SOME OBSERVATIONS ON THE RANDOM RESPONSE
OF LINEAR AND NONLINEAR DYNAMICAL SYSTEMS

Thesis by

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ABSTRACT

In this thesis, two problems are investigated which commonly arise in the study of the response of randomly excited discrete dynamical systems.

First examined is the problem of obtaining the nonstationary stochastic response of a nonlinear system subject to deterministically modulated stationary Gaussian random excitation. An extension of the generalized method of equivalent linearization is used to obtain an approximation to this response. The accuracy of this approximate technique is investigated by means of Monte Carlo simulation.

Attention is then turned to the first passage problem for the stationary response of a lightly damped linear oscillator excited by white noise. A method is developed to generate approximate values for the limiting decay rate of the corresponding first passage probability density. This method is extended so that an approximate first passage probability distribution may be calculated when the oscillator response is nonstationary. The accuracy of this approximate distribution is examined.

As a practical application, it is indicated how this technique may be used to determine an earthquake-like random process which generates a response spectrum consistent with given data. The accuracy and range of validity of the procedure are indicated by a simulation study.

The approximate solution of the first passage problem is combined with the equivalent linearization technique to yield a procedure for computing approximate first passage probabilities of a weakly nonlinear oscillator. The errors introduced by this procedure are investigated.

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CHAPTER I

Introduction

The destructive effects of strong ground shaking are a matter of significant concern in all seismically active regions of the globe. In such regions, the importance of incorporating seismic considerations into engineering design criteria is generally well recognized.

The analytical tools employed in the design of structures and equipment to withstand earthquake motions range from the very simple to the highly sophisticated. Building codes commonly recommend an "equivalent" static analysis for estimating the earthquake response of a structure. (See, for example, reference [37].) Though easily applied and computationally economical, the accuracy of such an approach may prove inadequate in the presence of severe design constraints. Among the most accurate methods for determining the response of a system to seismic excitation is that of numerical integration of the equations of motion. Time histories of the system response to several characteristic earthquake accelerograms are generated, from which the design adequacy may be surmised. This method is also one of the most costly, so that it is often used only as a check of the final design configuration of the most important structures.

The use of linear response spectra represents a compromise between these two approaches. The displacement response spectrum is a representation of maximum response displacement attained by a single-degree-of-freedom linear oscillator excited

by a particular earthquake. Velocity and acceleration response spectra are defined analogously. For a given damping ratio, the maximum response is plotted as a function of the natural frequency or period of the oscillator.

For a multidegree-of-freedom linear system, response spectrum information may be used to determine the maximum response magnitude of any particular mode of the system. The modal contributions to the maximum response of the system may then be combined by any of several methods [38]. Depending on the method used, an upper bound or a most probable value for the maximum response will be obtained.

When it is not possible to disregard the nonlinearities of a dynamical system, the information contained in a linear response spectrum is not strictly applicable. Though this fact has long been recognized, response spectra are still quite often the only data provided as seismic design criteria.

An approach for circumventing this difficulty is proposed herein. Essentially, two separate elements are involved. The first of these is to determine a random process that generates response spectra consistent with the data provided, and that has sample functions reasonably similar to actual earthquake excitation. To find such a process, it is necessary to have the capability of computing the probability that the response of a randomly excited linear oscillator will remain below the corresponding response spectrum value. In the study of stochastic processes, the calculation of such probabilities falls into the category of "first passage" problems.

The second element of the suggested approach is to determine the stochastic response of the nonlinear system excited by the appropriate artificial earthquake process. This, and the first passage problem for the nonstationary response of a linear oscillator comprise the two basic analytical problems involved in the use of linear response spectrum data to estimate the seismic response of a nonlinear system.

It is the objective of this thesis to develop approximate methods for solving these two problems. The earthquake model chosen is a stationary Gaussian random process multiplied by a modulating envelope function. This model is considered to be relatively realistic [19, 32-35], yet it is simple enough to remain analytically tractable in the problems to be examined.

The stochastic response of a nonlinear multidegree-of-freedom system subject to deterministically modulated stationary Gaussian noise is considered in Chapter II. This problem is of considerable practical importance in a number of fields. Hence, it is treated first, and without special reference to the particular application in mind.

An extension of the generalized method of equivalent linearization is used to develop a set of nonlinear integral-differential equations for the elements of the covariance matrix of the response. The existence of solutions to these equations is discussed and a numerical example is performed so as to indicate the degree of accuracy and efficiency that may be expected from the method.

In Chapter III, the first passage problem for the stationary response of a linear oscillator excited by stationary white noise is examined. A semi-empirical technique is developed to calculate the approximate distribution of the time to first excursion across a symmetric double barrier. This method is then extended to the case of nonstationary oscillator response to modulated stationary Gaussian noise with non-white spectral density. A procedure for using this method to obtain an artificial earthquake process that is consistent with given response spectrum data is then discussed. Finally, it is demonstrated how the techniques developed in Chapters II and III may be combined to obtain first passage probabilities for a weakly nonlinear oscillator. A number of simulation studies throughout Chapter III serve to indicate the range of validity and degree of accuracy of the methods discussed.

The numerical examples chosen in this investigation are intended to illustrate the basic principles involved in the analyses performed. As a consequence, the examples are relatively simple and do not necessarily reflect the actual problems arising from a specific engineering application. However, it is felt that these examples do represent non-trivial problems and hence demonstrate the usefulness of the approximate methods developed.

CHAPTER II

The Nonstationary Response of a Nonlinear System Subject
to Deterministically Modulated Stationary Gaussian Noise

2.1 Introduction

In many situations of engineering interest, a dynamical sys-
tem is subject to excitation of a random nature, and the system
response can only be described probabilistically. In such cases,
it is often adequate to model the input as a stationary Gaussian
process multiplied by a deterministic modulating time function.
If the system is linear, it is well-known that the response
to such excitation will itself be a Gaussian process [1] which is
completely specified, in the stochastic sense, by its mean value
vector and its cross-correlation matrix. For the linear case,
these two functions may be expressed explicitly and hence, the
theory for this special case is complete.

If the system is nonlinear, the problem is considerably more
complicated and the general theory is not sufficiently well devel-
oped to obtain exact statistical response characteristics for most
cases.

When the excitation is a white noise process, the problem
may be formulated in terms of a Fokker-Planck-Kolmogorov
equation for the transition probability density of the response
process. Although explicit solutions to this equation are rare for
nonlinear systems, this formulation does serve to unify the theory
for the white noise case. A paper by Caughey [2] provides an
exhaustive review of the cases for which exact solutions to the

Fokker-Planck-Kolmogorov equation have been found.

When exact response statistics cannot be explicitly obtained, one must generally rely on approximate techniques. When the excitation is of constant intensity in time, (i.e., the modulating function is a constant), one may be interested in the stationary system response. Iwan and Yang [3] have developed an equivalent linearization technique for obtaining the approximate stationary response of multidegree-of-freedom nonlinear systems subject to stationary Gaussian noise of arbitrary spectral density. With the additional restriction that the nonlinear terms in the equations of motion must be small, Crandall [4] obtains an approximate solution to the same problem by means of classical perturbation techniques.

When the nonstationary response of the system is of interest, one must resort to slightly more sophisticated methods. Iwan and Spanos [5] have developed a technique for finding the approximate envelope response statistics of a narrow-band single-degree-of-freedom nonlinear oscillator subject to unmodulated white noise as it approaches steady-state from zero initial conditions. This method first uses equivalent linearization and the narrow-bandedness of the response to derive an approximate first-order differential equation for the envelope response. The associated Fokker-Planck-Kolmogorov equation is then solved by eigenfunction expansion for the transition probability density of the envelope response. The problem of the response of a multidegree-of-freedom system to modulated stationary Gaussian white noise was

approached by Kaul and Penzien [6] within the context of a particular type of nonlinear system. Without stating all of the assumptions, a time-varying equivalent linear system is found, and the associated Fokker-Planck-Kolmogorov equation is formulated. This is then used to obtain a Liapunov-type matrix differential equation for the covariance response of the equivalent linear system.

In this chapter, an equivalent linearization technique is used to develop a method for finding the approximate transient covariance response of a multidegree-of-freedom nonlinear system subject to deterministically modulated stationary Gaussian noise of arbitrarily specified spectral density. The simplifications that arise in certain special cases are noted, and the accuracy of the method is then checked by means of Monte-Carlo simulation.

2.2 Derivation and Solution of the Equations Governing the Parameters of the Equivalent Linear System

2.2.1 Review of the Generalized Method of Equivalent Linearization - Stationary Response

The equations of motion for a discrete n-degree-of-freedom dynamical system may be written in the general form

$$M\ddot{\underline{x}} + \underline{h}(\underline{x}, \dot{\underline{x}}) = \underline{f}(t) \qquad (2.1)$$

where $(\cdot) \equiv (d/dt)$. M is the $n \times n$ mass matrix, and $\underline{x}$, $\underline{h}$, and $\underline{f}$ are n-vectors, $\underline{x}$ representing generalized displacements. $h_i(\underline{x}, \dot{\underline{x}})$ is the total internal generalized force associated with the ith degree of freedom for $1 \leq i \leq n$, and $\underline{f}(t)$ is the external generalized force

vector, a function of time.

Suppose that a time-independent linear system with the same mass matrix as (2.1) is subjected to the same generalized force vector $\underline{f}(z)$ as in (2.1). That is

$$M\underline{\ddot{x}}_1 + C\underline{\dot{x}}_1 + K\underline{x}_1 = \underline{f}(t) \tag{2.2}$$

where C and K are the stiffness and damping matrices, respectively, and $\underline{x}_1$ is the generalized displacement vector of the linear system. Due to the wealth of theory on the response of linear systems, it is assumed that (2.2) may be solved explicitly for any given M, C, and K matrices, and $\underline{f}(t)$ specified. It is clear that generally $\underline{x}$ and $\underline{x}_1$ will differ. However, if one could choose C and K in such a way as to minimize some measure of the difference between the two systems, then $\underline{x}_1$ would be a candidate for an approximate solution to (2.1)

This is the essence of equivalent linearization techniques, and the aforementioned minimization is accomplished in the following way. The solution of (2.2), $\underline{x}_1(t)$, is expressed explicitly as a function of M, C, K, and t, and then substituted into the left-hand side of (2.1) in place of $\underline{x}(t)$ to obtain

$$M\underline{\ddot{x}}_1 + \underline{h}(\underline{x}_1, \underline{\dot{x}}_1) = \underline{f}(t) + \underline{d}(t) \tag{2.3}$$

In this representation, it is clear that $\underline{f}(t) + \underline{d}(t)$ is the input to the nonlinear system that would be necessary to cause its response to be $\underline{x}_1(t)$.

The difference between the nonlinear system (2.1) and the linear system (2.2) is defined to be $G(\|\underline{d}(t)\|)$, where $\|\cdot\|$ denotes a vector norm, and $G(\cdot)$ is an averaging operator, the choice of which is generally dictated by the physical situation. Since $G(\|\underline{d}(t)\|)$ is an explicit function of the elements of C and K, it may be minimized with respect to these in the usual fashion, i.e.,

$$\frac{\partial G(\|\underline{d}(t)\|)}{\partial c_{ij}} = 0$$

$$\forall_{i,j} \quad ; \quad i,j = 1,\ldots,n \qquad (2.4)$$

$$\frac{\partial G(\|\underline{d}(t)\|)}{\partial k_{ij}} = 0$$

At this point it should be remarked that in order to remain consistent with the assumption that C and K are constant in time, the averaging operator should possess the property that $G(\|\underline{d}(t)\|)$ is time independent. The interested reader is referred to [5] and [7] for a detailed discussion of this and other properties of this operator.

The expression in (2.4) constitutes a set of 2n simultaneous equations, linear in c_{ij} and k_{ij}. Spanos has derived necessary and sufficient conditions for the existence and uniqueness of solutions to these equations [8]. When a solution does exist, the resulting C and K matrices may be inserted into (2.2) and this set of differential equations are then referred to as an equivalent linear system, the solution of which may be found by standard

techniques applicable to autonomous linear systems.

In the next section, an equivalent linearization technique will be formulated in application to the stochastic response of a nonlinear system. In many aspects, it is similar to the generalized method of equivalent linearization just described; however, a significant difference results from allowing the equivalent linear system to be time-varying. The usefulness of time-varying equivalent linear systems for nonlinear problems with deterministic excitation is questionable since the solution of the equivalent linear system may require as much computational effort as would a numerical solution to the original nonlinear system of equations. However, for the stochastic problem it will be shown that its introduction produces a considerable savings in the computational effort required to obtain response statistics in comparison with other applicable techniques that are currently available.

2.2.2   The Equivalent Linear System for Nonstationary Response of a Nonlinear System

As described in (2.2.1), the strategy of equivalent linearization is to replace the nonlinear system with some member of a class of linear systems, the corresponding solutions of which are known. The member of the class which is chosen is one which minimizes some average measure of the equation difference. The solution to the linear system is then taken as an approximate solution to the original nonlinear system of equations.

It should be pointed out that the minimization of the equation difference with respect to the linear parameters does not necessarily guarantee that a minimization of the _solution_ difference has been achieved, and this may be considered a drawback of averaging methods. However, it is evident that, for a system with sufficiently small nonlinear terms, the approximate solution obtained by equivalent linearization will become asymptotic to the exact solution with decreasing nonlinearity.

For cases of steady-state response to periodic excitation and stationary random response, equation difference averaging is performed with respect to time, and the equivalent linear system is assumed to be time invariant. This procedure could also be applied to the nonstationary stochastic response of a nonlinear system by again assuming that the equivalent linear system is time invariant and integrating over some fixed interval of time. However, there are several observations which suggest that more accurate approximate solutions might be obtained if the class of systems to be considered as candidates for the equivalent linear system were permitted to vary with time.

For example, it is well known that the equivalent linear stiffness for the stationary response of a damped duffing oscillator is a monotonically increasing or decreasing function of the variance of the response, depending on whether the coefficient of the cubic term is positive or negative. For nonstationary response, the variance will change with time. Since the statistical response varies with time, and the effective system

parameters depend upon the response, it is only reasonable to propose that the system parameters should also be allowed to change with time.

With this somewhat heuristic justification, the class of linear systems from which the equivalent linear system will be chosen will be treated as time varying. However, the explicit dependence of the equivalent linear parameters is on the response statistics rather than on time. In order to emphasize this functional dependence, a special notation is introduced. Let $S(t)$ represent some statistical description of the response of the equivalent linear system at time t. Then, an expression of the form

$$B\left[S(t)\right]$$

denotes that the quantity B, which may be a scalar or a matrix, is an explicit function of the statistics of the response of the equivalent linear system, which are in turn functions of time.

With the foregoing discussion in mind, consider once again the nonlinear dynamical system described by

$$M\underset{\sim}{\ddot{x}} + \underset{\sim}{h}(\underset{\sim}{x}, \underset{\sim}{\dot{x}}) = \underset{\sim}{f}(t) \qquad (2.1)$$

with the initial conditions

$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$$
$$\underset{\sim}{\dot{x}}(t_0) = \underset{\sim}{v}_0 \qquad (2.5)$$

There is no loss of generality in assuming $t_0 = 0$, and this will

be done henceforth. Let $\underset{\sim}{f}(t)$ represent a random process with an infinite number of sample functions $^k\underset{\sim}{f}(t)$. The sample functions $^k\underset{\sim}{x}(t)$ of the response process $\underset{\sim}{x}(t)$ then obey

$$M^k\underset{\sim}{\ddot{x}} + \underset{\sim}{h}(^k\underset{\sim}{x}, {}^k\underset{\sim}{\dot{x}}) = {}^k\underset{\sim}{f}(t) \tag{2.6}$$

$$^k\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

$$^k\underset{\sim}{\dot{x}}(0) = \underset{\sim}{v}_0 \qquad ; \quad k = 1, 2, \ldots$$

The desired response statistics are, in general, non-stationary. For reasons already discussed, therefore, let the proposed equivalent linear system have the form

$$M^k\underset{\sim}{\ddot{x}} + C\left[S(t)\right]^k\underset{\sim}{\dot{x}} + \beta\left[S(t)\right]^k\underset{\sim}{x} = {}^k\underset{\sim}{f}(t) \tag{2.7}$$

$$^k\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

$$^k\underset{\sim}{\dot{x}}(0) = \underset{\sim}{v}_0 \qquad ; \quad k = 1, 2, \ldots$$

Let the vector $^k\underset{\sim}{\delta}$ be the difference between the left-hand side of equation (2.8) and that of equation (2.7). Written explicitly, then $^k\underset{\sim}{\delta}$ is given by

$$^k\underset{\sim}{\delta}\left[S(t)\right] \equiv C\left[S(t)\right]^k\underset{\sim}{\dot{x}} + \beta\left[S(t)\right]^k\underset{\sim}{x} - \underset{\sim}{h}(^k\underset{\sim}{x}, {}^k\underset{\sim}{\dot{x}}) \tag{2.8}$$

A measure of this equation difference is given by the square of the Euclidean norm of $^k\underset{\sim}{\delta}$ denoted by

$$\left\|^k\underset{\sim}{\delta}\right\|_2^2 = {}^k\underset{\sim}{\delta}^T {}^k\underset{\sim}{\delta} \tag{2.9}$$

in which $^k\underset{\sim}{\delta}^T$ denotes the transpose of the vector $^k\underset{\sim}{\delta}$.

Let $[0, \tau]$ be the time interval during which the transient response of (2.1) is of interest, where τ is required to be a constant. Furthermore, let β be a scalar random variable defined by

$$\beta(k) \equiv \frac{1}{\tau} \int_0^T {}^k\underline{\delta}(t)^T \, {}^k\underline{\delta}(t) \, dt \qquad (2.10)$$

then, $\beta(k)$ is the (temporal) average value of the instantaneous equation difference corresponding to ${}^k\underline{f}(t)$.

For an ensemble of M samples, the ensemble error $\mathcal{E}_M$ may be defined as

$$\mathcal{E}_M \equiv \frac{1}{M} \sum_{k=1}^{M} \int_0^{\tau} {}^k\underline{\delta}(t)^T \, {}^k\underline{\delta}(t) \, dt \qquad (2.11)$$

Hence,

$$E[\beta] = \lim_{M \to \infty} \frac{1}{\tau} \sum_{k=1}^{M} \int_0^T {}^k\underline{\delta}(t)^T \, {}^k\underline{\delta}(t) \, dt$$

$$= \lim_{M \to \infty} \frac{1}{\tau} \mathcal{E}_M$$

. (2.12

where the assumption is made that the limit exists. It is apparent that $E[\beta]$ provides a measure of the total equation difference resulting from replacing the original nonlinear system (2.6) by the linear system (2.7).

The equivalent linear system will be defined as that system which causes $E[\beta]$ to attain its minimum value. Thus, it is desired to find the time-varying matrices $C[S(t)]$ and $B[S(t)]$

such that

$$E[\beta]\Big|_{C,\beta} = \text{minimum} \quad \nabla^C[S(t)], \beta[S(t)] \tag{2.13}$$

Using (2.10) and the properties of the expectation operator, equation (2.13) may be written as

$$\int_0^T E\left[\|\underline{\varepsilon}\{C[S(t)], \beta[S(t)], t\}\|_2^2\right] dt = \text{minimum} \quad \nabla^C[S(t)], \beta[S(t)] \tag{2.14}$$

Expression (2.14) is the statement of a calculus of variations problem where the functional does not depend upon the derivatives of the varying functions, and the integration limits are not variable. The appropriate Euler equations for this problem are

$$\frac{\partial E\left[\|\underline{\varepsilon}(C, \beta, t)\|_2^2\right]}{\partial c_{ij}} = 0$$

$$\tag{2.15}$$

$$\frac{\partial E\left[\|\underline{\varepsilon}(C, \beta, t)\|_2^2\right]}{\partial b_{ij}} = 0$$

It should be pointed out that equations (2.15) are independent of the time interval $[0, T]$ over which the minimization is to be performed.

Using (2.9) and inverting the order of differentiation and expectation, (2.15) becomes

$$E\left[\frac{\partial}{\partial c_{ij}}\left(\sum_{m=1}^{n}\delta_m^2\right)\right] = 0$$

$$(2.16)$$

$$E\left[\frac{\partial}{\partial b_{ij}}\left(\sum_{m=1}^{n}\delta_m^2\right)\right] = 0$$

From (2.8), the components of $\underline{\delta}$ are given by

$$\delta_m = \left[\sum_{j=1}^{n}\left(c_{mj}x_j + b_{mj}x_j\right)\right] - h_m \qquad (2.17)$$

Therefore,

$$E\left[\frac{\partial}{\partial c_{ij}}\left(\delta_m^2\right)\right] = 0 \qquad ; \quad \forall i \neq m$$

$$(2.18)$$

$$E\left[\frac{\partial}{\partial b_{ij}}\left(\delta_m^2\right)\right] = 0 \qquad ; \quad \forall i \neq m$$

thus (2.16) is equivalent to

$$E\left[\frac{\partial\left(\delta_i^2\right)}{\partial c_{ij}}\right] = 0 \qquad ; \quad i,j=1,\ldots,n$$

$$(2.19)$$

$$E\left[\frac{\partial\left(\delta_i^2\right)}{\partial b_{ij}}\right] = 0 \qquad ; \quad i,j=1,\ldots,n$$

Performing the differentiation in (2.19) yields

$$E\left[\delta_i \frac{\partial \delta_i}{\partial c_{ij}}\right] = 0 \quad ; \quad i,j=1,\ldots,n$$

$$\text{(2.20)}$$

$$E\left[\delta_i \frac{\partial \delta_i}{\partial b_{ij}}\right] = 0 \quad ; \quad i,j=1,\ldots,n$$

Using (2.17) then gives

$$\frac{\partial \delta_i}{\partial c_{ij}} = \dot{x}_j \quad ; \quad j=1,\ldots,n$$

$$\text{(2.21)}$$

$$\frac{\partial \delta_i}{\partial b_{ij}} = x_j \quad ; \quad j=1,\ldots,n$$

Substituting from (2.21) and (2.17) into (2.20) yields

$$E\left[\left\{\sum_{m=1}^{n}\left(c_{im}\dot{x}_m + b_{im}x_m\right) - h_i\right\}\dot{x}_j\right] = 0 \quad ; \quad i,j=1,\ldots,n$$

$$\text{(2.22)}$$

$$E\left[\left\{\sum_{m=1}^{n}\left(c_{im}\dot{x}_m + b_{im}x_m\right) - h_i\right\}x_j\right] = 0 \quad ; \quad i,j=1,\ldots,n$$

Rewriting (2.22) in matrix notation

$$E\left[h_i \underset{\sim}{y}\right] = E\left[\underset{\sim}{y}\underset{\sim}{y}^T\right]\begin{bmatrix} b_{\sim i*}^T \\ c_{\sim i*}^T \end{bmatrix} \quad ; \quad i=1,\ldots,n \qquad \text{(2.23)}$$

where

$$\underset{\sim}{y} \equiv \left\{\begin{matrix} \underset{\sim}{x} \\ \underset{\sim}{\dot{x}} \end{matrix}\right\} \qquad \text{(2.24)}$$

and

$$\underline{b}_{i*} = \text{ith row vector of the } \mathbb{B}(t) \text{ matrix}$$

$$\underline{c}_{i*} = \text{ith row vector of the } \mathbb{C}(t) \text{ matrix}$$

Equivalently, (2.23) may be written in the form

$$E\left[\underline{y}\underline{h}^T(\underline{y})\right] = E\left[\underline{y}\underline{y}^T\right]\left[\mathbb{B}, \mathbb{C}\right]^T \qquad (2.25)$$

Expression (2.25) represents $2n^2$ equations linear in b_{ij} and c_{ij}, the elements of the $\mathbb{B}$ and $\mathbb{C}$ matrices, respectively. In [7] and [8], a set of equations are derived which are nearly identical to (2.25), the difference being that the expectation operator $E[\cdot]$, which may be time-dependent, is replaced by a general time-invariant averaging operator $G(\cdot)$. The assumption of the time-invariance of G, however, is not essential for several of the results derived in [8], and some of these results will be mentioned here.

Note that the expectation operator has the properties

$$E\left[a(t) + b(t)\right] = E\left[a(t)\right] + E\left[b(t)\right] \qquad (2.26)$$

and

$$E\left[a^2(t)\right] > 0 \quad ; \quad \forall\, a(t) \neq 0 \qquad (2.27)$$

It is proven in [8] for an averaging operator possessing the properties (2.26) and (2.27), that if a solution to (2.25) for $[\mathbb{B}, \mathbb{C}]$ exists, the corresponding value of $E[\underline{\delta}^T \underline{\delta}]$ is an absolute (global) minimum. There may be more than one such solution, but no

one of these will be any better (or worse), in the sense of the minimization criterion (2.13), than any other.

Furthermore, it is pointed out in [8] that equations of the form of (2.25) will have a unique solution only if $E[\underline{y}\underline{y}^T]$ is non-singular. When $E[\underline{y}\underline{y}^T]$ is singular, the existence of a solution is not guaranteed, and when a solution does exist, it is not unique.

There is a well-known result of probability theory which states that the covariance matrix of the probability distribution of an N-dimensional vector $\underline{y}$ is singular if and only if there exists a set of constants $\{a_i ; 1 \leq i \leq N\}$ such that

$$\Pr\left[a_1 y_1 + \ldots + a_N y_N = 0\right] = 1 \qquad (2.28)$$

where $\Pr[\cdot]$ denotes the probability of occurrence of the situation described inside the brackets. For a proof of this result, see Feller [16]. In words, (2.28) says that, with probability one, at least one of the elements of $\underline{y}$ is expressible as a deterministic linear combination of the rest of the elements. In terms of the response of a dynamical system, (2.28) can be true only if the governing equations of motion contain a redundant equation, or if an uncoupled degree of freedom is subject to deterministic excitation only. In such cases, the probability distribution of the response process $\underline{y}$ is said to be _degenerate_ in an N-dimensional space. It is always possible, in this event, to eliminate from the equations of motion the equation or equations irrelevent to the non-deterministic response of the system, thereby creating a

non-degenerate response process in a space of lower dimension.

Without sacrifice of generality, therefore, one may require the nonlinear system under consideration to have a non-degenerate response process. If this requirement is imposed, then the covariance matrix $E[y\chi^T]$ will never be singular so that a unique solution to (2.25) for $[\beta, C]$ always exists.

2.2.3 Special Case of Gaussian Distributed Excitation

In the study of random vibrations, much attention is given to sources of stochastic excitation that have a Gaussian probability density. There are several good reasons for this, not the least of which is that a number of important natural sources of random excitation, including strong-motion earthquakes, exhibit, at least approximately, a Gaussian distribution [9-12]. The excitation from such natural sources may often be considered as the resultant sum of many independent addtitive random effects, so that modeling such excitation as a Gaussian process is consistent with the central limit theorem. Further, the response of a linear system to Gaussian excitation is also a Gaussian process, which permits the distribution of the response process to be completely described statistically by its mean value vector and its covariance matrix.

In the context of equivalent linearization, there is an added attraction to restricting the random input to Gaussian processes. For the case of a jointly Gaussian vector process $y(t)$ with mean 0, Atalik and Utku have found an explicit solution to (2.25).

This solution is given by

$$b_{ij} = E\left[\frac{\partial h_i}{\partial x_j}\right]$$

$$(2.29)$$

$$c_{ij} = E\left[\frac{\partial h_i}{\partial \dot{x}_j}\right]$$

To make use of this solution form, it is necessary to assume that $\underline{h}(\underline{y})$ is sufficiently smooth that the partial derivatives in (2.29) exist. Atalik and Utku apply this solution to the case where $\underline{y}(t)$ is stationary, but the assumption of stationarity is not used in the derivation of the solution (2.29). It is therefore applicable to the nonstationary process $\underline{y}(t)$ in (2.25).

So that advantage may be taken of the solution from (2.29), only systems of the following form will be considered

$$M\ddot{\underline{x}} + \underline{h}(\underline{x}, \dot{\underline{x}}) = \hat{\underline{\theta}}(t)\, r(t) \qquad (2.30)$$

where $r(t)$ is a stationary Gaussian random process with zero mean and power spectral density $S(\omega)$, and $\hat{\underline{\theta}}(t)$ is a deterministic modulating vector function of time. It will also be required that h_i be an odd function of its arguments for $i = 1, \ldots, n$, and zero initial conditions will be imposed, i.e.

$$\underline{x}(0) = \underline{0}$$

and

$$(2.31)$$

$$\dot{\underline{x}}(0) = \underline{0}$$

It is clear from symmetry considerations that the oddness of $\underset{\sim}{h}$ and the zero initial conditions will ensure that the mean value of the response process is zero. Thus,

$$E\left[\underset{\sim}{x}(t)\right] = \underset{\sim}{0}$$

and $\qquad\qquad\qquad \forall t \geq 0 \qquad\qquad\qquad$ (2.32)

$$E\left[\underset{\sim}{\dot{x}}(t)\right] = \underset{\sim}{0}$$

Although the requirement that $\underset{\sim}{h}$ be odd does restrict somewhat the class of nonlinearities that can be analyzed, it should be pointed out that this property is quite commonly found in actual non-linearities arising in engineering applications. Further, the zero initial conditions reflect the most commonly assumed initial situation for engineering applications.

The equivalent linear system for (2.30) will be

$$M\underset{\sim}{\ddot{x}} + C\left[S(t)\right]\underset{\sim}{\dot{x}} + B\left[S(t)\right]\underset{\sim}{x} = \underset{\sim}{\hat{\theta}}(t)\, r(t) \qquad (2.33)$$

where C and B are given by equation (2.29).

It will be noted that the use of (2.29) in (2.33) generates a set of stochastic differential equations with coefficient matrices that are functions of the statics of the response process, which are, at this point, unknown. In order to transform (2.33) into an equation containing only the statistical parameters of the response process, it is necessary to make use of some results of the theory of linear stochastic differential equations.

2.3 Some Results of the Theory of Linear Stochastic Differential Equations

2.3.1 A Generalized Form of a Result Obtained by Caughey and Stumpf [12]

In reference [12], the response of a single-degree-of-freedom damped harmonic oscillator subject to stationary Gaussian random noise is considered. The oscillator is given arbitrarily specified initial conditions, and a solution is then obtained for the mean value and the variance of the response as a function of time. The following is a straightforward extension of this result to include time-varying multidegree-of-freedom linear systems subject to deterministically modulated stationary Gaussian random excitation.

Consider a set of N time-varying linear differential equations

$$\dot{\underline{y}} = A(t)\underline{y} + \underline{\theta}(t) r(t) \qquad \underline{y}(0) = \underline{c} \qquad (2.34)$$

where $A(t)$ is an $N \times N$ deterministic matrix continuous in time, $r(t)$ is a stationary Gaussian random process with zero mean and power spectral density $S(\omega)$, $\underline{\theta}(t)$ is a deterministic modulating N-vector time function, and $\underline{y}$ is the N-vector response process. Let $Y(t)$ be the fundamental solution matrix associated with (2.34), defined by

$$\dot{Y}(t) = A(t) Y(t)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.35)

$$Y(0) = I$$

where I is the $N \times N$ identity matrix. Then, the solution to

(2.34) may be written

$$\underset{\sim}{y}(t) = Y(t)\underset{\sim}{c} + Y(t) \int_0^t Y^{-1}(s)\underset{\sim}{\theta}(s) r(s) ds \qquad (2.36)$$

Taking expectations on both sides of (2.36), and noting that Y and $\underset{\sim}{\theta}$ are deterministic,

$$E\left[\underset{\sim}{y}(t)\right] = Y(t)\underset{\sim}{c} + Y(t) \int_0^t Y^{-1}(s)\underset{\sim}{\theta}(s) E\left[r(s)\right] ds \qquad (2.37)$$

By assumption,

$$E\left[r(s)\right] = 0 \qquad (2.38)$$

which leads to

$$E\left[\underset{\sim}{y}(t)\right] = Y(t)\underset{\sim}{c}$$

Hence, for zero initial conditions, It is clear that

$$E\left[\underset{\sim}{y}(t)\right] = \underset{\sim}{0} \qquad \forall t \geq 0 \qquad (2.40)$$

This confirms an earlier statement, and the zero start will be the case of interest henceforth.

It is necessary to know the covariance matrix $E\left[\underset{\sim}{y}\underset{\sim}{y}^T\right]$ in order to completely define the probability density of the response, $p(\underset{\sim}{y})$, at a given value of time for a Gaussian random process. The cross-correlation matrix $E\left[\underset{\sim}{y}(t_1) \underset{\sim}{y}^T(t_2)\right]$ may be obtained in a straightforward fashion. Using (2.36) gives

$$\underline{y}(t_1)\underline{y}^T(t_2) = \left\{ Y(t_1)\int_0^{t_1} Y^{-1}(s_1)\underline{\theta}(s_1)r(s_1)ds_1 \right\}$$

$$\cdot \left\{ Y(t_2)\int_0^{t_2} Y^{-1}(s_2)\underline{\theta}(s_2)r(s_2)ds_2 \right\}^T$$

$$= \left\{ Y(t_1)\int_0^{t_1} Y^{-1}(s_1)\underline{\theta}(s_1)r(s_1)ds_1 \right\}$$

$$\cdot \left\{ \int_0^{t_2} r(s_2)\underline{\theta}^T(s_2)Y^{-1^T}(s_2)ds_2 \cdot Y^T(s_2) \right\},$$

$$(2.41)$$

or

$$\underline{y}(t_1)\underline{y}^T(t_2) =$$

$$Y(t_1)\left\{ \int_0^{t_1}\int_0^{t_2} Y^{-1}(s_1)\underline{\theta}(s_1)\underline{\theta}^T(s_2)Y^{-1^T}(s_2)r(s_1)r(s_2)ds_2 ds_1 \right\}Y^T(s_2)$$

$$(2.42)$$

Taking expectations gives

$$E\left[\underline{y}(t_1)\underline{y}^T(t_2)\right]$$

$$Y(t_1)\left\{ \int_0^{t_1}\int_0^{t_2} Y^{-1}(s_1)\underline{\theta}(s_1)\underline{\theta}^T(s_2)Y^{-1^T}(s_2)E\left[r(s_1)r(s_2)\right]ds_2 ds_1 \right\}Y^T(s_2)$$

$$(2.43)$$

The autocorrelation function R of the random process r(t) is

given by

$$R(s_1, s_2) \equiv E\left[r(s_1)r(s_2)\right] \qquad (2.44)$$

By assumption, $r(t)$ is a stationary process, so $R(s_1, s_2)$ can only be a function of the difference between s_1 and s_2, and not on the actual values themselves. That is

$$R(s_1, s_2) = R(s_1 - s_2) = R(s) \qquad (2.45)$$

where

$$s \equiv s_1 - s_2 \qquad (2.46)$$

Using the fact that the autocorrelation function of a stationary process and the power spectral density of the process are Fourier transform pairs, $R(s)$ may be written

$$R(s) = \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega s}\, d\omega \qquad (2.47)$$

From (2.46), it is obvious that

$$e^{i\omega s} = e^{i\omega s_1}\, e^{-i\omega s_2} \qquad (2.48)$$

Thus,

$$E\left[r(s_1)r(s_2)\right] = \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega s_1}\, e^{-i\omega s_2}\, d\omega \qquad (2.49)$$

Substituting (2.49) into (2.43) yields

$$E\left[\underline{y}(t_1)\underline{y}^T(t_2)\right] = \int_{-\infty}^{\infty} \underline{F}(\omega, t_1)\underline{F}^{*T}(\omega, t_2)\, S(\omega)\, d\omega \qquad (2.50)$$

where

$$\underset{\sim}{F}(\omega,t) \equiv Y(t) \int_0^t Y^{-1}(s) \underset{\sim}{\theta}(s) e^{i\omega s} ds \qquad (2.51)$$

and $\underset{\sim}{F}^*$ denotes the complex conjugate of $\underset{\sim}{F}$.

The covariance matrix $Q(t)$ is given by

$$Q(t) = E\left[\underset{\sim}{y}(t)\underset{\sim}{y}^T(t)\right] \qquad (2.52)$$

Letting $t_1 = t_2 = t$ in (2.50) gives

$$Q(t) = \int_{-\infty}^{\infty} \underset{\sim}{F}(\omega,t) \underset{\sim}{F}^{*T}(\omega,t) S(\omega) d\omega \qquad (2.53)$$

It is of interest to note that the N-vector quantity $\underset{\sim}{F}(\omega,t)$ as defined in (2.51) satisfies the differential equation

$$\underset{\sim}{\dot{F}} = A(t)\underset{\sim}{F} + \underset{\sim}{\theta}(t) e^{i\omega t} \qquad (2.54)$$

with the initial condition

$$\underset{\sim}{F}(0) = \underset{\sim}{0} \qquad (2.55)$$

Thus, $\underset{\sim}{F}(\omega,t)$ has an immediate physical significance. It is simply the response of the linear system of equations (2.34), except that the excitation $r(t)$ is now harmonic and deterministic rather than random.

Once $Q(t)$ is found, the probability density $p(y(t))$ may be written explicitly. For a jointly Gaussian distributed N-vector variable $\underset{\sim}{y}$, with mean $\underline{0}$ and covariance matrix Q, the probability density will be given by

$$p(\underline{y}) = \frac{1}{\left[(2\pi)^N \det(Q)\right]^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\underline{y}^T Q^{-1} \underline{y}\right) \qquad (2.56)$$

2.3.2 Simplifications Resulting from Gaussian White Noise Excitation

Consider the special case where the power spectral density $S(\omega)$ is a constant in ω. That is

$$S(\omega) = S_0 \quad ; \quad \forall \omega, \ -\infty < \omega < \infty \qquad (2.57)$$

The autocorrelation of the process $r(t)$ corresponding to (2.57) is given by the inverse Fourier transform of $S(\omega)$. Using (2.47)

$$R(s) = S_0 \int_{-\infty}^{\infty} e^{i\omega s} d\omega = 2\pi S_0 \delta(s) \qquad (2.58)$$

where $\delta(s)$ is the Dirac delta function. Recalling the definition (2.44) of $R(s)$, (2.58) may be written as

$$E\left[r(s_1) r(s_2)\right] = 2\pi S_0 \delta(s_1 - s_2) \qquad (2.59)$$

Substituting from (2.59) into (2.43), then yields

$$E\left[y(t_1) y^T(t_2)\right]$$

$$= (2\pi S_0) Y(t_1) \left\{ \int_0^{t_1} \int_0^{t_2} Y^{-1}(s_1) \theta(s_1) \theta^T(s_2) Y^{-1T}(s_2) \delta(s_1 - s_2) ds_2 ds_1 \right\} Y^T(t_2)$$
$$(2.60)$$

From a well-known property of the Dirac delta function, the inner integral may be expressed as

$$\int_0^{t_2} Y^{-1}(s_1)\underline{\theta}(s_1)\underline{\theta}^T(s_2)Y^{-1T}(s_2)\delta(s_1-s_2)ds_2 = Y^{-1}(s_1)\underline{\theta}(s_1)\underline{\theta}^T(s_1)Y^{-1T}(s_1)$$

$$(2.61)$$

Letting $t_1 = t_2 = t$ in (2.60) and (2.61) yields

$$Q(t) = 2\pi S_0 Y(t) G(t) Y^T(t) \qquad (2.62)$$

where

$$G(t) \equiv \int_0^t Y^{-1}(s)\underline{\theta}(s)\underline{\theta}^T(s)Y^{-1T}(s)ds \qquad (2.63)$$

Expressions (2.62) and (2.63) constitute an explicit representation of $Q(t)$ and may be evaluated for any value of t once the fundamental solution matrix $Y(t)$ is known. It is also possible to solve for $Q(t)$ directly without first computing $Y(t)$, which may prove convenient in some applications. This may be accomplished as follows.

Differentiating both sides of (2.62), one obtains

$$\dot{Q} = 2\pi S_0\left(\dot{Y}GY^T + Y\dot{G}Y^T + YG\dot{Y}^T\right) \qquad (2.64)$$

Substituting from (2.37) for $\dot{Y}$ and using (2.63), (2.64) becomes

$$Q = 2\pi S_0\left(AYGY^T + Y\left[Y^{-1}\underline{\theta}\,\underline{\theta}^T Y^{-1T}\right]Y^T + YGY^T A^T\right) \qquad (2.65)$$

Simplifying this, and using (2.62) yields

$$\dot{Q}(t) = A(t)Q(t) + \left[A(t)Q(t)\right]^T + 2\pi S_0\underline{\theta}(t)\underline{\theta}^T(t) \qquad (2.66)$$

where the fact that Q(t) is symmetrical has been used. The Liapunov type matrix differential equation (2.66) must be solved with the initial condition

$$Q(0) = 0 \qquad\qquad (2.67)$$

It should be remarked that equation (2.66) can be derived in a somewhat less direct fashion from the Fokker-Planck-Kolmogorov equation that governs the transition probability density of the response process $\underline{y}$. In the absence of other criteria, the choice of derivations was here made in favor of the more intuitive approach.

It is appropriate at this point, to remark on the relative merits of using (2.66) versus (2.62) and (2.63) for the purpose of computing Q(t). Expressions (2.62) and (2.63) involve the fundamental solution matrix Y(t), which is obtained by solving a system of N^2 first order time-varying differential equations. Further, (2.63) requires the evaluation of the integrals of $N(N + 1)/2$ time functions. Alternatively, (2.66) involves only the solution of $N(N + 1)/2$ first order time-varying differential equations, generating Q(t) directly. In the most general case, one must expect that any of the above procedures would have to be performed numerically, in which case, the use of (2.66) is considerably less time consuming. However, in the special case that Y(t) may be expressed explicitly, the use of (2.62) and (2.63) may be more expedient.

2.3.3 Remarks on the Use of the Results

Throughout sections (2.3.1) and (2.3.2), the matrix A has been written as an explicit function of time. Due to the fashion in which the results from these two sections will be used, it is of importance to consider whether expressions (2.53) and (2.51) for the covariance matrix $Q(t)$ are still valid if the matrix $A(t)$ is replaced by the implicitly time-varying matrix $A[S(t)]$.

Upon examination of the foregoing analysis, it is apparent that the validity of these expressions depends critically upon the existence of the fundamental solution matrix $Y(t)$. Once the existence of such a matrix is established, all the results in (2.3.1) and (2.3.2) follow.

When the A matrix contains functional dependence upon $S(t)$, $Y(t)$ must satisfy the equations

$$\dot{Y}(t) = A[S(t)]Y$$

and

$$Y(0) = I$$

(2.68)

It is a well-known result from the theory of ordinary differential equations that if A is continuous in time, then a unique solution to (2.68) exists. (See, for instance, reference [17]). Therefore, if $S(t)$ is such that, given any arbitrarily small positive constant ϵ, and any value of t in the interval of interest $[0, \tau]$, there exists a positive constant ν such that

$$|a_{ij}[S(t+\nu)] - a_{ij}[S(t)]| \leq \epsilon \qquad \forall_{i,j} \ ; \ 1 \leq i,j \leq N \ , \qquad (2.69)$$

then a unique solution to (2.68) for Y(t) exists.

Since $S(t)$ depends upon the solution of the equivalent linear system, there is no way to know a priori whether $A[S(t)]$ satisfies (2.69) or not. Therefore, when applying the results of (2.3.1) and (2.3.2) to an equivalent linear system, it will be initially assumed that $A[S(t)]$ is continuous in time. This being the case, the equations governing $Q(t)$ in (2.3.1) and (2.3.2) will be valid. As will be discussed in section (2.4), this will lead to a set of nonlinear equations for $Q(t)$. For the case of Gaussian excitation, knowing $Q(t)$ is equivalent to knowing $S(t)$, and hence, $A[S(t)]$. If a solution to these nonlinear equations exists such that $A[S(t)]$ is indeed continuous in time, it will be taken as the approximate solution for $Q(t)$. If no such solution exists, then the initial assumption of the continuity of $A[S(t)]$ was incorrect, and the method is not applicable.

The applicability of the method will depend upon the nature of the nonlinear function $\underline{h}(\underline{y})$. It may be possible to determine the most general class of such functions for which the method applies; however, such a task seems formidable, and deviates from the objectives of this thesis. In practice, the final check on the continuity of $A[S(t)]$ should serve as an adequate criterion for determining whether or not the method is applicable.

2.4 A Method for Generating Approximate Statistics for Nonlinear Systems

At this point, all the results necessary for finding a set of approximate statistics for the response of a multidegree-of-freedom

system under Gaussian excitation have been developed. All that remains is to tie these results together so as to define a clear procedure for generating these statistics.

Recall that statistical characteristics are sought for the non-linear problem defined by (2.30) and (2.31) as

$$M\ddot{\underline{x}} + \hat{\underline{h}}(\underline{x}, \dot{\underline{x}}) = \hat{\underline{\theta}}(t) \, r(t) \tag{2.30}$$

$$\begin{aligned} \underline{x}(0) &= \underline{0} \\ \dot{\underline{x}}(0) &= \underline{0} \end{aligned} \tag{2.31}$$

The proposed equivalent linear system is given by (2.35) and (2.33) as

$$M\ddot{\underline{x}} + \mathcal{C}\big[S(t)\big]\dot{\underline{x}} + \mathcal{B}\big[S(t)\big]\underline{x} = \hat{\underline{\theta}}(t) \, r(t) \tag{2.35}$$

$$\begin{aligned} \underline{x}(0) &= \underline{0} \\ \dot{\underline{x}}(0) &= \underline{0} \end{aligned} \tag{2.33}$$

It should be pointed out that (2.30) and (2.33) were formulated as second order differential equations only because this form makes the application to dynamical systems more readily apparent. For the purpose of conciseness, let these now be rewritten as first order equations. Thus (2.30) and (2.31) become

$$\dot{\underline{y}} = \underline{h}(\underline{y}) + \underline{\theta}(t) \, r(t) \tag{2.70}$$

$$\underline{y}(0) = \underline{0} \tag{2.71}$$

where

$$\underline{y} \equiv \left\{ \begin{array}{c} \underline{x} \\ \dot{\underline{x}} \end{array} \right\} \tag{2.72}$$

$$\underline{\theta}(t) \equiv \left\{ \begin{array}{c} \underline{0} \\ M^{-1}\hat{\underline{\theta}}(t) \end{array} \right\} \tag{2.73}$$

$$\underline{h}(\underline{y}) \equiv \left[\begin{array}{cc} 0 & I_n \\ 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \underline{x} \\ \dot{\underline{x}} \end{array} \right\} - \left\{ \begin{array}{c} \underline{0} \\ M^{-1}\hat{\underline{h}}(\underline{x}, \dot{\underline{x}}) \end{array} \right\} \tag{2.74}$$

In (2.74), I_n denotes the $n \times n$ identity matrix. The equivalent linear system described by (2.33) and (2.31) becomes

$$\dot{\underline{y}} = A\left[S(t)\right]\underline{y} + \underline{\theta}(t)\, r(t) \tag{2.75}$$

$$\underline{y}(0) = \underline{0} \tag{2.76}$$

where

$$A\left[S(t)\right] = \left[\begin{array}{cc} 0 & I_n \\ -M^{-1}\mathcal{B}\left[S(t)\right] & -M^{-1}\mathcal{C}\left[S(t)\right] \end{array} \right] \tag{2.77}$$

As shown in section (2.22), the elements of the $\mathcal{C}\left[S(t)\right]$ and $\mathcal{B}\left[S(t)\right]$ matrices must satisfy (2.25),

$$E\left[\underline{y}\,\hat{\underline{h}}^T(\underline{y})\right] = E\left[\underline{y}\,\underline{y}^T\right]\left[\mathcal{B},\, \mathcal{C}\right]^T \tag{2.25}$$

Since the input excitation to the linear system (2.75) is a Gaussian random process, the response $\underline{y}(t)$ will also be a Gaussian process. Hence, the solution for $\mathcal{C}$ and $\mathcal{B}$ given by equations (2.29) is applicable. Rewriting (2.29) in 2N-space notation, gives

$$a_{ij} = E\left[\frac{\partial h_i}{\delta y_j}\right] \quad ; \quad i,j=1,\ldots,2n \qquad (2.78)$$

where (a_{ij}) are the elements of the A matrix.

It has been established that the mean value of the response process will be $\underset{\sim}{0}$ for the duration of the response so only the co-variance matrix $E[\underset{\sim}{y}\underset{\sim}{y}^T]$ must be found in order to determine the probability density of the response process at each instant. Using the notation of (2.3.1),

$$Q(t) \equiv E\left[\underset{\sim}{y}(t)\underset{\sim}{y}^T(t)\right] \qquad (2.52)$$

and $Q(t)$ is given by

$$Q(t) = \int_{-\infty}^{\infty} \underset{\sim}{F}(\omega,t)\, \underset{\sim}{F}^{*T}(\omega,t)\, S(\omega)\, d\omega \qquad (2.53)$$

where $\underset{\sim}{F}(\omega,t)$ satisfies

$$\dot{\underset{\sim}{F}}(\omega,t) = A\left[S(t)\right]\underset{\sim}{F}(\omega,t) + \underset{\sim}{\theta}(t)\, e^{i\omega t} \qquad (2.54)$$

$$\underset{\sim}{F}(\omega,0) = \underset{\sim}{0} \qquad (2.55)$$

For the special situation when $r(t)$ is a white noise process, a simpler method for determining $Q(t)$ is to solve equation (2.66)

$$\dot{Q} = A\left[S(t)\right]Q + \left(A\left[S(t)\right]Q\right)^T + 2\pi S_0\, \underset{\sim}{\theta}(t)\, \underset{\sim}{\theta}^T(t) \qquad (2.66)$$

with the initial condition

$$Q(0) = 0 \qquad (2.67)$$

Using the fact that $\underline{y}$ is a jointly Gaussian distributed vector, the elements of A may be expressed in terms of Q as

$$a_{ij} = E\left[\frac{\partial h_i}{\partial y_j}\right] = \int_{-\infty}^{\infty} \ldots (2n) \ldots \int_{-\infty}^{\infty} \frac{\partial h_i}{\partial y_j} p(\underline{y}) \, dy_1 \ldots dy_{2n} \qquad i, j = 1, \ldots, 2n \tag{2.79}$$

where

$$p(\underline{y}) = \frac{1}{\left[(2\pi)^{2n} \det(Q)\right]^{\frac{1}{2}}} \cdot \exp\left(-\tfrac{1}{2} \underline{y}^T Q^{-1} \underline{y}\right) \tag{2.56}$$

Since each element of A is an explicit function of the elements of Q, equation (2.66) is a set of $4n^2$ nonlinear ordinary differential equations which must be solved with the initial condition (2.67). Due to the symmetry of Q, only $n(2n+1)$ of the equations are distinct.

In the more general case of non-white excitation, equation (2.54) must be solved simultaneously with equation (2.53) to generate Q(t). In appendix A, a discussion is given concerning an approach to the solution of (2.53) and (2.54) using numerical techniques.

To summarize, the problem of finding approximate response statistics for the nonlinear system described by (2.30) and (2.31) has been reduced to the solution of the equations given by (2.53), (2.54), and (2.55). For the special case of modulated white noise excitation, (2.66) and (2.67) provide a more tractable formulation.

The usefulness of the approximate technique hinges on two important considerations. First is the effort required to solve

the appropriate nonlinear equations (i.e., (2.53), (2.54), and (2.55)) produced by equivalent linearization. With sufficient time and effort, accurate response statistics may be obtained by simulation of the original nonlinear system (2.30) and (2.31) on a digital or analog computer. If the effort required to obtain the approximate solution through equivalent linearization is not considerably less, then it is not really justifiable. Since there is no general method to determine a priori just how large an ensemble of time history simulations will be necessary to produce accurate statistics, this question is rather difficult to answer analytically. However, experience has shown that, in terms of computer time, the analytical solution tends to be much less time-consuming. It is not uncommon for the ratio of the time required for a simulation solution to that required for the analytic solution to be greater than 100.

The second consideration relating to the usefulness of the approximate method concerns the accuracy of the solutions obtained. Establishing useful analytical bounds for the error incurred by equivalent linearization techniques has proven to be a most difficult problem. Though some preliminary efforts have been made along these lines [14, 15], there are currently no analytical techniques for generating such bounds. As a consequence, the usual procedure for checking the accuracy of these methods is to compare the approximate solution with the numerically generated "exact" solution for a few specific cases. Obviously, one cannot draw general conclusions about the accuracy from this type of procedure, and

this would seem to be a major drawback of averaging techniques in general. However, due to the widespread usage of equivalent linearization methods, there exists a fairly large body of results from a variety of applications, the precision of which have been checked either by experiment or numerical simulation, usually with favorable conclusions. In fact, even for quite severely non-linear systems, the approximate solutions produced by equivalent linearization are often adequate for engineering applications.

2.5 Numerical Example - Duffing Oscillator

As an application of the techniques described in the previous sections, consider the example of a Duffing oscillator subject to modulated Gaussian white noise. The equation of motion is

$$m\ddot{\Psi} + c\dot{\Psi} + k(\Psi + \hat{\epsilon}\Psi^3) = \hat{\theta}(t)w(t) \quad ; \quad m, c, k > 0, \hat{\epsilon} \geq 0 \quad (2.80)$$

where $w(t)$ is Gaussian white noise with spectral density S_0. The parameter $\hat{\epsilon}$ is a measure of the degree of nonlinearity of the system (2.80). When $\hat{\epsilon} = 0$, the system is linear, and one may find the theoretically exact statistical solution.

Let $\sigma_{\psi m}$ be defined as

$$\sigma_{\psi m} \equiv \max\left\{ \left(E\left[\psi^2(t)\right]\right)^{\frac{1}{2}} \right\}\Big|_{\hat{\epsilon}=0} \quad \forall t \quad ; \quad 0 \leq t \leq \tau \quad (2.81)$$

where $[0, \tau]$ is the interval during which the response of (2.80) is of interest. Then, a dimensionless form of (2.80) may be obtained by introducing the variable x defined as

$$x \equiv \frac{\psi}{\sigma_{\psi m}} \tag{2.82}$$

Equation (2.80) may then be written as

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2(x + \epsilon x^3) = \theta(t)w(t) \tag{2.83}$$

where

$$\epsilon = \hat{\epsilon}\sigma_{\psi m}^2 \, ,$$

$$\theta = \frac{\hat{\theta}}{m\sigma_{\psi m}} \, ,$$

$$\omega_0 = \sqrt{\frac{k}{m}} \, ,$$

and

$$\zeta = \frac{c}{2\sqrt{km}}$$

To remain consistent with previous sections, (2.83) will be converted to the form of two first order equations. Let $\underset{\sim}{y}$ be defined by

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \tag{2.84}$$

Then (2.83) may be represented as

$$\dot{\underset{\sim}{y}} = \underset{\sim}{h}(\underset{\sim}{y}) + \underset{\sim}{\theta}(t)w(t) \tag{2.85}$$

where

$$h_1(\underline{y}) = y_2$$

(2.86)

$$h_2(\underline{y}) = -2\zeta\omega_0 y_2 - \omega_0^2(y_1 + \epsilon y_1^3)$$

and

$$\theta_1(t) = 0$$

(2.87)

$$\theta_2(t) = \theta(t)$$

The equivalent linear system for (2.85) will have the form

$$\dot{\underline{y}} = A[S(t)]\underline{y} + \underline{\theta}(t)w(t)$$

(2.88)

Since $\underline{y}$ is assumed to be Gaussian, the elements of A are given by equation (2.78). Evaluating these elements by means of (2.78) and (2.86) gives

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2\left(1 + 3\epsilon E\left[y_1^2\right]\right) & -2\zeta\omega_0 \end{bmatrix}$$

(2.89)

The covariance matrix Q(t) may be represented explicitly as

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} E\left[y_1^2\right] & E\left[y_1 y_2\right] \\ E\left[y_2 y_1\right] & E\left[y_2^2\right] \end{bmatrix}$$

(2.90)

Hence, from (2.90), (2.89) may be written as

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2(1 + 3\epsilon q_{11}) & -2\zeta\omega_0 \end{bmatrix}$$

(2.91)

Since the excitation in this example is modulated white noise, the covariance matrix satisfies the Liapunov-type differential equation (2.66). Applying equation (2.66) yields

$$\dot{q}_{11} = 2q_{12}$$

$$\dot{q}_{12} = q_{22} - \left[\omega_0^2 q_{11}(1 + 3\epsilon q_{11}) + 2\zeta\omega_0 q_{12}\right] \qquad (2.92)$$

$$\dot{q}_{22} = -2\left[\omega_0^2 q_{12}(1 + 3\epsilon q_{11}) + 2\zeta\omega_0 q_{22}\right] + 2\pi S_0 \theta^2(t)$$

From (2.90), it is obvious that $q_{21} = q_{12}$.

When $\theta(t)$ is a constant, the stationary response may be found by letting $\dot{q}_{ij} = 0$, for $i = 1, 2$ and $j = 1, 2$ in (2.92). Letting $\theta(t)$ equal unity, (2.92) then reduces to

$$q_{12} = 0$$

$$q_{22} - \omega_0^2 q_{11}(1 + 3\epsilon q_{11}) = 0 \qquad (2.93)$$

$$q_{22} = \frac{\pi S_0}{2\zeta\omega_0}$$

For this particular example, the exact covariance matrix corresponding to stationary response may be found by solving the stationary part of the associated Fokker-Planck-Kolmogorov equation as described in reference [2]. It will be noted that the expressions for q_{12} and q_{22} in (2.93) are actually exact in this

case.   Solving for q_{11} yields

$$q_{11} = \frac{1}{6\epsilon}\left[\sqrt{1 + 6\frac{\pi\epsilon S_0}{\zeta\omega_0^3}} - 1\right] \qquad (2.94)$$

This result was also obtained by Iwan and Yang [3].   The standard
deviation of displacement σ_x is defined by

$$\sigma_x \equiv \sqrt{q_{11}} \qquad (2.95)$$

In reference [3], it is observed that, even for arbitrarily large
nonlinearity, the value of σ_x given by (2.94) and (2.95) deviates
from the exact value by less than 7.5%.

Letting ϵ approach zero, (2.94) reduces to

$$q_{11} = \frac{\pi S_0}{2\zeta\omega_0^3} \qquad (2.96)$$

which is the exact solution for the linear case.

To determine the nonstationary response of (2.83), it is
necessary to solve the differential equations given by (2.92) with
zero initial conditions.   It is apparent that this cannot be done
analytically for nonzero ϵ.   The solutions to (2.94) presented
herein were obtained numerically.   The numerical method used
was a fourth-order Adams-Moulton predictor-corrector scheme
for solving initial value problems.

Because an important application of this approximate technique
is to earthquake engineering, the modulating function $\theta(t)$ was chosen

so as to resemble the "envelope" of a typical strong-motion ground acceleration record. Specific forms for such envelope functions have been suggested by a number of authors. (See, for instance [18, 19]). For the present example, the form chosen was

$$\theta(t) = t e^{-\gamma t} \qquad (2.97)$$

where γ is a free parameter which may be interpreted physically as the reciprocal of the time required for the excitation to build up to its maximum intensity. In this example, γ was chosen so that

$$\frac{\gamma}{\omega_0} = \frac{1}{10\pi}$$

For a linear system with a natural frequency of 2.0 Hz, this simulates roughly the envelope behavior of the North-South component of the Helena, Montana earthquake of October 31, 1935 [21].

As a check on the accuracy of the equivalent linearization technique, a Monte-Carlo simulation study of equation (2.83) was also performed. The white noise process w(t) was approximated by a model similar to that employed in reference [20]. In brief, an ensemble of m sample functions of a white noise process were generated over an interval [0, τ]. The interval was divided into n equal subintervals of duration Δt. For each sample function, a sequence of n + 1 normally distributed numbers $G_1, \ldots, G_{n+1}$ was generated. The members of this sequence were then assigned

to ordinates at equally spaced intervals according to the rule

$$w(t_0 + j\Delta t) = G_j \qquad (2.98)$$

where t_0 is a random variable with uniform density over the initial subinterval $[-\Delta t, 0]$, and density zero elsewhere. It was assumed that

$$w(t_0) = 0 \qquad (2.99)$$

and that w varies linearly over each subinterval.

This procedure may be repeated m times to obtain an ensemble of m sample functions. If each ordinate G_j is then multiplied by the constant $(2\pi S_0/\Delta t)^{\frac{1}{2}}$, the power spectral density $S(\omega)$ of the new process so generated is given by

$$S(\omega) = S_0 \frac{6 - 8 \cos (\omega \Delta t) + 2 \cos (2\omega \Delta t)}{(\omega \Delta t)^4} \qquad (2.100)$$

For ω approaching zero with constant Δt, $S(\omega)$ approaches S_0. As mentioned in reference [20], this expression for $S(\omega)$ remains within 5% of S_0 for $\omega \Delta t < 0.57$, and within 10% for $\omega \Delta t < 0.76$. Thus, if Δt is made sufficiently small, the simulated process may be made to have a power spectral density which approximates that of a white noise process to within a given error tolerance out to any desired frequency. In this study, Δt was set equal to T/100, where

$$T = 2\pi/\omega_0 \qquad (2.101)$$

That is, T is the natural undamped period of the oscillator (2.83) when $\epsilon = 0$.

The ensemble size m was 200 for this simulation study. For nonzero values of ϵ, equation (2.83) was solved numerically using the previously mentioned predictor-corrector scheme. The 200 time histories so obtained for each ϵ value were then used to compute the time dependent covariance matrix $E\left[\underline{y}(t)\underline{y}^T(t)\right]$.

In figures 2.1 through 2.5, the elements of the covariance matrix are plotted as a function of dimensionless time t/T for values of ϵ ranging from zero up to 1.0. The simulation results for the off-diagonal element, $E[xv] = q_{12}$, were not plotted. This is due to the fact that it is considerably more costly to obtain accurate values for $E[xv]$ by simulation than it is to obtain comparably accurate values of $E[x^2]$ or $E[v^2]$.

The reason for this becomes apparent when one realizes that $E[xv]$ is proportional to the derivative of $E[x^2]$. Specifically, it is recognized that

$$\frac{d\,E[x^2]}{dz} = 2\,E[xv] \qquad (2.102)$$

where $v = dx/dt$. Whereas an ensemble of m time histories may produce acceptable accuracy in the $E[x^2]$ function, to make this function smooth enough for its time derivative to be accurate would require a much larger ensemble. In an attempt to determine how large an ensemble would be needed to obtain convergence of the $E[xv]$ function, an ensemble of 2000 time histories was obtained

for the linear case ($\epsilon = 0.0$). Although the $E[x^2]$ function so produced was considerably smoother than that obtained with the ensemble of 200, its derivative still fluctuated chaotically enough to indicate that $E[xv]$ had not yet converged. The cost of increasing the ensemble size still another order of magnitude was prohibitive, even for the linear case, so dependable simulation results for $E[xv]$ remain unavailable.

Examining figures 2.1 - 2.5, it is apparent that the correlation between the simulation results and the approximate analytical solutions is very good, even for the case where $\epsilon = 1.0$, which represents a moderately large nonlinearity. It is observed that the approximate solutions have the slight tendency to err on the nonconservative side, but these errors are relatively small, and would not be significant in most engineering applications.

As a final remark, it is of interest to compare the relative amounts of computational effort required to obtain the simulation results versus that for the analytical solution. For a single non-zero value of ϵ, the CPU time required to produce 200 simulated time histories using an IBM 370/158 was approximately 40 minutes. To obtain the corresponding covariance response by the approximate analysis, using the same computer, required about 20 seconds of CPU time. It is concluded that, for the type of system in this example, the approximate analysis performed herein is quite accurate and highly economical.
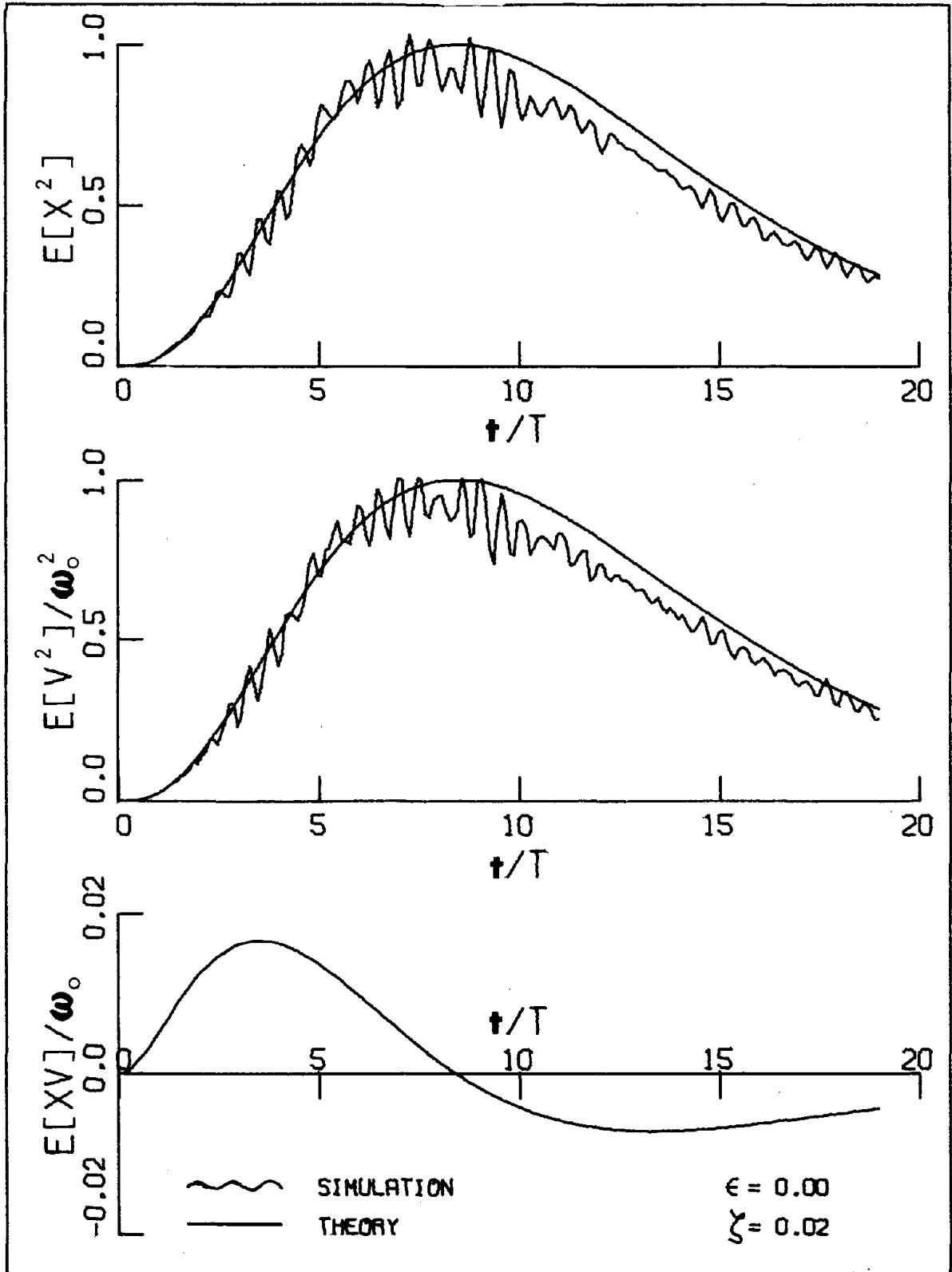
Fig. 2.1 Covariance of Nonstationary Response versus Time. Duffing Oscillator, $\epsilon = 0.0$ and $\zeta = 0.02$.
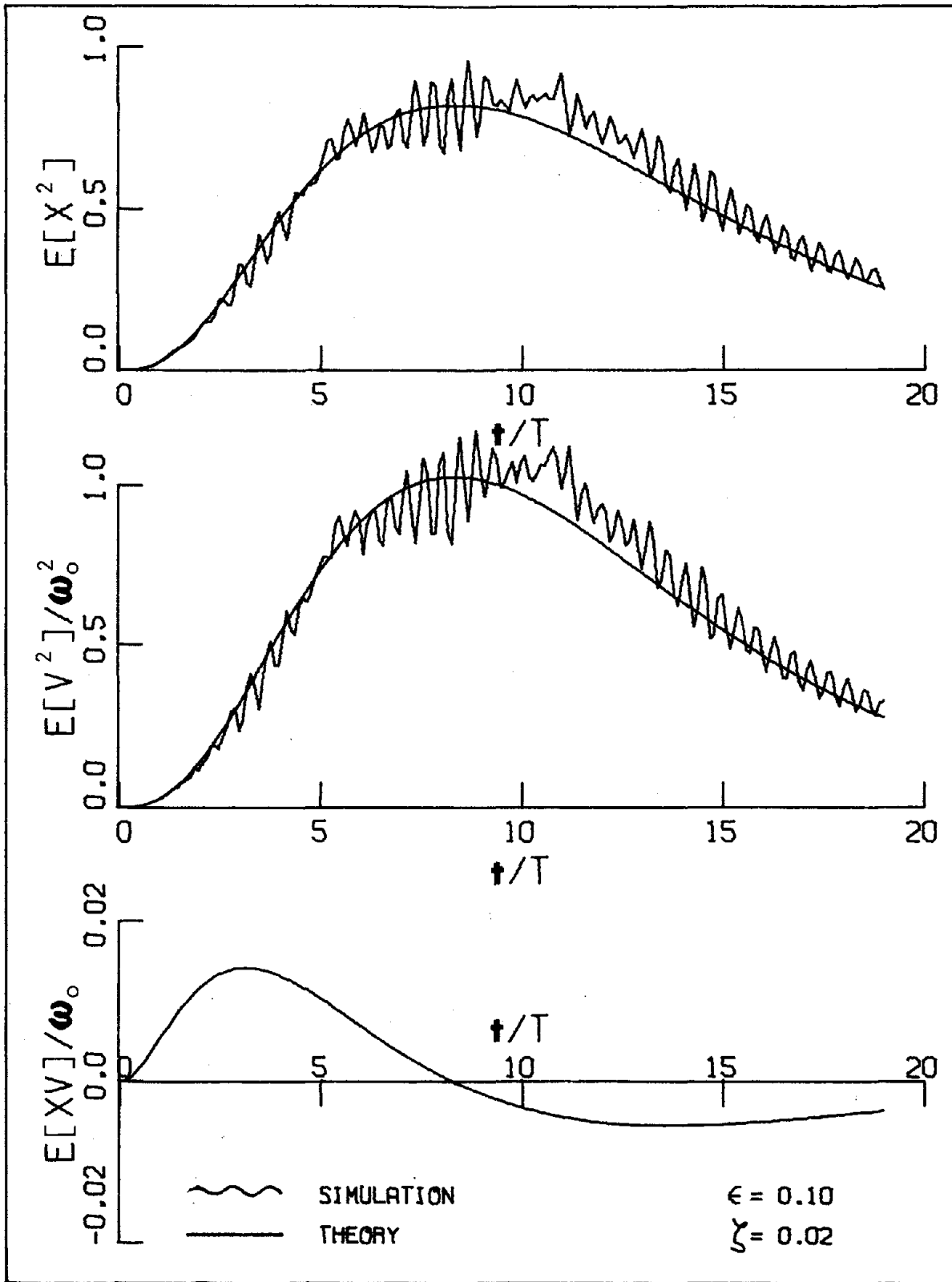
Fig. 2.2 Covariance of Nonstationary Response versus Time. Duffing Oscillator, $\epsilon = 0.1$ and $\zeta = 0.02$.
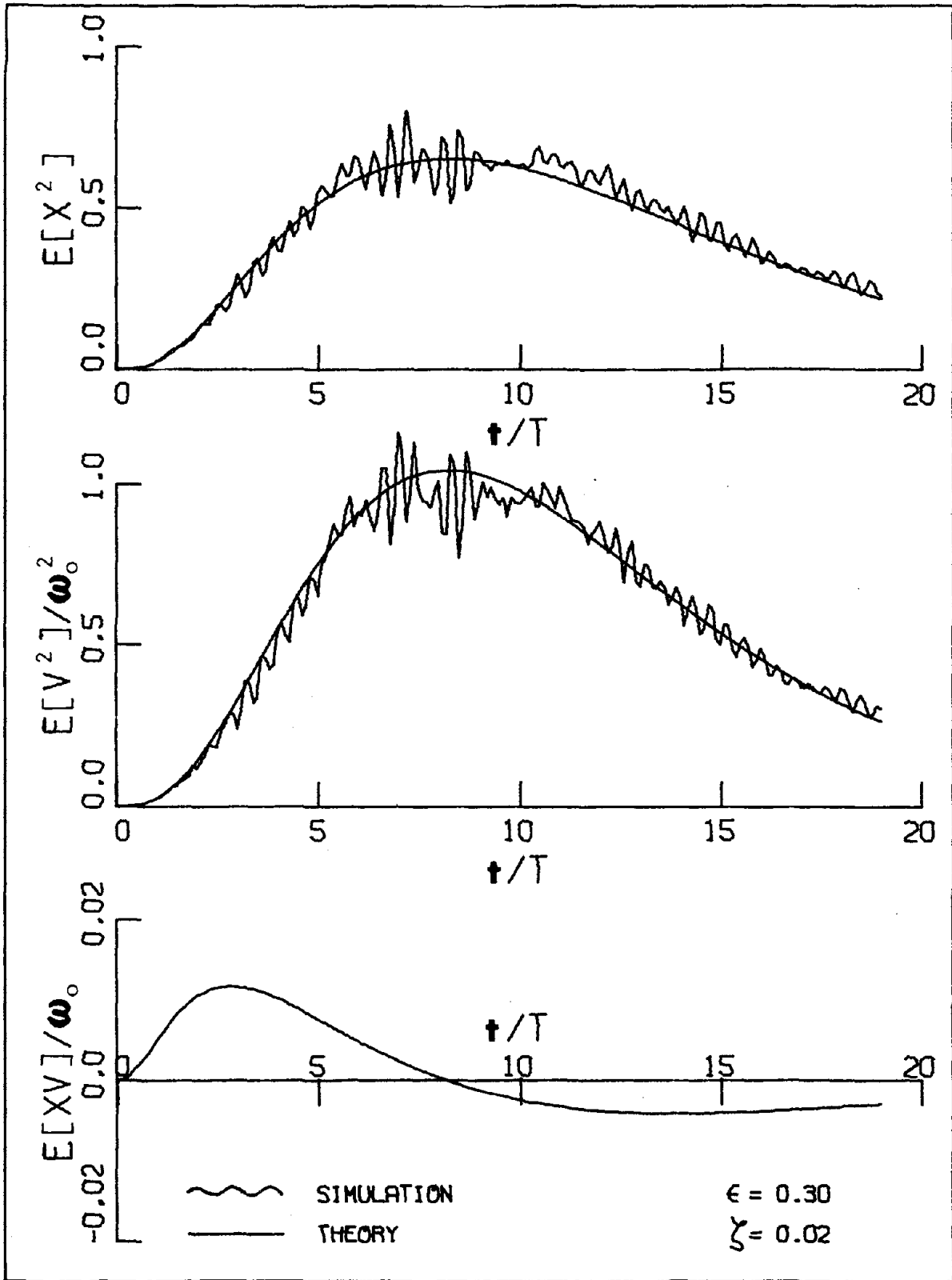
Fig. 2.3 Covariance of Nonstationary Response versus Time.
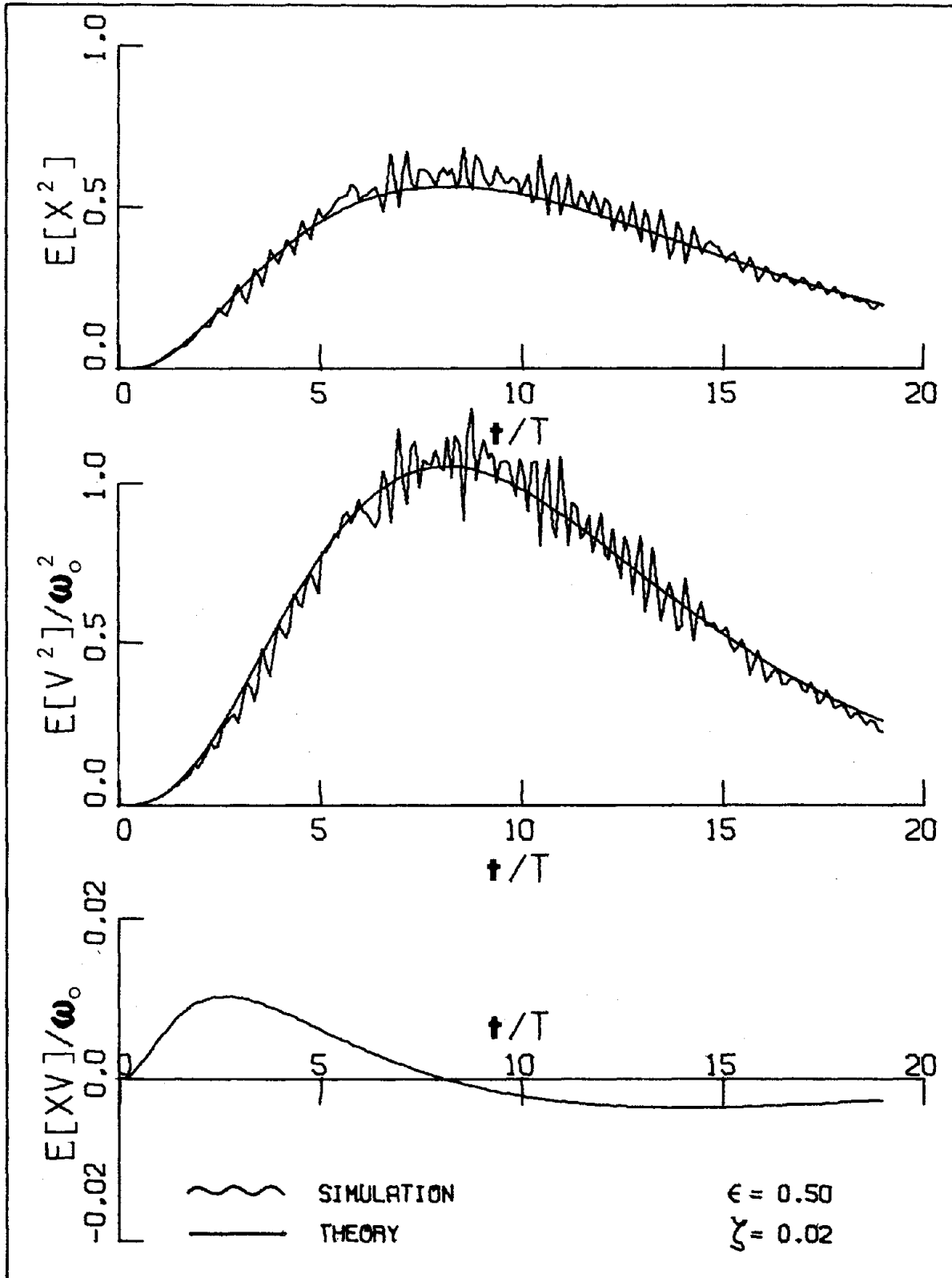Duffing Oscillator, $\epsilon = 0.3$ and $\zeta = 0.02$.

Fig. 2.4   Covariance of Nonstationary Response versus Time.
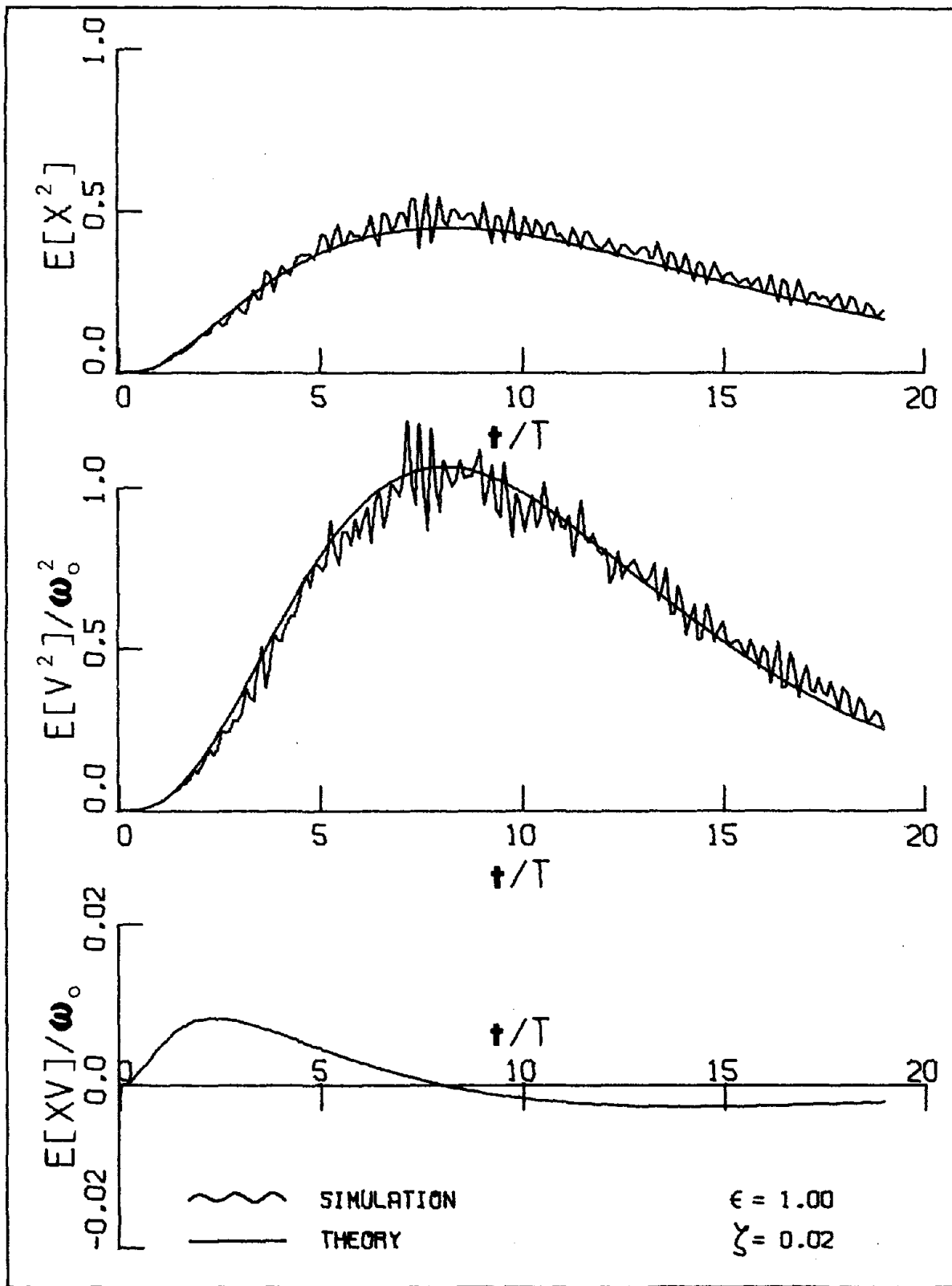Duffing Oscillator, $\epsilon = 0.5$ and $\zeta = 0.02$.

Fig. 2.5 Covariance of Nonstationary Response versus Time.
Duffing Oscillator, $\epsilon = 1.0$ and $\zeta = 0.02$.

CHAPTER III

The First Passage Problem for a Linear Oscillator

3.1 Introduction

In the study of the response of dynamical systems to random excitation, a classical problem is to determine the probability that the value of some response variable remains below a given threshold throughout a specified time interval. The probability distribution of the time required for the variable to first exceed the threshold is referred to as the first passage probability distribution. Knowledge of this distribution is of great practical importance in many engineering problems.

For an arbitrarily specified stochastic process, there is no generally applicable procedure for finding the first passage distribution or density, so a more restrictive specification of the problem must be made in order to proceed toward a solution. One of the simplest configurations of interest in the study of random vibrations is that of a linear oscillator subject to stationary Gaussian white noise. The equation of motion is

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = w(t) \tag{3.1}$$

where ζ is the fraction of critical damping, and ω_0 is the un-damped natural frequency.

Let $W(T)$ be the probability that the magnitude of x remains less than the threshold level b throughout the interval $[0, T]$, where b is a positive constant. W is therefore defined as

$$W(T) \equiv Pr\left[\left|x(t)\right|_{max} < b \; ; \; 0 \leq t \leq T\right] \qquad (3.2)$$

where $Pr[\cdot]$ denotes the probability that the bracketed expression is true. Rather than attempt to solve directly for $W(T)$, it is common to seek a solution for $p(T)$ where

$$p(T) \equiv -\frac{dW}{dT} \qquad (3.3)$$

Note that $p(T)\,dT$ is then just the probability that first passage occurs during the interval $[T, T + dT]$. The quantity $p(T)$ is called the first crossing probability density. To date, an exact analytical solution for $p(T)$ has not been found.

When the first passage problem is formulated as in (3.2), b is referred to as a type-D barrier. Often, it is desired to know the probability distribution of time to the first crossing of b by the variable x rather than $|x|$. In such situations, b is called a type-B barrier. A third type of barrier arises when one considers a threshold level for the envelope process $\hat{a}(t)$, where

$$\hat{a}^2 = x^2 + \frac{\dot{x}^2}{\omega_0^2} \qquad (3.4)$$

In this case first passage occurs when $\hat{a}$ first exceeds b. The level b is then known as a type-E barrier. Note that $\hat{a}^2$ is proportional to the total energy of the linear system (3.1). For future reference, when the variable of interest is below the specified threshold level, it is said to be in the "safe region".

Likewise, when it is above this level it is said to be in the
"unsafe region". This terminology obviously stems from applica-
tions where first passage represents failure or possible failure.
Although these terms are not really appropriate for all applications,
when interpreted literally, their meaning in the context of first
passage problems seems to be rather universally understood, and
hence, they will be used herein.

It is clear that there could be more varieties of threshold
levels, however only the aforementioned three types will be
considered herein. Of these, most attention will be given to the
type-D barrier as it is of perhaps the greatest importance in
earthquake engineering applications.

The first passage probability density for all of the barrier
types will be dependent upon the initial conditions imposed upon
equation (3.1). However, it has been observed by Crandall and
others that the effects of the initial conditions on this density tend
to die out as T becomes large compared with the natural period
of the oscillator [22]. Specifically, it has been observed
experimentally that $W(T)$ eventually approaches a decaying
exponential of the form

$$W(T) \sim e^{-\alpha T} \qquad (3.5)$$

regardless of the initial conditions [23].

In many applications, the mean time to first passage is
substantially greater than the interval during which the effects of

the initial conditions are important. Hence, (3.5) is often employed as an approximation for $W(T)$. The parameter α is called the limiting decay state of the first crossing density. Much of the effort in this chapter will be directed toward determining this decay rate. Since the value of α is unaffected by the initial conditions on (3.1), these conditions will often be left unspecified.

3.2 Problem Formulation in Terms of Transition Probability Density

It is well known that the response of a linear oscillator excited by white noise is a two-dimensional Markov process [24]. By this it is meant that, given only the state $(x, \dot{x})$ at time t_1, the future probability distribution $P_1(x, \dot{x}, t_2)$, where $t_2 > t_1$, will be the same as if the entire time history $[x(t), \dot{x}(t)]$ over the interval $[0 \le t \le t_1]$ were given. Thus, for a Markov process, once the present state is known, the states at all previous times become irrelevant to the future probability distribution. This property is quite important to the exact formulation of the first passage problem.

Referring to the type-D barriers, where the safe region is defined by $[x ; |x| < b]$, consider the two-dimensional phase space associated with the response of equation (3.1). Letting $v \equiv \dot{x}$, the ordered pair $[x(t), v(t)]$ trace out trajectories in the phase space as t varies. Suppose that, at time t_1, a trajectory, which has not yet left the safe region, is at (x_1, v_1). Let (x_2, v_2) also be in the safe region and let $q(x_1, v_1 | x_2, v_2 ; \Delta t)dx\, dv$ be the

infinitesimal probability of the trajectory reaching the differential element of area $dx \cdot dv$ centered at (x_2, v_2) at time $t_1 + \Delta t$ without leaving the safe region, given that it was at (x_1, v_1) at time t_1. For obvious reasons, q is called the conditional transition probability density. It is apparent that knowledge of q would provide sufficient information to solve the first passage problem. Specifically, if the oscillator had initial conditions (x_0, v_0) at time $t = 0$, one could write

$$W(t) = \int_{-\infty}^{\infty} \int_{-b}^{b} q(x_0, v_0 | x, v; t) dx \, dv \qquad (3.6)$$

A direct result of the Markovian nature of the response process $[x(t), v(t)]$ is that the conditional transition probability density satisfies the Smoluchowsky integral equation given by [24]

$$q(x_0, v_0 | x, v; t) = \int_{-\infty}^{\infty} \int_{-b}^{b} q(x_0, v_0 | \hat{x}, \hat{v}; t - \Delta t) \, q(x, v | \hat{x}, \hat{v}; \Delta t) d\hat{x} \, d\hat{v}$$

$$\qquad (3.7)$$

By standard techniques [24], one may derive from (3.7) a Fokker-Planck-Kolmogorov equation which governs q. This is given by

$$\frac{\partial q}{\partial t} = -v \frac{\partial q}{\partial x} + \frac{\partial}{\partial v} \left[\left(2\zeta \omega_0 v + \omega_0^2 x \right) q \right] + \pi S_0 \frac{\partial^2 q}{\partial v^2} \qquad (3.8)$$

where S_0 is the spectral density of $W(t)$.

Equation (3.8) must be solved with the initial condition

$$q\big|_{t=0} = \delta(x_0, v_0) \qquad (3.9)$$

and with the boundary conditions

$$q\left(x_0, v_0 \mid b, v ; t\right) = 0 \qquad ; \qquad -\infty < v \leq 0$$

$$\forall t \geq 0$$

$$q\left(x_0, v_0 \mid -b, v ; t\right) = 0 \qquad ; \qquad 0 \leq v < \infty$$

$$(3.10)$$

The boundary conditions (3.10) are required since it is obvious that no trajectory could cross +b with a negative velocity or -b with a positive velocity without having been outside the safe region. Figure 3.1 shows the phase plane with the safe region and the boundary conditions (3.10) indicated.

Though expressions (3.8), (3.9), and (3.10) constitute a precise formulation of the first passage problem, a solution to these equations has not been found, to date. One of the difficulties in effecting a solution becomes apparent when an attempt is made to separate variables in (3.8). Using the (x, v, t) coordinates, in which the boundary conditions are simple, only the time part of q separates out. This does, however, provide for an interesting observation.

Attempting a solution of the form

$$q = T(t) F(x, v) \qquad (3.11)$$

one obtains from (3.8)

$$\frac{\dot{T}}{T} = -v \frac{F_x}{F} + 2 \zeta \omega_0 \left(1 + v \frac{F_v}{F}\right) + \omega_0^2 x \frac{F_v}{F} + \pi S_0 \frac{F_{vv}}{F} = -\lambda$$
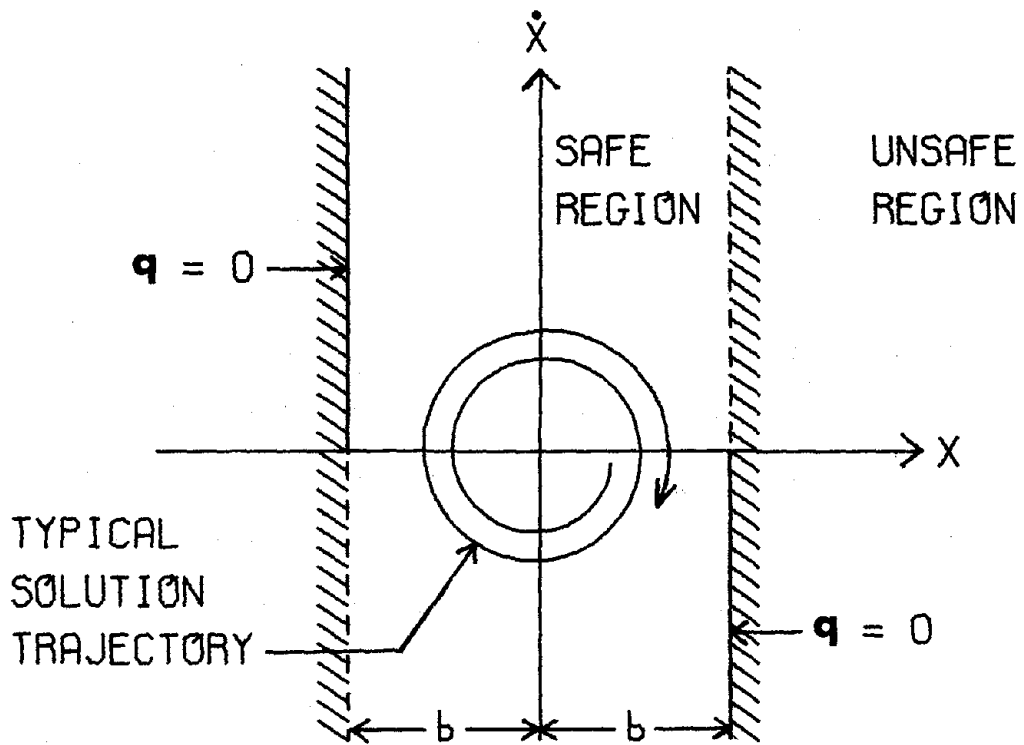
$$(3.12)$$

Fig. 3.1   Phase-Plane Domain for Transition Probability Density
of First Passage with Type-D Barrier.

Though little can be said about the "spatial" eigenfunctions $F(x, v)$, the form of the temporal eigenfunctions can be readily obtained. From (3.12), one may write

$$T_j(t) = e^{-\lambda_j t} \tag{3.13}$$

where $\lambda_j > 0$; $j = 1, 2, \ldots$, since q may not grow unbounded as $t \to \infty$. Hence, q may be written

$$q = \sum_{j=1}^{\infty} F_j(x, v) e^{-\lambda_j t} \tag{3.14}$$

Integrating (3.14) over the safe domain yields

$$W(t) = \sum_{j=1}^{\infty} A_j e^{-\lambda_j t} \tag{3.15}$$

where

$$A_j = \int_{-\infty}^{\infty} \int_{-b}^{b} F_j(x, v) dx \, dv \tag{3.16}$$

Let the eigenvalues be ordered such that

$$\lambda_1 < \lambda_2 < \ldots \tag{3.17}$$

As t increases, one would expect the principal eigenvalue to eventually dominate in (3.15). Thus, for large time,

$$W(t) \sim A_1 e^{-\lambda_1 t} \tag{3.18}$$

Letting $A_1 = 1$ and $\lambda_1 = \alpha$, this becomes identical with expression (3.5). In addition to the experimental observations, therefore, there is some theoretical justification for using (3.5) as an approximation for $W(t)$.

As a final note on theoretically exact formulations of the first passage problem, it is of some interest that if one considers a first order linear system instead of the harmonic oscillator (3.1), the problem may be solved exactly. A detailed discussion of this problem is given in reference [25].

3.3 Discussion of Some Currently Available Approximate Solutions

In the absence of an exact analytical solution to the first passage problem for the harmonic oscillator (3.1), numerous approximate solutions to the problem have been proposed. Of these, the most accurate schemes generally involve solving approximately for the conditional transition probability density governing first passage.

An obvious approach of this sort, though not especially elegant, would be to attempt numerical solution of (3.8) utilizing modern numerical analysis techniques. An interesting variation of this idea is based on a discretized version of the Smoluchowsky integral equation (3.7). This so-called "diffusion of probability" method has been employed by Crandall, et al. [23], to obtain approximately the limiting decay rate α for a number of damping values over a range of barrier levels [23]. For a sufficiently fine discretization, this procedure generates values of α that agree

well with those obtained by Monte-Carlo simulation.

For type-D thresholds, the best approximations for α currently available have been obtained by Mark [26]. Mark considers the case of the lightly damped oscillator ($\zeta \ll 1$). The observation is made that, for small enough damping, the sample functions of x(t) have an approximately sinusoidal appearance. The magnitudes of the peaks of these quasi-sinusoidal sample functions are treated as a one-dimensional, continuous-state, discrete time Markov process. The approximation that is made is that the peaks are separated by intervals of exactly $\Delta t = \pi/\omega_0$, an assumption which will become increasingly valid as the damping decreases. A conditional transition probability density function $\hat{p}(\hat{a}_0 | \hat{a}_1 ; \Delta t)$ for the magnitude of a peak $\hat{a}_1$, given the value of the preceding peak $\hat{a}_0$, is then derived. This leads to an integral equation, similar to the Smoluchowski equation, which is solved numerically.

For type-E barriers, an approximate analytical solution, which is accurate for small damping, was presented by Helstrom in 1959 [27]. For a lightly damped oscillator, it may be shown [28] that the envelope process $\hat{a}(t)$ defined by (3.4) satisfies approximately the differential equation

$$\dot{\hat{a}} + \zeta \omega_0 \left(\hat{a} - \frac{\sigma^2}{\hat{a}} \right) = \frac{w(t)}{\sqrt{2}\omega_0} \tag{3.19}$$

where σ^2 is the variance of the stationary response of x in (3.1).

The process described by equation (3.19) is a first order Markov process, and hence, the transition probability density $q(\hat{a}_0|\hat{a};t)$ satisfies the Fokker-Planck-Kolmogorov equation

$$\frac{\partial q}{\partial t} = \zeta \omega_0 \frac{\partial}{\partial \hat{a}} \left[\left(\hat{a} - \frac{\sigma^2}{\hat{a}} \right) q \right] + \sigma^2 \frac{\partial^2 q}{\partial \hat{a}^2} \qquad (3.20)$$

where $a(\hat{a}_0|\hat{a};t)\,d\hat{a}$ is the probability that the envelope which starts at $\hat{a}_0$ initially reaches the differential element centered at $\hat{a}$ a time t later without having left the safe region ($\hat{a} < b$). The initial condition

$$q(\hat{a}_0|\hat{a};0) = \delta(\hat{a} - \hat{a}_0) \qquad (3.21)$$

and the boundary condition

$$q(\hat{a}_0|b;t) = 0 \qquad \forall t > 0 \qquad (3.22)$$

must be imposed on (3.20). Helstrom obtains the Laplace transform of q in terms of confluent hypergeometric functions, and presents a series solution for q itself.

It has been suggested [1] that, when the damping is small, the limiting decay rate for a type-E barrier α_E should serve as a good approximation to the limiting rate for a type-D barrier α_D. The reasoning behind this is fairly straightforward. An E crossing indicates that the total energy of the system is sufficient to exit from the type-D safe region if there were no dissipation and no external force on the oscillator. If the damping is sufficiently small, the dissipation will be small and the oscillator response

will be narrow-band. Thus, the probability of an E crossing
being immediately followed by a D crossing will be near unity.
However, experiments have shown that for damping ratios as
low as 0.01, the approximation is not especially good [22].
Further, the deviation of α_D from α_E increases with increasing
barrier level b. Roberts [29] has recently presented some
analytical evidence to suggest that for barrier levels b in the
range $\{0 < b \leq 5\sigma\}$ reasonable correlation between α_D and α_E
may be expected provided $\zeta \leq 10^{-3}$. Such a restriction on ζ
will often prove too severe for practical applications. Hence,
this approximation will not be employed herein.

In order to apply either of the approximate analytical
solutions discussed thus far, it is necessary to perform a
substantial amount of numerical computation. The strictly
numerical schemes, such as Monte Carlo simulation and diffusion
of probability, require even more computational effort.

There does exist a number of approximate analytic solutions
for the limiting decay rate which, although not as accurate as
those already mentioned, are simpler and require much less effort
to implement. Of these, the simplest involves the assumption that
the barrier crossings are statistically independent events. This
will be approximately true when the barrier level b is large
compared with σ so that the average interval between successive
up-crossings of b becomes very long. If this assumption were
correct, the times at which such up-crossings occur would
constitute a Poisson process, and the intervals between

up-crossings would be exponentially distributed.

For a stationary random process with probability density $p_1(x, \dot{x})$, the average rate of up-crossing of the level $x = b$, denoted by ν_b is given by [11]

$$\nu_b = \int_0^\infty \dot{x} p_1(b, \dot{x}) \, d\dot{x} \tag{3.23}$$

For the case of the stationary oscillator response, p_1 is given by

$$p_1(x, \dot{x}) = \frac{1}{2\pi \omega_0 \sigma^2} \exp \left[-\frac{1}{2} \left(\frac{x^2}{\sigma^2} + \frac{\dot{x}^2}{\omega_0^2 \sigma^2} \right) \right] \tag{3.24}$$

Performing the integration in (3.23) yields

$$\nu_b = \frac{\omega_0}{2\pi} \exp \left[-\frac{1}{2} \frac{b^2}{\sigma^2} \right] \tag{3.25}$$

The rate of down-crossings of $-b$ will be the same as ν_b, due to the symmetry of $p_1(x, \dot{x})$. Thus, the average of the number of crossings out of the safe region per unit time is just $2\nu_b$ for type-D barriers. This gives the Poisson process average rate, so that

$$\alpha_D = 2\nu_b \tag{3.26}$$

Let $\hat{T}$ represent the length of the interval between two successive crossings out of the safe region. Then, the probability distribution of $\hat{T}$ is given by

$$\Pr[\hat{T} < T] = W(T) = e^{-2\nu_b T} \tag{3.27}$$

For barrier levels of practical interest, this approximation is often highly conservative, in the sense that it predicts average first-passage times much shorter than actually occur. For very high barrier levels, however, the approximation becomes quite good, and in fact, (3.26) becomes asymptotic to the actual limiting decay rate as $b \rightarrow \infty$ [30]. The Poisson average crossing rate $2\nu_b$ provides a convenient normalizing factor for other estimates of the limiting decay rate. Hence, such estimates are often presented in the non-dimensionalized form $\alpha/2\nu_b$.

For type-E barriers, an analogous estimate for α_E may be found by assuming that the up-crossings of b by the envelope process $\hat{a}$ are independent. This leads to a different Poisson average rate by the same procedure used to obtain ν_b for D-crossings. In this case, p_1 is given by [22]

$$p_1(\hat{a}, \dot{\hat{a}}) = \frac{\hat{a}}{\sqrt{2\pi}\,\sigma^2\beta} \exp\left[-\frac{1}{2}\left(\frac{\hat{a}^2}{\sigma^2} + \frac{\dot{\hat{a}}^2}{\beta^2}\right)\right] \tag{3.28}$$

where

$$\beta \equiv \omega_0\sigma\left[1 - \frac{1}{1-\zeta^2}\left\{1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}\right)\right\}^2\right]^{\frac{1}{2}} \tag{3.29}$$

One finds that the average frequency n_b of up-crossings of the level b by $\hat{a}$ is given by

$$n_b = \frac{1}{\sqrt{2\pi}} \frac{\beta b}{\sigma^2} \exp\left[-\tfrac{1}{2}\frac{b^2}{\sigma^2}\right] \qquad (3.30)$$

Vanmarcke [31] has proposed a modification of this latter scheme to obtain improved approximations to α_E and α_D from n_b. The stationary envelope response $\hat{a}$ is treated as a two state, continuous time, Markov process in which state 0 corresponds to $\hat{a} < b$, and state 1 corresponds to $\hat{a} > b$. The intervals T_0 and T_1 spent in states 0 and 1, respectively, are assumed to be independent random variables with exponential distributions. This leads to the approximation

$$\alpha_E = n_b \left(1 - \exp\left[-\tfrac{1}{2}\frac{b^2}{\sigma^2}\right]\right)^{-1} \qquad (3.31)$$

Vanmarcke [31] estimates the fraction of E-crossings that are immediately followed by a D-crossing on the basis of a physical argument. This generates the estimate

$$\alpha_D = 2\nu_b \frac{1 - \exp\left(-\tfrac{1}{2}\frac{n_b}{\nu_b}\right)}{1 - \exp\left(-\tfrac{1}{2}\frac{b^2}{\sigma^2}\right)} \qquad (3.32)$$
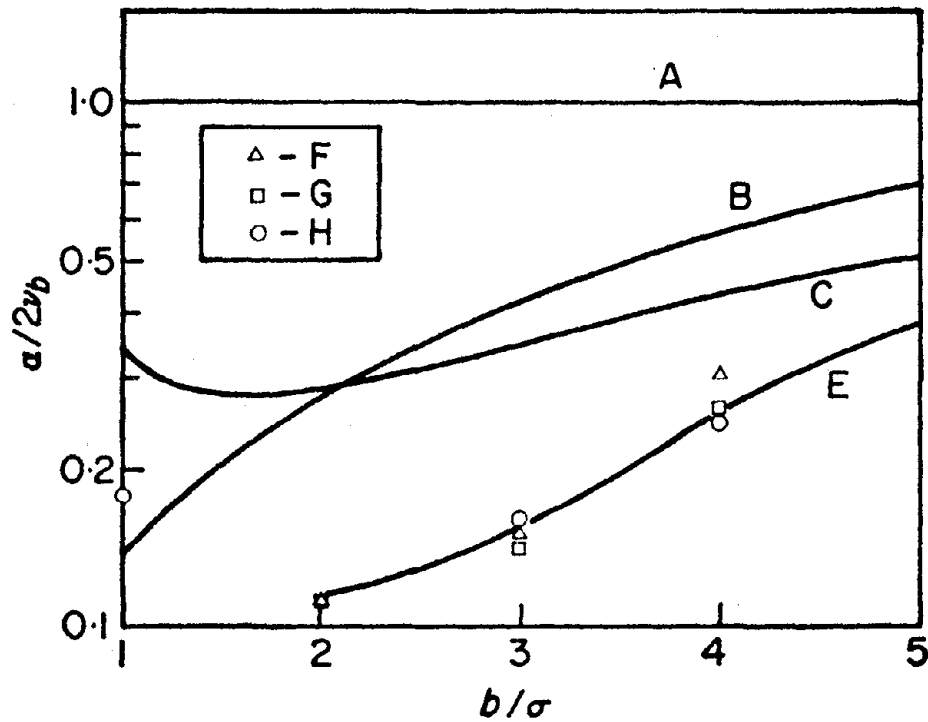
As mentioned earlier, the approximate methods which are based on solving approximately for the transition probability density of first passage are quite accurate for a lightly damped oscillator. The accuracy of the other methods varies with damping and barrier

level. Generally, as the barrier level increases, these simpler methods tend to give more accurate results. Figure 3.2 shows a comparison of the accuracy typical of the methods discussed for type-D barriers. Vanmarcke's method (C) is clearly more accurate than the Poisson process approximation (A), and predicts the correct qualitative behavior of α with variation in barrier level. For this reason, Vanmarcke's method may be considered the best simple approximation currently available.

3.4 A Semi-Empirical Approach

The object of this section is to develop an approximate method for generating the limiting decay rate for a type-D barrier that displays good accuracy for barrier levels in the range of engineering interest, while, at the same time, avoids excessive computational effort. In what follows, it will be assumed that the oscillator is lightly damped so that the response trajectories appear quasi-sinusoidal over one natural period of the oscillator. It will also be assumed that the oscillator has attained its stationary response distribution.

It is widely agreed that the basic reason the Poisson process approximation for α_D breaks down for low barrier levels is that the crossings out of the safe region are, in fact, not statistically independent events. Since the envelope varies slowly, when a peak occurs above the threshold level, the probability is higher than usual that the next peak will also be above the threshold. Thus, D-crossings tend to occur together in clumps. For

Fig. 3.2 Comparison of Analytical Estimates for Limiting
Decay Rates with Values Obtained Numerically
and Experimentally. Type-D Barriers, $\zeta = 0.01$.
From Reference [22].

A - Assumption of Independent D-Crossings.
B - Assumption of Independent E-Crossings.
C - Vanmarcke's Modified Two-State Markov Process.
E - Mark's Continuous State-Discrete Time Markov Process.
F - Digital Simulation.
G - Analog Simulation.
H - Numerical Diffusion of Probability.

decreasing damping ratio or barrier level, the tendency toward clumping will increase.

If allowance were made for the clumping tendency, one might expect a more accurate estimate for α_D to result. Before showing how this may be done, it is necessary to define some terms.

The clump size cs will be defined as the number of successive peaks that occur outside the safe region with no intervening peak inside the safe region. Since it has been assumed that the response is quasi-sinuoidal, successive D-crossings in a clump will be separated by an interval of approximately $1/2\nu_0$, where

$$\nu_0 \equiv \omega_0/2\pi \qquad (3.33)$$

Therefore, it is reasonable to define the clump duration T_1 as

$$T_1 \equiv cs/2\nu_0 \qquad (3.34)$$

Following the end of a clump will be an interval in which the oscillator response will remain in the safe region. Let the length of this interval, between the end of one clump and the beginning of the next, be denoted T_0. The sum of the sequential intervals T_0 and T_1 will be the length of the interval between the beginning of one clump and the beginning of the next. During this time, cs D-crossings will have occurred, so that the quantity

$$E\left[\frac{cs}{T_0 + T_1}\right]$$

will represent the average number of D-crossings per unit time.

If the response process is ergodic, this will be the same as $2\nu_b$, since ν_b was defined as the expected number of up-crossings of the +b level per unit time.

It will be assumed that

$$E\left[\frac{cs}{T_0 + T_1}\right] \approx \frac{E[cs]}{E[T_0 + T_1]} \qquad (3.35)$$

Hence,

$$E[T_0 + T_1] = \frac{E[cs]}{2\nu_b} \qquad (3.36)$$

Taking expectations in (3.34) yields

$$E[T_1] = \frac{E[cs]}{2\nu_0} \qquad (3.37)$$

Subtracting (3.37) from (3.36) then gives

$$E[T_0] = \frac{E[cs]}{2\nu_b}\left(1 - \frac{\nu_b}{\nu_0}\right) \qquad (3.38)$$

It is assumed that T_0 has an exponential distribution of the form of (3.5). Explicitly, this is written

$$W(T_0) = e^{-\alpha T_0} \qquad (3.39)$$

From (3.39), it therefore follows that

$$E[T_0] = \frac{1}{\alpha} \qquad (3.40)$$

Hence, (3.37), (3.38), and (3.40) imply

$$\alpha = \frac{\nu_b/\nu_0}{E[T_1]} \left(1 - \frac{\nu_b}{\nu_0}\right)^{-1} \tag{3.41}$$

The problem has thus been reduced to the determination of $E[T_1]$. Let the probability distribution of T_1 be denoted $P_{T_1}(t)$ where

$$P_{T_1}(t) = Pr[T_1 > t] \tag{3.42}$$

To determine the detailed structure of P_{T_1} would be very difficult. However, it is possible to surmise the qualitative behavior of $P_{T_1}(t)$ (particularly for large t) from physical considerations.

Consider the conditional probability

$$\hat{P}(n+1|n) \equiv Pr\left[T_1 > \frac{n+1}{2\nu_0} \mid T_1 > \frac{n}{2\nu_0}\right] = \frac{P_{T_1}\left(\frac{n+1}{2\nu_0}\right)}{P_{T_1}\left(\frac{n}{2\nu_0}\right)} \tag{3.43}$$

where $P[A|B]$ is the probability that A is true given that B is true. It is apparent that $\hat{P}(n+1|n)$ represents the probability that a clump, which already contains n D-crossings, will continue for at least one more D-crossing. The probability that a clump will contain exactly n D-crossings is given by

$$Pr[cs = n] = P_{T_1}\left(\frac{n}{2\nu_0}\right) - P_{T_1}\left(\frac{n+1}{2\nu_0}\right) \tag{3.44}$$

Dividing both sides of (3.44) by $P_{T_1}\left(\frac{n}{2\nu_0}\right)$ yields

$$\frac{\Pr[cs = n]}{P_{T_1}\left(\frac{n}{2\nu_0}\right)} = 1 - \hat{P}(n+1|n) \qquad (3.45)$$

Therefore, one obtains

$$\hat{P}(n+1|n) = 1 - \frac{\Pr[cs = n]}{P_{T_1}\left(\frac{n}{2\nu_0}\right)} \qquad (3.46)$$

The fact that clumps of D-crossings are more probable than isolated D-crossings may be interpreted as meaning that $\Pr[cs = n]$ is an increasing function of n for small enough n. Since $P_{T_1}\left(\frac{n}{2\nu_0}\right)$ is a monotone decreasing function of n, it is therefore certain, from (3.46), that $\hat{P}(n+1|n)$ will vary with n for sufficiently near 1.

As n becomes large one would expect $\hat{P}(n+1|n)$ to lose its dependence on n. That is, as the time from the beginning of a clump becomes large, the probability that the clump will continue becomes less and less influenced by the time of the initial D-crossing. Thus $\hat{P}(n+1|n)$ should approach a constant value as $n \to \infty$. Let this limiting value be denoted by P* where

$$P* \equiv \lim_{n \to \infty} \hat{P}(n+1|n) = \lim_{n \to \infty} \frac{P_{T_1}\left(\frac{n+1}{2\nu_0}\right)}{P_{T_1}\left(\frac{n}{2\nu_0}\right)} \qquad (3.47)$$

The probability density of T_1, $p_{T_1}(t)$ is defined as

$$P_{T_1}(t) \equiv -\frac{dP_{T_1}(t)}{dt} \qquad (3.48)$$

Suppose $p_{T_1}(t)$ were of the form $\beta \exp(-\beta t)$. Then $P_{T_1}(t)$ would be $\exp(-\beta t)$, and $\hat{P}(n+1|n)$ would be a constant for all n. However, this would not provide for the dependence of $\hat{P}$ upon n for n near 1. To allow for this dependence, let $p_{T_1}(t)$ be approximated by some member of the class of densities $f_\rho(t)$ given by

$$f_\rho(t) = \frac{1}{C_N} t^\rho \exp(-\beta t) \quad ; \quad \rho \geq 0, t \geq 0 \qquad (3.49)$$

where C_N is the normalizing constant. For $\rho > 0$, f_ρ increases initially but eventually decays toward zero as $t \to \infty$. This is the same qualitative behavior that would be expected of $p_{T_1}(t)$.

The probability distribution $F_\rho(t)$ associated with $f_\rho(t)$ has the form

$$F_\rho(t) = \frac{1}{C_N \beta} \exp(-\beta t) \left[t^\rho + \frac{\rho t^{\rho-1}}{\beta} + \frac{\rho(\rho-1)t^{\rho-2}}{\beta^2} + \dots \right] \qquad (3.50)$$

Hence, it may be easily shown that

$$\text{Lim} \frac{F_\rho\left(\frac{n+1}{2\nu_0}\right)}{F_\rho\left(\frac{n}{2\nu_0}\right)} = \exp\left(-\frac{\beta}{2\nu_0}\right) \qquad (3.51)$$

Since $P_{T_1}(t)$ is being approximated by $F_\rho(t)$, equations (3.51) and

and (3.47) imply that

$$P* = \exp\left(\frac{-\beta}{2\nu_0}\right) \tag{3.52}$$

For all $\rho > 0$, f_ρ and F_ρ will exhibit qualitative behavior consistent with what is known of the behavior of p_{T_1} and P_{T_1}, respectively. However, ρ has been left undetermined. It is proposed to determine the optimum value of ρ empirically as follows.

Let $p_{T_1}(t)$ be approximated by $f_\rho(t)$. Then $E[T_1]$ may be evaluated in the usual fashion as

$$E[T_1] = \frac{\rho + 1}{\beta} \tag{3.53}$$

If $P*$ could be evaluated analytically, β would be determined by equation (3.52). Equations (3.53) and (3.41) would then relate α to ρ. Available data on limiting decay rates could hence be used to determine the most consistent value of ρ.

To implement this scheme, it is necessary to find $P*$. Toward this end, let $q(x_0 | x_1 ; t)dx_1$ be the probability that a trajectory, which starts at x_0, reaches the differential element of measure dx_1 centered at x_1 a time t later. Therefore q is the transition probability density of x. Let ω_d be defined as

$$\omega_d \equiv \omega_0 \sqrt{1 - \zeta^2} \tag{3.54}$$

Then π/ω_d will be one half of a damped natural oscillator "period".

For $t = \pi/\omega_d$, q is given by [12]

$$q(x_0 | x ; \pi/\omega_d) = \frac{1}{\sqrt{2\pi}\,\sigma'} \exp\left[-\frac{1}{2}\left(\frac{x - \mu'}{\sigma'}\right)^2\right] \qquad (3.55)$$

where

$$(\sigma')^2 = \frac{\pi S_0}{2\zeta\omega_0^3}\left[1 - \exp\left(-2\pi\zeta\omega_0/\omega_d\right)\right] \qquad (3.56)$$

and

$$\mu' = -x_0 \exp\left(-\pi\zeta\omega_0/\omega_d\right) \qquad (3.57)$$

Suppose the oscillator is at a peak during a clump in which k D-crossings have already occurred. Let $p_k(r)$ be the conditional probability density of such peaks, given that $r \geq b$. That is, $p_k(r) = 0$ for $r < b$. Due to the narrow-bandedness of the response, it will be assumed that any two successive peaks will be separated by an interval of π/ω_d, and that x changes sign during this interval. With this assumption, $p_{k+1}(r)$ may be expressed as

$$p_{k+1}(r) = \lambda_k \int_b^\infty p_k(x)\, q(-x | r ; \pi/\omega_d)\, dx \quad ; \quad r \geq b$$

$$\qquad (3.58)$$

$$= 0 \qquad\qquad\qquad ; \quad r < b$$

The first argument of q is $-x$ rather than $+x$ to account for the assumed sign change of x, since r is always positive. The factor λ_k will be determined from the condition

$$\int_b^\infty p_{k+1}(r)\,dr = 1 \qquad\qquad (3.59)$$

Imposing this condition leads to

$$\frac{1}{\lambda_k} = \int_b^\infty \int_b^\infty p_k(x)\,q(-x|r\,;\,\pi/\omega_d)\,dx\,dr \qquad\qquad (3.60)$$

The quantity on the right of equation (3.60) will be recognized as the probability that the clump will continue for at least one more D-crossing given that k D-crossings have already occurred. Hence, from equation (3.43),

$$\frac{1}{\lambda_k} = \hat{P}(k+1|k) \qquad\qquad (3.61)$$

As $k \to \infty$, $\hat{P} \to P*$, and $p_k(r)$ approaches a stationary density $p_\infty(r)$. This stationary density must therefore satisfy

$$p_\infty(r) = \frac{1}{P*}\int_b^\infty p_\infty(x)\,q(-x|r\,;\,\pi/\omega_d)\,dx \qquad\qquad (3.62)$$

The integral equation (3.62) does not seem to be amenable to exact, closed-form analytical solution. However, the precise form of p_∞ is not really of interest here. It is necessary only to determine the eigenvalue $\frac{1}{P*}$. There exist several methods for determining approximate eigenvalues for integral equations of the form of (3.62). These include Galerkin's method, the collocation method, and Picard iteration. Of these methods, it was felt that Picard iteration was the easiest to implement for this particular

problem.

In Picard iteration, a function $f_0(r)$ is chosen to approximate the eigenfunction $p_\infty(r)$. A series of functions $f_i(r)$ are then generated using the regenerative scheme

$$f_{i+1}(r) = \lambda_i \int_b^\infty f_i(x)\, q(-x \mid r ; \pi/\omega_d)\, dx \qquad (3.63)$$

When the functions $f_i(r)$ and $f_{i+1}(r)$ become "sufficiently close" over the range $b \leq r < \infty$, the iteration is stopped. The eigenfunction is then approximated by $f_{i+1}(r)$, and the principal eigenvalue is approximated by λ_i. If the initial guess f_0 is a good approximation to p_∞, this method should converge very rapidly.

Consider the oscillator in stationary response with probability density p_s where

$$p_s(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\tfrac{1}{2}\frac{x^2}{\sigma^2}\right) \qquad (3.64)$$

Let it be given that at time t_1, $x(t_1) \geq b$, then the probability density of x becomes p_c where

$$p_c(x) = \frac{p_s(x)}{\displaystyle\int_b^\infty p_s(u)\, du} \quad ; \quad x \geq b$$

$$p_c(x) = 0 \qquad\qquad ; \quad x < b \qquad (3.65)$$

It is proposed that $p_c(x)$ provides a reasonable approximation to

$p_\infty(x)$.   Therefore, let $f_0(x) = p_c(x)$.   After some tedious algebra,
$f_1(x)$ may be evaluated from equation (3.62).   This yields

$$\frac{f_1(x)}{\lambda_0} = \frac{\exp\left[-\frac{1}{2}\frac{c_4}{\sigma^2}x^2\right] \cdot \frac{1}{2}\,\mathrm{erfc}\left[(b - c_3 x)\sqrt{c_2/2}\right]}{\sqrt{\frac{\pi c_2}{2}}\,\sigma\sigma'\,\mathrm{erfc}\left[\frac{b}{\sqrt{2}\,\sigma}\right]} \qquad (3.66)$$

where

$$c_1 = \exp\left(-\pi\zeta\omega_0/\omega_1\right)$$

$$c_2 = \frac{1 - c_1^2 + c_1^4}{\sigma^2(1 - c_1^2)}$$

$$\qquad (3.67)$$

$$c_3 = \frac{c_1^2}{1 - c_1^2 + c_1^4}$$

$$c_4 = \frac{1}{1 - c_1^2 + c_1^4}$$

To continue the iteration procedure, numerical integration
methods would be required to evaluate $f_i(x)$ for $i \geq 2$.   This would
be a very time-consuming procedure, which would only be
warranted if the accuracy of the first order approximation were
found to be inadequate.

Thus, for the present investigation, P^* will be approximated
by P_c where

$$P_c = \frac{1}{\lambda_0} \qquad (3.68)$$

Equation (3.59) requires that

$$\int_b^\infty f_1(x)\,dx = 1 \qquad (3.69)$$

Hence, one obtains

$$\int_b^\infty \frac{f_1(x)}{\lambda_0}\,dx = P_c \qquad (3.70)$$

Due to the complicated form of $\dfrac{f_1(x)}{\lambda_0}$, an exact evaluation of P_c cannot be performed analytically. Consequently, it must be evaluated either numerically or by some approximate analytical method. One such method is suggested by the shape of the curve $f_1(r)/\lambda_0$. Let the mean value of peak magnitude associated with $f_1(r)/\lambda_0$ be denoted $E_1[r]$. In the neighborhood of $E_1[r]$, $f_1(r)/\lambda_0$ strongly resembles a Gaussian density. Therefore, it will be undertaken to approximate (3.66) with the Gaussian density which has the same mean value and standard deviation as $f_1(r)/\lambda_0$. Let this Gaussian density be denoted $f_G(r)$, and let P_{CG} be given by

$$P_{CG} = \int_b^\infty f_G(r)\,dr \qquad (3.71)$$

To obtain an indication of how closely P_{CG} approximates P_C, these two quantities were evaluated for ζ ranging from 0.001 to 0.08. These comparisons are shown in figure 3.3. As can be
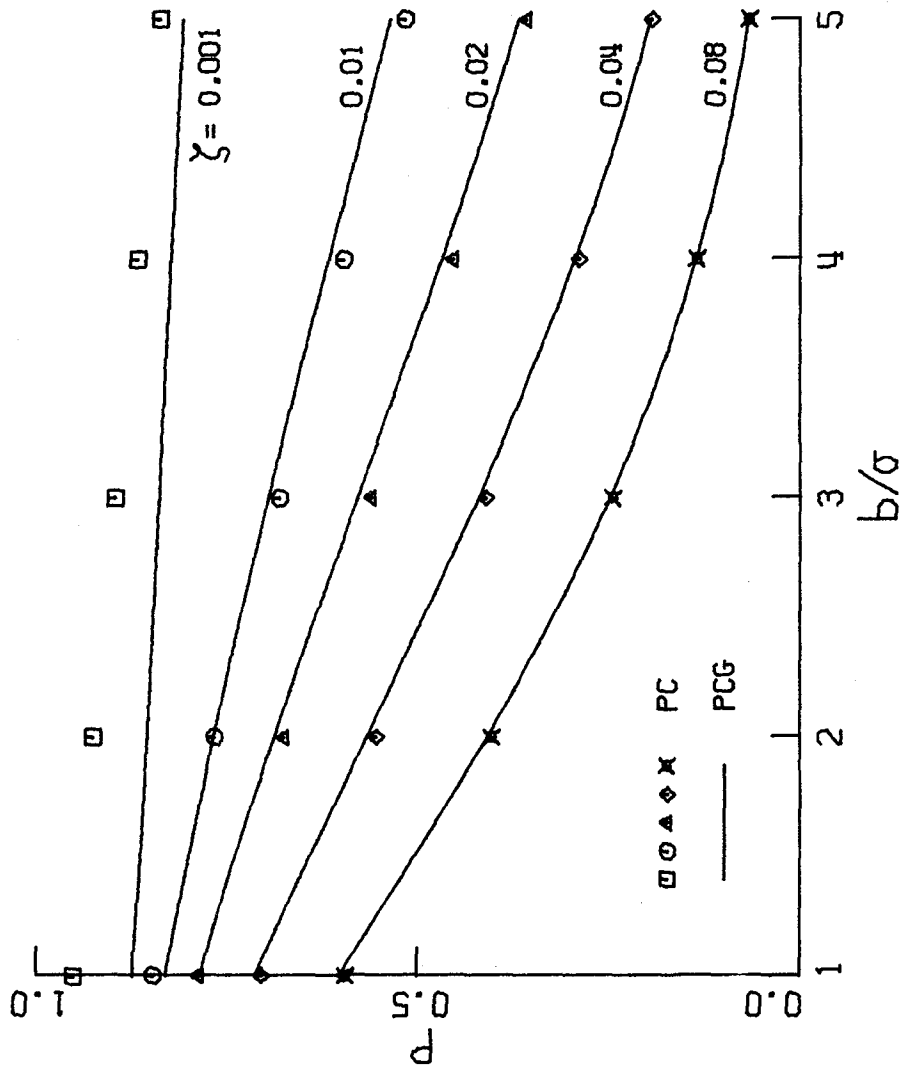
Fig. 3.3   Comparison of P_C and P_{CG} at Various Damping Ratios and Barrier Levels.

seen from the figure, P_{CG} is extremely close to P_C except for the case where $\zeta = 0.001$. Even then, the approximation seems to be quite good.

It should be mentioned that when P* is very close to zero or very close to 1, the approximations used for P* (i.e., P_{CG} or P_C) are not really adequate. The reason for this is that, near these two extremes, the limiting decay rate α is very sensitive to small changes in P*. This can be seen by solving for α in terms of P*.

From equations (3.52), (3.53), and (3.41), one obtains

$$\frac{\alpha}{2\nu_b} = \frac{-1}{1+\rho} \ln (P*) \left(1 - \frac{\nu_b}{\nu_0} \right)^{-1} \tag{3.72}$$

As P* approaches zero, the derivative $\frac{d}{dP*} [\ln (P*)] = \frac{1}{P*}$ becomes very large, and a small inaccuracy in the evaluation of P* can drastically affect the resulting value of α. Further, as $P* \rightarrow 1$, $-\ln P* \rightarrow 1 - P*$, so that the difference $1 - P*$ becomes important. A small percentage change in P* will then create a large percentage change in $\ln(P*)$ so, once again α becomes sensitive to small inaccuracies in P*.

For small damping ($\zeta \lesssim 1$), the situation where $P* \rightarrow 0$ will only occur when the barrier level is large. For such levels, the value for α given by equation (3.72) tends to err on the high side. Therefore, when (3.72) gives a value for $\frac{\alpha}{2\nu_b}$ which is greater than unity, it is disregarded, and α is set equal to $2\nu_b$ since $2\nu_b$ is the limit α approaches for large b.

The situation where $P* \rightarrow 1$ will occur when the damping is extremely small ($\zeta < .005$), since the correlation between the values of two consecutive peaks is then very high.

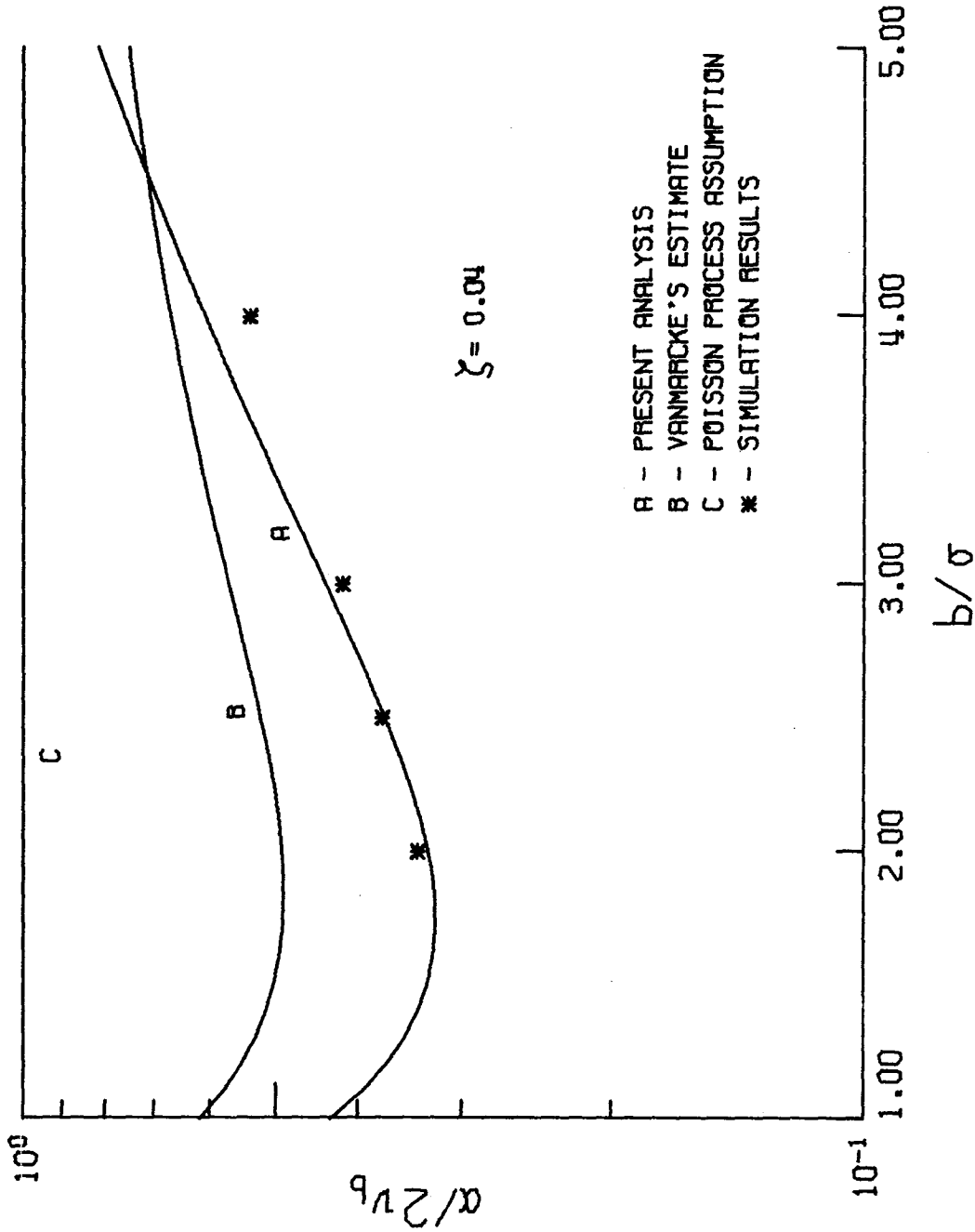From the available data [23], it was determined that letting

$$\rho = 1 \tag{3.73}$$

produces the most consistent results for damping ratios in the range $0.01 \leq \zeta \leq 0.10$, and barrier levels in the range $1 \leq \frac{b}{\sigma} \leq 5$. For ρ significantly less than 1, α tends to be overly conservative (i.e., too large). Conversely, if ρ is significantly larger than unity, the corresponding values of α tend to underestimate the first crossing rate.

In figures 3.4 and 3.5, the variation of $\frac{\alpha}{2\nu_b}$ with barrier level b/σ is displayed for two values of damping. For comparison, simulation results and the corresponding values from Vanmarcke's two-state Markov process approximation are also presented. The present analysis is seen to correspond well with the simulation results.

3.5 Application of Approximate Solution to Earthquake Engineering

3.5.1 Extension of Results to Nonstationary Response

The random processes involved in earthquake engineering are, in general, nonstationary. Therefore, the usefulness of an approximate method for generating first passage probabilities will be greatly enhanced if the method may be extended to such processes. Corotis [31] has extended, by analogy, Vanmarcke's

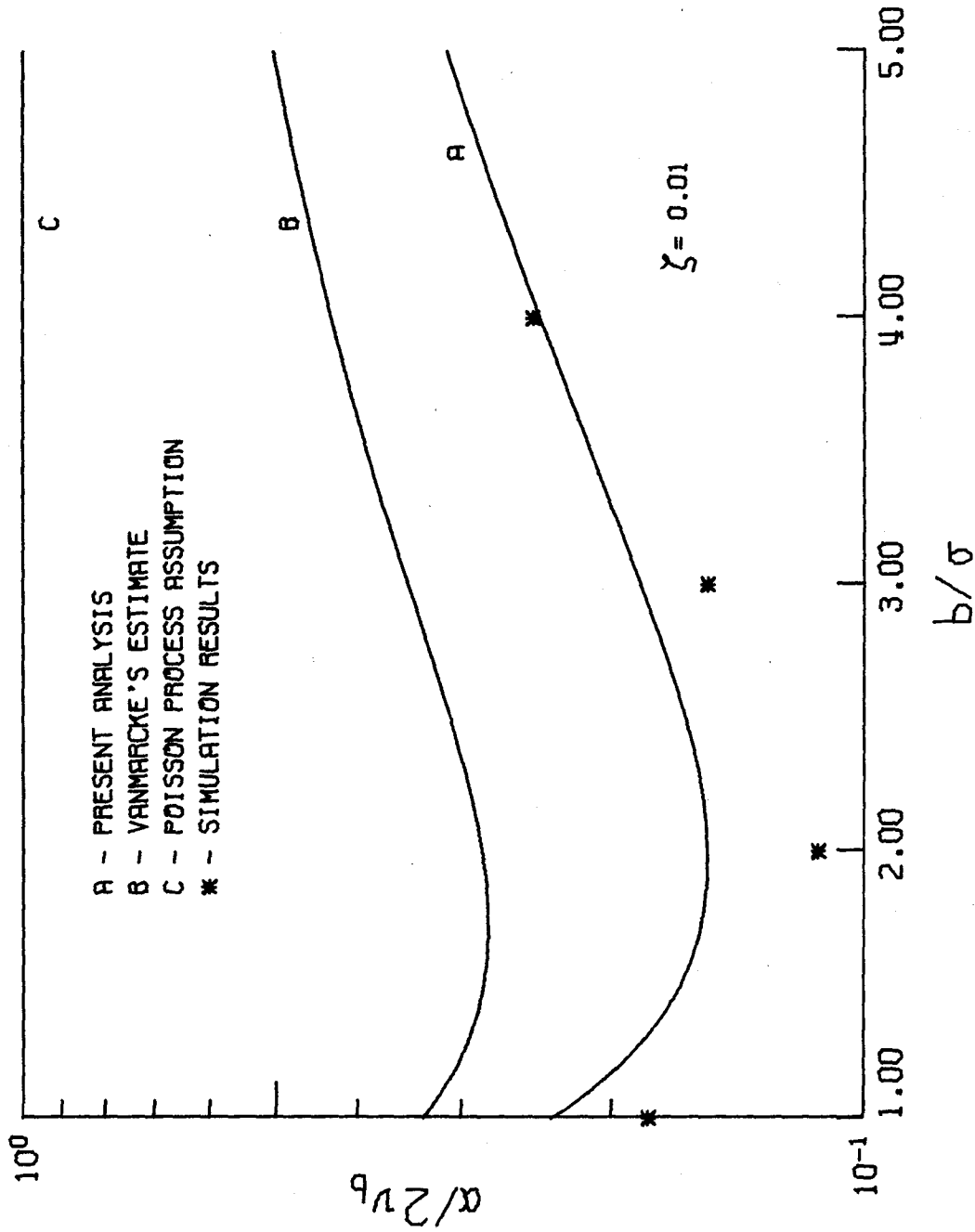Fig. 3.4   Limiting Decay Rate versus Barrier Level, $\zeta = 0.04$.

Fig. 3.5 Limiting Decay Rate versus Barrier Level, $\zeta = 0.01$.

approximate solution to include nonstationary response processes, with favorable results. This approach will also be taken herein.

Let x represent the response of a linear oscillator, initially at rest, subject to deterministically modulated stationary Gaussian white noise. The response is then governed by

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \theta(t)w(t)$$

(3.74)

$$x(0) = 0 \quad , \quad \dot{x}(0) = 0$$

where $\theta(t)$ is the modulating function. Let S_0 be the spectral density of $w(t)$. The Fourier transform of the autocorrelation of the excitation will then be $\theta^2(t)S_0$. This may be thought of as a time dependent intensity or "spectral density".

Let the nonstationary probability density of the response be denoted by p_r. This density will have the general form

$$p_r(x, \dot{x}, t) = \frac{1}{2\pi\sqrt{Det[Q(t)]}} \exp\left[-\tfrac{1}{2}\underline{x}^T Q^{-1}(t)\underline{x}\right]$$

(3.75)

where

$$\underline{x} \equiv \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

(3.76)

and

$$Q(t) = \left(q_{ij}(t)\right) = E\left[\underline{x}(t)\underline{x}^T(t)\right]$$

(3.77)

Suppose p_r varies slowly enough in time so that, over an interval of a single oscillator period $t, t + \dfrac{2\pi}{\omega_d}$, p_r may be

considered as approximately stationary. Then it would be reasonable to assume that the first D-crossings during this interval will occur with approximately the same frequency as if p_r were truly stationary. If this is assumed, an expression for the instantaneous first crossing rate $\alpha(t)$ may be derived in a fashion entirely analogous to the derivation of the limiting decay rate given by equation (3.72). For the nonstationary case, this expression takes the form

$$\frac{\alpha(t)}{2\nu_b(t)} = \frac{-1}{2} \ell n \, [\, P^*(t)\,] \left(1 - \frac{\nu_b(t)}{\nu_0} \right)^{-1} \tag{3.78}$$

Except for the time dependence of α, ν_b, and P^*, this expression is identical to (3.72).

As in the stationary case, ν_b is given by equation (3.23). However, the probability density in the integrand must be replaced by $p_r(x, \dot{x})$. Thus $\nu_b(t)$ may be expressed as

$$\nu_b(t) = \int_0^\infty \dot{x} \, p_r(b, \dot{x}, t) \, d\dot{x} \tag{3.79}$$

Let

$$\Phi(t) = \left(\varphi_{ij}(t) \right) = Q^{-1}(t) \tag{3.80}$$

Substituting from (3.75) into (3.79) then yields

$$\nu_b(t) = \frac{\sqrt{\mathrm{Det}\,(Q)}}{\pi q_{11}} \left[\exp\left(-\tfrac{1}{2}\varphi_{11}b^2\right) - b\varphi_{12}\sqrt{\frac{\pi}{2\varphi_{22}}} \exp\left(-\tfrac{1}{2}\frac{b^2}{q_{11}}\right) \mathrm{erfc}\left(\frac{b\varphi_{12}}{\sqrt{2\varphi_{22}}}\right) \right] \tag{3.81}$$

It should be mentioned that, as in the case of stationary response, one method for obtaining approximate first crossing probabilities is to assume that the D-crossings occur according to a Poisson process. In the case of nonstationary response, however, the Poisson process is inhomogeneous, with time-dependent rate $2\nu_b(t)$.

The procedure to obtain P*(t) is similar to that followed in section 3.4 to determine the corresponding quantity P*. However, since the excitation intensity is now $\theta^2(t)S_0$, expression (3.55) for the transition probability density q must be replaced by

$$q(x_0 \mid x ; \pi/\omega_d) = \frac{1}{\sqrt{2\pi}\, \sigma'\theta(t)} \exp\left[\left(-\tfrac{1}{2}\, \frac{x - \mu'}{\sigma'\theta(t)}\right)^2\right] \qquad (3.82)$$

where σ' and μ' are given in equations (3.56) and (3.57), respectively.

With $\nu_b(t)$ and P*(t) thus specified, equation (3.78) may be used to obtain approximate first passage probabilities for non-stationary response. The fraction of trajectories which realize a first D-crossing during the interval $[t, t + \Delta t]$ is $\alpha(t)\Delta t + \mathcal{O}(\Delta t^2)$. By definition, this is just the probability of first crossing during this infinitesimal interval. Recalling definition (3.2) of W(t), the first passage probability, one may write

$$\alpha(t)\Delta t = W(t) - W(t + \Delta t) + \mathcal{O}(\Delta t)^2 \qquad (3.83)$$

Regarding (3.80) yields

$$\frac{W(t + \Delta t) - W(t)}{\Delta t} = -\alpha(t) + \Theta(\Delta t) \qquad (3.84)$$

Taking limits as $\Delta t \to 0$ then gives

$$\frac{dW}{dt} = -\alpha(t) \qquad (3.85)$$

It is required that

$$W(0) = 1 \qquad (3.86)$$

Hence, it is obvious that

$$W(t) = \exp\left[-\int_0^t \alpha(s)\,ds\right] \qquad (3.87)$$

From (3.3) then, the first passage probability density p(t) is

$$p(t) = \alpha(t)\exp\left[-\int_0^t \alpha(s)ds\right] \qquad (3.88)$$

3.5.2 Numerical Example - Calculation of First Passage Probability Density

To obtain an indication of the accuracy of this extension, an example was performed in which

$$\Theta(t) = 1 \quad ; \quad \forall t \geq 0 \qquad (3.89)$$

The three independent elements of the covariance matrix Q may be attained by solving equation (2.66). In this case, they are given by

$$E[x^2(t)] = \frac{\pi S_0}{2\zeta\omega_0^3}\left[1 - \exp(-2\zeta\omega_0 t)\left\{\frac{1}{1-\zeta^2} + \frac{\zeta\omega_0}{\omega_d}\sin 2\omega_d t \right. \right.$$
$$\left.\left. - \frac{\zeta^2\omega_0^2}{\omega_d^2}\cos 2\omega_d t\right\}\right]$$

$$(3.90)$$

$$E[x(t)\dot{x}(t)] = \frac{\pi S_0}{\omega_d^2}\exp(-2\zeta\omega_0 t)\sin^2\omega_d t \qquad (3.91)$$

$$E[\dot{x}^2(t)] = \frac{\pi S_0}{2\zeta\omega_0}\left[1 - \exp(2\zeta\omega_0 t)\left\{\frac{1}{1-\zeta^2} - \frac{\zeta\omega_0}{\omega_d}\sin 2\omega_d t \right. \right.$$
$$\left.\left. - \frac{\zeta^2\omega_0^2}{\omega_d^2}\cos 2\omega_d t\right\}\right]$$

$$(3.92)$$

Figures 3.6 through 3.8 show the first passage probability density p(t) as a function of t/T where T is the undamped natural oscillator period. The integration in equation (3.88) was performed numerically, using Simpson's rule. For comparison, simulation results from reference [23] are also shown. Also displayed is the approximation to first passage density resulting from the Poisson process assumption. The accuracy of the present approximation is clearly far superior to the latter. It is interesting to note that the Poisson process assumption produces more accurate results for the more heavily damped case. This is to be expected since the correlation between D-crossings drops off as the damping increases.
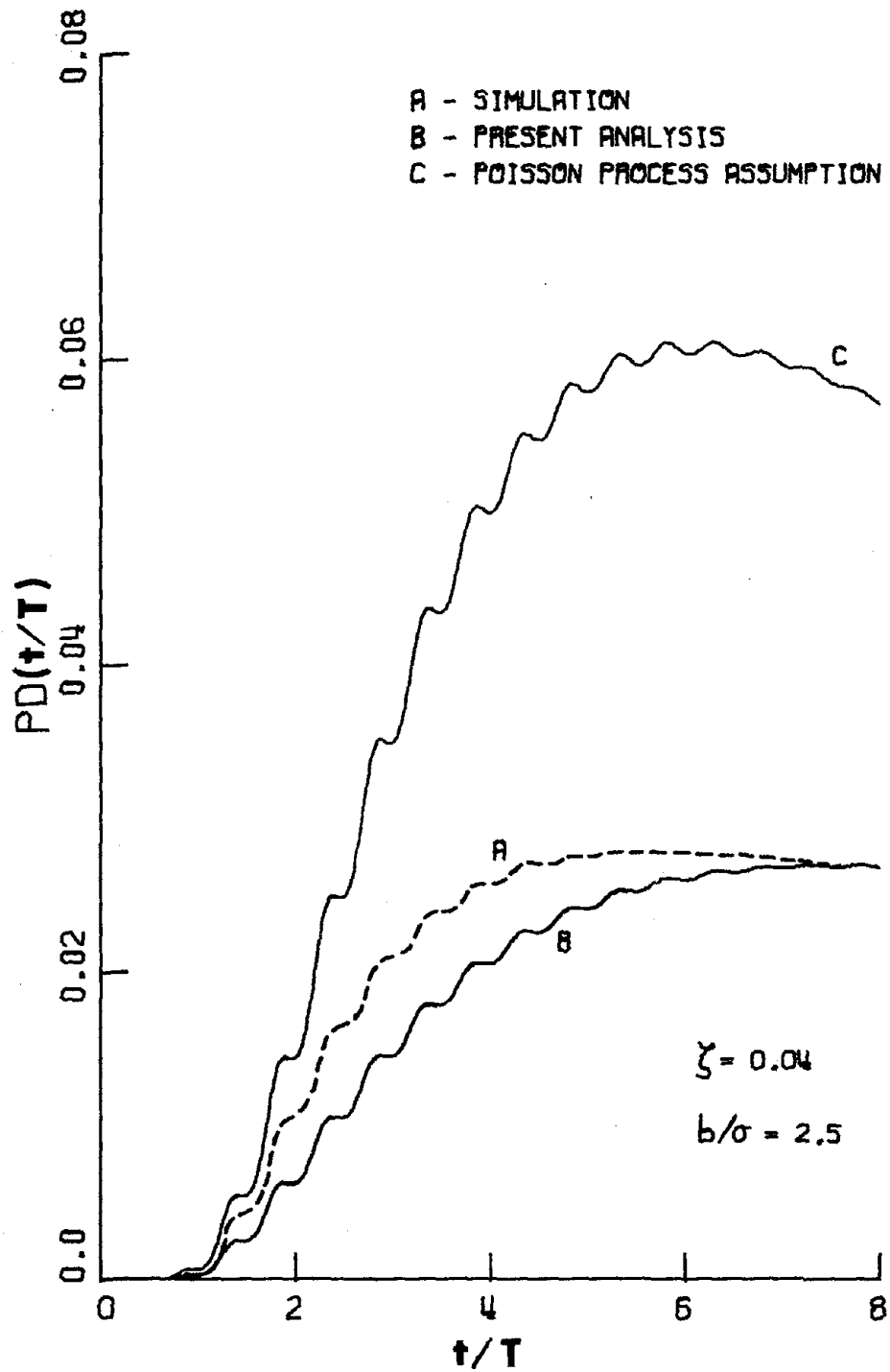
Fig. 3.6   First Passage Probability Density versus Time.
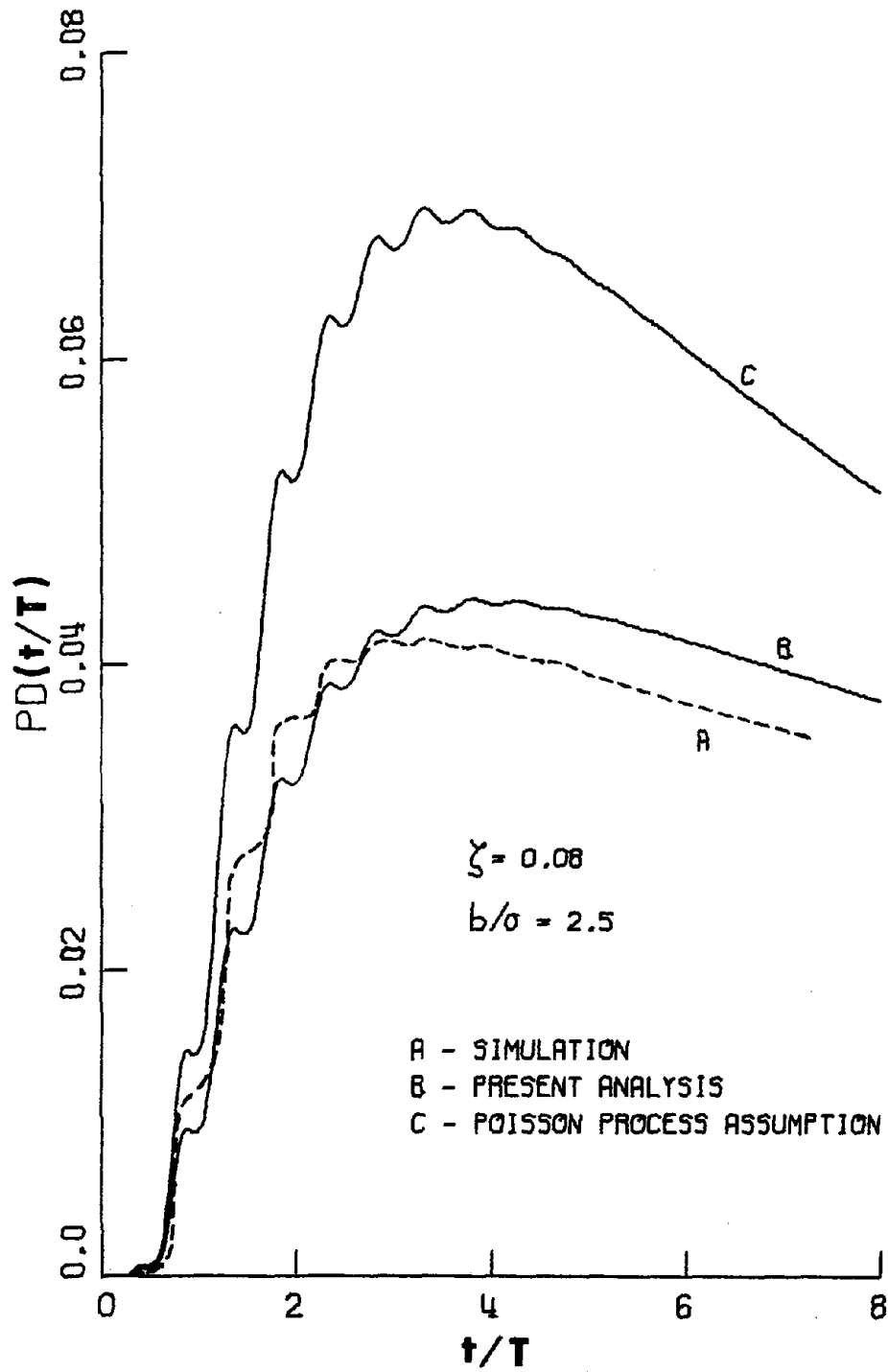Zero Start, $\zeta = 0.04$ and $b = 2.5\sigma$.

Fig. 3.7   First Passage Probability Density versus Time.
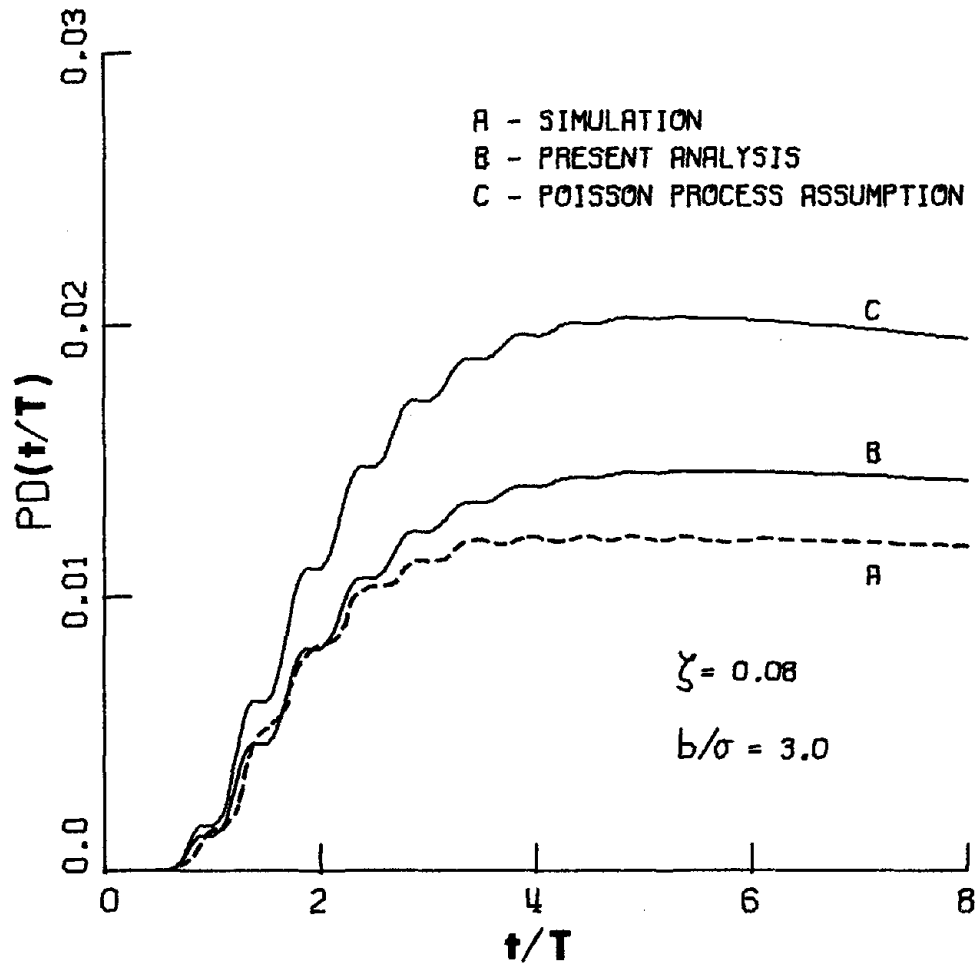Zero Start, ζ = 0.08 and b = 2.5σ.

Fig. 3.8   First Passage Probability Density versus Time.
Zero Start, $\zeta = 0.08$ and b = 3.0σ.

3.5.3 Applications Involving Response Spectrum Data

The practice of specifying the effects of seismic excitation by means of "design" response spectra has become increasingly common in recent years. When using response spectrum data for design purposes, a major difficulty arises if the system subject to the seismic excitation is nonlinear, since a response spectrum generally gives information about the maximum response of linear systems only. In such cases, it is desirable to "invert" the response spectrum, that is, to find a class of earthquake-like excitation time histories which produce the given design spectrum over some given range of frequencies $[\omega_\ell, \omega_u]$. These time histories may then be taken as possible inputs to the nonlinear system.

To approach this problem analytically, it is necessary to have an appropriate model for earthquake excitations. A relatively realistic model of this sort consists of a zero-mean stationary Gaussian random process multiplied by a deterministic envelope function. The appropriate spectral density $S(\omega)$ of the stationary process is usually inferred from site conditions and possibly records of previous ground acceleration time histories [10, 19, 32-35]. This model will serve to indicate the principle of the method to be described. However, it should be mentioned that this method could be adapted to accommodate more sophisticated models such as those described in references [9] and [36].

A linear oscillator subject to this artificial earthquake excitation will satisfy the equation

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = \theta(t)\, g(t) \qquad (3.93)$$

where $\theta(t)$ is the envelope function and $g(t)$ is a stationary Gaussian random process with spectral density function $S(\omega)$. It will be assumed that the oscillator starts from rest. Let τ' be the earthquake duration. Hence, for $t < 0$ and $t > \tau'$, $\theta(t) = 0$. Let $|x_m(\omega_0, \zeta)|$ represent the maximum displacement magnitude of the oscillator due to this excitation. Then, $|x_m(\omega_0, \zeta)|$ must occur during the interval $[0, \tau]$ where

$$\tau = \tau' + \pi/\omega_d \qquad (3.94)$$

For a single sample function of the excitation, $|x_m(\omega_0, \zeta)|$ will be a deterministic function of ζ and ω_0. For a constant value of damping ζ, the associated displacement response spectrum $SD_\zeta(\omega_0)$ will then be precisely defined as

$$SD_\zeta(\omega_0) = |x_m(\omega_0, \zeta)| \quad ; \quad \omega_\ell \leq \omega_0 < \omega_u \qquad (3.95)$$

However, the excitation is not just a single function of time, but rather an entire ensemble of possible time functions. Therefore, $|x_m(\omega_0, \zeta)|$ is actually a random variable, so that a probabilistic interpretation of response spectrum information is needed.

In accordance with the probabilistic nature of $|x_m(\omega_0, \zeta)|$, it will be required that a reliability level P_s be specified in addition to the damping ratio and natural frequency.

Then, the response spectrum $SD_\zeta(\omega_0, Ps)$ will be defined as the displacement magnitudes such that

$$\Pr\left[\,|\,x_m(\omega_0, \zeta)\,|\; \leq\; SD_\zeta(\omega_0, Ps)\right] \;=\; Ps \quad ; \quad \omega_\ell \leq \omega_0 \leq \omega_u$$

$$(3.96)$$

With this interpretation of response spectra, and with an earthquake model specified, it is possible to state more precisely the problem at hand. Given a response spectrum $SD_\zeta(\omega_0, Ps)$, where ζ and Ps are specified constants, it is desired to find an envelope function $\theta(t)$ and a spectral density function $S(\omega)$ such that the response of the oscillator (3.93) satisfies equation (3.96).

To obtain realistic artificial earthquake excitation, $\theta(t)$ should resemble actual earthquake envelopes. Let it also be required that $\theta(t)$ be smoothly varying during the interval of excitation $[0, \tau']$. Other than these restrictions, the choice of $\theta(t)$ may be made arbitrarily. Suggestions for actual envelope shapes may be found in references [19, 32-35]. Once an appropriate choice for $\theta(t)$ has been made, the problem is reduced to finding the corresponding spectral density function.

To determine $S(\omega)$, let equation (3.93) be rewritten in the vector form

$$\dot{\underline{x}} \;=\; A\underline{x} \;+\; \theta(t)\,g(t)\,\underline{e}_2$$

$$(3.97)$$

$$\underline{x}(0) \;=\; \underline{0}$$

where

$$\underline{x} \;=\; \left\{ \begin{array}{c} x \\ \dot{x} \end{array} \right\} ,$$

$$(3.98)$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}, \qquad (3.99)$$

and

$$\underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \qquad (3.100)$$

Since $\underset{\sim}{x}$ is Gaussian with mean $\underset{\sim}{0}$, the covariance matrix Q will determine the probability density of the response. Let $\underset{\sim}{F}(\omega, t)$ be the solution to the equation

$$\underset{\sim}{\dot{F}} = A\underset{\sim}{F} + \underset{\sim}{e}_2 \theta(t) \exp(i\omega t)$$

$$\qquad (3.101)$$

$$\underset{\sim}{F}(0) = \underset{\sim}{0}$$

Then, as shown in section (2.31), $Q(t)$ may be expressed as

$$Q(t) = \int_{-\infty}^{\infty} \underset{\sim}{F}(\omega, t)\, \underset{\sim}{F}^{*T}(\omega, t)\, S(\omega)d\omega \qquad (3.102)$$

When $\zeta \ll 1$, all the components of the matrix $\underset{\sim}{F}(\omega, t)\, \underset{\sim}{F}^{*T}(\omega, t)$ are sharply peaked at the resonant frequency $\omega = \omega_0$. In this case, it may be shown that [12]

$$Q(t) \approx S(\omega_0) \int_{-\infty}^{\infty} \underset{\sim}{F}(\omega, t)\, \underset{\sim}{F}^{*T}(\omega, t)\, d\omega \qquad (3.103)$$

provided $S(\omega)$ has no sharp peaks and is a smoothly varying function of ω in the neighborhood of $\omega = \omega_0$.

The approximation to $Q(t)$ given in (3.103) is the exact expression for the covariance matrix that would result if g(t), in

equation (3.93), represented white noise with constant spectral density $S_0 = S(\omega_0)$. Hence, for each value of ω_0, a good approximation to the stochastic response of (3.93) may be obtained by replacing g(t) with white noise of spectral density $S(\omega_0)$ provided ζ is sufficiently small. Since response spectra are usually constructed only for lightly damped systems, this will not be a severe practical restriction.

Since the methods for approximating first passage probabilities described in section (3.5.1) are designed for a lightly damped oscillator excited by modulated Gaussian white noise, they may now be applied to this problem. The appropriate expression for W(t) is given in equation (3.87), where W(t) is the probability that $|x|$ will not exceed b during the interval $[0, t]$. In this application, the interval during which the maximum response must occur is $[0, \tau]$. It has been specified, in equation (3.96), that the probability that $|x|$ will not exceed $SD_\zeta(\omega_0, Ps)$ is equal to Ps. Hence, for each value of ω_0 in the range $[\omega_\ell, \omega_u]$, the threshold level b will be set equal to $SD_\zeta(\omega_0, Ps)$, and the condition

$$W(\tau) = Ps \qquad (3.104)$$

will be imposed.

For a given envelope function $\theta(t)$, increasing the spectral density S_0 will cause $W(\tau)$ to decrease, and vice versa. Hence, for a given value of ω_0, it is a simple matter to vary the value of S_0 until equation (3.104) is satisfied to within some tolerance. When the agreement between $W(\tau)$ and Ps is sufficiently close,

S_0 is taken as the value of $S(\omega_0)$. Repeating this procedure for many different values of ω_0 in the range $[\omega_\ell, \omega_u]$ will thus generate, pointwise, the desired spectral density of g(t) appropriate to the given response spectrum and reliability value. Recall that an assumption essential to implementing the above procedure was that $S(\omega)$ varies smoothly with ω, and has no sharp peaks. Once $S(\omega)$ has been found, therefore, this assumption must be checked.

It is noted that, once an appropriate envelope function $\theta(t)$ is chosen and the corresponding spectral density $S(\omega)$ is determined, the method described in Chapter II may be applied to determine analytically the approximate response of a nonlinear system to such artificial earthquake excitation. By this procedure, therefore, a response spectrum may actually be used to estimate nonlinear response statistics.

3.5.4 Numerical Example - Response Spectrum Prediction

The practicality of the procedure described thus far will depend heavily upon how accurately the approximate method of section (3.5.1) can predict $W(\tau)$. To check this accuracy, a simulation study of equation (3.93) was performed. The excitation used in this study was the same as that used in the simulation study of section (2.5). The equation of motion simulated was thus

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = C\gamma t \exp(-\gamma t)\hat{w}(t) \quad ; \quad 0 \le t \le \tau'$$

$$(3.105)$$

where C is a dimensionless constant, and $\hat{w}(t)$ is the nearly-white

noise process described in section (2.5). The spectral density $S(\omega)$ of $\hat{w}(t)$ is approximately equal to a constant S_0 over the range of values of ω_0 studied. The earthquake duration τ' was taken to be $4/\gamma$. At $t = 4/\gamma$, the envelope function $C\gamma t \exp(-\gamma t)$ has decayed to approximately 20% of its maximum value. Typical values of τ' may be in the range of 10 to 30 seconds, depending on the severity of the earthquake.

Since (3.105) is a linear equation and the excitation consists of many piecewise linear segments, the theoretically exact solution may be found for each member of the ensemble of excitation time histories.

Nine values of ω_0 and three values of damping were considered. For each combination of ζ and ω_0, 250 time histories were simulated, and the maximum value of $|x|$ was recorded for each time history. These maxima were then arranged in the increasing order

$$\left\{ |x_m|_1, |x_m|_2, \ldots, |x_m|_{250} \right\}$$

For any level b such that $|x_m|_1 < b < |x_m|_{250}$, an integer n may be found such that

$$|x_m|_n \le b < |x_m|_{n+1} \tag{3.106}$$

Then, corresponding to the level b, Ps was taken to be

$$Ps = \frac{n}{250} \tag{3.107}$$

By this procedure, response spectra corresponding to Ps values of 0.5, 0.7, and 0.9 were computed for $\zeta = 0.02$. Also computed were the spectra corresponding to Ps = 0.5 for $\zeta = 0.04$ and $\zeta = 0.001$.

The analytical approximate technique described in this section was used to generate the corresponding theoretical response spectra. These results are compared in figures 3.9 and 3.10. The response spectra are plotted as a dimensionless pseudovelocity

$$\frac{PSV}{c\sqrt{S_0}\,\tau'}$$

where PSV is the usual pseudovelocity given by

$$PSV = \omega_0 SD \tag{3.108}$$

The horizontal axis in these figures is the dimensionless frequency $\tau' F$ where

$$F = \frac{\omega_0}{2\pi} \tag{3.109}$$

Figure 3.9 shows that the approximate analytical spectra corresponding to $\zeta = 0.02$ are in good agreement with the simulation results over the range

$$\tau' F \geq 10$$

for all the values of Ps considered. In all cases, however, there is a noticeable discrepancy at $\tau' F = 5$, the lowest frequency
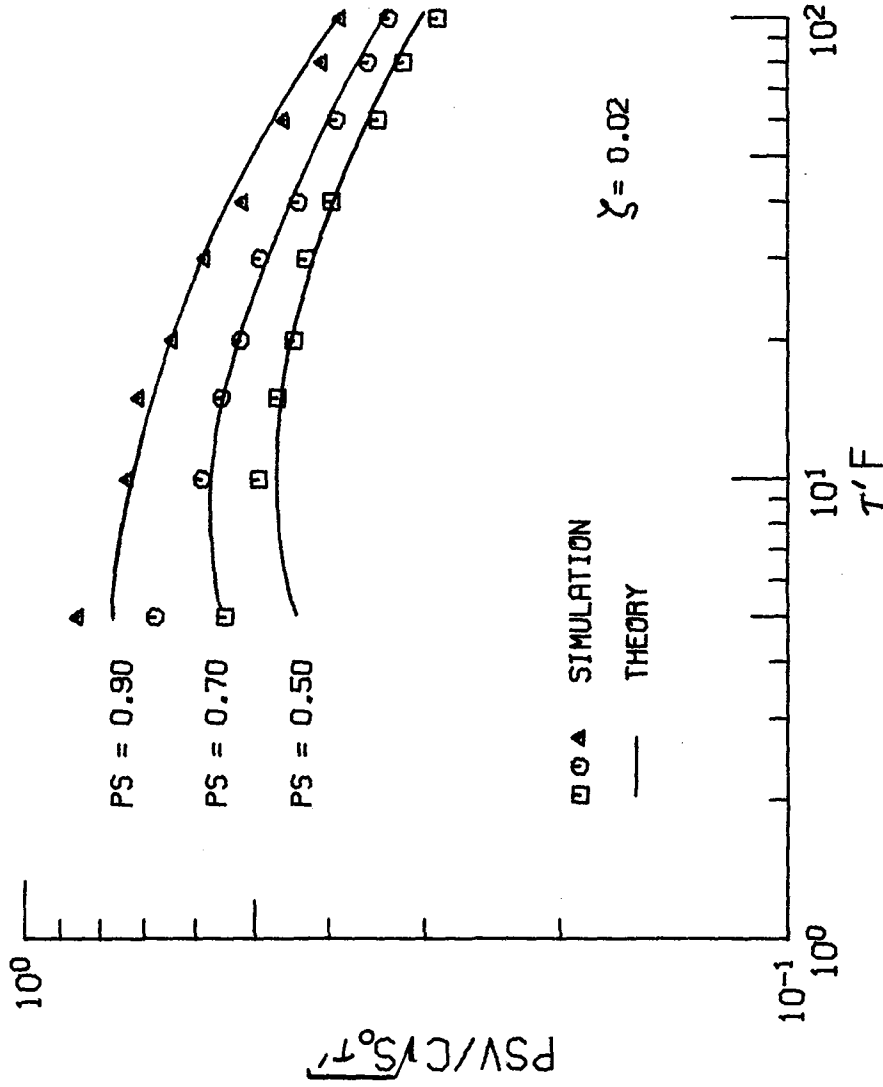
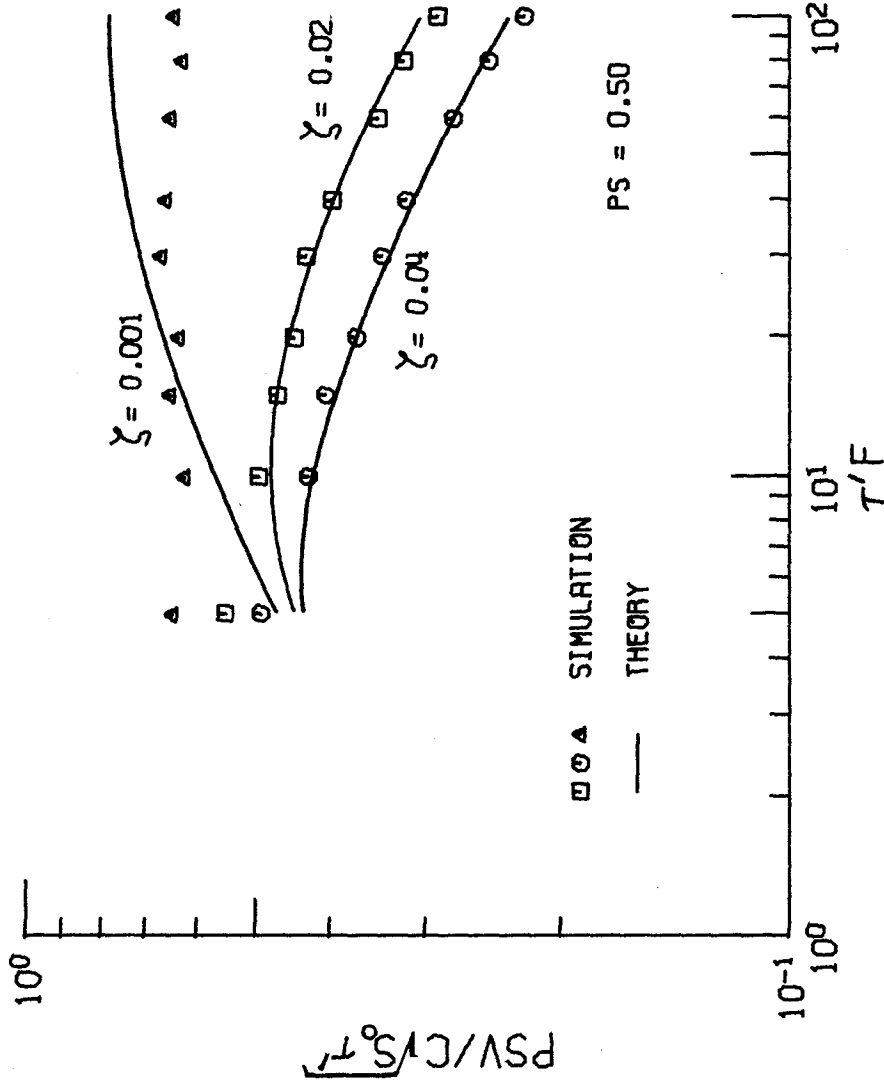Fig. 3.9 Response Spectra for Modulated White Noise Excitation; Variation with Reliability Value, $\zeta = 0.02$.

Fig. 3.10 Response Spectra for Modulated White Noise Excitation; Variation with Damping Ratio, Ps = 0.50.

considered in these response spectra. This is almost surely attributable to the fact that, at this frequency, the earthquake envelope exhibits relatively large changes over one natural oscillator period. Hence, the response probability density also changes rapidly over a cycle, which violates one of the assumptions of section (3.5.1).

Figure 3.10 shows the variation of response spectrum level with ζ for Ps = 0.5. It is seen that the theoretical and simulation results agree well for ζ = 0.04 and ζ = 0.02, when $\tau'F \geq 10$. For ζ = 0.001, however, it is apparent that the approximate analytical curve is relatively undependable. It is roughly 30% below the simulation results at the low-frequency end of the spectrum, and about the same percentage above at the high-frequency end. As discussed in section (3.4), appreciable error of this sort is always to be expected for exceedingly small damping ratios due to the resulting sensitivity of α to small inaccuracies in the evaluation of P*.

3.6   First Passage Probabilities for a Nonlinear Oscillator

Consider a nonlinear oscillator subject to modulated Gaussian white noise, with equation of motion

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x + \epsilon h(x,\dot{x}) = \theta(t)w(t) \qquad (3.110)$$

If this system is lightly damped and if the nonlinearity parameter ϵ is sufficiently small, the response trajectories of (3.110) will often be quasi-sinusoidal in appearance provided $\theta(t)$ does not vary

too rapidly in time. When this is the case, the methods of Chapter II may be used to create a relatively simple extension of the procedure described in section (3.5.1) to determine approximate first passage probabilities for the system (3.110).

The equivalent linear system for equation (3.110) may be written

$$\ddot{x} = A[S(t)]\underset{\sim}{x} + \underset{\sim}{\theta}(t)w(t) \qquad (3.111)$$

where

$$\underset{\sim}{x} = \left\{ \begin{array}{c} x \\ \dot{x} \end{array} \right\} , \qquad (3.112)$$

$$\underset{\sim}{\theta}(t) = \left\{ \begin{array}{c} 0 \\ \theta(t) \end{array} \right\} , \qquad (3.113)$$

and

$$A[S(t)] = \left[\begin{array}{cc} 0 & 1 \\ -\left(\omega_0^2 + \epsilon E\left[\frac{\partial h}{\partial x}\right]\right) & -\left(2\zeta\omega_0 + \epsilon E\left[\frac{\partial h}{\partial \dot{x}}\right]\right) \end{array} \right]$$
$$(3.114)$$

Hence, the covariance matrix Q(t) may be obtained by solving the corresponding matrix differential equation (2.66). The nonstationary probability density of the response $p_r(x, \dot{x}, t)$ will then be given by equation (3.75), and ν_b, the rate of up-crossings of the level b, will be given by equation (3.81).

Observing the form of equation (3.114), it is natural to define the instantaneous equivalent "natural" frequency ω_e and damping ratio ζ_e by

$$\omega_e^2 \equiv -a_{21} = \omega_0^2 + \epsilon E\left[\frac{\partial h}{\partial x}\right]$$

$$2\zeta_e \omega_e \equiv -a_{22} = 2\zeta\omega_0 + \epsilon E\left[\frac{\partial h}{\partial \dot{x}}\right]$$

(3.115)

Equations (3.12) imply

$$\omega_e = \left(\omega_0^2 + \epsilon E\left[\frac{\partial h}{\partial x}\right]\right)^{\frac{1}{2}}$$

(3.116)

and

$$\zeta_e = \frac{1}{\omega_e}\left(\zeta\omega_0 + \frac{\epsilon}{2} E\left[\frac{\partial h}{\partial \dot{x}}\right]\right)$$

(3.117)

Following section (3.5.1), suppose that p_r varies slowly in time so that it is nearly stationary over an interval of a single oscillator "period" $\left[t, t + \frac{2\pi}{\omega_e}\right]$. Then, over such an interval, ω_e and ζ_e will be nearly constant, and the system (3.110) will behave approximately like an autonomous linear oscillator with natural frequency ω_e and damping ratio ζ_e.

From this, a procedure analogous to that of section (3.5.1) follows directly. In the derivation of $P*(t)$, the transition probability density $q(x_0 | x; \pi/\omega_d)$ will still be given by equation (3.82), except that σ' and μ' must now be redefined as

$$\sigma' = \left[\frac{\pi S_0}{2\zeta_e \omega_e^3}\left(1 - \exp\left(-2\pi\zeta_e\omega_e/\omega_d\right)\right)\right]^{\frac{1}{2}}$$

(3.118)

$$\mu' = x_0 \exp\left(-\pi\zeta_e\omega_e/\omega_d\right)$$

(3.119)

where ω_d is given by

$$\omega_d = \omega_e \left(1 - \zeta_e^2 \right)^{\frac{1}{2}} \qquad (3.120)$$

P*(t) may hence be derived by the procedure of section (3.4).

Thus, equation (3.78) generates $\alpha(t)$ and the first passage probability may be found using equation (3.87).

To check the accuracy of this method, an example was performed with the cubic hardening system used in the simulation study of section (2.5). The equation of motion for this system is

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 (x + \epsilon x^3) = t \exp(-\gamma t) w(t) \qquad (3.121)$$

In this case

$$\omega_e = \omega_0 \left(1 + 3\epsilon q_{11} \right)^{\frac{1}{2}} \qquad (3.122)$$

and

$$\zeta_e = \zeta \qquad (3.123)$$

As in section (3.5), γ was chosen so that

$$\frac{\gamma}{\omega_0} = \frac{1}{10\pi}$$

and the duration τ' of the excitation was

$$\tau' = \frac{4}{\gamma}$$

The damping ratio ζ was 2%.

The same simulated time histories that were generated for the example of section (2.5) were used in this study to obtain the probability distributions of the maximum displacement $|x_m|$ corresponding to each value of ϵ. The ensemble size M was 200 for each ϵ value.

In analogy to the response spectrum notation, let $SD(\epsilon, Ps)$ represent the displacement magnitude which is exceeded by $(1 - Ps)M$ of the time histories in the ensemble having nonlinearity parameter ϵ. The value of $SD(\epsilon, Ps)$ was obtained from the simulation results for ϵ values ranging from 0 to 1.0, and Ps values of 0.5, 0.7, and 0.9. The corresponding values of SD were also computed by means of the approximate analytical method of this section.

It is noted that there are actually two independent sources of possible error inherent in this analytical technique. The approximations made to generate first passage probabilities of a linear oscillator provide one source, while the substitution of an equivalent linear system for the original nonlinear oscillator introduces another. The accuracy of the approximate first passage probabilities for the linear case has already been investigated in section (3.5.4). Hence, it would be desirable to uncouple the errors observed in section (3.5.4) from the new errors introduced by equivalent linearization. To do this, let $\Delta(\hat{\epsilon}, Ps)$ be defined as

$$\Delta(\hat{\epsilon}, Ps) = \frac{SD(\hat{\epsilon}, Ps) - SD(0, Ps)}{SD(0, Ps)} \qquad (3.124)$$

Thus, Δ represents the relative change observed in SD as ϵ varies from zero to $\hat{\epsilon}$. Note that

$$\Delta(0, Ps) \equiv 0 \qquad\qquad (3.125)$$

Thus, a comparison of the value of Δ obtained by simulation with that resulting from the approximate analysis should give a better indication of the magnitude of the errors attributable to the equivalent linearization approximation than would a direct comparison of actual SD values.

Figure 3.11 shows $\Delta(\epsilon, Ps)$ plotted as a function of ϵ for the three values of Ps. The approximate analytical results show the correct qualitative behavior and are in reasonably good agreement with the values obtained by simulation. It is noted, however, that the discrepancies appear to grow as Ps increases. This tendency is probably attributable to the detailed shape of the peaks of the nonlinear oscillator response. The response resembles a sine wave with slowly varying amplitude and phase, except that the largest amplitude peaks are somewhat "flattened" as a result of the stiffening restoring force. This flattening tends to reduce the value of the absolute maximum excursion more severely than it does the mean square displacement. The equivalent linearization method replaces the actual oscillator with a system that exhibits more nearly sinusoidal trajectories, but which attains very nearly the same mean square displacement as the original oscillator. As a result, when one attempts to determine a displacement level corresponding to a value of Ps near unity, that is, a level which

is exceeded infrequently, the approximate analytical technique will generally predict a slightly higher level than is actually observed.
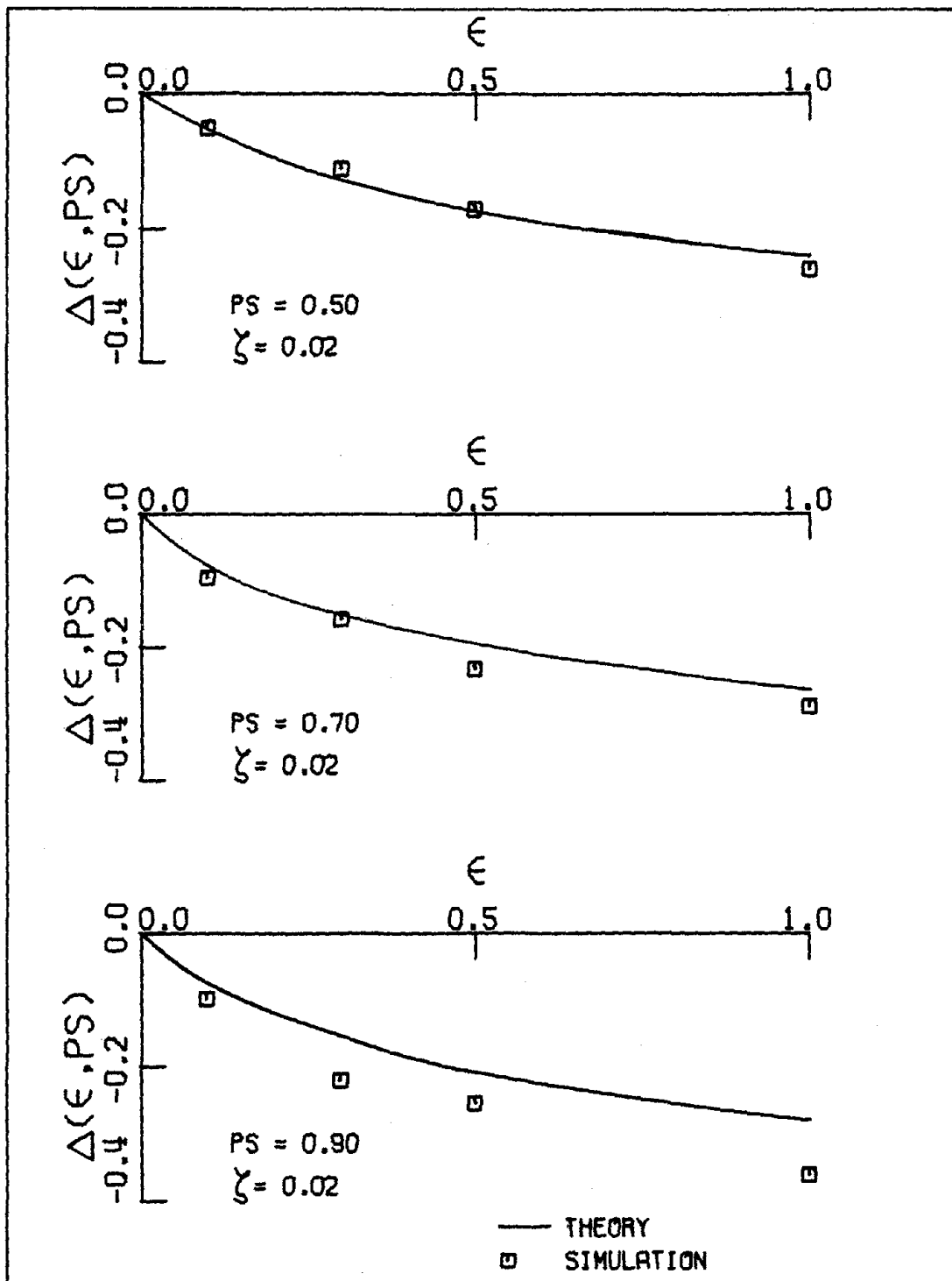
Fig. 3.11   Relative Change in Maximum Response versus
             Nonlinearity Parameter. Duffing Oscillator, ζ = 0.02.

CHAPTER IV

Summary and Conclusions

Considered in Chapter I is the problem of estimating the seismic response of a nonlinear dynamical system when the description of the input excitation is provided in the form of response spectra. A solution approach is proposed which breaks this problem down into two distinct parts:

(1) Find a random process that is amenable to analytical description which has reasonably earthquake-like sample functions, and which generates response spectra consistent with the data provided.

(2) Develop an analytical method to determine the stochastic response of the nonlinear system when it is excited by such a random process.

It is suggested that earthquake excitations may reasonably be modeled by a stationary Gaussian random process, with appropriate spectral density, multiplied by a judiciously chosen envelope function. With this model, problem (1) reduces to finding an envelope function and spectral density which generate a process that meets the requirements stated in (1). It is recognized that this requires a solution to the first passage problem for the nonstationary response of a linear oscillator.

Problem (2) becomes that of determining the nonstationary response of a nonlinear system excited by deterministically

modulated stationary Gaussian noise of arbitrarily specified spectral density. This problem is considered in Chapter II. To obtain an approximate solution, a method is proposed which is based on the concept of equivalent linearization. The equivalent linear system is treated conceptually as a time-varying system, though it is pointed out that the equivalent parameters are explicitly solution-dependent rather than time-dependent. This leads to a set of nonlinear integral-differential equations for the covariance matrix of the response. In the special case of white noise excitation, the problem reduces to the solution of a set of nonlinear differential equations.

In the last section of Chapter II, the method is illustrated by means of a specific example. The response of a Duffing oscillator to modulated white noise is considered. The method is used to compute the time history of the covariance matrix elements. The diagonal elements exhibit excellent agreement with the corresponding results of a Monte-Carlo simulation study. Although there are currently no analytical methods for generating useful error bounds for this approximate method, the evidence accumulated to date indicates that accuracy acceptable to engineering applications may generally be expected.

In Chapter III, the first passage problem for a linear oscillator excited by white noise is considered. The special case of the stationary response process is first considered. An approximate expression is developed for the limiting decay of the first passage probability density in terms of the expected "clump duration".

The probability distribution of clump durations is considered, and its qualitative behavior is surmised from physical considerations. It is undertaken to approximate this distribution with some member of a class of distributions which exhibit the proper qualitative behavior. The expected clump duration is then found to depend upon two undetermined quantities, one of which is considered a constant, to be evaluated empirically. The only other unknown is a conditional probability which is shown to be related to the eigenvalue of an integral equation having a non-symmetric kernel. Picard iteration is employed to determine the eigenvalue approximately.

This method is then extended to include the nonstationary response of a linear oscillator subject to modulated stationary Gaussian white noise. It is pointed out that, if the excitation is not white, the method may still be applied provided the spectral density of the excitation is sufficiently well-behaved.

The simulation studies described in Chapter III indicate that the method for generating limiting decay rates is relatively accurate provided the oscillator damping is not excessively small or the barrier level extremely high. The extension to nonstationary response demands the additional restriction that the response probability density may not change too extensively over a natural period of the oscillator. However, when the response density is slowly varying, and the damping ratio is a few percent, this analysis appears to predict first passage probabilities with accuracy sufficient for most applications.

The final section of Chapter III indicates how the equivalent linearization technique of Chapter II may be used to extend the approximate solution of the first passage problem to the response of a weakly nonlinear oscillator. A numerical example is performed with a hardening Duffing oscillator. The distribution of the maximum oscillator excursion is obtained both by the approximate analysis and from simulation results. The distributions obtained by these two methods are in good agreement over the range of exceedence probability investigated. It is noted, however, that the approximate analysis tends to be more in error as the exceedence probability becomes small. This tendency is interpreted as the effect of the deviation of the large-amplitude nonlinear oscillator peaks from sinusoidal appearance.

The principal motivation for using the approximate analytical methods described in this thesis is the savings in computational effort afforded. Indeed, the computer time required to obtain dependable results from a simulation study was generally several orders of magnitude greater than that which was required to obtain the corresponding approximate analytical result. Because of the generally favorable indications of these simulation studies, it is concluded that the approximate techniques developed in Chapters II and III constitute substantial progress toward a solution of the original problem posed in Chapter I.

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APPENDIX A

Efficiency Considerations in the Numerical Evaluation of
the Covariance Matrix

As shown in the text, when a nonlinear system is excited by modulated stationary Gaussian noise of arbitrarily specified spectral density $S(\omega)$, the covariance matrix is approximately described by the equation

$$Q(t) = \int_{-\infty}^{\infty} \underline{F}(\omega, t) \underline{F}^{*T}(\omega, t) S(\omega) \, d\omega \tag{A1}$$

where $\underline{F}(\omega, t)$ satisfies

$$\frac{d}{dt} [\underline{F}(\omega, t)] = A[S(t)] \underline{F}(\omega, t) + \underline{\theta}(t) \exp(i\omega t) \tag{A2}$$

$$\underline{F}(\omega, 0) = \underline{0} \tag{A3}$$

The matrix $A[S(t)]$ depends explicitly on the elements of Q as indicated by equations (2.79) and (2.56). Hence, numerical evaluation of the improper integral of equation (A1) would be a very costly numerical procedure since equation (A2) must be solved numerically over the range of ω for which $S(\omega) \neq 0$. Furthermore, the elements of $\underline{F}(\omega, t)$ tend to oscillate quite rapidly as t becomes large, thereby requiring more and more evaluations of $\underline{F}(\omega, t)$ as t increases in order to obtain accurate numerical values for the elements of $Q(t)$.

The purpose of this appendix is to derive a differential equation for $Q(t)$ which is more amenable to numerical solution than equations (A1), (A2), and (A3).

Let $\underset{\sim}{G}(\omega, t)$ be defined as

$$\underset{\sim}{G}(\omega, t) = [S(\omega)]^{\frac{1}{2}} \underset{\sim}{F}(\omega, t) \tag{A4}$$

Then equation (A1) implies

$$Q(t) = \int_{-\infty}^{\infty} \underset{\sim}{G}(\omega, t) \underset{\sim}{G}^{*T}(\omega, t) \, d\omega \tag{A5}$$

Clearly, $\underset{\sim}{G}(\omega, t)$ must satisfy

$$\frac{d}{dt} [\underset{\sim}{G}(\omega, t)] = A[S(t)] \underset{\sim}{G}(\omega, t) + \underset{\sim}{\theta}(t) [S(\omega)]^{\frac{1}{2}} \exp(i\omega t) \tag{A6}$$

Taking the complex conjugate transpose of equation (A6) yields

$$\frac{d}{dt}[\underset{\sim}{G}^{*T}] = \underset{\sim}{G}^{*T} A^T + \underset{\sim}{\theta}^T(t) [S(\omega)]^{\frac{1}{2}} \exp(-i\omega t) \tag{A7}$$

where the arguments of G and A have been deleted for convenience. Noting that

$$\frac{d}{dt}[\underset{\sim}{G}\underset{\sim}{G}^{*T}] = \frac{d}{dt}[\underset{\sim}{G}]\underset{\sim}{G}^{*T} + \underset{\sim}{G}\frac{d}{dt}[\underset{\sim}{G}^{*T}] , \tag{A8}$$

equations (A6) and (A7) lead to

$$\frac{d}{dt}[\underset{\sim}{G}\underset{\sim}{G}^{*T}] = A[\underset{\sim}{G}\underset{\sim}{G}^{*T}] + [\underset{\sim}{G}\underset{\sim}{G}^{*T}]A^T + [S(\omega)]^{\frac{1}{2}} \exp(i\omega t) \underset{\sim}{\theta}(t) \underset{\sim}{G}^{*T}$$

$$+ [S(\omega)]^{\frac{1}{2}} \exp(-i\omega t) \underset{\sim}{G} \underset{\sim}{\theta}^T(t) \tag{A9}$$

Integrating both sides of equation (A9) with respect to ω over the

interval $(-\infty, \infty)$ and using equations (A4) and (A5) yields

$$\frac{dQ}{dt} = AQ + (AQ)^T + \underline{\Phi}(t)\underline{\theta}^T(t) + \underline{\theta}(t)\underline{\Phi}^T(t) \tag{A10}$$

where

$$\underline{\Phi}(t) = \int_{-\infty}^{\infty} S(\omega) \exp(-i\omega t)\underline{F}(\omega,t)d\omega \tag{A11}$$

Let the fundamental solution matrix Y(t) be defined by

$$\frac{d}{dt}[Y(t)] = A[S(t)]Y(t)$$

$$\tag{A12}$$

$$Y(0) = I$$

Then, from equations (A2) and (A3), $F(\omega,t)$ may be expressed

$$\underline{F}(\omega,t) = Y(t)\int_{0}^{t} Y^{-1}(s)\underline{\theta}(s) \exp(i\omega s)ds \tag{A13}$$

Substituting from equation (A13) into equation (A11), and reversing the order of integration yields

$$\underline{\Phi}(t) = Y(t)\int_{0}^{t}\left[\int_{-\infty}^{\infty} S(\omega) \exp[i\omega(s-t)]d\omega\right] Y^{-1}(s)\underline{\theta}(s)ds \tag{A14}$$

It will be recognized that

$$\int_{-\infty}^{\infty} S(\omega) \exp[i\omega(s-t)]d\omega = R(s-t) \tag{A15}$$

where R is the autocorrelation function of the excitation. Hence, $\underline{\Phi}(t)$ may be expressed as

$$\underline{\Phi}(t) = Y(t) \int_0^t Y^{-1}(s)\,\underline{\theta}(s)\,R(s-t)\,ds \qquad (A16)$$

Equation (A10) may now be solved simultaneously with equations (A12) and (A16) to generate Q(t). The integration in equation (A16) must generally be performed numerically at each time step in the differential equation (A10). Thus, considerable computational effort will still be required to obtain Q(t); however, the procedure will not be as costly using this formulation as it would if the integration over the frequency domain in equation (A1) were attempted directly.