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MODAL ANALYSIS FOR STRUCTURES WITH FOUNDATION INTERACTION

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Presented at ASCE National Structural Engineering Meeting, Cleveland, Ohio, April, 1972

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Figure 1 shows a schematic model often used as the basis for dynamic analysis of a structure resting on a deformable foundation. The structure itself is represented by discrete masses connected by springs and dashpots. The soil or rock is replaced by two springs and corresponding dashpots: one set corresponding to swaying (horizontal motion) and the other to rocking. Generally, the damping for the soil springs, and especially for the swaying spring, is larger than the damping in the structure. Moreover, as will be discussed subsequently, the damping in some parts of this system is viscous in nature while in other parts the damping is more nearly hysteretic.

Because damping varies as to magnitude and type, classical modal analysis generally is not strictly applicable to a structure-soil system. For any given mathematical model, an exact solution can be obtained by working in the frequency domain: i.e., using Fourier analysis (8). However, such methods sacrifice the considerable advantages of modal superposition:

a) Most practicing engineers involved in the dynamic analysis of buildings are familiar with modal analysis and the proper interpretation of the ensuing results.

b) Modal analysis permits better visualization from the values of the natural frequencies and the modal shapes, of the significance

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of the flexible foundation on the response.

c) When the input is specified in the form of a response spectrum, modal analysis is a direct and logical procedure in spite of the inaccuracy involved in the combination of modal maxima.

Because of these practical advantages, it often will be desirable to use modal superposition even though it does not give entirely accurate results.

In order to apply modal superposition to the soil-structure system of Fig. 1, it is necessary to have a rule for assigning an equivalent fractional critical damping β_{ieq} to each mode. This paper proposes that β_{ieq} be computed by the following <u>weighted modal damping</u> rule:

$$\beta_{ieq} = \frac{1}{4\pi} \frac{\sum_j ER_{ij} E_{s_{ij}}}{\sum_j E_{s_{ij}}} \tag{1}$$

where $E_{s_{ij}}$ is the energy stored in the j^{th} component of the system when the system deforms in the i^{th} mode, and ER_{ij} is an <u>energy ratio</u>: the ratio of the energy dissipated in the j^{th} component to the energy stored in the component. The relation of ER_{ij} to the usual measures of damping, and the theoretical justification for this rule, will be discussed.

Use of modal superposition together with eq. (1) involves two types of errors.

a) The structure-soil system will not, in general, have normal modes in the classical sense, and this difficulty can at best only partly be overcome by using weighted modal damping. (It is possible to retain modal superposition by working in complex arithmetic, thus transforming the system into one with twice the number of degrees of freedom (1,4), but this procedure sacrifices the advantages cited above).

b) In order to carry out modal superposition in the time domain, the damping in each mode must be assumed to be viscous. However, the actual damping in parts of the system may be more nearly hysteretic, rather than viscous, in nature.

In the next section of this paper, the potential effect of the second of these errors is investigated by comparing the response of single-degree-of-freedom (1-DOF) systems with hysteretic and viscous damping. In the final section, the combined effects of both errors is studied for multi-DOF structure-soil systems, by comparing results obtained using weighted modal damping with results from exact solutions in the frequency domain.

HYSTERETIC AND VISCOUS DAMPING

Viscous Damping

The concept of viscous damping, as represented by a dashpot with a resistance proportional to the velocity, is a familiar one to engineers involved in dynamic analyses. It is worth noticing, however, that viscous damping is used in structural dynamics because of its mathematical simplicity (allowing analytical closed-form solutions), rather than on the basis of its physical significance.

The physical characteristic of viscous damping is the viscosity coefficient or dashpot constant c. The fraction of critical damping $\beta = c/2\sqrt{kM}$ for a 1-DOF system depends not only on the dashpot constant but also on the mass M and the spring constant k. Thus the same dashpot would produce different values of β for two systems with different masses but the same stiffness.

The significance of the fraction of critical damping is better understood by considering the energy E_d dissipated per cycle by a 1-DOF system with viscous damping, under a steady state harmonic motion at a frequency Ω. Calling u the displacement and $\dot{u}$ the velocity; and denoting the phase angle by θ:

$$u = A \sin (\Omega t + \theta) \qquad \dot{u} = A\,\Omega\cos (\Omega t + \theta) \tag{2}$$

and

$$E_d = \int_0 c\dot{u}^2 dt = \frac{T}{2}\, cA^2\Omega^2 = \pi c\Omega A^2 \tag{3}$$

The maximum strain energy E_s stored in the spring is $E_s = \frac{1}{2}\, kA^2$. Defining an energy ratio $ER = E_d/E_s$:

$$ER = 2\pi \frac{c}{k}\, \Omega = 4\pi\beta\frac{\Omega}{\omega} \tag{4}$$

where $\omega = \sqrt{k/M}$ is the natural frequency. This energy ratio varies directly as the frequency Ω.

Hysteretic Damping

Most engineering materials, including soils, exhibit a hysteretic stress-strain diagram when strained cyclically. In each cycle energy is dissipated, represented by the area within the hysteresis loop. This energy loss is a function of the amplitude, but experiments show (4) that it is to a large extent independent of frequency.

A rigorous solution of systems having hysteretic damping requires a nonlinear dynamic analysis in the time domain. Their behavior under a steady-state harmonic motion of fixed amplitude suggests, however, the idea of a type of damping, called linear hysteretic damping (3), that causes an energy dissipation (and energy ratio) independent of frequency. This formulation has appeared many times in the literature under many different names; the name structural damping has often been used (7). Using complex notation, the equation of motion for a 1-DOF system is:

$$M\ddot{u} + k(1+2Di)u = f(t) = Pe^{i\Omega t} \tag{5}$$

where D is a hysteretic damping ratio (also called loss factor).

The energy dissipated per cycle by this system is:

$$E_d = 2\pi k D A^2 \tag{6}$$

and the energy ratio is independent of frequency:

$$ER = 4\pi D \tag{7}$$

Thus either D or ER provides a physical characteristic defining damping in a system with hysteresis. While Eq.(5) is properly defined only for a steady-state harmonic motion, by decomposing the input into its frequency components through the Fourier transform, the response to a transient excitation may be evaluated.

Equivalent Damping

Making use of the relation $\dot{u} = i\Omega u$, eq. (5) can be rewritten as:

$$M\ddot{u} + 2kD\frac{1}{\Omega}\dot{u} + ku = Pe^{i\Omega t} \tag{8}$$

Eq. (8) is similar to that of a viscous system with a dashpot constant $c = 2kD\frac{1}{\Omega}$. The response of a hysteretic 1-DOF system is:

$$A = \frac{P}{(k-M\Omega^2) + 2Dki} = \frac{P/k}{(1-\frac{\Omega^2}{\omega^2}) + 2Di} \tag{9}$$

whereas for a viscous system the corresponding expression would be:

$$A = \frac{P}{(k-M\Omega^2) + ic\Omega} = \frac{P/k}{(1-\frac{\Omega^2}{\omega^2}) + 2\beta i\frac{\Omega}{\omega}} \tag{10}$$

Comparing the steady-state harmonic response of two 1-DOF systems, with the same mass and stiffness but one with viscous damping and the other with hysteretic damping, it can be seen that the motions will be equal at only one frequency. This is the frequency Ω, which provides the same energy ratio for both systems

$$4\pi D = 4\pi\beta\frac{\Omega}{\omega} \qquad \text{or} \qquad \Omega = \frac{D}{\beta}\,\omega \tag{11}$$

Since in most problems the damping ratios D are small, these systems behave essentially as narrow-band filters, and their response to a transient input is primarily controlled by the values of the transfer function in the neighborhood of resonance $(\frac{\Omega}{\omega} = 1)$. It would seem therefore logical to define them as equivalent when they produce the same energy ratio at resonance, that is, when

$$D = \beta \tag{12}$$

Fig. 2 shows the amplitude and phase angle of the transfer function (ratio $\frac{A}{P}$ previously defined) for two such systems with $D = \beta = 0.1$. While both functions are equal at resonance $(\Omega/\omega = 1)$, they differ for other values of Ω/ω. This difference is more noticeable for large values of Ω/ω, and particularly evident for the phase angle (notice however the logarithmic scale used for the amplitude).

In most cases, however, differences in the response to a transient excitation are much smaller than the differences in the transfer function. Fig. 3 shows the inverse Fourier transforms of the previous functions. For the viscous case this inverse transform represents properly the impulse response function, which would be used for a solution in the time domain with Duhamel's integral. For the hysteretic case it can be considered as an equivalent impulse response, although this is an approximation since the equation of motion is not properly defined in the time domain. The differences in these two functions are negligible and only distinguishable for $t \ll 2\pi/\omega$. Response spectra obtained by integration in the time domain for viscous systems with a fraction of critical damping β, and by integration in the frequency domain for hysteretic systems with a damping ratio $D = \beta$, are also practically identical, except for relatively long periods.

Thus the second error mentioned in the introduction, i.e., considering the modal damping as entirely viscous while it may be in part viscous and in part hysteretic, is of little practical importance, provided that the contributions of each component's damping to the modal values are properly computed.

SIMPLE TWO-DEGREE-OF-FREEDOM SYSTEM

Now consider a simple 2-DOF system formed by two identical 1-DOF systems. For each 1-DOF system individually, the natural frequency is $\omega = \sqrt{k/M}$. For the coupled system, the natural frequencies are:

$$\omega_1 = \frac{1}{2}\,\omega(\sqrt{5} - 1) \qquad \omega_2 = \frac{1}{2}\,\omega(\sqrt{5} + 1) \tag{13}$$

Modal superposition in the frequency domain rigorously applies to the coupled system for both viscous and hysteretic damping (4), but only for viscous damping can superposition be used in the time domain.

First let us assume that the 1-DOF systems each have viscous damping β. Then the fractions of critical damping for the two modes are:

$$\beta_1 = \beta\frac{\omega_1}{\omega} = \frac{\sqrt{5}-1}{2}\,\beta \qquad \beta_2 = \beta\frac{\omega_2}{\omega} = \frac{\sqrt{5}+1}{2}\,\beta \tag{14}$$

Thus the β_i for the two modes are different, and each is different from the β for the component 1-DOF systems.

On the other hand, now suppose that each 1-DOF system has hysteretic damping D. Then the damping ratio for each mode of the 2-DOF system also is D. If damping in a multi-DOF system is linear hysteretic and uniform, then the damping for the components is the same as the damping for the system.

For both viscous and hysteretic damping, the energy ratio is the same for each component and for the overall system. However, with viscous damping ER for both components and system varies with frequency, whereas with hysteretic damping ER is independent of frequency.

Since the energy ratios are the same for both component 1-DOF systems, Eq. (1) reduces to

$$\beta_{ieq} = \frac{1}{4\pi} ER = \beta \frac{\omega_i}{\omega} \text{ or } = D \qquad (15)$$

which predicts correctly the modal damping for both cases.

DEFINITION OF DAMPING IN SOIL-STRUCTURE SYSTEM

The most troublesome aspect of analyzing soil-structure interaction is defining the damping in the system in a useful, meaningful way. To use Eq. 1, it is necessary to know the energy ratio for each component.

Damping in Soil

Using results from the theory for a mass resting upon an elastic half-space (7), the force deflection equations for the soil-structure interface may be written, neglecting coupling terms, as:

$$\left.\begin{aligned} P &= k_h u + c_h \dot{u} \\ M_r &= k_r \phi + c_r \dot{\phi} \end{aligned}\right\} \qquad (16)$$

where P is a steady-state sinusoidal force applied to the foundation, M_r a steady-state sinusoidal moment, and u and ϕ the corresponding dispacement and rotation. The spring "constants" k_h and k_r and the dashpot "constants" c_n and c_r are, in reality, functions of frequency. However, k_h c_h and k_r vary smoothly with frequency and hence may be taken, without introducing appreciable error, as constants over the frequency range of interest. On the other hand, c_r has considerable variation, but the ratio c_r/k_r is small.

Adding now some internal dissipation of energy in the soil, of a hysteretic nature, represented by a damping ratio D, these equations become:

$$\begin{aligned} P &= k_h(1+2iD)u + c_h\dot{u} = k_h u + (c_h + \frac{2Dk_h}{\Omega})\ \dot{u} \\ M_r &= k_r(1+2iD)\phi + c_r\dot{\phi} = k_r\phi + (c_r + \frac{2Dk_r}{\Omega})\ \dot{\phi} \end{aligned} \qquad (17)$$

For the range of frequencies Ω of interest and typical values of D of about 0.05, the term c_h is much larger than $2Dk_h/\Omega$ and the latter can be neglected. On the other hand c_r is much smaller than $2Dk_r/\Omega$. For practical purposes it is thus reasonable to assume that the horizontal (swaying) spring is associated with a viscous dashpot c_h, whereas the rocking spring has a hysteretic dissipation of energy represented by the damping ratio D. Thus the equations simplify to:

$$\left.\begin{aligned} P &\approx k_h u + c_h\dot{u} \\ M_r &\approx k_r\phi + \frac{2Dk_r}{\Omega}\dot{\phi} \end{aligned}\right\} \qquad (18)$$

That is, the damping associated with swaying is essentially viscous, while the damping associated with rocking is primarily hysteretic.

Using these results, the meaning of critical damping ratios for the foundation may be understood. Since rocking damping is hysteretic, damping ratio is defined fully without reference to any mass. However, when it is said that the swaying spring has 20% of critical damping in a particular problem, it is necessary to define for what mass, or at which frequency. Notice, however, that as long as the fraction of critical damping β_h and the reference frequency ω_h are picked consistently, so that $\beta_h = \frac{1}{2}\frac{c_h}{k_h}\omega_h$, the results should always be the same.

Damping in Structure

If damping is assumed to be linear hysteretic and the same in all parts of the structure, the damping ratio D_i for each mode is the same as the damping ratio for each component $D_j = D$. The damping matrix C_H is then simply

$$C_H = 2DK \tag{19}$$

where K is the stiffness matrix, and the matrix equation for a steady-state motion is

$$M\ddot{U} + (K + iC_H)U = \text{exciting forces}$$

$$\text{or} \qquad M\ddot{U} + \frac{1}{\Omega} C_H\dot{U} + KU = \text{exciting forces} \tag{20}$$

If it is assumed, on the other hand, that the damping in each component is viscous, the matrix equation of motion is:

$$M\ddot{U} + C_V\dot{U} + KU = \text{exciting forces} \tag{21}$$

With this assumption it is possible to obtain solutions in the time domain, but it becomes very difficult to define the damping for each component in a meaningful way.

Two approaches can be followed to obtain the damping matrix C_V. In one approach which is not possible from the practical standpoint at this time, the component dampings are assumed directly and C_V is assembled as the stiffness or mass matrices. The resulting system will not have in general normal modes.

The second approach starts by assuming the existence of normal modes and assigning a fraction of critical damping β_i for each mode. The damping matrix is then given by (3, 10):

$$C_V = \sum_{j=\ell}^{\ell+n} a_j M(M^{-1}K)^j \tag{22}$$

where ℓ is an arbitrary integer and the coefficients a_j are obtained from the solution of the system of equations:

$$2\beta_i\omega_i = \sum_{j=\ell}^{\ell+n} a_j\omega_i^{2j} \tag{23}$$

Particular cases frequently used are

a) $$C_V = a_1K \tag{24}$$

The resulting modal dampings are then $\beta_i = \frac{1}{2} a_1\omega_i$, increasing linearly with frequency. This corresponds to the situation of the two-degree-of-freedom system used before as an example.

b) $$C_V = a_oM \tag{25}$$

The resulting modal dampings are then $\beta_i = \frac{1}{2} \frac{a_o}{\omega_i}$ and decrease with frequency. This corresponds to a case where each dashpot is associated with a mass (joining it to the base) rather than with a spring.

c) To approximate better a condition of constant modal damping, a linear combination of the two previous cases is used:

$$C_V = a_oM + a_1K \tag{26}$$

The resulting modal dampings are then $\beta_i = \frac{1}{2}\frac{a_o}{\omega_i} + \frac{1}{2} a_1\omega_i$.

In the first 2 cases the coefficients $a_1 a_0$ are selected so that the desired value of β is reached at a specified frequency. In the third case this value will also be obtained only at one frequency setting a relation between a_0 and a_1. A second relation can be obtained by specifying an additional property of the function $\beta(\omega) = \frac{1}{2}\frac{a_o}{\omega} + \frac{1}{2} a_1\omega$. The variation of this function is illustrated for the three cases in Fig. 4. It is worthwhile noticing that the third solution has the convenient, but possibly dangerous, property of rapidly filtering low and high frequency components.

Alternatively the matrix transformation

$$C_V = MQBQ^TM \tag{27}$$

where Q is the modal matrix containing the i^{th} modal shape, normalized, as its i^{th} column, and B is a diagonal matrix $B_{ii} = 2\beta_i\omega_i$, will also produce the desired nodal dampings in each mode. When all the modal shapes are known Eq. (27) provides a simpler solution than Eq. (22), but in both cases the physical meaning of the terms of the C_V matrix is difficult to visualize.

Combined Damping

For the remainder of this study, damping for the soil-structure system will be assumed to follow the pattern indicated in Fig. 1. While in this figure the structure is idealized as a close-coupled system (shear type building) this is not a requirement. Neither is it required that the structure have constant modal damping of a hysteretic nature. The basic assumption for the study is that the energy dissipation can be reproduced by associating an energy ratio (viscous or hysteretic) with each spring (or stiffness term). This implies that for the schematic system of Fig. 1 one can define for each spring j a hysteretic damping ratio D_j and a viscous dashpot c_j (or a fraction of critical damping β_j and a reference frequency ω_j) where the dashpot force is proportional to the rate of strain in the spring. The situation shown in the figure, with a constant modal damping for the structure (a constant hysteretic damping ratio for each structural element), a hysteretic damping ratio for the rocking spring and a viscous dashpot for the swaying spring is, however, probably the most realistic.

As pointed out earlier in defining the viscous damping for the swaying spring in the form of a fraction of critical damping β, it is necessary to specify also a mass or a reference frequency. In this study the value of β_h is defined as the one corresponding to the so-called swaying frequency, ω_h, which would be the natural frequency of a rigid structure with the same total mass and a rigid rocking spring. It corresponds thus to taking the total mass of the building and its foundation. In the same way the rocking frequency ω_r would be defined as the frequency of a rigid structure with the same total mass and base moment of inertia and a rigid swaying spring.

WEIGHTED MODAL DAMPING

Let then M represent the mass matrix and K the stiffness matrix of the soil-structure system. If ω_i are the natural frequencies and ϕ_i the corresponding modal shapes (normalized so that $\phi_i^T M \phi_i = 1$) computed by a regular dynamic analysis, it is possible to define for each spring j a modal strain Δ_{ij}, corresponding to a set of displacements $U = \phi_i$. Assuming a steady-state periodic motion at a frequency Ω, $U = A\phi_i \sin(\Omega t + \theta)$, and the

the maximum strain energy would be

$$E_{s_i} = A^2 \sum_j \frac{1}{2} k_j \Delta_{ij}^2 \tag{28}$$

and the energy dissipated per cycle

$$E_{d_i} = 2\pi A^2 (\Omega \sum_j k_j \beta_j \frac{1}{\omega_j} \Delta_{ij}^2 + \sum_j k_j D_j \Delta_{ij}^2) \tag{29}$$

The first summation represents the viscous dissipation of energy, the second the hysteretic losses. An energy ratio can thus be defined for the i^{th} mode

$$ER_i = 4\pi\Omega \frac{\sum_j k_j \beta_j \frac{1}{\omega_j} \Delta_{ij}^2}{\sum_j k_j \Delta_{ij}^2} + \frac{\sum_j k_j D_j \Delta_{ij}^2}{\sum_j k_j \Delta_{ij}^2} \tag{30}$$

Assuming on the other hand in each mode i a damping ratio D_i' and a fraction of critical viscous damping β_i' at the natural frequency ω_i

$$ER_i = 4\pi\beta_i' \frac{\Omega}{\omega_i} + 4\pi D_i' \tag{31}$$

Equating these 2 expressions

$$\beta_i' = \frac{\sum_j \beta_j \frac{\omega_i}{\omega_j} k_j \Delta_{ij}^2}{\sum_j k_j \Delta_{ij}^2} \tag{32}$$

and

$$D_i' = \frac{\sum_j D_j k_j \Delta_{ij}^2}{\sum_j k_j \Delta_{ij}^2} \tag{33}$$

Thus, if normal modes exist, each mode should have a viscous fraction of critical damping β_i' at its natural frequency ω_i and a hysteretic damping ratio D_i', as defined by the above equations.

For a one-degree-of-freedom system (as represented by a modal equation), interchanging a hysteretic damping ratio D by a fraction of critical viscous damping $\beta = D$ makes in general a negligible difference. Thus each mode can be considered to have an equivalent viscous damping β_{ieq}

$$\beta_{ieq} = \frac{\sum_j \left(\beta_j \frac{\omega_i}{\omega_j} + D_j\right) k_j \Delta_{ij}^2}{\sum_j k_j \Delta_{ij}^2} \tag{34}$$

Calling $E_{s_{ij}} = \frac{1}{2} k_j \Delta_{ij}^2 A^2$ the maximum strain energy in spring j under a steady-state periodic motion with the shape of the i^{th} mode:

$$\beta_{ieq} = \frac{\sum_j \left(\beta_j \frac{\omega_i}{\omega_j} + D_j\right) E_{s_{ij}}}{\sum_j E_{s_{ij}}} \tag{35}$$

This formula can also be interpreted as stating that the energy ratio in each mode at resonance ($\Omega = \omega_i$) is a weighted average of the energy ratios in each individual component at the same frequency, where the weighting factors are the individual energy terms $E_{s_{ij}}$.

Eq. (35) is the weighted modal damping rule foreseen by Eq. (1). This general form of rule had been suggested earlier by Biggs (6) who used terms only in the form of $D_j(\beta_j = 0)$. The earlier rule was thus equivalent to assuming only hysteretic damping. Eq. (35) extends it to include both viscous and hysteretic damping.

Relation to Normal Mode Theory

Combining Eqs. (20) and (21), the equations of motion for the soil-structure system can be written in a general form as:

$$\left.\begin{aligned} &M\ddot{U} + C_V\dot{U} + (K + iC_H)U = \text{exciting forces} \\ \text{or}\qquad &M\ddot{U} + \left(C_V + \frac{1}{\Omega} C_H\right)\dot{U} + KU = \text{exciting forces} \end{aligned}\right\} \tag{36}$$

where C_V and C_H are damping matrices, the first one corresponding to the viscous dissipation of energy, the second one to the hysteretic losses. For the conditions assumed, both matrices are assembled in the same way as the stiffness matrix K.

The condition for this system to have normal modes, in the classical sense, is that $\phi_i^T C_V \phi_j$ and $\phi_i^T C_H \phi_j = 0$ for $i \neq j$, or if Q is the modal matrix (with the modal vectors ϕ_i as columns), that the two matrices $Q^T C_V Q$ and $Q^T C_H Q$ be diagonal. If this condition is satisfied, each mode i will have a fraction of critical viscous damping β_i' and a hysteretic damping ratio D_i' given by

$$\left.\begin{array}{ll} \beta_i' = \frac{1}{2\omega_i} \phi_i^T C_V \phi_i & D_i' = \frac{1}{2\omega_i^2} \phi_i^T C_H \phi_i \\ \text{and} \quad \beta_{ieq} = \beta_i' + D_i' = \frac{1}{2} \left(\frac{1}{\omega_i} \phi_i^T C_V \phi_i + \frac{1}{\omega_i^2} \phi_i^T C_H \phi_i\right) & \end{array}\right\} \qquad (37)$$

It can be shown that Eqs. (37) are identical to Eqs. (32), (33), (34). The rule suggested provides therefore the correct values of modal damping if normal modes exist. For the general case, when normal modes do not exist, the rule is equivalent to neglecting the off-diagonal terms in the matrices $Q^T C_V Q$, $Q^T C_H Q$.

EXAMPLES

To determine the validity of the suggested rule, several typical cases were analyzed. For each case, three different analyses were performed; one in the frequency domain using the actual damping matrices C_V and C_H, providing what will be referred to as "exact" solution. In the second analysis normal modes were assumed, neglecting in effect the off-diagonal terms of matrices

$Q^T C_V Q$ and $Q^T C_H Q$; viscous and hysteretic damping terms were kept, however, separate, and the solution was again obtained in the frequency domain. This solution is referred to as modal superposition in the frequency domain. Finally in the third analysis a modal solution was obtained by using Eq. (34) and considering all the modal damping to be viscous. This is referred to as modal superposition in the time domain. The third type of analysis is the one which would be used in practice.

Comparison of results for the first and second analyses shows the error introduced by assuming normal modes. Comparison of the second and third analyses shows the additional error introduced by replacing hysteretic modal damping by viscous modal damping. Comparison of the first and third solutions indicates the overall error resulting from use of weighted modal damping.

Rigid Structures

A set of three rigid cylinders with variable height H were first considered. The excitation was an artificial earthquake with characteristics similar to those of the El Centro 1940 earthquake but a peak acceleration of 0.1g. Soil properties and dampings were as in Fig. 1. (The factor $\sqrt{2}$ in C_s is a strain reduction factor.) The acceleration at the top and at the bottom of the cylinder, and the corresponding transfer functions, were compared. Table 1 shows the values of the peak top accelerations for the three cylinders. The maximum difference is about 10%, and results mostly from the assumption of normal modes, rather than from the equivalencing of viscous and hysteretic terms.

5-Story Building

This building rested on a very soft soil with a shear wave velocity of $300/\sqrt{2}$ feet per second. Table 2 shows the natural frequencies of this

TABLE 1 - EXACT AND APPROXIMATE RESULTS - COUPLED ROCKING AND SWAYING - RIGID CYLINDERS

	Peak Acceleration at Top		
	H = 75 ft.	H = 225 ft.	H = 450 ft.
Exact	0.18g	0.27g	0.17g
Modal superposition in the frequency domain	0.16g	0.25g	0.17g
Modal superposition in the time domain	0.16g	0.245g	0.165g

system. f_R represents the rocking frequency and f_H the swaying frequency as defined previously. f_{RH}^{I} and f_{RH}^{II} are the coupled rocking-swaying frequencies (rigid structure) and f_1, f_2, and f_3 the first three natural frequencies of the soil-structure system. The values shown for rigid foundation would be the frequencies of the structure alone.

It can be seen that the flexibility of the foundation reduces the fundamental frequency by a factor of about 2, and that a considerable coupling should be expected between swaying and rocking since f_R and f_H are very close. Because of this large interaction and the considerable difference between damping values (50% viscous for the swaying spring at f_H, 5% hysteretic for the rocking spring, 3% hysteretic for all modes of the structure) this should be an unfavorable case for the suggested rule. The results from the three analyses, in terms of peak accelerations at the bottom of the foundation (swaying acceleration) and the top of the structure, and peak forces in the top and bottom springs, are shown in Table 3. The maximum difference occurs in the force in the top spring and is of the order of 20%. Differences in the accelerations and base shear are of the order of 10%.

15-Story Building

This shear-type building, shown in Fig. 1, was studied for a range of soils with a shear wave velocity from $800/\sqrt{2}$ to $2000/\sqrt{2}$ ft. per second. The corresponding frequencies are shown in Table 4 (f_{RH} representing the first of the two coupled rocking-swaying frequencies). As could be expected, the differences between the approximate and exact solution were negligible for the stiffer soils (little interaction), and for the softer soil the maximum error, again in the peak force in the top spring, was less than 5%. The results for this case are shown in Table 5.

TABLE 2 - FREQUENCIES (cps) FOR 5-STORY BUILDING WITH SOIL-STRUCTURE INTERACTION

	f_R	f_H	f_{RH}^{I}	f_{RH}^{II}	f_1	f_2	f_3
$C_s = 300/\sqrt{2}$ cps	2.4	2.2	1.8	4.4	1.6	4.9	9.0
rigid foundation					3.3	10.0	16.5

TABLE 3 - EXACT AND APPROXIMATE RESULTS - 5-STORY BUILDING WITH SOIL-STRUCTURE INTERACTION

	Peak Acceleration g		Peak Force - lb x 10^4	
	Bottom	Top	Top	Bottom
Exact	0.093	0.188	88	540
Modal superposition in the frequency domain	0.094	0.163	72	500
Modal superposition in the time domain	0.094	0.163	72	500

TABLE 4 - FREQUENCIES FOR 15-STORY BUILDING WITH SOIL STRUCTURE INTERACTION

	f_R	f_H	f_{RH}	f_1	f_2	f_3
$C_s = 800/\sqrt{2}$ fps	1.6	3.8	1.3	0.88	3.1	5.0
$C_s = 2000/\sqrt{2}$ fps	4.3	10.0	3.5	1.07	3.2	5.2
Rigid foundation				1.11	3.2	5.2

TABLE 5 - EXACT AND APPROXIMATE RESULTS - 15-STORY BUILDING WITH SOIL STRUCTURE INTERACTION $C_s = 800/\sqrt{2}$ fps

	Peak Acceleration at Top - g		Peak Force - lb x 10^6	
	Swaying	Rocking	Top	Bottom
Exact	0.10	0.12	0.25	1.14
Modal superposition in the frequency domain	0.102	0.125	0.24	1.13
Modal superposition in the time domain	0.102	0.125	0.24	1.13

WEIGHTED VS. UNIFORM MODAL DAMPING

In the seismic design of structures (and particularly nuclear reactors), including the effect of foundation flexibility, it is often assumed that a constant value of damping applies to each mode. This value is normally limited to 4 to 7%. The rule proposed here suggests on the other hand a weighted modal damping which would be different in each mode, and substantially higher for modes where the swaying motion plays a predominant role.

When a uniform modal damping is used, the effect of the swaying spring is greatly distorted, resulting in horizontal accelerations at the base of the structure considerably higher than those of the input earthquake (by a factor often of 2 or even more). This result is not in agreement with evidence from actual earthquakes (2).

Fig. 5 shows the soil-structure interaction effect, measured as the ratio of the base shear for the structure on flexible foundation, to the corresponding shear for a rigid foundation, for a nuclear reactor containment structure on a range of soils with different shear wave velocities. The shear is that at the base of the interior pedestal supporting the reactor itself. For the case of constant modal damping the interaction with the soil may produce amplifications of the base shear of as much as 50% (there might be an even larger peak in the neighborhood of that point). When weighted modal damping is used, on the other hand, the interaction effect is always beneficial.

CONCLUSIONS

From the cases studied, it would seem that modal analysis of structures with foundation interaction, with the modal dampings computed from

Eqs. (34) or (37), provides reasonably accurate results in spite of its lack of mathematical rigor. The approximation is good even when the damping values are very different for the various components, resulting in off-diagonal terms in the matrices $Q^T C_V Q$, $Q^T C_H Q$ which are by no means negligible. It deteriorates, however, in other kinds of problems (soil amplification studies for instance) when there is not only a substantial difference in the components' dampings but also in their stiffnesses (a couple of orders of magnitude or more). In these cases solution in the frequency domain is recommended. Further comparative studies may be appropriate to determine better the actual range of applicability of modal analysis when normal modes do not properly exist, but for the problem at hand it would seem to cover most practical situations.

The use of a constant modal damping in soil-structure interaction studies will distort the effect of the swaying spring and produce unrealistic amplifications of the base motion. A weighted modal damping reproduces much better the actual behavior and should thus be used. Otherwise it might be closer to reality to suppress the swaying spring considering it rigid (8).

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APPENDIX II - NOTATION

The following symbols are used in this paper:

A	amplitude of steady-state periodic motion
A_h A_r a_j	coefficients
B	diagonal matrix with viscous damping terms
β, β'	fraction of critical damping
β_{ieq}	equivalent fraction of critical damping for i^{th} mode
$C_V C_H$	viscous and hysteretic damping matrices
c, c_h, c_r	dashpot constant
C_s	shear wave velocity of soil
D D'	hysteretic damping ratios
$E_d E_s$	energy dissipated and strain energy in a steady-state periodic motion
ER	energy ratio
$f_H f_R f_{HR} f_i$	natural frequencies in cycles per second
H	diagonal matrix with hysteretic damping terms, height of structure
i	$\sqrt{-1}$, also denotes mode number
j	denotes component
K	stiffness matrix
k, k_r, k_h	spring constants
M	mass matrix, mass
M_r	sinusoidal moment applied at the foundation
P	sinusoidal horizontal force applied at the foundation
Q Q^T	modal matrix and its transpose
U, $\dot{U}$, $\ddot{U}$	

u, $\dot{u}$	displacement and velocity of a 1-DOF system
Δ_{ij}	strain in j^{th} component in i^{th} mode
$\phi_i \phi_u$	modal shapes for soil structure system
ϕ	phase angle and rotation
Ω ω	frequencies in radians/second
ω_i	natural frequency of i^{th} mode
ω_j	natural reference frequence for viscous damping in j^{th} component
ω_h, ω_r	swaying and rocking frequencies
Θ	phase angle

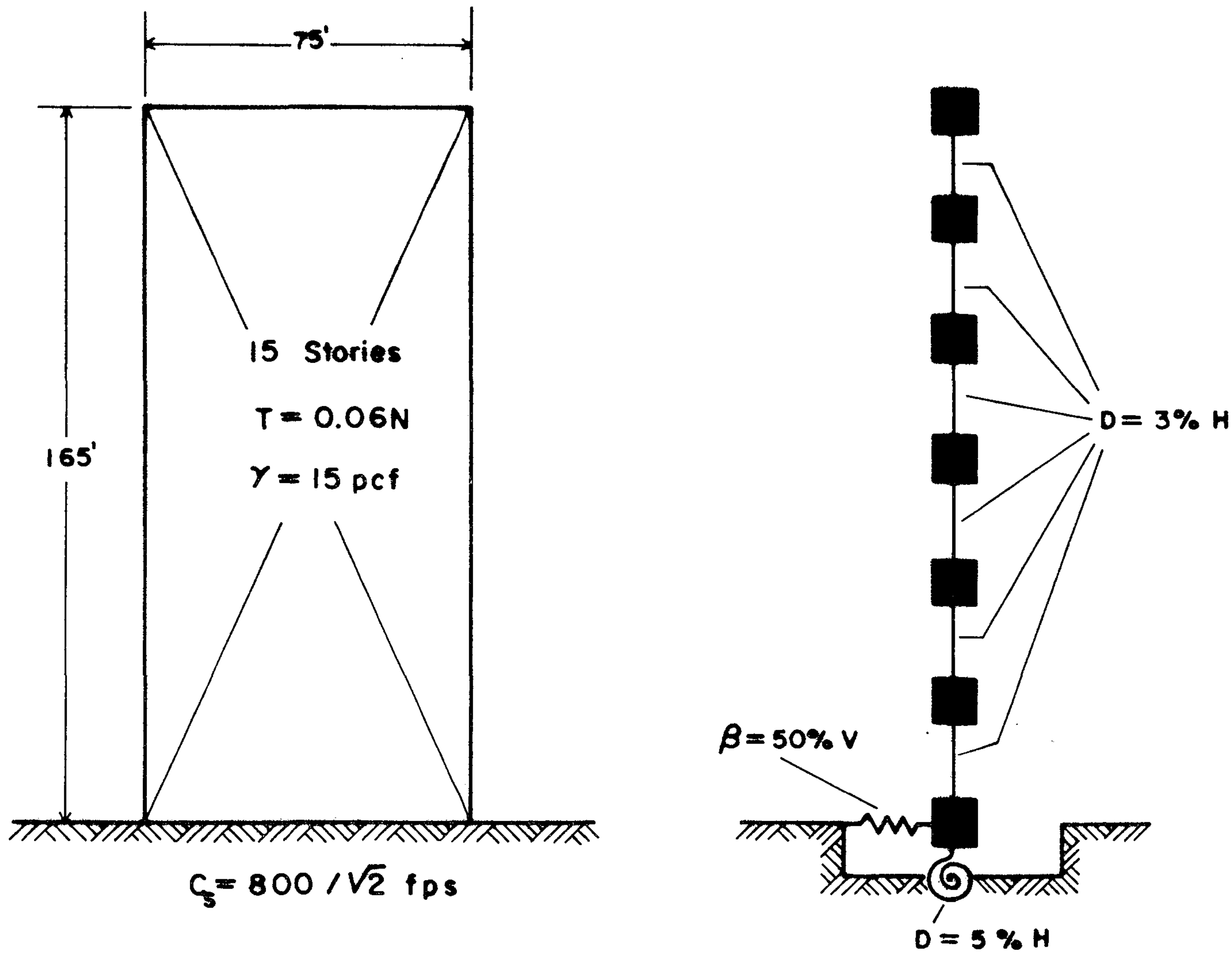

FIGURE 1 : TYPICAL SOIL STRUCTURE SYSTEM

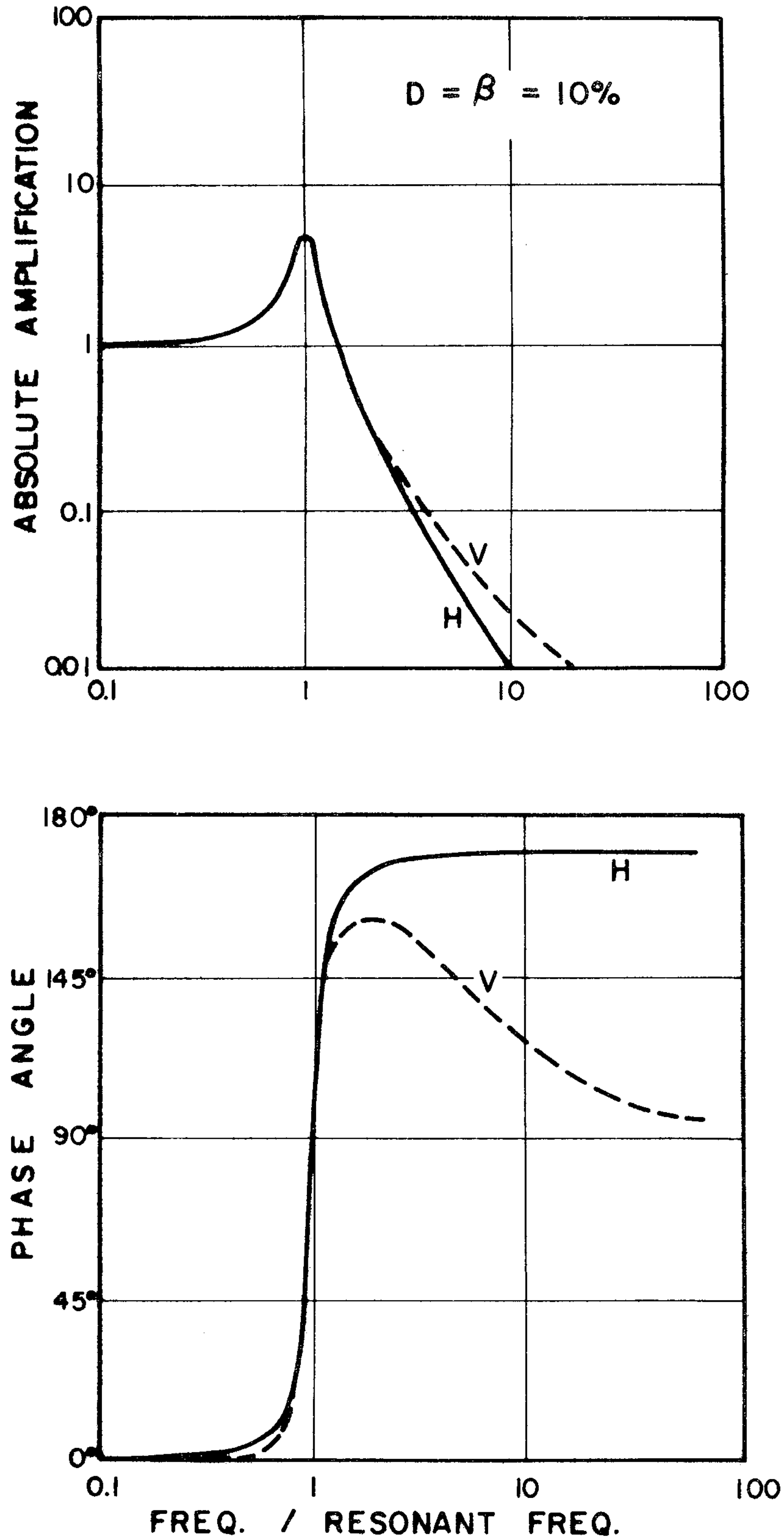

FIGURE 2: TRANSFER FUNCTIONS FOR VISCOUS AND HYSTERETIC SYSTEMS

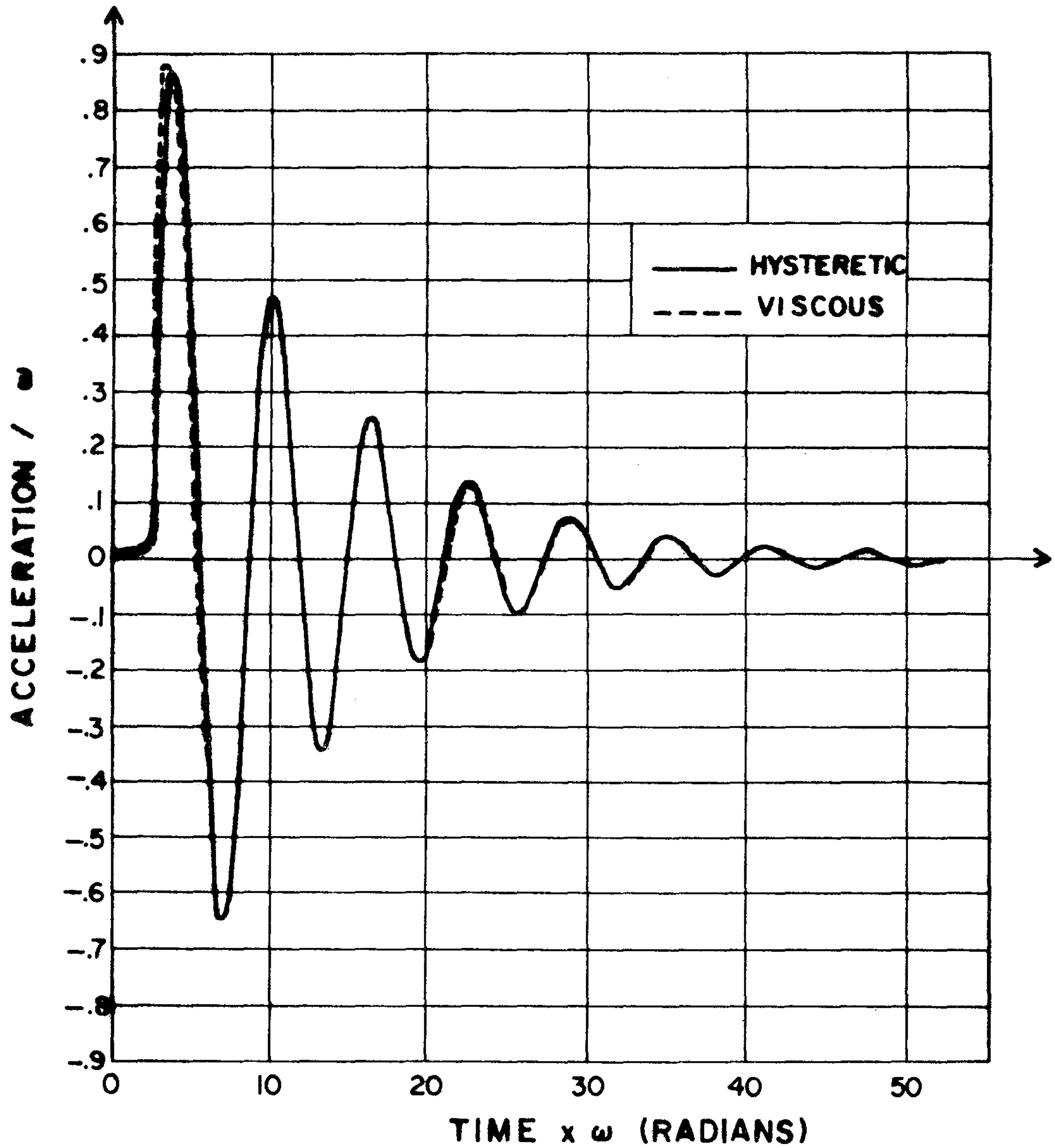

FIGURE 3: IMPULSE - RESPONSE FUNCTIONS OF EQUIVALENT LINEAR - HYSTERETIC AND VISCOUS 1-DOF SYSTEMS, $\beta = D = 0.10$

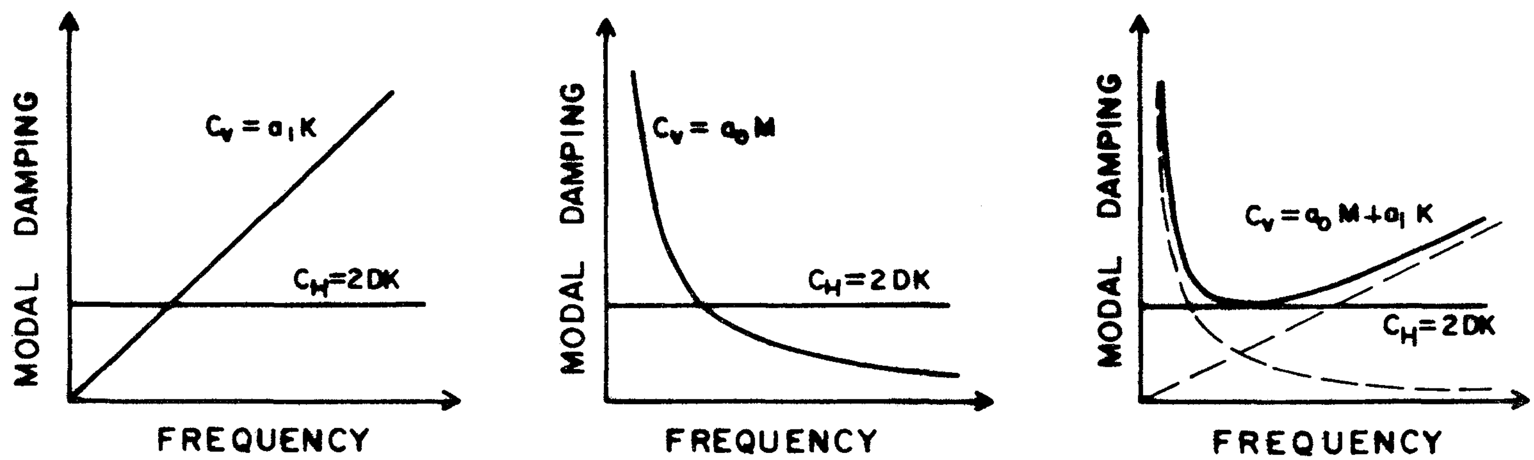

FIGURE 4 : DAMPING VS FREQUENCY

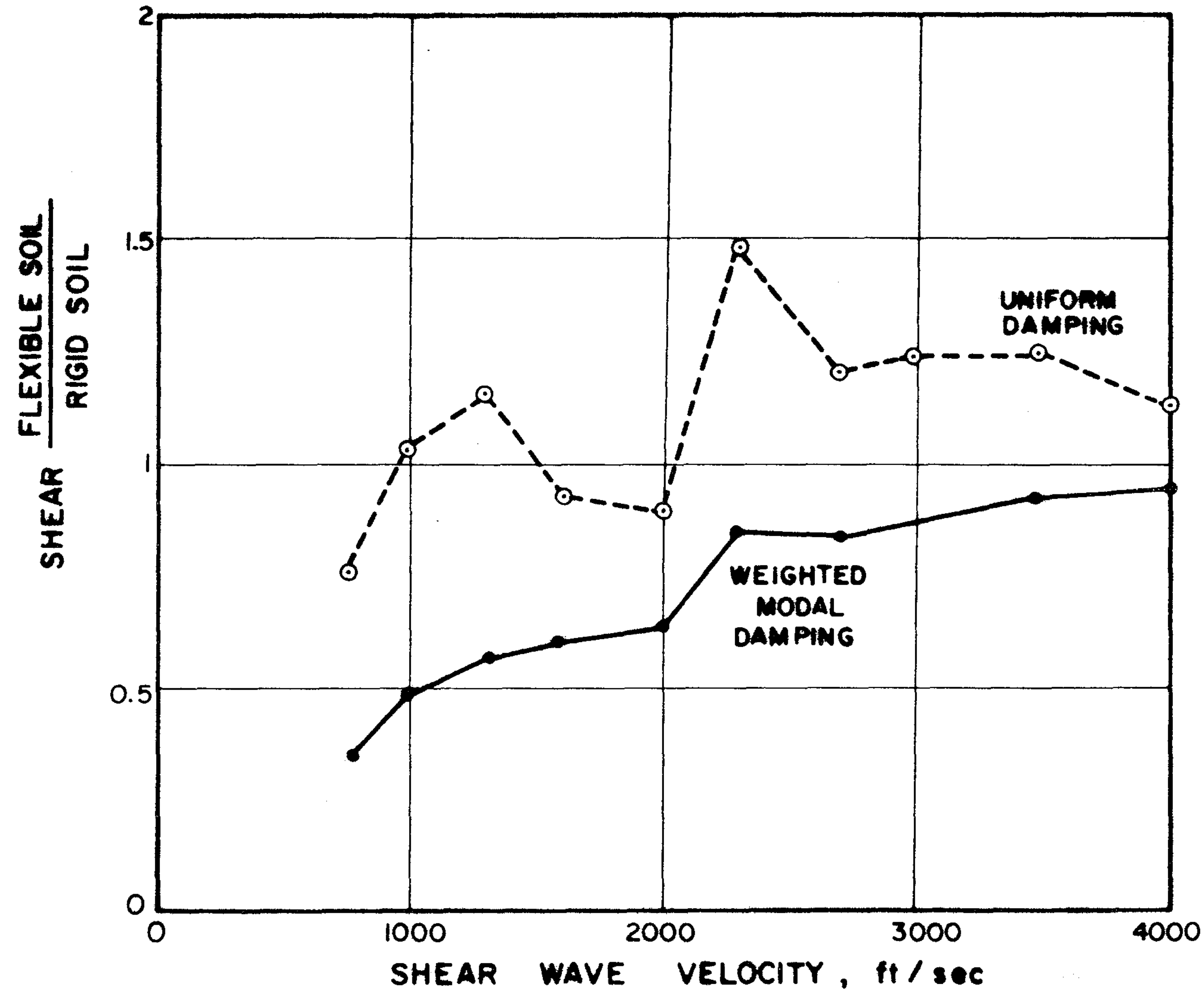

FIGURE 5 : SOIL STRUCTURE INTERACTION EFFECT USING UNIFORM AND WEIGHTED MODAL DAMPING