Seismic Vulnerability, Behavior and Design of Underground Piping Systems

Vibration Frequencies of Buried Pipelines

by

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Overall aims of this research are to develop a systematic way of assessing the adequacy and vulnerability of water/sewer distribution systems subjected to seismic loads and also to develop future design methodologies for water/sewer systems. This paper develops and provides the basic fundamentals of dynamics of buried pipelines. The dynamic fundamentals reported include the determination of fundamental frequencies of continuously elastic-supported straight pipelines subjected to axial, torsional, and flexural motions. Various boundary conditions, which can represent the actual construction, have been considered. Using a finite element and consistent mass approach, the matrix formulation of buried pipeline is developed.

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The research has been sponsored by the Earthquake Engineering Program of NSF-RANN under grant no. ENV76-14884 and Dr. S.C. Liu is the Program Manager of this Project in which Dr. Leon Ru-Liang Wang is the Principal Investigator. The overall aims of this research are to develop a systematic way of assessing the adequacy and vulnerability of water/sewer distribution systems subjected to seismic loads and to develop future design methodologies.

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Buried Pipelines; Dynamics; Lifeline Earthquake Engineering;

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ABSTRACT

To aid the research on seismic vulnerability, behavior and design of underground piping system, this paper developes and provides the basic fundamentals of dynamics of buried pipelines.

The dynamic fundamentals reported include the determination of fundamental frequencies of continuously elastic-supported straight pipelines subjected to axial, torsional and flexural motions. Various boundary conditions, which can represent the actual construction, have been considered.

Using a finite element and consistent mass approach, the matrix formulation of buried pipeline is developed.

INTRODUCTION

An earlier study[27] has shown that buried water/sewer pipelines have been damaged heavily by earthquakes. Other than the catastrophic failures caused by landslides or liquefaction of soil, substantial failures of buried pipelines reported were resulted from seismic shaking/vibration. Many papers on the dynamic analysis and design of above ground buildings can be found[3,25], but very little has been done for underground pipelines. Only until recently, several state of the art[5,27] and behavioral study[11,21] papers have been published.

To aid the research on seismic vulnerability, behavior and design of underground piping system, several aspects of the basic fundamentals of dynamics of buried pipelines have been studied[26]. Based on the report[26], this paper presents the fundamental frequencies of continuously elastic-supported pipelines subjected to axial, torsional and flexural motions. Using a finite element and consistent mass approach[4], the matrix formulation of a buried pipeline system is developed.

BACKGROUND

Continuously supported structures on ground or below ground may be analyzed by using the anology of beams on an elastic foundation[9]. Dynamically, soil resistant springs have been used to handle pile-soil foundation problems[18] and other underground structures[8,22]

In a recent paper, Parmelee and Ludtke[17] formulated the dynamic equation of motion for buried pipelines which were treated as a plane strain problem. The spring constant was obtained analytically using elastic half space theory originally developed by Mindlin and Cheng[15]. It was found that the value of the spring constant is a function of the Young's modulus at the site, diameter of pipe and buried depth. The static soil reaction modulus has been evaluated and shown in another paper [16].

In another paper, Sakurai and Takahashi [20] have studied the dynamic longitudinal stresses of underground pipelines during earthquakes theoretically as well as experimentally. In this study, the resistance to the motion was assumed to be the friction force between the pipe and the soil. They further assumed that the friction force was linearly proportional to the relative displacement. Their discussion was extended briefly to include lateral motion, but not vertical or torsional vibrations.

AXIAL VIBRATION FREQUENCY OF A STRAIGHT BURIED PIPE

A buried pipe restrained by friction forces surrounding the pipe and an elastic spring at the right end, subjected to axial motion is shown in Fig. 1. Using the notations shown in Fig. 1, the dynamic equilibrium equation of an undamped beam is

$$m\,dx\,\ddot{y} + f\,2\pi R_o dx = A d\sigma \tag{1}$$

where m, A, R_o are mass per unit length along the pipe, cross sectional area and outer radius of pipe; f, σ are frictional force per unit area and axial stress; y, absolute displacement of pipe. Note that the mass along the pipe may include the mass of water in pipe, and a portion of the soil mass that might move with the pipe as described by Parmelee [17] in addition to pipe mass itself. This mass may be expressed as

$$m = \lambda\, m_p \tag{2}$$

where λ is a constant; m_p is mass per unit length of pipe itself.

Assuming that the frictional force is proportional to the relative displacement, $u = y - y_s$, the equation of motion in terms of u becomes

$$\ddot{u} + \frac{2\pi R_o}{\lambda\, m_p} k_a\, u = \frac{A}{\lambda m_p} \frac{\partial \sigma}{\partial x} - \ddot{y}_s \tag{3}$$

where k_a is the frictional spring constant and y_s is the absolute displacement of soil medium.

Substituting the stress-strain ($\sigma = E\varepsilon$) and strain-displacement ($\varepsilon = \frac{\partial u}{\partial x}$) relationships, and eliminating the excitation function $\ddot{y}_s$ for the undamped frequency study, Eqn. (3) becomes:

$$\ddot{u} + \frac{2\pi R_o}{\lambda\, m_p} k_a\, u = \frac{EA}{\lambda m_p} \frac{\partial^2 u}{\partial x^2} \tag{4}$$

Letting $u(x,t) = \phi(x)\, f(t)$, Eqn. (4) can be transformed into two differential equations after separation of variables.

$$\phi''(x) + \alpha^2 \phi(x) = 0 \tag{5}$$

$$f(t) + \omega^2 f(t) = 0 \tag{6}$$

in which

$$\alpha^2 = \frac{a^2 \lambda m_p}{EA} \qquad \text{and} \qquad \omega^2 = \frac{2\pi R_o k_a}{\lambda\, m_p} + a^2 \tag{7}$$

Note that Eqn. (5) determines the modal shape function and Eqn. (6) determines the frequency of the system. Both equations are related by a constant a^2 shown in Eqn. (7). For frequency calculation, one needs only to solve Eqn. (5) to find a^2. Eqn. (7) will yield the frequency of the system.

The solution to Eqn. (5) is

$$\phi(x) = C_1 \sin\alpha x + C_2 \cos\alpha x \tag{8}$$

which depends on boundary conditions.

Axial Frequency of Buried Free-Free Pipe

First, let us examine the vibration of a buried free-free pipe with the following boundary conditions:

$$x = 0 \text{ and } x = L \qquad \varepsilon = \frac{\partial u}{\partial x} = \phi'(x) = 0 \tag{9}$$

The eigen value and the eigen function will be

$$\alpha L = n\pi \quad \text{and} \quad \phi(x) = \cos(n\pi x/L) \qquad n = 0,1,2,\ldots.$$

Substituting Eqn. (10) into Eqn. (7) and then Eqn. (8) the axial frequency of the system is found to be:

$$\omega_n = \sqrt{\frac{2\pi R_o k_a}{\lambda\, m_p} + \frac{n^2 \pi^2 EA}{\lambda\, L^2 m_p}} \qquad n = 0,1,2,\ldots. \tag{11}$$

where n determines the modal frequencies and shapes. When n = 0, the zero mode frequency is

$$\omega = \omega_o = \sqrt{\frac{2\pi R_o k_a}{\lambda\, m_p}} \tag{12}$$

which is the rigid motion of pipe.

Normalizing the frequencies by ω_o, Eqn. (11) becomes

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \frac{n^2 \pi^2}{L^2} \; \frac{EA}{2\pi R_o k_a}} \qquad n = 0,1,2,\ldots. \tag{13}$$

Using AWWA Standards[1,2], the periods for the rigid body modes for concrete and cast-iron pipes are shown in Fig. 2. One can see from Fig. 2 that the period decreases with increasing soil friction spring constant but increases with added masses. One will further note that the period increases with increasing pipe sizes. This is because the increase of pipe size and the added masses from soil and water for the rigid mode means an increase of mass of the system which in turn increases the period.

It is also noted that the periods shown fall into the ranges of observation by Sakurai and Takahashi[20].

The frequency ratios, ω_n/ω_o, versus a parameter, $EA/2\pi R_o L^2 k_a$ are plotted in Fig. 3 for n = 1 to 5 modes.

Axial Frequency of Buried Free-Fixed Pipe

Next, we examine the vibration of a buried free-fixed pipe. The boundary conditions are:

$$\begin{aligned} x = 0 \quad & \varepsilon = \frac{\partial u}{\partial x} = \phi'(0) = 0 \\ x = L \quad & u(L) = \phi(L) = 0 \end{aligned} \tag{14}$$

Using the solution shown in Eqn. (8) we find the eigen value and eigen function as follows:

$$\alpha L = \frac{n\pi}{2} \text{ and } \phi(x) = \cos\,(n\pi x/2L) \qquad n = 1,2,\ldots. \tag{15}$$

Therefore the frequency for the free-fixed pipe is

$$\omega_n = \sqrt{\frac{2\pi R_o}{\lambda\ m_p} k_a + \frac{n^2\pi^2 EA}{4\lambda L^2 m_p}} \qquad n = 1,2,\ldots. \tag{16}$$

and the frequency ratio as compared to the rigid body motion is

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \frac{n^2\pi^2\ EA}{8\pi R_o L^2 k_a}} \qquad n = 1,2,\ldots. \tag{17}$$

which is presented in Fig. 4 for n = 1 to 5 modes.

Axial Frequency of Buried Free-Spring Restrained Pipe

For general application of free-spring restrained pipe, we refer back to Fig. 1. The boundary conditions are:

$$\begin{aligned} x = 0 \qquad & \varepsilon = \frac{\partial u}{\partial x} = \phi'(0) = 0 \\ x = L \qquad & u(L) = \frac{R}{K} \end{aligned} \tag{18}$$

for which R is the resistant force from the spring and K is the end spring constant. For equilibrium, the resistence of the spring can be determined from the end force of the member, i.e.

$$R = -\,E\,\varepsilon(L)A = -\,EA\,\frac{\partial u}{\partial x} = -\,EA\,\phi'(L) \tag{19}$$

Substituting Eqns. (18) and (19) into Eqn. (8), the characteristic equation for the system becomes

$$\cot \alpha L = \frac{EA}{KL} \cdot \alpha L \tag{20}$$

which may be solved graphically or numerically for a given $\frac{EA}{KL}$ value. Once αL is solved, the frequencies of the system are obtained by Eqn. (7). Figs. 5 and 6 indicate the frequency ratios of first two modes for various $\frac{EA}{KL}$ values.

TORSIONAL VIBRATION OF A STRAIGHT BURIED PIPE

Due to rocky action of earthquakes, buried pipeline may subject to torsional vibration. Referring to Fig. 7, the dynamic equilibrium for the torque of the system can be written:

$$\rho\, I_p\, \ddot{\Phi}\, dx + f\; 2\pi R_o^2\, dx = d\Gamma$$

or

$$\rho\, I_p\, \ddot{\Phi} + f\, .\, 2\pi R_o^2 = \frac{d\Gamma}{dx} \tag{21}$$

where G, I_p, ρ are shear modulus, polar moment and mass density of pipe; Γ is the applied torque, f is the interface friction force per unit area.

Relating the friction resistance to R_o with a spring constant, k_t, Eqn. (21) becomes:

$$\ddot{\Phi} + \frac{2\pi R_o^3 k_t}{I_p}\Phi = \frac{G}{\rho}\Phi'' \qquad (22)$$

Following the same procedures of separation of variables as given for the axial vibration, Eqn. (22) is transformed into two differential equations

$$\Phi''(x) + \alpha^2(x) = 0 \qquad (23)$$

$$\ddot{f}(t) + \omega^2 f(t) = 0 \qquad (24)$$

Both equations are related by

$$\alpha^2 = \frac{a^2\rho}{G} \quad \text{and} \quad \omega^2 = \frac{2\pi R_o^3 k_t}{\rho I_p} + a^2 \qquad (25)$$

The solution of Eqn. (23) is

$$\phi(x) = C_1 \sin \alpha x + C_2 \cos \alpha x \qquad (26)$$

<u>Torsional Frequency of Buried Free-Free Pipe</u>

For free-free pipe, the boundary conditions are

$$x = 0 \text{ and } x = L; \quad \frac{d\phi}{dx} = 0 \qquad (27)$$

It is found that the eigen values and eigen functions are

$$\alpha = \frac{n\pi}{L} \text{ and } \phi(x) = \cos n\pi x/L \qquad n = 0,1,2,\ldots. \qquad (28)$$

The torsional frequency of the free-free buried pipe is

$$\omega_n = \sqrt{\frac{2\pi R_o^3 k_t}{\rho\ I_p} + \frac{n^2\pi^2 G}{L^2\ \rho}} \qquad n = 0,1,\ldots. \tag{29}$$

For n = 0

$$\omega_o = \sqrt{\frac{2\pi R_o^3 k_t}{\rho\ I_p}} \tag{30}$$

which is the frequency of the rigid body mode.

The ratios of frequency of various modes to the rigid body mode are

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \frac{n^2\pi^2\ G\ I_p}{2\pi R_o^3\ L^2 k_t}} \qquad n = 1,2,\ldots. \tag{31}$$

Note that if the frictional resistance on the surface of the pipe is the same for axial and torsional motions, the torsional frequency ratio for the free-free pipe will be the same as the axial frequency except for the conversion of E to G, k_a to k_t and A/R_o to I_p/R_o^3.

<u>Torsional Frequency of Buried Free-Fixed Pipe</u>

Without further discussion, the torsional frequency for the buried free-fixed pipe will be obtained from the axial vibration solution as

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \frac{n^2\pi^2\ G\ I_p}{8\pi R_o^3\ L^2 k_t}} \tag{32}$$

The graphical presentations for torsional frequencies will be the same as for axial frequencies except for the conversion of E to G, k_a to k_t and A/R_o to I_p/R_o^3 and thus will not be repeated.

FLEXURAL VIBRATION OF A STRAIGHT BURIED PIPE

Flexural vibration of a buried pipe may result from earth motion in vertical or lateral directions by earthquakes. The spring resistance from all sides of a buried pipe are assumed to be the same. Then, the problem can be considered as a common beam on an elastic foundation.

Referring to Fig. 8, the equation of flexural motion without forcing function of a buried pipe is formulated[6,23] as follows:

$$m\,\ddot{y} + k_f y + EI\,\frac{\partial^4 y}{\partial x^4} = 0 \tag{33}$$

where EI is the flexural stiffness of the pipe and m, k_f are mass and soil flexural spring constant per unit length. Note that the mass described here may include the mass of water and soil that are vibrating with the pipe, in addition to the mass of the pipe itself as defined by Eqn. (2).

Again by separation of variables, Eqn. (33) reduces to two ordinary differential equations

$$\phi^{IV}(x) - \alpha^4\phi(x) = 0 \tag{34}$$

$$\ddot{f}(t) + \omega^2 f(t) = 0 \tag{35}$$

The two equations are related by

$$\omega = \sqrt{\alpha^4\,\frac{EI}{\lambda m_p} + \frac{k_f}{\lambda\, m_p}} \tag{36}$$

in which α must be obtained after solving Eqn. (34). Once α is found, the frequency of the system is determined by Eqn. (36). The general solution of Eqn. (34) is

$$\phi(x) = A\sin\alpha x + B\cos\alpha x + C\sinh\alpha x + D\cosh\alpha x \tag{37}$$

which is governed by the boundary conditions:

Flexural Frequency of Buried Free-Free Pipe

The boundary conditions for the free-free pipe are:

$$
\begin{aligned}
x = 0;\ M(0) &= \phi''(0) = 0 \\
V(0) &= \phi'''(0) = 0 \\
x = L;\ M(L) &= \phi''(L) = 0 \\
V(L) &= \phi'''(L) = 0
\end{aligned}
\tag{38}
$$

The characteristics equation is obtained by substituting the above boundary conditions into Eqn. (37) as

$$\cosh\alpha L = \sec\alpha L \tag{39}$$

The solution of the above characteristic equation yields the eigen values and the eigen function:

$$\alpha L \tilde{=} \frac{n\pi}{2} \qquad n = 0,3,5 \text{ (odd numbers)}$$

$$\phi(x) = \sin\alpha x + \sinh\alpha x + \frac{\sin\alpha L - \sinh\alpha L}{\cosh\alpha L - \cos\alpha L}(\cos\alpha x + \cosh\alpha x) \tag{40}$$

When n = 0 which is the rigid body mode, the frequency is

$$\omega_o = \sqrt{\frac{k_f}{\lambda m_p}} \tag{41}$$

The shape function for the rigid body motion is obtained from Eqn. (34) without the presence of α as

$$\phi(x) = C_o + C_1 x + C_2 x^2 + C_3 x^3 \tag{42}$$

By applying the boundary conditions as shown in Eqn. (38) the rigid body mode has two arbitrary constants

$$\phi(x) = C_o + C_1 x \tag{43}$$

In other words, there will be two possible rigid body modes with the same frequency (Eqn. 41)), one changes its position without changing its slope and the other changes its slope without changing its position. These two mode shapes are shown in Fig. 9.

Using spring constants for fine grain soils observed experimentally by Howard[10], the rigid body mode periods for AWWA concrete and cast iron pipes are shown in Fig. 10 and Fig. 11 respectively. It is noted that these rigid mode periods for lateral or vertical motion are very similar to those for axial motion reported earlier.

The frequency for other modes are normalized by the rigid mode frequency to yield the frequency ratio:

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \frac{n^4 \pi^4}{2^4} \frac{EI}{k_f L^4}} \qquad n = 3,5 \qquad (44)$$

or

$$\frac{\omega_{\bar{n}}}{\omega_o} = \sqrt{1 + \pi^4 (\bar{n} + \frac{1}{2})^4 \frac{EI}{k_f L^4}} \qquad \bar{n} = 1,2,\ldots. \qquad (45)$$

where $\bar{n} = n - 2$

The frequency ratios for $\bar{n} = 1$ to 5 which represent 3rd to 7th modes are shown in Fig. 2.

Flexural Frequency of Free-Hinged Pipe

The boundary conditions for free-hinged pipes are

$$\begin{aligned} x = 0;\ & M(0) = \phi''(0) = 0 \\ & V(0) = \phi'''(0) = 0 \\ x = L;\ & y(L) = \phi(L) = 0 \\ & M(L) = \phi''(L) = 0 \end{aligned} \qquad (46)$$

The characteristics equation is found to be

$$\tan\alpha L = \tanh\alpha L \tag{47}$$

The eigen value and eigen function are

$$\alpha L = 0 \text{ and } \alpha L \simeq (n - \frac{3}{4}) \qquad n = 2,3,\ldots.$$

$$\phi(x) = \sin\alpha x + \sinh\alpha x + \frac{\sin\alpha L - \sinh\alpha L}{\cosh\alpha L - \cos\alpha L}(\cos\alpha x + \cosh\alpha x) \tag{48}$$

There is one rotational rigid mode rotating about the hinge and the frequency ratios for the higher modes are

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + (n - \frac{3}{4})^4 \pi^4 \frac{EI}{k_f L^4}} \qquad n = 2,3,\ldots. \tag{49}$$

or

$$\frac{\omega_{\bar{n}}}{\omega_o} = \sqrt{1 + \pi^4 (\bar{n} + \frac{1}{4})^4 \frac{EI}{k_f L^4}} \qquad \bar{n} = 1,2,\ldots. \tag{50}$$

where $\bar{n} = n - 1$

The frequency ratios for $\bar{n} = 1$ to 5 which represent 2nd to 6th modes are shown in Fig. 13.

Flexural Frequency of Free-Fixed Pipe

The boundary conditions are:

$$x = 0; \quad M(0) = \phi''(0) = 0 \tag{51}$$
$$V(0) = \phi'''(0) = 0$$

$$x = L; \quad y(L) = \phi(L) = 0$$
$$y'(L) = \phi'(L) = 0$$

The characteristics equation is found to be:

$$\cos\alpha L \cosh\alpha L + 1 = 0 \tag{52}$$

and the eigen value and eigen function are:

$$\alpha L \cong 1.875, \quad (n - \frac{1}{2})\pi \qquad n = 2,3,\ldots.$$

$$\phi(x) = \sin\alpha x + \sinh\alpha x + \frac{\cos\alpha L + \cosh\alpha L}{\sin\alpha L - \sinh\alpha L}(\cos\alpha x + \cosh\alpha x) \tag{53}$$

The frequency ratios are:

$$\frac{\omega_1}{\omega_o} = \sqrt{1 + 0.12688\,\frac{EI}{\pi^4 k_f L^4}} \tag{54}$$

and

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \pi^4 (n - \frac{1}{2})^4 \frac{EI}{k_f L^4}} \qquad n = 2,3,\ldots. \tag{55}$$

or

$$\frac{\omega_{\bar{n}}}{\omega_o} = \sqrt{1 + \pi^4 (\bar{n} + \frac{1}{2})^4 \frac{E2}{k_f L^4}} \qquad \bar{n} = 1,2,\ldots. \tag{56}$$

where $\bar{n} = n - 1$

Except ω_1 which is very close to the rigid mode frequency ω_o, the frequency ratios for $\bar{n} = 1$ to 5 which represent 2nd to 6th modes are also shown in Fig. 12.

<u>Flexural Frequency of Hinged-Hinged Pipe</u>

Boudary Conditions:

$$\begin{aligned} x = 0; \quad & y(0) = \phi(0) = 0 \\ & M(0) = \phi''(0) = 0 \\ x = L; \quad & y(L) = \phi(L) = 0 \\ & M(L) = \phi''(L) = 0 \end{aligned} \tag{57}$$

Characteristic Equation:

$$\sin\alpha x = 0 \tag{58}$$

Eigen value and Eigen function :

$$\alpha L = n\pi; \text{ and } \phi(x) = \sin\alpha x \qquad n = 1,2,.... \tag{59}$$

Frequency Ratio:

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + n^4\pi^4 \frac{EI}{k_f L^4}} \qquad n = 1,2,.... \tag{60}$$

The frequency and mode shape for n = 1 to 5 are shown in Fig. 14.

Flexural Frequency of Hinged-Fixed Pipe

Boundary Conditions:

$$\begin{aligned} x = 0;\ & y(0) = \phi(0) = 0 \\ & M(0) = \phi''(0) = 0 \\ x = L;\ & y(L) = \phi(L) = 0 \\ & y'(L) = \phi'(L) = 0 \end{aligned} \tag{61}$$

Characteristic Equation:

$$\tan\alpha L = \tanh\alpha L \tag{62}$$

Eigen value and eigen function

$$\begin{aligned} &\alpha L \doteq (n + \tfrac{1}{4})\pi; \qquad n = 1,2,.... \\ &\phi(x) = \sin\alpha x - \frac{\sin\alpha L}{\sinh\alpha L}\sinh\alpha x \end{aligned} \tag{63}$$

Frequency ratio

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \pi^4(n + \tfrac{1}{4})^4 \frac{EI}{k_f L^4}} \qquad n = 1,2,.... \tag{64}$$

Fig. 13 shows the frequencies for the hinged-fixed pipe.

Flexural Frequency of Fixed-Fixed Pipe

Boundary Conditions:

$$\begin{aligned} x = 0;\ & y(0) = \phi'(0) = 0 \\ & y'(0) = \phi'(0) = 0 \end{aligned} \tag{65}$$

$$\begin{aligned} x = L;\ & y(L) = \phi(L) = 0 \\ & y'(L) = \phi'(L) = 0 \end{aligned}$$

Characteristic Equation:

$$\cos\alpha L \cosh\alpha L = 1 \tag{66}$$

Eigen value and Eigen function:

$$\alpha L \doteq (n + \frac{1}{2}) \qquad n = 1,2,.... \tag{67}$$

$$\phi(x) = \sin\alpha x - \sinh\alpha x + \frac{\cos\alpha L - \cosh\alpha L}{\sin\alpha L + \sinh\alpha L}(\cos\alpha x - \cosh\alpha x) \tag{68}$$

Frequency Ratio

$$\frac{\omega_n}{\omega_o} = \sqrt{1 + \pi^4 (n + \frac{1}{2})^4 \frac{EI}{k_f L^4}} \qquad n = 1,2,....$$

Fig. 12 shows the frequencies for the fixed end pipe.

MATRIX FORMULATION OF BURIED PIPING SYSTEM

For the application of dynamic analysis to an actual water/sewer distribution system which may consist of several mains and many branches of different sizes, a simplified but accurate method that can be handled with reasonable amount of effort must be developed. Using the analytical continuous solution to the differential equation for a buried pipe described in previous sections, it will be very difficult, if not impossible, to get any reasonable solution for the large degrees of freedom system. This leads to the adoption of the well known matrix finite element approach[19,24].

Since seismic excitation may come from any direction to the piping system, it is anticipated that some pipes in the system will be dominated by the axial motion or torsional motion, while other pipes may be dominated by flexural motions either in vertical or lateral direction. Therefore for generality, there will be 12 degrees of freedom for each member with six degrees of freedom at each end.

Note that flexural vibration, axial vibration and torsional vibration are all uncoupled. The developments of element mass matrix or stiffness matrix can be worked out separately for each of the above mentioned motion.

For simplicity, the spring constants are assumed to be the same and uniform along the pipe in all directions.

Element Flexural Stiffness Matrix

A buried pipe with a distributed flexural resistant soil spring k_f is shown in Fig. 15. At each end, there is a linear displacement coordinate and a rotational displacement coordinate, denoted as Y_1 to Y_4 which are a function of time. The distributed displacement function of the pipe can be represented by the discrete modal displacement coordinates:

$$y(x,t) = Y_1(t)\ \phi_1(x) + Y_2(t)\ \phi_2(x) + Y_3(t)\ \phi_3(x) + Y_4(t)\ \phi_4(x) \qquad (69)$$

which may be written in a matrix form

$$y(x,t) = [\phi_1(x)\ \phi_2(x)\ \phi_3(x)\ \phi_4(x)] \begin{Bmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{Bmatrix} \tag{70}$$

$$= [\phi]^t \ \{Y\}$$

Note that the strain energy of the system consists of the energies from pipe and soil spring.

$$U = U_{pipe} + U_{spring} \tag{71}$$

The strain energy from pipe is

$$U_{pipe} = \frac{1}{2} \int EI(y'')^2 dx \tag{72}$$

in which EI is flexural stiffness of pipe and

$$y''(x,t) = [\phi''(x)]^t \ \{Y\} \tag{73}$$

Substituting Eqn. (71) into Eqn. (70), the strain energy of the pipe becomes

$$U_{pipe} = \frac{1}{2} [Y]^t \ (\int EI \ \{\phi''\} \ [\phi'']^t dx) \ \ \{Y\}^t$$

$$= \frac{1}{2} [Y]^t \ \ [K_p]_f \ \{Y\} \tag{74}$$

where $[K_p]_f$ is the stiffness matrix of a pipe element with its expanded form as

$$[K_p]_f = \int_o^L EI \begin{bmatrix} \phi_1'' \phi_1'' & \phi_1'' \phi_2'' & \phi_1'' \phi_3'' & \phi_1'' \phi_4'' \\ \phi_2'' \phi_1'' & \phi_2'' \phi_2'' & \phi_2'' \phi_3'' & \phi_2'' \phi_4'' \\ \phi_3'' \phi_1'' & \phi_3'' \phi_2'' & \phi_3'' \phi_3'' & \phi_3'' \phi_4'' \\ \phi_4'' \phi_1'' & \phi_4'' \phi_2'' & \phi_4'' \phi_3'' & \phi_4'' \phi_4'' \end{bmatrix} dx \tag{75}$$

Choosing following Hermitian polynomials which satisfy the boundary conditions:

$$\begin{aligned}
\phi_1(x) &= 2x^3/L^3 - 3x^3 - 3x^2/L^2 + 1 \\
\phi_2(x) &= L(x^3/L^3 - 2x^2/L^2 + x/L) \\
\phi_3(x) &= 3x^2/L^2 - 2x^3/L^3 \\
\phi_4(x) &= L(x^3/L^3 - x^2/L^2)
\end{aligned} \tag{76}$$

and working out the integrations, one will find the stiffness of the pipe element as the common flexural stiffness of a beam element.

$$[K_p]_f = \begin{bmatrix} \frac{12EI}{L^3} & & \text{symmetric} & \\ \frac{6EI}{L^2} & \frac{4EI}{L} & & \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{77}$$

The strain energy from the soil spring is

$$\begin{aligned}
U_{spring} &= \frac{1}{2} \int k_f \, y^2 \, dx \\
&= \frac{1}{2} [Y]^t \left(\int k_f \{\phi\} \; [\phi]^t \, dx \right) \; \{Y\}^t \\
&= \frac{1}{2} [Y]^t \, [K_s]_f \; \{Y\}
\end{aligned} \tag{78}$$

where $[K_s]_f$ is the stiffness matrix from the soil spring for flexural motion.

Using the Hermitian polynomials, Luk[13] worked out the details and reported as

$$[K_s]_f = \begin{bmatrix} \frac{13}{35} k_f L & & \text{Symmetric} & \\ \frac{11}{210} k_f L^2 & \frac{k_f L^3}{105} & & \\ \frac{9}{70} k_f L & \frac{13}{420} k_f L^2 & \frac{13}{35} k_f L & \\ -\frac{13}{420} k_f L^2 & \frac{-k_f 13}{140} & -\frac{11}{210} k_f L^2 & \frac{k_f L^3}{105} \end{bmatrix} \tag{79}$$

The flexural stiffness matrix of a buried pipe element will be

$$[K_{bp}]_f = [K_p]_f + [K_s]_f \tag{80}$$

Element Flexural Mass Matrix

Using the consistent mass approach[4] the mass matrix of a buried pipe element is developed.

Note that the kinetic energy of the system is

$$T = \frac{1}{2} \int_o^L \lambda m_p \dot{y}^2 \, dx \tag{81}$$

Using the Hermitian polynomials of Eqns. (76) and Eqn. (81) can be transformed to

$$T = \frac{1}{2} [\dot{Y}]^t \left(\int_o^L \lambda m_p \{\phi\} [\phi]^t \, dx \right) \{\dot{Y}\}$$

$$= \frac{1}{2} [\dot{Y}]^t [M_{bp}]_f \{\dot{Y}\} \tag{82}$$

Note that Eqn. (82) has the same form as the soil spring stiffness. Thus, by interchanging k_f to λm_p, in Eqn. (79) the flexural mass matrix of a buried pipe element is obtained.

Element Axial Stiffness Matrix

The Hermitian polynomials for the axial displacement are

$$\phi_1(x) = 1 - \frac{x}{L}$$
$$\phi_2(x) = \frac{x}{L} \tag{83}$$

The axial strain energy of pipe is

$$U_{pa} = \frac{1}{2} EA \int_o^L \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{84}$$

which, in turn, the axial stiffness of pipe is found,

$$[K_p]_a = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \tag{85}$$

The strain energy of soil spring in axial direction

$$U_{sa} = \frac{1}{2} \int 2\pi R_o k_a \, y^2 \, dx \tag{86}$$

The stiffness matrix from the soil spring is found by integrating the deflection function

$$[K_s]_a = 2\pi R_o \begin{bmatrix} \frac{k_a L}{3} & \frac{k_a L}{6} \\ \frac{k_a L}{6} & \frac{k_a L}{3} \end{bmatrix} \tag{87}$$

Element Axial Consistent Mass Matrix

Without further explanation, the element axial consistent mass matrix is

$$[M_{bp}]_a = \lambda \begin{bmatrix} \frac{m_p L}{3} & \frac{m_p L}{6} \\ \frac{m_p L}{6} & \frac{m_p L}{3} \end{bmatrix} \tag{88}$$

Element Torsional Stiffness and Mass Matrixes

$$[K_p]_t = \begin{bmatrix} \frac{G I_p}{L} & -\frac{G I_p}{L} \\ -\frac{G I_p}{L} & \frac{G I_p}{L} \end{bmatrix} \tag{89}$$

$$[K_s]_t = 2\pi R_o^3 \begin{bmatrix} \frac{k_t L}{3} & \frac{k_t L}{6} \\ \frac{k_t L}{6} & \frac{k_t L}{3} \end{bmatrix} \tag{90}$$

$$[M_{bp}]_t = \lambda\rho \begin{bmatrix} \frac{I_p L}{3} & \frac{I_p L}{6} \\ \frac{I_p L}{6} & \frac{I_p L}{3} \end{bmatrix} \tag{91}$$

General Buried Pipe Element Stiffness and Mass Matrices

By combining the contributions from flexural, axial and torsional motions, the generalized buried pipe element stiffness and consistent mass matrices are summarized in Table 1.

Formulation of Buried Piping System

Upon the determination of these buried pipe element stiffness and consistent mass matrices, they can be input into available generalized computer programs such as ICES-STRUDL[12] or NASTRAN[14] for solution. With the developments shown, it is not difficult at all to write a computer program to do the frequency analysis for the discrete system.

Using the above formulations, Davis[7] has been able to obtain frequency values for one and two pipe systems with AWWA Sections. For a straight pipe, the discrete frequencies and mode shape obtained by Davis are comparable to the solutions obtained by solving the differential equations given in this paper.

CONCLUSIONS

The vibration characteristics (axial, flexural and torsional) of buried pipelines are greatly influenced by the properties of both the pipe and the surrounding soil medium. For engineering practice, the problem can be successfully handled by the analogy of beams on an elastic foundation.

With the assumptions that both the pipe and the soil medium have uniform-continuous properties, analytical solutions of natural frequencies (natural periods) of buried straight pipes can be obtained. However, for buried piping systems, finite element approach, which converts a continuous system to a system of discrete coordinates, must be employed. Since both the stiffness and mass matrices for the buried piping system have been worked out in the paper, it is only a matter of computer program to obtain the numerical solutions.

ACKNOWLEDGEMENT

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Table 1

Non-zero Terms of Stiffness and Mass Matrices
of A Buried Pipe

Non-zero Term	$[K_p]$	$[K_s]$	$[M_{bp}]$
(1,1)	EA/L	$2\pi R_o \cdot \frac{1}{3} k_a L$	$\frac{1}{3} \lambda m_p L$
(1,7)=(7,1)	$-EA/L$	$2\pi R_o \cdot \frac{1}{6} k_a L$	$\frac{1}{6} \lambda m_p L$
(2,2)	$12EI/L^3$	$\frac{13}{35} k_f L$	$\frac{13}{35} \lambda m_p L$
(2,6)-(6,2)	$6EI/L^2$	$\frac{11}{210} k_f L^2$	$\frac{11}{210} \lambda m_p L^2$
(2,8)=(8,2)	$-12EI/L^3$	$\frac{9}{70} k_f L$	$\frac{9}{70} \lambda m_p L$
(2,12)=(12,2)	$6EI/L^2$	$-\frac{13}{420} k_f L^2$	$-\frac{13}{420} \lambda m_p L$
(3,3)	$12EI/L^3$	$\frac{13}{35} k_f L$	$\frac{13}{35} \lambda m_p L$
(3,5)=(5,3)	$-6EI/L^2$	$-\frac{11}{210} k_f L^2$	$-\frac{11}{210} \lambda m_p L^2$
(3,9)=(9,3)	$-12EI/L^3$	$\frac{9}{70} k_f L$	$\frac{9}{70} \lambda m_p L$
(3,11) = (11,3)	$-6EI/L^2$	$\frac{13}{420} k_f L^2$	$\frac{13}{420} \lambda m_p L^2$
(4,4)	GI_p/L	$2\pi R_o^3 \cdot \frac{1}{3} k_t L$	$\frac{1}{3} \rho \; I_p L$
(4,10)=(10,4)	$-GI_p/L$	$2\pi R_o^3 \cdot \frac{1}{6} k_t L$	$\frac{1}{6} \rho \; I_p L$
(5,5)	$4EI/L$	$\frac{1}{105} k_f L^3$	$\frac{1}{105} \lambda m_p L^3$
(5,9)=(9,5)	$6EI/L^2$	$-\frac{13}{420} k_f L^2$	$-\frac{13}{420} \lambda m_p L^2$
(5,11)=(11,5)	$2EI/L$	$-\frac{1}{140} k_f L^3$	$-\frac{1}{140} \lambda m_p L^3$
(6,6)	$4EI/L$	$\frac{1}{105} k_f L^3$	$\frac{1}{105} \lambda m_p L^3$
(6,8)=(8,6)	$-6EI/L^2$	$\frac{13}{420} k_f L^2$	$\frac{13}{420} \lambda m_p L^2$
(6,12)=(12,6)	$2EI/L$	$-\frac{1}{140} k_f L^3$	$-\frac{1}{140} \lambda m_p L^3$

Table 1 (Continued)

Non-zero Term	$[K_p]$	$[K_s]$	$[M_{bp}]$
(7,7)	EA/L	$2\pi R_o \cdot \frac{1}{3} k_a L$	$\frac{1}{3} \lambda m_p L$
(8,8)	$12EI/L^3$	$\frac{13}{35} k_f L$	$\frac{13}{35} \lambda m_p L$
(9,9)	$12EI/L^3$	$\frac{13}{35} k_f L$	$\frac{13}{35} \lambda m_p L$
(10,10)	GI_p/L	$2\pi R_o^3 \frac{1}{3} k_t L$	$\frac{1}{3} \rho\ I_p L$
(11,11)	$4EI/L$	$\frac{1}{105} k_f L^3$	$\frac{1}{105} \lambda m_p L^3$
(12,12)	$4EI/L$	$\frac{1}{105} k_f L^3$	$\frac{1}{105} \lambda m_p L^3$

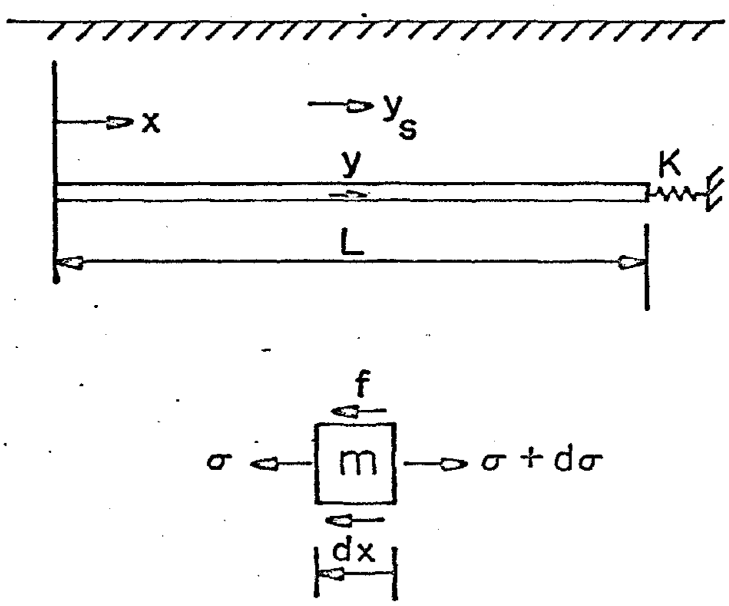


Fig. 1 Axial Vibration of a Buried Pipe

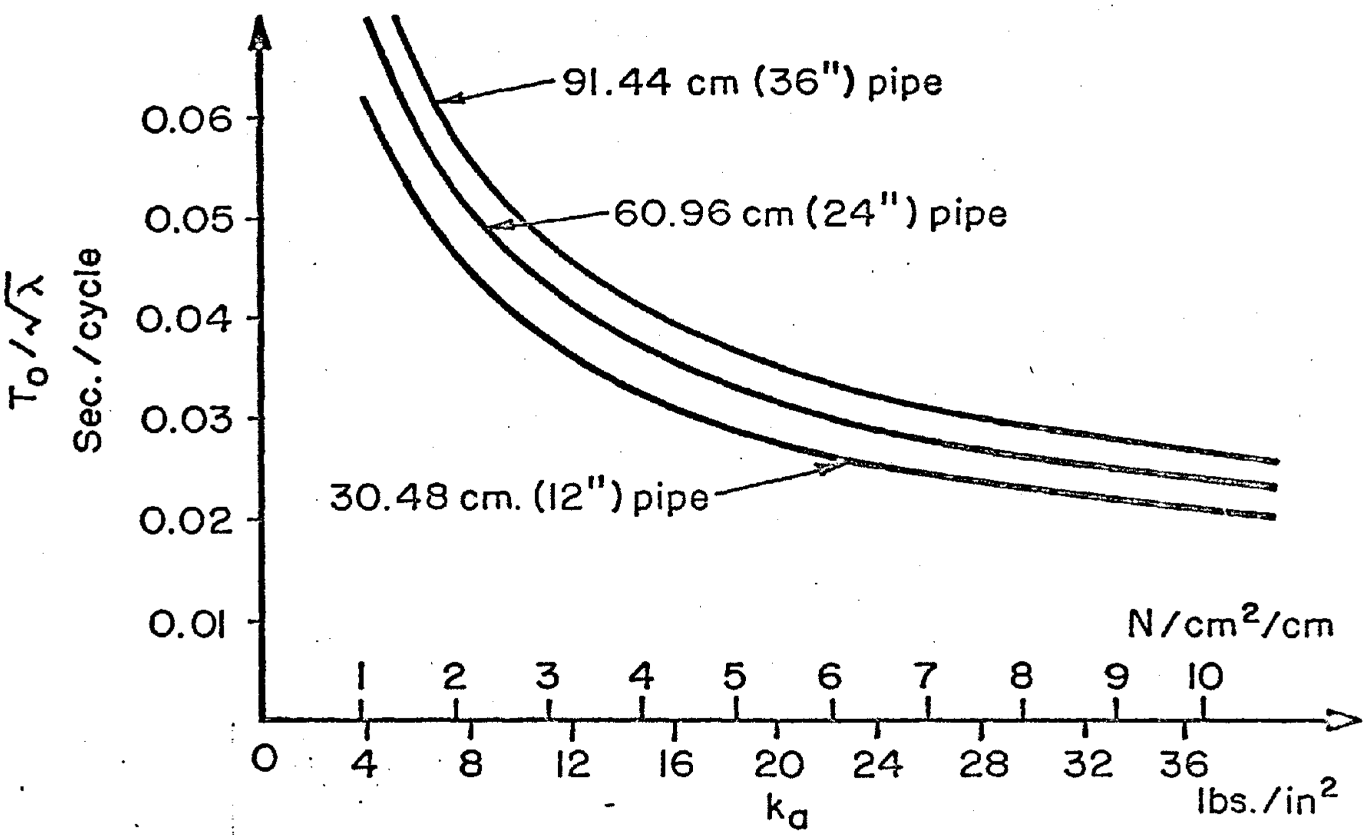


Fig 2a Rigid Mode Natural Period of Buried AWWA Concrete Pipe - Axial Motion

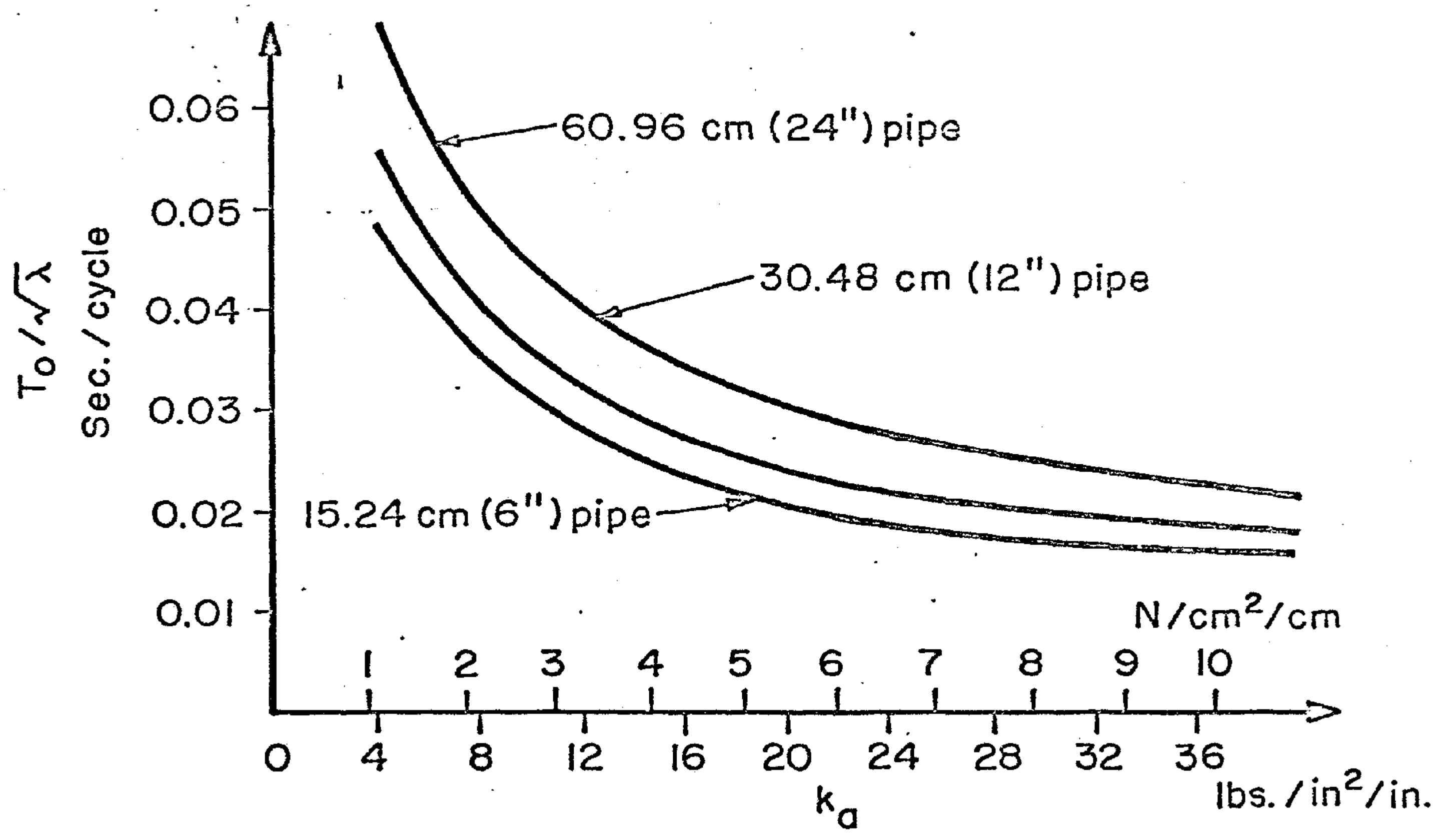


Fig. 2b - Rigid Mode Natural Period of Buried AWWA Cast Iron Pipe - Axial Motion

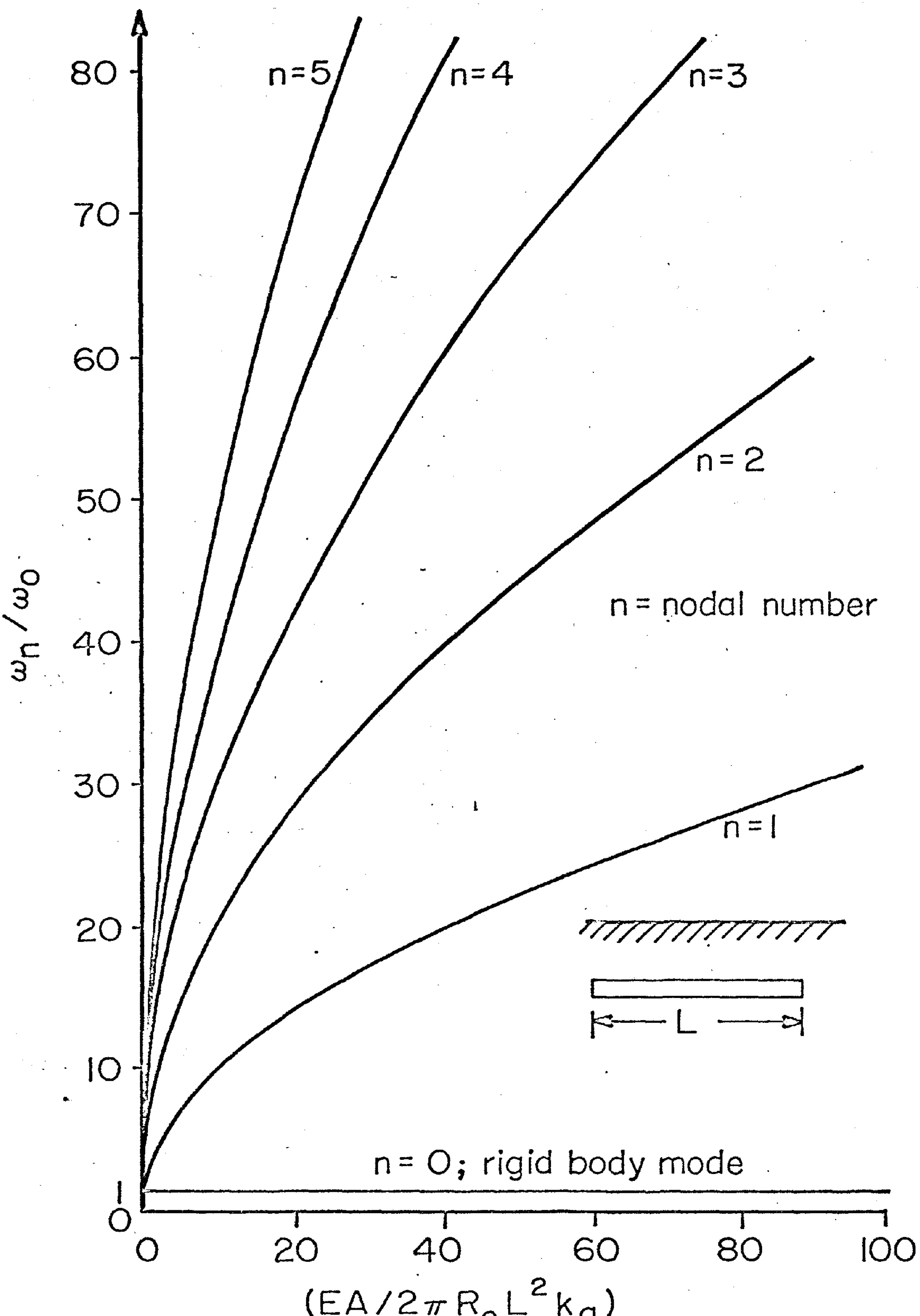


Fig. 3 - Axial Frequency Ratio of Buried Free-Free Pipe

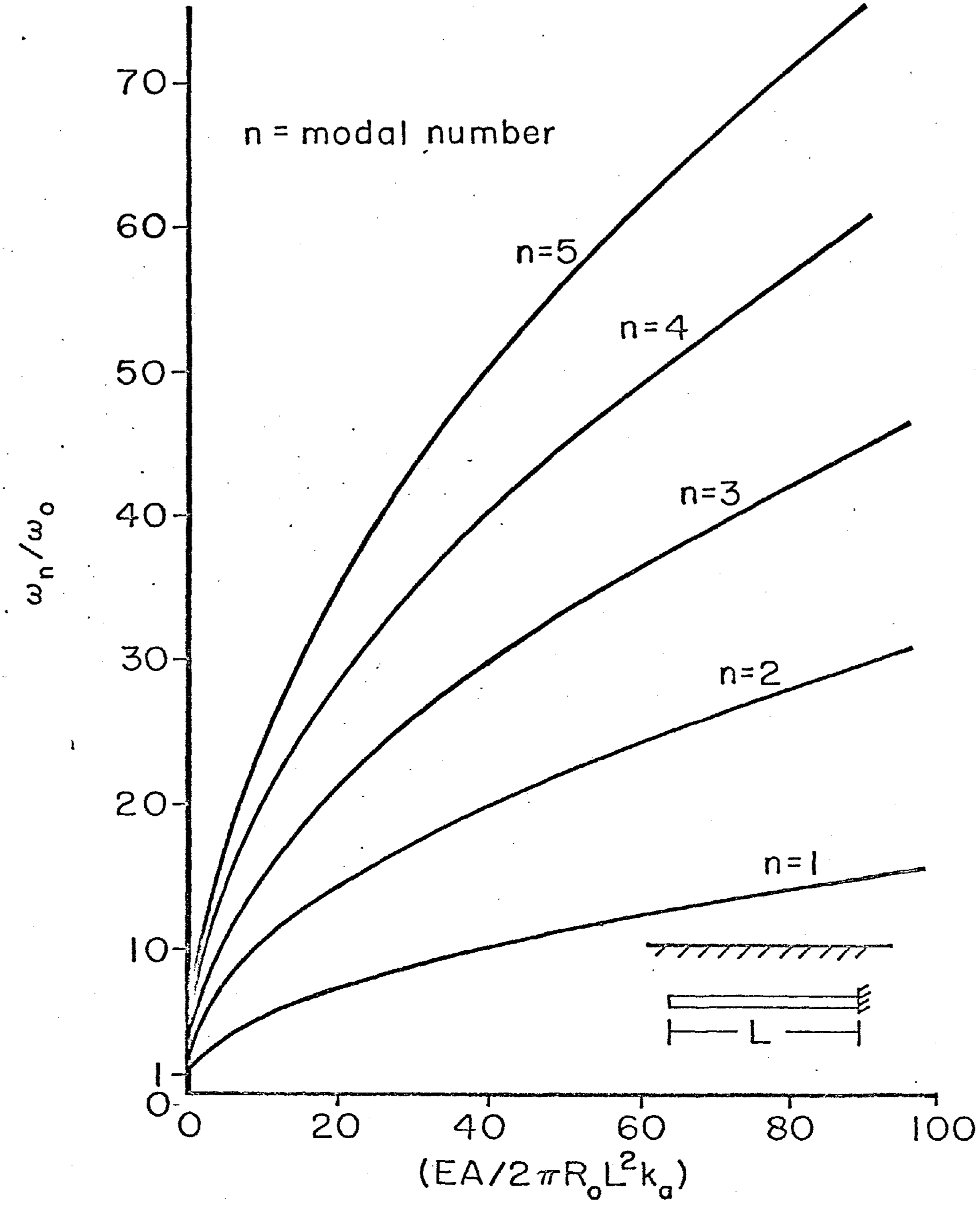


Fig. 4 - Axial Frequency Ratio of Buried Free-Fixed Pipe 32

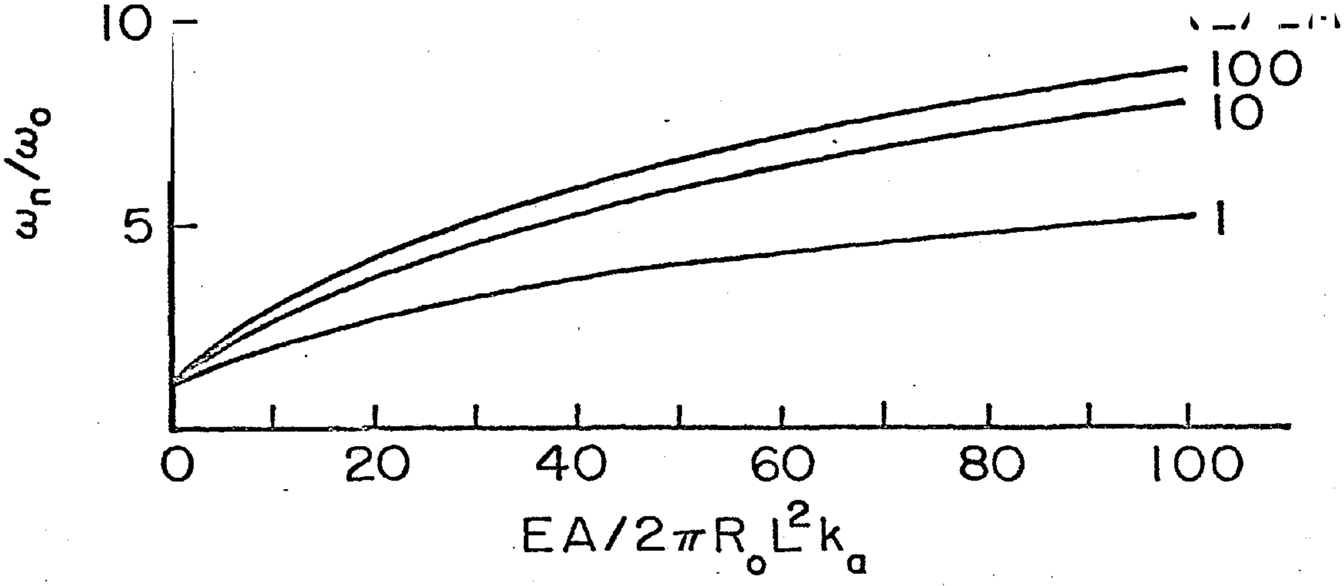


Fig. 5 - First Mode Axial Frequency Ratio of Buried Free-Spring Restrained Pipe

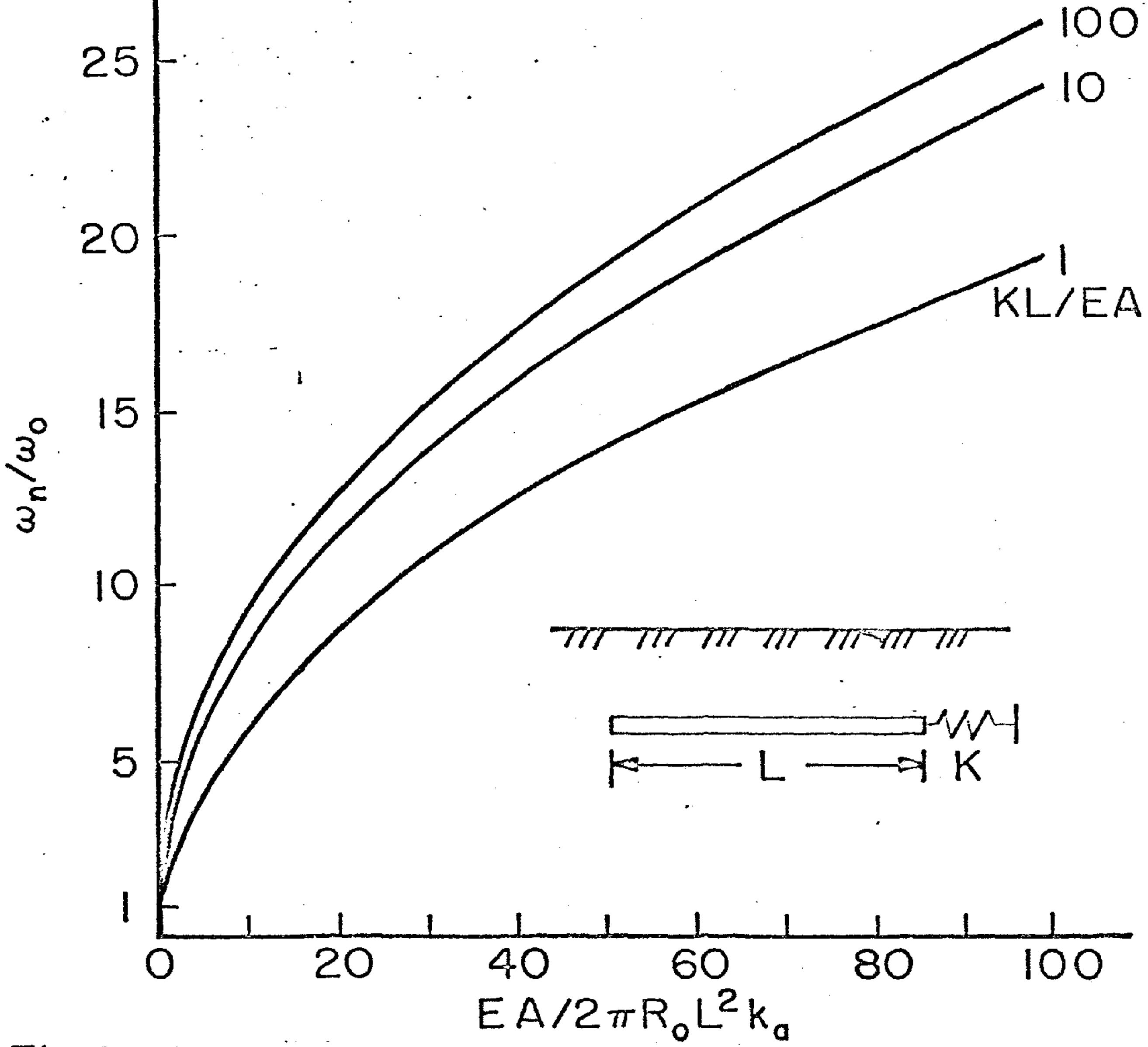


Fig. 6 - Second Mode Axial Frequency Ratio of Buried Free-Spring Restrained Pipe

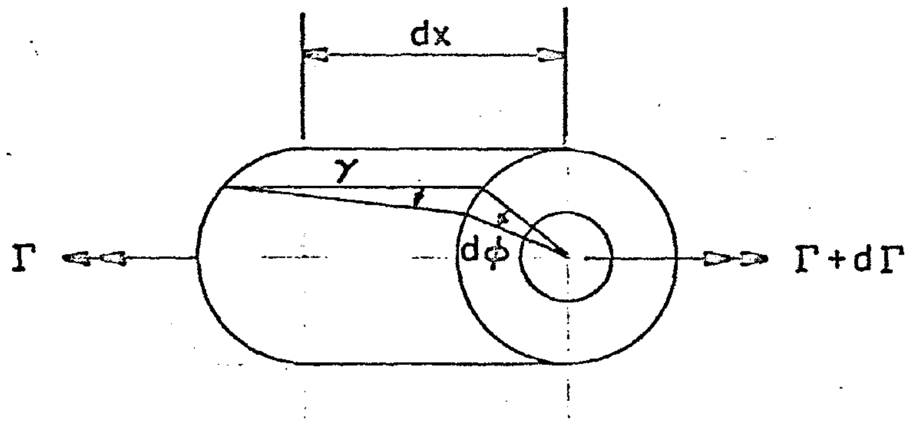


Fig. 7- Torsion in a Pipe Segment

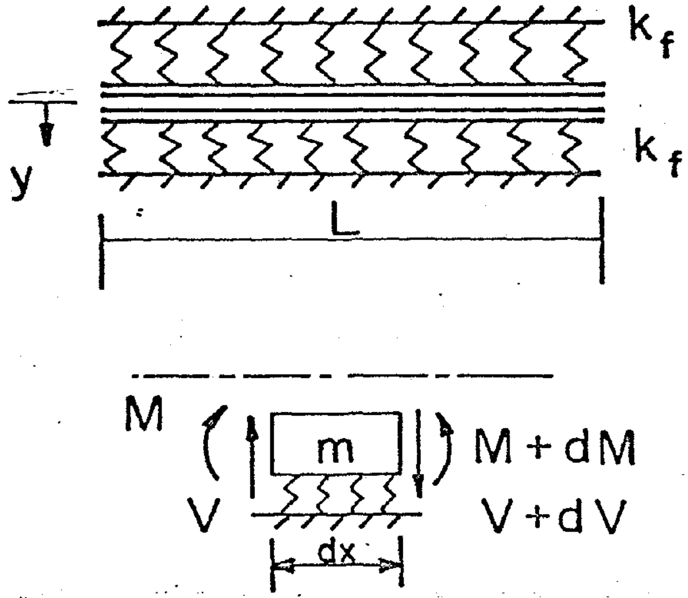
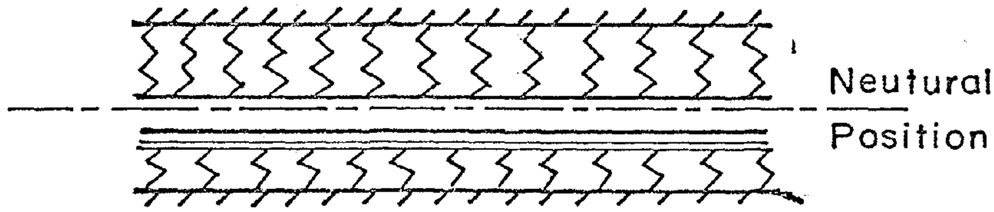
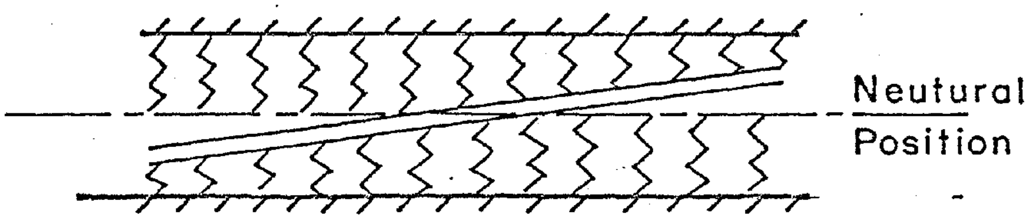


Fig. 8 - Flexural Vibration of Buried Pipe



(a) Deflection Rigid Mode



(b) Rotation Rigid Mode

Fig. 9 - Rigid Modes in Flexural Vibration

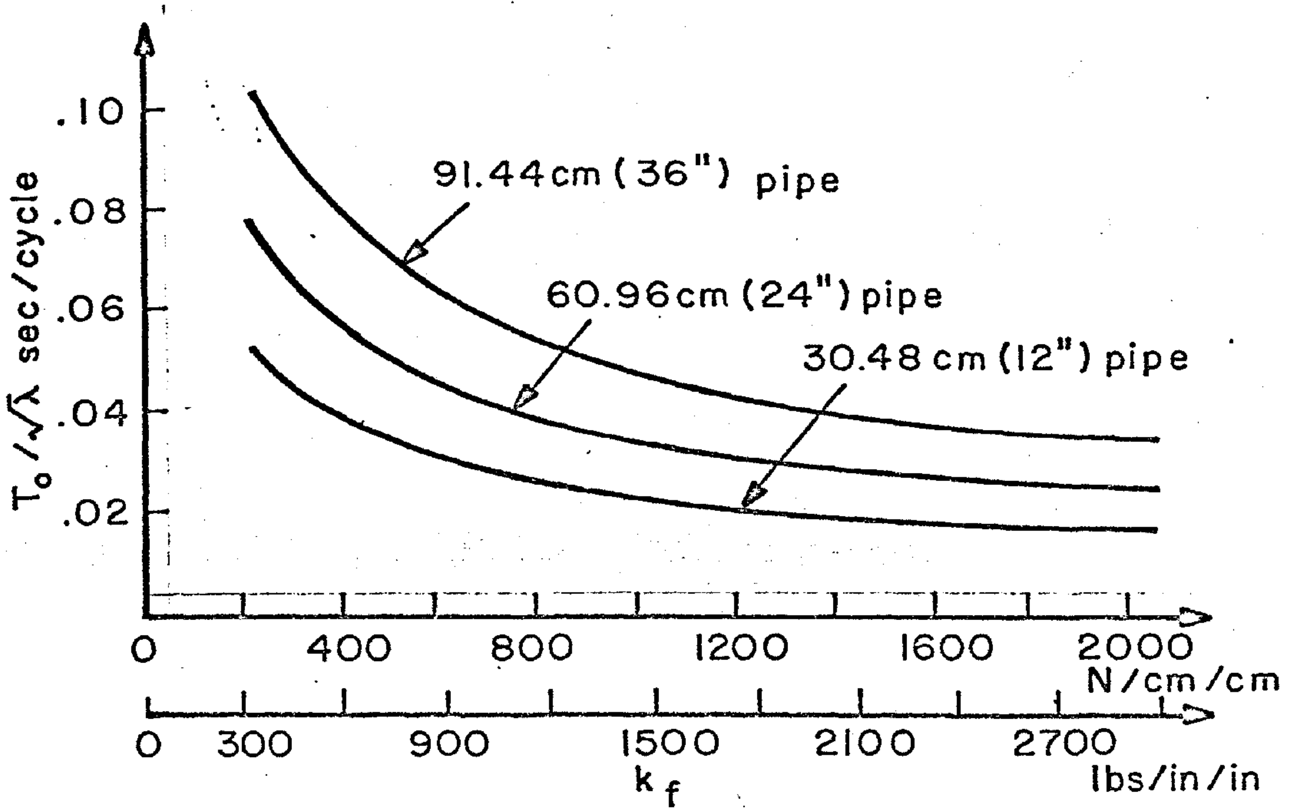


Fig.10- Rigid Mode Natural Period of Buried AWWA Concrete Pipe - Lateral Motion

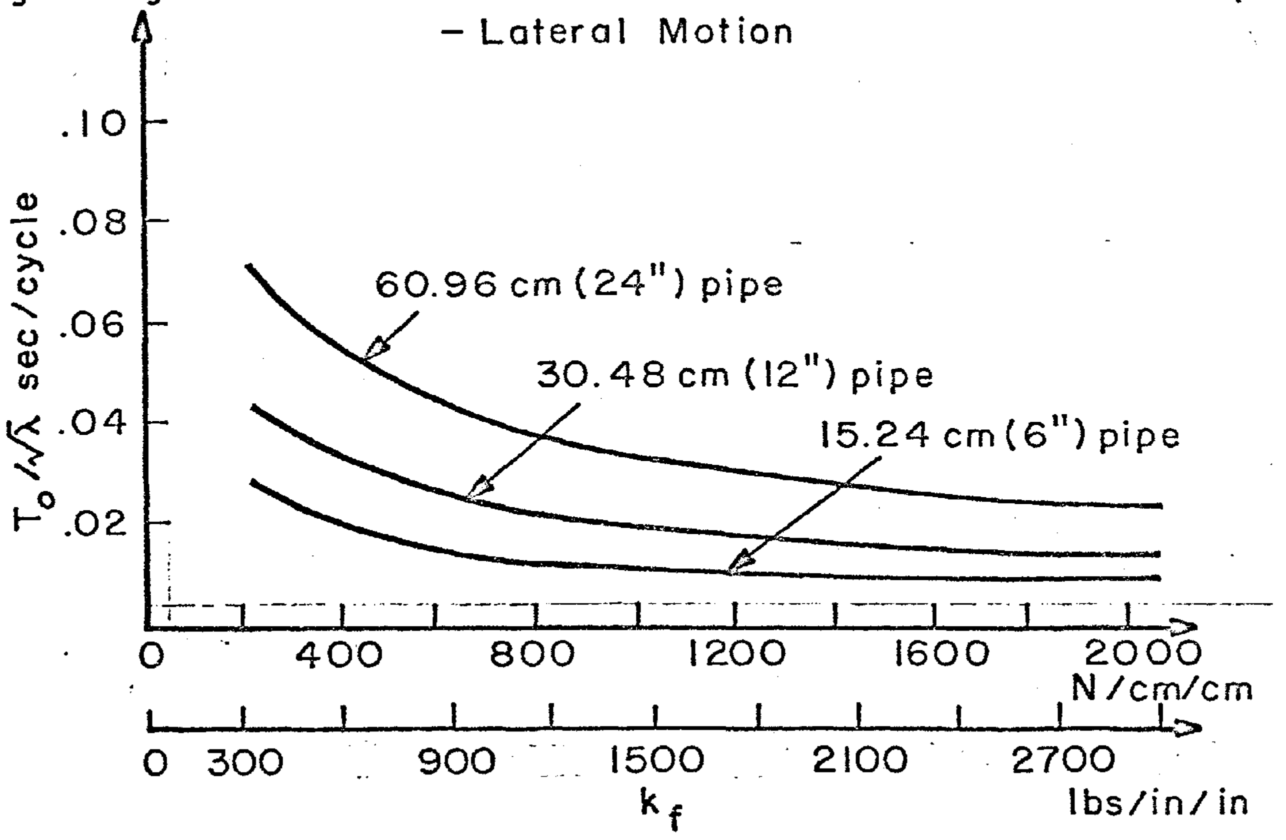


Fig.11 - Rigid Mode Natural Period of Buried AWWA Cast Iron Pipe - Lateral Motion

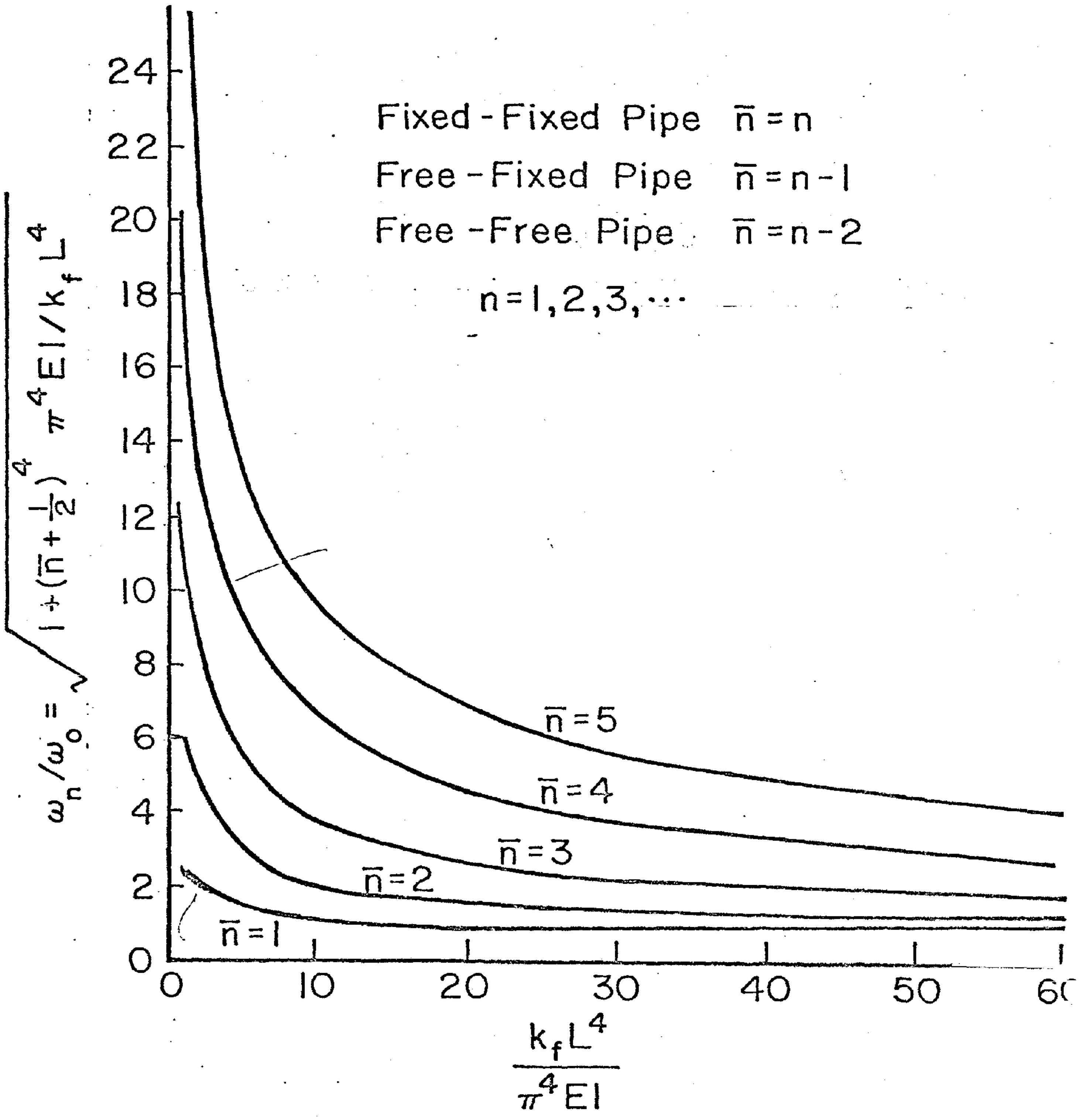


Fig. 12 - Flexural Frequency Ratio For Buried Fixed-Fixed Pipe, Free-Free Pipe, and Free-Fixed Pipe

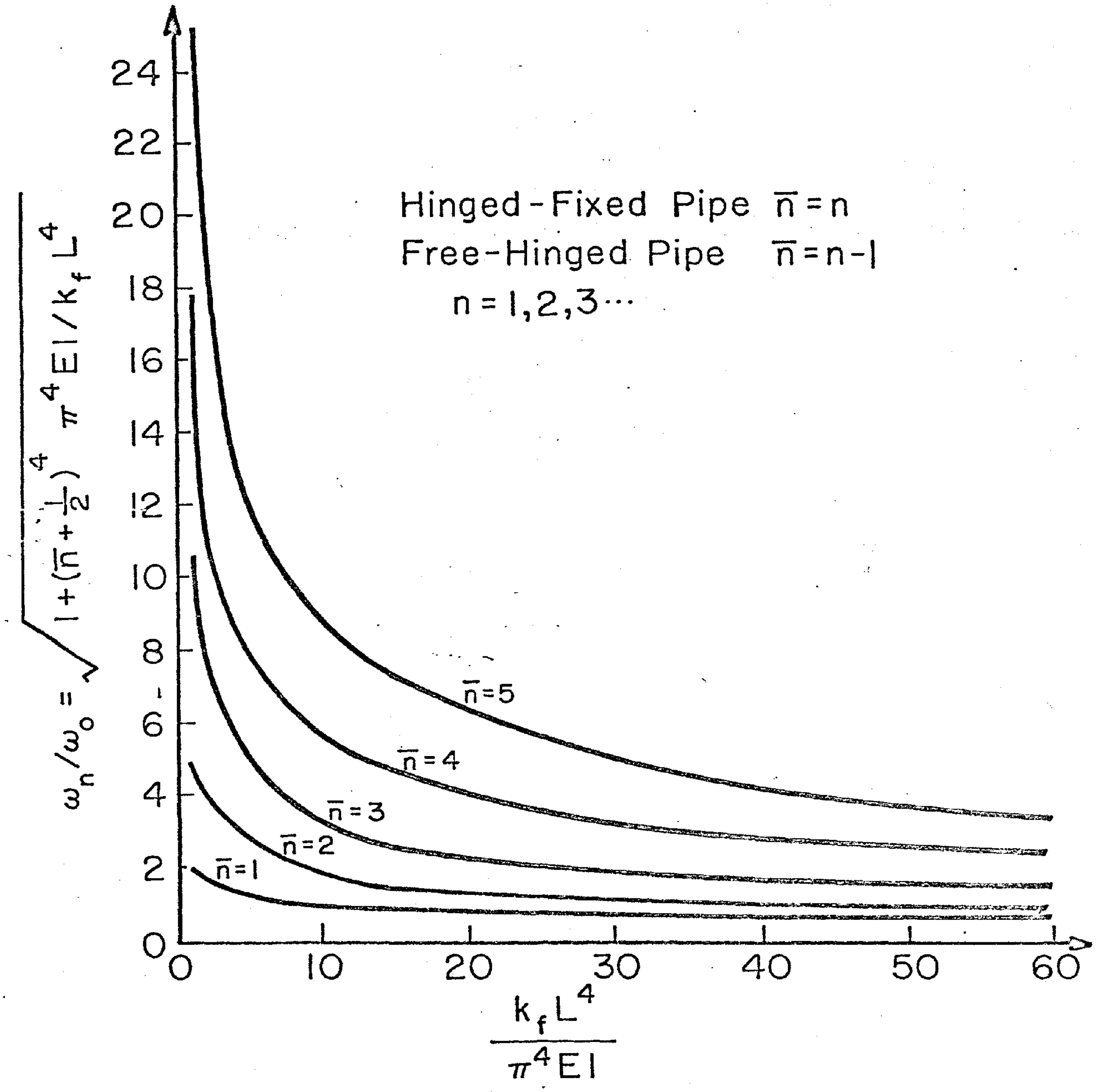


Fig. 13 - Flexual Frequency Ratio for Buried Hinged-Fixed Pipe and Free-Hinged Pipe

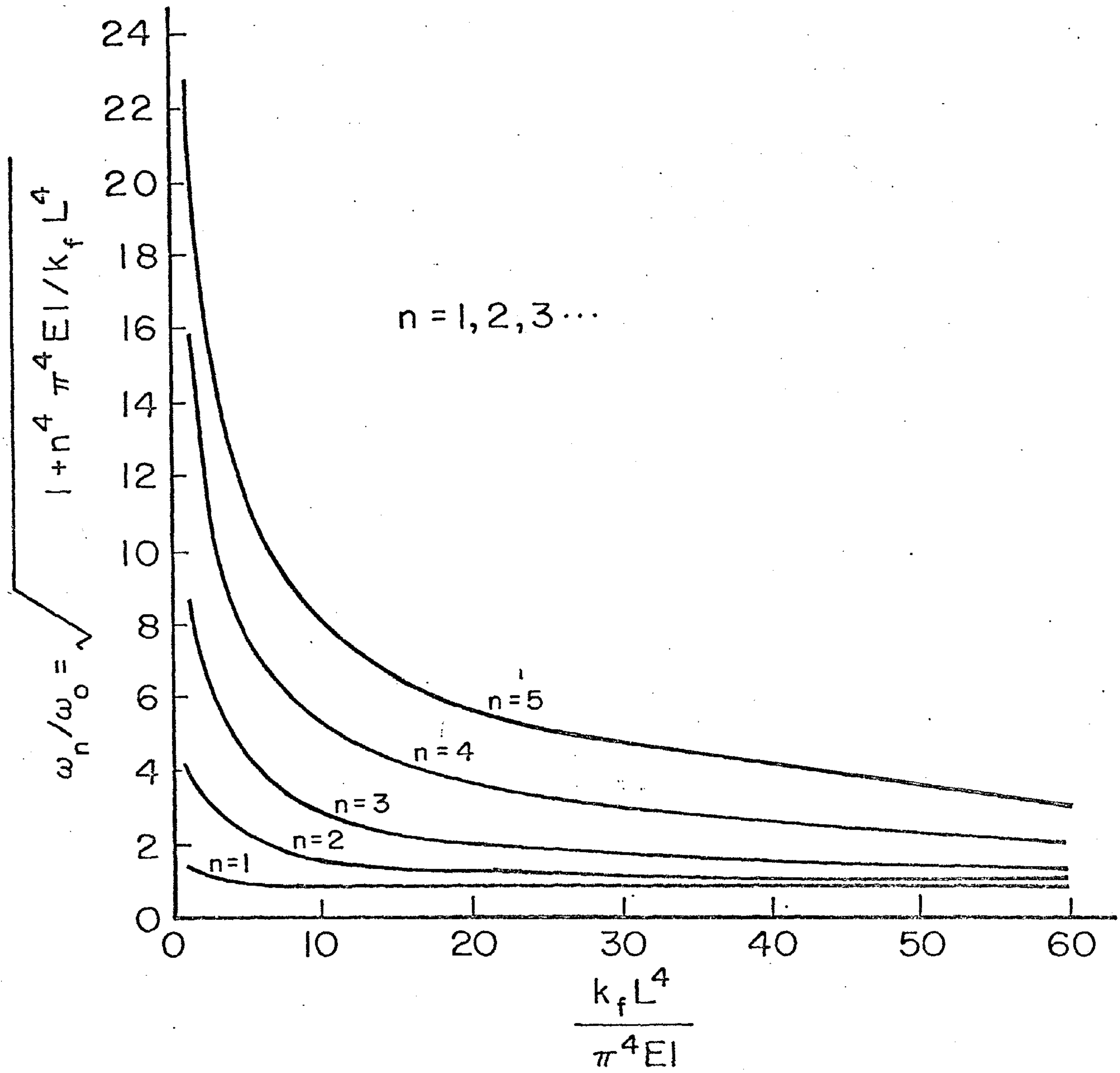


Fig. 14 - Frequency Ratio for Buried Simply Supported Pipe

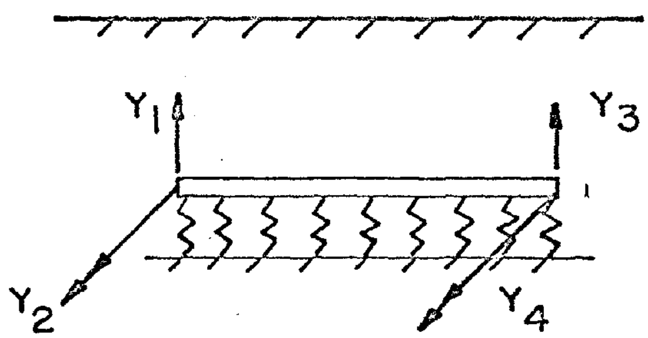


Fig. 15 – A Buried Pipe with Flexural Nodal Coordinates

NOTATION

A	Cross Sectional Area of Pipe
E	Young's Modulus of Pipe
f	Frictional Force Per Unit Area Between Soil and Pipe
G	Shear Modulus of Pipe
I	Flexural Moment of Inertia of Pipe
I_p	Polar Moment of Inertia of Pipe
k_a	Distributed Axial Soil Spring Constant (Friction)
k_f	Distributed Flexural Soil Spring Constant (Compression)
k_t	Distributed Torsional Soil Spring Constant (Friction)
K	Concentrated Axial Soil Spring Constant at End of Pipe
$[K_p]$	Stiffness Matrix of Pipe Element
$[K_s]$	Stiffness Matrix of Soil Medium
$[K_{bp}]$	Stiffness Matrix of Buried Pipe System
L	Length of Pipe
m	Mass Per Unit Length of Buried Pipe System
m_p	Mass Per Unit Length of Pipe Itself
M	Moment in Pipe
$[M_p]$	Mass Matrix of Pipe
$[M_s]$	Mass Matrix of Soil Medium
$[M_{bp}]$	Mass Matrix of Buried Pipe System
n	Modal Index
R	Axial Resistant Force From Concentrated Axial Soil Spring K
R_o	Outer Radius of Pipe
T	Kinetic Energy
u	Relative Displacement Coordinate Between Pipe and Soil Medium $(y-y_s)$

V	Shear Force in Pipe
x	Linear Space Coordinate
$y, \dot{y}, \ddot{y}$	Continuous Displacement, Velocity and Acceleration Coordinates of Pipe
$y_s, \dot{y}_s, \ddot{y}_s$	Continuous Displacement, Velocity and Acceleration Coordinates of Soil Medium
$\{Y\}$	Discrete Displacement Vector of Pipe
Γ	Torque in Pipe
ε	Axial Strain in Pipe
λ	Mass Ratio (m/m_p)
ρ	Mass Density of Pipe
σ	Axial Stress in Pipe
ϕ	Angle of Twist of Pipe
$\phi\ (x)$	Modal Function
ω	Circular Frequency of Buried Pipe System
ω_o	Rigid Mode Circular Frequency of Buried Pipe System