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ALGORITHM FOR THE ESTIMATION OF SEISMIC RISK

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This report presents an algorithm to calculate the probability characteristics of various effects of a number of earthquakes on a given object in a seismic zone. The algorithm is based on: (1) the sequence of earthquakes in space, time and energy; (2) spatial distribution of intensity of ground tremors from fixed earthquakes; and (3) effects that the given object experiences from ground tremors of a certain intensity. A computer version of the algorithm permits the use of practically any engineering, seismological and economic information necessary for estimates of effects of earthquakes. Definitions of problems and general solutions which may be calculated are given. The problem of calculating a generalized Poissonian distribution which is often encountered in earthquake statistics and in the mathematical theory of risk is discussed.

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PREFACE

The article translated in this report appeared originally in Computational Seismology, Volume 6, 1974 (in Russian). The publication was given to Professors Whitman and Cornell during their visit to Moscow in January 1974 and discussed extensively with the authors at that time. Following initial translation in Cambridge, the translation was edited first by Dr. Keilis-Borak and then by Betsy Schumacker.

A translation of the entire table of contents of Volume 6 appears on the pages following this preface. A translation has also been made of the first paper in the volume and is available as Internal Study Report No. 43.

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The algorithm presented here makes it possible to calculate the probability characteristics of various effects of a number of earthquakes on a given object in a seismic zone. The algorithm is based on the simplest models:*

1) the sequence of earthquakes in space, time, energy;

2) spatial distribution of intensity of ground tremors from fixed earthquakes;

3) effects that the given object experiences from ground tremors of a certain intensity.

A computer version of the algorithm permits the use of practically any engineering, seismological & economic information (see section 5) necessary for estimates of effects of earthquakes. It can find application in problems of insurance and optimization of systems of measures for protecting against the after-effects of earthquakes. Examples of problems which may be calculated are given in section 7. Specific calculations are given in [4, 10]. The present article is a continuation of [3, 10] ; our basic attention is concentrated on the computational side of the problem. Therefore, some facts and hypotheses introduced in [3] are given without proofs. Model 2, given here reflects the theoretical and statistical analysis of Italian isoseismals presented by the authors in [4] .

Contents of this article: in section 1 the formulation of the problem and basic definitions are given; in section 2 basic hypotheses; in section 3 the general solution of problems. In 2 - 4, the models 1 - 3 are constructed, and parametrization of basic seismological characteristics is introduced. Models and their parameters reflect the available data for real earthquakes. An algorithm, implemented on an electronic computer, is presented in section 5.6.

Two problems explored in section 5 may be of independent interest: the coding in an electronic computer of various kinds of maps of the earth's surface, and the algorithm permits one (in a program) to determine whether a given point lies within a closed region on the earth's surface. Both problems are important for work with catalogues of earthquakes.

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*See pages 5-8 of the translation of [11] and § 7 of this paper.

In section 6 is discussed the problem of calculating a generalized Poissonian distribution which is often encountered in statistics about earthquakes and in the mathematical theory of risk.

A discussion of algorithms is given in section 7.

SECTION 1. THE PROBLEM AND DEFINITIONS

Earthquakes are regarded as random events (t, g, e) in the space T x $\mathbf{G}$ x E, where t is the moment of occurrence of the event within the time interval T, g is the vector coordinate of the epicenter within the seismic region $\mathbf{G}$, e is the measure of energy (magnitude, energy class and so on) within the energy range E.

In $\mathbf{G}$ two subsets are distinguished -- object 0, and the zone of seismic danger $\mathbf{G}'$. Each subset can consist of a set of areas, lines and points. In the case of the object, this subset can be separate areas of the earth's surface, networks of roads, cities and buildings, and in the case of zone $\mathbf{G}'$ the subset is seismically active regions and faults threatening object 0. It is not necessary for the danger zone to overlap the object 0.

An effect* is associated with each earthquake -- some random value x(t, g, e), which characterizes the damage to the object caused by an earthquake (t, g, e).

The problem consists of calculating the statistical distribution function $F^{\Sigma}(x) = P(X_{\Sigma} < x)$ for the total effect

$$X_{\Sigma}(T) = \sum_{0 < t_i < T} x(t_i, g_i, e_i) \qquad (1.1)$$

caused by the sequence of events (t_i, g_i, e_i) in the volume T x $\mathbf{G}'$ x E.

The algorithm allows the calculation of practically any effect for which there are a minimum of necessary estimates of the value x(t, g, e) -- the effect of individual earthquakes.

We will calculate the following effects:**

1. Tremors of the object: the total size of those parts of the objects which experience tremors of a given intensity. Each part is counted as many

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* See pages 3-4 of the translation of [11].

**See page 4 of the translation of [11].

times as it experienced such tremors during a set of earthquakes. For regions this is the sum of the areas affected, for lines --- the sum of the lengths affected, for points --- the number of points affected.

2. Total damage to the object from seismic tremors: complete damage from earthquakes or that part of the damage which was averted thanks to antiseismic measures.

3. Size of population or other valuable commodity present in the zones of tremors of given intensities.

Let us explain the notation. All distribution functions are symbolized by the symbol F(.). The index below shows the random value to which the distribution applies: F_n (.) is the distribution of n. Functions like $\phi(x/y)$ are defined as functions of argument x for a fixed value of y.

SECTION 2. HYPOTHESES

The algorithm given below for calculating $F^{\Sigma}(x)$ is based on the following assumptions:

Hypothesis 1. Model of the sequence of earthquakes.

Events (t, g, e) form a Poissoian flow in the volume T x G x E, this flow is uniform in the time t. In other words, the number of events n(ΔV), appearing in the non- overlapping volumes $\Delta V = \Delta t \cdot \Delta g \cdot \Delta e$, are statistically independent and have a Poisson distribution:

$$P(n(\Delta V) = k) = \frac{\lambda^k(\Delta V)}{k!} e^{-\lambda(\Delta V)} \qquad (2.1)$$

with parameters

$$\lambda(\Delta V) = \Lambda \Delta t \iint_{\Delta g \, \Delta e} p(g) f(e|g) \, dg \, de. \qquad (2.2)$$

Parameter (2.2) defines the average number of events (t, g, e) in the volume ΔV; p(g) is the average density of the epicenters of earthquakes in G' for an energy range E; f(e|g) is the normalized frequency-of-occurrence law for events at a point $g \in G'$; in other words, f(e|g) Δe is the conditional probability of the appearance of an event with energy $e \in \Delta e$ on the condition that it appeared at point g; Λ is the average number of events in volume G' x E for a unit of time.

Hypothesis 2. Effects $x(t, g, e)$ from various fixed earthquakes are statistically independent.

Hypothesis 3. Earthquakes (t, g, e) cause at each point $g \in G_1$ tremors, the intensity of which is defined by the random function $I(\tilde{g}|g, e)$. In other words, tremors are of an intensity c and higher in the random areas $\sigma(c) = \{\tilde{g}: I(\tilde{g}|g, e) \geq c\}$ ($\sigma(c)$ are called below isoseismal areas).

Hypothesis 4. The effect of earthquake $x(t, g, e)$ is the sum of the effects caused by tremors on each point of the object.

Hypothesis 5. A tremor of intensity c causes at each point of an object $g \in O$ a determined effect, defined by the function

$$x(t, g, e) = p(t, g)x_0(g, c). \qquad (2.3)$$

where $x_0(g, c)$ is the effect of the tremor (g, c) at an initial moment in time, and $p(t, g)$ is a nonrandom function characterizing the dynamics of the effect in time.*

We will present two examples explaining the meaning of $p(t, g)$. Let X_Σ be the total number of people (within the given object) who experience tremors of a given intensity during time T. Then the multiplier $p(t, g)$ is equal to $\exp(a(g)t)$. It takes into consideration the change in population at point g from initial time t=o until the moment of occurrence at time t. The parameter $a(g)$ indicates the rate of the growth in population in a unit of time.

Second example: let X_Σ be the economic effectiveness of antiseismic measures. It is necessary to attribute their cost to some momemt of time $t = 0$, for example to the moment of completion of construction. The value $x_0(\tilde{g}, c)$ now measures prevented damage from tremors of intensity c at a moment t; $x_0 \cdot \exp(-\beta t)$ is the part of the cost of antiseismic measures that was returned during the tremor $(t, \tilde{g}, c)$. Parameter β is the normative indicator (bank interest) of investment effectiveness. If in addition, the damage in the period of the object's construction is considered, then for $t \leq 0$, $p(t, g) = \varphi(t, g) \cdot \exp(-\beta t)$, where the coefficient $\varphi(t, g)$ signifies that part of the damage $x_0(g, c)$ at the time t before the moment when the object will be completed.

In the frame of the given algorithm it is not difficult to calculate also the

*At the suggestion of Robert Whitman & Allin Cornell, the algorithm was modified in such a way as to define the effect by means of a damage probability matrix.

chance fluctuation in values $x_o(g, c)$. We limit ourselves to simple events since the outside probability models $x_o(g, c)$ depend to a large degree on specifics of effects, and in practice run into considerable difficulties connected with weak research on the question and limited statistical material.

SECTION 3. SOLUTION OF THE PROBLEM

Under hypotheses 1 & 2 (pages 3,4), the distribution function $F^{\Sigma}(x)$ of total effect (1.1) is a generalized Poissonian distribution ((3), p. 99):

$$F^{\Sigma}(x) = \sum_{n=0}^{\infty} \frac{(\Lambda T)^n}{n!} e^{-\Lambda T} F^{(n)}(x) \qquad (3.1)$$

where $F^{(n)}(x)$ is the n-th convolution of F(x). F(x) is the distribution of the effect of a single arbitrary earthquake in volume T x G' x E:

$$F(x) = \frac{1}{T} \int_T \int_{G'} \int_E F(x|t, g, e)p(g)f(e|g)dt\, dg\, de \qquad (3.2)$$

Function $F(x|t, g, e)$ defines the distribution of the effect $x(t, g, e)$.

Allowing for hypotheses 4 and 5 we find that effect $x(t, g, e)$ is

$$x(t, g, e) = \int_O p(t, g)x_o(\tilde{g}, I(\tilde{g}|g, e))d\tilde{g}, \qquad (3.3)$$

if the field of intensity of the tremor is $I(\tilde{g}|g, e)$. It follows from hypothesis 4 that the probability of (3.3) is equal to the probability of the realization of $I(\tilde{g}|g, e)$.

SECTION 4. SUPPLEMENTARY HYPOTHESES

In this section the models of isoseismals & the frequency-of-occurrence law (models 1,2 on page 1) are specified. For the functions involved, the parametrization shown is that which is used in the computer program. Parametrization is chosen according to available experimental data and can be easily changed.

1. MODEL OF ISOSEISMALS. Hypothesis 6. The function $I(\tilde{g}|g, e)$, defining the intensity of tremor I at point $\tilde{g}$ from an earthquake (g, e), is defined by:

$$I(\tilde{g}|g, e) = I_o(\tilde{g}|g, e) + \Delta I(\tilde{g}). \qquad (4.1)$$

Here $I_o(\tilde{g}|g, e)$ corresponds to the intensity of tremor at point $\tilde{g}$ for areas with "normal" soil, and $\Delta I(g)$ is a determined correction for I_o, established from local soil conditions ("correction for microseismic zoning").

Hypothesis 7. Analytic presentation of I_o. The isolines $I_o=c$ of function $I_o(\tilde{g}|g, e)$ are assumed to be concentric elipses with center at point g. We will define these elipses by three parameters: area Q, ratio of axes L, and azimuth A of the long axis.

Areas $Q_c(e|g)$, inside isoseismals $\sigma(c)$, are represented by the model

$$\lg Q_c(e|g) = \lg \hat{Q}_c(e|g) + \Delta Q(g), \qquad (4.2)$$

where $\lg \hat{Q}_c$ defines the mean value of the logarithm of area $Q_c(e|g)$, and $\Delta Q(g)$ is a random addition, defined by the distribution functions

$$F_{\Delta Q}(\cdot|g). \qquad (4.3)$$

If $Q_c(e|g)$ in (4.2) happens to be less than certain threshold values $\underline{Q}(c, g)$, then we assume $Q_c(e|g) = 0$.

Let us clarify and make more specific hypothesis 7. As is shown by investigation of facts collected in the field and some theoretical considerations (see [4] and the literature there), $\lg \hat{Q}_c$ and the magnitude of an earthquake M are connected by an approximately linear dependence, within some intervals of magnitude $(\underline{M}_c, \overline{M})$, that is

$$\lg \hat{Q}_c(M|g) = a_c(g) + b_c(g)M, \; M \geq \underline{M}_c. \qquad (4.4)$$
$$\hat{Q}_c(M|g) = o \text{ for } M < \underline{M}_c$$

The thresholds of areas $\underline{Q}(c, g)$ are equal to the value $\hat{Q}_c(\underline{M}\, c|g)$ in (4.4). We notice that everywhere we leave out the focal depth of an earthquake, which is considered fixed.

The distribution function of correction ΔQ in (4.3) is defined by the observed dispersion of empirical data relative to model (4.4). From here, the truncated normal distribution can serve as a natural model for distribution (4.3).

$$F_{\Delta Q}(x|g) = \frac{\Phi(x) - \Phi(-k)}{1 - 2\Phi(-k)}, \quad |x| < k \qquad (4.5)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{u^2}{2})du$; parameter k and dispersion parameter σ depend on g.

The dispersion $\sigma_{\Delta Q}$ of distribution (4.5) is connected with σ^2 in the following way:

$$\sigma^2_{\Delta Q} = P_3(k^2) \quad \sigma^2 / (1 - 2\Phi(-k)),$$

where $P_3(x)$ is the χ^2 distribution with 3 degrees of freedom. For k = 3 or 2.5 the relation is $\sigma_{\Delta Q}/\sigma = 0.986$ or 0.943.

The ratio of the major axes of an eliptical isoseismal is a nonrandom function

$$L(e, g) \geq 1. \tag{4.6}$$

The orientation of isoseismal areas is defined by the azimuth A of the largest axis. We assume that

$$A = \bar{A}(g) + \Delta A(g, e), \tag{4.7}$$

where $\bar{A}(g)$ is the average direction of extension of isoseismal areas, actually coinciding with the predominate direction of the major geological faults near the area of point g; $\Delta A(g, e)$ is the random correction for the average azimuth, depending on the energy and location of the earthquake. This is defined by distribution function $F_{\Delta A}(\cdot|g, e)$.

The dependence of corrections on energy is connected with the fact that for strong earthquakes the isoseismals are usually elongated along major faults, whereas the isoseismals of weak earthquakes can reflect the directions of smaller faults.

The introduction of corrections ΔI (hypothesis 6) results in the distortion of the correct (eliptical) form of the isoseismal; this explains many features of actual isoseismals.

Consequence. Hypotheses 6 and 7 result in the following expression for the probability of effect x(t, g, e) (formula (3.3)). If for earthquake (t,g,e) the correction to the area of the isoseismals (see (4.2)) happened to be ΔQ in (4.2), and the correction to the azimuth (see (4.7)) happened to be ΔA,

then the intensity $I_o(\tilde{g}|g, e)$ for "normal" soil is defined by the isoseismal $\sigma_c(c) = (\tilde{g}: I_o(\tilde{g}|g, e) \geq c)$. These isoseismals are of the following form:

$$((\tilde{x} - x)\sin(\bar{A} + \Delta A) + (\tilde{y} - y)\cos(\bar{A} + \Delta A))^2/L(g, e) +$$
$$((\tilde{x} - x)\cos(\bar{A} + \Delta A) - (\tilde{y} - y)\sin(\bar{A} + \Delta A))^2 \cdot L(g, e) \leq$$
$$2\pi^{-1}\hat{Q}_c(e|g)10^{\Delta Q}.$$

Here $(\tilde{x}, \tilde{y}) = \tilde{g}$, $(x, y) = g$ are local Cartesian coordinates of points $\tilde{g}$ and g. The probability of effect $x(t, g, e)$ in (3.3) is equal to

$$dF_{\Delta A}(\Delta A|g, e) \cdot dF_{\Delta Q}(\Delta Q|g). \qquad (4.9)$$

2. THE SEQUENCE OF EARTHQUAKES. The intensity of the flow of events (t, g, e) in an arbitrary point in volume $T \times G \times E$ is defined by the function $\lambda_g(e) = \lambda \cdot p(g) \cdot f(e|g)$, (see 2.2), which we usually call the frequency-of-occurrence law of earthquakes at point g.

The next hypothesis, which defines one of the possible parametrizations of $\lambda_g(e)$, is slightly broader than Gutenberg's linear law hypothesis, and takes care of the possibility of the attenuation of the frequency-of-occurrence curve at large energies [3].

Hypothesis 8. The frequency-of-occurrence law - $\lambda_g(M)$ is a function of magnitude M and has the appearance

$$\lambda(M|g) = \lambda_0(g)\,\psi(M|g), \qquad (4.10)$$

where

$$\lg \psi(M) = \begin{cases} -\gamma(M - M_1) & \underline{M} \leq M \leq M_1 \\ -\gamma_1(M - M_1) & M_1 \leq M \leq \bar{M}. \end{cases}$$

Parameters of slope γ and γ_1, point of bend M_1, and maximum magnitude $\bar{M}$ depend on point g. The lower threshold of magnitude $\underline{M}$ is defined by the range of e under consideration. The parameter $\lambda_0(g)$ defines the intensity of the flow of earthquakes of magnitude M_1 at point g.

SECTION 5. DESCRIPTION OF THE ALGORITHM.

Computation of the distribution of total effect is divided into 3 parts:

--- map of the region;

--- distribution of the effect of a single earthquake;

--- distribution of the total effect.

The first part includes the reading and compact coding of initial information; the second calculates the distribution function of the effect of a single event $F(x)$ in (3.2); and the third computes the final function $F^{\Sigma}(x)$ from (3.1). We will describe each part of the algorithm.

1. MAP OF THE REGION*. In each point g of the area G it is necessary to specify the following values:

1) $O(g)$ --- a sign (0 or 1) defining whether the point g belongs to the object O;

2) $G'(g)$ --- a sign (0 or 1) defining, analogously with 1), whether the point g belongs to the danger zone for the object;

3) $\Delta I(g)$ --- correction to the intensity of tremors for the soil conditions;

4) $\bar{A}(g)$ --- average azimuth of isoseismals;

5) $F_{\Delta A}(\cdot|e, g)$ --- distribution function of the correction to the azimuth;

6) $\lambda_c(g)$ --- intensity (frequency-of-occurrence) of earthquakes for some fixed energy e_o;

7) $F(e|g)$ --- type of distribution of the number of earthquakes by energy; it is defined by the parameters: $\gamma(g)$, $\gamma_1(g)$, $M_1(g)$, $\bar{M}(g)$ (formula (4.10)); it may also be defined by table ;

8) $x_c(g)$ --- effect from a tremor of intensity c at point g;

9) $p(t|g)$ --- dynamics of effect $x_o(g, e)$ (see formula (2.3));

10) $I_o(\tilde{g}|e, g)$ --- average model of isoseismals for normal soil. The model includes the parameters: $a_c(g)$, $b_c(g)$ defining the areas $Q_c(e, g)$ of isoseismals (see formula (4.4); $L(e, g)$ --- elogation of isoseismals;

11) $F_{\Delta Q}(\cdot|g)$ --- distribution function of logarithmic corrections for areas of isoseismic zones (see formula (4.5)).

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*It may be easier to read this section while looking at the specific example of the coding of information in [12].

The set of all these functions which depend on a point g we will call map of area G, and conditionally note by the vector

$$K(g) = (\varphi_1(g),\ldots,\varphi_s(g)), \quad s = 11. \qquad (5.1)$$

We will similarly call any function (vector or scalar) that permits us to find the values of these functions.

The basic goal of the first part of the program is to read into the computer the map of area G^E and to code it in a compact fashion in the computer memory. This can be done in several ways. In real situations, the set of possible values of each component of map K(g) is not large, because seismological and engineering information is limited and the models are very rough. For example, region G could actually be divided into a few subregions (2 - 3) differing only in the model of isoseismals $I_o(\tilde{g}|g, e)$ and in the distribution F_{M_0} for each sub region. Therefore, the reading of maps K(g) can be done in two stages. 1st stage. Reading in the value of the function $\varphi_i(g)$, that is of the sets

$$D_i = (\varphi : \varphi = \varphi_i(g), \; g \in G) \qquad (5.2)$$

Elements D_i can be figures, like in the case of functions $p_o(g)$; or functions, for instance types of the frequency-of-occurrence law of f(e|g); or vector-functions, for example models $I_o(\tilde{g}|e, g)$. After the ordering of elements D_i the map of area G is transformed into the function

$$k(g) = (n_1(g),\ldots,n_s(g)) \qquad (5.3)$$

where k(g) is a vector and each of its components is an interger. The component $n_i(g)$ is the ordinal number of the specific function $\varphi_i(g)$ in the list D_i. 2nd stage. Construction of the map k(g). A map of (5.3) turns out to be even more complicated than the function, and preparing it directly for each point g is rather difficult work. We construct maps k(g) by superposition of elementary maps.

Elementary map $k(g)$ is a vector function with integer components; it is equal to $(n_1,\ldots,n_s)$ in the subset $G \in \mathcal{G}$ and is equal to $(0,\ldots,0)$ outside $\mathcal{G}$, that is

$$k(g) = \begin{cases} (n_1,\ldots,n_s) & g \in G, \\ (0,\ldots 0) & g \notin G, \end{cases} \qquad (5.4)$$

The set G can be an area, a line or a point.

Elementary maps are superimposed by the following rule.

Let $k_i(g) = (n_1^i(g),\ldots,n_s^i(g))$, $i = 1, 2$ be some maps of region $\mathcal{G}$. Then the super position of these maps

$$k(g) = k_2(g) \circ k_1(g) = (n_1(g),\ldots,n_s(g))$$

is defined by the rule

$$n_i(g) = \begin{cases} n_i^2(g) \\ n_i^1(g), \end{cases} \quad \text{if} \quad \begin{matrix} n_i^2 \neq 0 \\ n_i^2 = 0 \end{matrix} \qquad (5.5)$$

In other words, the operation of superposition of maps keeps already stored information if the next map does not bring new information on the same parameters, and replaces old information by the new if there is new information. By superimposing elementary maps on another, one can achieve arbitrarily complicated maps. The contours G of the elementary maps are the isolines of the function specified by this map.

Reading in the groups of elementary maps. The elementary map $k(g)$ can be read in by specification of the contour G, and the value $k(g) = k$ within or on this contour (see Jordan's algorithm). The reading is considerably simplified because many contours of elementary maps have common parts. Therefore, maps $k(g)$ are divided into groups. For each group of maps $(\partial G, k)$, a family of lines $L = (l)$ can be defined, that is a set of the parts of the contours from which any contour can be constructed. The contour is a set of numbers

$$k(g) = (n_1,\ldots,n_p:k), \qquad (5.6)$$

where $(n_1,\ldots,n_p)$ are the ordered set of numbers of the lines which form contour L

and k shows which value is defined within the contour. The contour is followed in one direction; the sign of n_i is negative if the ith line is stored having opposite direction.

The line $l \in L$ is defined by the set of points $(g_1, \ldots g_{n(l)})$ which are used as knots of polygons on a sphere, that is the contour between neighboring points g_i, g_{i+1} is interpolated by a geodesic arc. The positive direction is assumed from g_i to g_{i+1}.

The information presented is enough to construct an arbitrary set of elementary maps. It remains to describe the algorithm which can define if a given point is inside the contour or not.

"Jordan's" algorithm. Let us look at the arbitrary, closed area G, surrounded by polygonal contours - continous and non-intersecting. The contour is defined on the half-sphere S_o by the knots (g_i). Below, the test function $\varphi(g)$, $g \in S_o$, will be defined: it is equal to 1 for all points inside area G, and equal to 0 for points outside G.

Let λ, φ be the spherical coordinates of points g, λ is the longitude, φ is the latitude. Let us find those sides of polygon G which cross the meridian, with λ passing through point g_o; that is those sides which are arcs $l_k = (g_{i_k}, g_{i_{k+1}})$, k = 1,...,n, for which

$$a_k = |\lambda_{i_k} - \lambda_o| \cdot |\lambda_{i_{k+1}} - \lambda_o| \cdot |\pi - |\lambda_{i_k} - \lambda_{i_{k+1}}|| \geq 0. \qquad (5.7)$$

Here $|x|$ is equal to 1,0 or - 1 if x is respectively larger than, equal to, or less than zero. We define for the trio of points g_1, g_2, g_3 from S_o the determinant

$$\det |g_1, g_2, g_3| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad (5.8)$$

where $(x, y, z) = (\cos\lambda \cdot \cos\varphi, \sin\lambda \cdot \cos\varphi, \sin\varphi)$ are the euclidian coordinates of point g. Let us find the numbers

$$d_k = \det |g_{i_k}, g_{i_{k+1}}, g_o| . \qquad (5.9)$$

Let us denote

$$\varphi_k = \begin{cases} (1 - a_k) \cdot \mathrm{sgnd}_k/2 & \text{if } d_k \neq 0 \\ 0 & \text{if } d_k = 0,\ b_k > 0 \\ \text{stop} & \text{if } d_k = 0,\ b_k \leq 0 \end{cases} \tag{5.10}$$

where, analogously to (5.7)

$$b_k = (\varphi i_k - \varphi_o) \cdot (\varphi i_{k+1} - \varphi_o). \tag{5.11}$$

Our test function is the following:

$$\varphi(g_o) = \begin{cases} \left| \sum_{k=1}^{n} \varphi_k \right| - 1 & \text{if all } \varphi_k \neq \text{stop}, n \neq o \\ 1 & \text{if one } \varphi_k = \text{stop}, n \neq o \\ 0 & \text{if } n = o \end{cases} \tag{5.12}$$

Proof. Let $(g(t),\ 0 \leq t \leq 1)$ be some smooth, single valued parametrization of contours, and $a(t)$ be a continuous function which defines the angle between rays (geodesic arcs) $v_o = (g_o, g(0))$ and $v_t = (g_o, g(t))$;

Here g_o is a fixed point from s_o. From the "argument principle" it is known that for contour G which is non-intersecting, $a(1) = 2\pi$ if g_o is inside G, and $a(1) = 0$ if g_o is outside G. Let $t=t_i$, $i = 0, 1,\ldots,n$ be consecutive moments of intersection of contours with direction v_o. Then $a(t_{i+1}) - a(t_i) = 2\pi\varepsilon_i$, where $\varepsilon_i = \pm 1$ or 0. Since $a(t)$ is continuous and is not equal to $a(t_i)$, $a(t_i) \pm 2\pi$, inside the interval (t_i, t_{i+1}), we have

$$a(t_{i+1}) - a(t_i) = \frac{\pi}{2} \left[\operatorname{sgn} \dot{\alpha}\ (t_{i+1} - 0) + \operatorname{sgn} \dot{\alpha}\ (t_i + 0) \right],$$

and therefore

$$a\ (1)/\pi = \sum_{i=1}^{n-1} \left| \operatorname{sgn} \dot{\alpha}\ (t_i + 0) + \operatorname{sgn} \dot{\alpha}\ (t_i - 0) \right| + \left[\operatorname{sgn} \dot{\alpha}\ (t_o + 0) + \operatorname{sgn} \dot{\alpha}\ (t_n - 0) \right],$$

where $t_o = 0$, $t_n = 1$.

We define the number φ_k by the condition: $\varphi_k = \pm 1, \pm 0.5$ or 0 if the sides of the polygon l_k respectively intersect the ray v_o, coincide with one of the end points of ray v_o, or do not intersect v_o; the sign of φ_k corresponds to the positive or negative increment of the amplitude a(t). Then we get

$$\sum \varphi_k = \begin{cases} 1 & g \in G \\ 0 & g \notin G \end{cases}$$

Let us consider the meridian passing through g_o, as the two rays v_{o1} and v_{o2}, leaving g_o in opposite directions, and let us sum the tests for both directions. Then we obtain the test (5.10) - (5.12). It remains to note that the sign of d_k in (5.9) defines in which direction the arc l_k crosses the meridian at point g_o. The case $d_k = 0$ arises when g_o lies on the arc l_k or on its extension. According to (5.7), in the first case $b_k \leq 0$. If $b_k > 0$, the arc l_k lies on the meridian of point g_o, $g_o \in l_k$ and consequently, should be excluded from consideration.

2. DISTRIBUTION OF EFFECTS OF A SINGLE EARTHQUAKE F (x)*. The algorithm for calculating the distribution function of the effect of a single event is described in Section 3 ("Solution of problems"), and comes to the numerical integration of function F(x|t, g, 1) over the volume T x G' x E (see 3.2)). F(x|t, g, 1) is defined by the formulae (3.3, 4.8, 4.9).

The corresponding computations are rather long in their general form. Therefore, some "specialization of an algorithm, using the most specific features of the particular problem, can significantly improve the speed and accuracy of the algorithm. Below, we will show a method of discretization of the maps of the region, and we will specify an algorithm in one important practical case.

Discretization of Maps. Region G is taken as a spherical rectangle with new spherical coordinates $\tilde{\varphi}, \tilde{\lambda}$; the long side $\mathcal{G}$ lies on the equator $\tilde{\varphi}=0$, $\tilde{\lambda} \geq 0$, and the smaller side lies on the zero meridian $\tilde{\lambda} = 0$, $\tilde{\varphi} \geq 0$.

Let us look at the transformation of area $\mathcal{G}$

$$(\tilde{\varphi}, \tilde{\lambda}) \longrightarrow (x,y) = (R\lambda \cos\varphi', R\varphi) \qquad (5.13)$$

*See page 10 of the translation of [11].

into the plane (x, y) where R is the radius of the Earth. It transforms the spherical axes $\tilde{\lambda} = 0$ and $\tilde{\varphi} = 0$ into Cartesian axes x = 0 and y = 0, and does not change the element of the area $ds = R^2 \cos\varphi \, d\varphi \, d\lambda$. The linear elements, except the direction of latitude, do not remain linear, and the distortion in point (5.13) is the larger, the larger is the value $\tilde{\lambda} \sin\tilde{\varphi} = x \operatorname{tg} y$. Consequently, in choosing a system of coordinates $(\tilde{\varphi}, \tilde{\lambda})$ in G, it is preferable to minimize the maxiumum latitude $|\tilde{\varphi}|$, than longitude $|\lambda|$, which follows from the inequality $x \operatorname{tg} y < y \operatorname{tg} x$ for $x > y$.

The presentation of the region in the plane (x, y) is convenient for computations, thanks to the simple relationship of the coordinates (5.13). We also notice that the isoseismic areas defined in (4.8) in the coordinate system (x, y) will keep on the sphere exactly the same area and almost the same elliptical form.

The region is discretized in the following manner. On the plane (x, y) the region is covered by a rectangular grid of knots $((i + 0.5)\delta, (j + 0.5)\delta) = (x_i, y_j)$ with spacing δ; these knots correspond to the points on a sphere:

$$\tilde{\varphi}_j = (j + 0.5)\delta / R, \quad \tilde{\lambda}_i = (i + 0.5)\delta / (R \cos\tilde{\varphi}_j) \tag{5.14}$$

The map of the region is approximated by constants inside the squares $\delta \times \delta$ with centers at knots (x_i, y_j). Point objects are transferred to the closest knots, and linear objects are divided into elementary segments of length δ, each segment is replaced by the knot closest to its center. In this manner, all the initial information is concentrated in the knots (5.14).

Distribution function F(x). Let us consider the case when the coefficient p(t, g), which defines the dynamics of effect (2.3), is not dependent on g, that is

$$p(t, g) \equiv p(t) \tag{5.15}$$

Then in this case $x(t, g, e) = p(t) \cdot x_0(g, e)$, and if $F_0(x)$ is the distribution

of $x_o(g, e)$ averaged by g and e, then

$$F(x) = \int_0^1 F_o\left(\frac{x}{p(T\tau)}\right) d\tau . \qquad (5.16)$$

We will illustrate the calculations of $F_o(x)$ with the following limitations.

Let the energy interval E be divided into several intervals $\Delta_i E_i$ in each interval: a) the distribution of random corrections to the azimuth $F_{\Delta A}(\cdot|g)$, and the elongation of the isoseismals L(g) do not depend on the energy; b) the parameter $b_c(g)$ of the area of the isoseismal in model (4.4) does not depend on the intensity c. In practice these conditions are not too limiting. Moreover, as a compensation, hypothesis "b" allows a nonlinear dependence $\lg\hat{Q}_c$ on the magnitude (see (4.4)).

Let us introduce the notation: let g or $\tilde{g}$ be the knots (5.14) and δ be the distance between them. From (4.10) we define the integral

$$I(g, \Delta E) = \int_{\Delta M} \psi(M|g)\, dM , \qquad M = \lg e . \qquad (5.17)$$

Then

$$\Lambda = \sum_{g \in G} I(g, E)\, \lambda_o(g)\, \delta^2 \qquad (5.18)$$

is the intensity of the flow of earthquakes with epicenters in zone G and with energy in the range E;

$$p(g) = \lambda_o(g) I(g, E)\, \delta^2 / \Lambda \qquad (5.19)$$

is the probability of an earthquake with energy in range E in the knot g;

$$p(\Delta E_i | g) = \frac{I(g, \Delta E_i)}{I(g, E)} \qquad (5.20)$$

is the probability of an earthquake with energy in the i^{th} energy interval in the knot g;

$$F_g(\cdot \mid g, \Delta E_i) \tag{5.21}$$

is the distribution of the random value $q = 10^{b(g)M+\Delta Q}$, where M has the density of probability $\psi(M|g)/I(g,\Delta_i E)$ in interval $\Delta_i E$, and ΔQ, independent of M, is a random value with distribution $F_{\Delta Q}(\cdot|g)$ (see (4.3)).

If $\psi(M|g)$ is defined by (4.10) and $F_{\Delta Q}$ is defined by (4.5), then $F_g(\cdot|g, \Delta_i E)$ can be calculated by explicit formula through a Gaussian distribution function.

The algorithm consists of consecutive choosing of all possible combinations of arguments in the following order.

1. Knots $g \in G'$, that is, the sources of the earthquake. Probability of an earthquake is defined by $p(g)$.

2. Energy intervals ΔE_i. An earthquake with energy in the interval ΔE_i has a probability $p(\Delta E_i|g)$. For given g and ΔE_i the model of isoseismals is defined, that is, the distribution of the areas $\hat{Q}_c(\Delta E_i, g)$ with parameters $a_c(g)$ and $b(g,\Delta E_i)$ (see (4.4)), the elongation of the isoseismal $L(g|\Delta E_i)$, the orientation $\bar{A}(g)$ and distribution function $F_{\Delta A}(\cdot|g,\Delta E_i)$ for azimuth corrections.

3. Azimuth corrections ΔA_i. Probabilities of azimuth corrections $P_{\Delta A_i}$, are defined by the distribution $F_{\Delta\Delta}$, which we discretize.

4. Points of an object $\tilde{g} \in O$, making a contribution to the total effect of an earthquake with epicenter at g.

The set of points $\tilde{g}$ is defined by the largest possible isoseismal for earthquakes with center at g and with energy $e \in \Delta E_i$.

Let $R(g, \tilde{g})$ be the left side of the inequality (4.8). We define the numbers

$$q_c(\tilde{g}) = \max(\pi 10^{\alpha_c(g)} R(g, \tilde{g}), \quad Q_c(g)), \tag{5.22}$$

where $Q_c(g)$ is the area of the smallest isoseismal of intensity c. The values $q_c(\tilde{g})$ have the following meaning. Suppose that at point g an earthquake with magnitude M and random correction to the area of isoseismal ΔQ occurred, so that

$$q = 10^{b(g)M+\Delta Q}$$

belongs to the interval (q_c, q_{c+1}). Then at point $\tilde{g}$, according to (4.8) and (4.9), the intensity of tremors will be $c + \Delta I(\tilde{g})$, where ΔI is a correction to intensity for local soil conditions. Consequently, with probability

$$P = p(g) \cdot p(\Delta_i E | g) \cdot p_{\Delta A}(g) \cdot dF_q(p | g, \Delta E_i) \tag{5.23}$$

the effect of the earthquake on point g is equal to

$$x_o = \sum_{\tilde{g}, c} x(\tilde{g},\ c + \Delta I(\tilde{g}))\ [q_c(\tilde{g}) < q < q_{c+1}(\tilde{g})] . \tag{5.24}$$

where $[a < x < b]$ is equal to 1 if $a < x < b$, and is equal to 0 in the opposite case; $x(\tilde{g}, c)$ is the effect of the tremor at point $\tilde{g}$. In this way, the distribution function $F_o(x)$ is defined by the set of numbers p_o and x_o.

The described version of the algorithm for calculating functions F(x) has a number of operations proportional to

$$S'/\delta^2 \cdot \bar{S}_c/\delta^2 \cdot N_M \cdot N_A \cdot \bar{N}_q \tag{5.25}$$

where S' is the area of danger zones of volume G'; $\bar{S}_c$ is the average area of the largest isoseismal in the energy intervals ΔE_i; N_M is the number of energy intervals; N_A is the average number of azimuth corrections; N_q is the average number of steps for a variable q in intervals ΔE_i, $\bar{N}_q$ defines the discretization of distribution $F_q(\cdot | g, \Delta_i E)$.

The last part of the algorithm --- distribution of total effect --- is necessary to describe in more detail; we will do it in the next section.

SECTION 6. TOTAL EFFECT $F^{\Sigma}(x)$

We denote by

$$\Pi(x | t, F) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-t} F^{(n)}(x) \tag{6.1}$$

a generalized Poissonian distribution, defined by parameter t and "step function" F(x). (In the future we shall always assume that F(x) is con-

centrated on the half-axis $x \geq 0$.)

From (6.1) it follows that, if F(x) is a distribution function of the effect from a single earthquake in volume $T \times G' \times E$, then the distribution of the total effect is defined by the formulae

$$F^{\Sigma}(x) = \Pi(x|\Lambda T, F) \tag{6.2}$$

or

$$F^{\Sigma}(x) = \Pi(x|\Lambda_0 T, \hat{F}) \tag{6.3}$$

where

$$\Lambda_0 = \Lambda \cdot (1 - F(0)) \tag{6.4}$$

and

$$\hat{F}(x) = \frac{F(x) - F(0)}{1 - F(0)} . \tag{6.5}$$

The identity of (6.2) and (6.3) becomes obvious, if they are transformed by a Laplace transform.

In the last presentation of $F^{\Sigma}(x)$ only those earthquakes which give a non-zero effect are included. Parameter Λ_0 defines the average number of such earthquakes in the volume $T \times G' \times E$ and $\hat{F}(x)$ is the distribution of the non-zero individual effect of a single earthquake.

In connection with problems of insurance, various asymptotic formulae were obtained for distribution (6.1) with the assumption that parameter t is large and F(x) does not depend on t (see the review in [8]). These results are applicable to our situation, if we assume that parameters $\Lambda_0 T$ are large and the dependence F(x) on T for large T is weak. This means the following. Inasmuch as F(x) depends on time through the coefficient p(t, g), which defines a dynamic effect (see (2.3)), then it is necessary to assume that p(t, g) is limited from above and below: $0 < c_1 < p < c_2 < \infty$ as $T \to \infty$. For example, if $p(t, g) = \exp(-\beta t)$, then $\beta = 0(1/T)$.

These conditions are not always fulfilled. This is especially true for the effect of strong earthquakes, for which $\Lambda_0 T$ is small. Therefore, it is necessary in actual problems to calculate the generalized Poissonian distribution in a sufficiently broad range of the values of parameter t.

Below are presented (some in a more general form) the asymptotic expansions of $\pi(x|t, F)$ for $t\to\infty$, found by Kramer [8], and also the direct method of calculation of $\pi(x|t, F)$, suitable for moderately large and moderately small values of t.

1. ASYMPTOTIC FORMULAE [8]. The distribution $\pi(x|t, F)$ has an average value $m_\Sigma = tm_1$, and dispersion $\sigma_\Sigma = \sqrt{tm_2}$, where m_k is the moment of the k^{th} order of the distribution F(x). As $t\to\infty$, the distribution $\pi(x|t, F)$ is asymptotically normal with the same parameters --- average m_Σ and dispersion σ_Σ (see (6.20)).

Statement 1 that follows below gives an asymptotic expansion of the distribution $\pi(x|t, F)$, which defines more precisely the convergence of $\pi(x|t, F)$ to a normal distribution as $t\to\infty$. This expansion is an analogue of the Edgeworth series known in the theory of summation of independent random values.

Statement 1. Assume that the following conditions are satisfied:

a) F(x) includes an absolutely continuous component;

b) F(x) has r + 1 finite moments

$$m_k = \int_0^\infty x^k dF(x) < \infty), \quad K = 1, \ldots, r+1. \tag{6.6}$$

In this case even for all real x

$$\pi(tm_1 + \sqrt{tm_2}x|t, F) = \Phi(x) - \varphi(x)\sum_{k=3}^{r} t^{-\frac{k}{2}+1} R_k(x) + O(t^{-\frac{r-1}{2}}). \tag{6.7}$$

Here $\Phi(x)$ is the standard normal distribution and $\varphi(x)$ is its density; $R_k(x)$ are polynomials, defined for $r = 3\div 6$ as:

$$R_3(x) = c_3 H_3(x), \tag{6.8}$$

$$R_4(x) = c_4 H_4(x) + \frac{c_3^2}{2} H_6(x),$$

$$R_5(x) = c_5 H_5(x) + c_3 c_4 H_7(x) + \frac{c_3^3}{6} H_9(x),$$

$$R_6(x) = c_6 H_6(x) + \left(c_3 c_5 + \frac{c_4^2}{2}\right) H_8(x) + \frac{c_3^2}{2} c_4 H_{10}(x) + \frac{c_3^4}{24} H_{12}(x).$$

Here $H_k(x)$ are Hermite polynomials of order $k - 1$, defined by the recursive relation

$$H_{k+1}(x) = xH_k(x) - (k - 1)H_{k-1}(x), \qquad H_1(x) \equiv 1, \tag{6.9}$$

and for coefficients c_k we have

$$c_k = \frac{1}{k!}\, m_k m_2^{-\frac{k}{2}} \tag{6.10}$$

Observation 1. In the case when $p(t, g) = \exp(-\beta t)$, the distribution $F(x)$ has the form (5.16). In this case, if μ_k are the moments of distribution $F_o(x)$, then

$$m_k = \mu_k \cdot \frac{1 - \exp(-\beta k T)}{\beta k T}. \tag{6.11}$$

The asymptotic expansion (6.7) represents $\pi(x|t, F)$ in the best way near central values of x, that is, in the interval $\{x: |x - tm_1| = 0(\sqrt{t})\}$.

For large values, when $|x - tm_1| = 0(t)$, another expansion is preferable:

Statement 2. If the conditions of the preceding observation are satisfied, and the function is defined

$$p(h) = \int_o^\infty e^{hx} dF(x) \tag{6.12}$$

for $h < h_o$, $h_o > 0$, then the asymptotic expansion holds:

$$\pi(p_i(h)t|t, F) = \begin{cases} e^{-\beta(h)t} \Gamma(-h\sqrt{p_2(h)t}), & h < 0 \\ 1 - e^{-\beta(h)t} \Gamma(h\sqrt{p_2(h)t}), & h > 0 \end{cases} \tag{6.13}$$

where

$$p_k(h) = \int_o^\infty x^k e^{hx} dF(x) = \frac{d^k}{dh^k} p(h) \tag{6.14}$$

$$\beta(h) = 1 - p(h) + hp_1(h) \tag{6.15}$$

and

$$\Gamma(u) = \tilde{H}_0(u) + \sum_{k=3}^{r} t^{-\frac{k}{2}+1} \tilde{R}_k(u)\,[-\operatorname{sgn} h]^k + O\left(t^{-[\frac{r}{2}]}\right) \tag{6.16}$$

Functions $\tilde{R}_k(u)$ and $\tilde{H}_k(u)$ are defined by (6.8) with coefficients $\tilde{c}_k$ instead of c_k, where $\tilde{c}_k$ are:

$$\tilde{c}_k = \frac{1}{k!}\,\Gamma_k(h) \cdot \Gamma_2^{-\frac{k}{2}}(h) \,. \tag{6.17}$$

$\tilde{H}_k(u)$ is defined by the recursive relation

$$\tilde{H}_{k+1}(u) = u\tilde{H}_k(u) - \tilde{H}_{k-1}(0) \cdot (k-1) \tag{6.18}$$

with initial values

$$\tilde{H}_0(u) = e^{u^2/2}\,\Phi(-u), \qquad \tilde{H}_{-1}(0) = -\frac{1}{\sqrt{2\pi}} \,. \tag{6.19}$$

Observation 2. The Berry-Esseen theorem [2,6] gives a simple estimate of error for the approximation $\pi(x|t, F)$ of the first term of expansion (6.7) and (6.13):

$$\left|\pi\left(t m_1 + \sqrt{t m_2}\, x \,|\, t, F\right) - \Phi(x)\right| < \frac{0{,}82}{\sqrt{t}} \cdot \frac{m_3}{m_2^{3/2}} \tag{6.20}$$

and when $h > 0$

$$\left|\pi\left(\Gamma_1(h)\,t \,|\, t, F\right) - 1 + e^{-\beta(h)t + \frac{h^2}{2}\Gamma_2(h)t}\,\Phi\left(h\sqrt{\Gamma_2(h)\,t}\right)\right| \le$$

$$e^{-\beta(h)t} \min\left(1, \frac{1{,}64}{\sqrt{t}} \cdot \frac{\Gamma_3(h)}{\Gamma_2^{3/2}(h)}\right) . \tag{6.21}$$

Estimate (6.21) follows easily from the formulae given in [1,2].

For distribution F(x) of the form (5.16), with $\beta > 0$ the marginal distribution $\pi(x|\beta) = \lim \pi(x|t, F)$ exists for $t = \Lambda_0\, T \to \infty$; its moments of the

k^{th} order are: $\lim m_k \cdot t = \mu_k \Lambda_0 / (\beta k)$ "Observation 1").

Here $P(h)\ t = 1 + \frac{\Lambda_0}{\beta} \int_0^h \frac{\phi_0(x) - 1}{x}\, dx$, where $\phi_0(-u)$ is the Laplace transformation of distribution $F_0(x)$. Allowing for this, formulae (6.7, 6.13, 6.20 and 6.21) can easily be rearranged into an expansion of distribution $\pi(x|\beta)$ by parameter $\beta \downarrow 0$. In particular from (6.20) we find

$$\left| \pi\left(\mu_1 \frac{\Lambda_0}{\beta} \sqrt{\frac{\mu_2 \Lambda_0}{2\beta}}\, x \,|\, \beta\right) - \phi(x) \right| \le 0{,}7731 \sqrt{\frac{\beta}{\Lambda_0}}\, \frac{\mu_3}{\mu_2^{1.5}/2}\,, \qquad (6.22)$$

A similar estimate is found in [9] with constant 5.4.

2. DIRECT METHOD. For small values of parameter t, the series (6.1) is a natural expansion of $\pi(x|t, F)$ by t and it seems convenient enough for calculations. The basic difficulty is the calculation of convolutions of high order, because of the fast accumulation of errors from numerical integration of $F^{(n)}(x)$. We replace the functions $F^{(n)}(x)$ for large n by their normal approximation; then we come up with the following approximate formula:

$$\pi(x|t,F) = \sum_{n=0}^{N_1} \frac{t^n}{n!} e^{-t} F^{(n)}(x) + \sum_{n=N_1+1}^{N_2} \tilde{F}_n\left(\frac{x - m_1 n}{\sigma\sqrt{n}}\right) + \varepsilon_1(N_1) + \varepsilon_2(N_2) + \varepsilon_3 \qquad (6.23)$$

where m_1 and $\sigma = \sqrt{m_2 - m_1^2}$ are the average and dispersion of F(x), and $\tilde{F}_n(x)$ is a section of the Edgeworth series for convolutions of the n^{th} order, that is

$$\tilde{F}_n(y) = \phi(y) - \phi(y) \sum_{n=3}^{r} n^{-\frac{k}{2}+1} \hat{R}_k(y)\,. \qquad (6.24)$$

Here $\hat{R}_k(x)$ is expressed by the polynomials $H_k(x)$ in (6.9) according to formulae (6.8) with coefficients $\hat{c}_k$, instead of c_k. They are defined below by (6.25) for $k = 3 \div 6$.

Let m_{ok} be the central moments of the function F(x), then

$$\hat{c}_3 = \hat{\mu}_3,\ \hat{c}_4 = \hat{\mu}_4 - 1/8,\ \hat{c}_5 = \hat{\mu}_5 - \hat{c}_3/2,\ \hat{c}_6 = \hat{\mu}_6 - \hat{c}_4/2 - \hat{c}_3^2/2 - 1/48 \quad (6.25)$$

and

$$\hat{\mu}_k = \frac{1}{k!} \cdot m_{ok}\, \sigma^{-k} \quad . \qquad (6.26)$$

Errors in formulae (6.23). The error $\varepsilon_2(N_2)$ proceeds from the truncation of the series (6.23) on the N_2-component, that is

$$\varepsilon_2(N_2) = \sum_{N_2+1}^{\infty} \frac{t^n}{n!} e^{-t} F^{(n)}(x) < \sum_{N_2+1}^{\infty} \frac{t^n}{n!} e^{-t} = P_{2(N_2+1)}(2t) \qquad (6.27)$$

where $P_f(x)$ is the χ^2 distribution with f degrees of freedom. The second error ε_3 is due to the replacing of convolutions by their normal approximations. It is known [6] that an error of such approximations

$$\delta_n = \max_x \left| F^{(n)}(x) - \tilde{F}_n\left(\frac{x - m_1 n}{\sigma\sqrt{n}}\right)\right|$$

approaches zero and has the order o $(n^{-k/2+i})$. If we assume that when n increases δ_n monotonically approaches zero, than

$$|\varepsilon_3| < \delta_{N_1} \sum_{n=N_1+1}^{N_2} \frac{t^n}{n!} e^{-t} = \delta_{N_1}\left(P_{2(N_1+1)}(2t) - P_{2(N_2+1)}(2t)\right) \quad , \qquad (6.28)$$

since $\delta_{N_1+k} < \delta_{N_1}$ as assumed. The value of δ_{N_1} is easy to estimate in the process of calculations.

The error $\varepsilon_1(N_1)$ is connected with the numerical integration of convolutions $F^{(n)}(x)$, $n \le N_1$. For its estimation, we first note that if $\tilde{F}(x)$ is some approximation of F(x), with error

$$\varepsilon_f = \max\left|F(x) - \tilde{F}(x)\right| . \qquad (6.29)$$

then

$$\left|F^{(m)}(x) - \tilde{F}^{(m)}(x)\right| < n\varepsilon_F. \tag{6.30}$$

Actually

$$F^{(n)}(x) - \tilde{F}^{(n)}(x) = \left(F(x) - \tilde{F}(x)\right) * \left[\sum_{j=0}^{n-1} F^{(j)} * \tilde{F}^{(n-1-j)}\right](x),$$

where * is the symbol of the convolution. Estimate (6.30) follows from (6.29) and from the fact that the items standing under the summation sign are the distribution functions, and consequently cannot exceed 1. For the numerical calculation of convolutions it is convenient to take for $\tilde{F}(x)$ the step function

$$\tilde{F}(x) = (F(kh), \ |x - kh| < h/2, \quad k = 0, 1, 2... \tag{6.31}$$

In this case

$$\varepsilon_F \le \max_k \left(F((k+1)h) - F(kh)\right). \tag{6.32}$$

Allowing for (6.30) we find an estimate of the error $\varepsilon_1(N_1)$:

$$\left|\varepsilon_1(N_1)\right| \le \max_x \left|\sum_{k=1}^{N_1} \frac{t^n}{n!} e^{-t} \left(F^{(k)}(x) - \tilde{F}^{(k)}(x)\right)\right| \le \varepsilon_F \sum_{n=1}^{N_1} \frac{t^n}{(n-1)!} e^{-t}$$

that is

$$\left|\varepsilon_1(N_1)\right| \le \varepsilon_F \, t\left(1 - P_{2N_1}(2t)\right). \tag{6.33}$$

Another method of estimation of $\varepsilon_1(N_1)$ for an arbitrary interval of variable x is given in [3].

Observation 3. The direct method (6.23) requires a considerable volume of calculations for moderate values of t, when parameter N_1 should be chosen relatively large. The following method, based on the possibility of the

unlimited division of distribution $\pi(x|t, F)$ [6]:
$\pi(x|t_1+t_2, F) = \pi(x|t_1, F) * \pi(x|t_2, F)$, allows a shortening of the calculations. Let

$$\pi_k(x) = \pi(x|t \cdot 2^{-k}, F) \tag{6.34}$$

and the error term of the calculation $\pi(x|t, F)$ is $\varepsilon(t)$. Then $\pi(x|t, F) = \pi_0(x)$ can be obtained from $\pi_k(x)$ in k steps; these steps are the consecutive operations of "self-convolution":

$$\pi_{n-1} = \pi_n * \pi_n \qquad n = k, k-1, \ldots, 1 \tag{6.35}$$

Allowing for (6.30), the calculation error is

$$\varepsilon(t) = 2^k \cdot \varepsilon(t \cdot 2^{-k}) \,. \tag{6.36}$$

3. NUMERICAL EXAMPLES. The methods of calculating the generalized Poissonian distribution $\pi(x|t, F)$ are illustrated below for the example function.

$$F(x) = \begin{cases} p - p \cdot \exp(-x), & 0 \le x \le 0.53143 \\ 1 - p \cdot \exp(-x), & x > 0.53143 \end{cases} \tag{6.37}$$

for p = 1 and 0.75.

The following values of the parameters were taken: The order of approximation r in formulae (6.7, 6.24) equals 6, and in (6.13) r = 4. In (6.24) function F(x) was given in 200 points with spacing $\Delta = 0.05$, the number of calculated convolutions $N_1 = 4$.

The results of calculations $(1-\pi(x|t, F))10^5$ are presented in a table, where comparisons are made of: a) three methods of calculation for large parameter t(t = 16, p = 1); c, d) methods (6.7) and (6.24) respectively for "moderate" values of t(t = 8; 4 and p = 0.75). In table "c" calculations of $\pi(x|t, F)$ by the direct method (6.24) are given for "small" parameters

t = 1(p = 1; 0.75), when the asymptotic method is non-applicable.

In the example given, all three methods are practically equal for the "large" parameter t = 16; method (6.24) also gives a fairly good result for small t = 1.

For practical use, formulae (6.7) and (6.24) are more convenient. For large parameters, method (6.13) is more accurate, although it requires the numerical calculation of Laplace-Stiltess transformation of the function F(x). In this way, even more accurate methods of calculating $\pi(x|t, F)$ for the whole range of parameter t can be expected, if the modern technique of Laplace transform is used. Let us clarify this. If.

$$L(F) = \int_0^\infty e^{-sx}\, dF(x), \tag{6.38}$$

and L^{-1} is the opposite operation, then

$$\pi(\ \ |t, F) = L^{-1}\ (e^{t(LF-1)}).$$

However, considerable difficulty arises here in relation to the necessity of insuring a sufficiently high accuracy of the calculation of LF.

Table

Таблица

а) $t = 16$, $x = 0$ (4) 40

$p = 1$,			
точно	I ($r = 6$)	II ($r = 4$)	III ($r=6$)
100000	99983	—	100000
99658	99680	99659	99654
93961	93951	93967	93963
74614	74615	74630	74615
46460	46460	46474	46459
22613	22614	22624	22613
8828	8823	8832	8828
2850	2858	2850	2850
782	781	782	784
186	180	186	190
39	42	40	40

б) $t = 1$, $x = 0$ (1) 10

$p = 0.75$		$p = 1$	
точно	III	точно	III
63212	63212	63212	63212
30818	30800	34574	34579
14666	14649	18258	18262
7098	7093	9386	9389
3382	3381	4723	4726
1589	1592	2335	2338
738	743	1137	1142
340	346	547	553
155	163	260	267
70	80	122	131
31	41	57	66

Таблица (окончание

в) $t = 8$, $p = 0.75$, $x = 0$ (2) 24

точно	I	III
99967	100267	99775
95915	95812	95971
80012	80025	80035
56723	56763	56748
34818	34851	34842
18965	18982	18996
9363	9337	9371
4260	4309	4268
1810	1870	1816
725	669	732
276	241	285
101	120	111
35	50	46

г) $t = 4$, $p = 0.75$, $x = 0$ (1,5)18

точно	I	III
98168	98153	98075
77891	77703	77976
50596	50657	50554
29160	29202	29170
15378	15319	15374
7576	7449	7566
3536	3906	3596
1579	1601	1568
639	374	670
283	283	299
115	210	113
45	69	62
18	12	34

Note. Function $1 - \eta(x|t, F)$ is for F defined by (6.37). The line divides the values of the function into intervals $|x - m_\pi| \lesssim \sigma_\pi$. Here m_π and σ_π are the average and dispersion of $\pi(x|t, F)$; I, II, III correspond to the methods of calculating π by formulae (6.13), (6.7), and (6.24) respectively.

SECTION 7. DISCUSSION OF THE ALGORITHM

The advantages of the suggested algorithm are:

a) a broad possibility for using practically all presently available seismological and engineering information;

b) universality --- that is, the possibility of calculating a great range of economic as well as seismological effects of earthquakes.

We will list here the aspects of earthquake effects for which we calculated distribution functions with the help of the program that we described.* All calculations refer to the given interval of time:

a) tremors of the object: total size of those parts of the object that experienced tremors of a certain intensity; each part is counted as many times during earthquakes as it experienced tremors. Regions, cities, roads and buildings can serve as objects;

b) total amount of population that felt tremors of a certain intensity in the given region;

c) total cost of valuables or production of a given type, present in the zone of tremors of certain intensity in the given region;

d) the damage to the object from seismic tremor of the object --- this can be total loss, or that part of the damage which was avoided due to anti-seismic measures.

The distribution of the given characteristics presents not only seismo-engineering interest. They can be applied to questions of insurance, planning the building in seismic regions, optimization of building codes (see [11]).

The shortcomings of the suggested algorithm are a strong idealization of earthquake sequence and the macroeffects of earthquakes. In most cases, this is due to an insufficiency of data about these questions. Above all, a hypothesis about the independence of earthquakes in close space-time volumes is doubtful. For example, such phenomena dependency as after- and fore-shocks of strong earthquakes and chains of earthquakes of similar energy are known. However, only a relatively small part of earthquakes is joined in chains (see [5]). More significant parts of related events --- aftershocks ---

* See also [11, 4 10].

can be allowed for in the framework of the described program by including the effect of aftershocks in the effect of the basic earthquake (for a discussion of this question see [3]).

Construction of models of isoseismals demands serious attention. The suggested model is sufficiently flexible, but all the same, further examination of the data is needed.

We note that in the framework of the suggested algorithm, the influence of the focal depth can be taken into consideration. For this, a minimum of data is necessary --- distribution of the density of the hypocenters by depth and dependence of the parameters of isoseismals on the focal depth. In the absence of these data, the depth effect is allowed for indirectly since in the construction of the probabilistic models of isoseismals we average the data for all depths at which earthquakes actually took place. In the framework of the suggested algorithm, a random variation of the deterministic characteristics $\Delta I(g)$, $x_c(g)$, $L(e, g)$, $x_c(\tilde{g})$ is possible.

Due to the approximate nature of the models and the roughness of the actual information, more accurate answers can be expected in those problems where the objects are considerably extended in space, and the effects are studied for long intervals of time. In such cases, the effect of tremors $x_c(\hat{g})$ is a more homogeneous function of $\dot{g}$, and the approximate character of the models and the local variations of them are smoothed out.

To this class of problems belong problems of insurance, and also the general problem of optimization of the whole system of anti-earthquake measures.

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