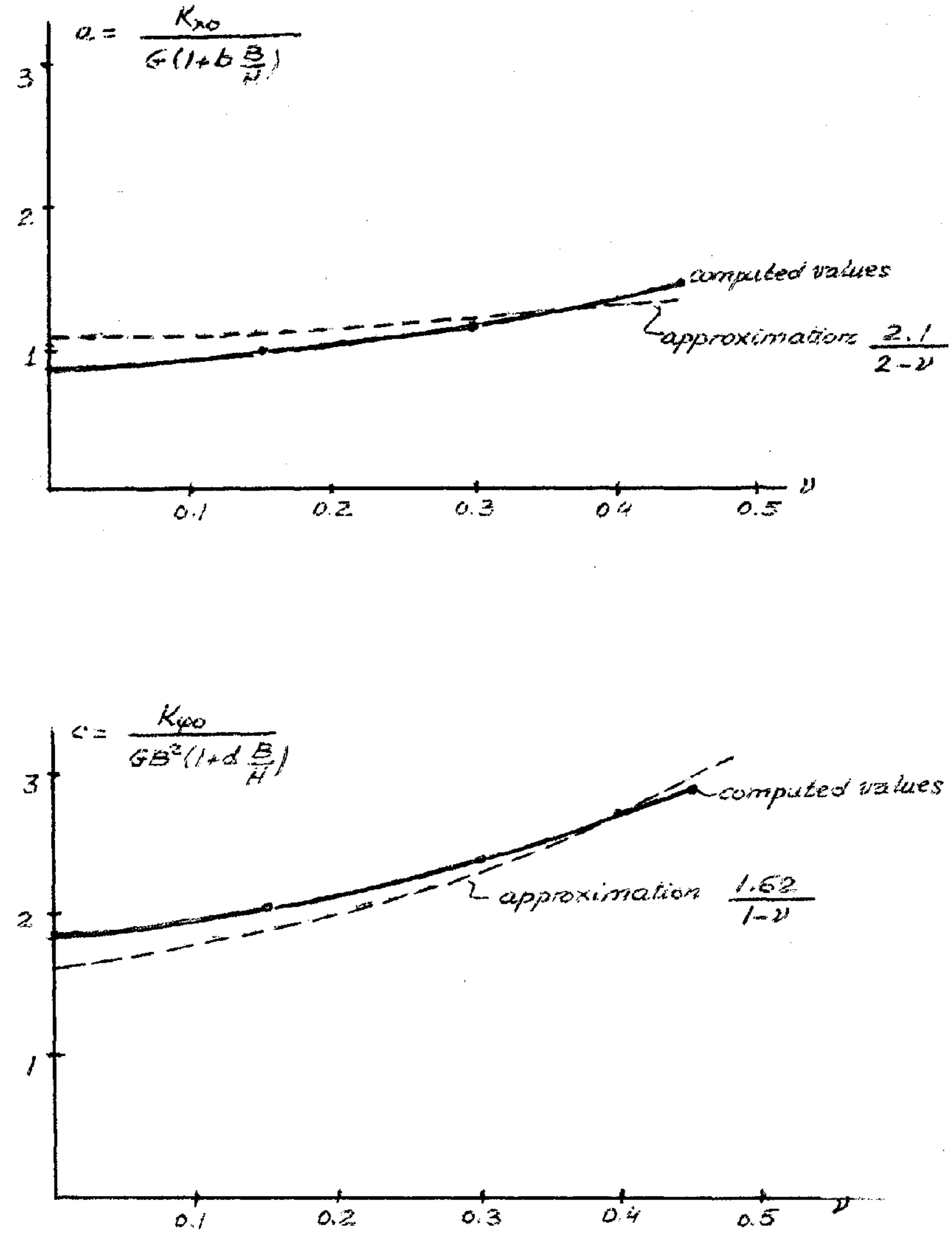



FIGURE 5. VARIATION OF STATIC STIFFNESSES WITH POISSON'S RATIO $\nu$

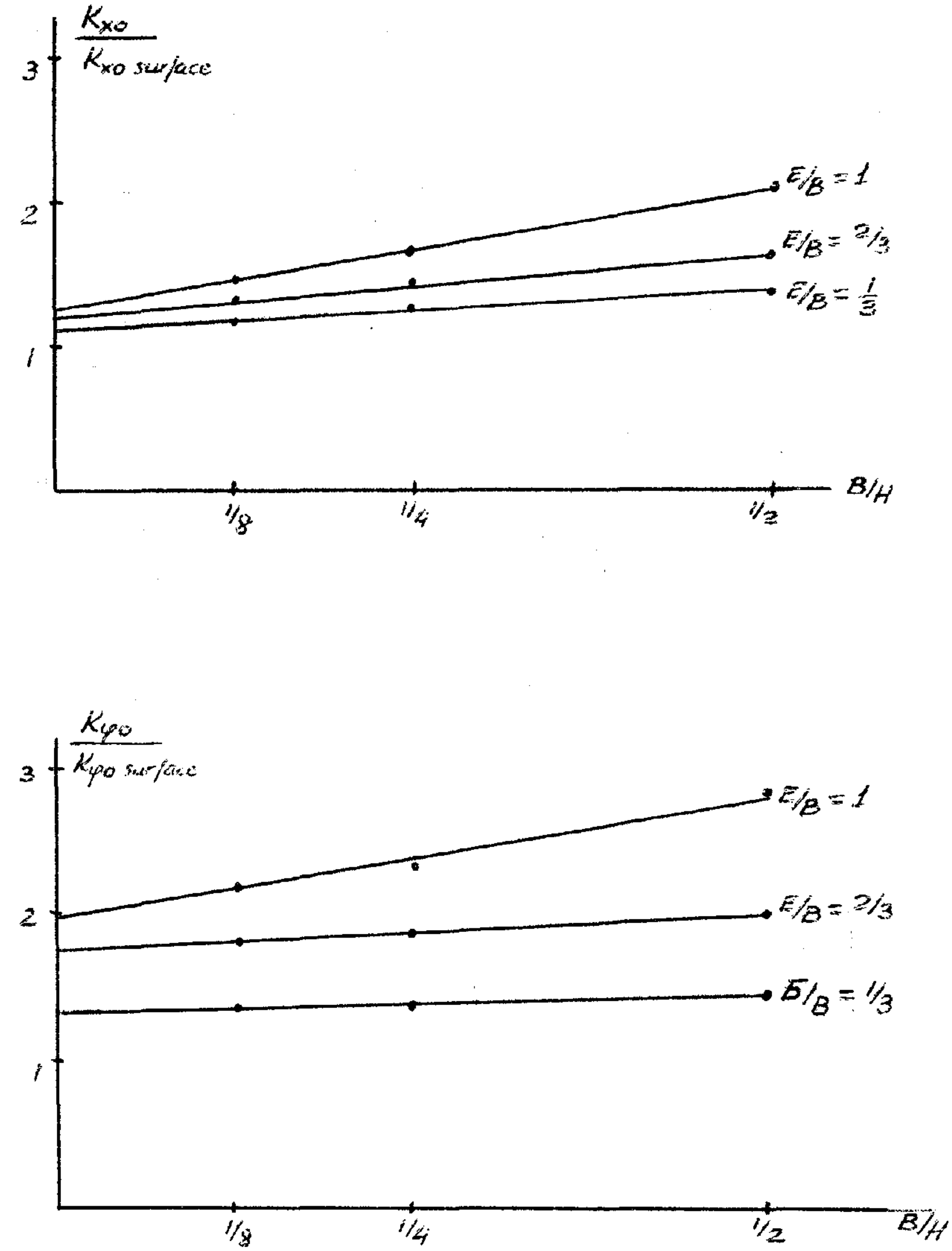



FIGURE 6. STATIC STIFFNESSES OF EMBEDDED FOUNDATIONS

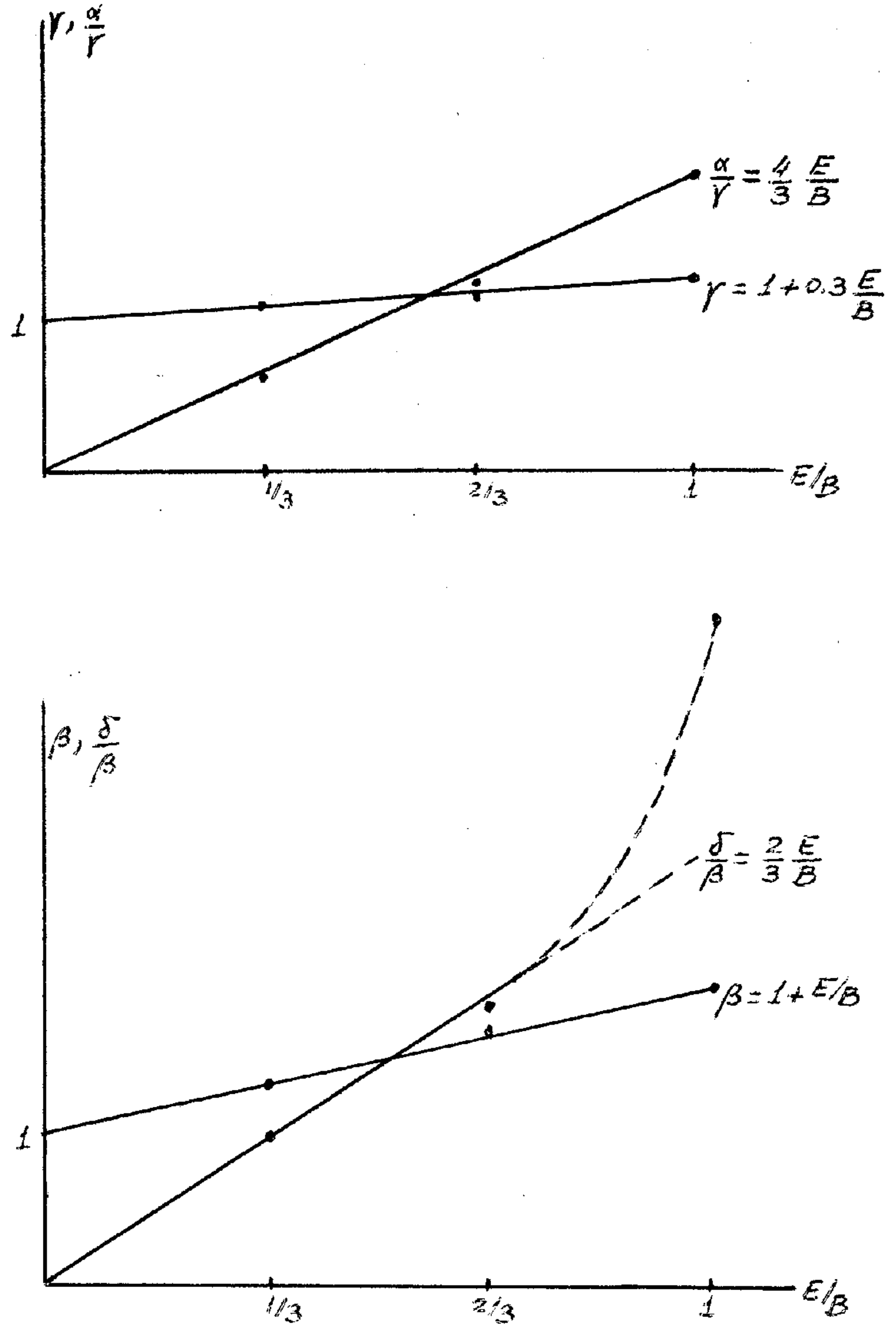



FIGURE 7. COEFFICIENTS FOR EMBEDDED FOUNDATIONS

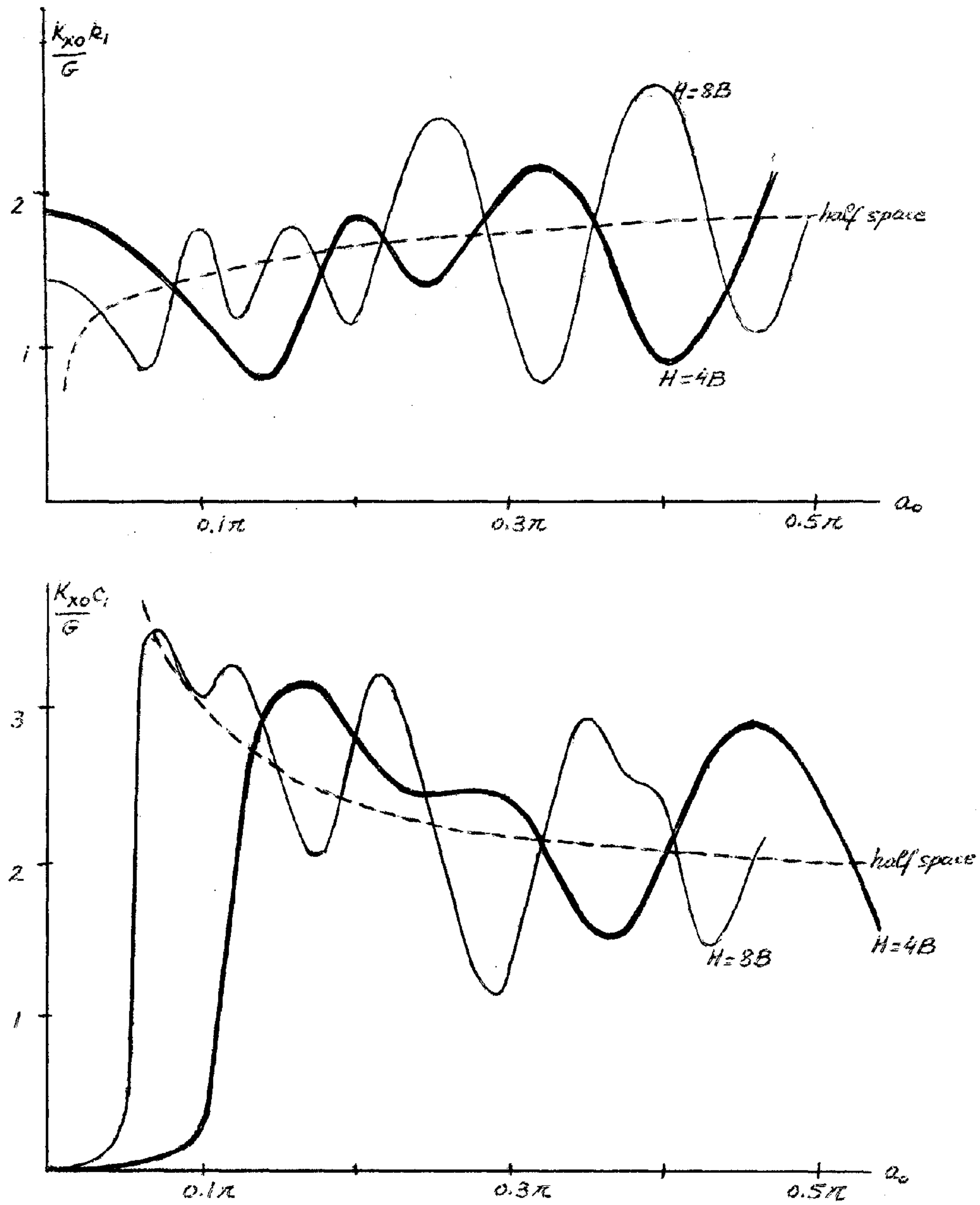



FIGURE 8. HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS
$D = 0.05$

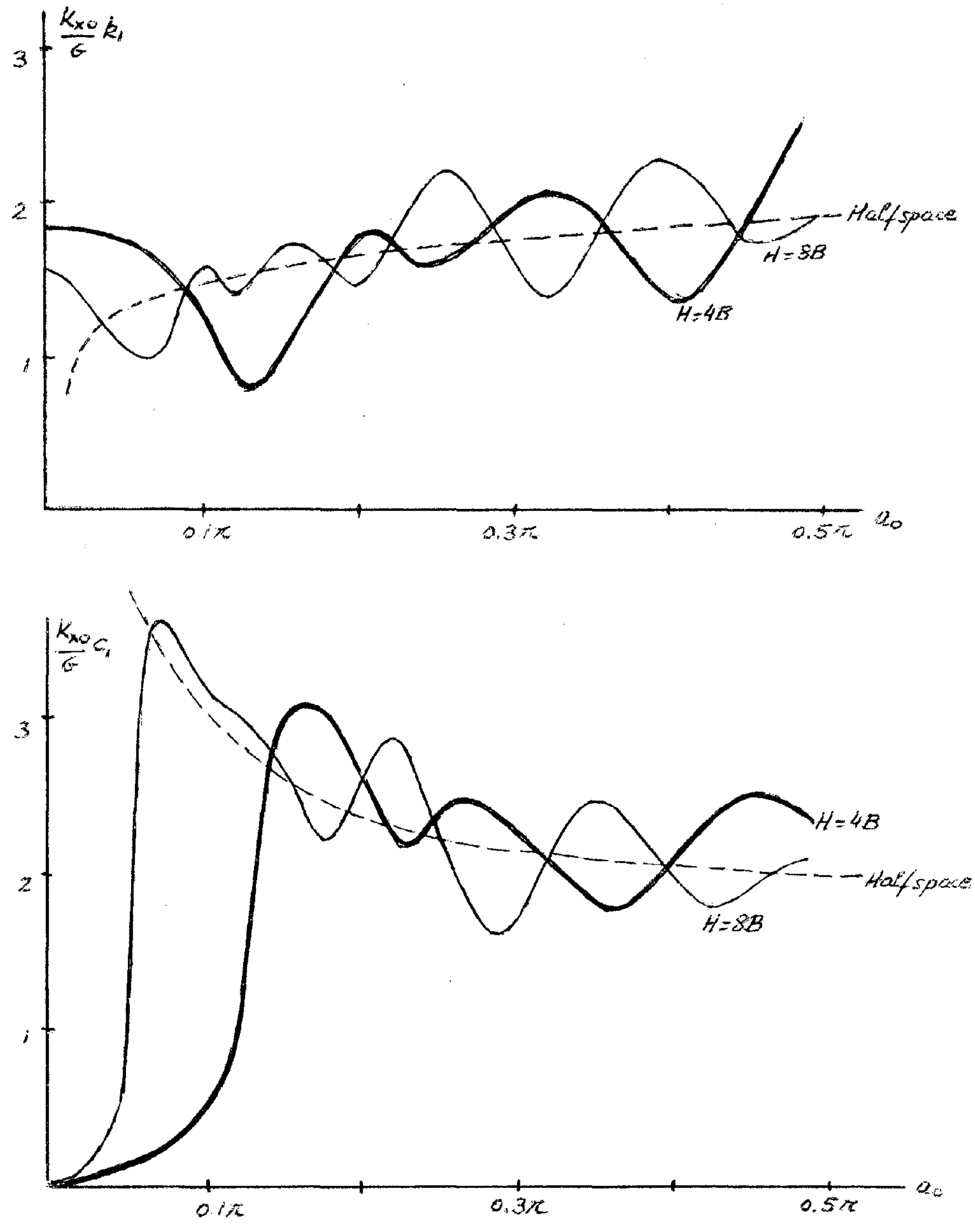
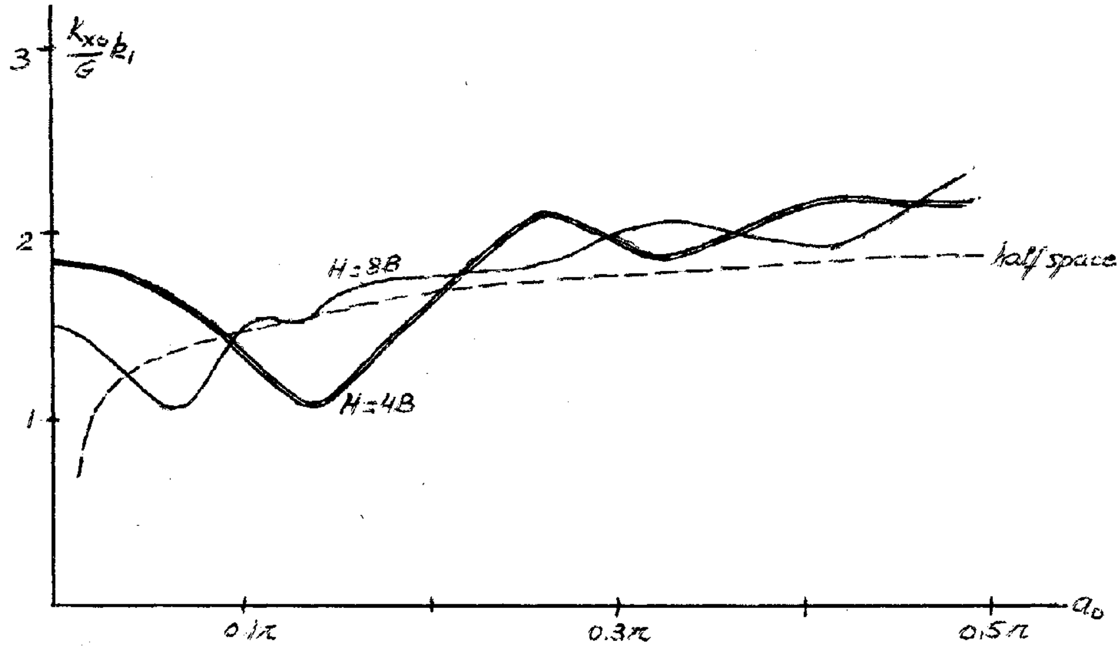



FIGURE 9. HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS

D = 0.10



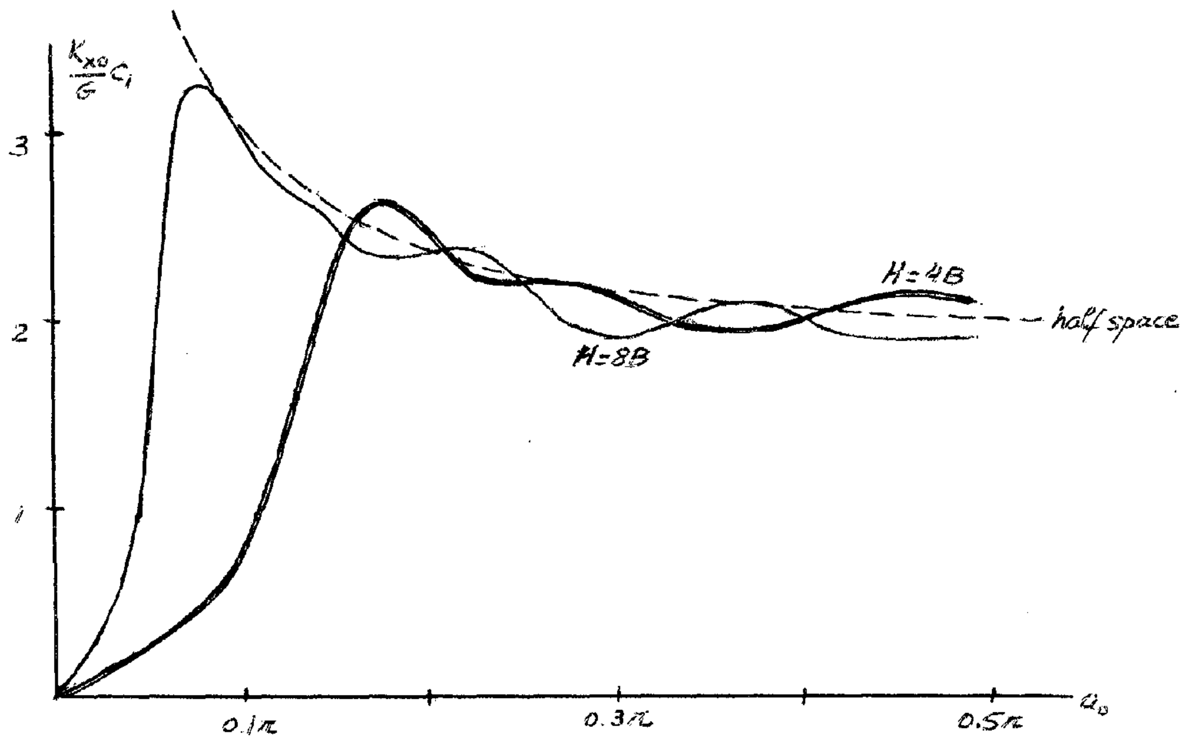




FIGURE 10 HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS

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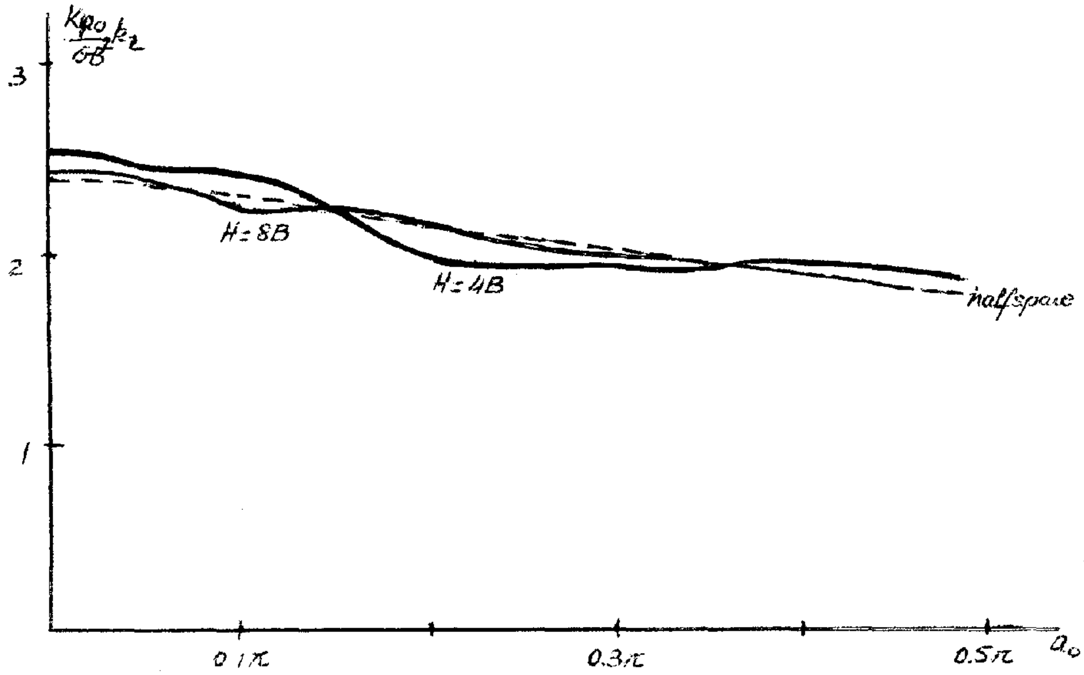


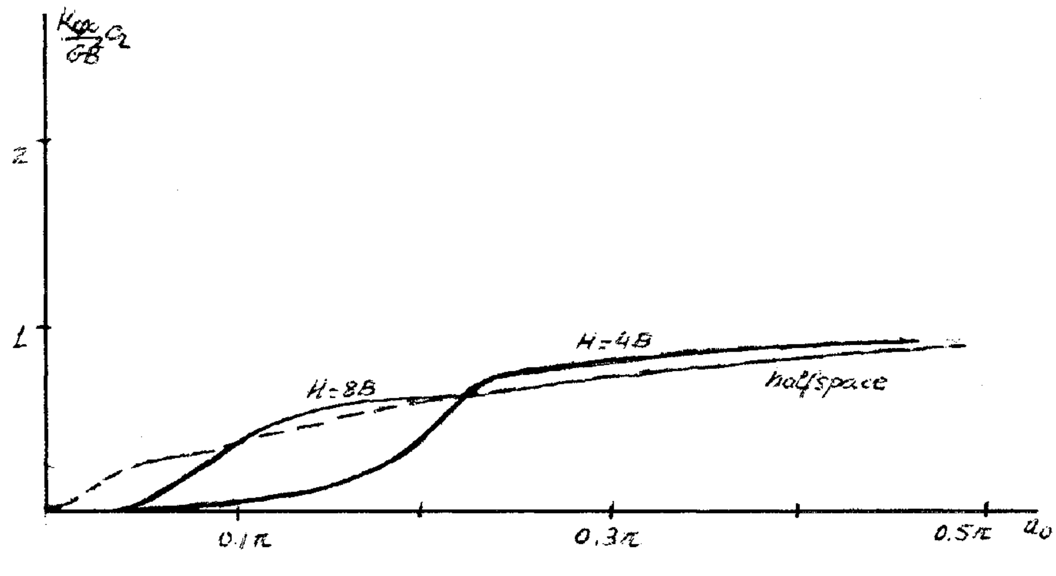




FIGURE 11 - ROCKING DYNAMIC STIFFNESS COEFFICIENTS

$D = 0.05$

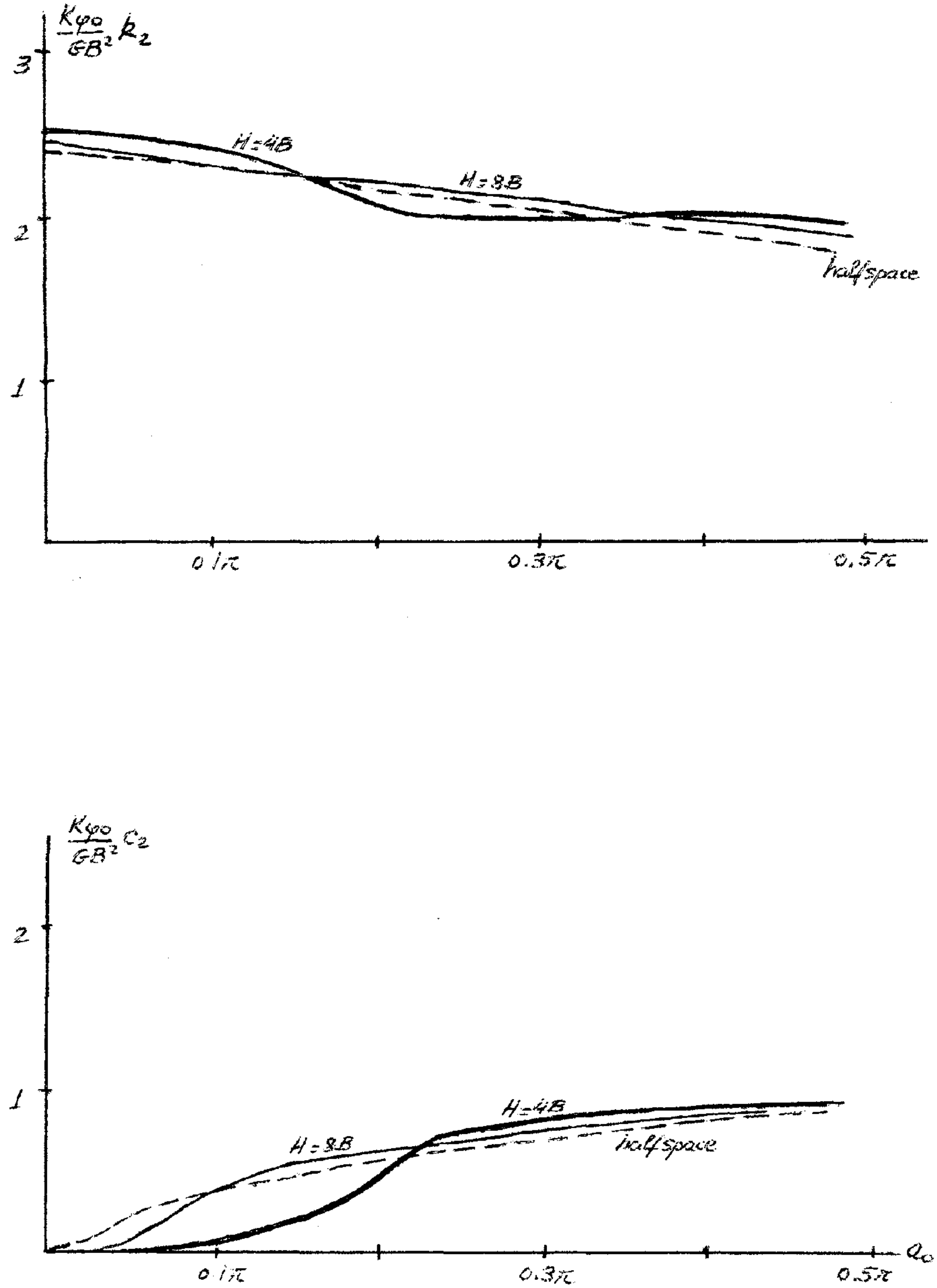
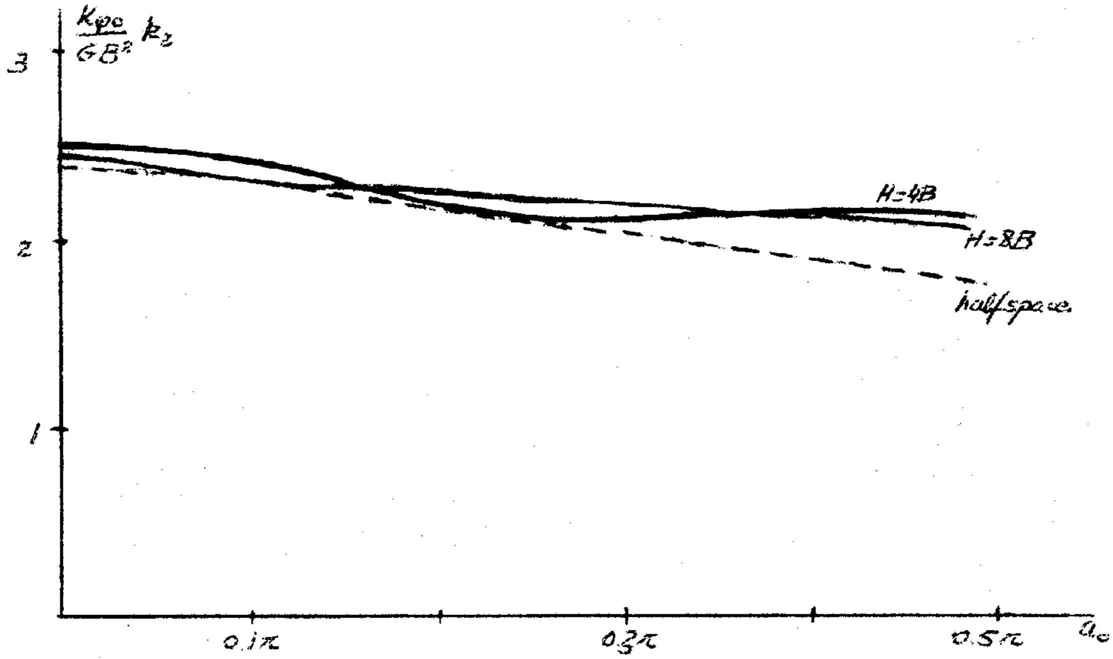



FIGURE 12 ROCKING DYNAMIC STIFFNESS COEFFICIENTS

D = 0.10



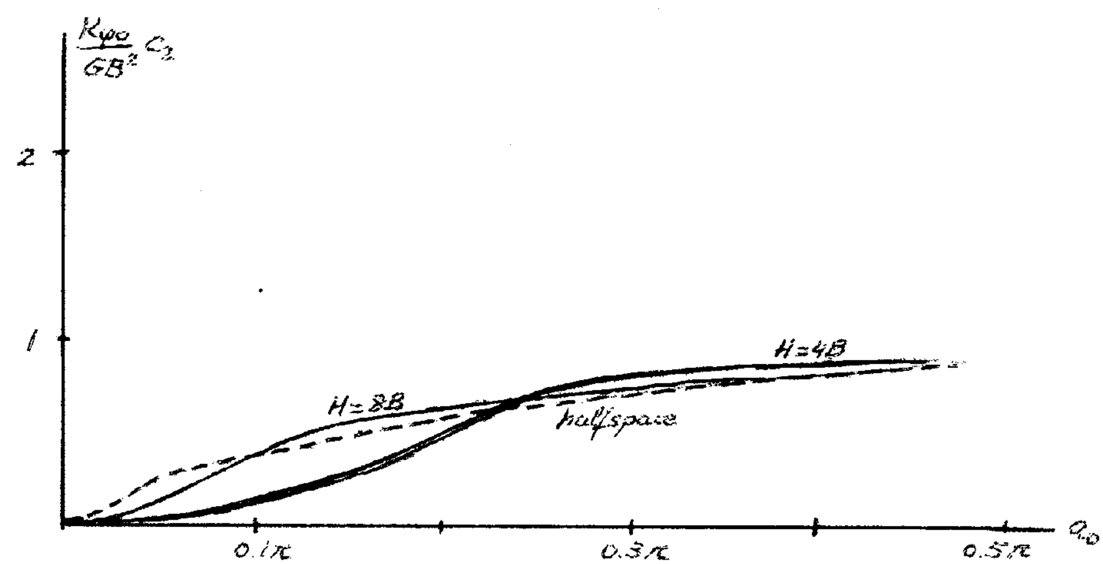
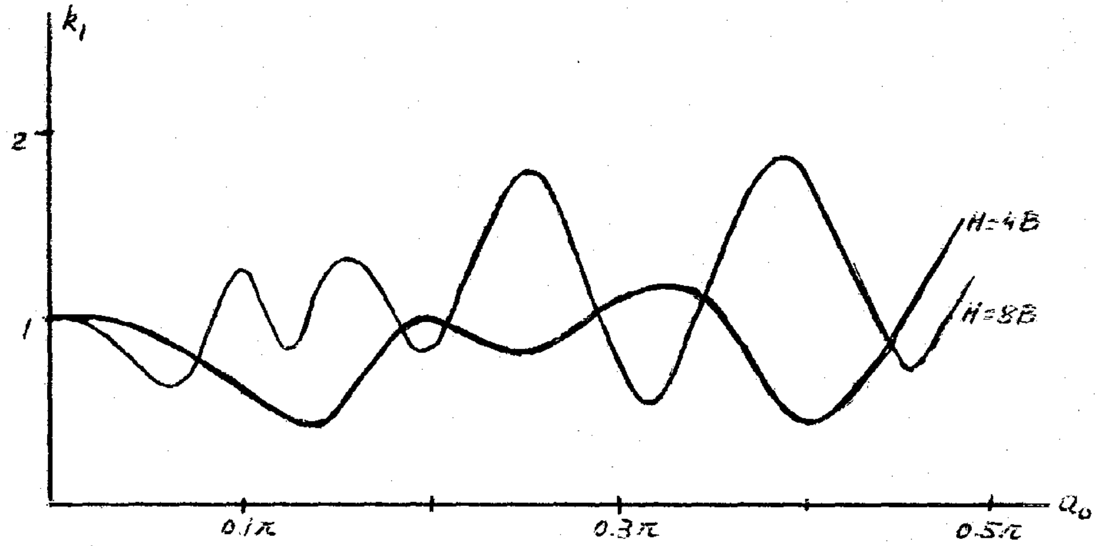




FIGURE 13 ROCKING DYNAMIC STIFFNESS COEFFICIENTS
$D = 0.20$



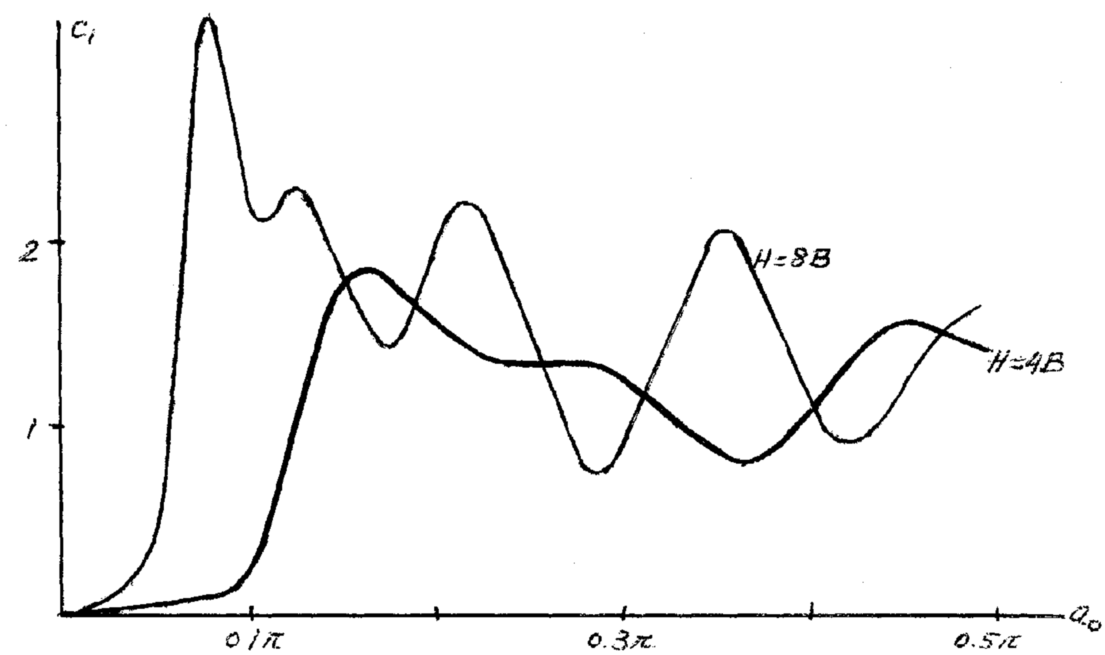




FIGURE 14. HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS
D = 0.05

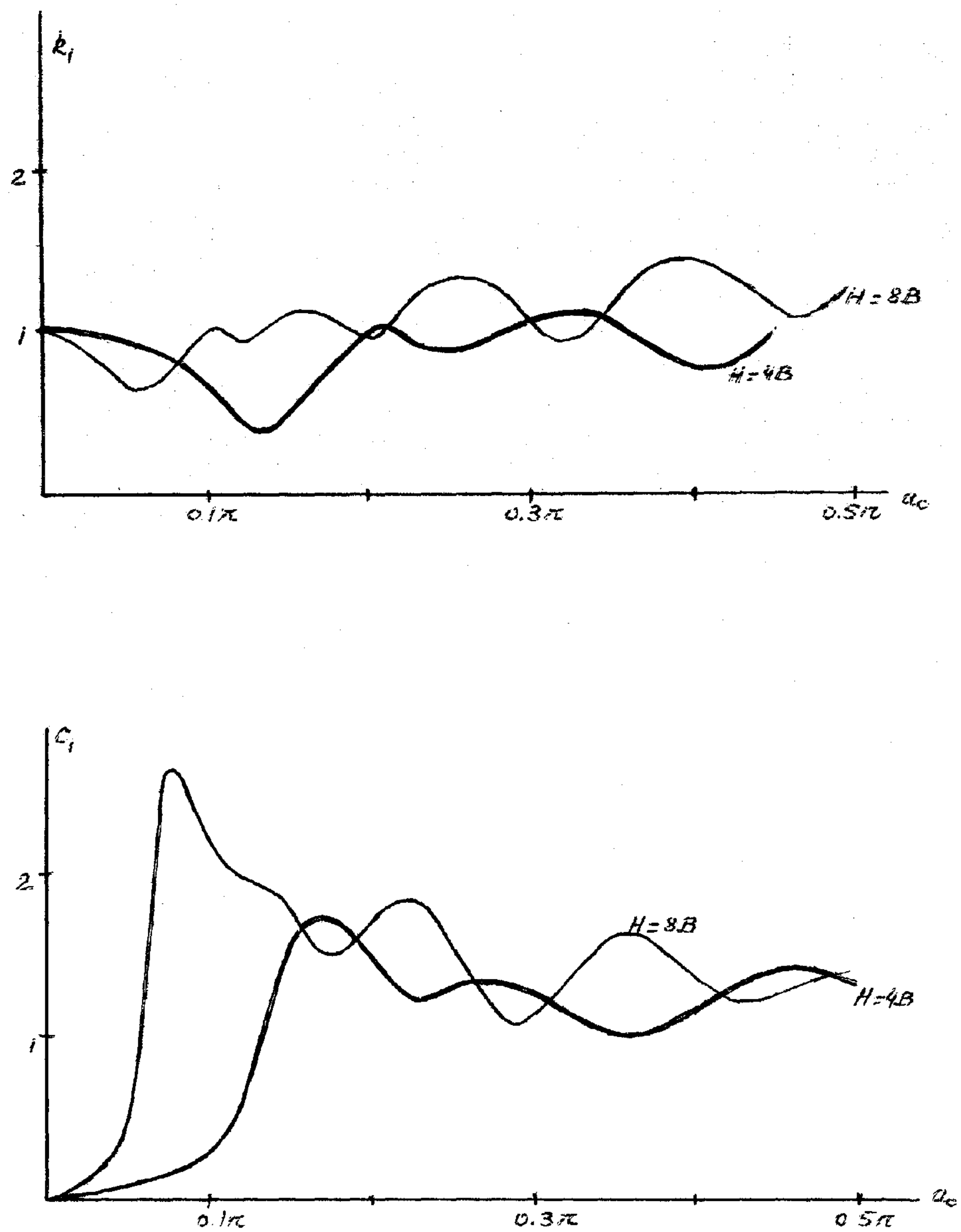



FIGURE 15 - HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS
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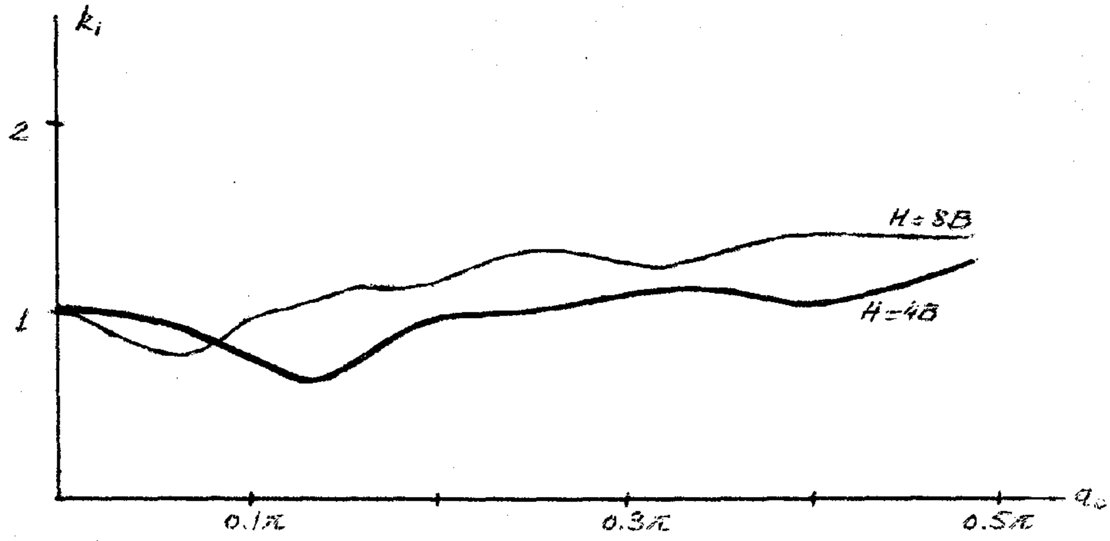


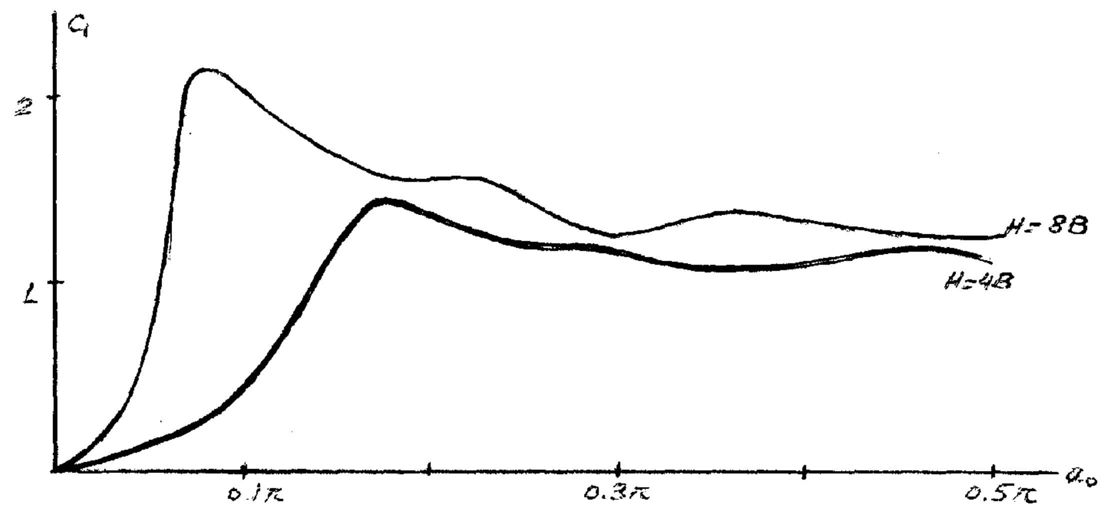
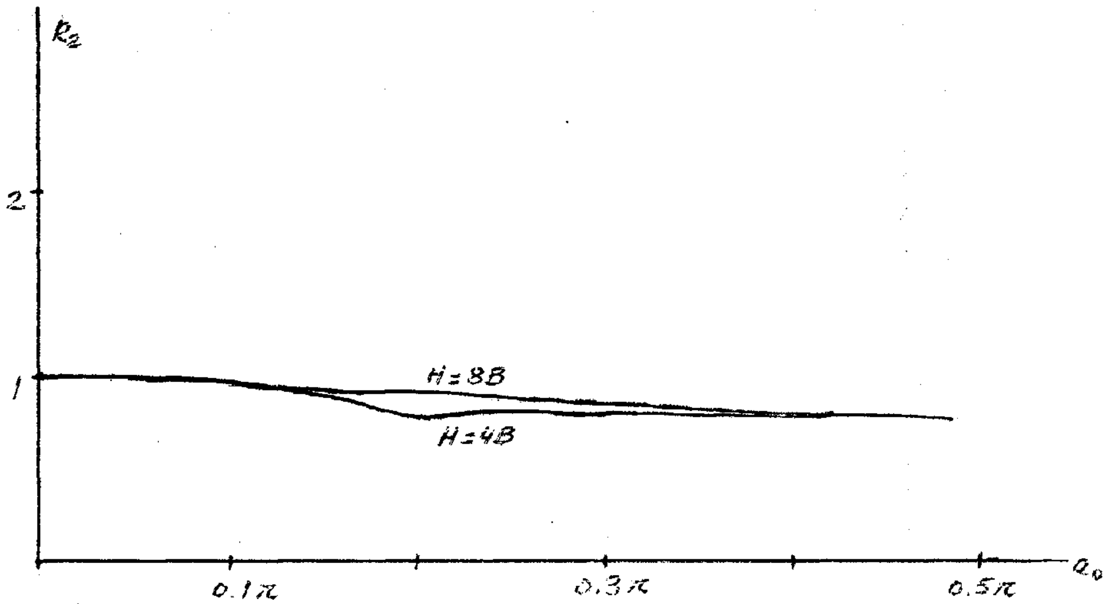




FIGURE 16. HORIZONTAL DYNAMIC STIFFNESS COEFFICIENTS

$D = 0.20$



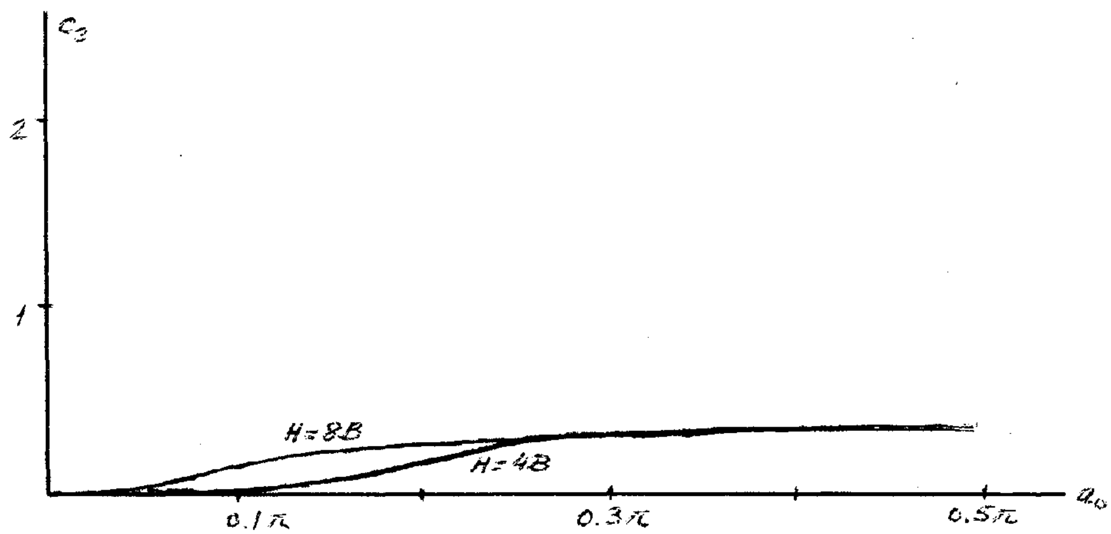




FIGURE 17. ROCKING DYNAMIC STIFFNESS COEFFICIENTS

D = 0.05

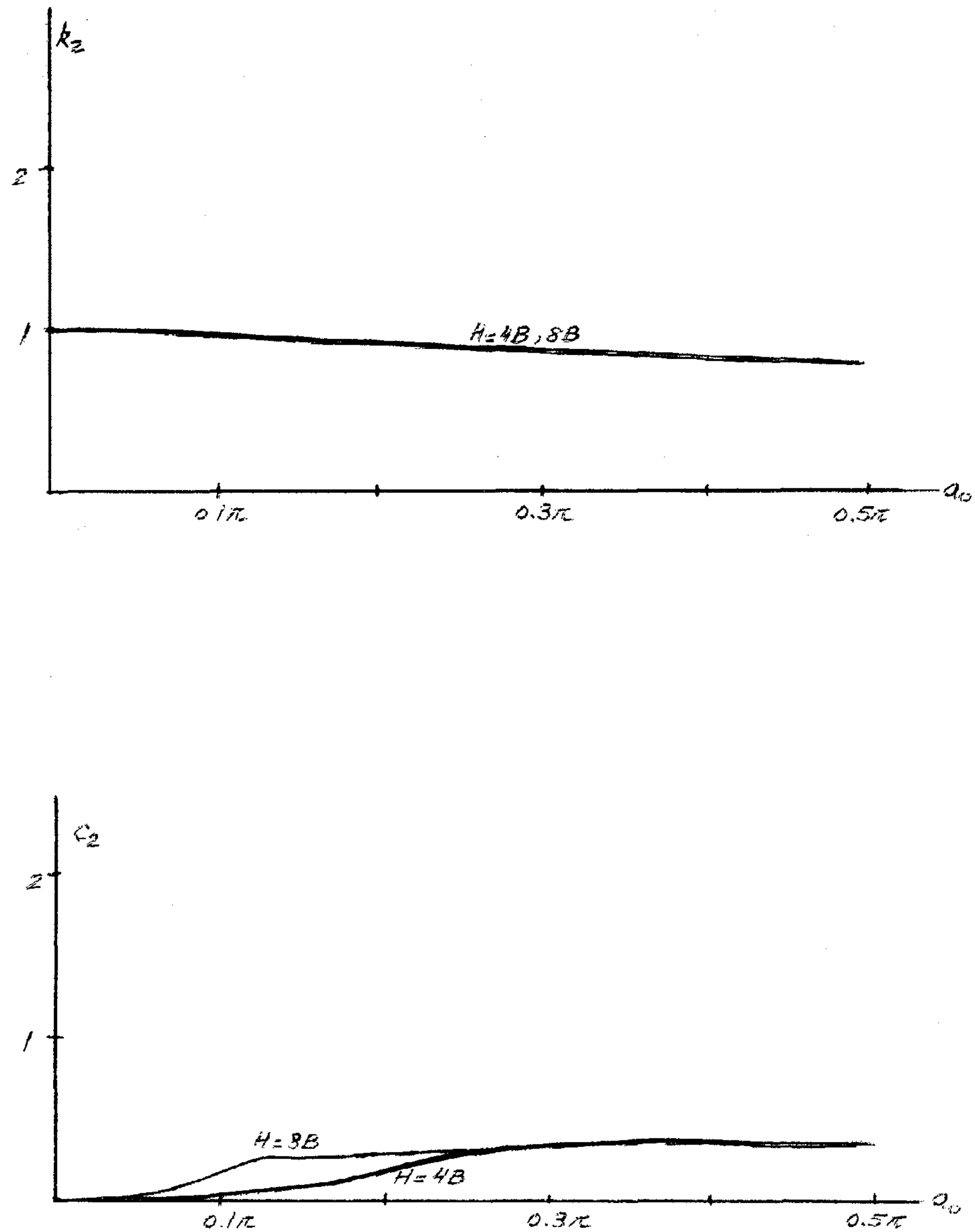



FIGURE 18 ROCKING DYNAMIC STIFFNESS COEFFICIENTS
D = 0.10

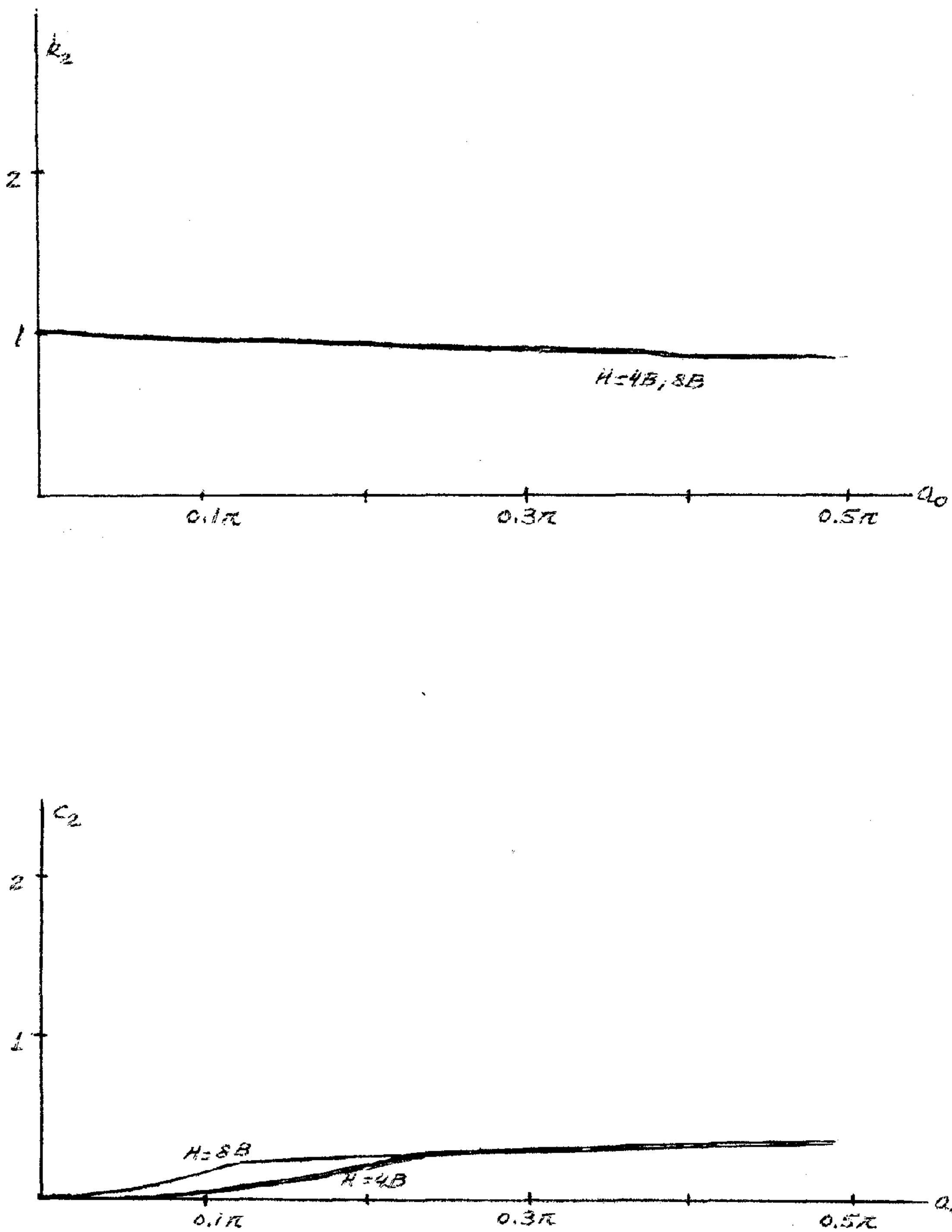



FIGURE 19 - ROCKING DYNAMIC STIFFNESS COEFFICIENTS

D= 0.20

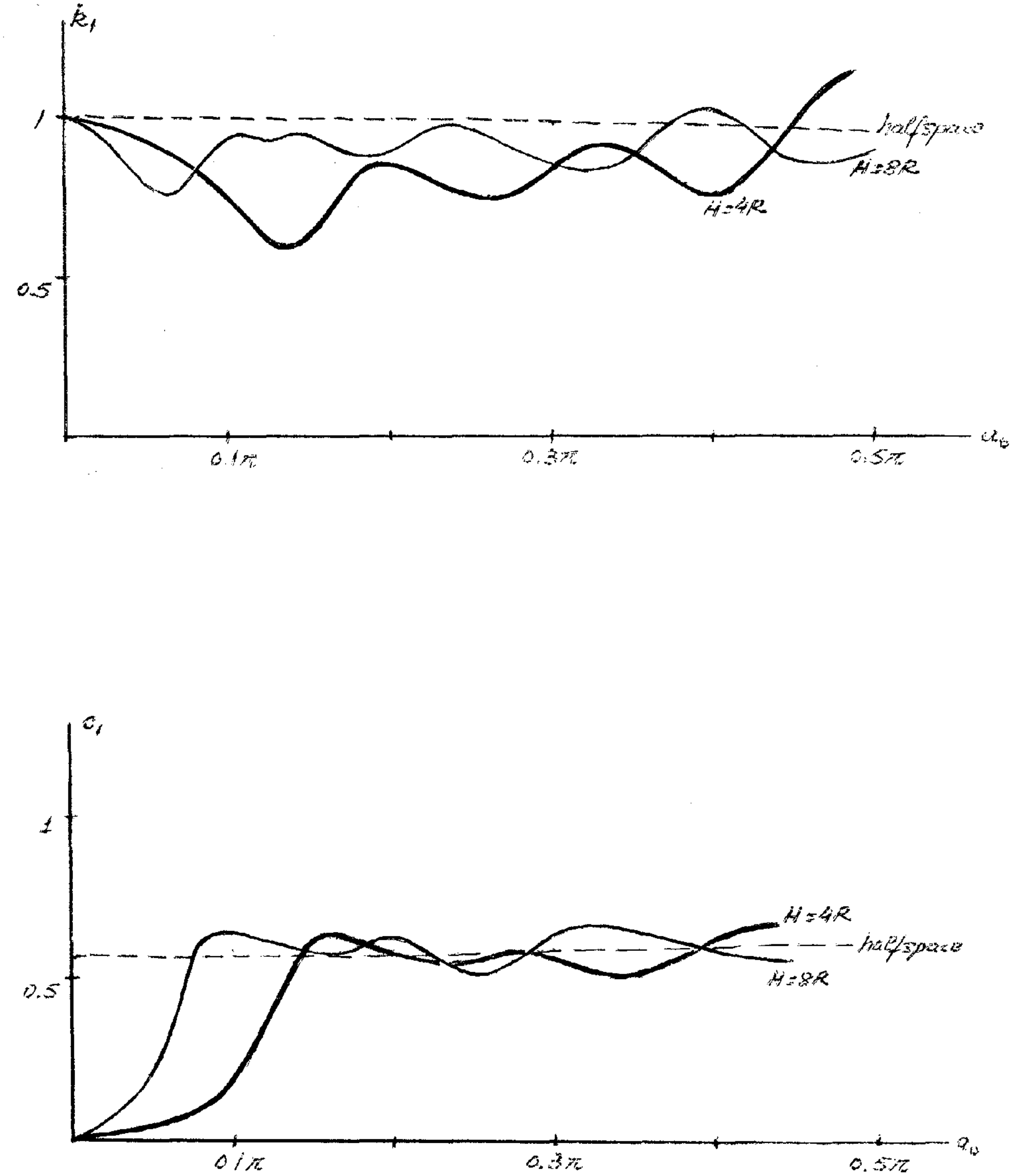



FIGURE 20 HORIZONTAL STIFFNESS COEFFICIENTS CIRCULAR FOUNDATION (after Kausel)