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BASELINE CORRECTION OF EARTHQUAKE RECORDS IN
THE FREQUENCY DOMAIN

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Abstract

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The report examines various integration schemes for discrete functions in the frequency domain, and develops procedures to perform a parabolic baseline correction to earthquake records in this domain. Only for acceleration records having zero mean value over the interval of definition does the simplest integration scheme analyzed (the "pseudo-continuous method") coincide with the well known "integration" procedure of dividing the Fourier Transform of the record by $\Omega \sqrt{-1}$. Records with no-zero mean, on the other hand, require additional terms to achieve consistency with time domain procedures such as the trapezoidal rule. The formulae developed are then applied in the last section of the report to the Golden Gate Earthquake of March 22, 1957, and compared against each other.
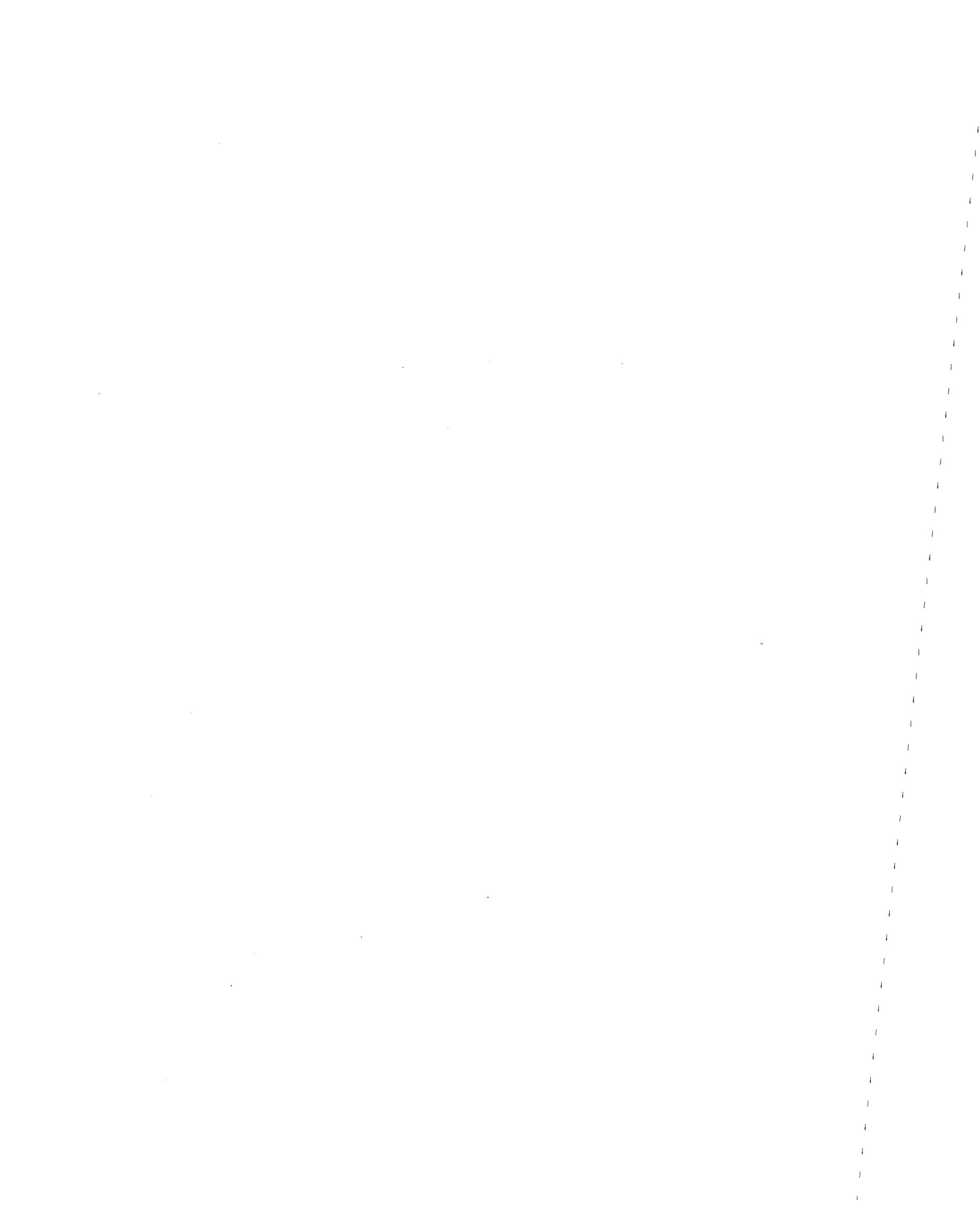
Preface

The work described in this report represents part of a research effort on Dynamic Soil-Structure Interaction carried out in the Civil Engineering Department at M.I.T. under the sponsorship of the National Science Foundation, Division of Advanced Environmental Research and Technology, through Grant ENV 77-18339.

This is the fourth of a series of research reports published under this grant. The previous ones were:

R78-20, "Dynamic Stiffness of Rectangular Foundations," by José Dominguez and J. M. Roesset, August 1978.

R78-24, "Response of Embedded Foundations to Travelling Waves," by José Dominguez and J. M. Roesset, August 1978.

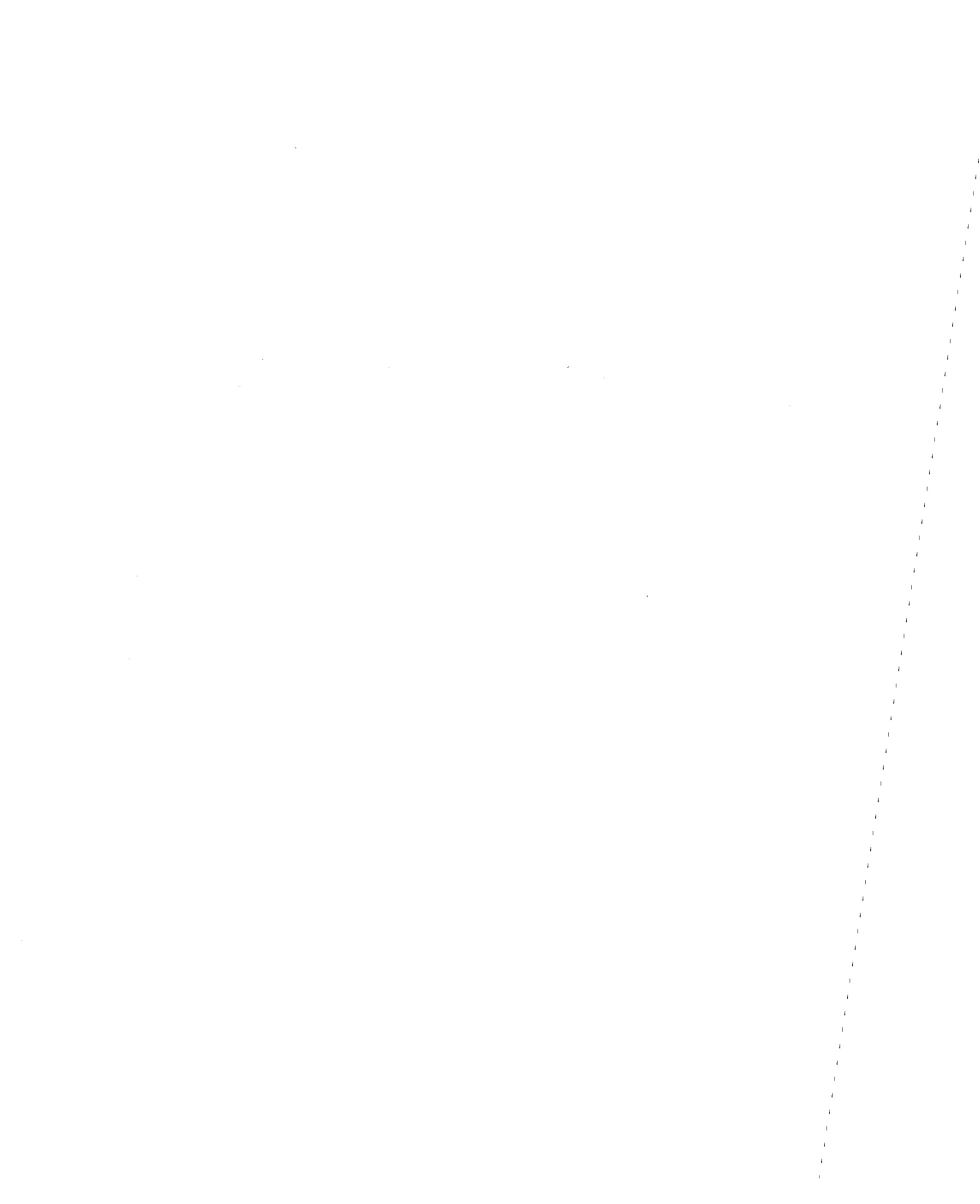R79-6, "Vertical and Torsional Stiffnesses of Cylindrical Footings," by E. Kausel and R. Ushijima, February 1979.

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Introduction

After the discovery of the Fast Fourier Transform (FFT) algorithm in 1965 by Cooley and Tuckey, it became computationally attractive to apply frequency domain techniques to the solution of physical systems involving linear differential equations, and in particular, to determine the response of mechanical systems to transient loads. Such procedures are particularly convenient, for instance, in seismic analyses incorporating soil-structure interaction effects as well as linear hysteretic damping. As is well known, the method is based on the convolution of the Fourier Transform of the input record with the frequency response functions (transfer functions) of the dynamic system under consideration. In a typical analysis, these transfer functions are determined only for a selected number of frequencies, with the spacing depending on the characteristics of the dynamic system. Additional values, if needed, are then found by interpolation.

Fourier Transform techniques are also very useful in the manipulation of earthquake records, such as changing the time step without adding spurious high frequencies (trigonometric interpolation), altering the frequency contents, or removing selected frequencies (filtering). The latter can be done either deliberately, or unwittingly when the transfer functions are terminated at a frequency lower than the cutoff (Nyquist or folding) frequency of the record. By necessity, all these manipulations destroy to a lesser or greater extent the baseline of the input motion, even before they are convolved with the transfer functions. Thus, it is desirable in many cases to develop procedures to restore a baseline without having to return first to the time domain. Such a procedure would afford also the opportunity to perform a baseline correction without addition of undesirable frequencies. The method presented in this work is based on the formulation of a second degree parabola in the frequency domain, with consideration of the discrete and periodic nature of the FFT algorithm.

Continuous versus Discrete Fourier Series

Let $f(t)$ be a real, single valued, piecewise continuous function of t defined in the interval $(0,T)$ which has only a finite number of discontinuities, a finite number of maxima and minima, and satisfies the Dirichlet condition

$$\int_0^T |f(t)|\, dt = \text{finite}$$

This function can then be expanded in the continuous Fourier Series (CFS)

$$f(t) = \sum_{j=-\infty}^{\infty} h_j\, e^{i\Omega_j t}, \quad \Omega_j = j\, \frac{2\pi}{T} \tag{1}$$

in which $i = \sqrt{-1}$, and e = base of natural logarithms. Alternatively, if the function is defined only at 2N equally spaced points,

$$f(t) = f_k = f(k\Delta t), \quad k = 0, 1, \ldots 2N-1, \ \Delta t = \frac{T}{2N}$$

it may be expanded into the Discrete Fourier Series (DFS)

$$f_k = \sum_{j=-N}^{N} H_j\, e^{i\frac{\pi}{N} jk} \tag{2}$$

The symbol $\sum$ indicates that the first and last elements in the series must be multiplied by 0.5. When the coefficients H_j are obtained from a real function f_k via equation (4) below, these elements are real and equal, and a conventional summation from $-(N-1)$ to N (or from $-N$ to $N-1$) could be employed as well. However, when they are obtained by alteration of a spectrum (such as multiplying a Fourier spectrum by a complex transfer function), H_{-N}, H_N are complex conjugates. Hence, the summation must be performed as indicated in order to define a real function f_k with equation (2) above.

For the continuous and discrete cases, the (spectral) coefficients h_j, H_j follow from

$$h_j = \frac{1}{T} \int_0^T f(t)\, e^{-i\Omega_j t}\, dt, \quad \Omega_j = j\, \frac{2\pi}{T} \tag{3}$$

and

$$H_j = \frac{1}{2N} \sum_{k=0}^{2N-1} f_k\, e^{-i\frac{\pi}{N} jk} \tag{4}$$

It can also be shown that

$$H_j = \sum_{m=-\infty}^{\infty} h_{j+2Nm} \tag{5}$$

provided that the function f(t) is not discontinuous at t=0, T, etc. Should this be the case, then equations (4) and (5) will agree only if the first value (f_0) of the discrete summation 4) is taken equal to the average value $\frac{1}{2}(f(0^-) + f(0^+))$, at the discontinuity. Equation (5) is the mathematical expression for the aliasing phenomenon.

Finally, if the function f(t) is defined zero outside the interval (0,T), then it has a continuous Fourier transform $F(\Omega)$ and discrete (fast) Fourier transform FFT (Ω) which are related to the CFS and DFS by

$$F(\Omega_j) = T\, h_j$$

and
$$FFT(\Omega_j) = T\, H_j \tag{6}$$

Evidently, $F(\Omega)$ is also defined for the continuous spectrum of frequencies Ω which contains the discrete spectrum Ω_j as a subset. Hence, adding zeroes in the time domain may be used to interpolate in the frequency domain, and vice versa.

Velocity vs. Acceleration in Frequency Domain, Continuous Case (CFS).

If c(t), a(t) denote corresponding velocity and acceleration time histories for an event defined in the interval $0 \le t \le T$, they satisfy the relationship

$$c(t) = c_0 + \int_0^t a(\tau)d\tau \tag{7}$$

where c_0 is the initial velocity. Multiplication by $e^{-i\Omega_j t}$ $(\Omega_j = \frac{2\pi}{T} j)$ and integration over the interval T yields the CFS (periodic in T)

$$\frac{1}{T}\int_0^T c(t)\, e^{-i\Omega_j t}\, dt = c_0 \frac{1}{T}\int_0^T e^{-i\Omega_j t}\, dt + \frac{1}{T}\int_0^T e^{-i\Omega_j t}\int_0^t a(\tau)\, d\tau\, dt \tag{8}$$

Integrating by parts, with $\Omega_j \ne 0$

$$\frac{1}{T}\int_0^T c(t)e^{-i\Omega_j t}\, dt = c_0 \left(\frac{1-e^{-i\Omega_j T}}{i\Omega_j T}\right) + \frac{1}{i\Omega_j T}\int_0^T a(t)\, e^{-i\Omega_j t}\, dt$$

$$- \frac{e^{-i\Omega_j T}}{i\Omega_j T}\int_0^T a(t)dt \tag{9}$$

Now, $\Omega_j T = 2\pi j$, so that $e^{-i\Omega_j T} = 1$. Also,

$$v_j = \frac{1}{T} \int_0^T c(t)\, e^{-i\Omega_j t}\, dt = \text{CFS of velocity}$$

$$g_j = \frac{1}{T} \int_0^T a(t)\, e^{-i\Omega_j t}\, dt = \text{CFS of acceleration} \tag{10}$$

In particular,

$$g_0 = g_{j=0} = \frac{1}{T} \int_0^T a(t)\, dt = \text{mean acceleration}$$

Therefore, equation (9) reduces to

$$v_j = \frac{g_j - g_0}{i\Omega_j} \qquad , \qquad \begin{array}{c} \Omega_j \neq 0 \\ (j \neq 0) \end{array} \tag{11}$$

which relates the CFS of the velocity and acceleration time histories for an event with non-zero mean acceleration ($g_0 \neq 0$). On the other hand, from equation (7), we obtain the final velocity at $t = T$ as

$$c_T = c_0 + \int_0^T a(t)\, dt = c_0 + T\, g_0 \tag{12}$$

Unless $g_0 = 0$, the velocity will display a discontinuity of $t = 0$, T, $2T$, etc. Since the CFS converges in the mean, the corresponding value at $t = 0$ and $t = T$ will be

$$\bar{c}_0 = \frac{c_0 + c_T}{2} = \frac{c_0 + c_0 + T\, g_0}{2} = c_0 + \frac{T}{2}\, g_0 \tag{13}$$

Also, from equation (1) we obtain with $t = 0$

$$\bar{c}_0 = \sum_{j=-\infty}^{\infty} v_j$$

$$= v_0 + 2 \sum_{j=1}^{\infty} \text{Re}(v_j) \tag{14}$$

Combination of (14) and (13) yields finally

$$v_o = c_o + \frac{T}{2} g_o - 2 \sum_1^\infty \text{Re} (v_j) \Bigg\}$$

$$= c_o + \frac{T}{2} g_o - 2 \sum_1^\infty \frac{\text{Im} (g_j)}{\Omega_j} \Bigg\}$$

(15)

(since $\text{Re} (v_j) = \text{Re} [(g_j - g_o)/i\Omega_j] = \text{Im} [g_j/\Omega_j]$).

Equations (11) and (15) relate then the CFS of the velocity to the CFS of the acceleration. A similar relationship exists between spectral displacements and velocities. Whenever these formulas are used to derive spectral accelerations from spectral velocities rather than vice versa, it may be necessary to compute the mean acceleration from the auxiliary equation $g_o = (c_T - c_o)/T$. The spectral values are not sufficient, since the information on initial and final velocity has been lost.

In most practical cases, the acceleration record has (or should have) zero mean, since a particle starting from rest must return to rest at the end of the event. This implies that $g_o = 0$, $c_o = 0$.

Velocity vs. Acceleration in Frequency Domain, Discrete Case (DFS)

The procedures involved in the correction for baseline in the discrete case are similar to the ones needed in the continuous case. However, we must define first the relationship between velocity and acceleration when they are not continuous functions, since we shall be imposing conditions in terms of velocities.

From the definition of the DFS of the velocity

$$c_k = \sum_{-N}^N V_j \, e^{i \frac{\pi}{N} jk}$$

it follows that the initial velocity c_o is given by

$$c_o = \sum_{-N}^N V_j$$

implying that the mean velocity V_o must be

$$V_o = c_o - 2 \sum_1^{N-1} \text{Re} (V_j) - \text{Re} (V_N)$$

(16)

This expression is similar to the one given by Eq. (15), when v_j is replaced by V_j. The only discrepancy is in the term $TG_0/2$, which accounts for the convergence of the CFS in the mean (i.e., for the average value of initial and final velocity). The "final velocity" corresponding to $k = 2N$ is not, however, a datum in the discrete velocity record (which consists of the 2N points $k = 0, 1, \ldots 2N-1$). A definition of DFS spectral amplitudes consistent with the continuous case would require an expression for V_j of the form

$$V_j = \frac{1}{2N} \sum_0^{2N} c_k \, e^{-i \frac{\pi}{N} jk}, \quad c_{2N} = c_0 + TG_0 = \text{"final velocity"}$$

instead of the customary

$$V_j = \frac{1}{2N} \sum_0^{2N-1} c_k \, e^{-i \frac{\pi}{N} jk} \tag{17}$$

This implies that DFS spectral amplitudes obtained by aliasing of a CFS (using equation (5)) need to be corrected for proper initial conditions, subtracting from each term V_j the constant $g_0 T/4N$. This correction is not necessary, of course, if the amplitudes V_j are obtained directly from equation (17) above.

To relate the spectral velocities and accelerations for the case $j \neq 0$, several procedures may be used. We examine below four different alternatives.

1) Discrete spectral velocity by aliasing

In this alternative, we attempt to match continuous and discrete velocity records at every time step, using the aliasing relationship (Eq. 5)), and correcting for initial conditions as described above:

$$V_j = \sum_{m=-\infty}^{\infty} v_{j+2Nm} - \frac{g_0 T}{4N} = \sum_{-\infty}^{\infty} \frac{g_{j+2Nm} - g_0}{i \Omega_{j+2Nm}} - \frac{g_0 T}{4N} \tag{18}$$

Since the continuous spectral accelerations g_j are not available in actual situations, this definition is not convenient, unless the record contains no frequency higher than the Nyquist (folding) frequency $(G_j = \sum_{-\infty}^{\infty} g_{j+2Nm} = g_j)$.

In this latter case, however, equation (18) transforms into

$$V_j = \sum_{m=-\infty}^{\infty} \frac{G_j - G_o}{i\Omega_{j}+2Nm} - \frac{G_o T}{4N}$$

$$= \frac{G_j}{i\Omega_j} - G_o \sum_{-\infty}^{\infty} \frac{1}{i\Omega_{j}+2Nm} - \frac{G_o T}{4N}$$

$$V_j = \frac{G_j - j\alpha\cot j\alpha\, G_o}{i\Omega_j} - G_o \frac{T}{4N} \quad , \quad \alpha = \frac{\pi}{2N} \; , \quad j \neq 0 \tag{19}$$

(The evaluation of the sum $\frac{2\pi}{T} \sum_{-\infty}^{\infty} \frac{1}{\Omega_{j}+2Nm} = \sum_{-\infty}^{\infty} \frac{1}{j+2Nm} = \alpha\cot j\alpha$ may be found in Appendix I).

Note that only in the case of zero mean acceleration ($G_o = 0$) does equation (19) agree with the naive derivation

$$a(t) = \frac{d\, c(t)}{dt} = \frac{d}{dt} \sum_{-N}^{N} V_j\, e^{i\Omega_j t} = \sum_{-N}^{N} (i\Omega_j V_j) e^{i\Omega_j t}$$

yielding $G_j = i\Omega_j V_j$. The discrepancy follows from the fact that the derivation of the series does not account for the discontinuities.

Equations (16) and (19) define then the discrete spectral velocities in terms of the spectral accelerations.

2) Pseudo-continuous method.

In this alternative, we postulate the relationship between discrete spectral accelerations and velocities to be the same as in the continuous case, except for the correction for initial conditions. Thus, we have from equations (11) and (16) (changing $g_j \to G_j$ and $v_j \to V_j$, and adding the correction $- G_o T/4N$)

$$V_j = \frac{G_j - G_o}{i\Omega_j} - \frac{G_o T}{4N}$$

$$V_o = c_o - 2 \sum_{1}^{N-1} \mathrm{Re}(V_j) - \mathrm{Re}(V_N) \tag{20}$$

While equations (20) are simple and convenient, their interpretation as

difference (or integration) operators in the time domain is not immediately apparent. The formulation corresponds to some form of "trigonometric differentiation," the nature of which is explored below.

We begin with the already familiar expressions

$$V_j = \frac{1}{2N} \sum_{\ell=0}^{2N-1} c_\ell \, e^{-i \frac{\pi}{N} j\ell} \tag{21}$$

$$a_k = \sum_{j=-N}^{N} G_j \, e^{i \frac{\pi}{N} jk}$$

$$= \sum_{j=-(N-1)}^{N-1} G_j \, e^{i \frac{\pi}{N} jk} + \frac{1}{2}(G_{-N} + G_N)(-1)^k \tag{22}$$

But from equation (20)

$$\frac{1}{2}(G_{-N} + G_N) = \frac{1}{2}[i\Omega_N(V_N + G_0 T/4N) + G_0 + i\Omega_{-N}(V_{-N} + G_0 T/4N) + G_0]$$

$$= G_0$$

since $\Omega_{-N} = -\Omega_N$, and $V_{-N} = V_N$ (by equation (21)). Hence

$$a_k = \sum_{j=-(N-1)}^{N-1} [i\Omega_j(V_j + \frac{G_0 T}{4N}) + G_0] \, e^{i \frac{\pi}{N} jk} + G_0(-1)^k$$

$$= \sum_{-(N-1)}^{N-1} i\Omega_j(V_j + \frac{G_0 T}{4N}) + G_0 \sum_{-N}^{N} e^{i \frac{\pi}{N} jk} \tag{23}$$

Substituting equation (21) into equation (23), we obtain

$$a_k = \frac{2\pi i}{T} \sum_{j=-(N-1)}^{N-1} j \left\{ \frac{1}{2N} \sum_{\ell=0}^{2N-1} c_\ell \, e^{-i \frac{\pi}{N} j\ell} + \frac{G_0 T}{4N} \right\} e^{i \frac{\pi}{N} jk} + 2NG_0 \delta_{ok}$$

(with $\delta_{ok} = 0$ for $k \neq 0$, $\delta_{oo} = 1$; see also Appendix 1).   Then,

$$a_k = \frac{\pi i}{NT} \left\{ \sum_{\ell=0}^{2N-1} c_\ell \sum_{j=-(N-1)}^{N-1} j \, e^{i\frac{\pi}{N}j(k-\ell)} + \frac{G_0 T}{2} \sum_{j=-(N-1)}^{N-1} j \, e^{i\frac{\pi}{N}jk} \right\} + 2NG_0 \delta_{ok}$$

$$= \sum_{\ell=0}^{2N-1} c_\ell \, S_{k\ell} + \frac{G_0 T}{2} S_{ko} + 2NG_0 \, \delta_{ko} \tag{24}$$

where $S_{k\ell}$ $(S_{ko} = S_{k,\ell=0})$ is the result of the sum

$$S_{k\ell} = \frac{\pi i}{NT} \sum_{j=-(N-1)}^{N-1} j \, e^{i\frac{\pi}{N}j(k-\ell)} \tag{25}$$

Evaluation of this sum yields (for derivation, see Appendix 1):

$$S_{k\ell} = \begin{cases} \frac{\pi}{T} (-1)^{k-\ell} \cot(k-\ell)\alpha \, , & \alpha = \pi/2N, \quad k-\ell \neq 0 \\ 0 & k-\ell = 0 \end{cases} \tag{26}$$

In the previous derivation, it was implicitly assumed that equation (22) was valid at all points, and particularly at $t = 0$ ($k = 0$). However, since the velocity changes abruptly from $c_0 + G_0 T = c_0 + 2N \, G_0 \Delta t$ to c_0 in the transition from $t = -\Delta t$ to $t = 0$, we should have an additional "singularity" in the acceleration record of magnitude $-2NG_0$ at $t = 0$ ($k = 0$). Consideration of this singularity would thus cancel the term $2NG_0 \delta_{ok}$ in equation (24). Also, if we define

$$\bar{c}_0 = \bar{c}_{2N} = c_0 + \frac{G_0 T}{2} \quad (\text{" initial velocity in the mean"})$$
$$\bar{c}_\ell = c_\ell \quad (\ell \neq 0)$$
$$\bar{c}_{-\ell} = \bar{c}_{2N-\ell}$$

implying a periodic velocity record, we can write equation (24) as

$$a_k = \frac{\pi}{T} \sum_{\ell=1}^{N} (\bar{c}_{k+\ell} - \bar{c}_{k-\ell})(-1)^{\ell-1} \cot \ell\alpha, \quad \alpha = \pi/2N$$

or

$$a_k = \alpha \sum_{\ell=1}^{N} \frac{\bar{c}_{k+\ell} - \bar{c}_{k-\ell}}{\Delta t} (-1)^{\ell-1} \cot \ell\alpha \tag{27}$$

$$\alpha = \pi/2N, \quad \Delta t = T/2N$$

which constitutes the time-domain equivalent (differentiation rule) of equation (20); in particular, eq. (27) is the equivalent of the formula $G_j = i\Omega_j V_j$ in the case of zero mean acceleration. As a differentiation formula, it is obviously not useful, but it serves to illustrate the nature of the pseudo-continuous method. Because of the alternating signs in the series, it converges slowly to the desired value. In the neighborhood of k, this formula behaves (for reasonably large N) as

$$a_k \sim \sum \frac{c_{k+\ell} - c_{k-\ell}}{\ell \Delta t}(-1)^{\ell-1}$$

so that the first term in the series is approximately twice the value given by the central difference formula.

3) Finite (central) differences scheme

In this procedure, acceleration and velocity records are related by some numerical difference formula. For instance, using the central difference formula, we write (from eq. 2)

$$c_{k+1} = \sum_{-N}^{N} V_j e^{i \frac{\pi}{N} j(k+1)}$$

$$c_{k-1} = \sum_{-N}^{N} V_j e^{i \frac{\pi}{N} j(k-1)}$$

so that

$$a_k = (c_{k+1} - c_{k-1})/2\Delta t = \frac{2N}{T} \sum_{-N}^{N} (iV_j \sin \frac{\pi}{N} j)e^{i \frac{\pi}{N} jk}$$

It follows that

$$G_j = \frac{2N}{T} i \sin 2j\alpha \, V_j \, , \quad \alpha = \frac{\pi}{2N} \tag{28}$$

For low values of j (j << N), the expression above approximates

$$G_j \approx i\Omega_j V_j \tag{29}$$

as in the continuous (pseudo-continuous) case with zero mean. It should be noted that the derivation of Eq. (28) is not valid if the velocity record is not continuous at k = 0 , k = 2N, i.e., if $G_0 \neq 0$. Also, the central difference formula cannot resolve components of velocity with constant amplitude and alternating signs. Therefore, the formula implies $G_N = 0$ as well.

If equation (28) is used to define V_j starting from G_j,

$$V_j = \frac{G_j}{\frac{2N}{T} i \sin 2j\alpha}$$ (30)

and if G_N = real number $\neq 0$, the velocity spectrum will show purely imaginary singularities V_{-N}, V_N. These singularities, however, are equal and opposite in sign, and cancel out in the summation of the series. Thus V_N may be set equal to zero, even if G_N is not.

4) Linear acceleration method

In this alternative, we define the velocity record by numerical integration of the acceleration record, assuming a linear variation of acceleration between time steps (trapezoidal rule). The velocity at $t = k\Delta t$ is then

$$c_k = c_{k-1} + \frac{a_k + a_{k-1}}{2} \Delta t$$

$$c_{k-1} = c_{k-2} + \frac{a_{k-1} + a_{k-2}}{2} \Delta t$$

$$\vdots$$

$$c_1 = c_0 + \frac{a_1 + a_0}{2} \Delta t$$

Addition of all terms yields then

$$c_k = c_0 + \left\{ \frac{1}{2} a_0 + (a_1 + a_2 + \ldots a_{k-1}) + \frac{1}{2} a_k \right\} \Delta t$$

$$= c_0 + \Delta t \sum_{\ell=0}^{\ell=k} a_\ell$$ (31)

$$c_k = c_0 + \Delta t \left\{ \sum_{\ell=0}^{\ell=k} a_\ell - \frac{1}{2}(a_0 + a_k) \right\}$$ (32)

Multiplying both sides by $\frac{1}{2N} e^{-i \frac{\pi}{N} jk}$ and summing up over k, we obtain

$$V_j = \frac{1}{2N} \sum_{k=0}^{2N-1} e^{-i \frac{\pi}{N} jk} \left\{ c_0 + \Delta t \left[\sum_{\ell=0}^{\ell=k} a_\ell - \frac{1}{2}(a_0 + a_k) \right] \right\}$$ (33)

But
$$\frac{1}{2N} \sum_{k=o}^{2N-1} e^{-i\frac{\pi}{N}jk} = \delta_{oj} = \begin{cases} 1 \text{ if } j=0 \\ 0 \text{ otherwise} \end{cases}$$

Also,
$$\frac{1}{2N} \sum_{k=o}^{2N-1} a_k e^{-i\frac{\pi}{N}jk} = G_j, \qquad a_o = \sum_{-N}^{N} G_m$$

Hence
$$V_j = (c_o - \frac{1}{2}\Delta t \sum_{-N}^{N} G_m)\delta_{oj} - \frac{1}{2}\Delta t\, G_j + \frac{\Delta t}{2N} \sum_{k=o}^{2N-1} e^{-i\frac{\pi}{N}jk} \sum_{\ell=o}^{\ell=k} a_\ell \quad (34)$$

It remains to evaluate the last double summation:

$$S_j = \frac{1}{2N} \sum_{k=o}^{2N-1} e^{-i\frac{\pi}{N}jk} \sum_{\ell=o}^{\ell=k} a_\ell \qquad (35)$$

Let's define $z = e^{-i\frac{\pi}{N}j}$, so that $z^k = e^{-i\frac{\pi}{N}jk}$. Hence,

$$S_j = \frac{1}{2N} \sum_{k=o}^{2N-1} z^k \sum_{o}^{k} a_\ell \qquad (36)$$

$$= \frac{1}{2N} \left\{ z^o a_o + z^1(a_o + a_1) + z^2(a_o + a_1 + a_2) + \right.$$
$$\left. +\ldots z^{2N-1}(a_o + \cdots a_{2N-1}) \right\}$$

$$S_j = \frac{1}{2N} \left\{ a_o(1 + z + z^2 + \ldots z^{2N-1}) + a_1(z + z^2 + \ldots z^{2N-1}) + \right.$$
$$\left. +\ldots a_{2N-1} z^{2N-1} \right\} \qquad (37)$$

Now,
$$z^k + z^{k+1} + \ldots z^{2N-1} = \frac{z^k - z^{2N}}{1 - z} \quad \text{when } z \neq 1 \quad (j \neq 0)$$

Also,
$$z^{2N} = 1. \text{ It follows then } (\text{for } j \neq o)$$

$$S_j = \frac{1}{2N} \left\{ a_o \frac{1-1}{1-z} + a_1 \frac{z-1}{1-z} + a_2 \frac{z^2-1}{1-z} + \cdots a_{2N-1} \frac{z^{2N-1}-1}{1-z} \right\}$$

$$= \frac{1}{1-z} \left\{ \sum_{k=o}^{2N-1} a_k z^k - \sum_{k=o}^{2N-1} a_k \right\}$$

$$S_j = \frac{1}{1-z} (G_j - G_0) \qquad (38)$$

The spectral velocity is then

$$V_j = -\frac{1}{2} \Delta t\, G_j + \frac{\Delta t}{1-z} (G_j - G_0)$$

$$= \frac{\Delta t}{2} \left\{ \frac{1+z}{1-z} G_j - \frac{2}{1-z} G_0 \right\} \qquad j \neq 0 \qquad (39)$$

Substituting $z = e^{-i \frac{\pi}{N} j}$, we obtain after some straightforward transformations

$$V_j = \frac{\Delta t}{2i} \cot j\alpha\, (G_j - G_0) - \frac{\Delta t}{2} G_0 \qquad (40)$$

with $\alpha = \pi/2N$ and $j \neq 0$.

Since $\Delta t = \frac{T}{2N}$, the expression above behaves for large N (and small j) like

$$V_j \sim \frac{G_j - G_0}{i\Omega_j} - \frac{G_0 T}{4N}$$

as in the pseudo-continuous case.

The term corresponding to $j = 0$ must be derived in a somewhat different manner. In this case, $z = 1$, so that equation (37) reduces to

$$S_0 = \frac{1}{2N} \left\{ 2Na_0 + (2N-1)\, a_1 + \quad \cdots\ (2N - \ell)a_\ell + \cdots\ a_{2N-1} \right\}$$

$$= \sum_0^{2N-1} a_\ell - \frac{1}{2N} \sum_1^{2N-1} \ell a_\ell \qquad (41)$$

But

$$\sum_0^{2N-1} a_\ell = 2NG_0 \qquad (42)$$

and

$$a_\ell = \sum_{-N}^{N} G_m e^{i \frac{\pi}{N} m\ell} = \sum_{-N-1}^{N-1} G_m e^{i \frac{\pi}{N} m\ell} + \frac{1}{2} (G_{-N} + G_N)(-1)^\ell$$

Then

$$\frac{1}{2N} \sum_{1}^{2N-1} \ell\, a_\ell = \frac{1}{2N} \sum_{1}^{2N-1} \ell \left\{ \sum_{-(N-1)}^{N-1} G_m\, e^{i\frac{\pi}{N}m\ell} + \frac{1}{2}(G_{-N} + G_N)(-1)^\ell \right\}$$

$$= \sum_{-(N-1)}^{N-1} G_m \frac{1}{2N} \sum_{1}^{2N-1} \ell\, e^{i\frac{\pi}{N}m\ell} + \frac{1}{4N}(G_{-N} + G_N) \sum_{1}^{2N-1} \ell\, (-1)^\ell \qquad (43)$$

Defining $\quad Q_m = \dfrac{1}{2N} \displaystyle\sum_{1}^{2N-1} \ell\, e^{i\frac{\pi}{N}m\ell} = \begin{cases} -\dfrac{1}{2}\left(1 + i \cot \dfrac{\pi}{2N} m\right) & m \neq 0 \\[2mm] N - 1/2 & m = 0 \end{cases} \qquad (44)$

(For the evaluation of the sum Q_m, see Appendix 1).

Also, $\qquad \displaystyle\sum_{1}^{2N-1} \ell\, (-1)^\ell = -N, \qquad$ so that

$$\frac{1}{2N} \sum_{1}^{2N-1} \ell\, a_\ell = \sum_{-(N-1)}^{N-1} G_m\, Q_m - \frac{1}{4}(G_{-N} - G_N) \qquad (45)$$

Therefore, $\qquad S_0 = 2N G_0 + \dfrac{1}{4}(G_{-N} + G_N) - \displaystyle\sum_{-(N-1)}^{(N-1)} G_m\, Q_m \qquad (46)$

and substituting into (34),

$$V_0 = c_0 - \frac{1}{2}\Delta t \left\{ G_0 + 2\mathrm{Re} \sum_{1}^{N-1} G_m + \frac{1}{2}(G_{-N} + G_N) \right\} - \frac{1}{2}\Delta t\, G_0 +$$

$$+ \Delta t \left\{ 2N G_0 + \frac{1}{4}(G_{-N} + G_N) - G_0\left(N - \frac{1}{2}\right) + \mathrm{Re} \sum_{1}^{N-1} G_m\left(1 + i \cot \frac{\pi}{2N} m\right) \right\}$$

$$V_0 = c_0 + \frac{T}{2} G_0 - \frac{\Delta t}{2} \left\{ G_0 + 2 \sum_{1}^{N-1} \mathrm{Im}\,(G_j) \cot \frac{\pi}{2N} j \right\} \qquad (47)$$

Using equations (40) and (47), we can verify that they satisfy indeed the relation

$$\sum_{-N}^{N} V_j = c_0 \qquad (48)$$

Table 1

	Case	Frequency Domain	Time Domain
1	DFS obtained by aliasing of CFS (with $G_j \equiv g_j$)	$V_j = \dfrac{G_j - j\alpha \cot j\alpha\, G_0}{i\Omega_j} - \dfrac{G_0 \Delta t}{2}$	Exact match at every time step between "continuous" and "discrete" records.
2	Pseudo-continuous method	$V_j = \dfrac{G_j - G_0}{i\Omega_j} - \dfrac{G_0 \Delta t}{2}$	$a_k = \alpha \sum\limits_{\ell=1}^{N} \dfrac{\overline{c}_{k+\ell} - \overline{c}_{k-\ell}}{\Delta t}(-1)^{\ell-1} \cot \ell\alpha$ $\begin{cases} \overline{c}_\ell = c_\ell \ (\ell \neq 0) \\ \overline{c}_0 = c_0 + G_0 T/2 = \overline{c}_{2N}, \ \overline{c}_{-\ell} = \overline{c}_{2N-\ell} \end{cases}$
3	Central difference formula	$V_j = \dfrac{G_j \Delta t}{i \sin 2j\alpha}$ $(G_0 = 0)$	$a_k = \dfrac{c_{k+1} - c_{k-1}}{2\Delta t}$
4	Linear acceleration method	$V_j = \dfrac{\Delta t}{2}\left\{ \dfrac{G_j - G_0}{i} \cot j\alpha - G_0 \right\}$	$c_k = c_{k=1} + \dfrac{a_{k-1} + a_k}{2} \Delta t$

In all four cases, the formulas are for $j \neq 0$. The case $j = 0$ is given by

$$V_0 = c_0 - 2 \sum_{1}^{N-1} \mathrm{Re}(V_j) - \mathrm{Re}\, V_N, \qquad c_0 = \text{initial velocity}$$

Also,

$$\alpha = \pi/2N, \qquad \Delta t = T/2N$$

This shows that the expression for V_o could have been derived in a much simpler way, simply using eqs. (40) and (48) to solve for V_o. The alternate derivation, however, emphasizes the consistency of the formulation.

5) Summary of formulas relating velocity to acceleration, discrete case.

We explored in the previous pages some procedures to relate velocity and acceleration spectra when the records are defined only at discrete points in time. Some of these methods are more convenient than others, but the basic differences should be kept in mind when attempting to reconcile results obtained with frequency domain and time domain procedures.

The formulas derived previously are summarized in table 1.

Fourier Series of Baseline Parabola

If $f(t) = a + bt + ct^2$, for $0 < t < T$, the CFS is (substituting $f(t)$ into Eq. (3) and integrating):

$$h_j = \begin{cases} a + \dfrac{T}{2}b + \dfrac{T^2}{3}c & , \quad j = 0 \\[3mm] \dfrac{iTb}{2\pi}\dfrac{1}{j} + \dfrac{T^2c}{2\pi^2}\left(\dfrac{1}{j^2} + \dfrac{\pi i}{j}\right) & , \quad j \neq 0 \end{cases} \tag{49}$$

The DFS can be obtained from the CFS using the aliasing relation (5) plus the correction for initial conditions. This involves determination of the following infinite sums (see Appendix 1 for derivation):

$$\left. \begin{aligned} S_j^{(1)} &= \sum_{m=-\infty}^{\infty} \frac{1}{j + 2Nm} = \frac{\pi}{2N}\cot j\alpha \\[3mm] S_j^{(2)} &= \sum_{-\infty}^{\infty} \frac{1}{(j + 2Nm)^2} = \left(\frac{\pi}{2N}\right)^2 \frac{1}{\sin^2 j\alpha} \end{aligned} \right\} \quad \alpha = \frac{\pi}{2N}, \quad j \neq 0$$

$$\left. \begin{aligned} S_0^{(1)} &= 0 \\[3mm] S_0^{(2)} &= 2\sum_{m=1}^{\infty} \frac{1}{(2Nm)^2} = \frac{1}{3}\left(\frac{\pi}{2N}\right)^2 \end{aligned} \right\} \quad j = 0 \tag{50}$$

The correction for initial conditions, on the other hand, consists of the term

$$E = \frac{1}{2N} \frac{f(T) - f(0)}{2} = \frac{a + bT + cT^2 - a}{4N} = \frac{bT + cT^2}{4N}$$

It then follows

$$H_o = a + \frac{T}{2} b + \frac{T^2}{3} c + \frac{T^2 c}{2\pi^2} S_o^{(2)} - E$$

$$H_j = \frac{iTb}{2} S_j^{(1)} + \frac{T^2 c}{2\pi^2} [S_j^{(2)} + \pi i\, S_j^{(1)}] - E$$

that is,

$$H_o = a + \frac{T}{2} b\, [1 - \frac{1}{2N}] + \frac{T^2 c}{3}[1 - \frac{3}{4N} + \frac{1}{8N^2}]$$

$$H_j = \frac{(b + cT)T}{4N} (-1 + i \cot j\alpha) + \frac{cT^2}{8N^2} \frac{1}{\sin^2 j\alpha}$$

$$(51)$$

with $\alpha = \pi/2N$

It can be shown that Eqs. (51) agree with the direct evaluation of the DFS for the "parabola"

$$f_k = a + b\Delta t\, k + b(\Delta t)^2 k^2 \quad, \qquad k = 0, 1, \ldots\, 2N-1$$

The terms in brackets in H_o are for most practical purposes equal to one, and can be dropped without significant error:

$$H_o = a + \frac{T}{2} b + \frac{T^2 c}{3}$$

$$H_j = \frac{(b + cT)T}{4N} (-1 + i \cot j\alpha) + \frac{cT^2}{8N^2} \frac{1}{\sin^2 j\alpha}$$

$$(52)$$

Equations (52) can then be used to perform the correction for baseline in the discrete case.

Baseline Correction

Many criteria can be used to perform the correction for baseline. Hence, the conditions chosen below are only one of many alternatives. These are:

1) zero mean acceleration
2) zero initial velocity
3) zero mean velocity
4) minimum mean squared velocity

The first condition implies that the final velocity is equal to the initial velocity, which by condition 2) will be zero. Therefore, the particle will start from rest and return to rest at the end of the time history. This does not necessarily correspond to an actual situation because of the delay in tripping the trigger of the recording instrument. However, it is a convenient assumption in many practical computations. The third condition, on the other hand, implies that initial and final displacements are equal. This is more mathematical convenience than physical necessity, since the ground may experience permanent dislocations. The fourth condition, finally, minimizes the energy contents of the ground record. On the other hand, expansion of the ground acceleration time history in Fourier series implies that the motion is periodic, but the period is generally greater than the duration of the motion as a result of the "quiet" zone added at the end of the record. Hence, the correction performed using the procedure described here will incorporate the effect of the quiet zone on the baseline.

a) Continuous case

As before, we denote by g_j, v_j the spectral amplitudes of the acceleration and velocity records, respectively. The first condition is satisfied if

$$g_0 + h_0 = 0$$

that is, if

$$a + \frac{1}{2} bT + \frac{1}{3} cT^2 = - g_0 \tag{53}$$

The second condition requires

$$\sum_{-\infty}^{\infty} v_j = 0 \tag{54}$$

But $\quad v_j = \dfrac{g_j + h_j}{i\Omega_j}\quad$ for $j \neq 0$; then, considering symmetry conditions,

equation (54) transforms into

$$v_o + 2 \sum_{1}^{\infty} \frac{I_m(g_j + h_j)}{\Omega_j} = 0 \tag{55}$$

Introducing equations (46) as well as the third condition ($v_o = 0$) we obtain

$$2 \left\{ \frac{T}{2\pi} \sum_{1}^{\infty} \frac{\text{Im } (g_j)}{j} + \frac{T}{2\pi} \left(\frac{Tb}{2\pi} + \frac{T^2 c}{2\pi}\right) \sum_{1}^{\infty} \frac{1}{j^2} \right\} = 0$$

which by expansion yields

$$T(b + cT) = -\frac{12}{\pi} \sum_{1}^{\infty} \frac{\text{Im } (g_j)}{j} \tag{56}$$

The fourth condition may be enforced using Parseval's theorem which states that

$$I = \frac{1}{T} \int_0^T c^2(t)dt = \sum_{-\infty}^{\infty} |v(\Omega_j)|^2$$

where $v(\Omega_j)$ is the CFS of $c(t)$. In our case, (with $v_o = 0$),

$$I = 2 \sum_{1}^{\infty} \frac{(g_j + h_j)(g_j + h_j)^*}{\Omega_j^2} \tag{57}$$

The superscript asterisk denotes the complex conjugate. The minimum condition $I = $ min. implies that

$$dI = 2 \sum_{1}^{\infty} \left\{ dh_j(g_j + h_j)^* + (g_j + h_j)dh_j^* \right\}/\Omega_j^2 = 0 \tag{58}$$

Now

$$dh_j = \frac{\partial h_j}{\partial a} da + \frac{\partial h_j}{\partial b} db + \frac{\partial h_j}{\partial c} dc \tag{59}$$

Also, for $j \neq 0$ we have from Eq. (49):

$$\frac{\partial h_j}{\partial a} = 0$$

$$\frac{\partial h_j}{\partial b} = i \frac{T}{2\pi j}$$

$$\frac{h_j}{\partial c} = \frac{T^2}{2\pi^2} \left\{ \frac{1}{j^2} + \frac{\pi i}{j} \right\}$$

(60)

On the other hand, from eq. (56) we derive

$$db + Tdc = 0$$

(61)

Introducing (60) and (61) into (59) yields then

$$dh_j = - i \frac{T^2}{2\pi j} dc + \frac{T^2}{2\pi^2} \left(\frac{1}{j^2} + \frac{\pi i}{j}\right) dc$$

$$= \frac{T^2}{2\pi^2} \frac{1}{j^2} dc = dh_j^*$$

(62)

Substituting (62) into (58) gives then the condition

$$dI = \frac{2T^2}{2\pi^2} dc \sum_1^\infty \frac{(g_j + h_j)^* + (g_j + h_j)}{j^2 \Omega_j^2} = 0$$

implying that

$$\sum_1^\infty \frac{Re(g_j + h_j)}{j^4} = 0$$

(63)

On the other hand

$$\sum_1^\infty \frac{Re (h_j)}{j^4} = \frac{T_c^2}{2\pi^2} \sum_1^\infty \frac{1}{j^6} = \frac{T_c^2}{2\pi^2} \cdot \frac{\pi^6}{945}$$

(64)

Hence, from (63) and (64) we can solve for c:

$$cT^2 = - \frac{1890}{\pi^4} \sum_1^\infty \frac{Re (g_j)}{j^4}$$

(65)

Equations (53), (56) and (65) are then sufficient to determine the three constants c, b, a (in that order).   For convenience, we repeat

below the first two equations:

$$bT = - \left[cT^2 + \frac{12}{\pi} \sum_1^\infty \frac{Im\ (g_j)}{j} \right]$$

$$a = - (g_0 + \frac{bT}{2} + \frac{cT^2}{3})$$

(66)

b) Discrete case

The coefficients a, b, c for the discrete baseline parabola are obtained in manners which parallel the procedure for the continuous case. As before, the final condition requires

$$G_0 + H_0 = 0 \tag{67}$$

which by the first of the equations (52) implies

$$a + \frac{1}{2} bT + \frac{1}{3} cT^2 + - G_0 \tag{68}$$

The second condition requires

$$\sum_{-N}^{N} V_j = 0$$

that is

$$V_0 + 2 \sum_1^{N-1} Re\ V_j + Re\ V_N = 0 \tag{69}$$

The third condition specifies that $V_0 = 0$. Also, in all four cases listed in table 1 (i.e., in all definitions of spectral velocities vs. acceleration), the last element V_N is purely imaginary (because $G_N + H_N$ is purely real), and $G_0 + H_0 = 0$. Therefore,

$$Re(V_N) = 0 \tag{70}$$

Equation (69) transforms then into

$$\sum_1^{N-1} Re(V_j) = 0 \tag{71}$$

The fourth condition, on the other hand, is expressed by

$$\sum_{-N}^{N} |V_j|^2 = minimum$$

that is,

$$V_0^2 + 2 \sum_1^{N-1} V_j V_j^* + V_N V_N^* = min. \qquad (72)$$

The asterisk denotes again the complex conjugate. The minimum condition requires then

$$2 \sum_1^{N-1} (dV_j V_j^* + V_j dV_j^*) + dV_N V_N^* + V_N dV_N^* = 0 \qquad (73)$$

The form of the differentials dV_j, dV_j^* depends on the particular definition of spectral velocity used. However, in all cases they are functions of the differential dH_j:

$$dH_j = \frac{\partial H_j}{\partial a} da + \frac{\partial H_j}{\partial b} db + \frac{\partial H_j}{\partial c} dc \qquad (74)$$

For $j \neq 0$, the various terms are:

$$\left. \begin{array}{l} \dfrac{\partial H_j}{\partial a} = 0 \\[3mm] \dfrac{\partial H_j}{\partial b} = \dfrac{T}{4N} (-1 + i \cot j\alpha) \\[3mm] \dfrac{\partial H_j}{\partial c} = \dfrac{T^2}{4N} (-1 + i \cot j\alpha) + \dfrac{T^2}{8N^2} \dfrac{1}{\sin^2 j\alpha} \end{array} \right\} \qquad (75)$$

Furthermore, expansion of equation (71) will result in an expression of the form $b + cT = constant$, so that $db + Tdc = 0$. It follows that equation (74) reduces to

$$dH_j = 0 + [\frac{T}{4N}(-1 + i \cot j\alpha)](-Tdc) + [\frac{T^2}{4N}(-1 + i \cot j\alpha) + \frac{T^2}{8N^2} \frac{1}{\sin^2 j\alpha}] dc$$

that is,

$$dH_j = \frac{T^2}{8N^2} \frac{1}{\sin^2 j\alpha} dc = dH_j^* \qquad (76)$$

1) Pseudocontinuous method

In table 1, we replace G_0 by $G_0 + H_0 = 0$, and G_j by $G_j + H_j$. It follows that the first two methods in this table give identical results:

$$V_j = \frac{G_j + H_j}{i\Omega_j} \tag{77}$$

The differentials are then

$$dV_j = \frac{dH_j}{i\Omega_j} , \qquad dV_j^* = \left(\frac{dH_j}{i\Omega_j}\right)^* = -\frac{dH_j}{i\Omega_j} = -dV_j$$

so that

$$dV_j V_j^* + V_j dV_j^* = \frac{\{(G_j + H_j)^* + (G_j + H_j)\}dH_j}{\Omega_j^2}$$

$$= \frac{2 \, \mathrm{Re} \, (G_j + H_j)}{\Omega_j^2} \, dH_j$$

$$= \frac{2\left(\frac{T^2}{8N^2}\right)}{\left(\frac{2\pi}{T}\right)^2} \, \frac{\mathrm{Re}(G_j + H_j)}{j^2 \sin^2 j\alpha} \, dc \tag{78}$$

Substitution of (78) into (73) results then in (with $\sin \alpha N = 1$)

$$\sum_1^{N-1} \frac{\mathrm{Re} \, (G_j + H_j)}{j^2 \sin^2 j\alpha} + \frac{1}{2} \frac{\mathrm{Re} \, (G_N + H_N)}{N^2} = 0 \tag{79}$$

But

$$\mathrm{Re} \, (H_j) = -\frac{(b + cT)T}{4N} + \frac{cT^2}{8N^2} \frac{1}{\sin^2 j\alpha}$$

Hence

$$-\frac{(b + cT)T}{4N} \left\{ \sum_1^{N-1} \frac{1}{j^2 \sin^2 j\alpha} + \frac{1}{2} \frac{1}{N^2} \right\} +$$

$$+ \frac{cT^2}{8N^2} \left\{ \sum_1^{N-1} \frac{1}{j^2 \sin^4 j\alpha} + \frac{1}{2} \frac{1}{N^2} \right\} +$$

$$+ \left\{ \sum_1^{N-1} \frac{\mathrm{Re} \, G_j}{j^2 \sin^2 j\alpha} + \frac{1}{2} \frac{\mathrm{Re} \, G_N}{N^2} \right\} = 0. \tag{80}$$

For reasonably large values of N, the first two parentheses in (80) approximate the asymptotic values (see table 2)

$$\sum_{1}^{N-1} \frac{1}{j^2 \sin^2 j\alpha} + \frac{1}{2} \frac{1}{N^2} \approx \frac{2N^2\pi^2}{45}$$

$$\sum_{1}^{N-1} \frac{1}{j^2 \sin^4 j\alpha} + \frac{1}{2} \frac{1}{N^2} \approx \frac{16 \ N^4\pi^2}{945}$$

Hence,

$$-(b + cT)T \cdot \frac{N\pi^2}{90} + \frac{cT^2 \cdot 2N^2\pi^2}{945} + \sum_{1}^{N-1} \frac{Re \ G_j}{j^2\sin^2 j\alpha} + \frac{1}{2} \frac{Re \ (G_N)}{N^2} = 0 \qquad (81)$$

from which we write

$$cT^2 = \frac{21}{4N} (b + cT)T - \frac{945}{2\pi^2 N^2} \left\{ \sum_{1}^{N-1} \frac{Re \ G_j}{j^2\sin^2 j\alpha} + \frac{1}{2} \frac{Re \ G_N}{N^2} \right\} \qquad (82)$$

(The term b + cT is left for convenience, as shall be seen). Finally, equation (71) results in

$$\sum_{1}^{N-1} Re(V_j) = \sum_{1}^{N-1} \frac{Im \ (G_j + H_j)}{\Omega_j} = 0$$

But
$$Im(H_j) = \cot j\alpha \ (b + cT) \frac{T}{4N}$$

Hence

$$\sum_{1}^{N-1} \frac{Im \ (G_j)}{j} + (b + cT) \frac{T}{4N} \sum_{1}^{N-1} \frac{\cot j\alpha}{j} = 0 \qquad (83)$$

Again, for reasonably large N (see table 2), the asymptotic value for the second sum is

$$\sum_{1}^{N-1} \frac{\cot j\alpha}{j} \approx \frac{N\pi}{3}$$

Therefore
$$(b + cT)T = - \frac{12}{\pi} \sum_{1}^{N-1} \frac{Im \ G_j}{j} \qquad (84)$$

Table 2

$V = \pi/2N$	N				
	4	8	16	32	64
$\dfrac{45}{2N^2\pi^2}\left\{\displaystyle\sum_1^{N-1}\dfrac{1}{j^2\sin^2 j\alpha} + \dfrac{1}{2}\dfrac{1}{N^2}\right\}$	1.06718	1.01824	1.00473	1.00120	1.00030
$\dfrac{945}{16N^4\pi^2}\left\{\displaystyle\sum_1^{N-1}\dfrac{1}{j^2\sin^4 j\alpha} + \dfrac{1}{2}\dfrac{1}{N^2}\right\}$	1.11764	1.02790	1.00687	1.00171	1.00043
$\dfrac{3}{N\pi}\displaystyle\sum_1^{N=1}\dfrac{\cot j\alpha}{j}$	0.72868	0.86116	0.92969	0.96461	0.98225
$\dfrac{3}{2N^2}\displaystyle\sum_1^{N-1}\dfrac{1}{\sin^2 j\alpha}$	0.93750	0.98438	0.99609	0.99902	0.99976
$\dfrac{945}{16N^6}\displaystyle\sum_1^{N=1}\dfrac{1}{\sin^2 2j\alpha\,\sin^4 j\alpha}$	1.44196	1.08958	1.02099	1.00516	1.00128
$\dfrac{45}{2N^4}\displaystyle\sum_1^{N-1}\dfrac{1}{\sin^2 2j\alpha\,\sin^2 j\alpha}$	1.58203	1.15356	1.03889	1.00976	1.00244
$\dfrac{3}{2N^2}\displaystyle\sum_1^{N-1}\cot^2 j\alpha$	0.65625	0.82031	0.90820	0.95361	0.97668
$\dfrac{945}{64N^6}\displaystyle\sum_1^{N-1}\dfrac{\cot^2 j\alpha}{\sin^4 j\alpha}$	0.99495	0.99968	0.99998	1.00000	1.00000
$\dfrac{45}{8N^4}\displaystyle\sum_1^{N-1}\dfrac{\cot^2 j\alpha}{\sin^2 j\alpha}$	0.92285	0.98053	0.99512	0.99878	0.99969

$$\sum_1^\infty \dfrac{1}{j^2} = \dfrac{\pi^2}{6}\ , \qquad \sum_1^\infty \dfrac{1}{j^4} = \dfrac{\pi^4}{90}\ , \qquad \sum_1^\infty \dfrac{1}{j^6} = \dfrac{\pi^6}{945}$$

Equations (68), (82), (84) can be then used to determine the three constants a, b, c.

2) Central difference formula:

The procedure is completely analogous to the pseudocontinuous case, merely requiring to change equation (77) by

$$
\begin{cases}
V_j = \dfrac{G_j + H_j}{i \, \sin 2j\alpha} \, \Delta t \quad , & \alpha = \dfrac{\pi}{2N} \\
\\
V_N = 0
\end{cases}
\tag{85}
$$

The second and third conditions result then in (compare with (83))

$$
\sum_1^{N-1} \frac{Im \, (G_j)}{\sin 2j\alpha} + \frac{T}{4N}(b+cT) \sum_1^{N-1} \frac{\cot j\alpha}{\sin 2j\alpha} = 0
\tag{86}
$$

But

$$
\sum_1^{N-1} \frac{\cot j\alpha}{\sin 2j\alpha} = \frac{1}{2} \sum_1^{N-1} \frac{1}{\sin^2 j\alpha} \approx \frac{N^2}{3} \qquad \text{(table 2)}
$$

Hence

$$
(b+cT)T = - \frac{12}{N} \sum_1^{N-1} \frac{Im \, (G_j)}{\sin 2j\alpha}
\tag{87}
$$

Finally, the last two conditions demand that (compare with (80))

$$
\sum_1^{N-1} \frac{Re \, (G_j)}{\sin^2 j\alpha \, \sin^2 2j\alpha} + \frac{cT^2}{8N^2} \sum_1^{N-1} \frac{1}{\sin^4 j\alpha \, \sin^2 2j\alpha} +
$$

$$
- \frac{(b+cT)T}{4N} \sum_1^{N-1} \frac{1}{\sin^2 j\alpha \, \sin^2 2j\alpha} = 0
\tag{88}
$$

Again, we replace the last two sums by their asymptotic values (table 2):

$$
\sum_1^{N-1} \frac{1}{\sin^4 j\alpha \, \sin^2 2j\alpha} \approx \frac{16N^6}{945}
$$

$$
\sum_1^{N-1} \frac{1}{\sin^2 j\alpha \, \sin^2 2j\alpha} \approx \frac{2N^4}{45}
$$

It follows that

$$cT^2 = \frac{21}{4N}(b+cT)T - \frac{945}{2N^4} \sum_1^{N-1} \frac{\text{Re}(G_j)}{\sin^2 j\alpha \, \sin^2 2j\alpha} \qquad (89)$$

(compare with eq. (82)).

In this case, equations (68), (87) and (89) are used to determine a, b, c.

3) Linear acceleration method

This time, the spectral velocity is given by

$$V_j = \Delta t \, \frac{G_j + H_j}{2i} \cot j\alpha \qquad (90)$$

The second and third conditions give then (compare with (83), (87)):

$$\sum_1^{N-1} \text{Im}(G_j) \cot j\alpha + \frac{T}{4N}(b+cT) \sum_1^{N-1} \cot^2 j\alpha = 0 \qquad (91)$$

But

$$\sum_1^{N-1} \cot^2 j\alpha \approx \frac{2}{3} N^2 \qquad \text{(table 2)}$$

Hence

$$(b+cT)T = -\frac{6}{N} \sum_1^{N-1} \text{Im}(G_j) \cot j\alpha \qquad (92)$$

Also, the third and fourth conditions give now (compare with eqs. (80), (88):

$$\sum_1^{N-1} \frac{\text{Re}(G_j)}{\sin^2 j\alpha} \cot^2 j\alpha + \frac{cT^2}{8N^2} \sum_1^{N-1} \frac{\cot^2 j\alpha}{\sin^4 j\alpha} +$$

$$-\frac{(b+cT)T}{4N} \sum_1^{N-1} \frac{\cot^2 j\alpha}{\sin^2 j\alpha} = 0 \qquad (93)$$

Also, (table 2)

$$\sum_1^{N-1} \frac{\cot^2 j\alpha}{\sin^4 j\alpha} \approx \frac{64N^6}{945}$$

$$\sum_1^{N-1} \frac{\cot^2 j\alpha}{\sin^2 j\alpha} \approx \frac{8N^4}{45}$$

Therefore

$$cT^2 = \frac{21}{4N}(b+cT)T - \frac{945}{8N^4}\sum_1^{N-1} Re(G_j)\left(\frac{\cot^2 j\alpha}{\sin^2 j\alpha}\right) \tag{94}$$

which should be compared to eqs. (82), (89). The coefficients a, b, c follow now from eqs. (68), (92) and (94).

Summary of equations for baseline parabola

$$y = a + bt + ct^2 , \qquad t = k\Delta t$$

$$= \bar{a} + \bar{b}\left(\frac{k}{2N}\right) + \bar{c}\left(\frac{k}{2N}\right)^2 , \qquad \Delta t = T/2N$$

$$\bar{a} = a, \qquad \bar{b} = bT, \qquad \bar{c} = cT^2$$

$$\bar{a} + \frac{1}{2}\bar{b} + \frac{1}{3}\bar{c} = -G_0$$

$$\bar{b} + \bar{c} = -p$$

$$\bar{c} = -(rp + q)$$

with coefficients p, r, q given below for the various cases.

TABLE 3

	p	r	q
Continuous	$\frac{12}{\pi}\sum_1^\infty \frac{Im(g_j)}{j}$	0	$\frac{1890}{\pi^4}\sum_1^\infty \frac{Re(g_j)}{j^4}$
Pseudo continuous	$\frac{12}{\pi}\sum_1^{N-1}\frac{Im(G_j)}{j}$	$\frac{21}{4N}$	$\frac{945}{2N^2\pi^2}\left\{\sum_1^{N-1}\frac{Re(G_j)}{j^2\sin^2 j\alpha} + \frac{1}{2}\frac{Re(G_N)}{N^2}\right\}$
Central Difference	$\frac{12}{N}\sum_1^{N-1}\frac{Im(G_j)}{\sin 2j\alpha}$	$\frac{21}{4N}$	$\frac{945}{2N^4}\sum_1^{N-1}\frac{Re(G_j)}{\sin^2 j\alpha \sin^2 2j\alpha}$
Linear acceleration method.	$\frac{6}{N}\sum_1^{N-1}Im(G_j)\cot j\alpha$	$\frac{21}{4N}$	$\frac{945}{8N^4}\sum_1^{N-1}\frac{Re(G_j)\cot^2 j\alpha}{\sin^2 j\alpha}$

$$(\alpha = \pi/2N)$$

Limited Parabolic Baseline Correction

The baseline correction technique described above extends over the duration of the earthquake $(0 \leq t \leq t_d)$ and during the quiet zone added at the end of the record $(t_d < t < T)$. As a result, an acceleration pattern will be introduced during the quiet zone, where acceleration was originally non-existent in the real earthquake.

The length of the quiet zone will be dependent upon the earthquake data and the algorithm used to compute the Fast Fourier Transform. If this zone is sufficiently large, it may exercise an unduly large effect on the baseline parabola used to correct the earthquake. Thus, an alternate method is explored below.

Redefining the shape of the baseline correction in the form

$$f(t) = a \cdot x(t) + b \cdot y(t) + c \cdot z(t) \quad , \tag{95}$$

where $x(t) = 1$ $y(t) = t$ $z(t) = t^2$ for $0 \leq t \leq z_d$

$ = 0 = 0 = 0 t_d < t < T \quad ,$

then a parabolic correction is introduced that will not have the spurious acceleration in the quiet zone.

The discontinuity of the function at time t_d prevents a concise analytical expression to be written for the funcitons h_j and H_j, the CFS and DFS of $f(t)$. Nevertheless, H_j may be expressed in the form

$$H_j = aX_j + bY_j + cZ_j \quad , \tag{96}$$

where X_j, Y_j, and Z_j are the DFS of the functions $x(t)$, $y(t)$ and $z(t)$, respectively. X_j, Y_j and Z_j can be evaluated numerically from equation (4), using the FFT algorithm.

Restricting our attention to the discrete case, the first criterion of zero mean acceleration requires that

$$aX_o + bY_o + cZ_o = - G_o \quad . \tag{97}$$

As stated previously, by imposing the second and third conditions of zero initial and mean velocities, equation (71) can be derived. That is,

$$\sum_{1}^{N-1} Re(V_j) = 0 \quad .$$

For notational convenience, given the spectral acceleration coefficient A_j, then the velocity spectral components shall now be denoted as $\overline{A}_j$. The coefficients $\overline{A}_j$ are evaluated using the formulae in Table 1. Using this notation and realizing that the formulae in Table 1 are linear transformations, it follows that

$$\begin{aligned}
V_j &= (\overline{G_j + H_j}) \\
&= (\overline{G_j + aX_j + bY_j + cZ_j}) \\
&= (\overline{G}_j + a\overline{X}_j + b\overline{Y}_j + c\overline{Z}_j \quad .
\end{aligned} \qquad (98)$$

Substituting this result into equation (71) yields

$$a \sum_{1}^{N-1} Re \ \overline{X}_j + b \sum_{1}^{N-1} Re \ \overline{Y}_j + c \sum_{1}^{N-1} Re \ \overline{Z}_j = - \sum_{1}^{N-1} Re \ \overline{G}_j \qquad (99)$$

Using the results of equations (97) and (99), one obtains

$$X_o \ da + Y_o \ db + Z_o \ dc = 0$$

$$\sum_{1}^{N-1} Re \ \overline{X}_j da + \sum_{1}^{N-1} Re \ \overline{Y}_j db + \sum_{1}^{N-1} Re \ \overline{Z}_j dc = 0 \quad .$$

Solving these two equations for da and db in terms of dc yields

$$da = Rdc \qquad \text{and} \qquad db = Sdc \quad , \qquad (100)$$

where

$$R = \left\{ (Y_o \sum_{1}^{N-1} Re \ \overline{Z}_j - Z_o \sum_{1}^{N-1} Re \ \overline{Y}_j) / (X_o \sum_{1}^{N-1} Re \ \overline{Y}_j - Y_o \sum_{1}^{N-1} Re \ \overline{X}_j) \right\}$$

$$\qquad (101)$$

$$S = \left\{ (Z_o \sum_{1}^{N-1} \overline{X}_j - X_o \sum_{1}^{N-1} Re \ \overline{Z}_j) / (X_o \sum_{1}^{N-1} Re \ \overline{Y}_j - Y_o \sum_{1}^{N-1} Re \ \overline{X}_j) \right\}$$

The final condition of mimimizing the mean squared velocity has been demonstrated (in equation (73)) to require that

$$2 \sum_{1}^{N-1} (dV_j \, V_j^* + V_j \, dV_j^*) + dV_N \, V_N^* + V_N \, dV_N^* = 0 \quad .$$

As a further notational simplification, this requirement shall be denoted as

$$\sum_{1}^{N} (dV_j \, V_j^* + V_j \, dV_j^*) = 0 \tag{102}$$

But from equation (98)

$$dV_j = \overline{X}_j da + \overline{Y}_j db + \overline{Z}_j dc \quad .$$

Using equations (100) and (101)

$$V_j = (R\overline{X}_j + S\overline{Y}_j + \overline{Z}_j)dc = U_j dc \quad . \tag{103}$$

In a similar manner

$$dV_j^* = \overline{G}_j^* + a\overline{X}_j^* + b\overline{Y}_j^* + c\overline{Z}_j^* \tag{104}$$

and

$$dV_j^* = U_j^* \, dc \tag{105}$$

Substituting equations (98), (103), (104) and (105) into equation (102) results in

$$\sum_{1}^{N} \left\{ (U_j dc)(\overline{G}_j^* + a\overline{X}_j^* + b\overline{Y}_j^* + c\overline{Z}_j^*) + (\overline{G}_j + a\overline{X}_j + b\overline{Y}_j + c\overline{Z}_j)(U_j^* \, dc) \right\} = 0$$

and by expansion

$$\left\{ a \sum_{1}^{N} (U_j \overline{X}_j^* + \overline{X}_j U_j^*) + b \sum_{1}^{N} (U_j \overline{Y}_j^* + \overline{Y}_j U_j^*) + c \sum_{1}^{N} (U_j \overline{Z}_j^* + \overline{Z}_j U_j^* + \right.$$

$$\left. + \sum_{1}^{N} (U_j \overline{G}_j^* + \overline{G}_j U_j^*) \right\} dc = 0$$

Consequently, the fourth condition results in

$$a \sum_{1}^{N} (U_j \overline{X}_j^* + \overline{X}_j U_j^*) + b \sum_{1}^{N} (U_j \overline{Y}_j^* + \overline{Y}_j U_j^*) + c \sum_{1}^{N} (U_j \overline{Z}_j^* + \overline{Z}_j + U_j^*) =$$

$$- \sum_{1}^{N} (U_j \overline{G}_j^* + \overline{G}_j U_j^*) \qquad (106)$$

Using equations (97), (99), (103) and the real part of equation (106), the correction coefficients a, b and c may be uniquely determined. It should be noted that this formulation is valid for any three arbitrarily chosen correction functions x(t), y(t) and z(t).

Numerical Example--The Golden Gate Earthquake

The frequency domain integration techniques and the baseline correction methods described in this paper, are applied to the Golden Gate Earthquake S80E of March 22, 1957. The earthquake lasted 13.03 seconds, with a peak acceleration of .12837g occurring at t = 1.45 seconds. The accelerations were recorded at intervals of Δt = .01 seconds (1304 acceleration values). This corresponds to a Nyquist folding frequency of 50 Hz. To compute the Fast Fourier Transform, a value of 2N = 2048 was chosen (T = 20.48 seconds). The coefficients for the baseline parabola calculated according to the various methods described in this report are listed in Table 4.

Graphical results of this investigation are presented in figures 1.1.1 through 4.3.3. The three digits for the figures have the following meaning:

a) First digit: Integration method.

    1. - Continuous (Aliasing)

    2. - Pseudocontinuous

    3. - Central Difference

    4. - Linear acceleration method

b) Second digit: Baseline correction method (see also Table 3).

    1. - No baseline correction

    2. - Parabolic baseline correction (period of baseline = T)

    3. - Limited baseline correction (period of baseline = t_d)

c) Third digit: Record component.

   1. - Acceleration
   2. - Velocity
   3. - Displacement

TABLE 4 - BASELINE CORRECTION COEFFICIENTS

Full Parabolic Base-line Correction	$f(t) = a + bt + ct^2$		$0 \leq t < T$
Method	a	b	c
Continuous (aliasing)	3.607798×10^{-3}	-1.160212×10^{-3}	4.792435×10^{-5}
Pseudo-continuous	3.604624×10^{-3}	-1.159281×10^{-3}	4.787889×10^{-5}
Central difference	3.637456×10^{-3}	-1.164434×10^{-3}	4.787838×10^{-5}
Linear acceleration	3.614157×10^{-3}	-1.160223×10^{-3}	4.787967×10^{-5}

Limited Parabolic Base-line Correction	$f(t) = ax(t) + by(t) + cz(t)$		$0 < t < t_d$
Method	a	b	c
Continuous (aliasing)	-2.360215×10^{-3}	2.636531×10^{-3}	-3.053304×10^{-4}
Pseudo-continuous	-2.360080×10^{-3}	2.636439×10^{-3}	-3.052234×10^{-4}
Central difference	-2.264105×10^{-3}	2.627296×10^{-3}	-3.059660×10^{-4}
Linear acceleration	-2.356620×10^{-3}	2.640890×10^{-3}	-3.058959×10^{-4}

The velocity and displacement records were computed (and plotted on the same graph) both in the frequency and in the time domains. The computation in the frequency domain was made according to the integration formulas listed in Table 2, while the time domain integration corresponded to the linear acceleration method with zero initial conditions; inspection of the graphs reveals an excellent agreement in all cases, with the exception of the uncorrected record using the central difference formula (figs. 3.1.2, 3.1.3). The discrepancy in this case follows from the inability of the formula to handle the discontinuity of the end of the record; once the correction is performed (which removes the discontinuity), the agreement improves substantially (figs 3.2.2, 3.2.3, 3.3.2, 3.3.3).

GOLDEN GATE EARTHQUAKE   ACCELERATION TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
NO BASELINE CORRECTION

Figure 1.1.1

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GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD
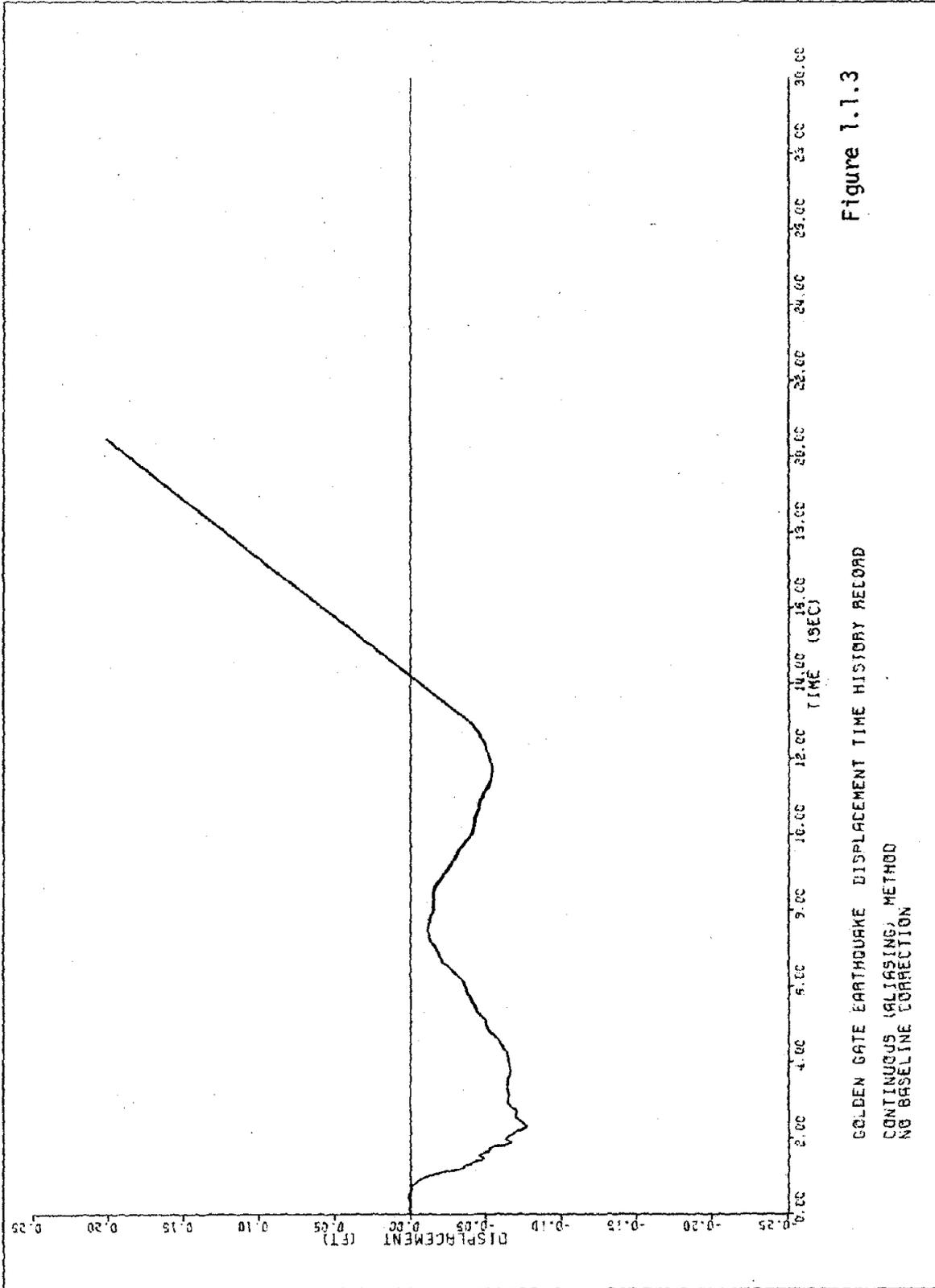
CONTINUOUS (ALIASING) METHOD
NO BASELINE CORRECTION

Figure 1.1.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
NO BASELINE CORRECTION
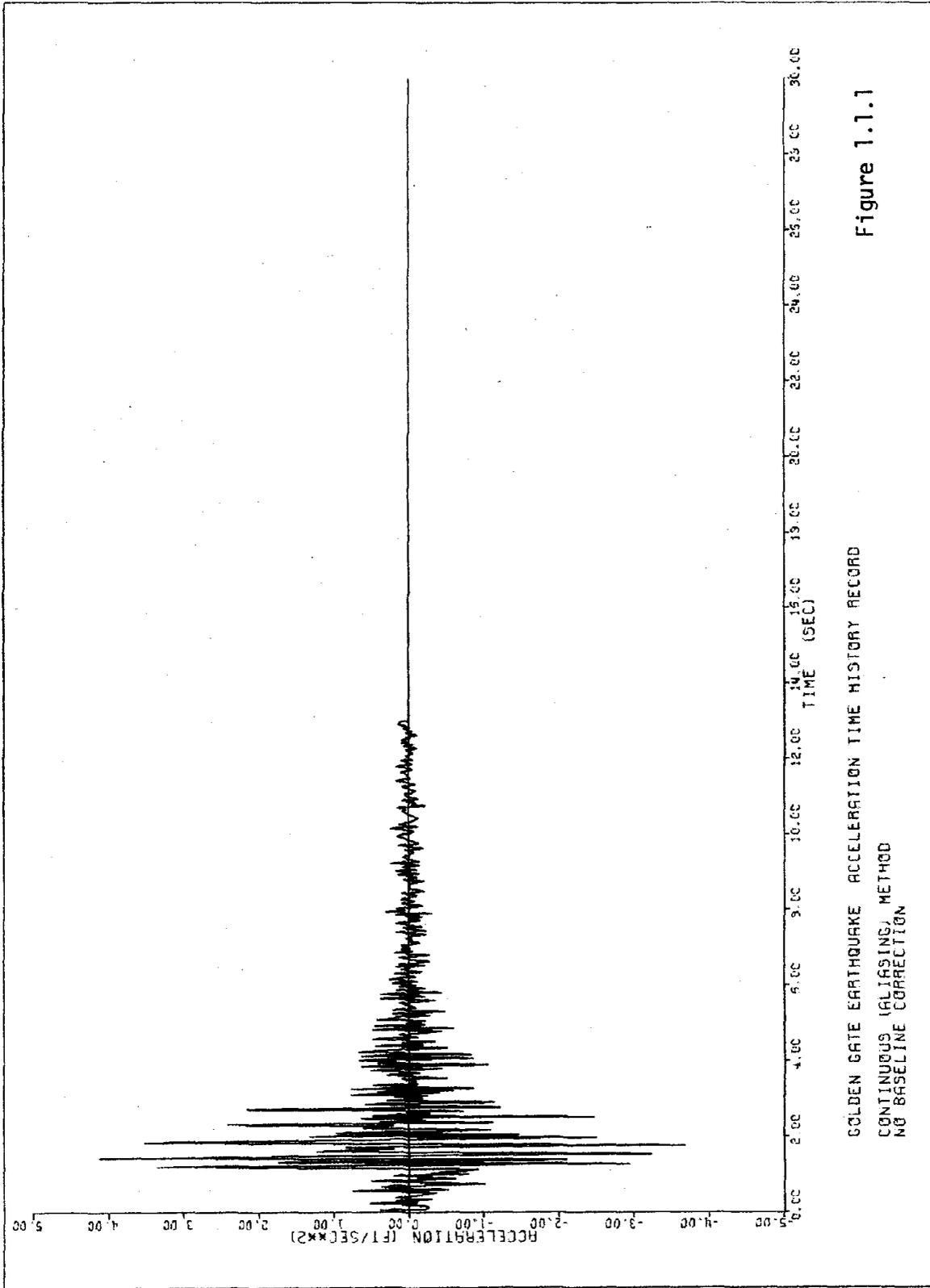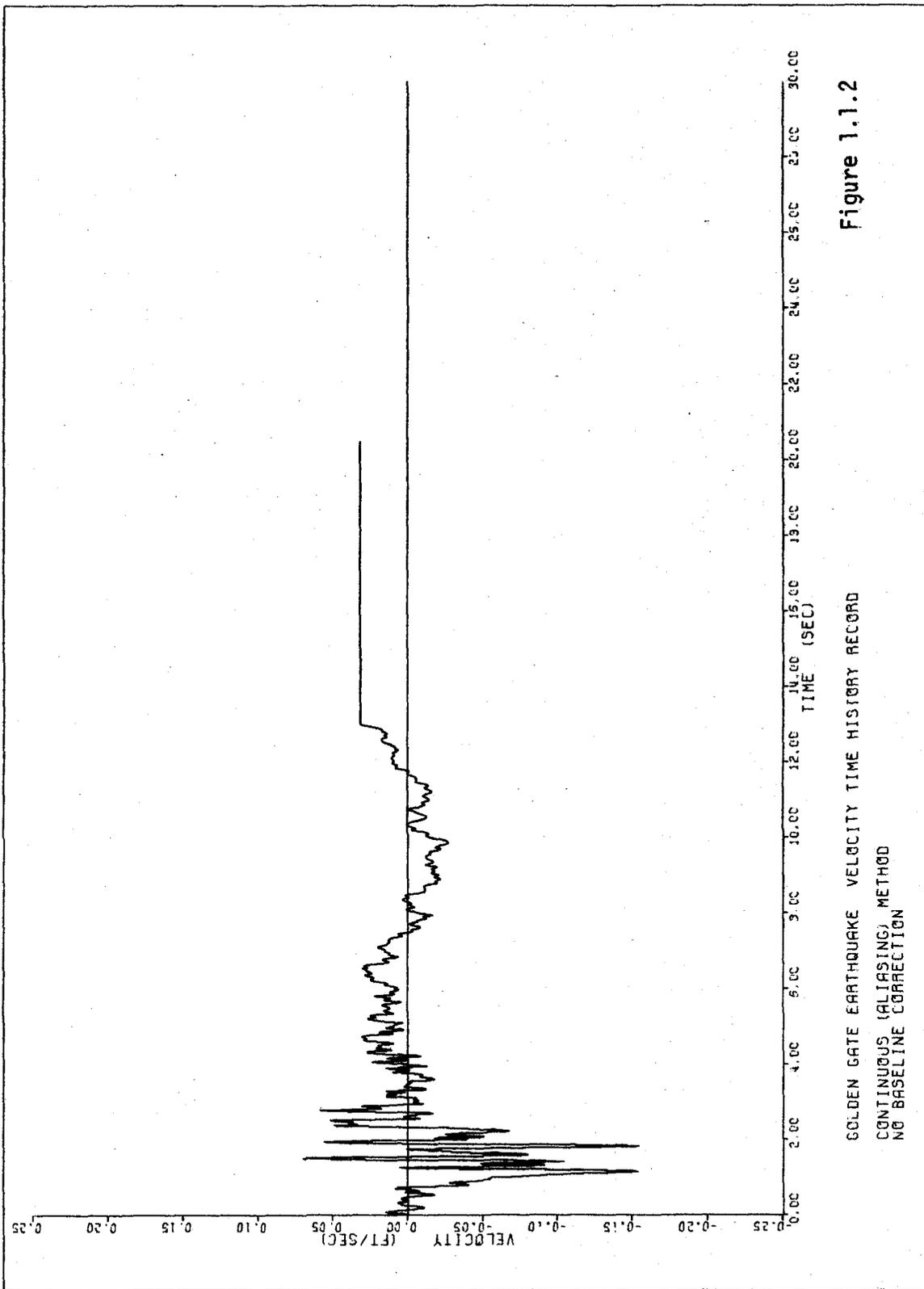
Figure 1.1.3

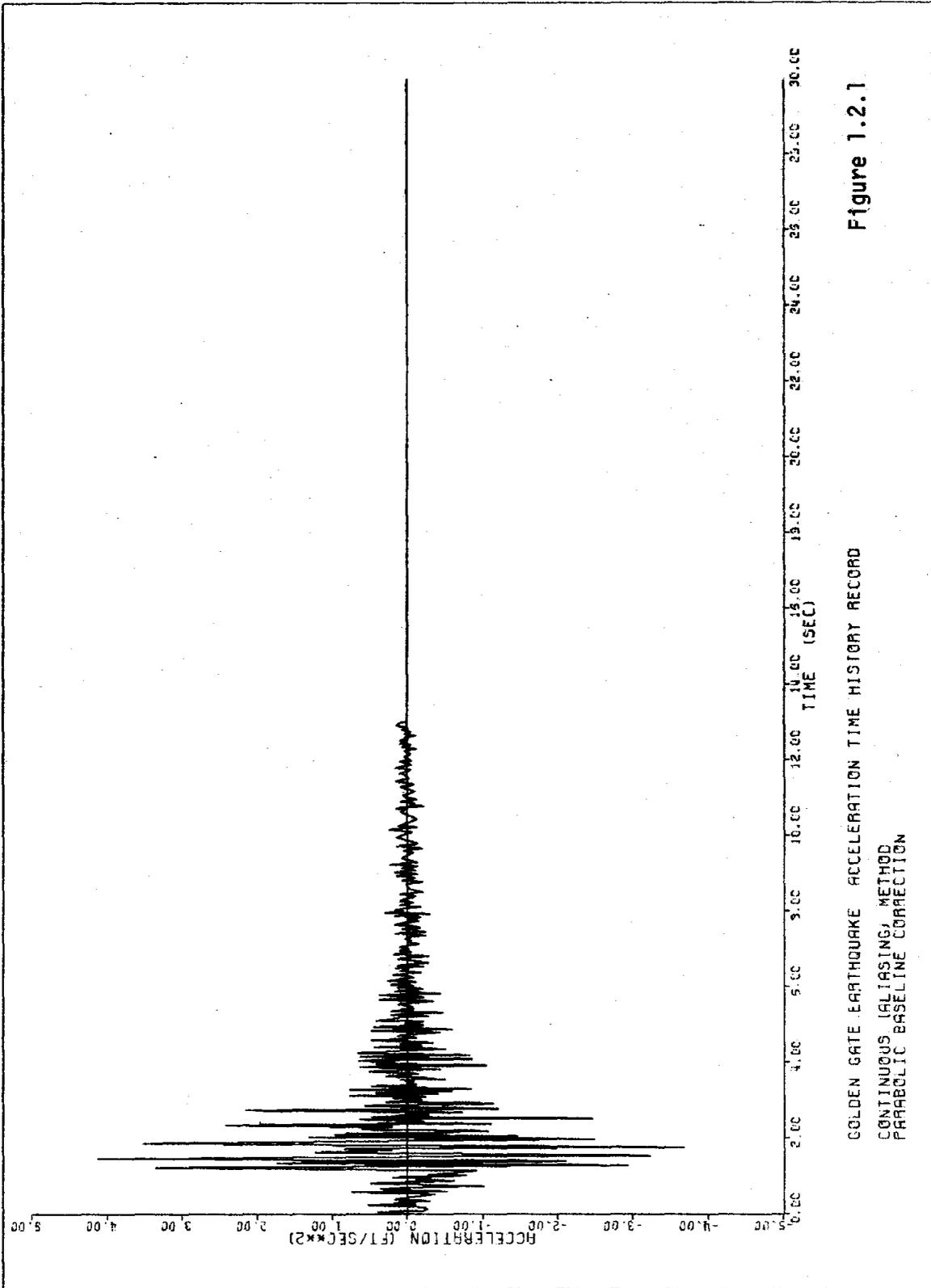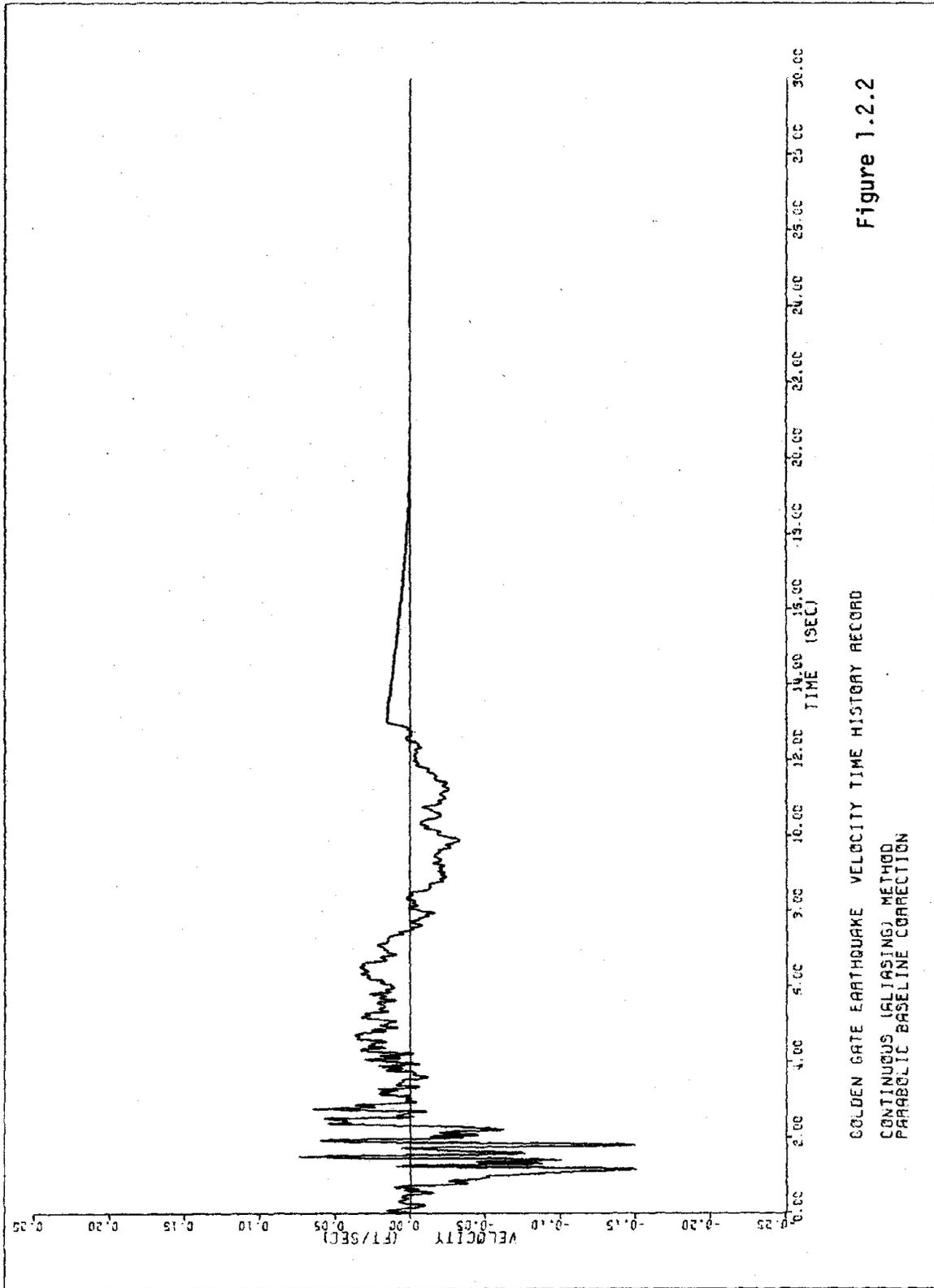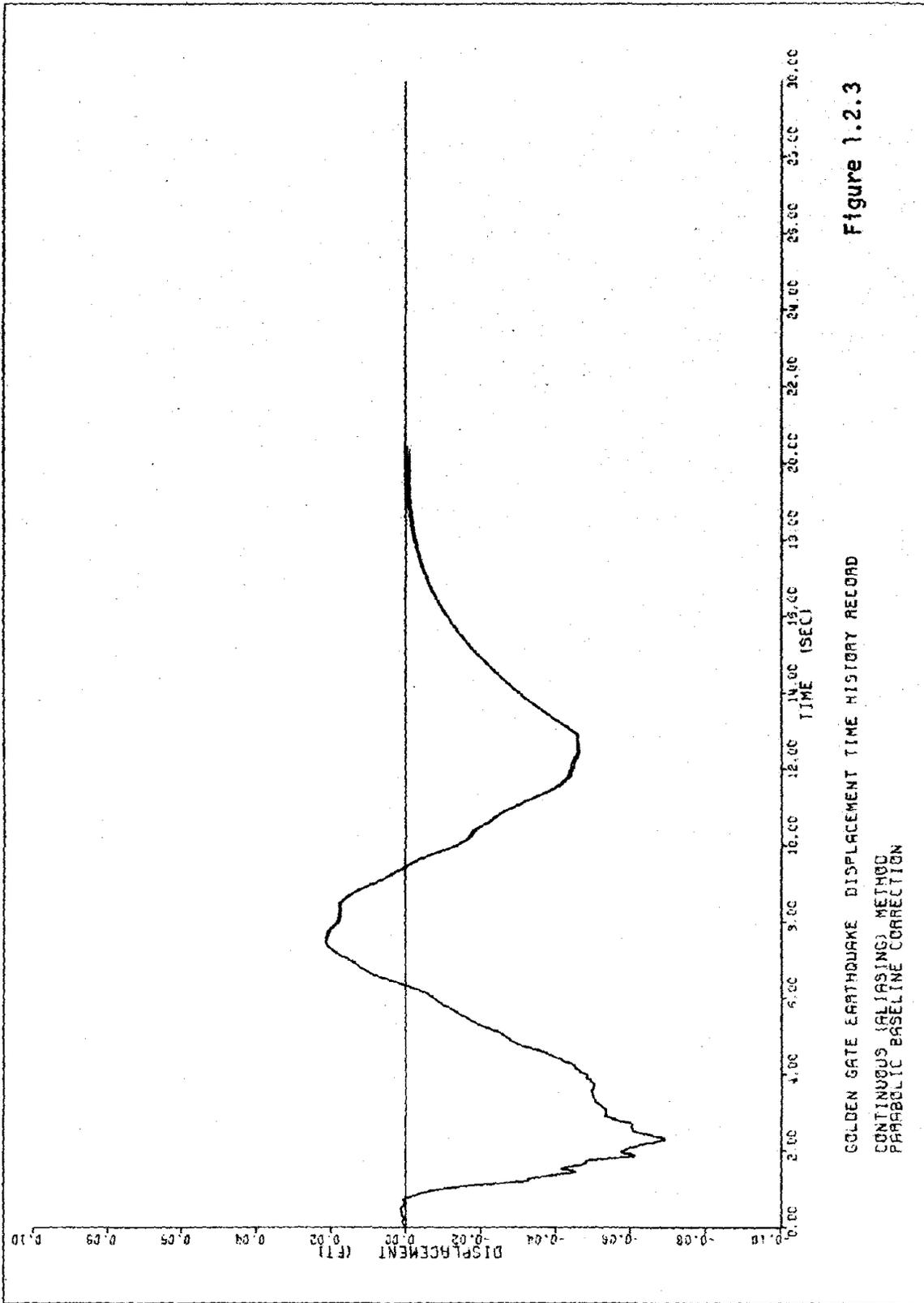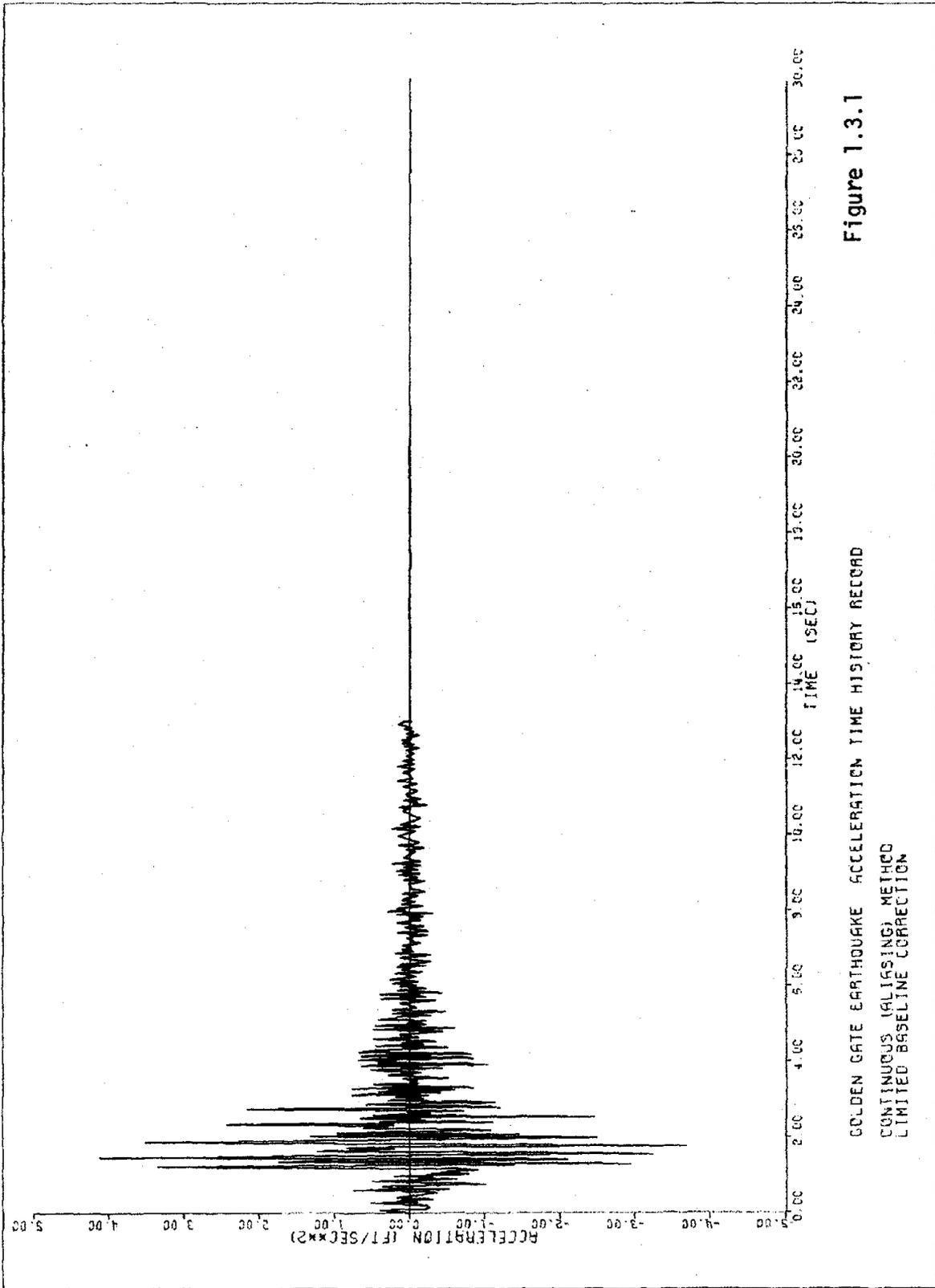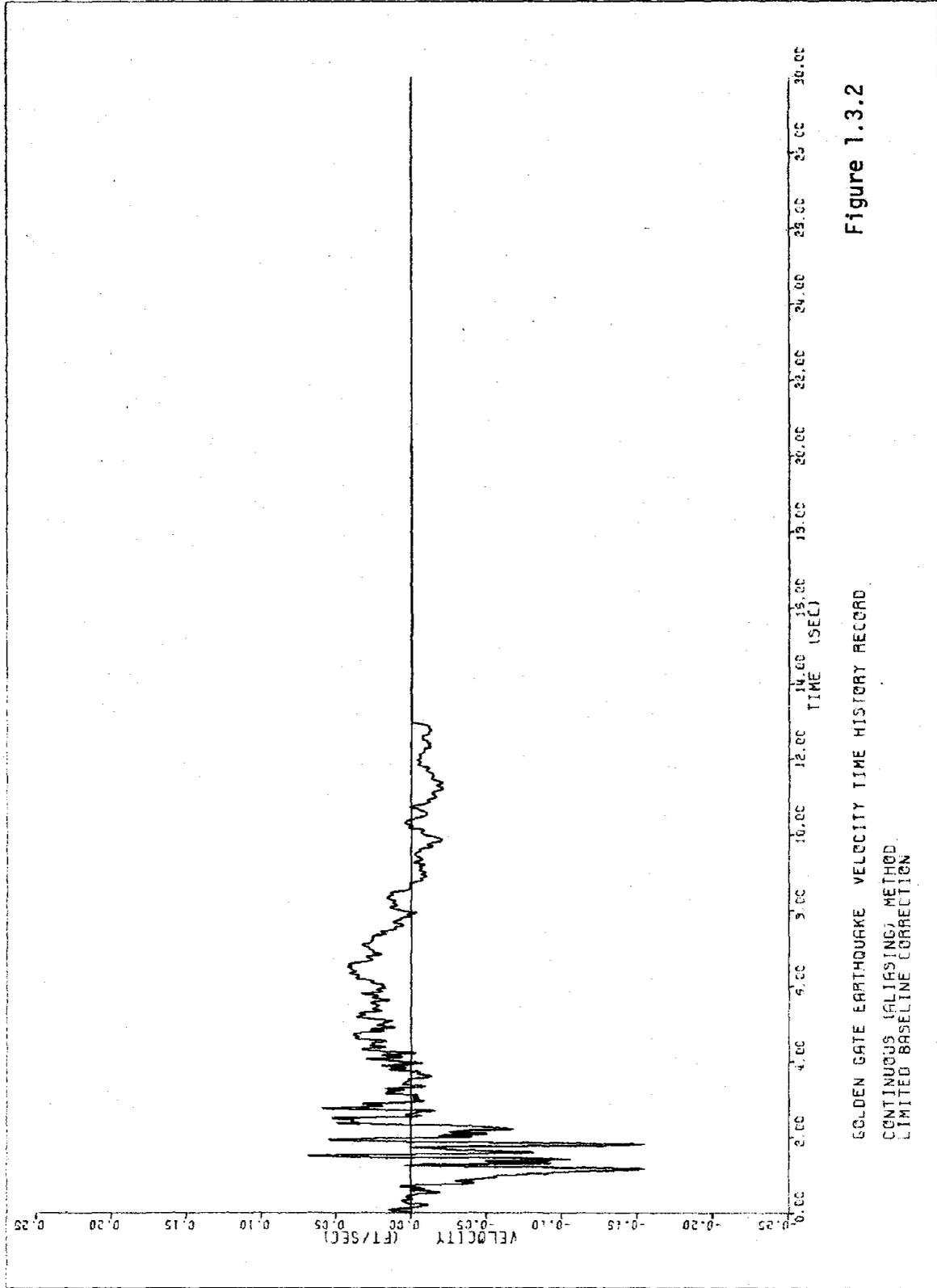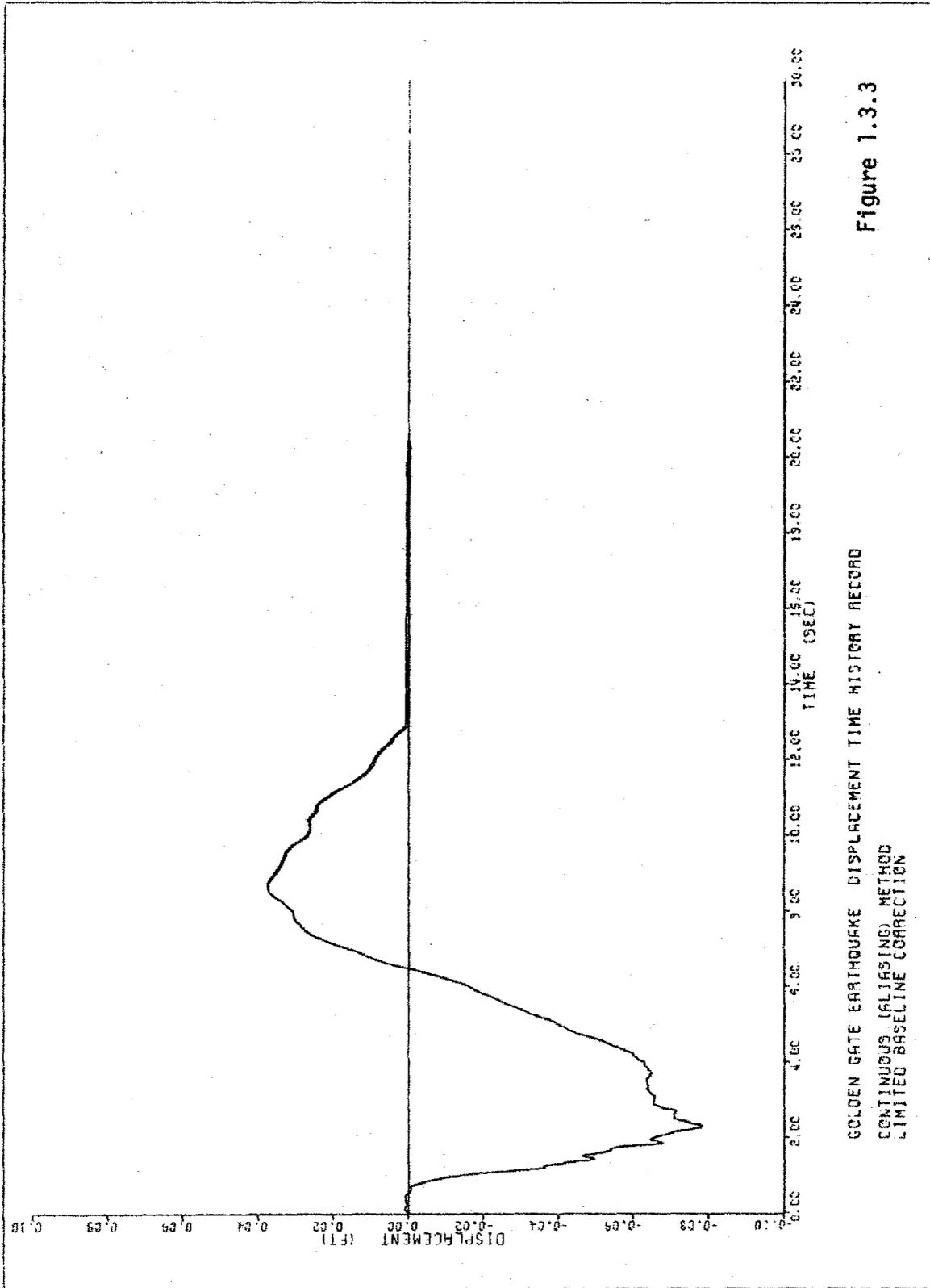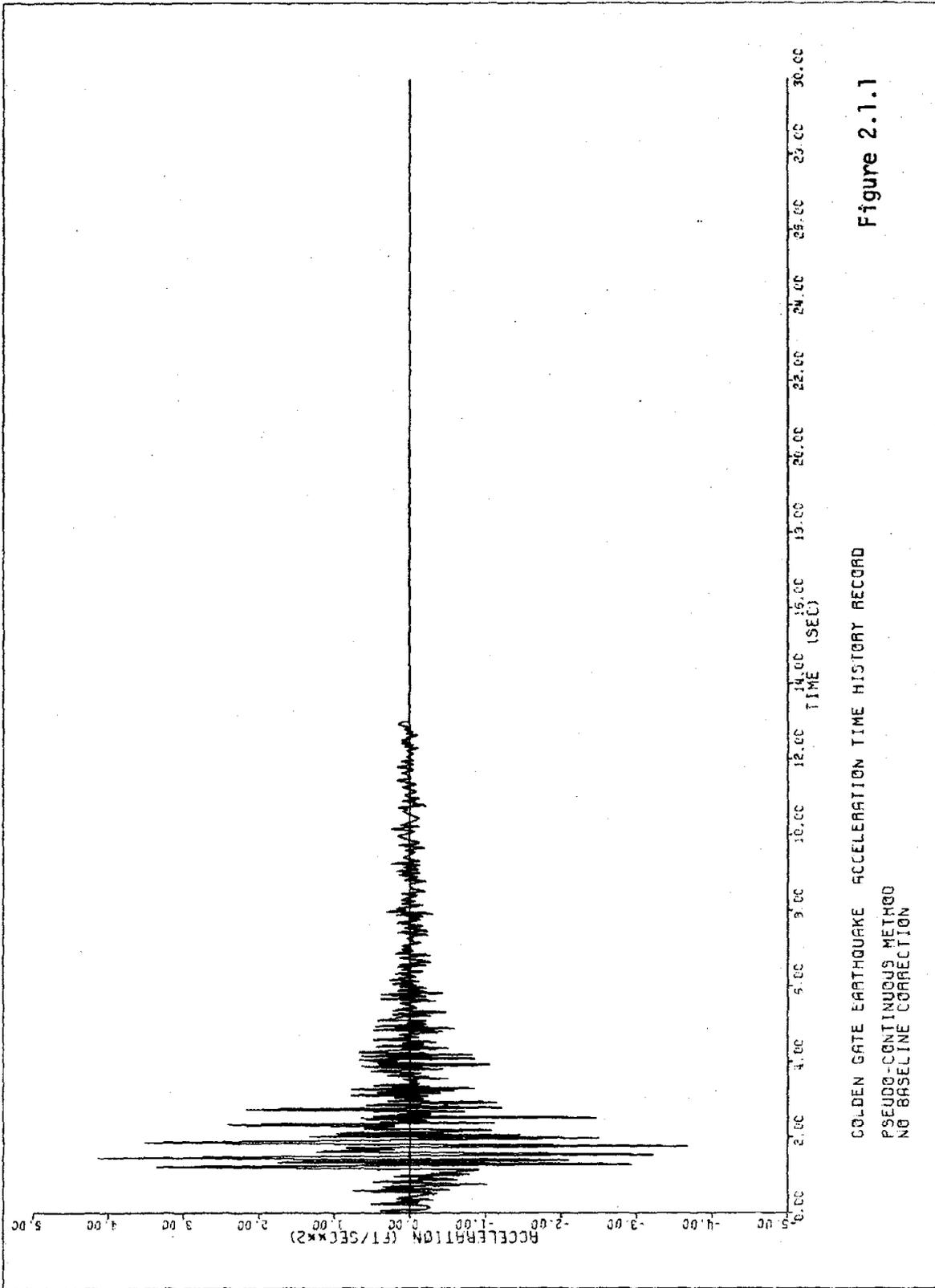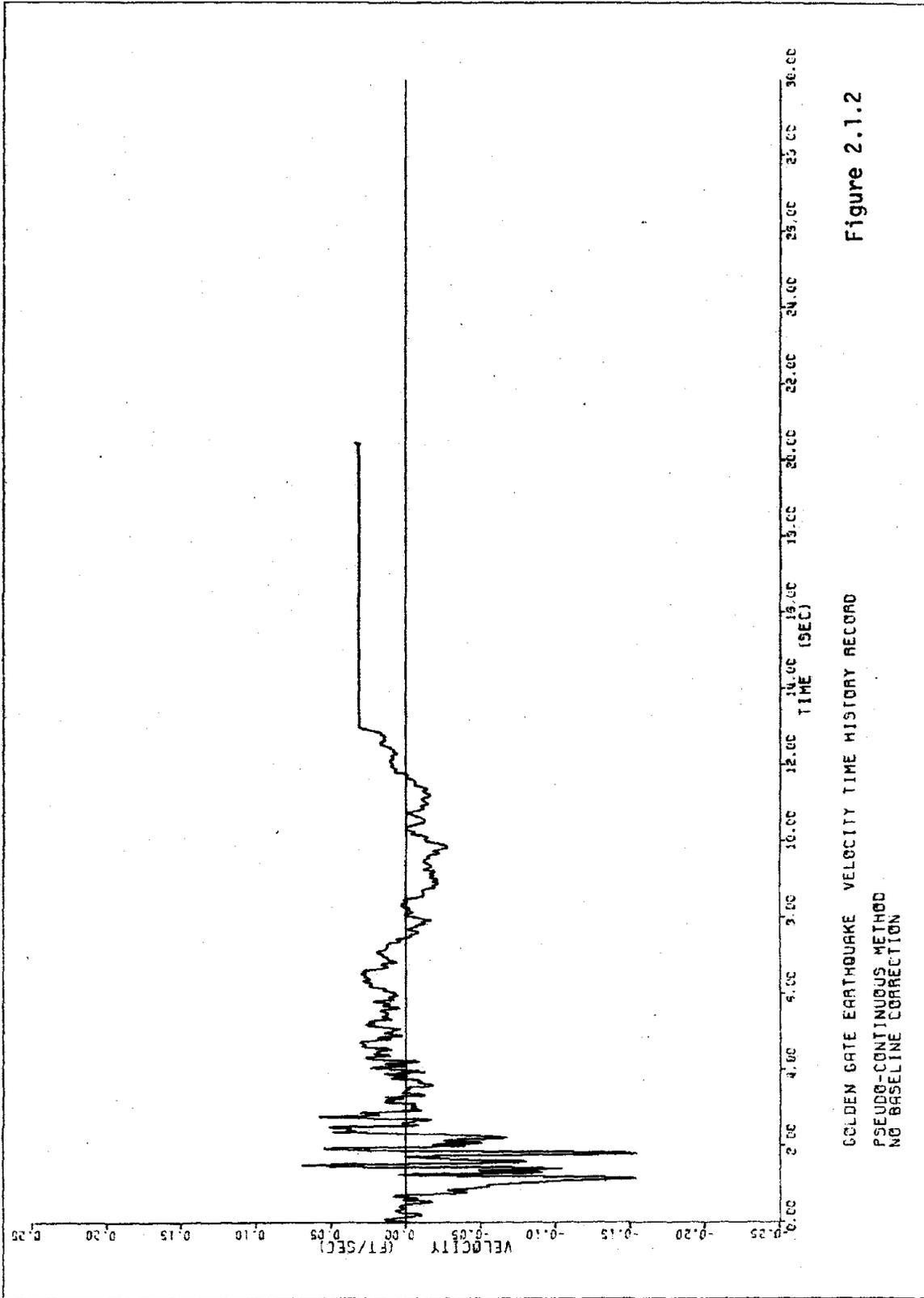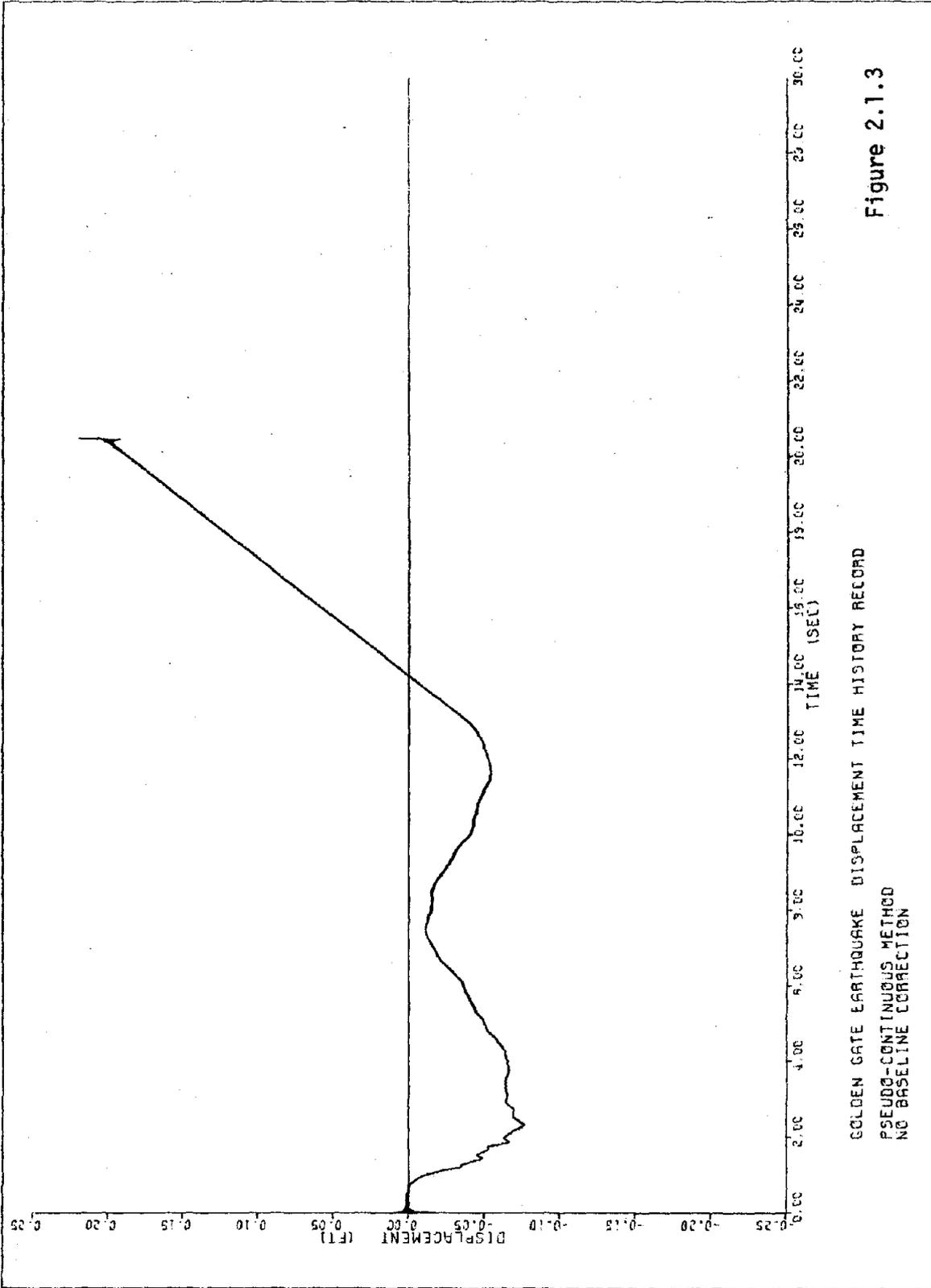GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
PARABOLIC BASELINE CORRECTION

Figure 1.2.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

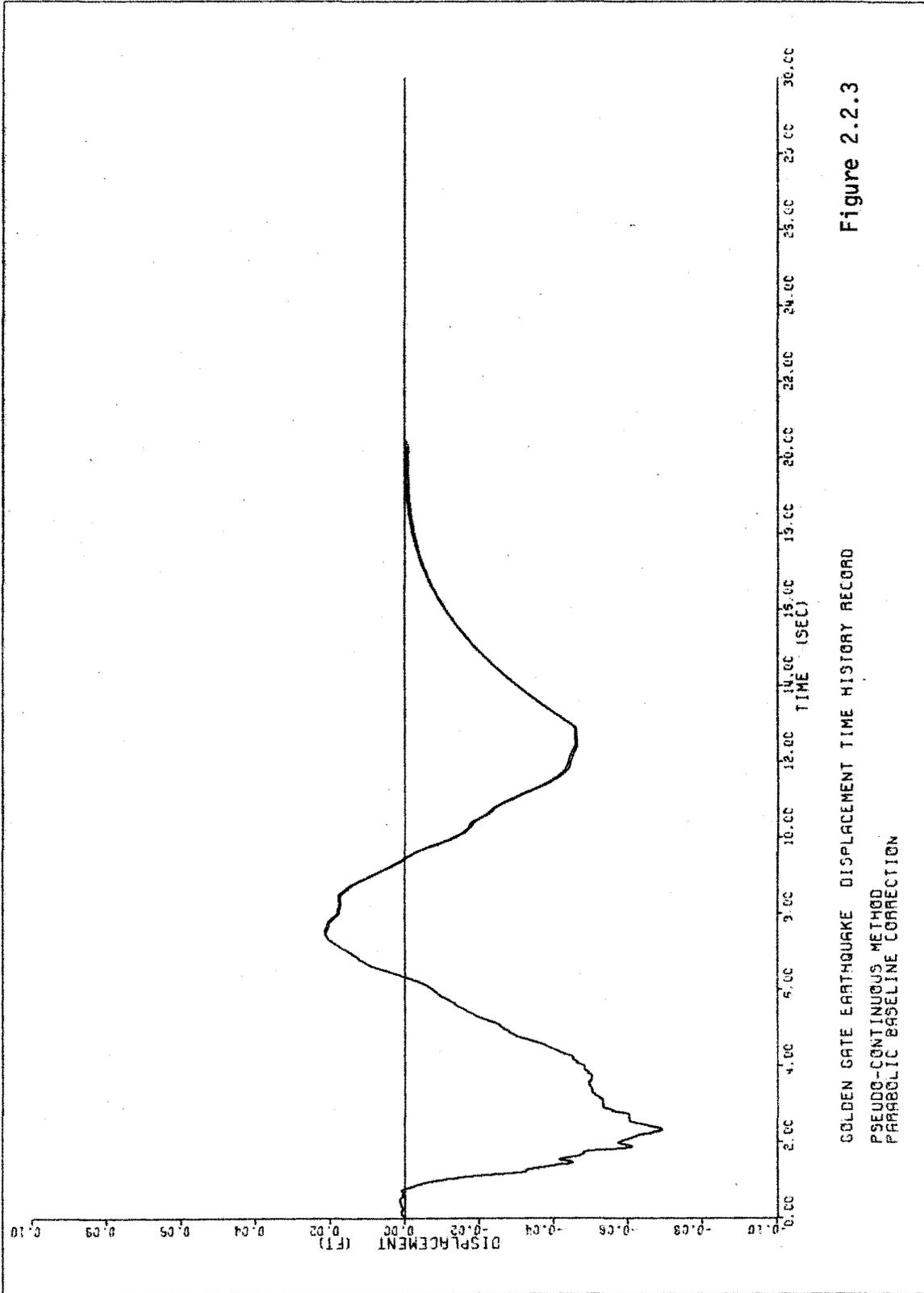CONTINUOUS (ALIASING) METHOD
PARABOLIC BASELINE CORRECTION

Figure 1.2.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
PARABOLIC BASELINE CORRECTION

Figure 1.2.3

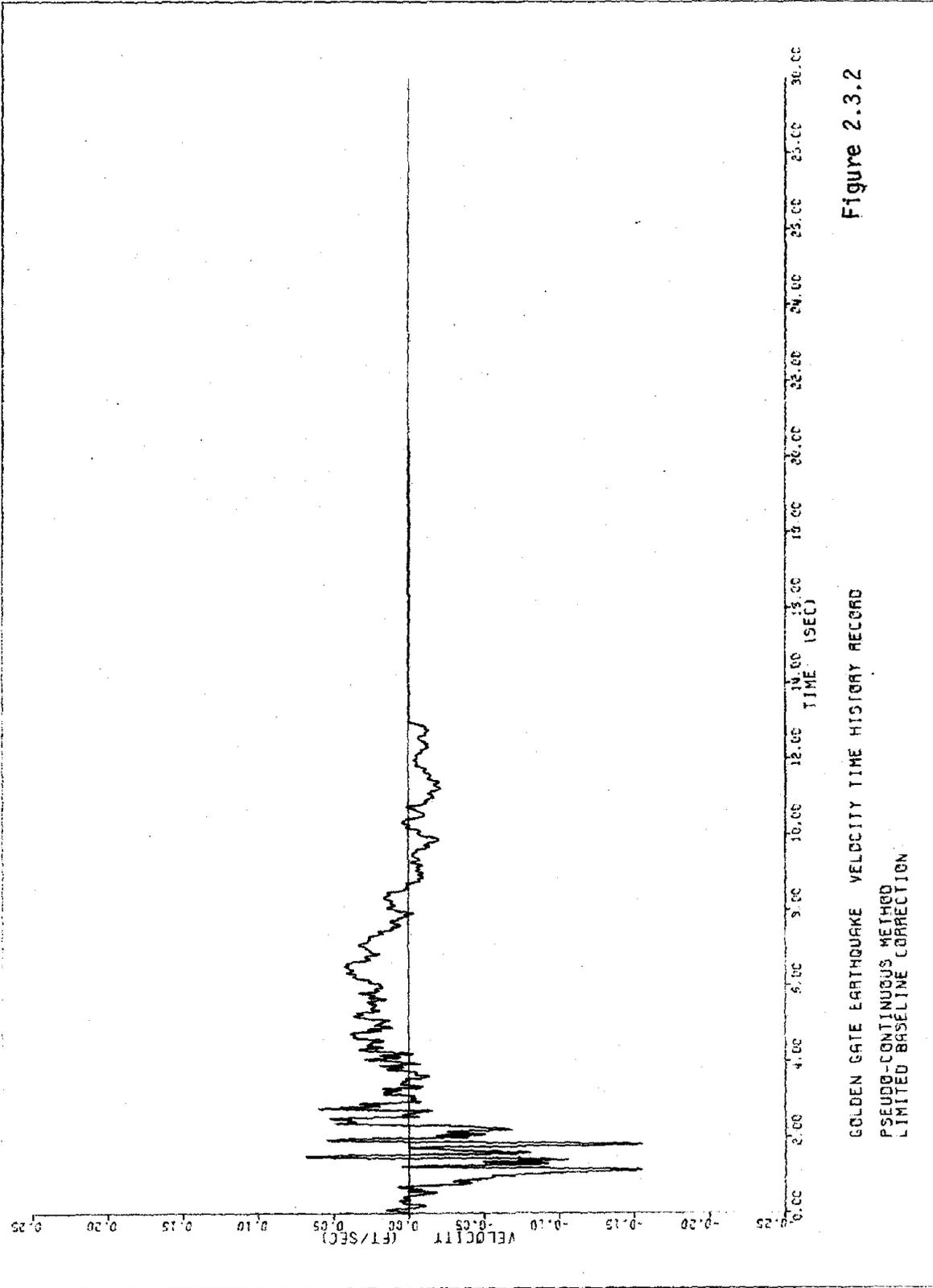GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
LIMITED BASELINE CORRECTION

Figure 1.3.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
LIMITED BASELINE CORRECTION

Figure 1.3.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

CONTINUOUS (ALIASING) METHOD
LIMITED BASELINE CORRECTION

Figure 1.3.3

GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

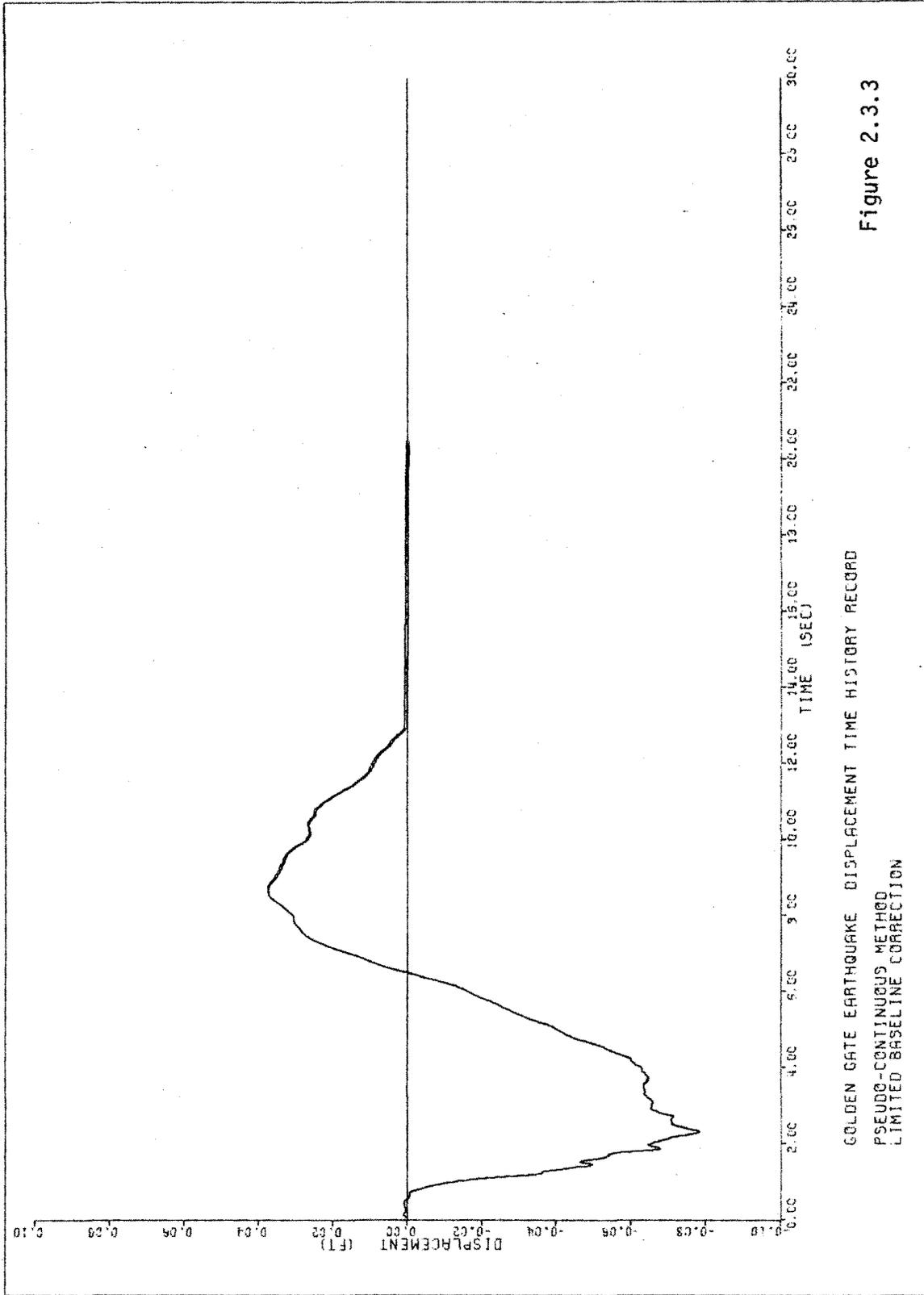PSEUDO-CONTINUOUS METHOD
NO BASELINE CORRECTION

Figure 2.1.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
NO BASELINE CORRECTION

Figure 2.1.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
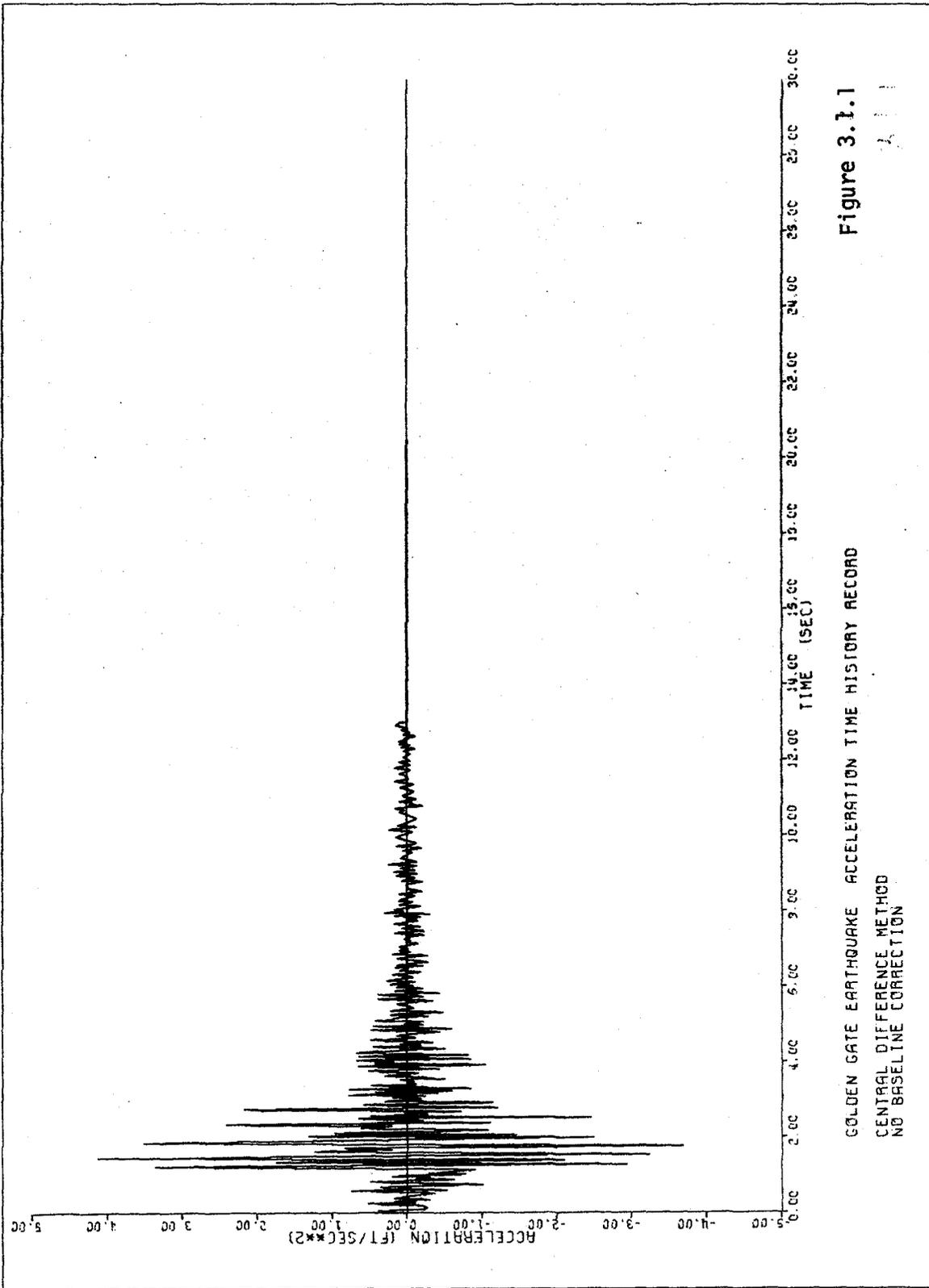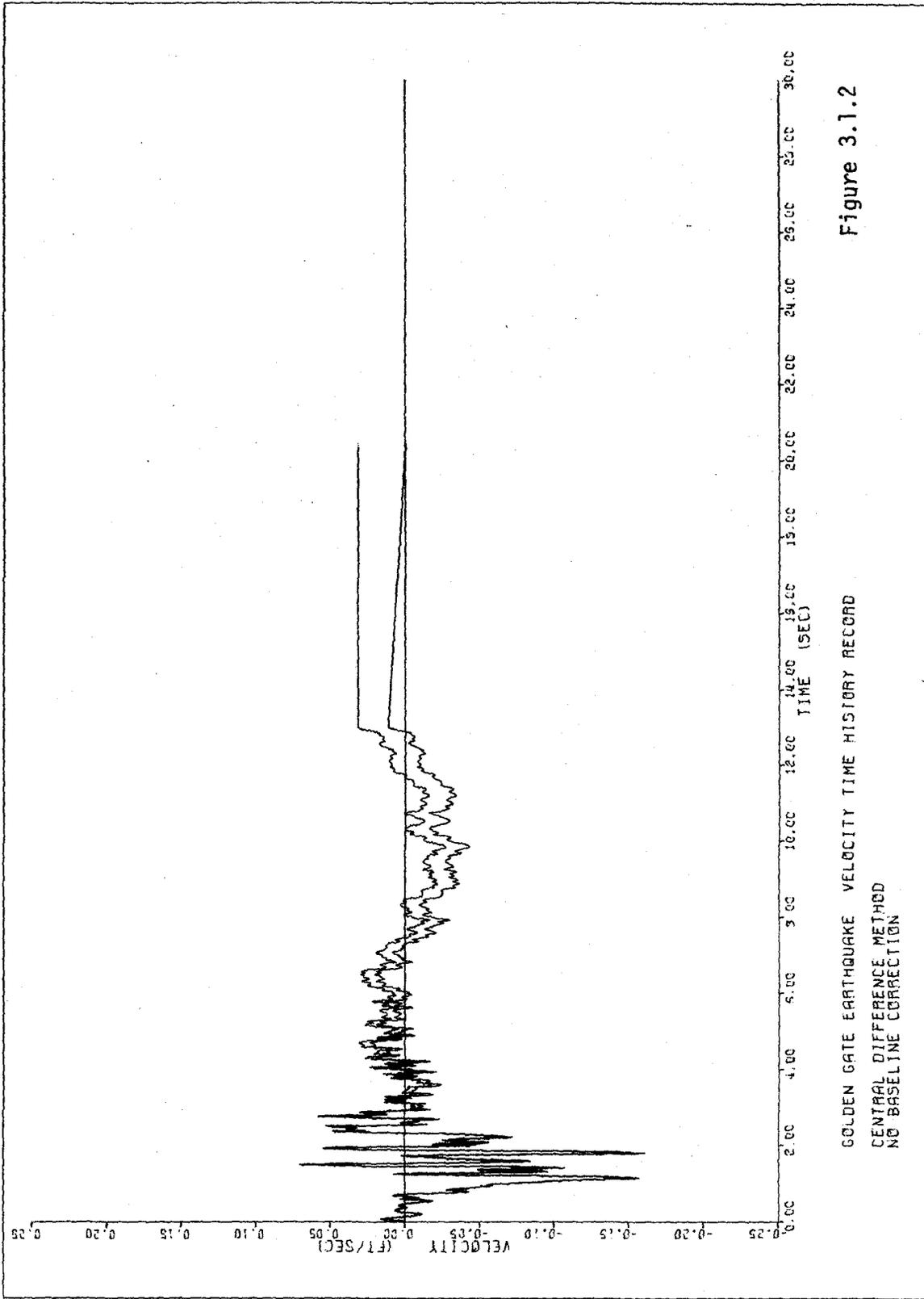NO BASELINE CORRECTION

Figure 2.1.3

GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
PARABOLIC BASELINE CORRECTION

Figure 2.2.1

GOLDEN GATE EARTHQUAKE   VELOCITY TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
PARABOLIC BASELINE CORRECTION

Figure 2.2.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
PARABOLIC BASELINE CORRECTION

Figure 2.2.3

GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

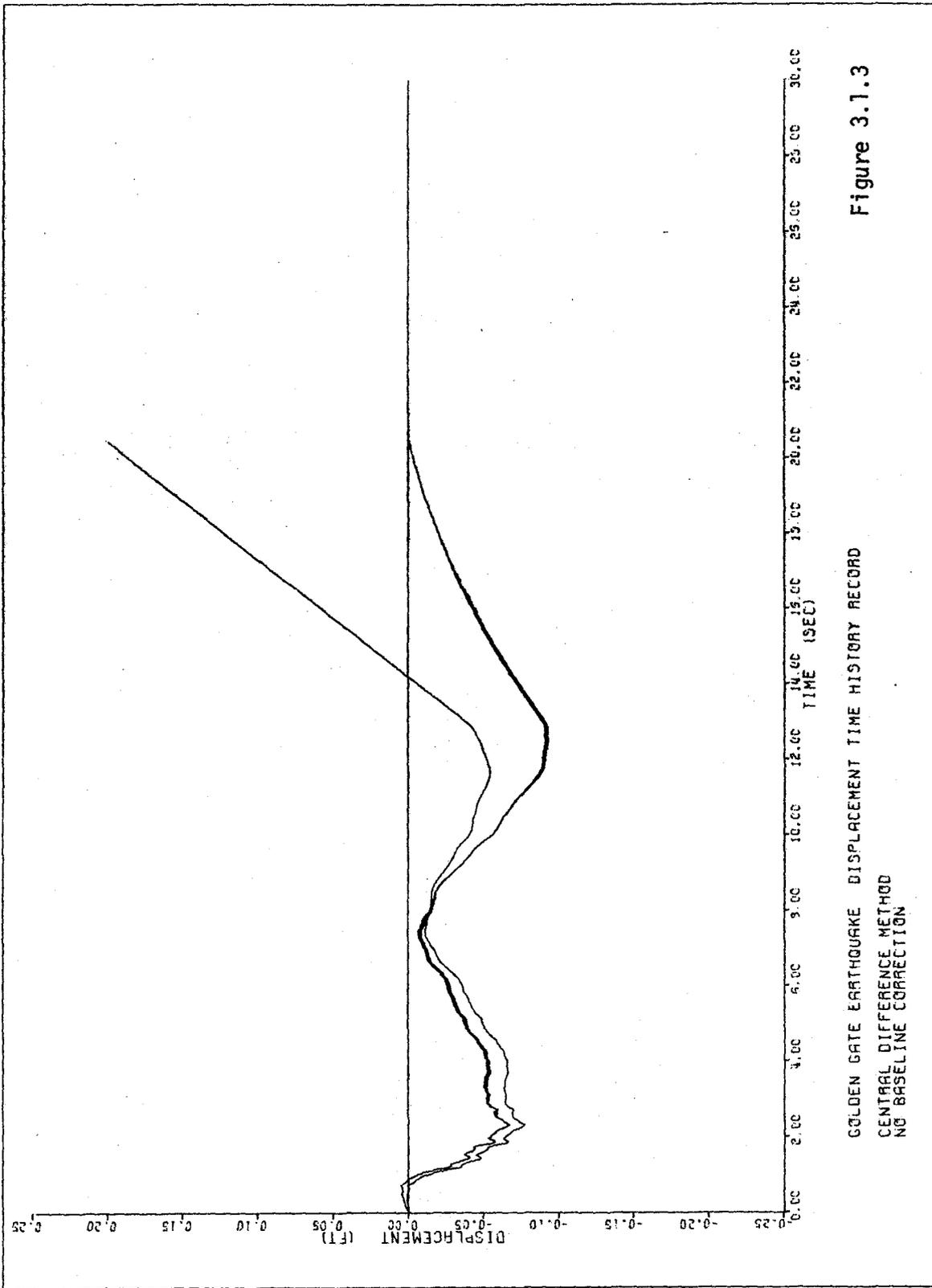PSEUDO-CONTINUOUS METHOD
LIMITED BASELINE CORRECTION

Figure 2.3.1

GOLDEN GATE EARTHQUAKE   VELOCITY TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
LIMITED BASELINE CORRECTION
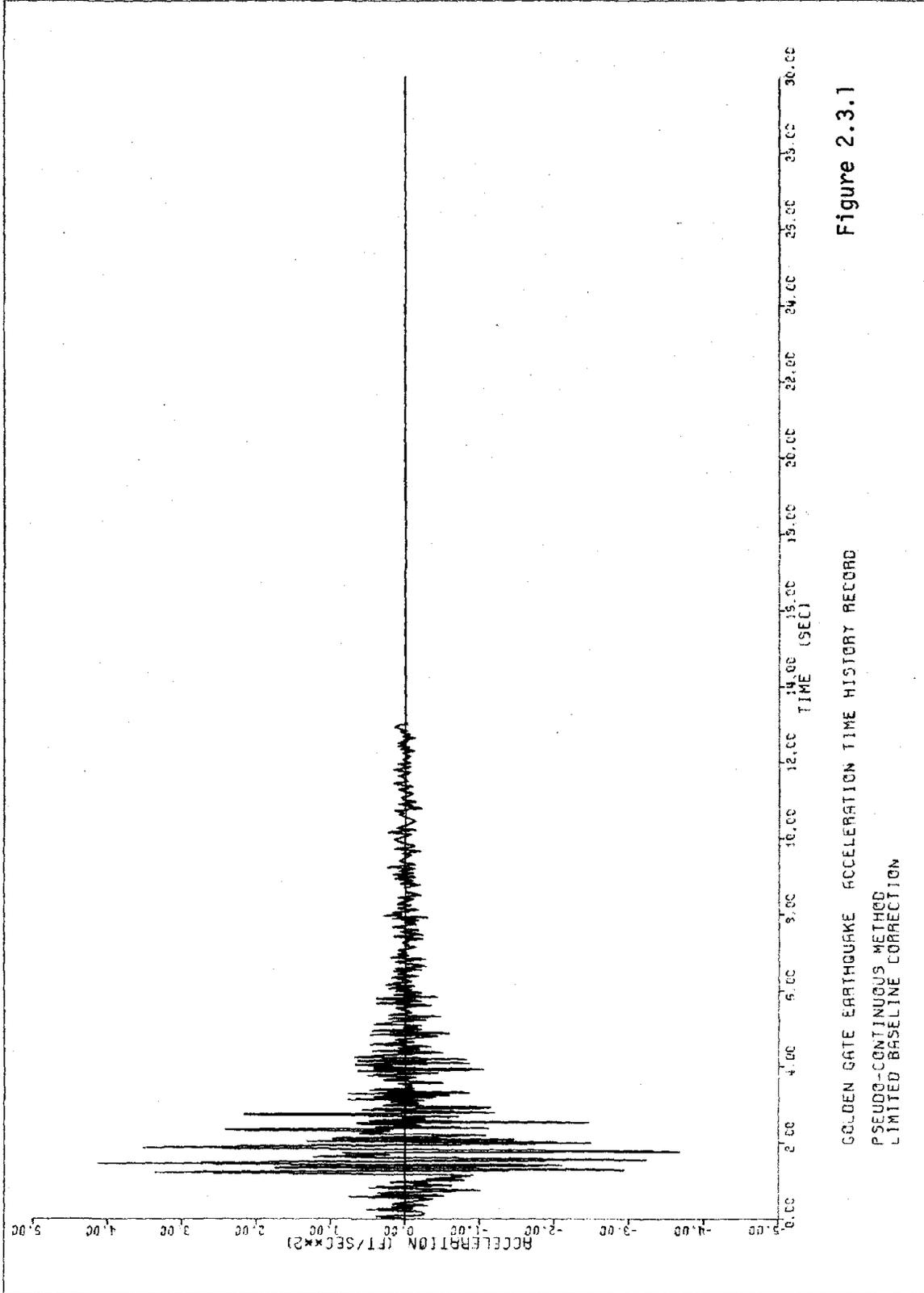
Figure 2.3.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

PSEUDO-CONTINUOUS METHOD
LIMITED BASELINE CORRECTION
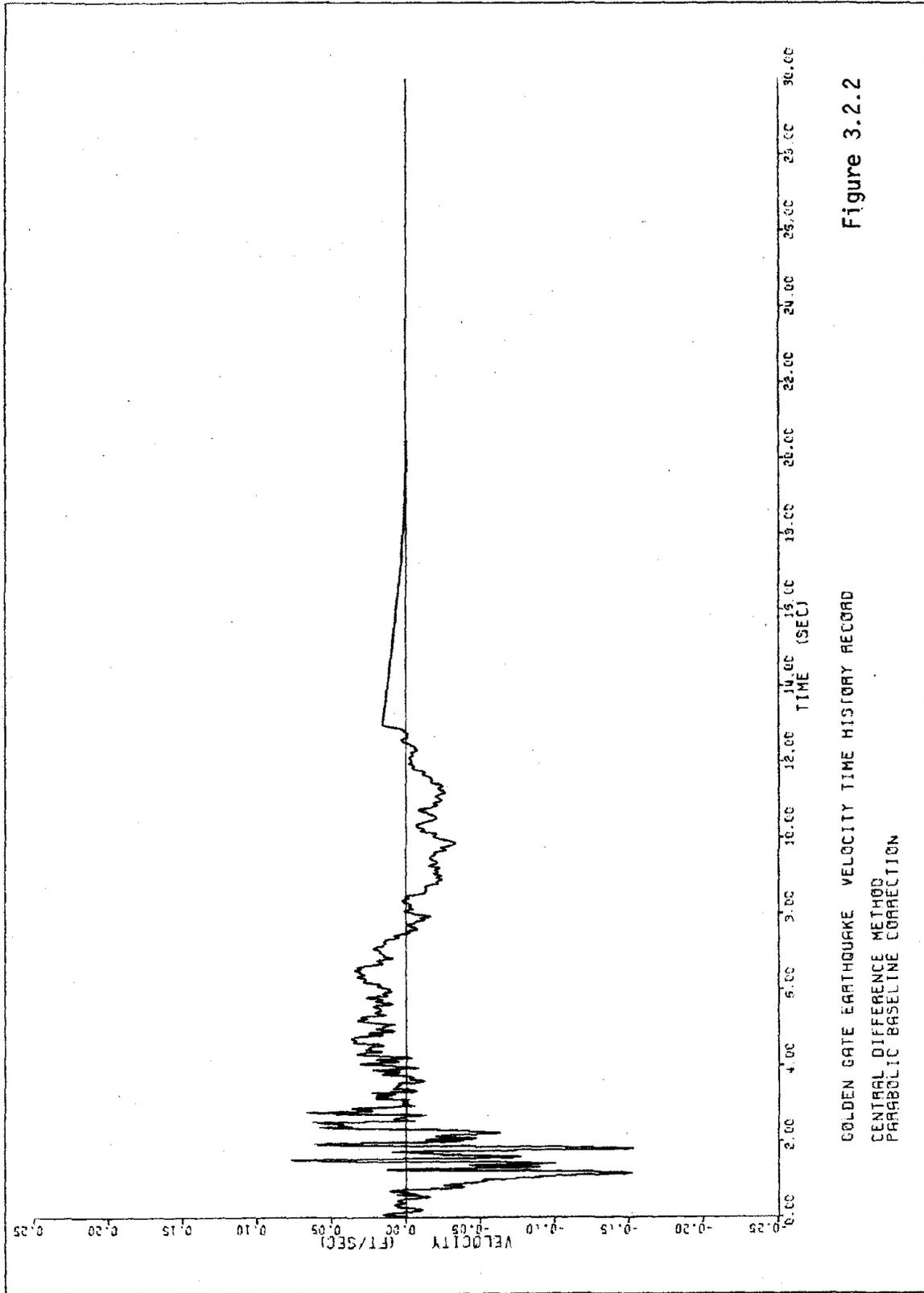
Figure 2.3.3

GOLDEN GATE EARTHQUAKE   ACCELERATION TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
NO BASELINE CORRECTION

Figure 3.1.1

GOLDEN GATE EARTHQUAKE   VELOCITY TIME HISTORY RECORD
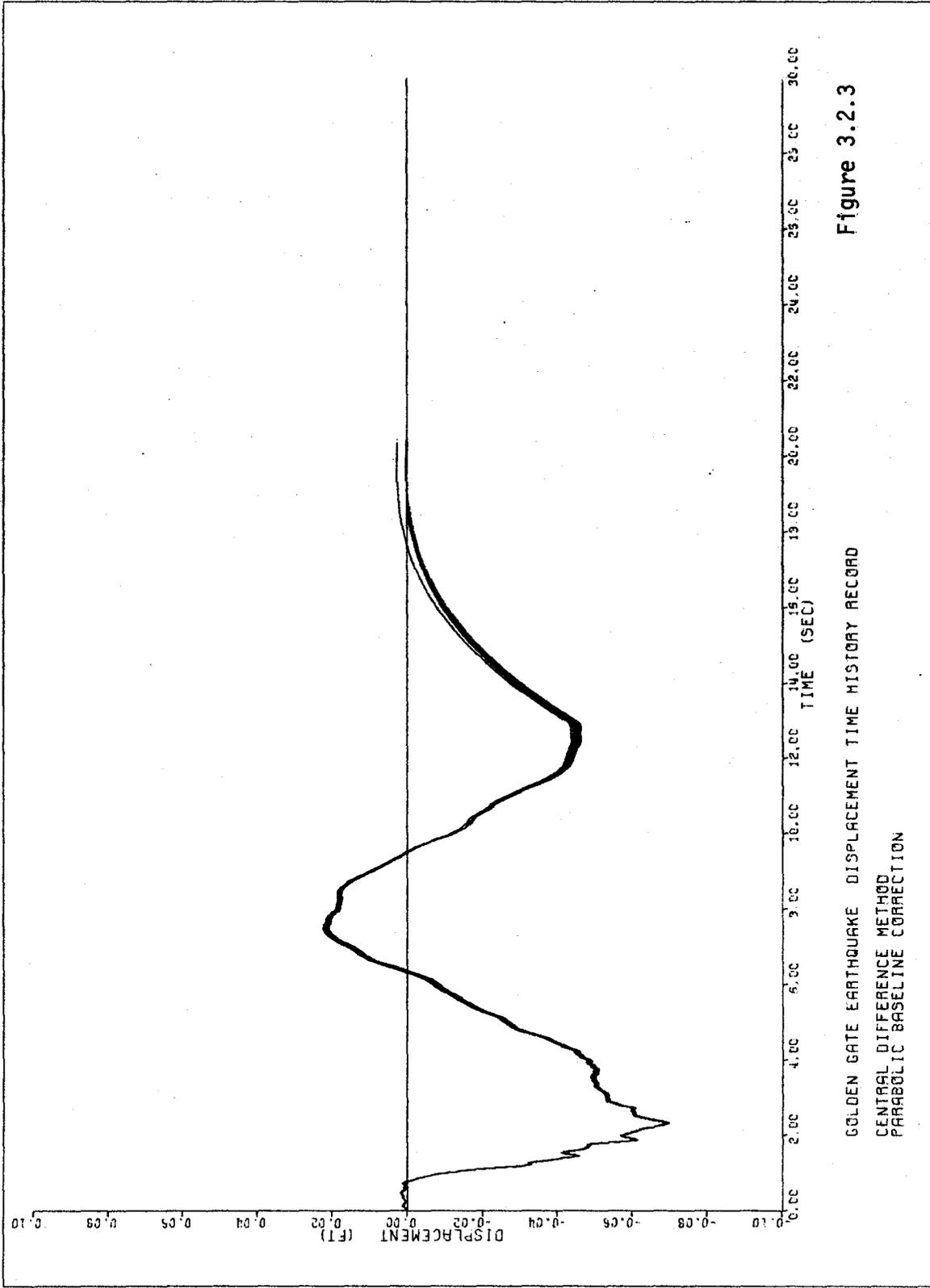
CENTRAL DIFFERENCE METHOD
NO BASELINE CORRECTION

Figure 3.1.2

GOLDEN GATE EARTHQUAKE   DISPLACEMENT TIME HISTORY RECORD
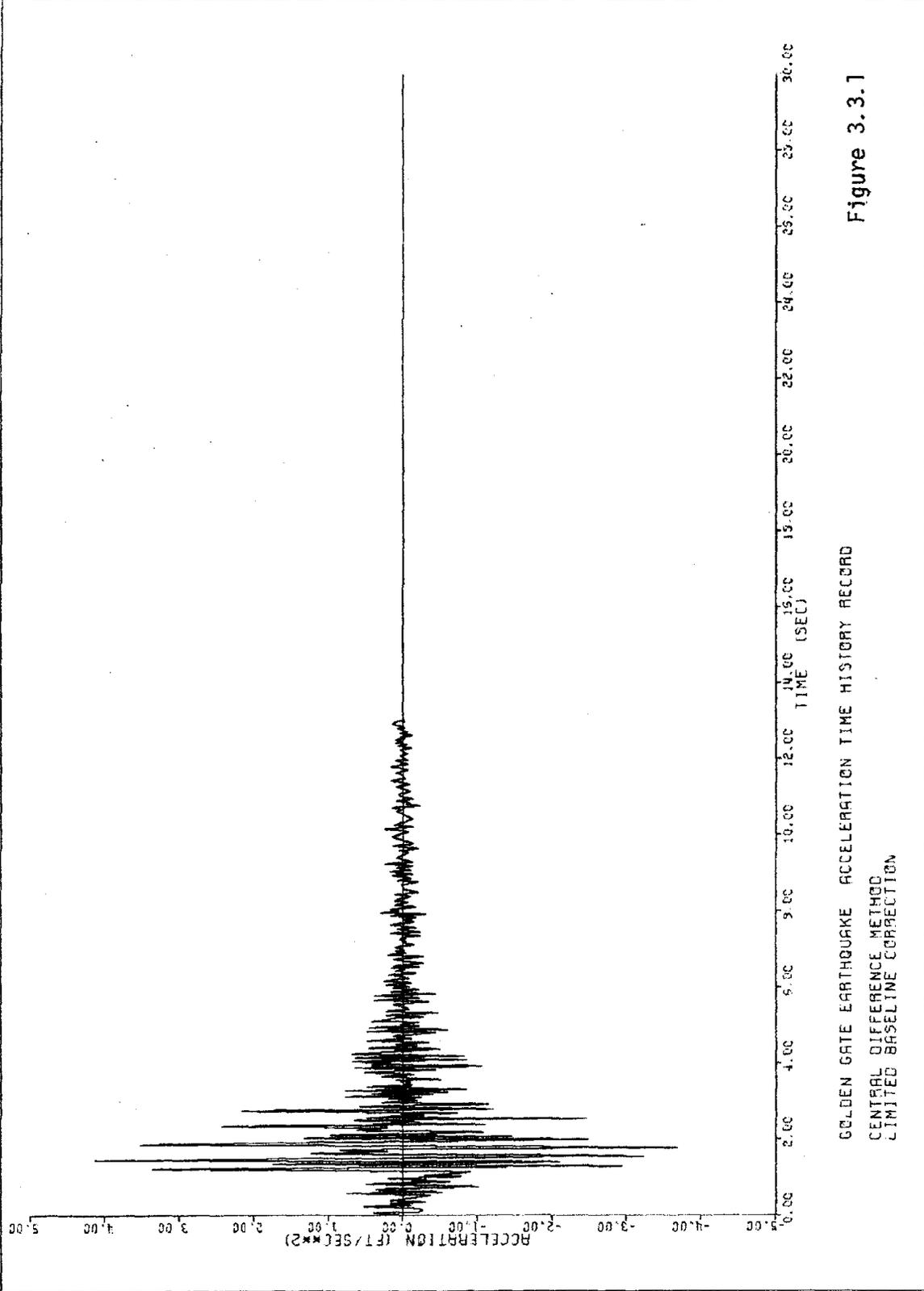
CENTRAL DIFFERENCE METHOD
NO BASELINE CORRECTION

Figure 3.1.3

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GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
PARABOLIC BASELINE CORRECTION

Figure 3.2.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
PARABOLIC BASELINE CORRECTION

Figure 3.2.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD
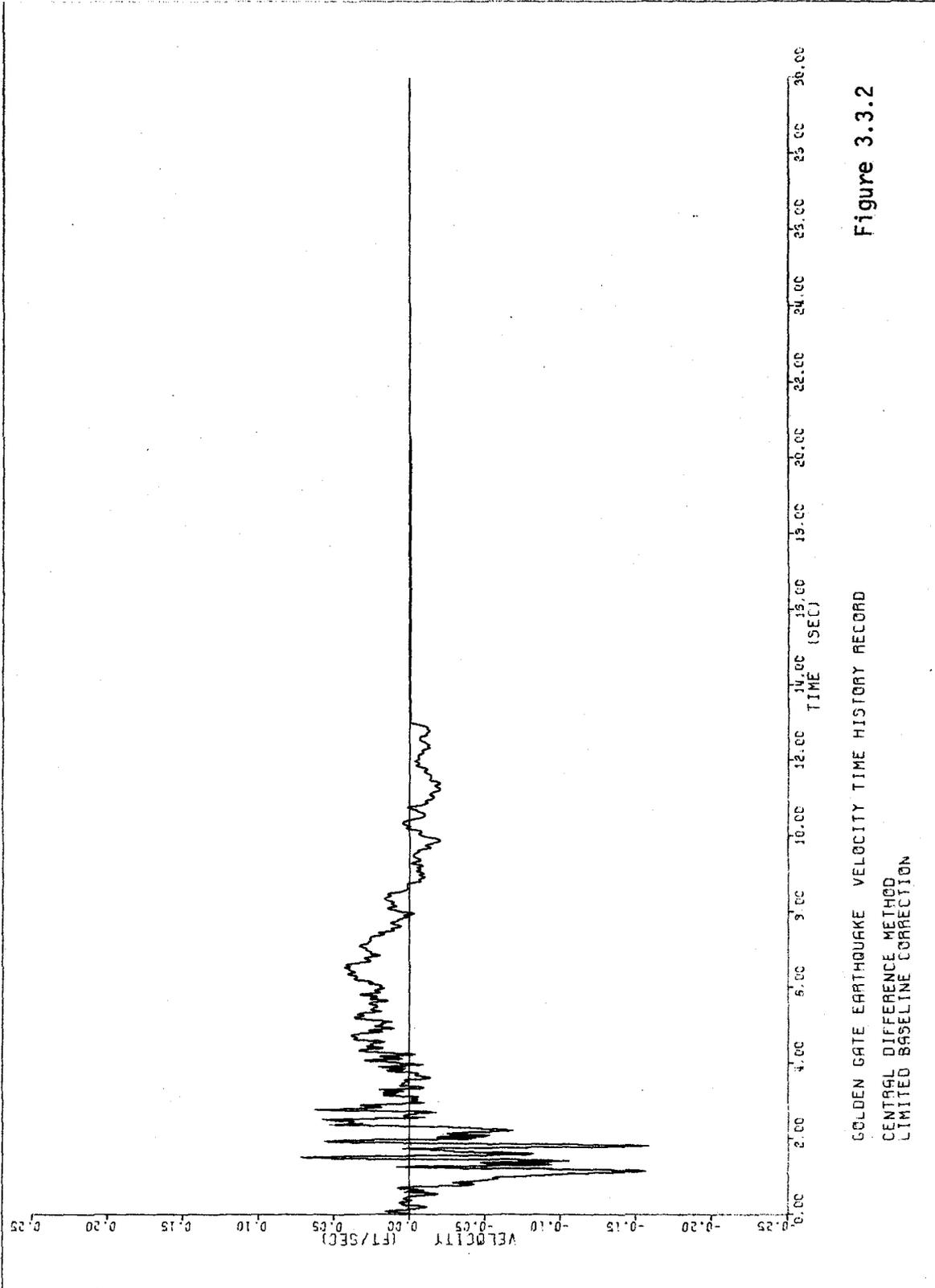
CENTRAL DIFFERENCE METHOD
PARABOLIC BASELINE CORRECTION

Figure 3.2.3

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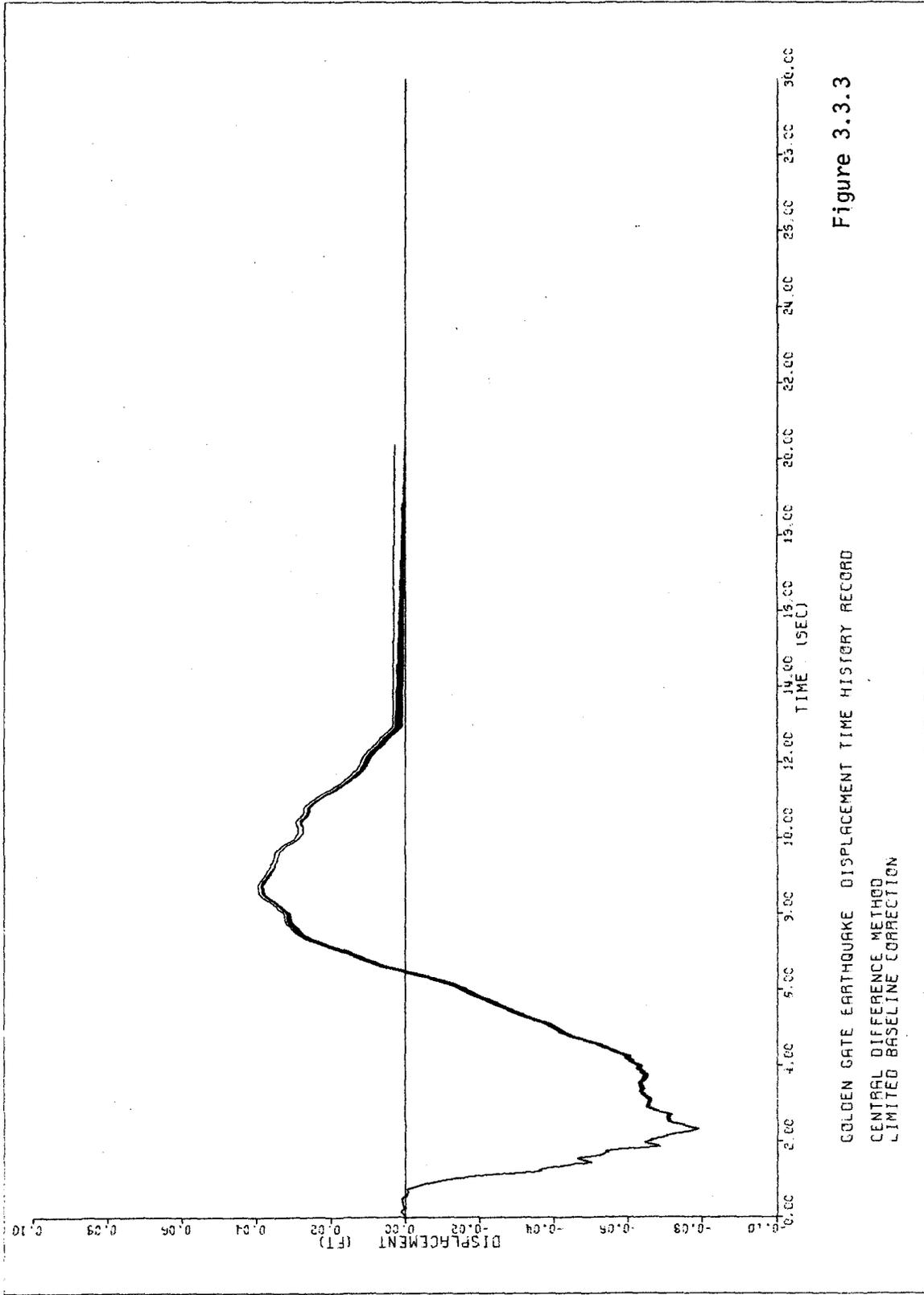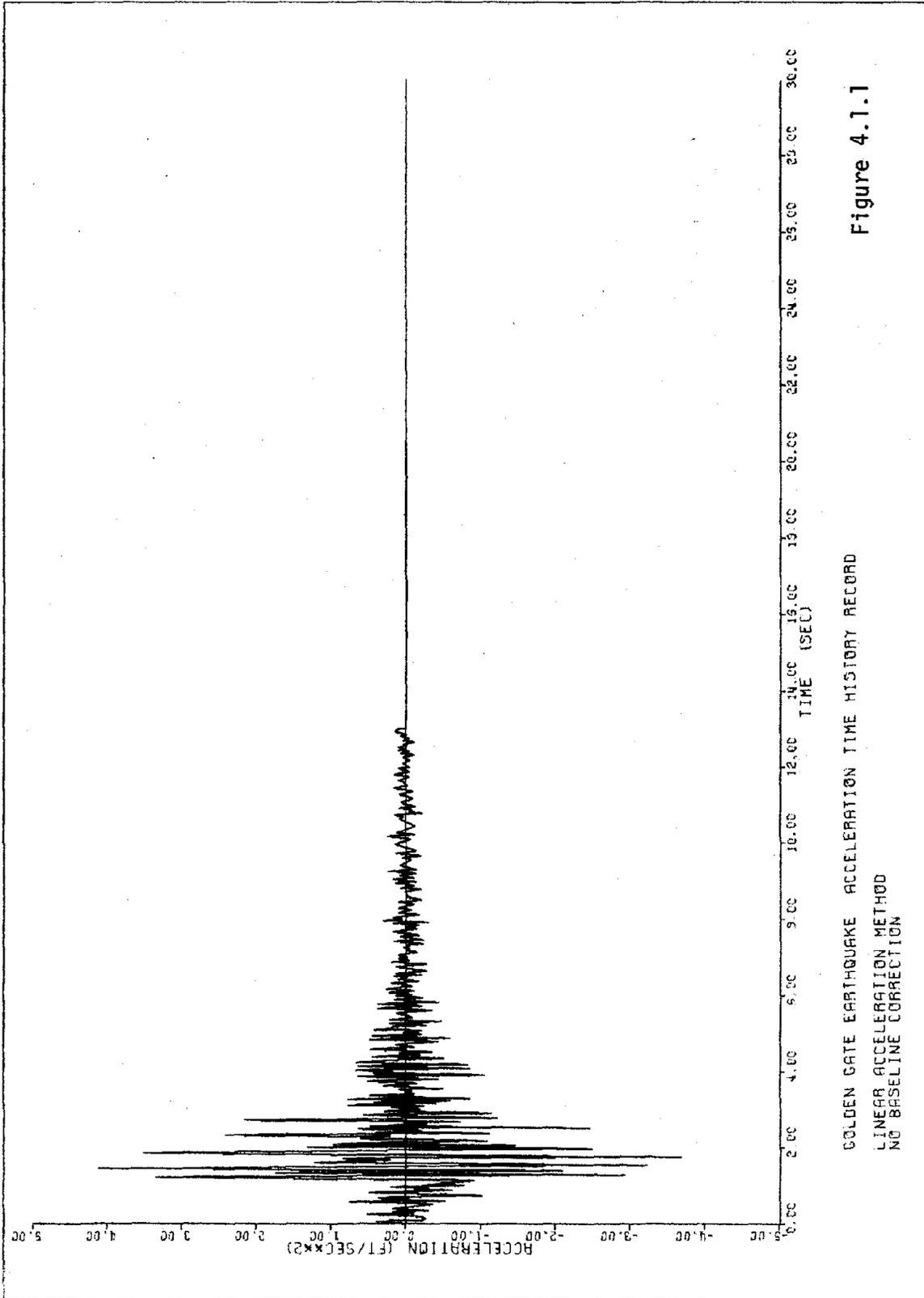GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
LIMITED BASELINE CORRECTION

Figure 3.3.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
LIMITED BASELINE CORRECTION

Figure 3.3.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

CENTRAL DIFFERENCE METHOD
LIMITED BASELINE CORRECTION

Figure 3.3.3

GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD
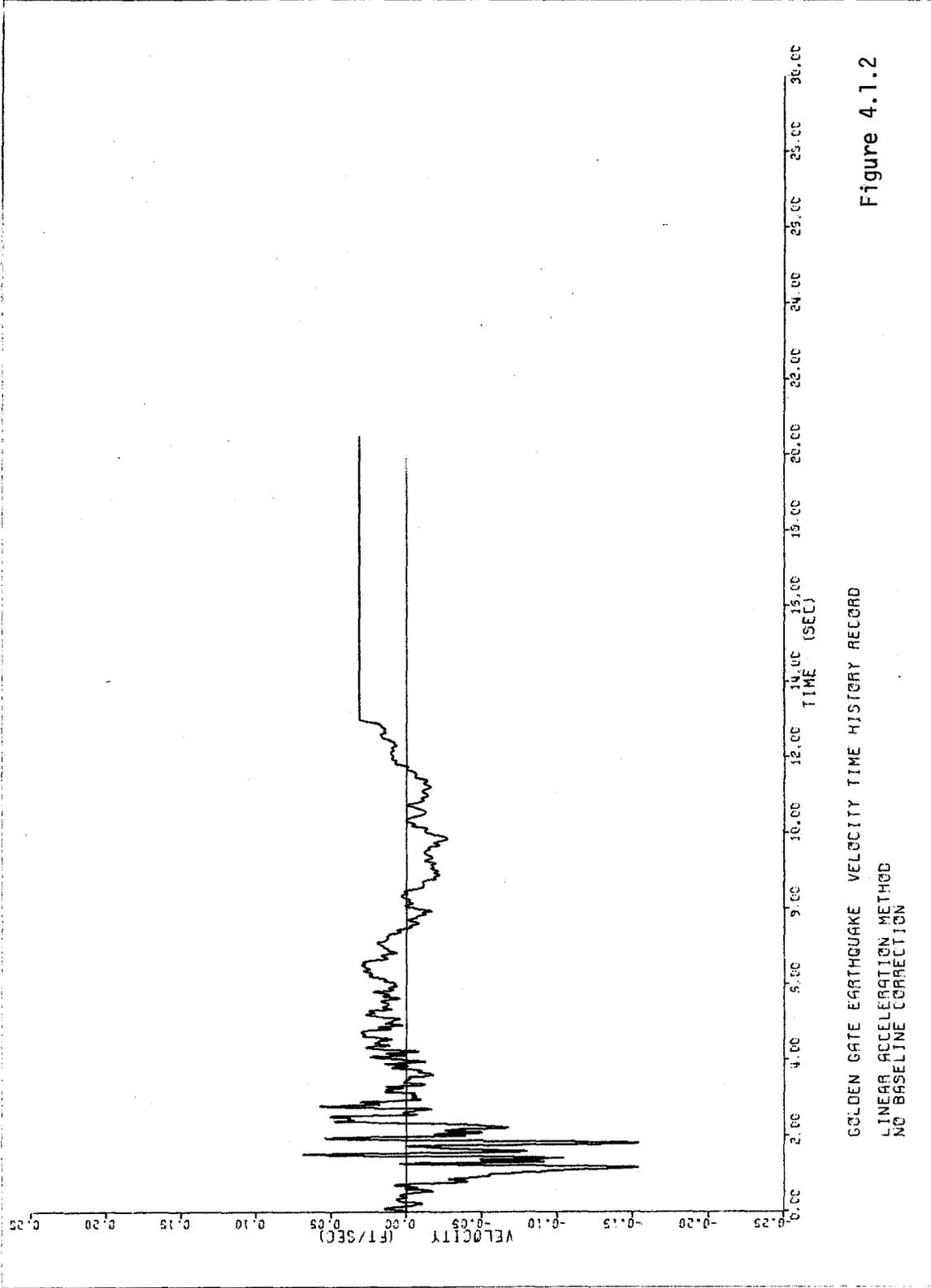
LINEAR ACCELERATION METHOD
NO BASELINE CORRECTION

Figure 4.1.1

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GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD

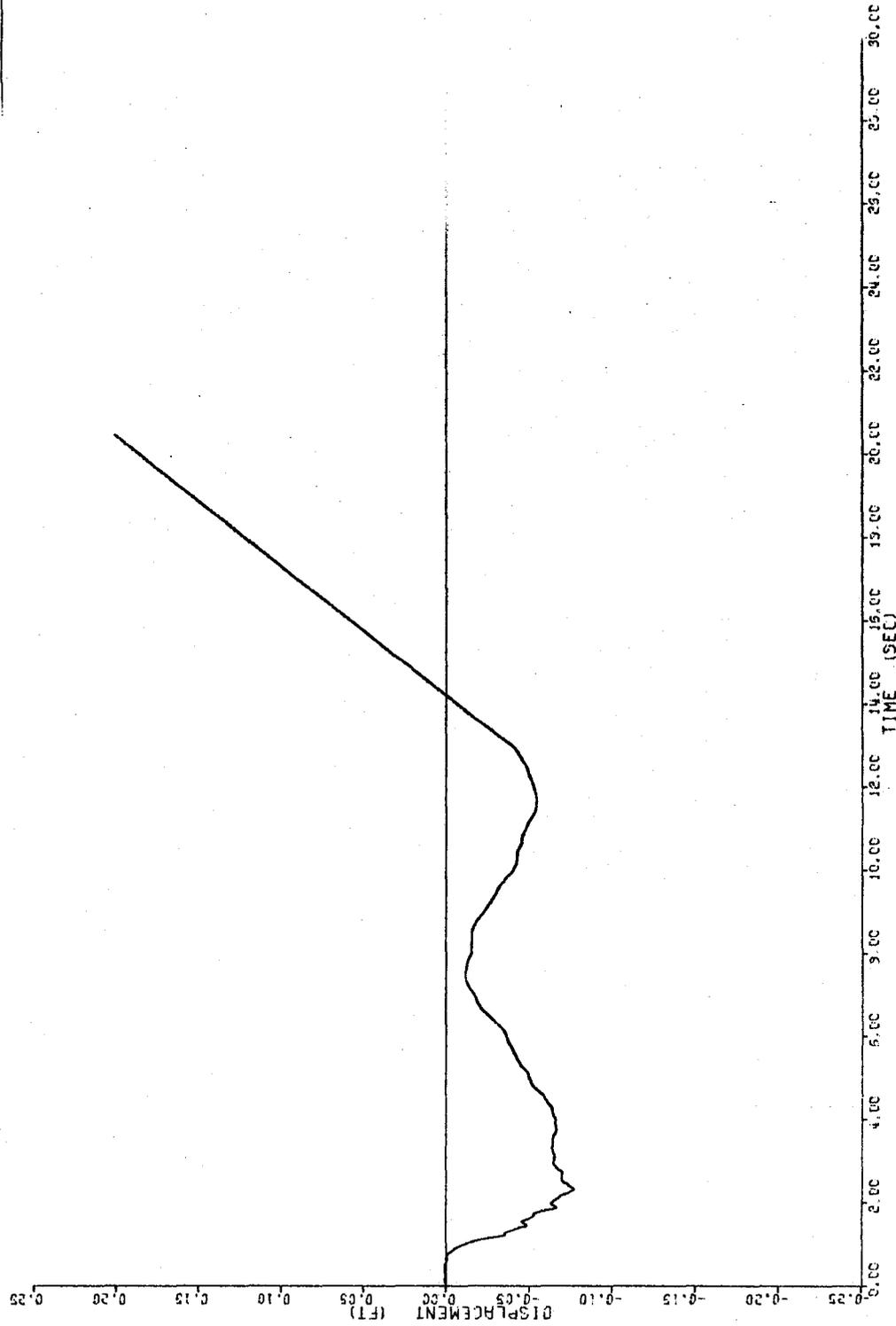LINEAR ACCELERATION METHOD
NO BASELINE CORRECTION

Figure 4.1.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

LINEAR ACCELERATION METHOD
NO BASELINE CORRECTION

Figure 4.1.3

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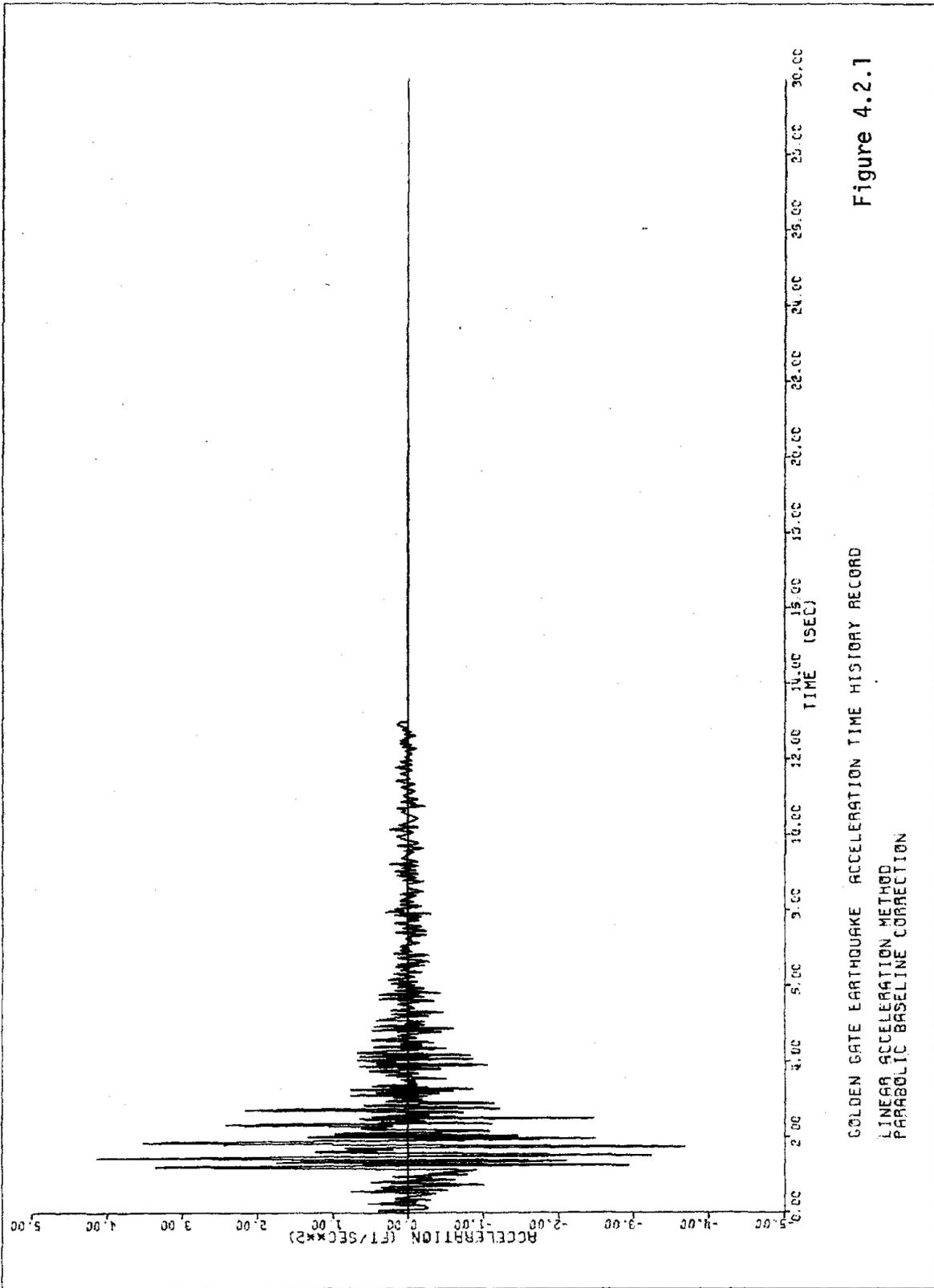Figure 4.2.1

GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

LINEAR ACCELERATION METHOD
PARABOLIC BASELINE CORRECTION

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD
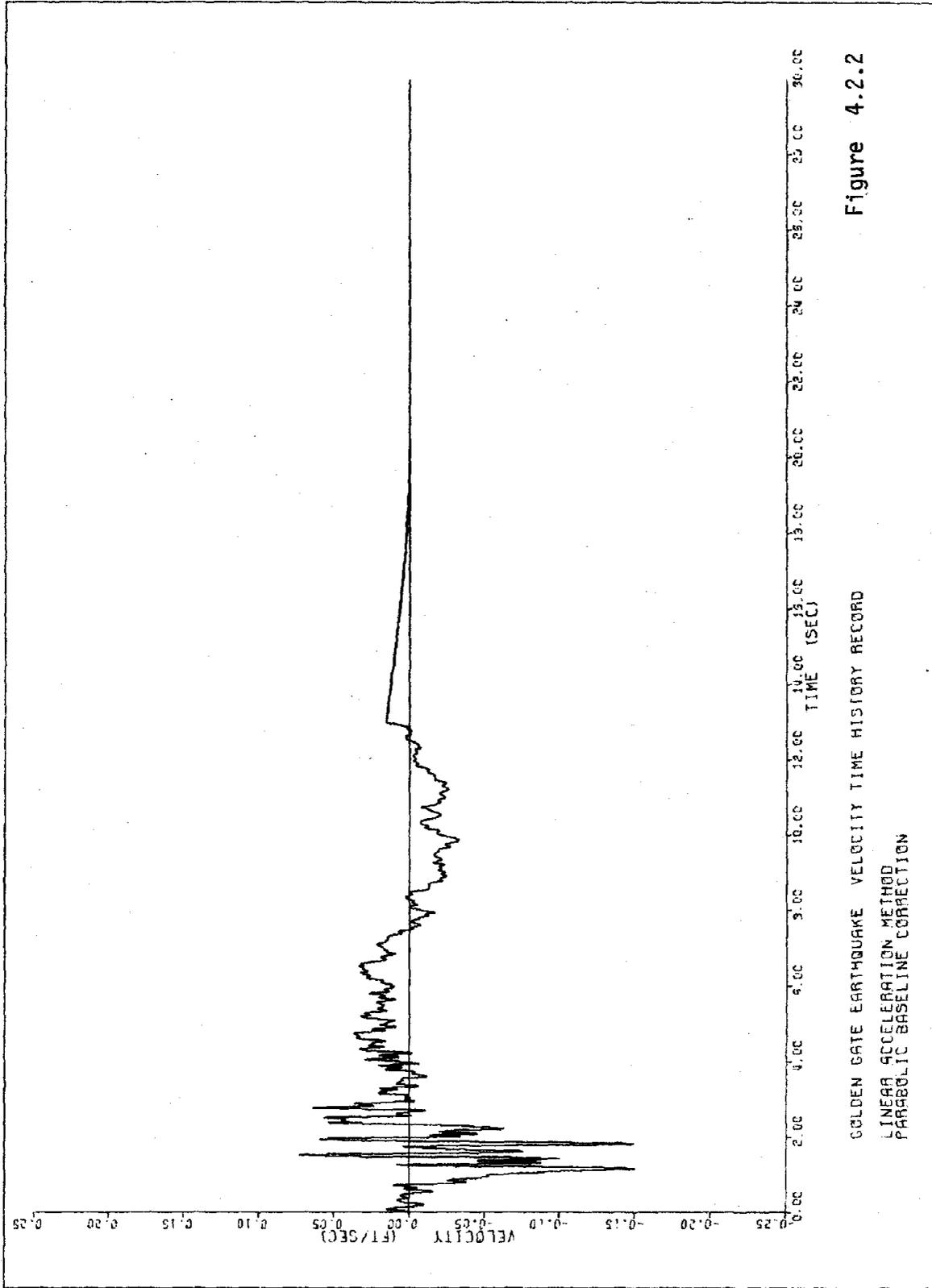
LINEAR ACCELERATION METHOD
PARABOLIC BASELINE CORRECTION
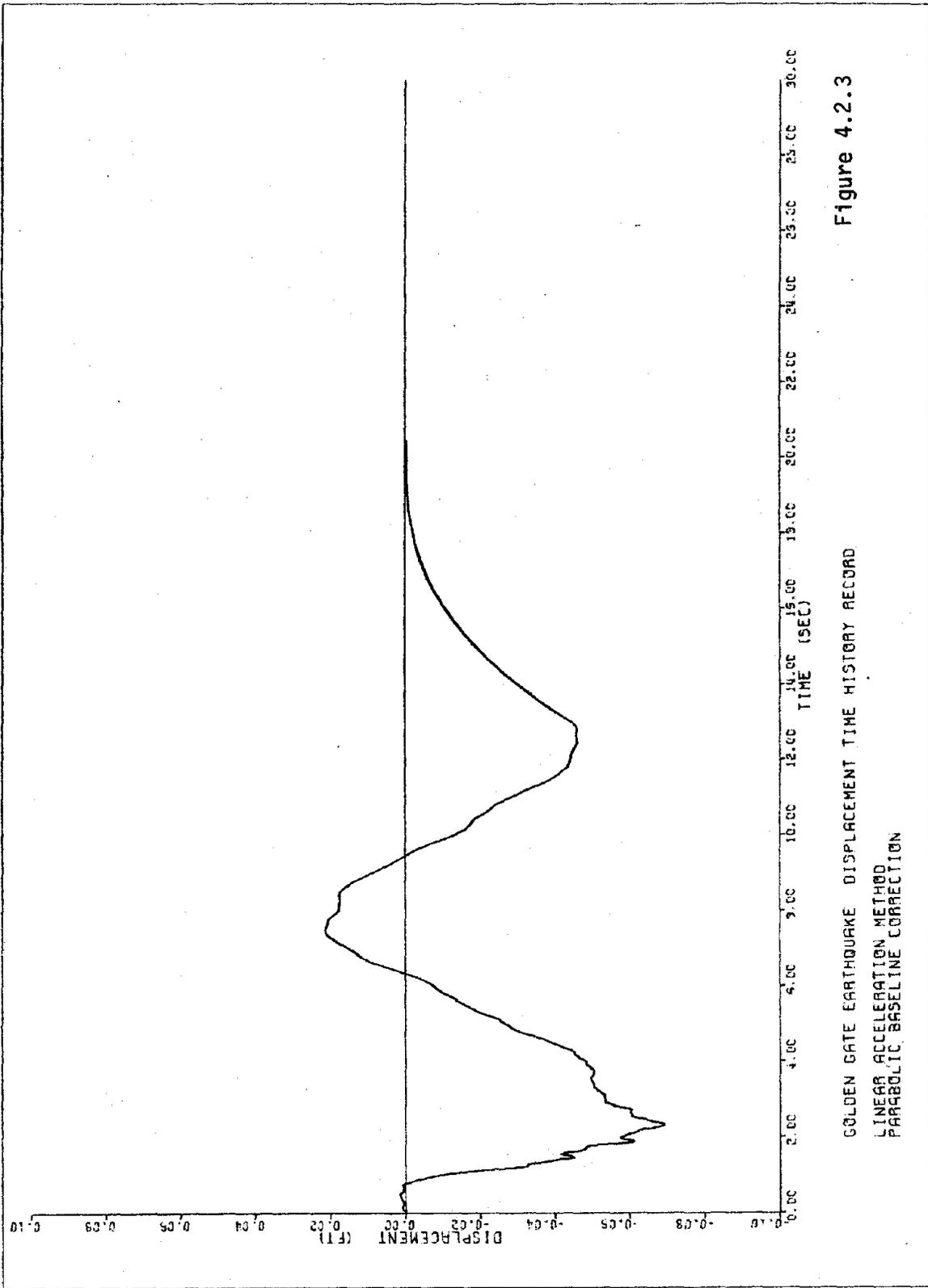
Figure 4.2.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

LINEAR ACCELERATION METHOD
PARABOLIC BASELINE CORRECTION

Figure 4.2.3

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GOLDEN GATE EARTHQUAKE ACCELERATION TIME HISTORY RECORD

LINEAR ACCELERATION METHOD
LIMITED BASELINE CORRECTION

Figure 4.3.1

GOLDEN GATE EARTHQUAKE VELOCITY TIME HISTORY RECORD
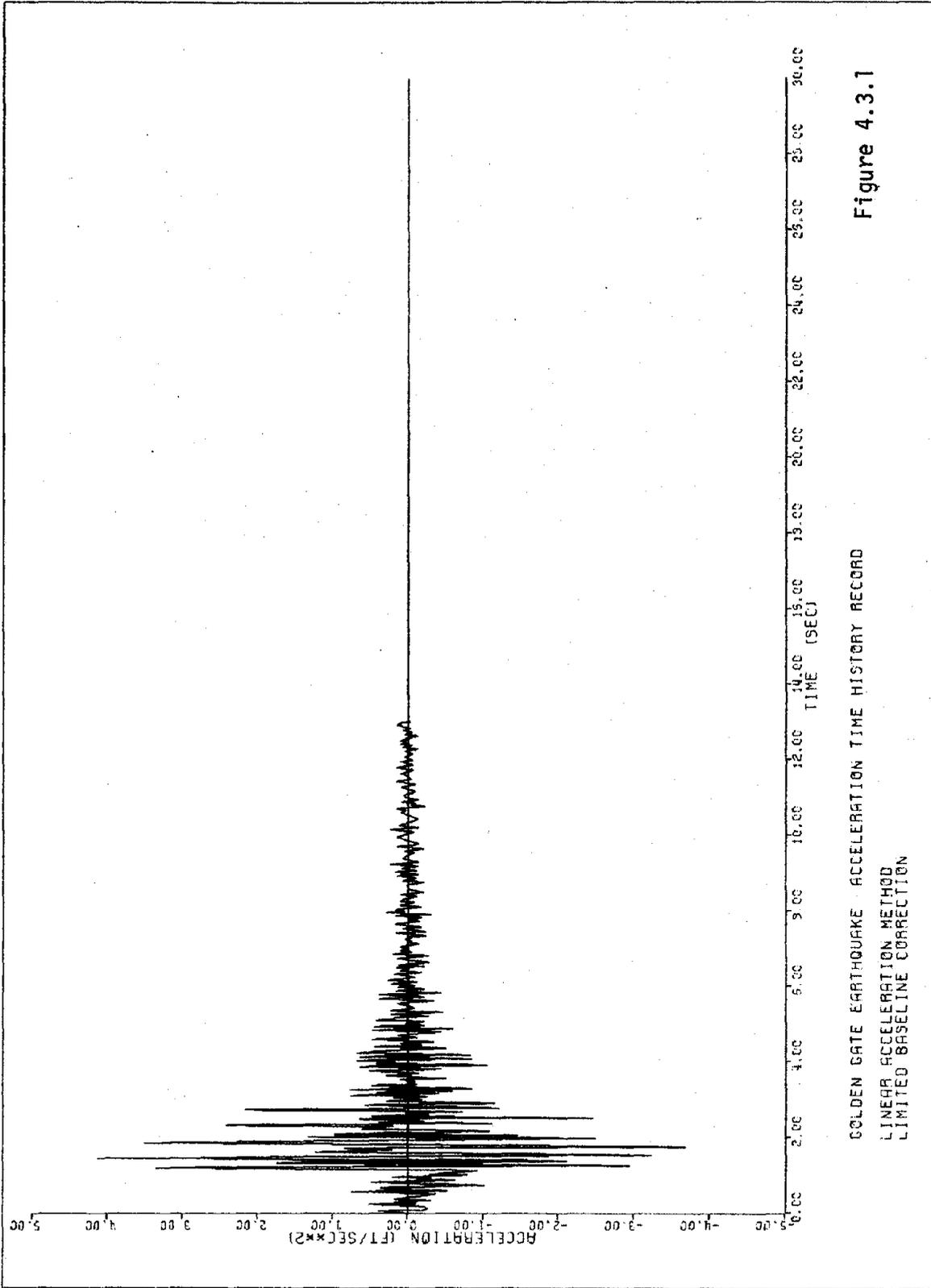
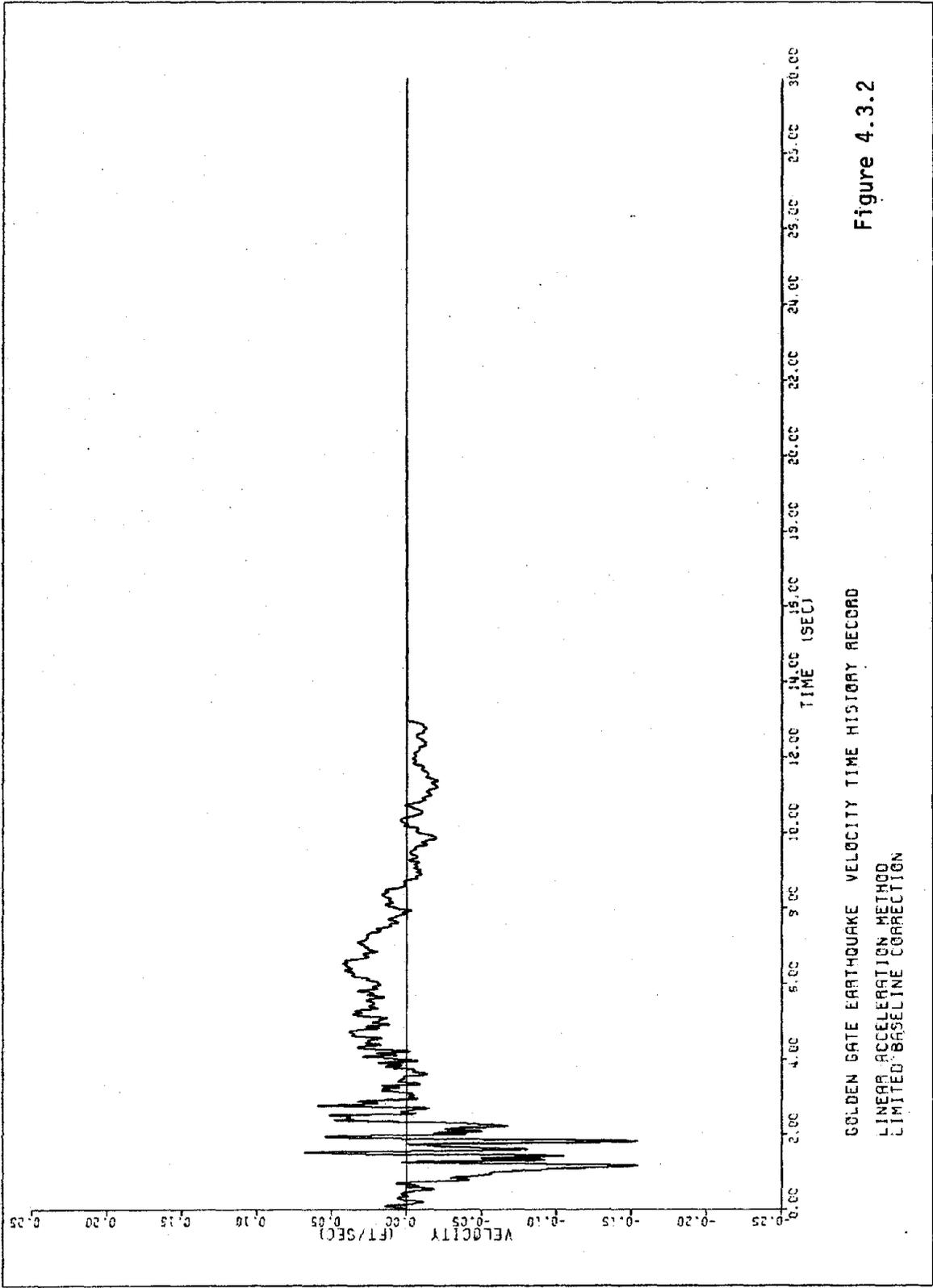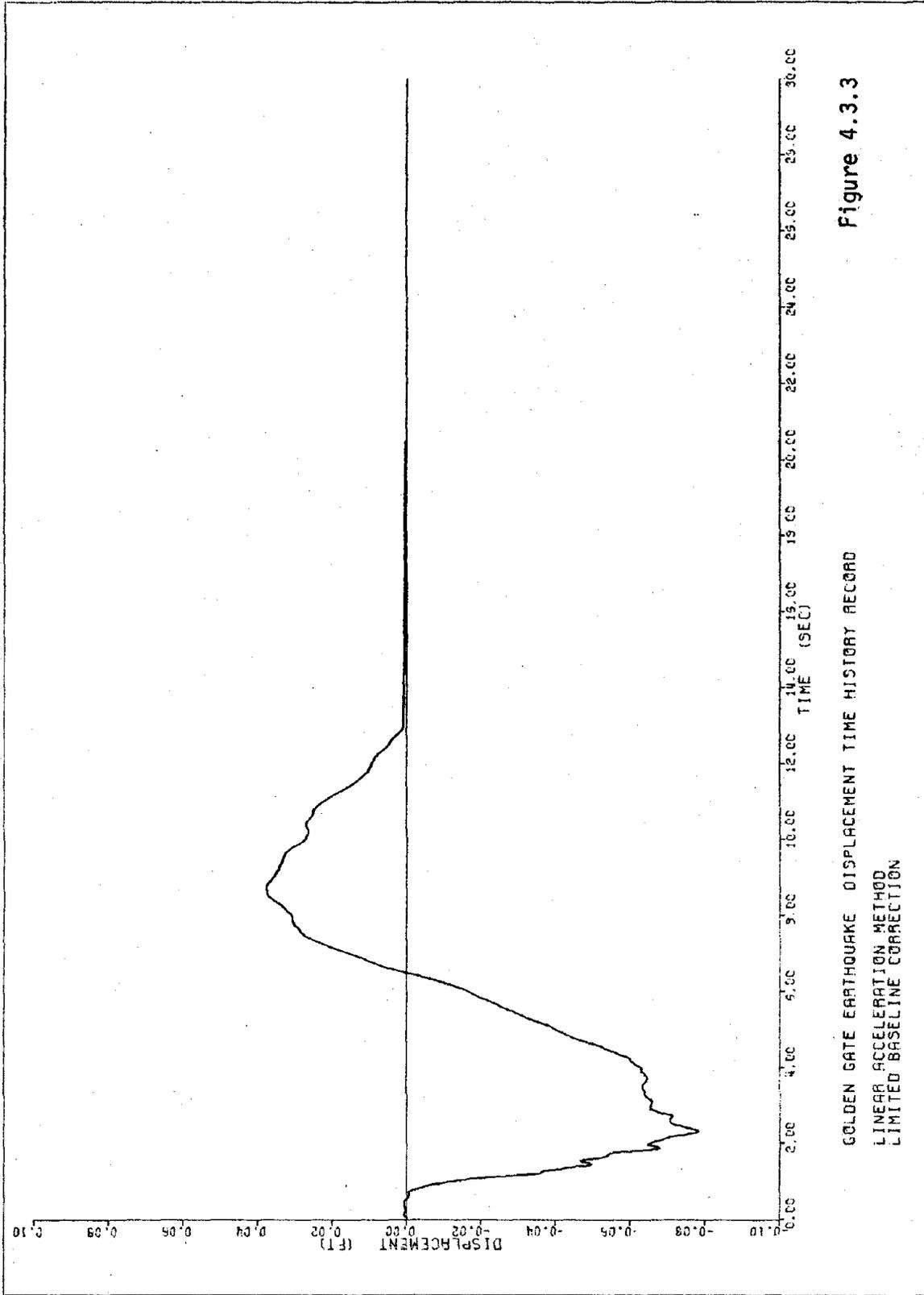LINEAR ACCELERATION METHOD
LIMITED BASELINE CORRECTION

Figure 4.3.2

GOLDEN GATE EARTHQUAKE DISPLACEMENT TIME HISTORY RECORD

LINEAR ACCELERATION METHOD
LIMITED BASELINE CORRECTION

Figure 4.3.3

The four methods of frequency domain integration produce highly consistent results. The computed motions are virtually indistinguishable from those computed in the time domain. Of the four methods presented, the central difference is the one that produces the poorest results. On the basis of these observations, it is recommended to use the simplest of all, namely the pseudocontinuous method.

The limited baseline correction technique, on the other hand, may be more suitably employed when a large discrepancy occurs between the physical length of the record and the augmented record length (T). This prevents the spurious acceleration pattern in the quiet zone and confines the correction to velocities and displacements. This can be seen by comparing figs. x.2.2 and x.2.3 with x.3.2 and x.3.3 (x = 1, 2, 3, 4).

Since the functions $x(t)$, $y(t)$ and $z(t)$ are chosen arbitrarily, the nature of the correction may be altered by adjusting x, y and z. As an example, x, y and z may be defined only during the last quarter of the ground shaking to limit the duration and the location of the baseline correction.

Conclusion

Various integration schemes in the frequency domain were examined in this report and compared to the trapezoidal integration rule ("linear acceleration method") in the time domain. These methods were then applied to a real earthquake record, to which a baseline correction parabola was added. It was found that the various methods yield consistent results, with excellent agreement between the time domain and frequency domain analyses. It is thus recommended to use the simplest of all integration methods, which is the one labeled "pseudocontinuous" in Table 1. The coefficients of the baseline parabola, on the other hand, are best computed with the "continuous" method in Table 3 (the inconsistency in the choice of methods has no practical consequences). Alternatively, one may use arbitrary correction functions x, y, z and apply the technique described in the last section of this report, computing the spectral amplitudes of the correction numerically with the FFT algorithm.

APPENDIX I

PART I) The derivation of several equations mentioned in the main text of this paper can be more readily demonstrated by first evaluating the following expressions.

a) $S_j = \sum\limits_{\alpha}^{\beta} k \, e^{i \frac{\pi}{N} jk}$ $(j \neq 0)$

Let $z = e^{i \frac{\pi}{N} j} \neq 1$ such that

$$S_j = z \sum\limits_{\alpha}^{\beta} k \, z^{k-1}$$

$$= z \frac{d}{dz} \left(\sum\limits_{\alpha}^{\beta} z^k \right)$$

$$= z \frac{d}{dz} \left(\frac{z^{\alpha} - z^{\beta+1}}{1-z} \right)$$

$$= \frac{\alpha z^{\alpha} - (\beta+1) z^{\beta+1}}{1-z} + \frac{z(z^{\alpha} - z^{\beta+1})}{(1-z)^2}$$

On the other hand, for $j = 0$ $(z = 1)$, we have

$$S_0 = \sum\limits_{\alpha}^{\beta} k = \frac{\beta(\beta+1)}{2} - \frac{(\alpha-1)\alpha}{2}$$

1) For $\alpha = -(N-1)$ and $\beta = N-1$, observe that $z^{\alpha} = z(-1)^j$ and $z^{\beta+1} = (-1)^j$; so for $j \neq 1$

$$S_j = \frac{-N(1+z)}{(1-z)} (-1)^j$$

$$= -N(-1)^j \left\{ \frac{z^{-1/2} + z^{1/2}}{z^{-1/2} - z^{1/2}} \right\}$$

$$= -N(-1)^j \left\{ \frac{\cosH i \frac{\pi}{2N} j}{\sinH i \frac{\pi}{2N} j} \right\}$$

$$= \frac{(-1)^j}{i} \cot \frac{\pi}{2N} j$$

Consequently:

$$\sum_{-(N-1)}^{N-1} k\, e^{i\frac{\pi}{N}jk} = \frac{N(-1)^j}{i} \cos \frac{\pi}{2N} j \qquad (j \neq 0)$$

$$= 0 \qquad (j = 0) \tag{107}$$

Also, one can observe from equation (107) (taking the complex conjugate) that

$$\sum_{-(N-1)}^{N-1} k\, e^{-i\frac{\pi}{N}jk} = -\frac{N(-1)^j}{i} \cos \frac{\pi}{2N} j \qquad (j \neq 0)$$

$$= 0 \qquad (j = 0) \tag{108}$$

2) For $\alpha=0$ and $\beta=2N-1$, we observe that $z^{\beta+1} = 1$ and $z^{\alpha} = 1$.

So for $j \neq 0$

$$S_j = \frac{-2N}{1-z}$$

$$= \frac{-2N\, z^{-1/2}}{z^{-1/2} - z^{1/2}}$$

$$= \frac{N(\cos \frac{\pi}{2N} j - i\, \sin \frac{\pi}{2N} j)}{i\, \sin \frac{\pi}{2N} j}$$

$$= -N(1 + i\, \cos \frac{\pi}{2N} j)$$

Consequently

$$\sum_0^{2N-1} k\, e^{i\frac{\pi}{N}jk} = -N(1 + i\, \cot \frac{\pi}{2N} j) \qquad (j \neq 0)$$

$$= 2N^2 - N \qquad (j = 0) \tag{109}$$

And similarly

$$\sum_0^{2N-1} k\, e^{-i\frac{\pi}{N}jk} = -N(1 - i\, \cot \frac{\pi}{2N} j) \qquad (j \neq 0)$$

$$= 2N^2 - N \qquad (j = 0) \tag{110}$$

b) $T_j = \sum\limits_{\alpha}^{\beta} k^2 e^{i \frac{\pi}{N} jk}$

As in Part a), let us denote $z = e^{i \frac{\pi}{N} j}$

$$T_j = \sum\limits_{\alpha}^{\beta} k^2 z^k$$

$$= \sum\limits_{\alpha}^{\beta} k(k-1)z^k + \sum\limits_{\alpha}^{\beta} kz^k$$

$$= z^2 \frac{d^2}{dz^2} (\sum\limits_{\alpha}^{\beta} z^k) + z \frac{d}{dz} \sum\limits_{\alpha}^{\beta} z^k$$

$$= Q_j + S_j$$

Observe that S_j has been evaluated in Part a). So, considering only the first term

$$Q_j = z^2 \frac{d^2}{dz^2} (\frac{z^{\alpha} - z^{\beta+1}}{(1-z)}$$

$$= \frac{\alpha(\alpha-1)z^{\alpha} + \beta(\beta+1)z^{\beta+1}}{1-z} + \frac{2z(\alpha z^{\alpha} - (\beta+1)z^{\beta+1})}{(1-z)^2} + \frac{2z^2(z^{\alpha} - z^{\beta+1})}{(1-z)^3}$$

Consequently

$$T_j = \frac{\alpha^2 z^{\alpha} - (\beta+1)^{\beta+1}}{1-z} + \frac{z((2\alpha+1)z^{\alpha} - (2\beta+3)z^{\beta+1})}{(1-z)^2} + \frac{2z^2(z\alpha - z^{\beta+1})}{(1-z)^3}$$

For the limits $\alpha=0$ to $\beta=2N-1$, with $z^{\alpha}=1$ and $z^{\beta+1} = 1$, then

$$T_j = \frac{-2N}{1-z} (2N + \frac{2z}{1-z})$$

$$= \frac{-2N z^{-1/2}}{z^{-1/2} - z^{1/2}} \left(2N + \frac{2z^{1/2}}{z^{-1/2} - z^{1/2}} \right)$$

$$= \left(\frac{N(\cos \frac{\pi}{2N} j - i \sin \frac{\pi}{2N} j)}{i \sin \frac{\pi}{2N} j} \right) \left(2N - \frac{\cos \frac{\pi}{2N} j + i \sin \frac{\pi}{2N} j}{i \sin \frac{\pi}{2N} j} \right)$$

$$= -2N^2(1 + i \cot \frac{\pi}{2N} j) + \frac{N}{\sin^2 \frac{\pi}{2N} j}$$

Also, it can be proved by mathematical induction that

$$\sum_{1}^{n} k^2 = \frac{n}{6}(2n+1)(n+1)$$

and in particular

$$\sum_{0}^{2N-1} k^2 = \frac{N}{3}(8N^2 - 6N+1)$$

Consequently

$$\sum_{0}^{2N-1} k^2 e^{i\frac{\pi}{N}jk} = -2N^2(1+i\cot\frac{\pi}{2N}j) + \frac{N}{\sin^2\frac{\pi}{2N}j} \qquad (j \neq 0)$$

$$= \frac{N}{3}(8N^2 - 6N+1) \qquad (j = 0)$$

$\qquad\qquad$ (111)

Similarly

$$\sum_{0}^{2N-1} k^2 e^{-i\frac{\pi}{N}jk} = -2N^2(1-i\cot\frac{\pi}{2N}j) + \frac{N}{\sin^2\frac{\pi}{2N}j} \qquad (j \neq 0)$$

$$= \frac{N}{3}(8N^2 - 6N+1) \qquad (j = 0)$$

$\qquad\qquad$ (112)

PART II) In reference to equations mentioned with the text.

a) In reference to equation 19

$$\sum_{m=-\infty}^{\infty} \frac{1}{j + 2Nm} = \alpha\cot j\alpha \qquad \text{where } \alpha = \frac{\pi}{2N}, \quad j \neq 0$$

$$S = \sum_{m=-\infty}^{\infty} \frac{1}{j + 2Nm}$$

$$= \frac{1}{j} + 2j\sum_{m=1}^{\infty} \frac{1}{j^2 - (2Nm)^2}$$

$$= \alpha\left\{\frac{1}{j\alpha} + 2j\alpha\sum_{m=1}^{\infty} \frac{1}{(j\alpha)^2 - \pi^2 m^2}\right\}$$

$$= \alpha\cot j\alpha$$

(See: Abramowitz, M. and Stegun, I., EDS Handbook of Mathematical Functions, New York,, Dover Publications, 1965, Page 75, Equation 4.3.91).

b) In reference to equations(23-24):

$$\sum_{j=-N}^{N} e^{i \frac{\pi}{N} jk} = 2N \, \delta_{ok}$$

Proof:

$$S = \sum_{-N}^{N} e^{i \frac{\pi}{N} jk}$$

$$= \sum_{-(N-1)}^{N} e^{i \frac{\pi}{N} jk}$$

For k = 0

$$S = \sum_{-(N-1)}^{N} (1) = 2N$$

For $k \neq 0$, let's denote $z = e^{i \frac{\pi}{N} k} \neq 1$, then

$$S = \sum_{-(N-1)}^{N} z^{j}$$

$$= \frac{1}{1-z} (z^{-(N-1)} - z^{N+1})$$

$$= \frac{z}{1-z} (z^{-N} - z^{N})$$

$$= 0 \quad (\text{since } z^{-N} = z^{N} = e^{\pm i\pi k} = (-1)^{k})$$

c) In reference to equations (25-26):

$$S_{k\ell} = \frac{\pi i}{NT} \sum_{j=-(N-1)}^{N-1} j \, e^{i \frac{\pi}{N} j(k-\ell)} =$$

$$= \begin{cases} \frac{\pi}{T}(-1)^{k-\ell} \cot (k-\ell)\alpha & k - \ell \neq 0 \\ \\ 0 & k - \ell = 0 \end{cases}$$

This follows directly from equation (107).

d) In reference to equation (44);

$$Q_m = \frac{1}{2N} \sum_{\ell=1}^{2N-1} \ell e^{i \frac{\pi}{N} m\ell} = \begin{cases} -\frac{1}{2} (1 + i \cot \frac{\pi}{2N} m) & m \neq 0 \\ \\ N - 1/2 & m = 0 \end{cases}$$

This follows directly from equation (109).

e) In reference to equation (50)

1) $\quad S_j^{(1)} = \sum_{m=-\infty}^{\infty} \frac{1}{j+2Nm} = \alpha \cot j\alpha \qquad \alpha = \frac{\pi}{2N}, \, j \neq 0$

(See part IIa, above).

2) $\quad S_j^{(2)} = \sum_{m=-\infty}^{\infty} \frac{1}{(j+2Nm)^2} = \alpha^2 \frac{1}{\sin^2 \alpha j}$

(This follows directly by taking the derivatives with respect to $j\alpha$ on both sides of the equation for $S_j^{(1)}$ from above).

3) $\quad S_0^{(1)} = \sum_{m=-\infty}^{\infty} \frac{1}{2Nm} = 0 \qquad$ (by antisymmetry)

4) $\quad S_0^{(2)} = 2 \sum_1^{\infty} \frac{1}{(2Nm)^2}$

$\qquad = \frac{2}{(2N)^2} \sum_{m=1}^{\infty} \frac{1}{m^2}$

$\qquad = \frac{1}{3} (\frac{\pi}{2N})^2 \qquad .$

PART III) The Discrete Fourier Series of a Parabola.

An alternative derivation to equation (51) is presented. An expression for H_j is formulated directly from equation (4) without using the aliasing relation of equation (5).

Beginning with equation (4)

$$H_j = \frac{1}{2N} \sum_{k=0}^{2N-1} f_k e^{-i \frac{\pi}{N} jk}$$

But for a parabola

$$f_k = a + \frac{bT}{2N} k + \frac{cT^2}{(2N)^2} k^2$$

so that

$$H_j = \frac{a}{2N} \sum_0^{2N-1} e^{-i \frac{\pi}{N} jk} + \frac{bT}{(2N)^2} \sum_0^{2N-1} k\, e^{-i \frac{\pi}{N} jk}$$

$$+ \frac{cT^2}{(2N)^3} \sum_0^{2N-1} k^2 e^{-i \frac{\pi}{N} jk} \tag{113}$$

But

$$R_j = \sum_0^{2N-1} e^{-i \frac{\pi}{N} jk} = 2N\, \delta_{oj}$$

Substituting (110), (112) and (114) into equation (113) yields

$$H_o = \frac{a}{2N}(2N) + \frac{bT}{(2N)^2}(2N^2-N) + \frac{cT^2}{(2N)^3} \frac{N}{3}(8N^2 - 6N+1)$$

$$= a + \frac{bT}{2}\left(1 - \frac{1}{2N}\right) + \frac{cT^2}{3}\left(1 - \frac{3}{4}N + \frac{1}{8N^2}\right)$$

and

$$H_j = \frac{bT}{(2N)^2}\left(-N\left(1-i \cos \frac{\pi}{2N} j\right)\right) +$$

$$+ \frac{cT^2}{(2N)^3}\left[\left(-2N^2\left(1 - i \cot \frac{\pi}{2N} j\right) + \frac{N}{\sin^2 \frac{\pi}{2N} j}\right]$$

$$= \frac{(b + cT)T}{4N}\left(-1 + i \cot \frac{\pi}{2N} j\right) + \frac{cT^2}{8N^2}\frac{1}{\sin^2 \frac{\pi}{2N} j}\ .$$

which agrees with the relationship obtained by aliasing.