AN ANALYTIC METHOD FOR STRONG MOTION STUDIES IN LAYERED MEDIA

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An analytic method is presented for calculating strong motion spectra and the response to arbitrary input in layered media. The method is based on the removal of secular terms at resonance of the equations with polynomial linearity. Through a convenient parametrization of the frequency, the procedure allows one to deal with linear equations and permits the extension of the method to multilayer systems by the use of transfer matrices. Competitive analytical methods and those of the present authors lead to nonlinear algebraic equations for the amplitudes of oscillation and are untractable in multilayer systems. This method is applied to wave amplification studies in geotechnical engineering. The scheme is based on a method appropriate for non-linear phenomena, and the computational task remains at the order of that of the linear analyses. The report includes the computer program, references, and figures.

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AN ANALYTIC METHOD FOR STRONG MOTION STUDIES IN LAYERED MEDIA†

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An analytic method is presented for calculating strong motion spectra and response to arbitrary input in layered media. The method is based on the removal of secular terms at resonance of the equations with polynomial nonlinearity. The nonlinear effects are introduced by the frequency shifts calculated from the secular term according to the method by Millman and Keller. The procedure, through a convenient parametrization of the frequency, allows one to deal with linear equations. This possibility permits the extension of the method to multilayer systems by the use of transfer matrices. The response to an arbitrary input motion is obtained from the response spectrum in the frequency domain by the use of (Fast) Fourier Transform. The competitive analytical methods such as Ritz-Kantorowich's, Krylov-Bogoliubov-Mitrapolsky's and the extension of the Duffing method by Ablowitz and the present authors lead to nonlinear algebraic equations for the amplitudes. These methods would therefore be untractable in multilayer systems as they would require the solution of large coupled nonlinear algebraic equations. The method developed here is applied to wave amplification studies in geotechnical engineering. The constitutive laws are defined by the Romberg-Osgood relation as a backbone curve along with hysteretic damping. The scheme here is based on a method appropriate for nonlinear phenomena and the computational task remains at the order of that of the linear analysis.

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I. INTRODUCTION

There are a wealth of phenomena such as shifts in frequency, dispersion due to amplitude, generation of harmonics, removal of resonance singularities, jump from a state to another ... which are primarily nonlinear in nature. This paper studies the aforementioned phenomena as it pertains to the forced shear oscillations of an elastic layer with a nonlinear stress-strain law of polynomial type. Various methods such as Ritz-Kantorowich [1], Krylov-Bogoliubov-Mitropolsky [2], extension by the present authors of the classical Duffing solution [3,4] have been used for the solution of this class of problems. All these methods lead to nonlinear algebraic equations for the amplitudes of oscillation and would be extremely difficult to apply to multilayer systems. In fact, the requirement of continuity in the displacement and stress across the interfaces between layers couples the motions of the layers. This would in the methods in [1-4] lead to rather complicated coupled nonlinear algebraic equations for the amplitudes of oscillation in the layers. The procedure

here through a convenient parametrization allows to deal with linear equations. Consequently the continuity requirements across the layer interfaces lead to linear algebraic equations for the amplitudes. These linear equations offer the attractive alternative to use a transfer matrix formalism familiar in the literature in layered media [5].

The basis of the method is the work of Millman and Keller's [6]. An analysis of this method and extensive calculations for a single layer can be found in work by the present authors [3,4]. Nevertheless, the single layer case is presented here as this solution is needed for the

multilayer system according to the transfer matrix formalism. Once the amplification spectrum is obtained from the solutions, it can be used in the same manner as in other methods (see for example the SHAKE procedure [7] that is widely used in earthquake studies) through a Fourier analysis for obtaining the response to an arbitrary input.

The motivation for this work was to obtain the strong motion response of soil layers. Similar problems exist in the finite amplitude vibrations of laminates, composite plates, water waves in a basin, etc. The nonlinear stress-strain relation and the corresponding field equations studied in this paper are:

$$\tau = G_o \frac{\partial v}{\partial x} + \zeta \frac{\partial^2 v}{\partial t \partial x} + G_1 \left(\frac{\partial v}{\partial x}\right)^3$$

$$G_o \frac{\partial^2 v}{\partial x^2} + \zeta \frac{\partial^3 v}{\partial t \partial x^2} + 3G_1 \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2 v}{\partial x^2} = \rho \frac{\partial^2 v}{\partial t^2} \tag{1.1}$$

Above τ is the stress, v the displacement, ρ the density, G_o , G_1 are respectively the linear and nonlinear shear moduli and ζ is the damping coefficient. The connection between these equations and the Ramberg-Osgood model for soil under cyclic loading is discussed in Sec. 5.

2. SPECTRUM OF A SINGLE LAYER

Before going into the solution for a multilayer system, the method is introduced in some detail for a single layer of thickness d which is forced sinusoidally with frequency ω at $x = d$ and is traction free at $x = 0$. The problem is defined by the equation (1.1) and the following boundary and the periodic initial conditions:

$$\tau\Big|_{x=0}=0 \qquad v\Big|_{x=d}=a\cos\omega t \qquad v(x,\,\omega t) = v(x,\,\omega t + 2\pi) \tag{2.1}$$

Introducing the dimensionless time s and the dissipation coefficient κ

$$s = \omega t \qquad \kappa = \zeta\omega/G_0 \tag{2.2}$$

The equations (1.1) and (2.1) read:

$$G_0\left(\frac{\partial^2 v}{\partial x^2} + \kappa\,\frac{\partial^3 v}{\partial s\,\partial x^2}\right) - \rho\omega^2\,\frac{\partial^2 v}{\partial s^2} + \lambda 3G_1\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial^2 v}{\partial x^2} = 0$$

$$G_0\left(\frac{\partial v}{\partial x} + \kappa\,\frac{\partial^2 v}{\partial s\,\partial x}\right) + \lambda G_1\left(\frac{\partial v}{\partial x}\right)^3\Big|_{x=0}=0 \qquad v\Big|_{x=d}=a\cos s$$

$$v(x,\,s + 2\pi) = v(x,\,s) \tag{2.3}$$

Above $\lambda = 1$ is inserted for ordering the nonlinear terms. With the expansions

$$v(x,\,s;\,\lambda) = v_0(x,\,s) + \lambda v_1(x,\,s) + \ldots$$

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \ldots \tag{2.4}$$

and the separation of the terms in the various powers of λ in (2.3), one has:

$$Lv_o \equiv G_o\left(\frac{\partial^2 v_o}{\partial x^2} + \kappa \frac{\partial^3 v_o}{\partial s \partial x^2}\right) - \rho \omega_o^2 \frac{\partial^2 v_o}{\partial s^2} = 0$$

$$G_o\left(\frac{\partial v_o}{\partial x} + \kappa \frac{\partial^2 v_o}{\partial s \partial x}\right)\Big|_{x=0} = 0 \qquad v_o\Big|_{x=d} = a \cos s \tag{2.5}$$

and

$$Lv_1 = 2\rho\, \omega_o \omega_1 \frac{\partial^2 v_o}{\partial s^2} - 3G_1\left(\frac{\partial v_o}{\partial x}\right)^2 \frac{\partial^2 v_o}{\partial x^2}$$

$$G_o\left(\frac{\partial v_1}{\partial x} + \kappa \frac{\partial^2 v_1}{\partial s \partial x}\right) + G_1\left(\frac{\partial v_o}{\partial x}\right)^3\Big|_{x=0} = 0 \qquad v_1\Big|_{x=d} = 0 \tag{2.6}$$

The periodic solutions in s are expressed conveniently in complex terms. Proceeding with v_o, we have:

$$v_o(x, s) = V_{o1}(x)e^{is} + V_{o1}^*(x)e^{-is} \tag{2.7}$$

With (2.7), (2.5) reduces to

$$LV_{o1} \equiv (1 + i\kappa)G_o\, V_{o1}'' + \rho\omega_o^2\, V_{01} = 0$$

$$(1 + i\kappa)G_o\, V_{o1}'\Big|_{x=0} = 0 \qquad V_{o1}\Big|_{x=d} = \frac{1}{2}a \tag{2.8}$$

Consequently, the solution for the system in (2.8) is readily found to be

$$V_{o1} = A \cos(Q_o x/d) \qquad A = a/2 \cos Q_o \tag{2.9}$$

where Q_o is a dimensionless wave number defined as:

$$Q_o = (\rho/(1 + i\kappa)G_o)^{1/2}\ \omega_o d \tag{2.10}$$

The shift in frequency is determined by requiring v_o to be orthogonal to the forcing term for the equation for v_1 [6b]. This procedure extracts the secular terms that would otherwise cause the scheme to diverge. Thus, the orthogonality condition in (2.6) yields:

$$\int_{s=0}^{2\pi}\int_{x=0}^{d} \left(2\rho\omega_o\omega_1 \frac{\partial^2 v_o}{\partial s^2} - 3G_1\left(\frac{\partial v_o}{\partial x}\right)^2 \frac{\partial^2 v_o}{\partial x^2}\right) v_o \, ds dx = 0 \tag{2.11}$$

By the substitution of v_o according to (2.7) and the integration over s , (2.11) yields:

$$4\rho\omega_o\omega_1 \int_{x=0}^{d} V_{o1}V^*_{o1}\, dx + 3G_1 \int_{x=0}^{d} [V'^2_{o1}\, V^{*''}_{o1}\, V^*_{o1}$$

$$+ 2V'_{o1}\, V^{*'}_{o1}(V''_{o1}\, V^*_{o1} + V^{*''}_{o1}\, V_{o1}) + V^{*'2}_{o1}\, V''_{o1}\, V_{o1}]dx = 0 \tag{2.12}$$

With the substitution of V''_{o1} and $V^{*''}_{o1}$ from (2.8), the above equation becomes:

$$4\rho\omega_o\omega_1 \int_{x=0}^{d} V_{o1}\, V^*_{o1}dx = 3G_1\left(\frac{\rho\omega_o^2}{G_o}\right) \int_{x=0}^{d} [\frac{1}{1 - i\kappa} V'^2_{o1}\, V^{*2}_{o1}$$

$$2\left(\frac{1}{1 + i\kappa} + \frac{1}{1 - i\kappa}\right)V'_{o1}\, V^{*'}_{o1}\, V_{o1}\, V^*_{o1} + \frac{1}{1 + i\kappa} V^{*'2}_{o1}V^2_{o1}]dx \tag{2.13}$$

A rearrangement of (2.13) yields:

$$4\rho\omega_o\omega_1 \int_0^d V_{o1} V_{o1}^* dx = 3G_1 \frac{1}{d^2} \int_0^d [Q_o^{*2} V_{o1}'^2 V_{o1}^{*2} + Q_o^2 V_{o1}^{*'2} V_{o1}^2$$

$$+ 2(Q_o^2 + Q_o^{*2}) V_{o1}' V_{o1}^{*'} V_{o1} V_{o1}^*] dx \tag{2.14}$$

A convenient expression for $\omega_o\omega_1$ is obtained by introducing the following definitions:

$$I_1 = \frac{d^2}{(AA^*Q_oQ_o^*)^2} \frac{1}{d} \int_0^d (Q_o^{*2} V_{o1}'^2 V_{o1}^{*2} + Q_o^2 V_{o1}^{*'2} V_{01}^2) dx$$

$$I_2 = 2\frac{d^2}{(AA^*Q_oQ_o^*)^2} (Q_o^2 + Q_o^{*2}) \frac{1}{d} \int_0^d V_{o1}' V_{o1}^{*'} V_{o1} V_{o1}^* dx$$

$$I_3 = \frac{1}{AA^*} \frac{1}{d} \int_0^d V_{o1} V_{o1}^* dx \tag{2.15}$$

With (2.15), $\omega_o\omega_1$ is found from (2.14) as:

$$\omega_o\omega_1 = \frac{3}{4} \frac{G_1}{\rho d^2} \frac{AA^*}{d^2} (Q_oQ_o^*)^2 \frac{(I_1 + I_2)}{I_3} \tag{2.16}$$

Using the expression for A in (2.9) one has:

$$\omega_o\omega_1 = \frac{3}{16} \frac{G_1}{\rho d^2} \left(\frac{a}{d}\right)^2 \frac{(Q_oQ_o^*)^2}{\cos Q_o \cos Q_o^*} \left(\frac{I_1 + I_2}{I_3}\right) \tag{2.17}$$

With the substitution of V_{o1} given by (2.9), the integrals in (2.15) read:

$$I_1 = \frac{1}{2}\int_0^1 (1 - \cos 2Q_o y \cos 2Q_o^* y)dy = \frac{1}{2}(1 - f_2)$$

$$I_2 = \frac{1}{(1+\kappa^2)^{1/2}}\int_0^1 \sin 2Q_o y \sin 2Q_o^* y \, dy = \frac{1}{(1+\kappa^2)^{1/2}} g_2$$

$$I_3 = \int_0^1 \cos Q_o y \cos Q_o^* y \, dy = f_1 \qquad (2.18)$$

where $y = x/d$ and

$$f_m = \int_0^1 \cos mQ_o y \cos mQ_o^* y dy = \frac{1}{2}\left[\frac{\sin m(Q_o - Q_o^*)}{m(Q_o - Q_o^*)} + \frac{\sin m(Q_o + Q_o^*)}{m(Q_o + Q_o^*)}\right]$$

$$g_m = \int_0^1 \sin mQ_o y \sin mQ_o^* y dy = \frac{1}{2}\left[\frac{\sin m(Q_o - Q_o^*)}{m(Q_o - Q_o^*)} - \frac{\sin m(Q_o + Q_o^*)}{m(Q_o + Q_o^*)}\right] \qquad (2.19)$$

Consequently, the wave amplification is expressed parametrically by (2.9) and (2.17); i.e.

$$V_{o1}(0)/V_{o1}(d) = 1/\cos Q_o$$

$$\omega = \omega_o\left(1 + \frac{3}{32}\frac{G_1}{G_o}\left(\frac{a}{d}\right)^2 \frac{Q_o Q_o^*}{\cos Q_o \cos Q_o^*}\frac{1}{(1+\kappa^2)^{1/2}} f(Q_o, Q_o^*)\right) \qquad (2.20)$$

where

$$f(Q_o,Q_o^*) = \frac{2(I_1 + I_2)}{I_3} = \frac{1 - f_2 + 2\, g_2/(1 + \kappa^2)^{1/2}}{f_1} \tag{2.21}$$

It should be noted also that for

$$\lim_{\kappa\to 0} Q_o = \lim_{\kappa\to 0} Q_o^* = q_o \equiv (\rho/G_o)^{1/2}\omega_o d \tag{2.22}$$

In this case, (2.17) becomes:

$$\frac{\omega_1}{\omega_o} = \frac{9}{32}\,\frac{G_1}{G_o}\,(\frac{a}{d})^2\,\frac{q_o^2}{\cos^2 q_o}\,f(q_o) \tag{2.23}$$

with

$$f(q_o) = [1 - \frac{\sin 4q_o}{4q_o}]/[1 + \frac{\sin q_o}{q_o}] \tag{2.24}$$

This result is also obtained by direct solution of (2.3) after setting $\kappa = 0$. This observation indicates therefore that the solution in (2.17) is uniformly valid in κ .

Above, ω_o is a convenient nonphysical parameter for expressing the solution. The connection between this method and a more conventional method similar to Duffing solution may be seen in Ref. [3,4]. The solution for the amplitude A in (2.9) as a function of the physical parameter ω may be obtained by the elimination (numerical or graphical) of ω_o . Fig. 1 illustrates graphically the elimination of ω_o between

$A(\omega_0)$ and $\omega(\omega_0)$ to yield $A(\omega)$. These figures and the process of eliminating ω_0 are discussed in Section 5b.

3. MULTI-LAYER SYSTEM

In the preceeding section , the problem of a single layer is solved. In this section, the solution is extended to a multi-layer system with (N-1) layers and N interfaces. The displacement and stresses are taken to be continuous across the interfaces. The notation is presented in Fig. 2. The boundary conditions are prescribed on the 1^{st} and N^{th} face. For the k^{th} layer, the upper face is the k^{th} interface, and the lower face is the $(k-1)^{st}$ interface. On the k^{th} interface, the displacement and stress are labelled with the index k and are denoted respectively as V^k and T^k . For representing the solution local coordinates are used for each layer such that $x = 0$ and $x = d_k$ define the lower and upper faces of the k^{th} interface.

As a preparation for the solution for a multi-layer system, we first consider a typical layer under arbitrary boundary conditions. In this case great flexibility is gained by formulating the problem as an initial value problems in space in the usual manner [5]. To this purpose the differential equations in (2.8) are still valid while the boundary conditions for V_{o1} are substituted by:

$$V_{o1}\big|_{x=0} = V_{k-1} \qquad (1 + i\kappa)G_o V'_{o1}\big|_{x=0} = T_{k-1} \tag{3.1}$$

With this notation the solution replacing (2.9) becomes:

$$V_{o1} = V_{k-1}\cos(Q_o x/d_k) + \frac{d_k T_{k-1}}{(1+i\kappa)G_o Q_o}\sin(Q_o x/d_k) \tag{3.2}$$

The displacement and stress on the upper face of the k^{th} layer are obtained by setting $x = d_k$ in (3.2). The resulting expressions are then represented in matrix form as:

$$\underset{\sim}{X}_k = \underline{\underline{A}}_k \cdot \underset{\sim}{X}_{k-1} \tag{3.3}$$

where

$$\underset{\sim}{X}_{k-1} = (V_{k-1} , T_{k-1}) \qquad \underset{\sim}{X}_k = (V_k , T_k)$$

$$\underline{\underline{A}}_k = \begin{bmatrix} \cos Q_o & \dfrac{d_k}{(1 + i\kappa)G_o Q_o} \sin Q_o \\ \\ -\dfrac{(1 + i\kappa)G_o Q_o}{d_k} \sin Q_o & \cos Q_o \end{bmatrix} \tag{3.4}$$

Above, the matrix $\underline{\underline{A}}_k$ carries the information from the $(k-1)^{st}$ face to the k^{th} and is commonly called as the "Transfer Matrix" [5]. The problem for a system with N interfaces (i.e. $N-1$ layers) has $2(N-1)$ unknowns. These are determined by the $2(N-2)$ continuity conditions at the inner interfaces and the two prescribed conditions, one at each of the outer faces. However, the transfer matrix method reduces the problem to the solution of a single equation. In fact, the transfer matrix is utilized by carrying the information from the face 1 to the face 2, from face 2 to face 3, and so on until the face N. Thus for the k^{th} face

$$\underset{\sim}{X}_k = \underline{\underline{B}}_k \cdot \underset{\sim}{X}_1 \tag{3.5}$$

where

$$\underline{\underline{B}}_k = \underline{\underline{A}}_k \cdot \underline{\underline{B}}_{k-1} \qquad k = 2, \ldots N \tag{3.6}$$

For $k = 2$, the convention $\underline{\underline{B}}_1 = \underline{\underline{I}}$ (I, the identity matrix) is adapted. For $k = N$ in (3.5), in explicit form one has:

$$\begin{bmatrix} V_N \\ \\ T_N \end{bmatrix} = \begin{bmatrix} B_{11,N} & B_{12,N} \\ \\ B_{21,N} & B_{22,N} \end{bmatrix} \begin{bmatrix} V_1 \\ \\ T_1 \end{bmatrix} \tag{3.7}$$

If V_1 and V_N are the prescribed conditions, T_1 is given by the first of the two equations in (3.7). Similarly, if V_1 and T_N are the prescribed conditions T_1 is found from the second of the two equations in (3.7). Once, the component of $\underset{\sim}{X}$ not given as a boundary condition is determined, the complete solution is generated by (3.3) or (3.5).

The shift in frequency is calculated again along the same reasoning as in Section 3. The only difference is that v_o and the PDE for v_1 here are expressed piecewise. Consequently the integral for the inner product of v_o with the right hand side of the PDE for v_1 has to be calculated with the corresponding expression in each interval. Thus, (2.14) is replaced by:

$$4\omega_o\omega_1 \sum_{k=2}^{N} \rho_k \int_0^{d_k} V_{o1}\, V^*_{o1} dx = 3 \sum_{k=2}^{N} \rho_k \frac{G_{1k}}{G_{ok}} \omega_o^2 \int_0^{d_k} \left[\frac{1}{1-i\kappa_k} V'^2_{o1}\, V^{*2}_{o1} \right.$$

$$\left. + 2\left(\frac{1}{1+i\kappa_k} + \frac{1}{1-i\kappa_k}\right) V'_{o1}\, V^{*\prime}_{o1}\, V_{o1}\, V^*_{o1} + \frac{1}{1+i\kappa_k} V^{*\prime 2}_{o1} V^2_{o1} \right] dx \tag{3.8}$$

For a convenient expression of (3.8) let us again introduce the definitions in (2.15) where the integrals are evaluated with the appropriate parameters for each layer and A is substituted with V_{k-1}. Clearly in this case V_{01} has the expression in (3.2) and the values of integrals are different from those in (2.18). With (2.15) interpreted as above, (3.8) yields:

$$\omega_o\omega_1 = \frac{3}{4}\sum_{k=2}^{N}\frac{G_{1k}}{d_k}(V_{k-1}V_{k-1}^*Q_{ok}Q_{ok}^*)^2(I_{1k}+I_{2k})/\sum_{k=2}^{N}\rho_k d_k V_{k-1}V_{k-1}^* I_{3k} \tag{3.9}$$

For evaluating the integrals in (2.15), let us rearrange the expression for V_{01} in (3.2) as:

$$V_{01} = V_{k-1}(\cos(Q_{ok}x/d_k) + t_{k-1}\sin(Q_{ok}x/d_k)) \tag{3.10}$$

where

$$t_{k-1} = \frac{T_{k-1}}{V_{k-1}(1+i\kappa_k)G_{ok}}\frac{d_k}{Q_{ok}} \tag{3.11}$$

Then I_{1k} - I_{3k} in (2.15) read (Q_o denotes Q_{ok}) :

$$I_{1k} = \int_0^1 (-\sin Q_o y + t_{k-1}\cos Q_o y)^2(\cos Q_o^* y + t_{k-1}^*\sin Q_o^* y)^2 dy$$

+ complex conjugate

$$I_{2k} = \frac{4}{(1+\kappa^2)^{1/2}}\int_0^1 (-\sin Q_o y + t_{k-1}\cos Q_o y)(-\sin Q_o^* y + t_{k-1}^*\cos Q_o^* y)$$

$$\times(\cos Q_o y + t_{k-1}\sin Q_o y)(\cos Q_o^* y + t_{k-1}^*\sin Q_o^* y)dy$$

$$I_{3k} = \int_0^1 (\cos Q_o y + t_{k-1}\sin Q_o y)(\cos Q_o^* y + t_{k-1}^* \sin Q_o^* y)dy \tag{3.12}$$

The evaluation of the integrals in (3.12) gives:

$$I_{1k} = \frac{1}{2}\,[(1+t_{k-1}^2)(1+t_{k-1}^{*2}) - (1-t_{k-1}^2)(1-t_{k-1}^{*2})\ f_2 - 4t_{k-1}t_{k-1}^* g_2$$

$$- 2t_{k-1}(1-t_{k-1}^*)h_2 - 2t_{k-1}^*(1-t_{k-1}^2)\ h_2^*]$$

$$I_{2k} = \frac{1}{(1+\kappa_k^2)^{1/2}}[(1-t_{k-1}^2)(1-t_{k-1}^{*2})\ g_2 + 4t_{k-1}t_{k-1}^*\ f_2$$

$$- 2t_{k-1}^*(1-t_{k-1}^2)h_2 - 2t_{k-1}(1-t_{k-1}^{*2})h_2^*]$$

$$I_{3k} = f_1 + t_{k-1}t_{k-1}^*\ g_1 + t_{k-1}\ h_1 + t_{k-1}^*\ h_1^* \tag{3.13}$$

where f_m and $g_m (m = 1,2)$ have the same definitions as in (2.19) and the remaining coefficients $h_m (m = 1,2)$ are defined as:

$$h_m = \int_0^1 \sin mQ_o y \cos mQ_o^* y dy = \frac{1}{2}\left[\frac{1 - \cos m(Q_o+Q_o^*)}{m(Q_o+Q_o^*)} + \frac{1 - \cos m(Q_o-Q_o^*)}{m(Q_o-Q_o^*)}\right] \tag{3.14}$$

It should be noted that the solution for a single layer in (2.17) is obtained from (3.9) and (3.13) by setting $t_{k-1} = 0$ and $V_{k-1} = A = a/2\cos Q_o$. Figure 2 presents the amplification results for the multilayer system with the parameters as indicated in Fig. 2a.

4. ARBITRARY INPUT

The result of the preceding sections allow to determine the amplitude on the surface. Using the frequency-amplitude dispersion relation obtained with the proper frequency shift, one can obtain by a Fourier Transform (fast or discrete Fourier Transform) the response to an arbitrary input given in terms of its Fourier components. A higher degree of approximation would introduce additional nonlinear effects such as a further correction on the frequency shift, generation of higher harmonics and higher order mode coupling. However these effects are of smaller magnitude than the shifts in the frequency-amplitude spectrum at the leading order and are neglected. A theoretical justification for neglecting these higher order resonances may be found in Ref. [8]. Basically a discrete Fourier Transform requires equally spaced discrete frequencies such as $\omega_\ell = \ell\Delta\omega$ with $\Delta\omega$ being a small increment. The above analysis however is formulated in terms of the parameters $\omega_{0\ell}$* . In the calculations therefore, first the $\omega_{0\ell}$ corresponding to a given ω_ℓ is needed to be determined. This is achieved by standard application of the "secant method". In the calculations, the choice of the starting value is facilitated since, the calculations are done for an increasing set of values for ω_ℓ . The converged solution of $\omega_{0\ell}$ provides a good starting value for the evaluation of $\omega_{0(\ell+1)}$. Thus for a forcing at the base as

$$V\Big|_{\text{base}} \equiv V_1 = \sum_{\ell=-L}^{L} a_\ell e^{i\omega_\ell t} \tag{4.1}$$

one first finds the response for each frequency ω_ℓ and then recombines these for obtaining the total response to the prescribed input. Thus for $A(\omega_\ell)$ being the amplification factor for the component of the input

*The index ℓ is added ω_0 of the formulation in the preceeding sections to distinguish between the various frequencies.

at the frequency ω_ℓ , the response to the forcing in (4.1) becomes:

$$V_{top} = \sum_{\ell=-L}^{L} a_\ell A(\omega_\ell) e^{i\omega_\ell t} \tag{4.2}$$

In doing these calculations it should be borne in mind that since the amplitudes at each frequency are different, a different order of nonlinearity is induced for each Fourier component. Figures 3 to 6 give comparisons of the linear and nonlinear responses to a Gaussian and a real earthquake forcing (N21E component of the 1952 Taft strong motion record [9]) at the base of single and multilayer systems.

5. DISCUSSION

a. Connection With Soil Mechanics

Several investigators have shown that the relationship between shear modulous and strain in soils is nonlinear and that the modulus is a decreasing function of the strain [10]. Among these we adopt the Ramberg-Osgood constitutive relation as a backbone curve with $G/G_{max} = 1/[1 + \alpha(\tau/\tau_y)^{R-1}]$. Here τ = shearing stress; τ_y = a yield or reference shearing stress; and α and R are parameters which determine the shape of the curve. It is found that for a large variety of soils the Ramberg-Osgood relationship fits the data quite reasonably with $\alpha = 1$, $R = 3$ and $\tau_y = 0.4\ S_u$ where S_u is the undrained shearing strength [11]. The strain-stress relationship of Ramberg-Osgood with $\alpha = 1$ and $R = 3$ reads: $\tau/\tau_y = (G_{max}\gamma/\tau_y)/[1 + (\tau/\tau_y)^2]$. To put this equation in a more conventional form, substitute τ/τ_y iteratively to get $\tau/\tau_y = (G_{max}\gamma/\tau_y)[(1 + G_{max}\gamma/\tau_y)^2/(1 + \tau/\tau_y)^2]^{-1}$. Expanding the (-1) power by the binomial formula, as an approximation to the Ramberg-Osgood relationship, one obtains:

$$\frac{\tau}{\tau_y} = \frac{G_{max}}{\tau_y}\gamma[1 - (\frac{G_{max}}{\tau_y})^2\gamma^2] \qquad \frac{G}{G_{max}} = 1 - (\frac{G_{max}}{\tau_y})^2\gamma^2 \tag{5.1}$$

A simple study shows that the form of G as proposed here is a very good approximation of the Ramberg-Osgood relationship for a large range of strains corresponding to $\tau/\tau_y \simeq 1$ while also being of convention form for the applications.

At this stage we deviate from the more conventional uses of the Ramberg-Osgood relation which are in the realm of plasticity theory. Our analysis uses the Ramberg-Osgood relation as the backbone curve and the damping is introduced through a linear term in the strain rate, as is experimentally suggested [10]. Thus we take:

$$\tau = G_{max}\gamma[1 - (\frac{G_{max}}{\tau_y})^2\gamma^2] + \zeta\,\frac{\partial\gamma}{\partial t} \tag{5.2}$$

For the cyclic loading with $\gamma = \gamma_0 \cos\omega t$, the fundamental part of the stress (i.e. the part in the stress with the frequency ω) is:

$$\tau = \gamma_0 G_{max}[1 - \frac{3}{4}(\frac{G_{max}}{\tau_y})^2\gamma_0{}^2]\cos\omega t - \gamma_0\zeta\omega\sin\omega t \tag{5.3}$$

For soils it is observed that the nature of the damping is hysteretic and thus is independent of the frequency of oscillation. It is known that the choice $\kappa = \zeta\omega/G_{max}$ (κ = constant) provides a reasonable description of the damping [7]. The compliance is obtained by substituting $\cos\omega t$ and $\sin\omega t$ in (5.3) by their complex representation. For $G = |G|e^{i\delta}$, (5.3) yields:

$$|G| = G_{max}\{[1 - \frac{3}{4}(\frac{G_{max}}{\tau_y})^2\gamma_0{}^2]^2 + \kappa^2\}^{1/2}$$

$$\delta = \tan^{-1}\{\kappa/[1 - \frac{3}{4}(\frac{G_{max}}{\tau_y})^2\gamma_0{}^2]\} \tag{5.4}$$

It is seen that $|G|$ decreases nonlinearly with the strain while the phase angle increases with it as is observed in the experiments [10].

Introducing the definitions

$$G_0 = G_{max}$$

$$G_1 = -\left(\frac{G_{max}}{\tau_y}\right)^2 G_{max} \qquad (5.5)$$

one obtains the equations in (1.1) studied above.

It may be taught that our model may be adequate for the description of soil behavior in general. However the use of backbone curves along with a hysteretic damping has proven as a successful model for the behavior in soils under cyclic loading [12,13]. In this spirit bears resemblances to the soil model used by Seed and co-workers [7]. Our analytical method of calculation is one appropriate to nonlinear phenomena. While equivalent linearization the procedures are iterative, ours are not. More critically, at each iterative step of the equivalent linearization calculations, the backbone curve is not followed but provides only a means to determine the end point of the straight line drawn from the origin in the (τ,γ) space to the state reached at the end of the deformation. In our calculations, however, the nonlinear path on the backbone curve in the (τ,γ) space is followed.

b. Discussion of the Calculations

Figure 1 illustrates the basic idea in the calculation procedure. For a clear description of the effect of nonlinearity, let us consider first the case with no dissipation. In this case the spectrum for the amplification in the displacement as well as those for the velocity, acceleration and the energy have singularities due to the term $1/\cos q_0$ as $q_0 \equiv \sqrt{\frac{\rho}{G_0}}\ \omega_0 d \rightarrow (2n+1)\pi/2$. For the linear analysis $\omega = \omega_0$ so that the same singularities exist in the frequency ω . However, for the nonlinear

analysis q_0 or equivalently ω_0 are merely convenient parameters and the pair of equations $A = A(\omega_0)$ (Figure 1a) and $\omega = \omega(\omega_0)$ (Figure 1b) are the parametric expressions for the physical relationship $A = A(\omega)$. The desired response of the system is expressed by the relation $A = A(\omega)$ which is obtained by the elimination of ω_0 between $A(\omega_0)$ and $\omega(\omega_0)$. In Figure 1, the points a_0 and a_0' are the values corresponding to $\omega_0 = \Omega_0$ and $\omega_0 = \Omega_0'$. The frequencies $\Omega = \omega(\Omega_0)$ and $\Omega' = \omega(\Omega_0')$ are seen to be smaller respectively than Ω_0 and Ω_0' due to the "softening" of the material with increasing amplitude. The points a and a' on the $A(\omega)$ curve are obtained respectively by simply carrying the points a_0 and a_0' to correspond to the values Ω and Ω' . We thus see that the nonlinearity bends the linear response curves to the left and removes the singularity. The analysis can certainly be persued to evaluate the higher harmonics. However the neglect of the higher harmonics are of a smaller consequence than those due to the frequency shifts.

When damping is present, the preceeding analysis basically remains the same. For this latter case, at resonance, the two branches with vertical asymptote in the linear analysis join. The amplitude though finite, is nevertheless large. The same bending of the curves occur as a result of the softening according to the nonlinear analysis. The result of this process is a lowering of the amplification and consequently nonlinear softening exhibits itself as a sort of further effective damping of the waves. Figures 2b-e show the amplification coefficient for a single and multilayer system without and with hysteretic damping.

For the arbitrary input, the amplification results of the calculations are displayed in Figures 3 to 6 . In these figures the

results of the linear analysis are also given for comparison. Results are presented for a single layer (Figure 3 and 4) and a multilayer system (Figure 5 and 6). In Figures 3 and 5 the input motions at the base rock are taken as the Gaussian function: $V(0,t)=A\exp(-(t-t_0)^2/\sigma^2)$ with $A=20\text{cm}$, $\sigma = 1$ sec. and $t_0 = 5$ sec.; while in Figures 4 and 6 the input motions at the base rock are taken as the record of an earthquake (N21E component of the 1952 Taft strong motion record [9].) For observing the effect of the nonlinearity, the amplitudes are augmented respectively by the factors 3/2 and 5/2 in Figure 3b and 3c as compared to that for Figure 3a; and similarly by the factors 2 and 4 in Figure 4b and 4c as compared to that for Figure 4a. Figures 3(a', b', c') and 4 (a', b', c') are the amplitudes at the top of the layer in the Fourier transform domain. Figures 5 and 6 show the response of the multilayer system in Figure 2a for the input motions used in Figures 3 and 4. In all of the figures it is seen as expected, that the softening due to nonlinearity has decreased the amplitudes from those of the linear analysis. The effect is, again as expected, more pronounced with increasing amplitude. In all of the above calculations, the damping coefficient κ is taken as 0.1 .

c. Concluding Remarks

The scheme presented here is based on a method appropriate for nonlinear phenomena. It is non iterative and the computational task is of the same order as for the linear analysis. The only additional price is the evaluation of the frequency shift which requires only a summation as in Eq. (3.9). The scheme from this viewpoint is expected to be several times faster than iterative methods such as the one used in SHAKE [7] or the direct integration of the nonlinear equations in the original coordinates [14] or characteristic coordinates such as CHARSOIL [11]. Due to the unavailability of appropriate multidimensional constitutive laws the calculations here have been kept to one dimensional studies. However the method is applicable in higher dimensions and irregular geometries when coupled with numerical procedures in the spacial coordinates for any problem involving nonlinear partial differential equations with analytic nonlinearities. Because the procedure extracts the dominant nonlinear effect through a convenient parameterization of the frequency, the computational effort is kept at the level of that of the linear analysis.

APPENDIX

```
      DIMENSION F11(401),R11(401),R22(401),RR (401),A(401),T11(401),
     1D11(401),D22(401),DD (401),WK(401),IWK(401),B(401)
     2,X01(5),Y01(5),Q(5),QQ(5),A11( 401),A22( 401)
     1,D(5),RO(5),GO(5),G1(5),ZETA(5),B111(5),B112(5),B121(5),B122(5)
      LOGICAL  LL(401)
      COMPLEX*16 A,B,ZAO,ZA1,ZA2,ZA3,ZA4,AK2,QQ,Q,QC,QK,Q1,Q2,S,SC,
     1S1,S2,C,CC,C1,C2,AI1,AI2,AA,AAC,BB,BBC,X01,Y01,B111,B112,B121,
     2B122,QQ2,A11,A22,A1,A2,A3,A4,A5,A7,A8,R,F1
      REAL*8PI,F,AAA,RO,D,GO,G1,ZETA,T1,TO,DT,T2,DQ,DQ1,FACT,OM,OMO,OM1
      EQUIVALENCE (IWK(1),WK(1),LL(1))
      CALL INDUMP
      THIS PROGRAM CALCULATES THE RESPONSE AND THE DISPLACEMENTS AT THE
      TOP OF THE SOIL LAYER OR LAYERS FOR A GIVEN INPUT MOTION AT THE
      BED-ROCK. IN ITS PRESENT FORM DATA FOR THREE KINDS OF INPUT MO-
      TIONS ARE INCLUDED. THESE ARE HARMONIC MOTION(AO*COS(W*T)),

      GAUSSIAN MOTION(AO*EXP(-(T-TO)**2)) AND REAL EARTHQUAKE MOTION.
      IF JKL EQUALS 1,2,3 THE INPUT CORRESPONDS RESPECTIVELY TO
      HARMONIC, GAUSSIAN PULSE AND REAL EARTHQUAKE MOTIONS. REMEMBER
      THAT YOU MUST CHANGE SOME DIMENSIONS ACCORDING TO THE NUMBER
      OF THE LAYERS OR THE NUMBER OF THE POINTS WHICH ARE USED IN THE
      FOURIER TRANSFORM. THE PROGRAM HAS THREE SUBROUTINES:
      (FFTP),TRANS, AND SHIFT. FFTP CALCULATES FOURIER TRANSFORM OF A
      GIVEN TIME DEPENDENT FUNCTION OR INVERSE FOURIER TRANSFORM. TO GET
      REAL FOURIER TRANSFORM COEFFICIENTS OF A GIVEN TIME DEPENDENT
      FUNCTION, YOU MUST USE THE COMPLEX CONJUGATE OF THE COEFFICIENTS
      GIVEN BY FFTP, AND MULTIPLY THEM WITH THE TIME INCREMENT(DT). THUS
      YOU CAN USE THE HALF OF THE COEFFICIENTS(1 TO N/2). TO GET INVERSE
      FOURIER TRANSFORM FROM KNOWN FOURIER COEFFICIENTS, YOU MUST DIVIDE
      THE COEFFICIENTS BY THE PERIOD TIME T. THE SUBROUTINE "TRANS"
      CALCULATES THE FOURIER COEFFICIENTS AT THE TOP USING TRANSFER MAT-
      RICES FOR A GIVEN MOTION AT THE BOTTOM.
      THE SUBROUTINE "SHIFT" CALCULATES THE FREQUENCY SHIFT ACCORDING TO
      THE UNDERLYING THEORY.
      ******************************************************************

       DATA FOR EACH LAYER
      JJ=NUMBER OF LAYERS   PLUS 1
      RO=DENSITY(GR/CM3)
      D=THICKNESS OF LAYER(CM)
      ZETA=DAMPING  RATIO
      GO= MAXIMUM SHEAR MODULOUS(DYN/CM2)
      G1= NONLINEAR ELASTICITY COEFFICIENT(DYN/CM2)
      FACT=(GO/TY)**2 WHERE TY IS THE YIELD STRESS
      N IS THE NUMBER OF DISCRETE FREQUENCIES FOR CASE JKL=1 AND IS THE
      NUMBER OF DISCRETE TIME INTERVALS FOR CASES JKL=2,3. OUR EXPERI-
      ENCES SHOW THAT N=200,400,(400-800) RESPECTIVELY FOR CASES JKL=1,2
      ,3 ARE GOOD CHOICES FOR THE PARAMETERS OF THE SYSTEM STUDIED HERE.
      ******************************************************************
       DEFINITIONS OF PHYSICALLY RELEVANT PARAMETERS
      A(I) ARE THE FOURIER COEFFICIENTS OF INPUT MOTION AT BED-ROCK.
      B(I) ARE THE INVERSE FOURIER TRANSFORM COEFFICIENTS OF RESPONSE AT
      THE TOP FOR LINEAR AND NONLINEAR CASES.
      A11(I) IS THE FOURIER COEFFICIENT OF RESPONSE AT THE SURFACE FOR
      LINEAR CASE.
      A22(I) IS THE FOURIER COEFFICIENT OF RESPONSE AT THE SURFACE FOR
      NONLINEAR CASE.
      X01(J) IS THE FOURIER COEFFICIENT OF RESPONSE AT J TH INTERFACE
      FOR A(I).
      RR(I) IS THE MODULOUS OF A(I).
      R11(I) IS THE MODULOUS OF A11(I).
      R22(I) IS THE MODULOUS OF A22(I).
      DD(I) IS THE INPUT DISPLACEMENT AT THE BED-ROCK.
      D11(I) IS THE DISPLACEMENT AT THE TOP FOR LINEAR CASE.
      D22(I) IS THE DISPLACEMENT AT THE TOP FOR NONLINEAR CASE.
      ALL DISPLACEMENTS ARE IN CM.
      J=JJ IS THE SURFACE
C      Y01(J) IS THE FOURIER COEFFICIENT OF STRESS AT J TH INTERFACE IN
C     DYN/CM2 FOR A(I).
C
C     ******************************************************************
```

```
      PI=3.14159265
      JJ=5
       PARAMETERS FOR INPUT DATA
      READ INPUT DATA
      FACT=250000.
      READ 10,(D(J),RO(J),GO(J),ZETA(J),J=2,JJ)
   10 FORMAT(4E15.4)
      N=400
      T IS THE PERIOD IN SECONDS.
      T=100.
      DT IS THE STEP SIZE OF THE TIME.
      DT=T/N
      DQ IS THE STEP SIZE OF THE FREQUENCY.
      DQ=2.*PI/T
      R IS THE IMAGINARY NUMBER I.
      R=(0.,1.)
      DO 20 J=2,JJ
      G1(J)=-FACT*GO(J)
      PRINT 30 ,D(J),RO(J),GO(J),G1(J),ZETA(J)
   30 FORMAT(5X,5E14.6)
   20 QQ(J)=D(J)*CDSQRT(RO(J)/(GO(J)*(1.+R*ZETA(J))))
      JKL=1,2,3
      ***************************************************************
      JKL=3
      ***************************************************************
      IF(JKL.NE.1) GO TO 40
      N2=N
      AO=10.
      AO IS THE AMPLITUDE OF HARMONIC MOTION
      DO 35 I=1,N2
   35 A(I)=AO/2.
      GO TO 50
   40 IF(JKL.NE.2) GO TO 60
      NE IS THE EFFECTIVE NUMBER OF THE DISCRETE POINTS FOR NONINFINITE-
      SIMAL AMPLITUDES IN INPUT. IT MUST BE CHANGED ACCORDING TO THE
      VALUE OF DT SO THAT EXPONENTIAL BECOMES LESS THAN 10.**(-69).
      NE=100
      DO 70 I=1,NE


      AO=10.
      TO=5.
      T1=(I-1.)*DT
      T2=-(T1-TO)**2
   70 A(I)=AO*DEXP(T2)
      GO TO 80
   60 CONTINUE
      NK IS THE NUMBER OF THE EARTHQUAKE DISPLACEMENT DATA.
      NK=100
      READ 90,( DD(I),I=1,NK)
   90 FORMAT(10F8.4)
      DO 100 I=1,N
  100 A(I)= DD(I)
   80 CONTINUE
      N1=N/2+1
      N2=N/2-1
       FAST FOURIER TRANSFORM OF INPUT MOTION
      A(I) ARE THE FOURIER TRANSFORM COEFFICIENTS OF THE INPUT FUNCTION.
      CALL FFTP(A,N,IWK,WK,LL)
      DO 110 I=1,N
      A(I)=DT*DCONJG(A(I))
  110 RR(I)=DSQRT((DREAL(A(I)))**2+(DIMAG(A(I)))**2)
   50 CONTINUE
      ***************************************************************
      LINEAR CASE
      ***************************************************************
      JJ=5
      DO 120 I=2,N2
      OMO=(I-1.)*DQ
      CALL  TRANS(I,JJ,OMO,QC,Q,R,ZETA,D,GO,A,XO1,YO1)
      R11(I)=DSQRT(((DREAL(XO1(JJ)))**2+(DIMAG(XO1(JJ)))**2)
      F11(I)=OM
  120 A11(I)=XO1(JJ)
      IF(JKL.EQ.1) GO TO 130
C
C     INVERSE TRANSFORM FOR THE LINEAR CASE
```

```
      DO 140  I=1,N1
140   B(I)=A11(I)/T
      DO 150 I=1,N2
150   B(N1+I)=DCONJG(B(N1-I))
      CALL FFTP(B,N,IWK,WK,LL)
      DO 160 I=1,N
      T11 IS THE TIME.
      T11(I)=DT*(I-1.)
160   D11(I)=B(I)
130   CONTINUE

       THE END OF THE LINEAR CASE
      ******************************************************************

      ******************************************************************
       NONLINEAR CASE

      ITERATIONS FOR DETERMINING OMO CORRESPONDING TO A GIVEN OM.
      KK IS THE MAXIMUM NUMBER OF ITERATIONS.
      KK=99
      DO 170 I=2,N2
      OM=(I-1.)*DQ
      K=1
      DQ1= DQ
      THE FIRST VALUE OF OMO FOR ITERATION IS TAKEN TO BE OM.
      OMO=OM
200   CONTINUE
      CALL SHIFT(I,JJ,OMO,QQ,Q,R,ZETA,D,GO,A,X01,Y01,B111,B112,
     1B121,B122,OM1,G1,RO)
      F IS THE DIFFERENCE BETWEEN THE FREQUENCIES IN THIS AND PREVIOUS
      STEPS.
      F=OMO+OM1-OM
      ERR=0.001
      ERR IS THE TOLERANCE OF THE ERROR OF OMO
      IF(DABS(F).LT.ERR) GO TO 180
      IF(K.EQ.1) GO TO 190
      IF(K.GT.KK) GO TO 180
      IF(F*AAA.GT.0.0) GO TO 190
      DQ1=-DQ1/2.
190   AAA=F
      K=K+1
      OMO=OMO+DQ1
      GO TO 200
180   CONTINUE
      A22(I)=X01(JJ)
      R22(I)=DSQRT(((DREAL(X01(JJ)))**2+(DIMAG(X01(JJ)))**2))
      PRINT 210,I,K,OM,OMO,F,A(I),A11(I),A22(I)
210   FORMAT(2X,2I4,2X,2F8.2,2X,7E12.4)
170   CONTINUE
      IF(JKL.EQ.1) GO TO 260
      THE INVERSE FOURIER TRANSFORM FOR THE NONLINEAR CASE
      DO 220 I=1,N1
220   B(I)=A22(I)/T
      DO 230 I=1,N2
230   B(N1+I)=DCONJG(B(N1-I))
      CALL FFTP(B,N,IWK,WK,LL)
      DO 240 I=1,N


240   D22(I)=B(I)
      PRINT 250,(T11(I),DD(I),D11(I),D22(I),I=1,N)
250   FORMAT(5X,4E14.4)
       THE END OF NONLINEAR CASE
      ******************************************************************

      DISPLACEMENT CURVES
      CALL DFIPS1(T11,D11,N ,01,10.)
      CALL DFIPS2(T11,D22,N ,12)
      CALL DFIPS2(T11,DD ,N ,01)
260   CONTINUE

       RESPONSE CURVES
      F11(1)=0.0
      RR(1)=A(1)
      R11(1)=RR(1)
      R22(1)=RR(1)
      CALL DFIPS1(F11,R11,N2,01,10.)
      CALL DFIPS2(F11,R22,N2,12)
      CALL DFIPS2(F11,RR,N2,01)
      STOP
      END
```

```
      SUBROUTINE SHIFT(I,JJ,OMO,QQ,Q,R,ZETA,D,GO,A,XO1,YO1,B111,B112,
     1B121,B122,OM1,G1,RO)
      DIMENSION Q(5),QQ(5),D(5),GO(5),ZETA(5),B111(5),B112(5),B121(5),
     1B122(5),XO1(5),YO1(5),A(401) ,RO(5),G1(5)
      REAL*8D,GO,ZETA,OMO,OM1,QQ1,RO,G1
      COMPLEX*16Q,QQ,QQ2,QK,QC,  R,B111,B112,B121,B122,XO1,YO1,S,C,
     1A,Q1,Q2,S1,S2,C1,C2,SC,CC,AI1,AI2,BB,BBC,AA,AAC,A1,A2,A3,A4,A5,
     3A6,A7,A8
      CALL   TRANS(I,JJ,OMO,QQ,Q,R,ZETA,D,GO,A,XO1,YO1)
C     INITIAL VALUES OF THE INTEGRALS
      AI1=0.0
      AI2=0.0
      DO 10 J =2,JJ
      Q(J)=OMO*QQ(J)
      QK=Q(J)
      F1=D(J)/(GO(J)*(1.+R*ZETA(J))*QK)
      QC=DCONJG(QK)
      Q1=QK+QC
      Q2=QK-QC
      S=CDSIN(QK)
      S1=CDSIN(Q1)
      S2=CDSIN(Q2)
      C=CDCOS(QK)
      C1=CDCOS(Q1)
      C2=CDCOS(Q2)
      SC=DCONJG(S)
      CC=DCONJG(C)
      BB=YO1(J-1)*F1
      BBC=DCONJG(BB)
      AA=XO1(J-1)
      AAC=DCONJG(AA)
      QQ2=3.*G1(J)*QK*QK*QC/(4.*D(J)**3)
      QQ1=RO(J)*OMO*D(J)
      A1=BB*BB-AA*AA
      A2=BBC*BBC-AAC*AAC
      A3=4.*AA*BB*AAC*BBC
      A4=2.*AA*BB
      A5=2.*AAC*BBC
      A7=AA*AA+BB*BB
      A8=AAC*AAC+BBC*BBC
      AI1=AI1+QQ1*(S1*(AA*AAC-BB*BBC)/Q1+S2*  (AA*AAC+BB*BBC)/Q2+(1.-C1)
     1*(AA*BBC+BB*AAC)/Q1+(1.-C2)*(BB*AAC-AA*BBC )/Q2)
      AI2=AI2+QQ2*(QK*(S1*(C1*(A3-A1*A2)+S1*(A1*A5+A2*A4))/Q1+
     1S2*(C2*(A3+A1*A2)+S2*(A1*A5-A2*A4))/Q2)+QC*(A7*(A8+SC*(-CC*
     2A2+A5*SC)/QC)+A8*S*(A1*C-A4*S)/QK+S1*(C1*(-A1*A2+A4*A5)+S1*
     3(A1*A5+A2*A4))/(2.*Q1)+S2*(-S2*(A1*A5-A2*A4)-C2*(A1*A2+A4*A5))/
     4(2.*Q2)))
   10 CONTINUE
C     FREQUENCY SHIFT
      OM1=DREAL(AI2)/DREAL(AI1)
      RETURN
      END
```

```
      SUBROUTINE TRANS(I,JJ,OMO,QQ,Q,R,ZETA,D,GO,A,XO1,YO1)
      DIMENSION Q(5),QQ(5),D(5),GO(5),ZETA(5),B111(5),B112(5),B121(5),
     1B122(5),XO1(5),YO1(5),A(401)
      REAL*8D,GO,OMO,ZETA
      COMPLEX*16Q,QQ,QK,QC,F1,R,B111,B112,B121,B122,XO1,YO1,S,C,A
      DO 10 J=2,JJ
      Q(J)=OMO*QQ(J)
       QK=Q(J)
      QC=DCONJG(QK)
      F1=D(J)/(GO(J)*(1.+R*ZETA(J))*QK)

      INITIAL VALUES OF THE TRANSFER MATRRIX ELEMENTS

      B111(1  )=1.0
      B122(1  )=1.0
      B112(1  )=0.0
      B121(1  )=0.0

      TRANSFER MATRICES


      S=CDSIN(QK)
      C=CDCOS(QK)
      B111(J)=B111(J-1)*C  +B121(J-1)*F1*S
      B112(J)=B112(J-1)*C  +B122(J-1)*F1*S
      B121(J)=-B111(J-1)*S  /F1+B121(J-1)*C
      B122(J)=-B112(J-1)*S  /F1+B122(J-1)*C
   10 CONTINUE
      XO1(1)=A(I)
      YO1(1)=-XO1(1)*B121(JJ)/B122(JJ)
      DO 20 J=2,JJ
      XO1(J)=B111(J)*XO1(1)+B112(J)*YO1(1)
   20 YO1(J)=B121(J)*XO1(1)+B122(J)*YO1(1)
      RETURN
      END
```

```
                                                                      FFTP0020
FFTP----------D-------LIBRARY 1--------------------------------------FFTP0030
                                                                      FFTP0040
  FUNCTION            - TO COMPUTE THE FAST FOURIER TRANSFORM OF A    FFTP0050
                          DATA VECTOR.                                FFTP0060
  USAGE               - CALL FFTP(A,N,IWK,WK,LL)                      FFTP0070
  PARAMETERS   A      - COMPLEX VECTOR OF LENGTH N WHICH CONTAINS ON  FFTP0080
                          INPUT THE SEQUENCE OF DATA TO BE            FFTP0090
                          TRANSFORMED. ON OUTPUT A CONTAINS THE       FFTP0100
                          FOURIER COEFFICIENTS.                       FFTP0110
               N      - N IS THE NUMBER OF DATA POINTS TO BE TRANS-   FFTP0120
                          FORMED. N MAY BE ANY POSITIVE INTEGER.      FFTP0130
               IWK    - WORK VECTOR OF LENGTH 6*N+150.                FFTP0140
                          (SEE PROGRAMMING NOTES FOR FURTHER DETAILS) FFTP0150
               WK     - SAME WORK VECTOR AS IWK.                      FFTP0160
                          (SEE PROGRAMMING NOTES)                     FFTP0170
               LL     - SAME WORK VECTOR AS IWK.                      FFTP0180
                          (SEE PROGRAMMING NOTES)                     FFTP0190
  PRECISION           - SINGLE/DOUBLE                                 FFTP0200
  LANGUAGE            - FORTRAN                                       FFTP0210
---------------------------------------------------------------------FFTP0220
  LATEST REVISION     - NOVEMBER 10, 1975                             FFTP0230
                                                                      FFTP0240
                                                                      FFTP0260
                                   RAD=2*PI  C30=COS(PI/6)            FFTP0270
                                   DETERMINE THE SQUARE FACTORS OF N  FFTP0510
      SUBROUTINE FFTP(A,N,IWK,WK,LL)                                  FFTP0250
      DIMENSION          IWK(1),WK(1),Z0(2),Z1(2),Z2(2),Z3(2),Z4(2)   FFTP0280
      LOGICAL            L1,LL(1)                                     FFTP0290
      COMPLX*16          A(N),ZA0,ZA1,ZA2,ZA3,ZA4,AK2
      DOUBLE PRECISION   CM,SM,C1,C2,C3,S1,S2,S3,C30,RAD,WK,A0,A1,A4,B4,FFTP0310
     *                   A2,A3,B0,B1,B2,B3,ZERO,HALF,ONE,TWO,Z0,Z1,Z2, FFTP0320
     *                   Z3,Z4                                        FFTP0330
      EQUIVALENCE        (ZA0,Z0(1)),(ZA1,Z1(1)),(ZA2,Z2(1)),(ZA3,Z3(1))FFTP0350
     *                   ,(A0,Z0(1)),(B0,Z0(2)),(A1,Z1(1)),(B1,Z1(2)), FFTP0360
     *                   (A2,Z2(1)),(B2,Z2(2)),(A3,Z3(1)),(B3,Z3(2)), FFTP0370
     *                   (ZA4,Z4(1)),(Z4(1),A4),(Z4(2),B4)            FFTP0380
      DATA               RAD/6.283185307179586D0/,                    FFTP0390
     *                   C30/.8660254037844386D0/                     FFTP0400
      DATA               ZERO,HALF,ONE,TWO/0.0D0,0.5D0,1.0D0,2.0D0/   FFTP043
      IF (N .EQ. 1) GO TO 9005                                        FFTP045
      K = N                                                           FFTP046
      M = 0                                                           FFTP047
      J = 2                                                           FFTP048
      JJ = 4                                                          FFTP049
      JF = 0                                                          FFTP050
      IWK(1) = 1                                                      FFTP052
    5 I = K/JJ                                                        FFTP053
      IF (I*JJ .NE. K) GO TO 10                                       FFTP054
      M = M+1                                                         FFTP055
      IWK(M+1) = J                                                    FFTP056
      K = I                                                           FFTP057
      GO TO 5                                                         FFTP058
   10 J = J + 2                                                       FFTP059
      IF (J .EQ. 4) J = 3                                             FFTP060
      JJ = J * J                                                      FFTP061
      IF (JJ .LE. K) GO TO 5                                          FFTP062
      KT = M                                                          FFTP063
                                   DETERMINE THE REMAINING FACTORS OF N FFTP064
      J = 2                                                           FFTP065
   15 I = K / J                                                       FFTP066
      IF (I*J .NE. K) GO TO 20                                        FFTP067
      M = M + 1                                                       FFTP068
      IWK(M+1) = J                                                    FFTP069
      K = I                                                           FFTP070
      GO TO 15                                                        FFTP071
   20 J = J + 1                                                       FFTP072
      IF (J .EQ. 3) GO TO 15                                          FFTP073
      J = J + 1                                                       FFTP074
      IF(J.LE.K) GO TO 15                                             FFTP075
      K = IWK(M+1)                                                    FFTP076
      IF (IWK(KT+1) .GT. IWK(M+1)) K = IWK(KT+1)                      FFTP077
      IF(KT.LE.0) GO TO 30                                            FFTP078
      KTP = KT + 2                                                    FFTP079
      DO 25  I = 1,KT                                                 FFTP080
         J = KTP - I                                                  FFTP081
         M = M+1                                                      FFTP082
         IWK(M+1) = IWK(J)                                            FFTP083
   25 CONTINUE                                                        FFTP084
   30 MP = M+1                                                        FFTP085
```

```
      IC = MP+1
      ID = IC+MP
      ILL = ID+MP
      IRD = ILL+MP+1
      ICC = IRD+MP
      ISS = ICC+MP
      ICK = ISS+MP
      ISK = ICK+K
      ICF = ISK+K
      ISF = ICF+K
      IAP = ISF+K
      KD2 = (K-1) / 2 + 1
      IBP = IAP + KD2
      IAM = IBP + KD2
      IBM = IAM + KD2
      MM1 = M-1
      I=1
   35 L = MP - I
      J = IC - I
      LL(ILL+L) = (IWK(J-1) + IWK(J)) .EQ. 4
      IF (.NOT. LL(ILL+L)) GO TO 40
      I = I + 1
      L = L - 1
      LL(ILL+L) = .FALSE.
   40 I = I + 1
      IF(I.LE.MM1) GO TO 35
      LL(ILL+1) = .FALSE.
      LL(ILL+MP) = .FALSE.
      IWK(IC) = 1
      IWK(ID) = N
      DO 45  J = 1,M
         K = IWK(J+1)
         IWK(IC+J) = IWK(IC+J-1) * K
         IWK(ID+J) = IWK(ID+J-1) / K
         WK(IRD+J) = RAD/IWK(IC+J)
         C1 = RAD/K
         IF (K .LE. 2) GO TO 45
         WK(ICC+J) = DCOS(C1)
         WK(ISS+J) = DSIN(C1)
   45 CONTINUE
      MM = M
      IF (LL(ILL+M)) MM = M - 1
      IF (MM .LE. 1) GO TO 50
      SM = IWK(IC+MM-2) * WK(IRD+M)
      CM = DCOS(SM)
      SM = DSIN(SM)
   50 KB = 0
      KN = N
      JJ = 0
      I = 1
      C1 = ONE
      S1 = ZERO
      L1 = .TRUE.
   55 IF (LL(ILL+I+1)) GO TO 60
      KF = IWK(I+1)
      GO TO 65
   60 KF = 4
      I = I+1
   65 ISP = IWK(ID+I)
      IF (L1) GO TO 70
      S1 = JJ * WK(IRD+I)
      C1 = DCOS(S1)
      S1 = DSIN(S1)
C                                  FACTORS OF 2, 3, AND 4 ARE
C                                  HANDLED SEPARATELY.
   70 IF (KF .GT. 4) GO TO 140
      GO TO (75,75,90,115), KF
   75 K0 = KB + ISP
      K2 = K0 + ISP
      IF (L1) GO TO 85
   80 K0 = K0 - 1
      IF (K0 .LT. KB) GO TO 190
      K2 = K2 - 1
      ZA4 = A(K2+1)
      A0 = A4*C1-B4*S1
      B0 = A4*S1+B4*C1
      A(K2+1) = A(K0+1)-ZA0
      A(K0+1) = A(K0+1)+ZA0
      GO TO 80
```

```
 85 KO = KO - 1
    IF (KO .LT. KB) GO TO 190
    K2 = K2 - 1
    AK2 = A(K2+1)
    A(K2+1) = A(KO+1)-AK2
    A(KO+1) = A(KO+1)+AK2
    GO TO 85
 90 IF (L1) GO TO 95
    C2 = C1 * C1 - S1 * S1
    S2 = TWO * C1 * S1
 95 JA = KB + ISP - 1
    KA = JA + KB
    IKB = KB+1
    IJA = JA+1
    DO 110 II = IKB,IJA


        KO = KA - II + 1
        K1 = KO + ISP
        K2 = K1 + ISP
        ZAO = A(KO+1)
        IF (L1) GO TO 100
        ZA4 = A(K1+1)
        A1 = A4*C1-B4*S1
        B1 = A4*S1+B4*C1
        ZA4 = A(K2+1)
        A2 = A4*C2-B4*S2
        B2 = A4*S2+B4*C2
        GO TO 105
100     ZA1 = A(K1+1)
        ZA2 = A(K2+1)
105     A(KO+1) = DCMPLX(AO+A1+A2,BO+B1+B2)
        AO = -HALF * (A1+A2) + AO
        A1 = (A1-A2) * C30
        BO = -HALF * (B1+B2) + BO
        B1 = (B1-B2) * C30
        A(K1+1) = DCMPLX(AO-B1,BO+A1)
        A(K2+1) = DCMPLX(AO+B1,BO-A1)
110 CONTINUE
    GO TO 190
115 IF (L1) GO TO 120
    C2 = C1 * C1 - S1 * S1
    S2 = TWO * C1 * S1
    C3 = C1 * C2 - S1 * S2
    S3 = S1 * C2 + C1 * S2
120 JA = KB + ISP - 1
    KA = JA + KB
    IKB = KB+1
    IJA = JA+1
    DO 135 II = IKB,IJA
        KO = KA - II + 1
        K1 = KO + ISP
        K2 = K1 + ISP
        K3 = K2 + ISP
        ZAO = A(KO+1)
        IF (L1) GO TO 125
        ZA4 = A(K1+1)
        A1 = A4*C1-B4*S1
        B1 = A4*S1+B4*C1
        ZA4 = A(K2+1)
        A2 = A4*C2-B4*S2
        B2 = A4*S2+B4*C2
        ZA4 = A(K3+1)
        A3 = A4*C3-B4*S3
        B3 = A4*S3+B4*C3
        GO TO 130
125     ZA1 = A(K1+1)
        ZA2 = A(K2+1)
        ZA3 = A(K3+1)
130     A(KO+1) = DCMPLX(AO+A2+A1+A3,BO+B2+B1+B3)
        A(K1+1) = DCMPLX(AO+A2-A1-A3,BO+B2-B1-B3)
        A(K2+1) = DCMPLX(AO-A2-B1+B3,BO-B2+A1-A3)
        A(K3+1) = DCMPLX(AO-A2+B1-B3,BO-B2-A1+A3)
```

```
135 CONTINUE
    GO TO 190
140 JK = KF - 1
    KH = JK/2
    K3 = IWK(ID+I-1)
    KO = KB + ISP
    IF (L1) GO TO 150
    K = JK - 1
    WK(ICF+1) = C1
    WK(ISF+1) = S1
    DO 145 J = 1,K
       WK(ICF+J+1) = WK(ICF+J) * C1 - WK(ISF+J) * S1
       WK(ISF+J+1) = WK(ICF+J) * S1 + WK(ISF+J) * C1
145 CONTINUE
150 IF (KF .EQ. JF) GO TO 160
    C2 = WK(ICC+I)
    WK(ICK+1) = C2
    WK(ICK+JK) = C2
    S2 = WK(ISS+I)
    WK(ISK+1) = S2
    WK(ISK+JK) = -S2
    DO 155 J = 1,KH
       K = JK - J
       WK(ICK+K) = WK(ICK+J) * C2 - WK(ISK+J) * S2
       WK(ICK+J+1) = WK(ICK+K)
       WK(ISK+J+1) = WK(ICK+J) * S2 + WK(ISK+J) * C2
       WK(ISK+K) = -WK(ISK+J+1)
155 CONTINUE
160 KO = KO - 1
    K1 = KO
    K2 = KO + K3
    ZA0 = A(KO+1)
    A3 = A0
    B3 = B0
    DO 175 J = 1,KH
       K1 = K1 + ISP
       K2 = K2 - ISP
       IF (L1) GO TO 165


       K = KF - J
       ZA4 = A(K1+1)
       A1 = A4*WK(ICF+J)-B4*WK(ISF+J)
       B1 = A4*WK(ISF+J)+B4*WK(ICF+J)
       ZA4 = A(K2+1)
       A2 = A4*WK(ICF+K)-B4*WK(ISF+K)
       B2 = A4*WK(ISF+K)+B4*WK(ICF+K)
       GO TO 170
165    ZA1 = A(K1+1)
       ZA2 = A(K2+1)
170    WK(IAP+J) = A1 + A2
       WK(IAM+J) = A1 - A2
       WK(IBP+J) = B1 + B2
       WK(IBM+J) = B1 - B2
       A3 = A1 + A2 + A3
       B3 = B1 + B2 + B3
175 CONTINUE
    A(KO+1) = DCMPLX(A3,B3)
    K1 = KO
    K2 = KO + K3
    DO 185 J = 1,KH
       K1 = K1 + ISP
       K2 = K2 - ISP
       JK = J
       A1 = A0
       B1 = B0
       A2 = ZERO
       B2 = ZERO
       DO 180  K = 1,KH
          A1 = A1 + WK(IAP+K) * WK(ICK+JK)
          A2 = A2 + WK(IAM+K) * WK(ISK+JK)
          B1 = B1 + WK(IBP+K) * WK(ICK+JK)
          B2 = B2 + WK(IBM+K) * WK(ISK+JK)
          JK = JK + J
          IF (JK .GE. KF) JK = JK - KF
180    CONTINUE
       A(K1+1) = DCMPLX(A1-B2,B1+A2)
       A(K2+1) = DCMPLX(A1+B2,B1-A2)
185 CONTINUE
    IF (KO .GT. KB) GO TO 160
    JF = KF
190 IF ( I .GE. MM ) GO TO 195
    I = I + 1
```

```
      GO TO 55
  195 I = MM
      L1 = .FALSE.
      KB = IWK(ID+I-1) + KB
      IF (KB .GE. KN) GO TO 215
  200 JJ = IWK(IC+I-2) + JJ
      IF (JJ .LT. IWK(IC+I-1)) GO TO 205
      I = I - 1
      JJ = JJ - IWK(IC+I)
      GO TO 200
  205 IF (I .NE. MM) GO TO 210
      C2 = C1
      C1 = CM * C1 - SM * S1
      S1 = SM * C2 + CM * S1
      GO TO 70
  210 IF(LL(ILL+I)) I = I + 1
      GO TO 55
  215 I = 1
      JA = KT - 1
      KA = JA + 1
      IF(JA.LT.1) GO TO 225
      DO 220  II = 1,JA
         J = KA - II
         IWK(J+1) = IWK(J+1) - 1
         I = IWK(J+1) + I
  220 CONTINUE
                                          THE RESULT IS NOW PERMUTED TO
                                          NORMAL ORDER.
  225 IF (KT .LE. 0) GO TO 270
      J = 1
      I = 0
      KB = 0
  230 K2 = IWK(ID+J) + KB
      K3 = K2
      JJ = IWK(IC+J-1)
      JK = JJ
      K0 = KB + JJ
      ISP = IWK(IC+J) - JJ
  235 K = K0 + JJ
  240 ZA4 = A(K0+1)
      A(K0+1) = A(K2+1)
      A(K2+1) = ZA4
      K0 = K0 + 1
      K2 = K2 + 1
      IF (K0 .LT. K) GO TO 240
      K0 = K0 + ISP
      K2 = K2 + ISP
      IF (K0 .LT. K3) GO TO 235
      IF (K0 .GE. K3 + ISP) GO TO 245
      K0 = K0 - IWK(ID+J) + JJ
      GO TO 235

  245 K3 = IWK(ID+J) + K3
      IF (K3 - KB .GE. IWK(ID+J-1)) GO TO 250
      K2 = K3 + JK
      JK = JK + JJ
      K0 = K3 - IWK(ID+J) + JK
      GO TO 235
  250 IF (J .GE. KT) GO TO 260
      K = IWK(J+1) + I
      J = J + 1
  255 I = I + 1
      IWK(ILL+I) = J
      IF (I .LT. K) GO TO 255
      GO TO 230
  260 KB = K3
      IF (I .LE. 0) GO TO 265
      J = IWK(ILL+I)
      I = I - 1
      GO TO 230
  265 IF (KB .GE. N) GO TO 270
      J = 1
      GO TO 230
  270 JK = IWK(IC+KT)
      ISP = IWK(ID+KT)
      M = M - KT
      KB = ISP/JK-2
      IF (KT .GE. M-1 ) GO TO 9005
      ITA = ILL+KB+1
      ITB = ITA+JK
      IDM1 = ID-1
      IKT = KT+1
      IM = M+1
```

```
      DO 275 J = IKT,IM
         IWK(IDM1+J) = IWK(IDM1+J)/JK
  275 CONTINUE
      JJ = 0
      DO 290 J = 1,KB
         K = KT
  280    JJ = IWK(ID+K+1) + JJ
         IF (JJ .LT. IWK(ID+K)) GO TO 285
         JJ = JJ - IWK(ID+K)
         K = K + 1
         GO TO 280
  285    IWK(ILL+J) = JJ
         IF (JJ .EQ. J) IWK(ILL+J) = -J
  290 CONTINUE
C                                  DETERMINE THE PERMUTATION CYCLES
C                                  OF LENGTH GREATER THAN OR EQUAL
C                                  TO TWO.
      DO 300  J = 1,KB
         IF (IWK(ILL+J) .LE. 0) GO TO 300
         K2 = J
  295    K2 = IABS(IWK(ILL+K2))
         IF (K2 .EQ. J) GO TO 300
         IWK(ILL+K2) = -IWK(ILL+K2)
         GO TO 295
  300 CONTINUE
C                                  REORDER A FOLLOWING THE
C                                  PERMUTATION CYCLES
      I = 0
      J = 0
      KB = 0
      KN = N
  305 J = J + 1
      IF (IWK(ILL+J) .LT. 0) GO TO 305
      K = IWK(ILL+J)
      KO = JK * K + KB
  310 ZA4 = A(KO+I+1)
      WK(ITA+I) = A4
      WK(ITB+I) = B4
      I = I + 1
      IF (I .LT. JK) GO TO 310
      I = 0
  315 K = -IWK(ILL+K)
      JJ = KO
      KO = JK * K + KB
  320 A(JJ+I+1) = A(KO+I+1)
      I = I + 1
      IF (I .LT. JK) GO TO 320
      I = 0
      IF (K .NE. J) GO TO 315
  325 A(KO+I+1) = DCMPLX(WK(ITA+I),WK(ITB+I))
      I = I + 1
      IF (I .LT. JK) GO TO 325
      I = 0
      IF (J .LT. K2) GO TO 305
      J = 0
      KB = KB + ISP
      IF (KB .LT. KN) GO TO 305
  005 RETURN
      END
```

DATA FOR THE SOIL LAYERS

.1364E+06	0.2500E+01	0.1125E+12	0.1000E+00
.1174E+06	0.2400E+01	0.5290E+12	0.1000E+00
.4910E+05	0.2350E+01	0.4369E+12	0.1000E+00
.4600E+05	0.2150E+01	0.6395E+11	0.1000E+00

DATA FOR THE REAL EARTHQUAKE MOTION

.79	-.555	-.305	-.058	.203	.396	.593	.889	1.23	1.557
.854	2.136	2.375	2.599	2.851	3.108	3.294	3.413	3.645	3.811
016	4.202	4.279	4.234	4.279	4.471	4.665	4.837	5.033	5.045
.778	4.347	3.607	2.532	1.142	-.249	-1.454	-2.409	-2.41	-1.897
-1.289	-.697	-.281	-0.534	-.859	-.91	-.555	-.333	-.34	-.63
-.595	-.612	-.683	-.786	-.97	-.78	-.351	-.162	-.231	-.655
-3.235	-3.131	-3.036	-3.214	-3.265	-2.845	-2.739	-2.815	-2.956	-3.048
-3.622	-4.042	-3.48	-2.992	-2.389	-1.42	-0.737	-.835	-1.429	-1.293
-1.205	-1.94	-2.397	-2.174	-1.849	-1.4	-0.607	-.586	-1.472	-2.366
2.521	-2.049	-1.753	-2.056	-2.094	-2.062	-2.282	-2.568	-2.705	-2.832

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FIGURE CAPTIONS

Figure 1 Elimination of ω_0 between $A = A(\omega_0)$ and $\omega = \omega(\omega_0)$ for obtaining $A = A(\omega)$.

Figure 2 Amplification spectra (a) Parameters ρ , G_0 and layer thicknesses d for the multilayered system, (b, c) Amplification spectrum for the lowest layer; (d, e) Amplification spectrum for the multi layer system

Figure 3 Response of a single layer to the Gaussian input motion at its base as $v = A \exp(-(t-t_0)^2/\sigma^2)$ with $\sigma = 1$ sec., $t_0 = 5$ sec. The parameters for the layer are those of the top layer in Figure 2a and $\kappa = 0.1$. (a', b', c') are the Fourier amplitudes corresponding to cases (a, b, c).

Figure 4 Response of the single layer system to the earthquake forcing applied at its base. The parameters for the layer are those of the top layer in Figure 2a with $\kappa = 0.1$. The input motion is the first 100 time intervals of the Taft 1952 earthquake [9]. The amplitude of the motion is taken as (a) the actual value (b) two times the actual value (c) four times the actual value. (a', b', c') are the Fourier amplitudes corresponding to cases (a, b, c).

Figure 5 Response of the multilayer system in Figure 2 w with $\kappa = 0.1$ to the Gaussian input motion at its base as $v = A \exp(-(t-t_0)^2/\sigma^2)$ with $\sigma = 1$ sec., $t_0 = 5$ sec. (a', b') are the Fourier amplitudes corresponding to cases (a, b).

Figure 6 Response of the multilayer system in Figure 2a with $\kappa = 0.1$ to the earthquake forcing applied at its base. The input motion is the first 100 time intervals of the Taft 1952 earthquake [9]. The amplitude of the motion is taken as (a) the actual value (b) two times the actual value; (a', b') are the Fourier amplitudes corresponding to cases (a, b).

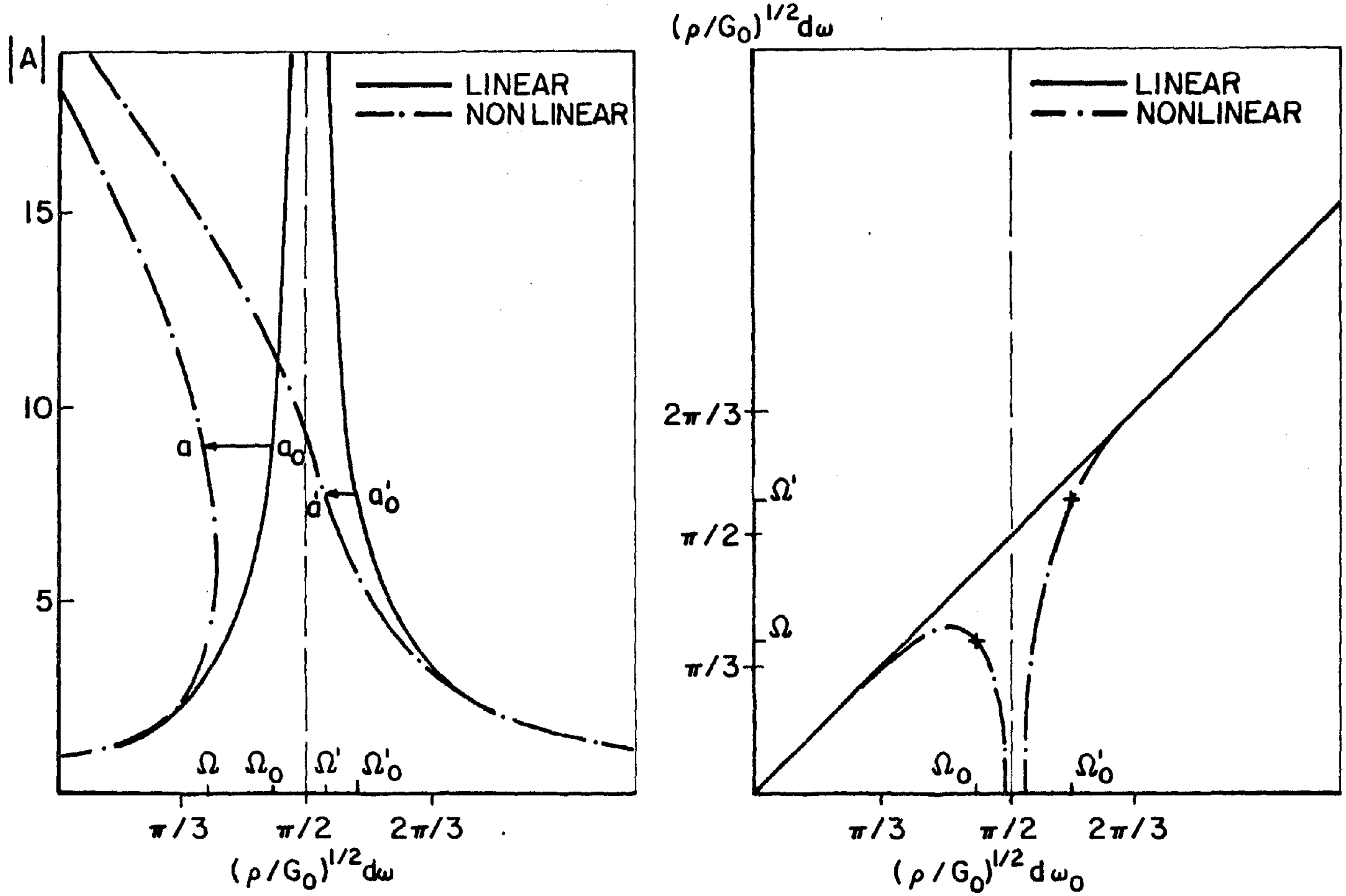
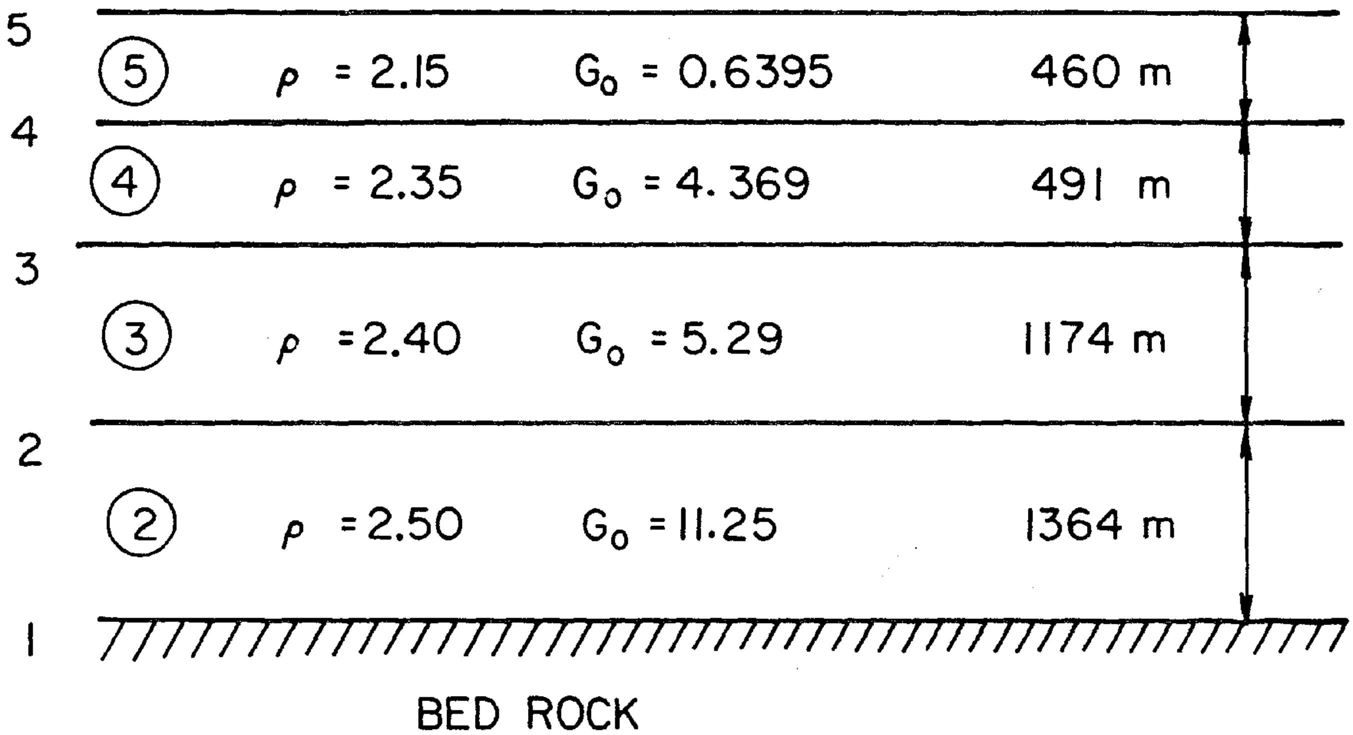

|A|
15
10
5
LINEAR
NON LINEAR
a
a0
a'
a'0
Ω
Ω0
Ω'
Ω'0
π/3
π/2
2π/3
(ρ/G0)1/2dω
(ρ/G0)1/2dω
LINEAR
NONLINEAR
2π/3
Ω'
π/2
Ω
π/3
Ω0
Ω'0
π/3
π/2
2π/3
(ρ/G0)1/2dω0

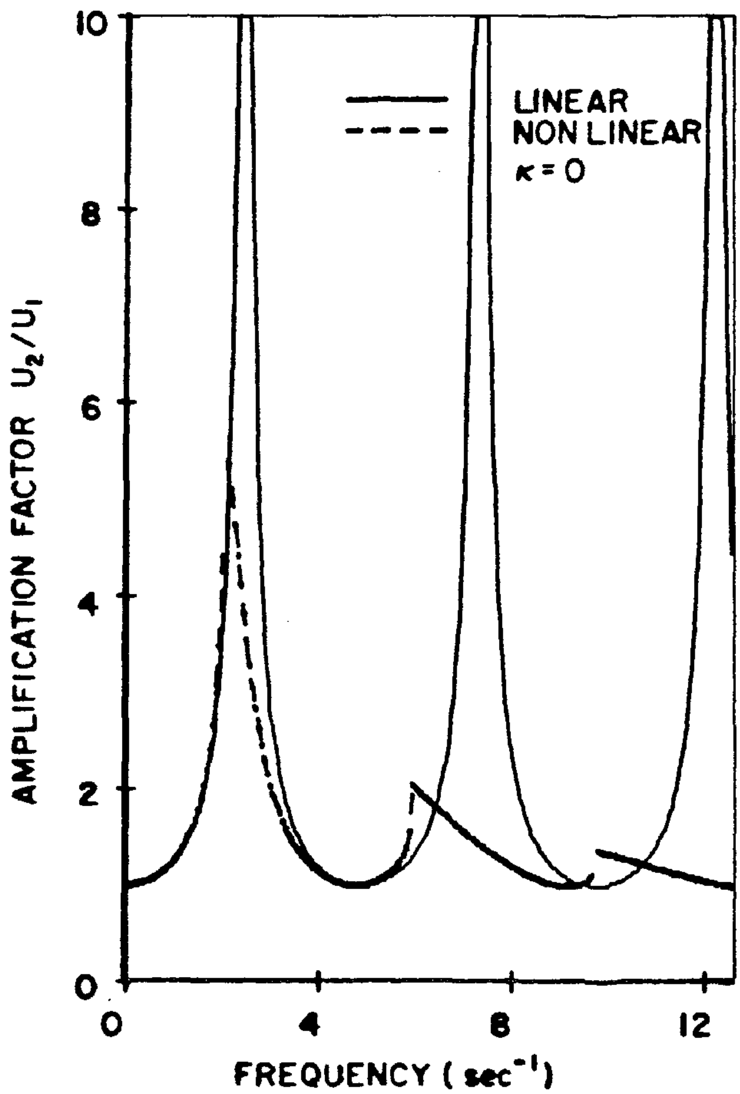
FIGURE 1



ρ (gr/cm^3) G_o (10^{10} dy/cm^2)

$G_1 = -\frac{1}{4} 10^6 G_O$ IN EACH LAYER

FIGURE 2a



10
8
6
4
2
0
AMPLIFICATION FACTOR U2/U1
LINEAR
NON LINEAR
κ = 0
0
4
8
12
FREQUENCY (sec⁻¹)

FIGURE 2b

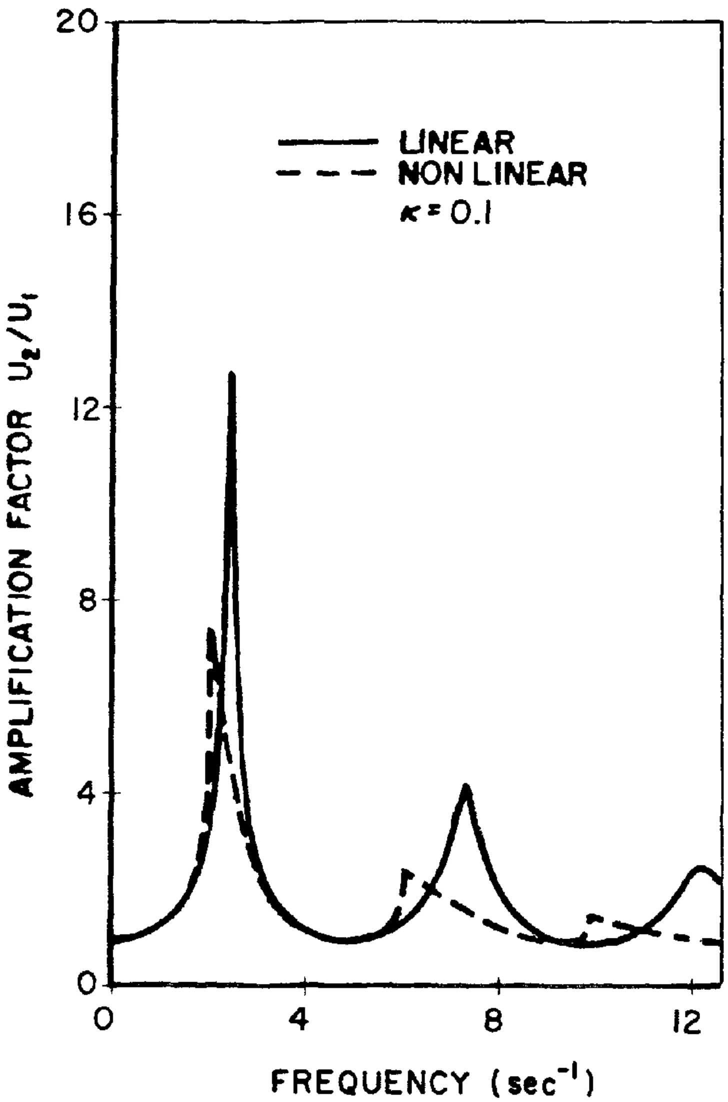
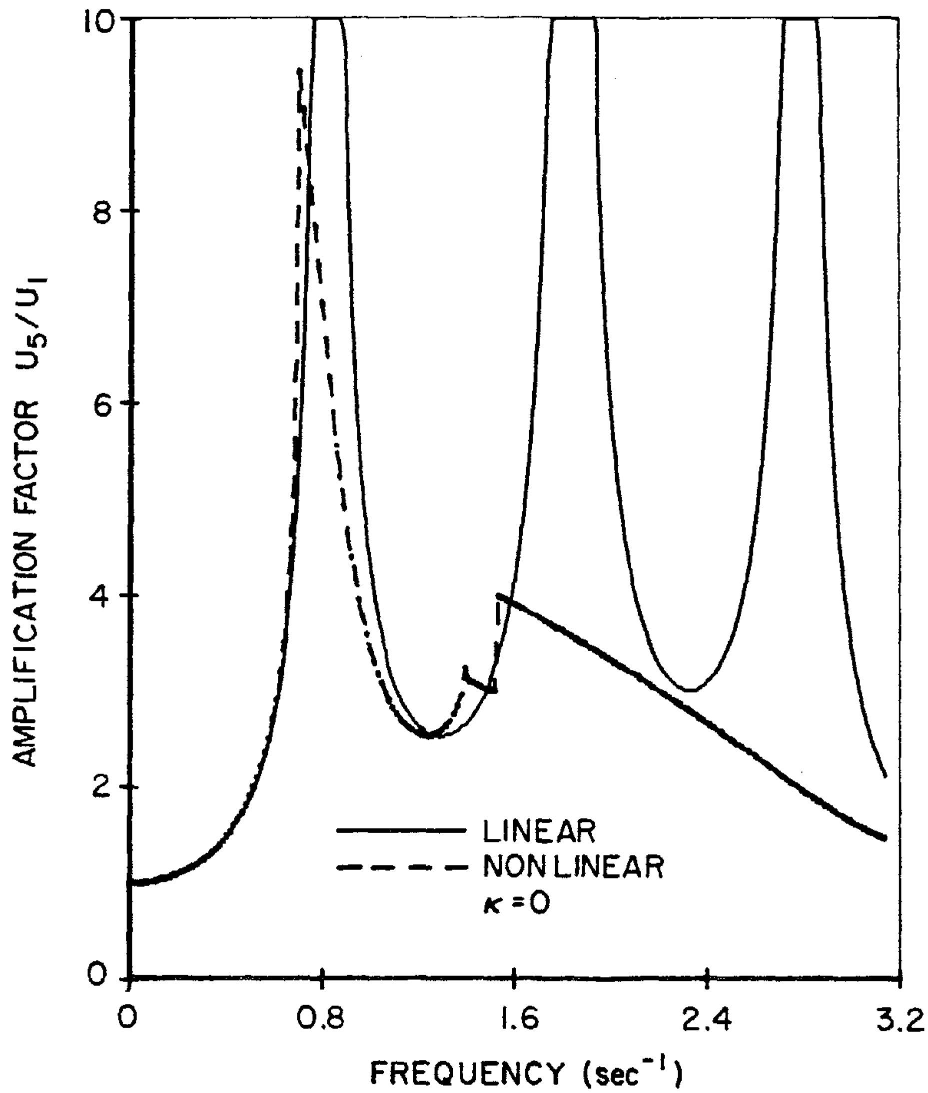
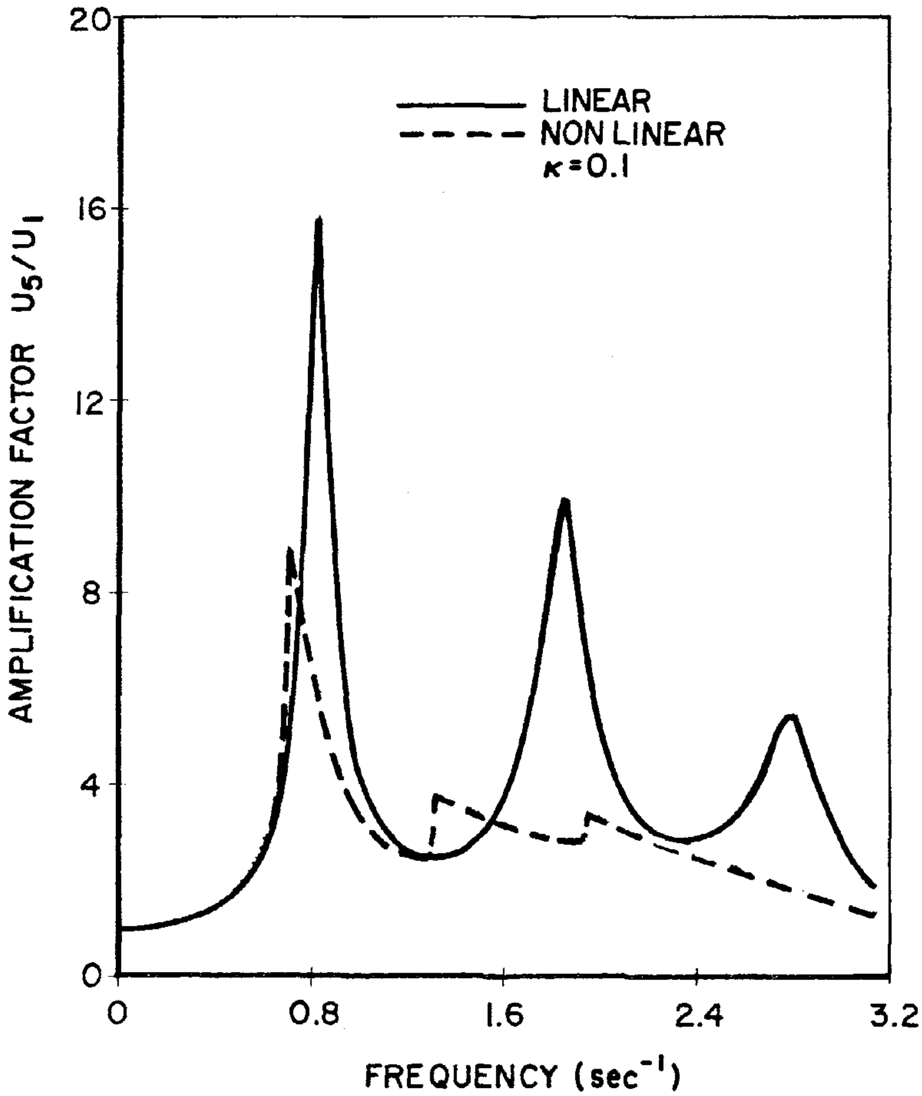

LINEAR
NON LINEAR
κ = 0.1
AMPLIFICATION FACTOR U2/U1
FREQUENCY (sec-1)
0
4
8
12
16
20
0
4
8
12

FIGURE 2c



10
8
6
4
2
0
AMPLIFICATION FACTOR U5/U1
0
0.8
1.6
2.4
3.2
FREQUENCY (sec⁻¹)
LINEAR
NON LINEAR
κ = 0

FIGURE 2d



LINEAR
NON LINEAR
κ=0.1
AMPLIFICATION FACTOR U5/U1
20
16
12
8
4
0
0
0.8
1.6
2.4
3.2
FREQUENCY (sec⁻¹)

FIGURE 2e

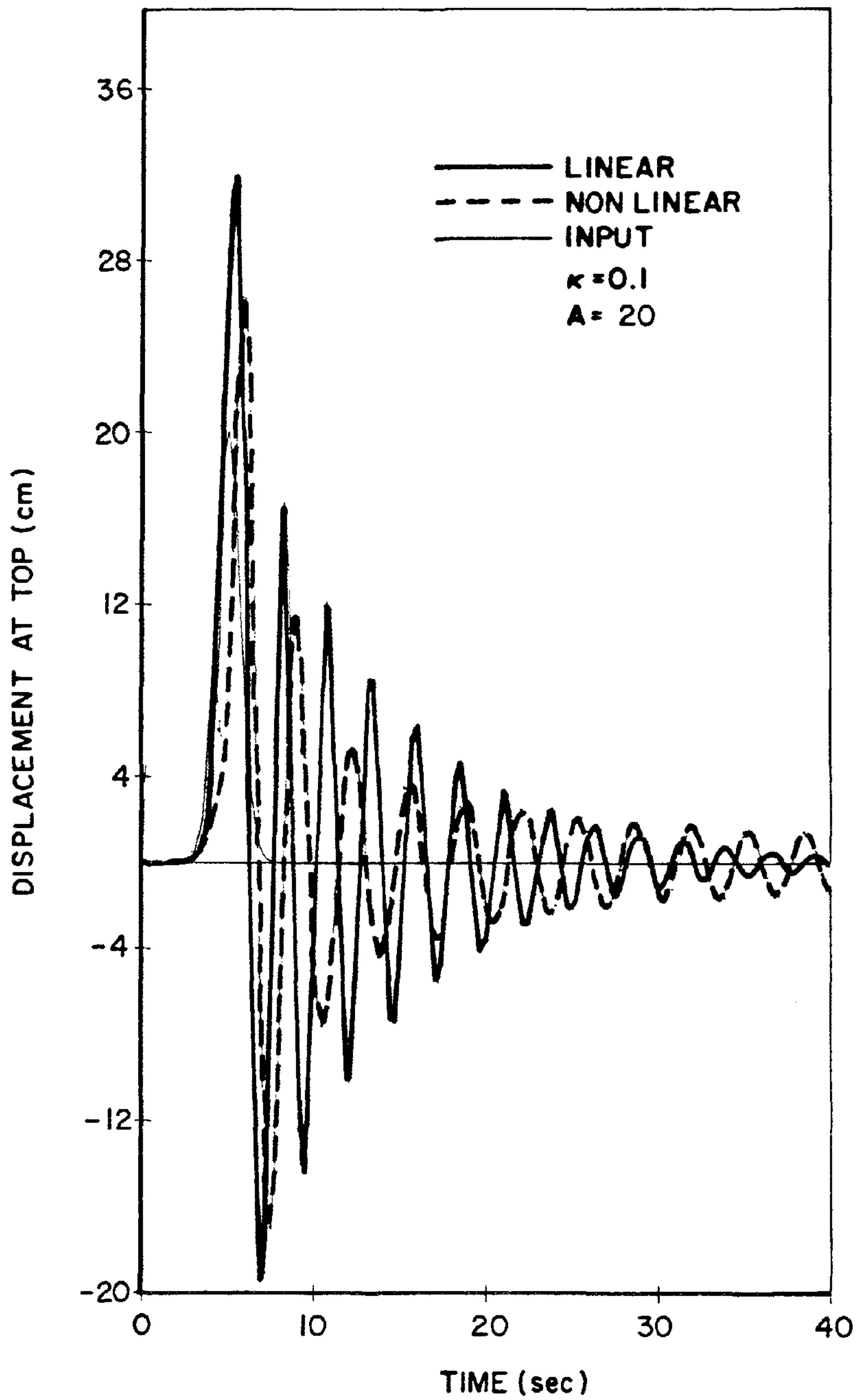

LINEAR
NON LINEAR
INPUT
κ=0.1
A= 20
36
28
20
12
4
-4
-12
-20
0
10
20
30
40
DISPLACEMENT AT TOP (cm)
TIME (sec)

FIGURE 3a

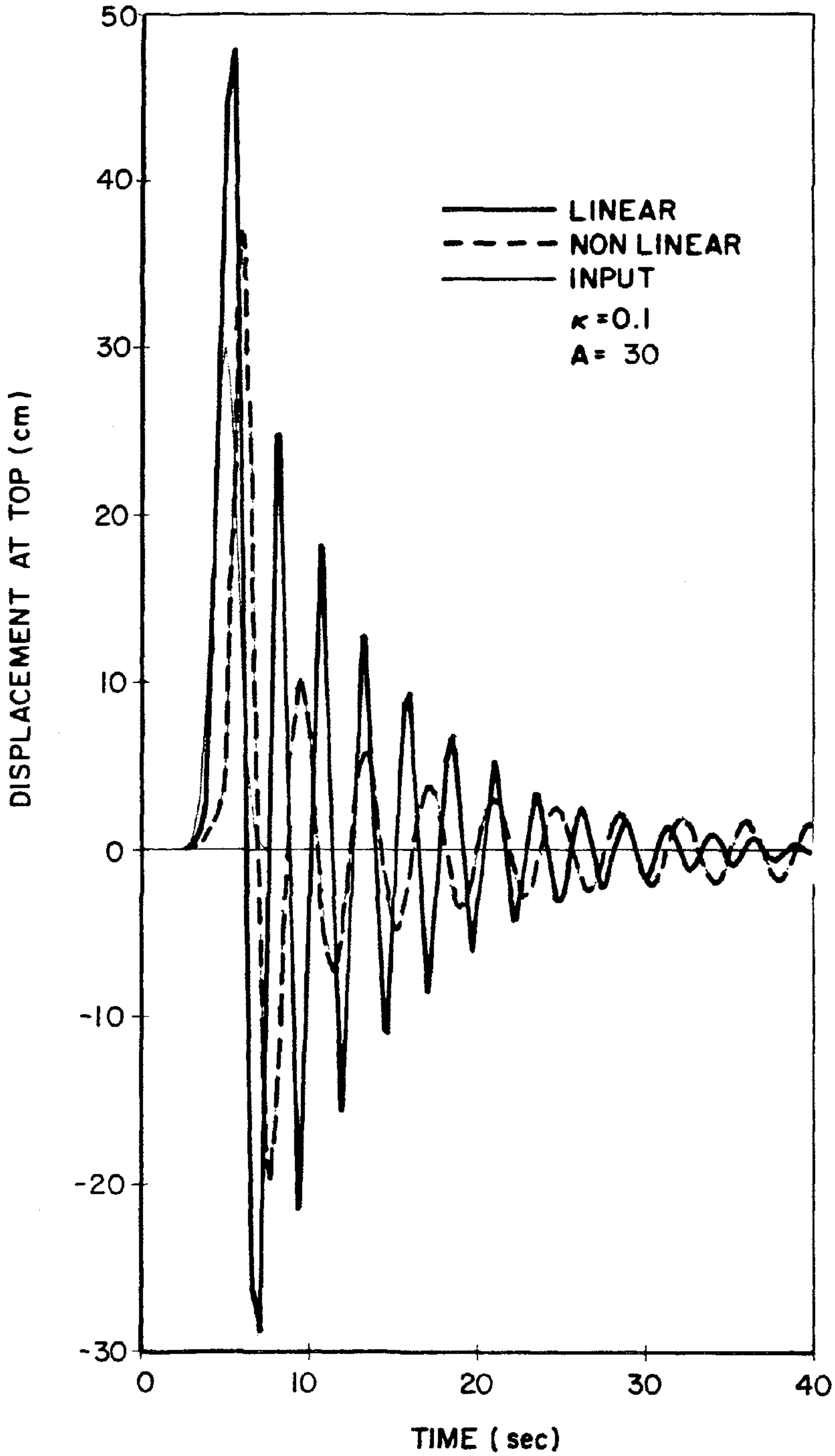
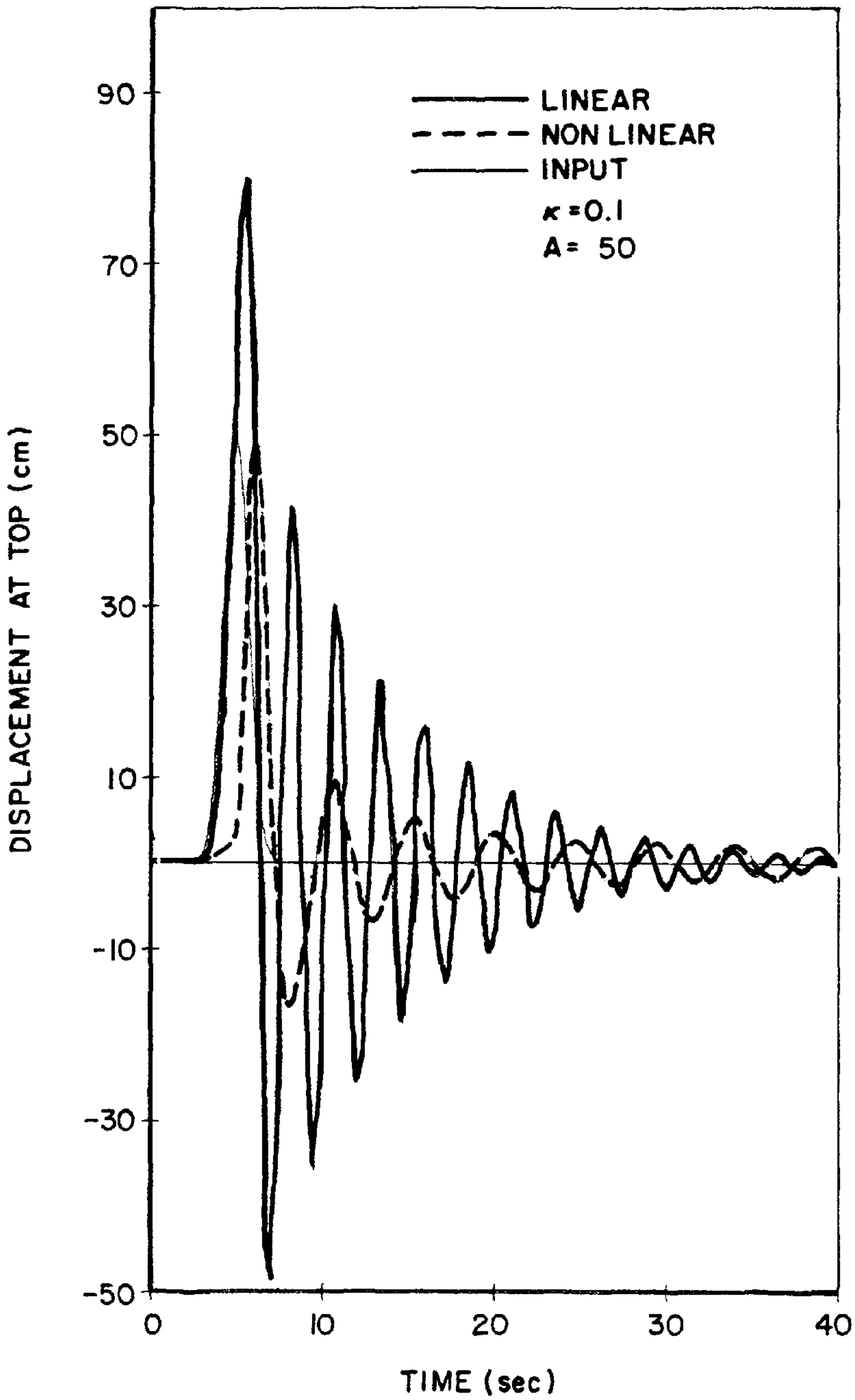
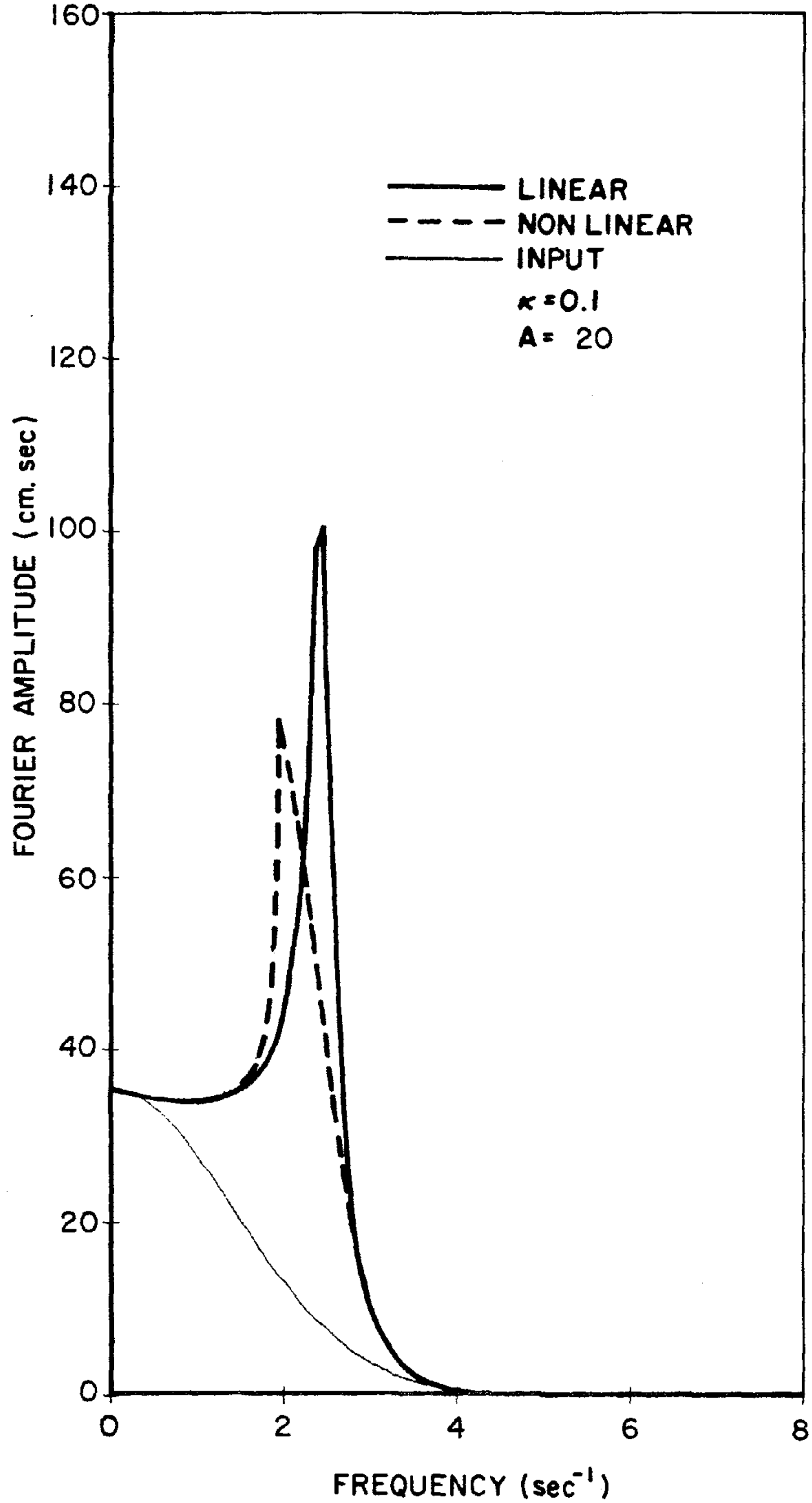

50
40
30
20
10
0
-10
-20
-30
DISPLACEMENT AT TOP (cm)
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 30
0
10
20
30
40
TIME (sec)

FIGURE 3b



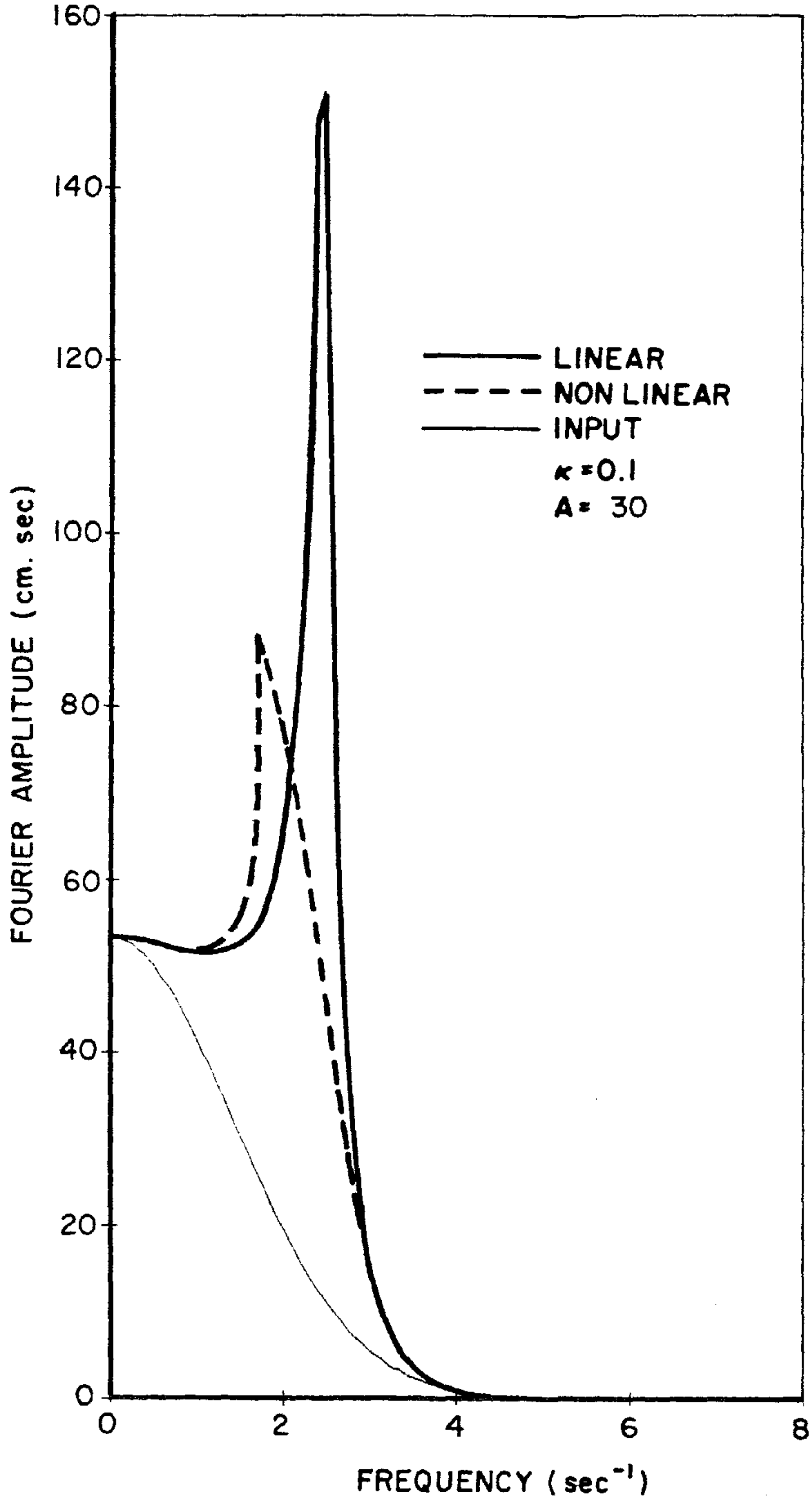
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 50
90
70
50
30
10
-10
-30
-50
DISPLACEMENT AT TOP (cm)
0
10
20
30
40
TIME (sec)

FIGURE 3c



160
140
120
100
80
60
40
20
0
FOURIER AMPLITUDE (cm. sec)
LINEAR
NON LINEAR
INPUT
κ=0.1
A= 20
0
2
4
6
8
FREQUENCY (sec⁻¹)

FIGURE 3a'



160
140
120
100
80
60
40
20
0
0
2
4
6
8
FOURIER AMPLITUDE (cm. sec)
FREQUENCY (sec^{-1})
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 30

FIGURE 3b'

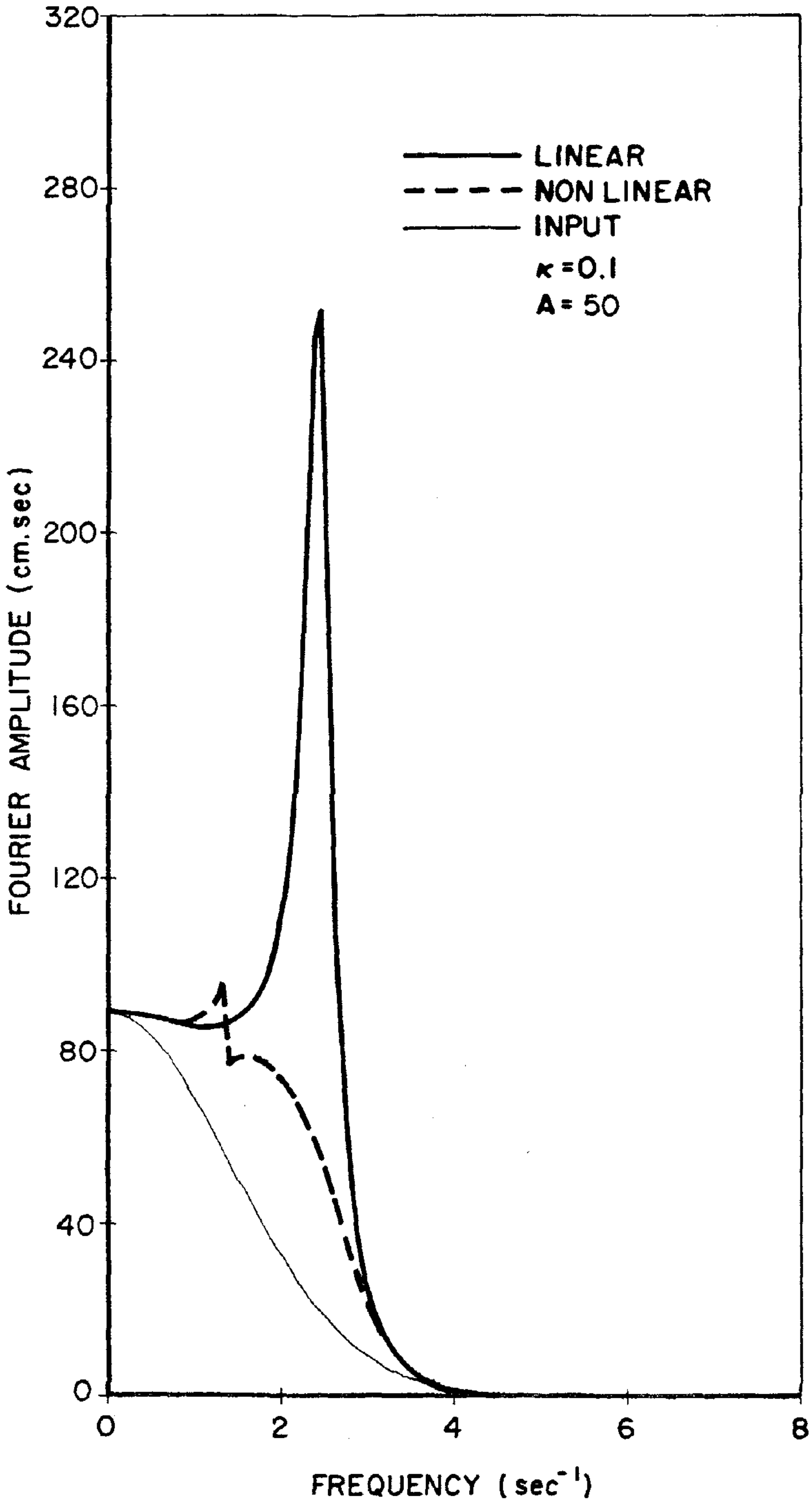
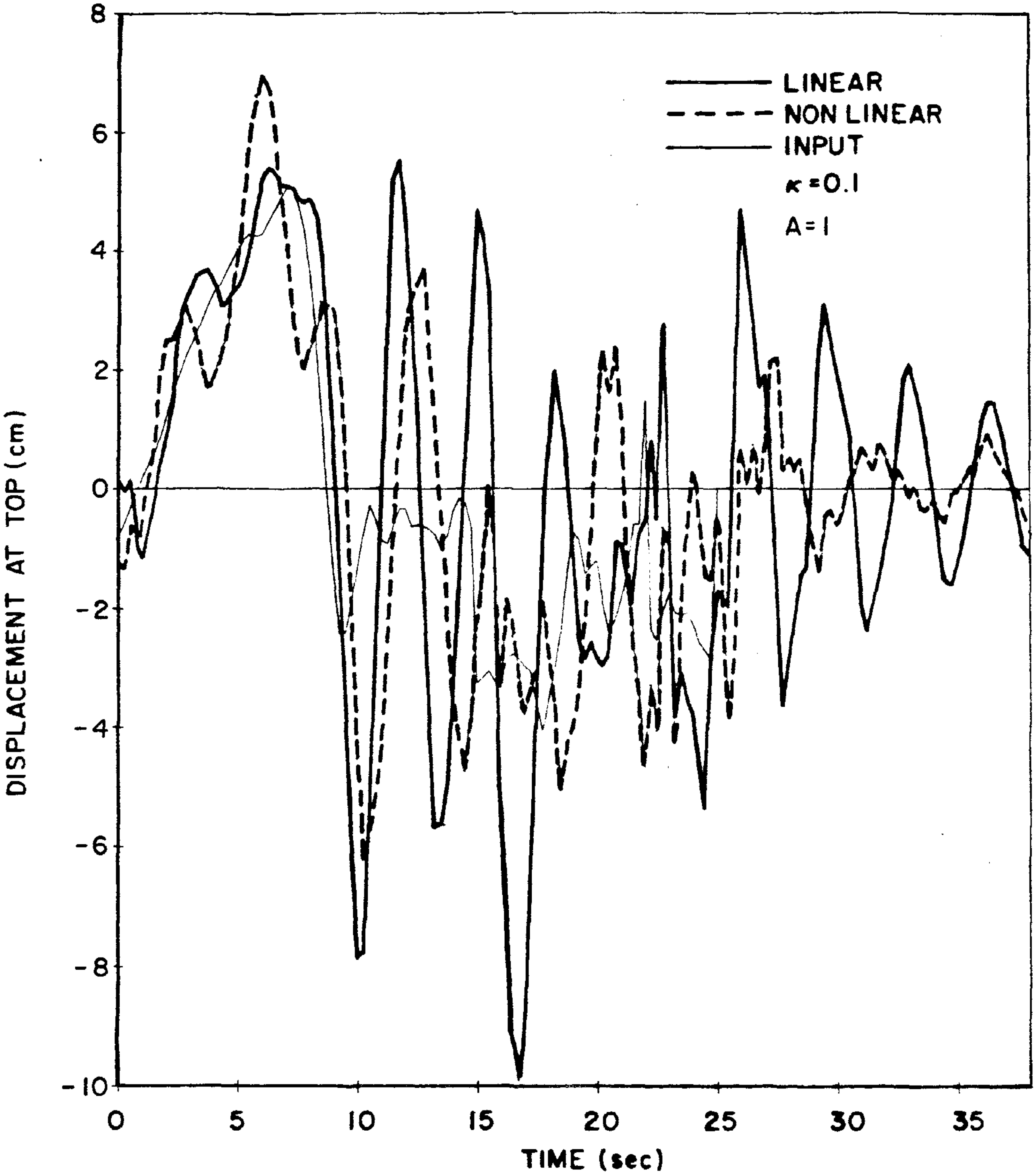
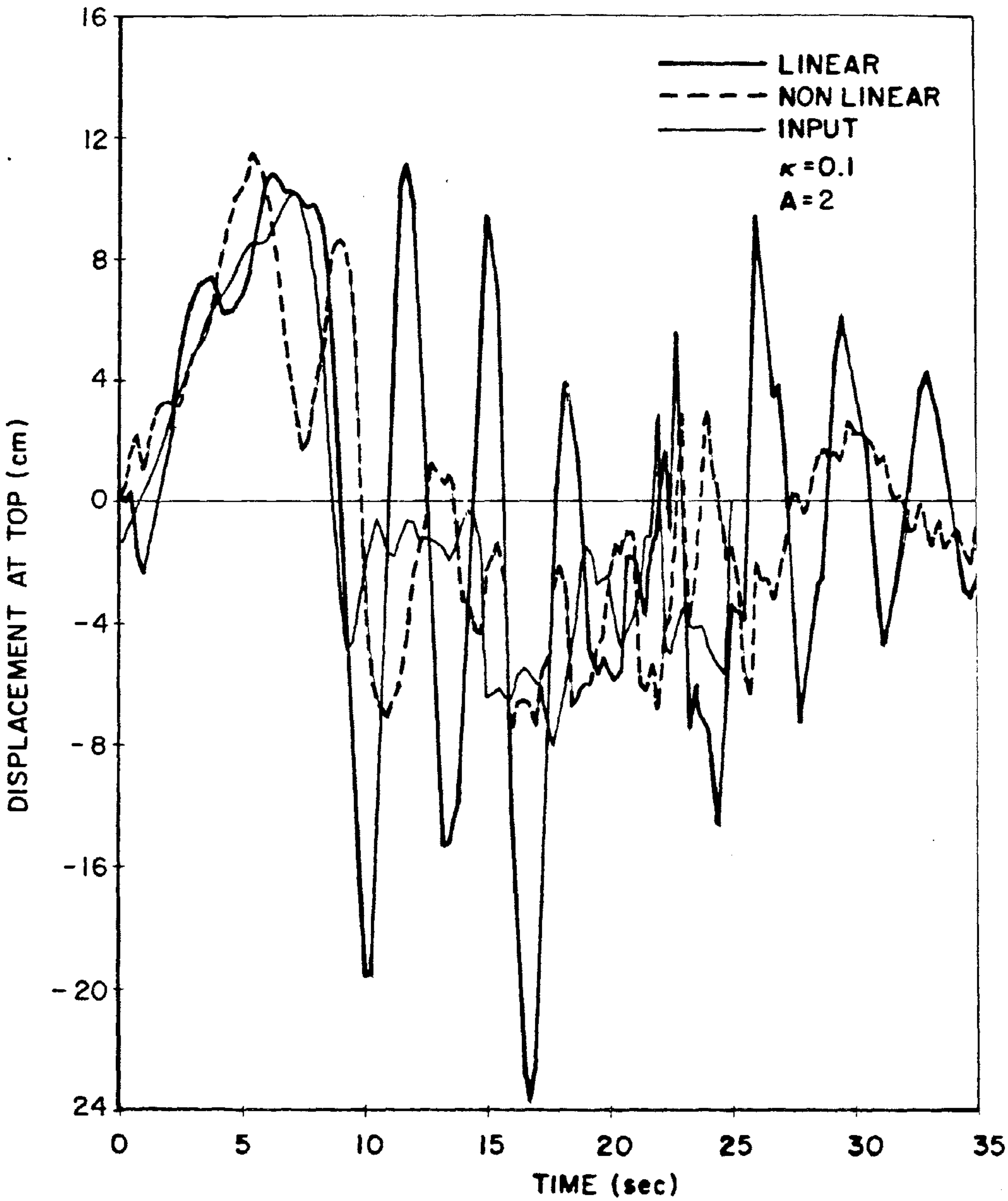

LINEAR
NON LINEAR
INPUT
κ=0.1
A= 50
FOURIER AMPLITUDE (cm.sec)
FREQUENCY (sec⁻¹)
0
40
80
120
160
200
240
280
320
0
2
4
6
8

FIGURE 3c'



LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 1
DISPLACEMENT AT TOP (cm)
8
6
4
2
0
-2
-4
-6
-8
-10
0
5
10
15
20
25
30
35
TIME (sec)

FIGURE 4a



LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 2
16
12
8
4
0
-4
-8
-16
-20
24
DISPLACEMENT AT TOP (cm)
0
5
10
15
20
25
30
35
TIME (sec)

FIGURE 4b

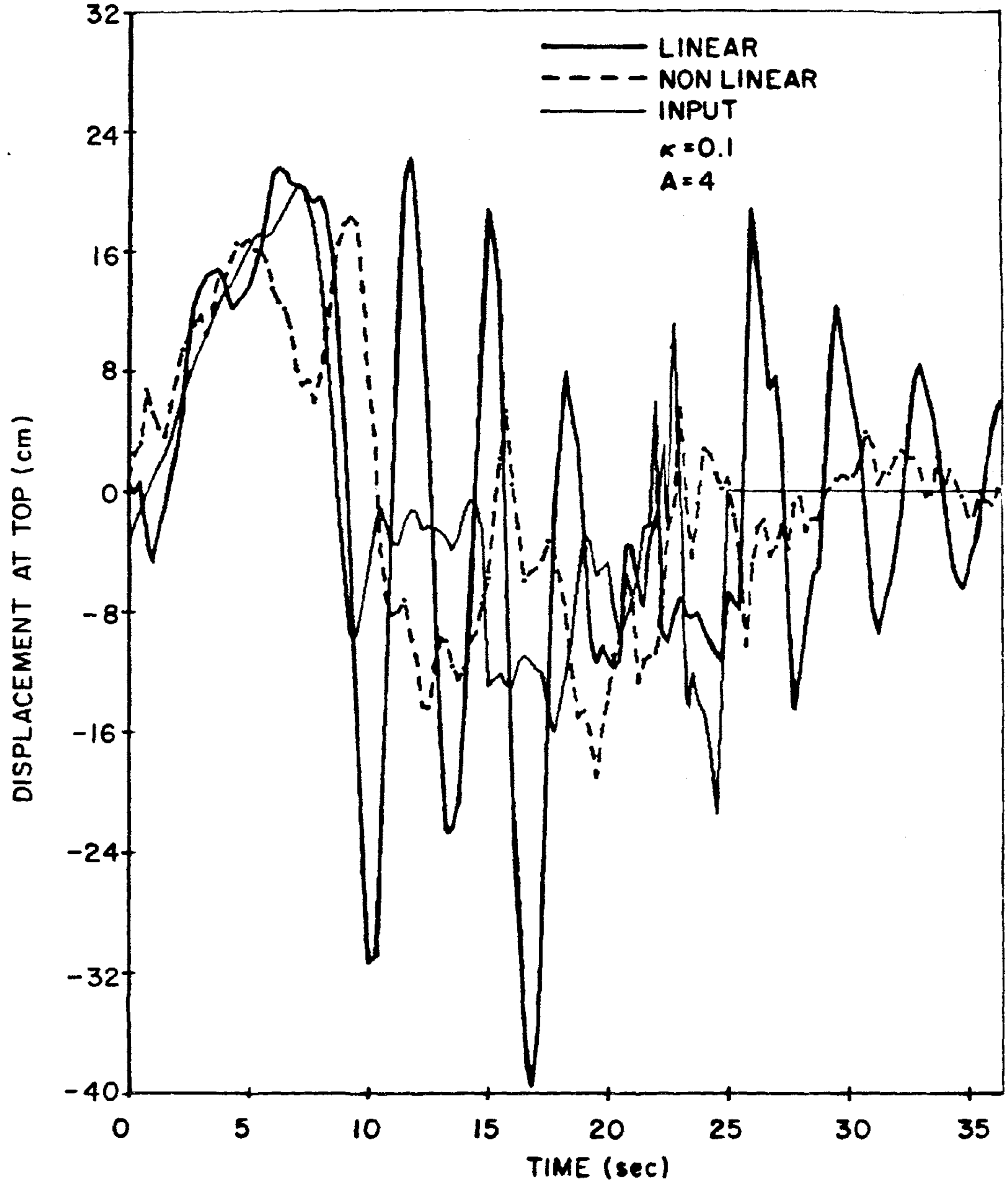

LINEAR
NON LINEAR
INPUT
κ=0.1
A=4
32
24
16
8
0
-8
-16
-24
-32
-40
DISPLACEMENT AT TOP (cm)
0
5
10
15
20
25
30
35
TIME (sec)

FIGURE 4c

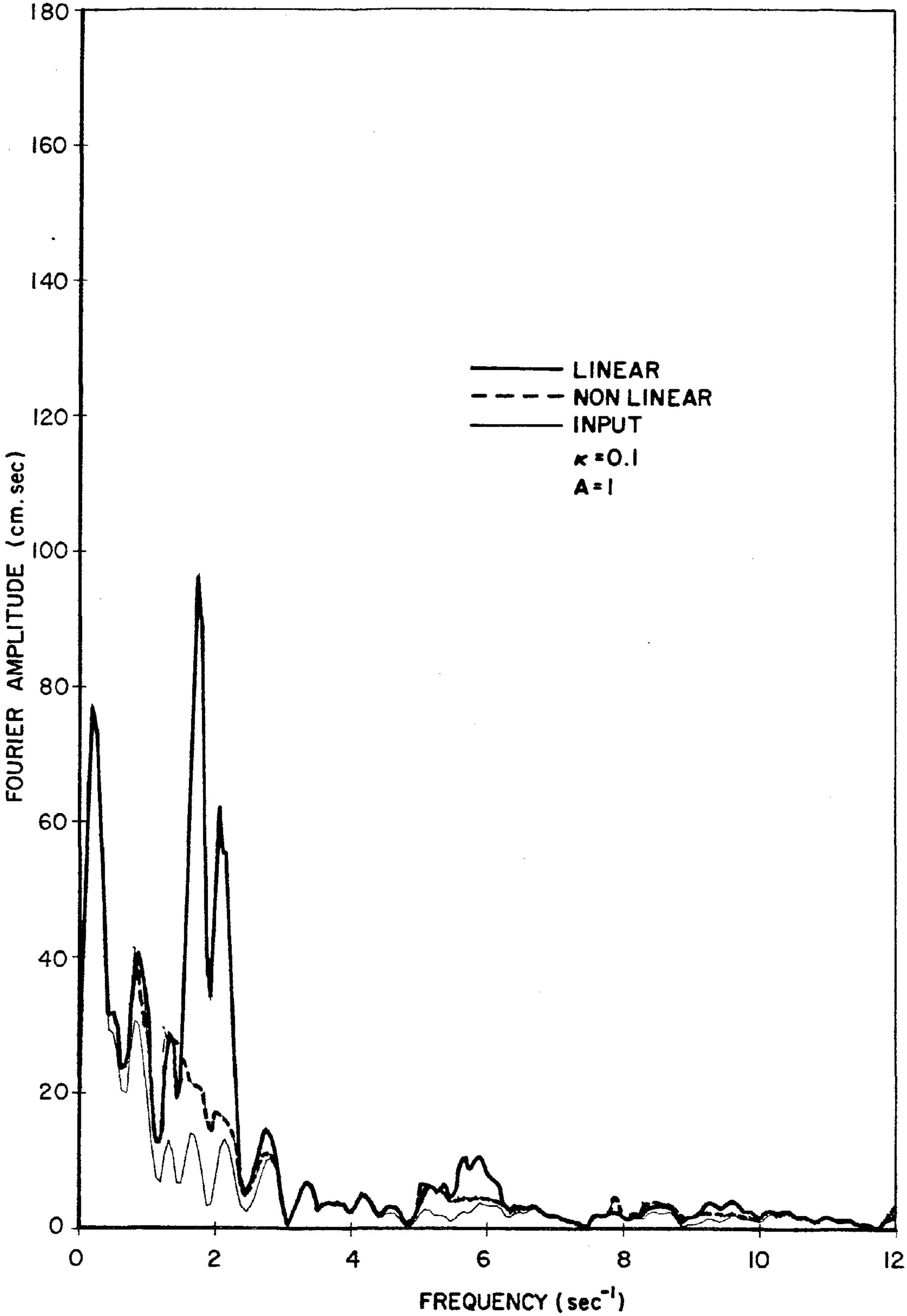
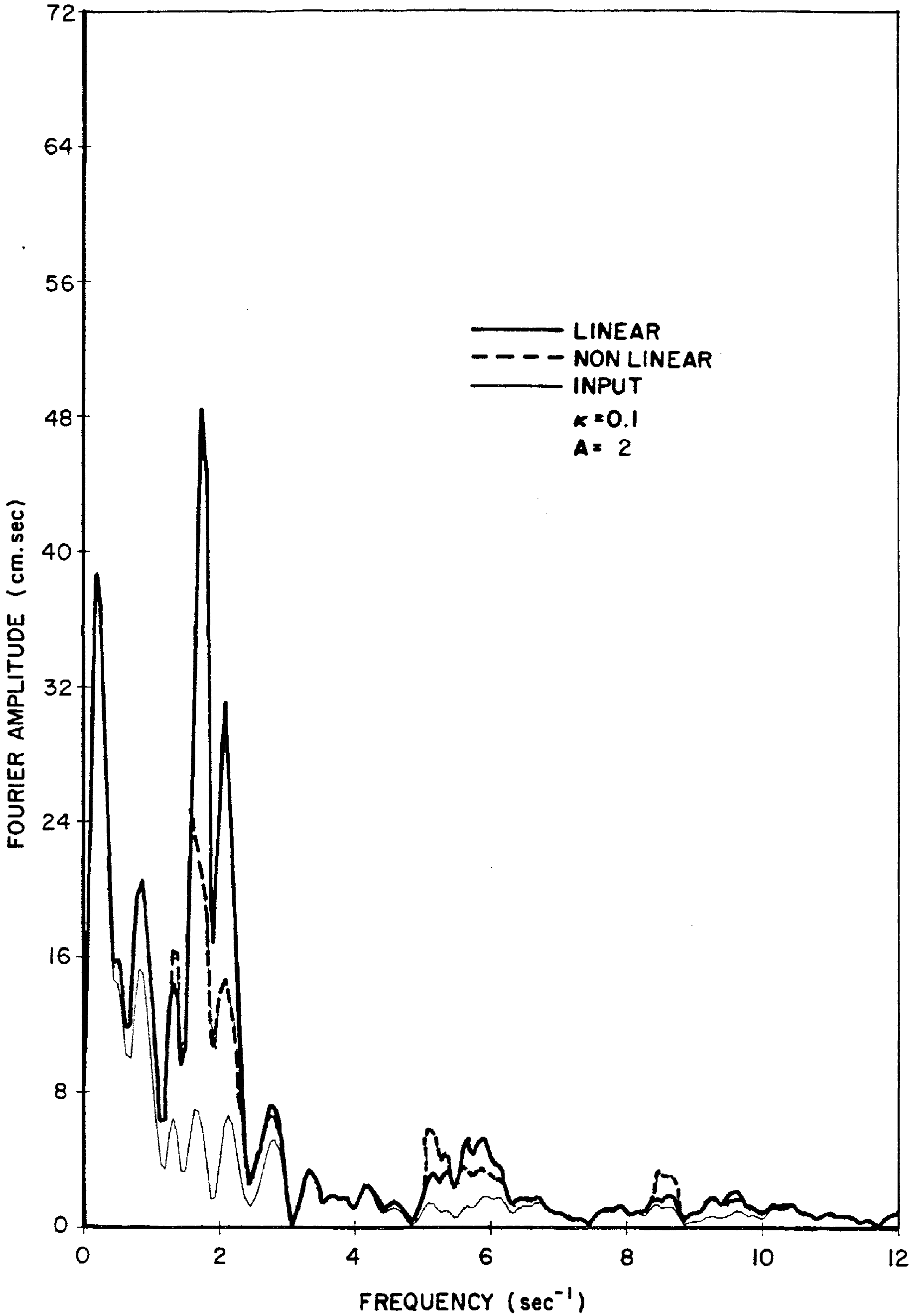
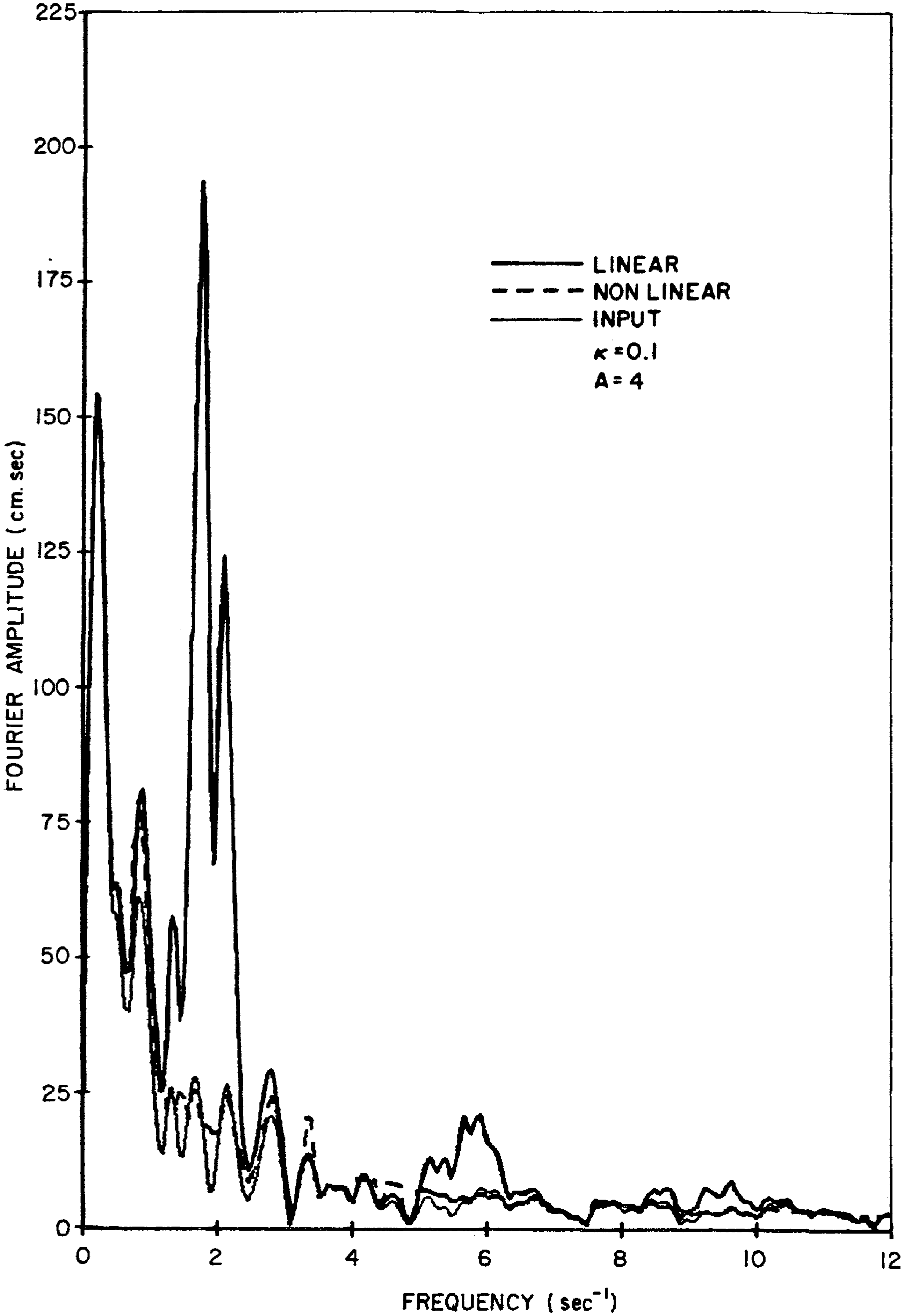

180
160
140
120
100
80
60
40
20
0
FOURIER AMPLITUDE (cm. sec)
LINEAR
NON LINEAR
INPUT
$\kappa = 0.1$
A = 1
0
2
4
6
8
10
12
FREQUENCY (sec^{-1})

FIGURE 4a'



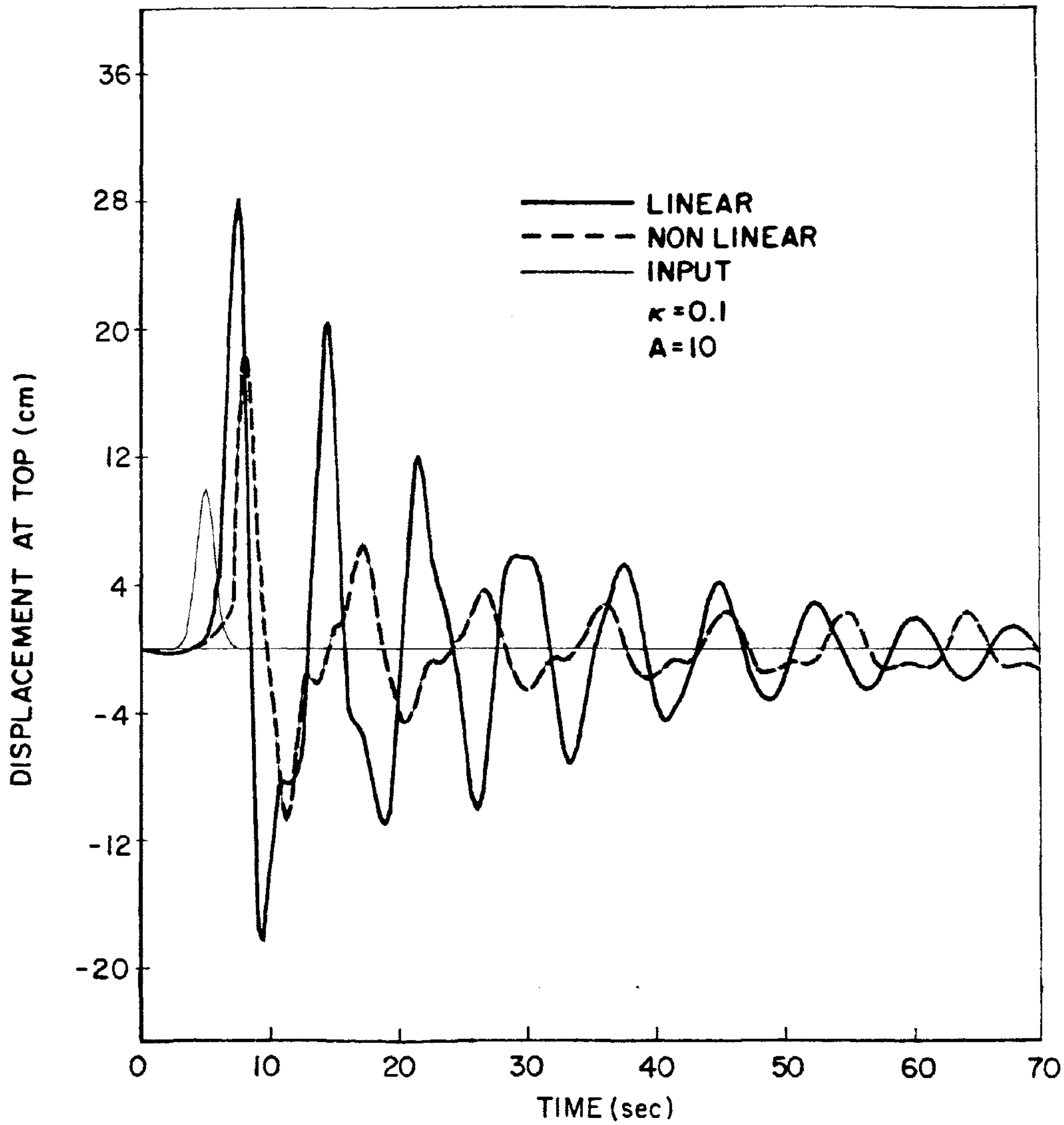
72
64
56
48
40
32
24
16
8
0
FOURIER AMPLITUDE (cm.sec)
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 2
0
2
4
6
8
10
12
FREQUENCY (sec⁻¹)

FIGURE 4b'



LINEAR
NON LINEAR
INPUT
κ=0.1
A=4
FOURIER AMPLITUDE (cm. sec)
FREQUENCY (sec⁻¹)
225
200
175
150
125
100
75
50
25
0
0
2
4
6
8
10
12

FIGURE 4c'



36
28
20
12
4
-4
-12
-20
DISPLACEMENT AT TOP (cm)
0
10
20
30
40
50
60
70
TIME (sec)
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 10

FIGURE 5a

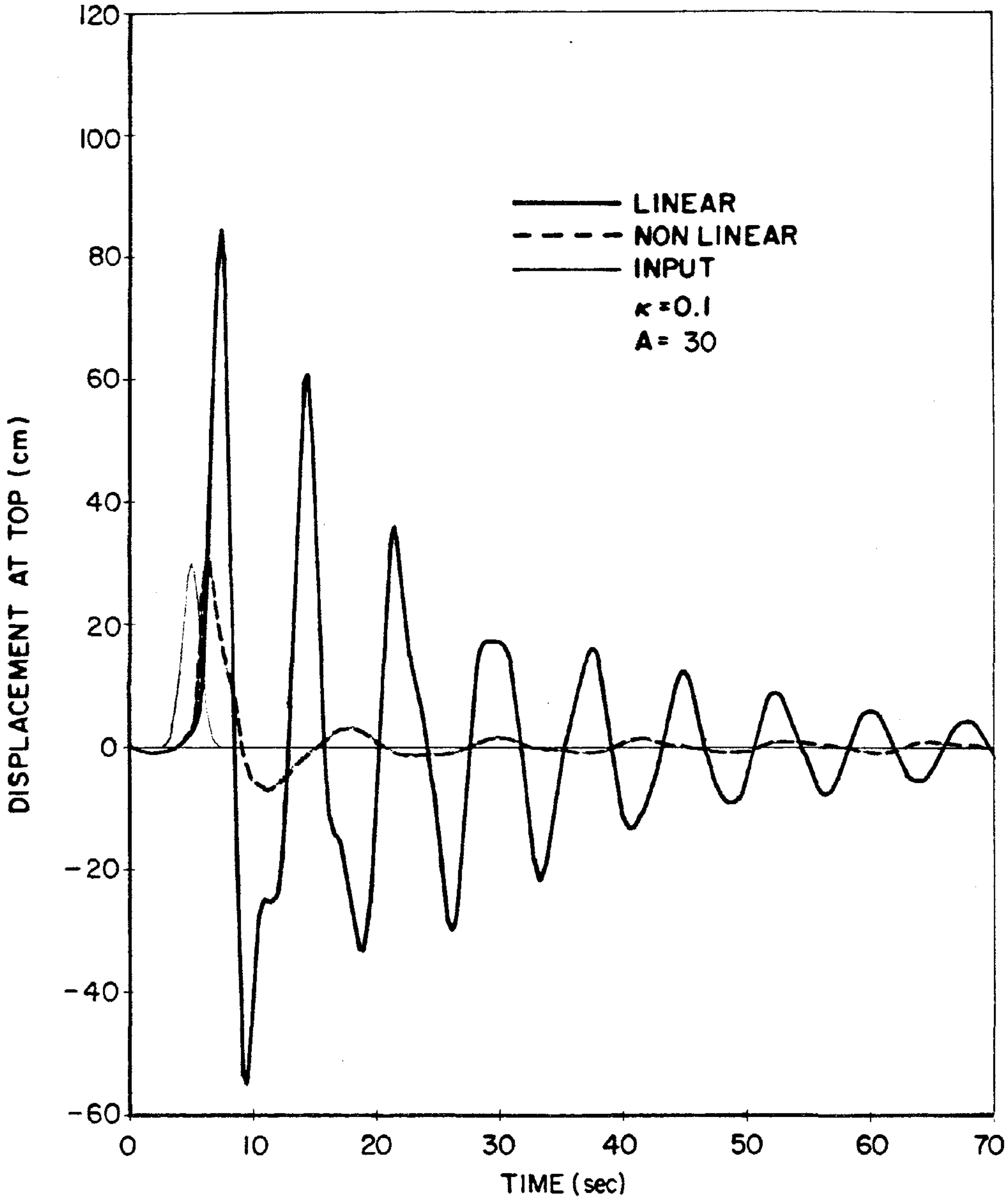

120
100
80
60
40
20
0
-20
-40
-60
DISPLACEMENT AT TOP (cm)
LINEAR
NON LINEAR
INPUT
κ=0.1
A= 30
0
10
20
30
40
50
60
70
TIME (sec)

FIGURE 5b

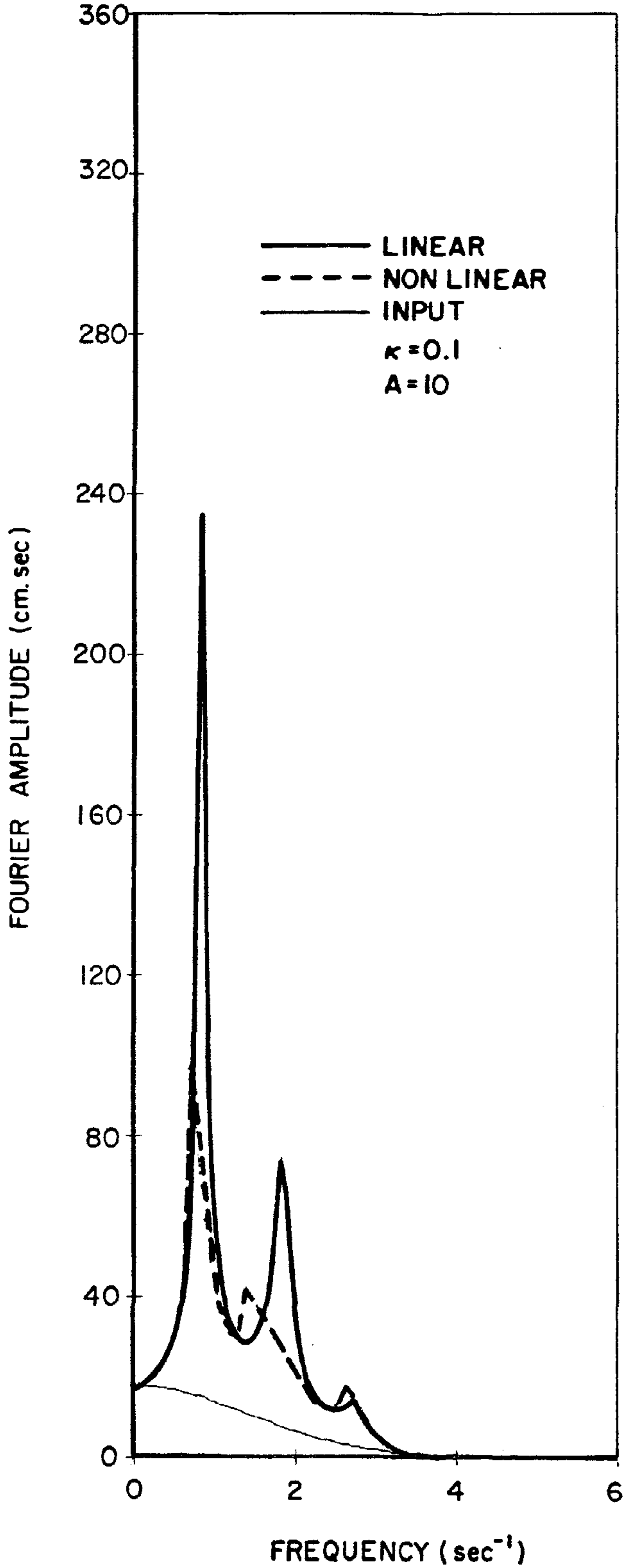
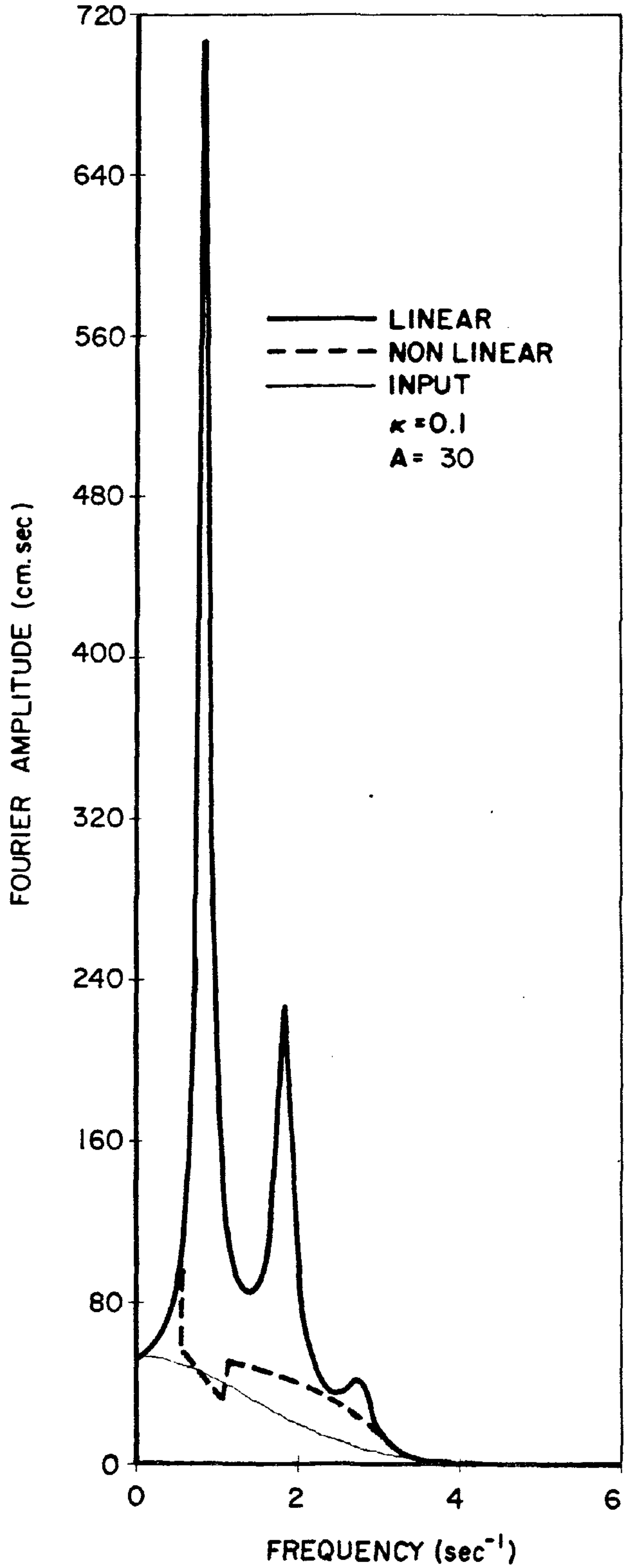

360
320
280
240
200
160
120
80
40
0
FOURIER AMPLITUDE (cm. sec)
LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 10
0
2
4
6
FREQUENCY (sec⁻¹)

FIGURE 5a'



720
640
560
480
400
320
240
160
80
0
FOURIER AMPLITUDE (cm. sec)
LINEAR
NON LINEAR
INPUT
κ =0.1
A= 30
0
2
4
6
FREQUENCY (sec⁻¹)

FIGURE 5b'

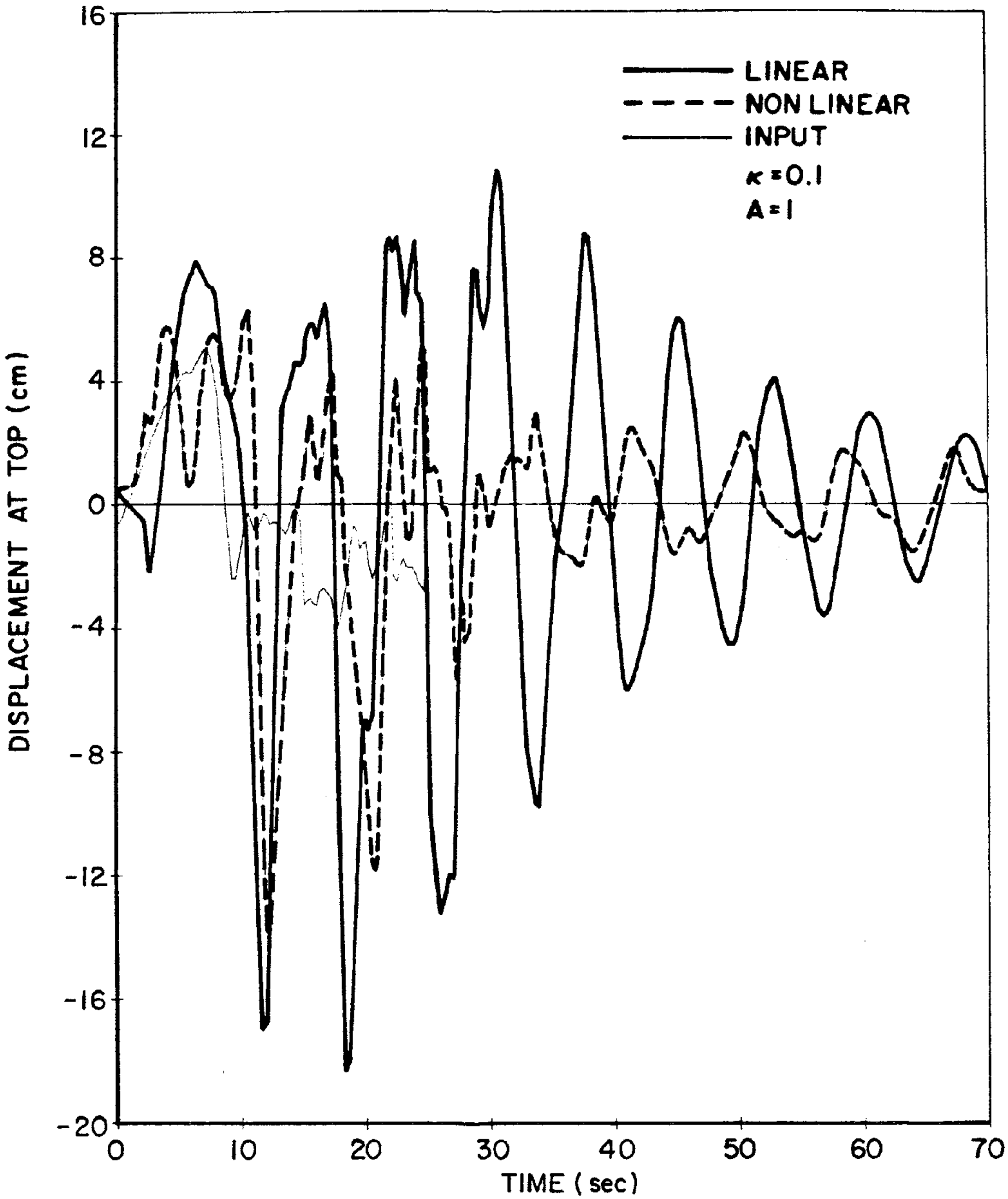

LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 1
DISPLACEMENT AT TOP (cm)
TIME (sec)
16
12
8
4
0
-4
-8
-12
-16
-20
0
10
20
30
40
50
60
70

FIGURE 6a

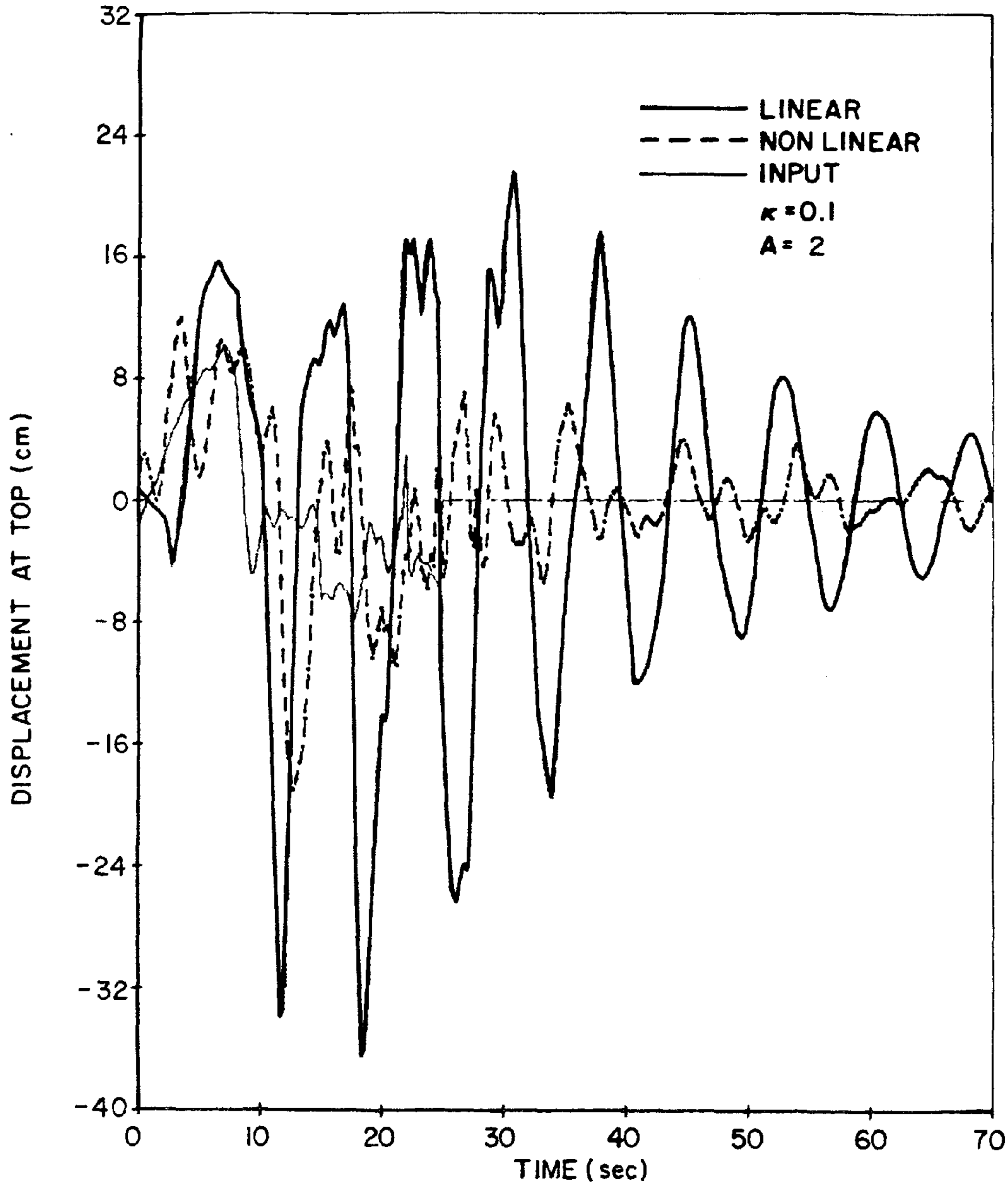

LINEAR
NON LINEAR
INPUT
κ = 0.1
A = 2
DISPLACEMENT AT TOP (cm)
TIME (sec)
32
24
16
8
0
-8
-16
-24
-32
-40
0
10
20
30
40
50
60
70

FIGURE 6b

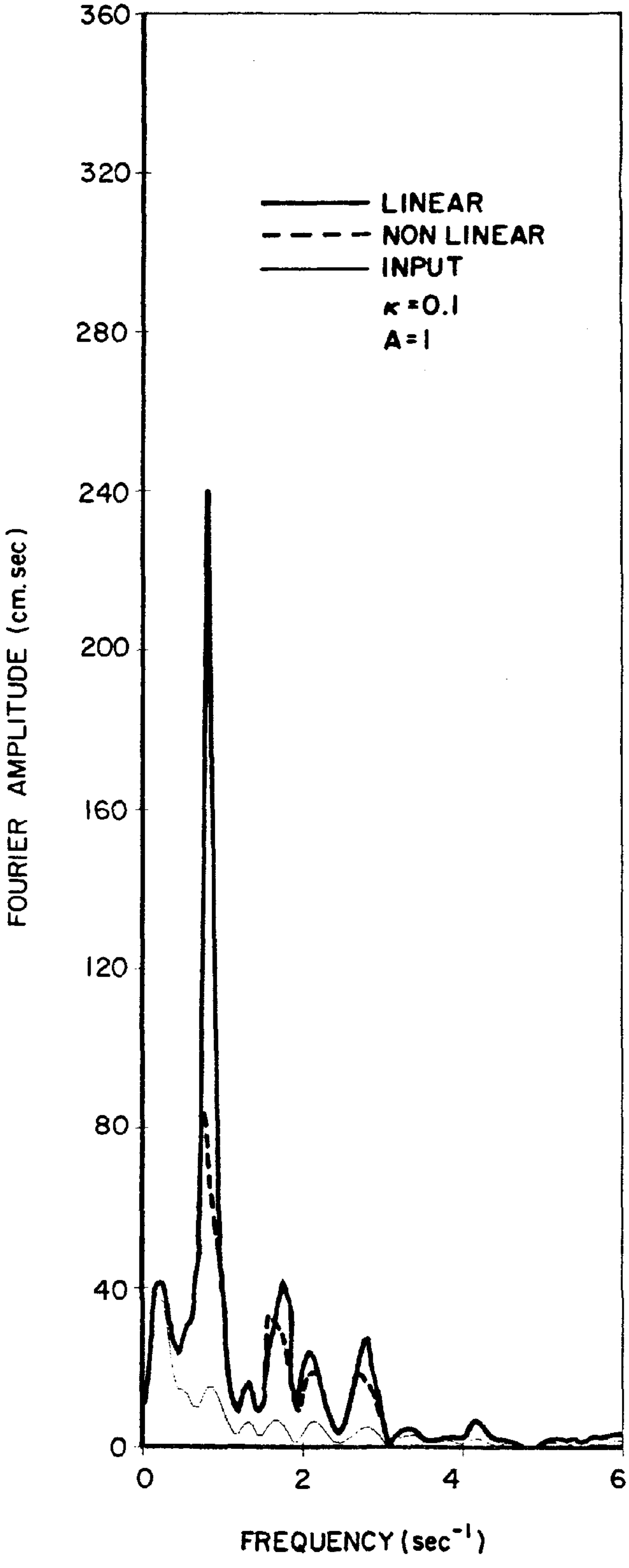
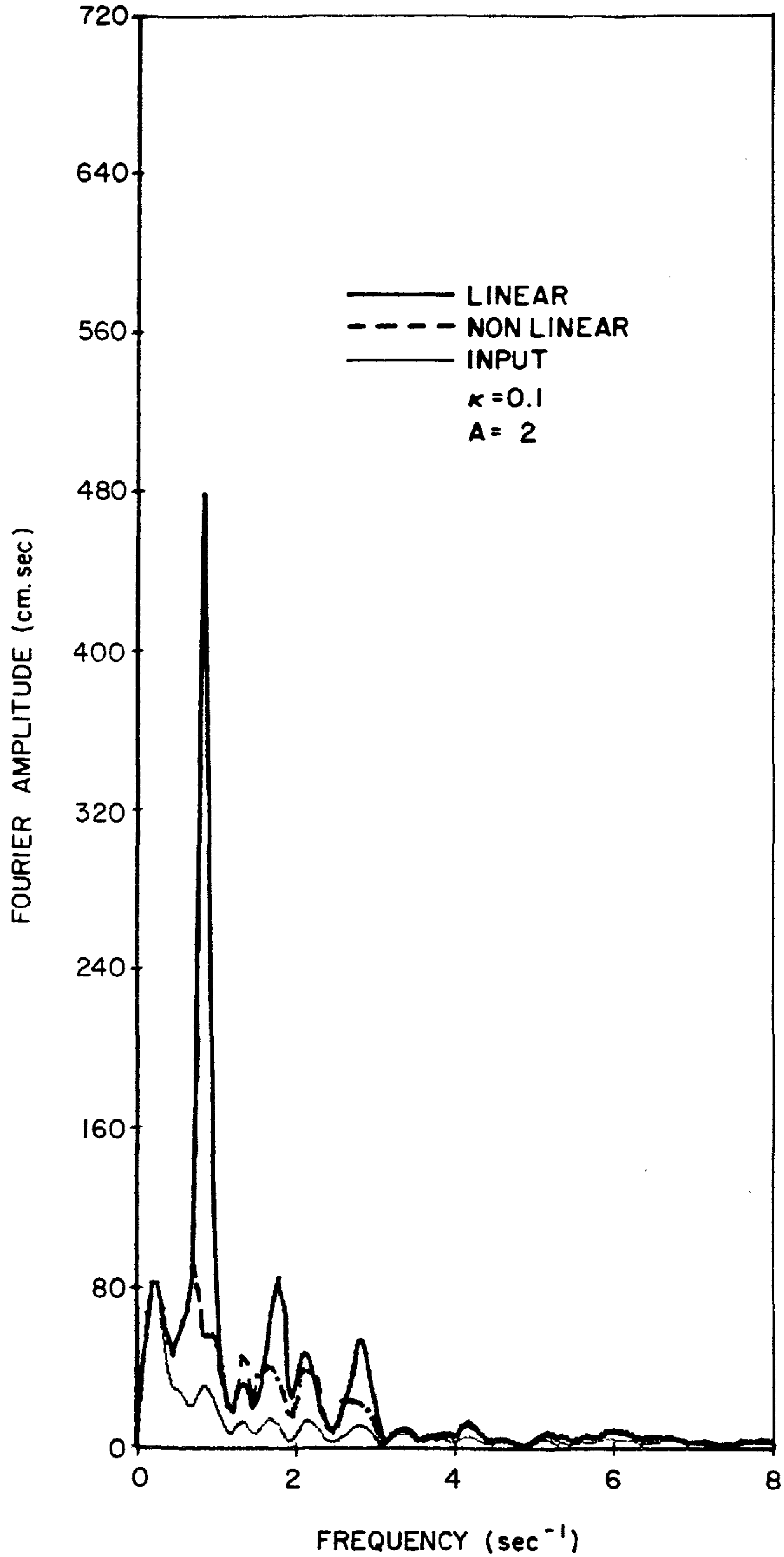

LINEAR
NON LINEAR
INPUT
κ=0.1
A=1
FOURIER AMPLITUDE (cm. sec)
FREQUENCY (sec⁻¹)
0
40
80
120
160
200
240
280
320
360
0
2
4
6

FIGURE 6a'



720
640
560
480
400
320
240
160
80
0
0
2
4
6
8
LINEAR
NON LINEAR
INPUT
κ =0.1
A= 2
FOURIER AMPLITUDE (cm. sec)
FREQUENCY (sec^{-1})

FIGURE 6b'