EARTHQUAKE WAVES IN A RANDOM MEDIUM

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EARTHQUAKE WAVES IN A RANDOM MEDIUM†

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ABSTRACT

Measurements are conducted with small samples in the laboratory and thus for all practical purposes the medium is macroscopically homogeneous. On the other hand the uncertainties and the irregular changes in situ are macroscopic inhomogeneities. This work is an effort for accounting for these stochastic changes in the elastic properties and density in a rational manner. The method used is that of Karal and Keller which is based on the use of the Green's function and neglect of third order correlations. The resulting integral equations are solved by Laplace transform. The analysis indicates that the energy decay in the mean motion through random mode coupling introduces damping into even a purely elastic medium and enhances the damping in a significant manner in a hysteretic viscoelastic medium. This consideration is important in relating the damping and dispersion characteristics of waves in situ to those measured in the laboratory. The formulation is extended to multilayer systems through transfer matrices and to arbitrary inputs by Fourier transform. Sample calculations are presented for single and multilayer systems to obtain response spectra and for the response to Gaussian and actual earthquake input motions.

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1. INTRODUCTION

Earthquake investigators have been aware of the need to account for uncertainty in the parameters appearing in deterministically based forecasting equations in a quantitative and rational manner. Applications of stochastic methods in earthquake engineering have usually been in the form of a random input into a deterministic system. However, the geological configurations and the material properties of soil deposits such as density, elastic moduli and damping coefficients are not always known with sufficient accuracy to justify a deterministic analysis. Consequently, uncertainties in the properties of the medium will result in uncertainties in the response spectra. In addition, through random mode coupling, the wave energy is distributed into various modes at the expense of the energy in the mean motion. This energy decay from the motion introduces damping into even an elastic medium and enhances the damping in a significant manner for a viscoelastic medium. Indeed for a set of calculations presented in Figure 3, increases in the damping coefficient of the order of 100% are observed due to overall stochastic changes in the density of 10-20% from the mean. For establishing the relation between the damping coefficient measured in situ and in the laboratory, this consideration may be quite significant. In fact, since measurements are conducted with small samples in the laboratory, the medium is a homogeneous one for all practical purposes. The distances over which inhomogeneities occur are of microscopic size as compared to the wave lengths involved. This analysis indicates that the use of the damping characteristics measured in the laboratory may significantly underestimate the actual damping in situ.

The basic idea in the derivations here follows those of Karal and Keller [1] who studied elastic waves, electromagnetic waves and diffusion problems in random media of infinite extent. The method basically obtains an equation for the average displacements. Classical perturbation theory fails in this problem due to the generation of secular terms by the correlation functions of the stochastic properties, which introduce non-local interactions. The resulting field equations for the dynamics of the wave propagation are integro-differential equations, similar to those in non-local continuum theories introduced by Edelen and Eringen [2,3]. Similar considerations to those in this paper have been applied by Beran and McCoy [4,5] for the study of composites and by Tang and Pinder [6] to aquifer hydrology problems.

2. EQUATIONS FOR THE AVERAGED DISPLACEMENT

The goal here is to study the amplification of shear waves in a layer forced harmonically at one face and is traction free at the other. We consider that the shear coefficient μ and the density ρ are stochastic functions of space and the probability variable is p. Without loss in generality, these may be expressed as:

$$\mu = \mu_0 + \mu_1(x, p) \qquad \rho = \rho_0 + \rho_1(x, p) \tag{2.1}$$

In this representation μ_0 and ρ_0 are the average values of μ and ρ and consequently $\bar{\mu}_1 = \bar{p}_1 = 0$ with the bar $(-)$ indicating statistical averages of the ensemble with probability density P: $\bar{f}(x) = \int_{-\infty}^{\infty} f(x, p)P(p)dp$. As one of the goals of this research is to study the enhancement in the damping of the medium, we take the damping coefficient ζ to be deterministic. The stress-strain relation is hence taken as: $T = \mu \frac{\partial v}{\partial x} + \zeta \frac{\partial^2 v}{\partial t \partial x}$ where v is the displacement and x is the spatial coordinate. Then, the dynamics is described by the momentum equation:

$$\frac{\partial T}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2} \tag{2.2}$$

Substitution of T above yields:

$$\frac{\partial}{\partial x} (\mu \frac{\partial v}{\partial x}) + \zeta \frac{\partial^3 v}{\partial t \partial x^2} = \rho \frac{\partial^2 v}{\partial t^2} \tag{2.3}$$

For the harmonic waves as $v = ue^{i\omega t}$, the reduced wave equation becomes:

$$\frac{d}{dx}\left(\mu \frac{du}{dx}\right) + i\zeta\omega \frac{d^2u}{dx^2} + \omega^2\rho u = 0 \tag{2.4}$$

With (2.1) the above equation is rearranged as:

$$\left(\frac{d^2}{dx^2} + q_o^2\right)u + \left[(1 + i\kappa)^{-1} \frac{d}{dx} n_2 \frac{d}{dx} + q_o^2 n_1\right]u = 0 \tag{2.5}$$

where

$$q_o = \left(\frac{\rho_o}{\mu_o(1+i\kappa)}\right)^{1/2}\omega \qquad n_1 = \frac{\rho_1}{\rho_o} \qquad n_2 = \frac{\mu_1}{\mu_o} \qquad \kappa = \frac{\zeta\omega}{\mu_o} \tag{2.6}$$

Similarly the stress-strain relation with $T(x, t) = \tau(x)e^{i\omega t}$ reads:

$$\tau = \mu_o(1 + i\kappa)\left(\frac{du}{dx} + n_2 \frac{du}{dx}\right) \tag{2.7}$$

In the form in (2.5) the equation appears as $L\,u + R\,u = 0$ where L is the deterministic operator and R is the random operator below:

$$L = \frac{d^2}{dx^2} + q_o^2 \qquad\qquad R = (1 + i\kappa)^{-1} \frac{d}{dx} n_2 \frac{d}{dx} + q_o^2 n \tag{2.8}$$

By their definitions

$$\bar{L} = L \qquad\qquad \bar{R} = 0 \tag{2.9}$$

With the above definitions, equivalent to (2.5), one also has

$$u = X_o G(x, o) + Y_o \frac{\partial G}{\partial x}(x, o)$$

$$- \int_{x=0}^{x} G(x, x')[(1 + i\kappa)^{-1} \frac{d}{dx}(n_2' \frac{du'}{dx'}) + q_o^2 n_1' u']dx' \tag{2.10}$$

Where $G(x, x')$ is the Green's function associated with L and $X_o = u(o)$, $Y_o = u'(0)$ are initial conditions at $x = 0$. The primes on the dependent variables, in (2.10) indicate that their argument is x': $u' = u(x')$, $n_1' = n_1(x')$ and $n_2' = n_2(x')$. The Green's function used here is one sided and is defined as:

$$\frac{d^2G}{dx^2} + q_o^2 G = 0$$

$$G\Big|_{x=x'} = 0 \qquad\qquad \frac{dG}{dx}\Big|_{x=x'} = 1 \tag{2.11}$$

The problem could also be formulated in terms of the Green's function for the boundary value problem. However, the formulation as a boundary value problem is quite inconvenient since one would need a two-sided Green's function defined piecewise. The formulation here is aimed to constructing the general solution when the initial conditions X_o , Y_o are viewed as integration constants. One can thus obtain the solution to the boundary value problem from this general solution by determining the constants X_o , Y_o from the prescribed boundary conditions. As a preparation for obtaining the terms $n_2 \frac{du}{dx}$ and $n_1 u$, integration by parts of the first integral yields:

$$u = X_o G(x, o) + Y_o \frac{\partial G}{\partial x}(x, o) + (1 + i\kappa)^{-1} G(x, o) n_2(o) Y_o$$

$$- (1 + i\kappa)^{-1} \int_{x'=o}^{x} \frac{\partial G}{\partial x} n_2' \frac{du'}{dx'} dx' - q_o^2 \int_{x'=o}^{x} G\, n_1' u' dx' \qquad (2.12)$$

In obtaining the above expression use is made of the first boundary condition in (2.11) and of the antisymmetry of the Green's function

$$\frac{\partial G}{\partial x} = - \frac{\partial G}{\partial x'} \qquad (2.13)$$

From (2.13), by the use of Leibnitz' rule*, and the substitution $\partial^2 G/\partial x^2 = -q_o^2\, G$ according to (2.11), one obtains:

$$\frac{du}{dx} = X_o \frac{\partial G}{\partial x}(x, o) - Y_o q_o^2 G(x, o) + (1 + i\kappa)^{-1} \frac{\partial G}{\partial x}(x, o) n_2(o) Y_o$$

$$- (1 + i\kappa)^{-1} n_2(x) \frac{du}{dx} + (1 + i\kappa)^{-1} q_o^2 \int_{x'=0}^{x} G\, n_2' \frac{du'}{dx'} dx'$$

$$- q_o^2 \int_{x'=0}^{x} \frac{\partial G}{\partial x} n_1' u' dx' \qquad (2.14)$$

$$^* \quad \frac{d}{dx} \int_{A(x)}^{B(x)} F(x,x') dx' = \int_{A(x)}^{B(x)} \frac{\partial F(x,x')}{\partial x} dx' + F(x,x')\Big|_{x'=B(x)} \frac{dB}{dx} - F(x,x')\Big|_{x'=A(x)} \frac{dA}{dx}$$

Finally, by the substitution of (2.14) into (2.7), the stress is obtained as:

$$\tau = \mu_o\{(1 + i\kappa)\frac{du}{dx} + (1 + i\kappa)[n_2 \chi_o \frac{\partial G(x,o)}{\partial x} - n_1 \gamma_o q_o^2 G(x,o)]$$

$$- n_2 n_2 \frac{du}{dx} + \frac{\partial G(x,o)}{\partial x} n_2 \, n_2'(o) \, \gamma_o$$

$$+ q_o^2 \int_{x'=o}^{x} [G(x,s')n_2 n_2' \frac{du'}{dx'} - (1 + i\kappa)\frac{\partial G}{\partial x} n_2 n_2' u'] \, dx'\} \tag{2.15}$$

The stress-strain relation above is exactly equivalent to that in (2.7) and no approximation has been made thus far. In obtaining the equations for the averaged quantities, the following approximations are adopted [1,8]:

$$\overline{n_i(x,p)n_j(x',p)u(x',p)} \simeq \overline{n_i(x,p)n_j('x,p)} \quad \overline{u(x',p)} \tag{2.16}$$

This approximation amounts to assuming that the correlations between $n_i(x,p)$ and $n_i(x',p)$ in the random operator R are stronger than those between $n_i(x,p)$ and $u(x,p)$. Furthermore introducing the correlation functions

$$\overline{n_i(x,p)n_j(x',p)} = N_{ij}(|x-x'|) \tag{2.17}$$

and noting that $\bar{n}_i = 0$, Eq. (2.15) yields:

$$\bar{\tau} = \mu_o\{(1 + i\kappa - N_{22}(o)) \frac{d\bar{u}}{dx} + S_2(x)Y_o + q_o^2 \int_{x'=o}^{x} (K_{22}(x-x') \frac{d\bar{u}'}{dx'} - (1 + i\kappa)K_{12}(x-x')\bar{u}')dx' \qquad (2.18)$$

Above:

$$S_2(x) = \frac{\partial G(x-x')}{\partial x} N_{22}(x-x')\Big|_{x'=0}$$

$$K_{22}(x-x') = G(x-x')\, N_{22}(x-x')$$

$$K_{12}(x-x') = \frac{\partial G(x-x')}{\partial x} N_{12}(x-x') \qquad (2.19)$$

In the stress-strain relation above, the first term with $N_{22}(0)$ indicates a local contribution, the term $S_2(x)$ a surface energy contribution and the last term under the integral a non-local contribution. The form of the stress-strain relation above is similar to the one derived formally by using the axioms of nonlocal continua [2,3]. In these latter the kernel K_{12} , which represents the correlations between the stochastic changes in the shear modulous and density, is absent. It appears that a formal derivation from a non-local continuum view point could also yield the same term. For noncorrelated density and modulous fluctuation, clearly $K_{12} = 0$. Similarly, if the shear modulous is deterministic on the sufrace, i.e. $n_2(0) = 0$ surface energy contribution $S_2(x)$ vanishes. In the derivations here, all of the terms are carried.

Finally, for the term $\partial\bar{\tau}/\partial x$ in the field equation, the integral terms in (2.18) are rearranged as follows:

$$1: \quad \frac{d}{dx}\int_{x'=0}^{x} K_{22}\,\frac{d\bar{u}'}{dx'}\,dx' = -\int_{x'=0}^{x} \frac{\partial K_{22}}{\partial x'}\,\frac{d\bar{u}'}{dx'}\,dx'$$

$$2: \quad \frac{d}{dx}\int_{x'=0}^{x} K_{12}\,\bar{u}'\,dx' = N_{12}(0)\bar{u} - \int_{x'=0}^{x} \frac{\partial K_{12}}{\partial x'}\,\bar{u}'\,dx' \qquad (2.20)$$

Above use is made of Leibnitz' formula and the following properties of K_{22} and K_{12} according to (2.19):

$$K_{22}(0) = 0 \qquad\qquad K_{12}(0) = N_{12}(0)$$

$$\frac{\partial K_{22}}{\partial x'} = -\frac{\partial K_{22}}{\partial x} \qquad\qquad \frac{\partial K_{12}}{\partial x'} = -\frac{\partial K_{12}}{\partial x} \qquad (2.21)$$

By the integrations by parts of the integrals in (2.20) and again use of the definitions in (2.19), one obtains respectively

$$1: \quad S_4(x)\,Y_o + \int_{x'=0}^{x} K_{22}(x-x')\,\frac{d^2\bar{u}'}{dx'^2}\,dx'$$

$$2: \quad S_3(x)\,X_o + \int_{x'=0}^{x} K_{12}(x-x')\,\frac{d\bar{u}'}{dx'}\,dx' \qquad (2.22)$$

Above $S_3(x)$ and $S_4(x)$ are surface correlation terms similar to $S_1(x)$:

$$S_3(x) = \frac{\partial G(x-x')}{\partial x} N_{12}(x-x')\Big|_{x'=0}$$

$$S_4(x) = G(x-x')\, N_{22}(x-x')\Big|_{x'=0} \tag{2.23}$$

Therefore with the use of (2.22) in (2.18) one obtains:

$$\frac{d\bar{\tau}}{dx} = \mu_o\{(1 - N_{22}(0)\ ) + i\kappa)\ \frac{d^2\bar{u}}{dx^2} - q_o^2(1 + i\kappa)S_3(x)X_o$$

$$+ \left(\frac{dS_2}{dx} + q_o^2 S_4(x)\ \right)Y_o$$

$$+ q_o^2 \int_{x'=0}^{x} \left(K_{22}\ \frac{d^2\bar{u}}{dx'^2} - (1 + i\kappa)K_{12}\ \frac{d\bar{u}'}{dx'}\right) dx'\}$$

In obtaining the final equations, the term $\rho = \rho_o(u + n_1 u)$ is also needed. With (2.12) used in the second term,

$$\rho_o(u + n_1 u) = \rho_o\{u + n_1(G(x,o)X_o + \frac{\partial G(x,o)}{\partial x}\ Y_o)$$

$$+ (1 + i\kappa)^{-1} G(x,o) n_1(x) n_2(o)\ Y_o$$

$$- (1 + i\kappa)^{-1} \int_{x'=0}^{x} \frac{\partial G}{\partial x}\, n_1 n_2'\, \frac{du'}{dx'}\, dx' - q_o^2 \int_{x'=0}^{x} G\, n_1 n_1'\, u'\, dx'\} \tag{2.25}$$

The statistical average of (2.25) yields:

$$\rho_o(\bar{u} + \overline{n_1 u}) = \rho_o\{\bar{u} + (1 + i\kappa)^{-1} S_1(x)\ Y_o - \int_{x'=0}^{x} (\ (1 + i\kappa)^{-1}\ K_{12}\ \frac{du'}{dx'} + q_o^2\ K_{11}\ u')\ dx'\} \qquad (2.26)$$

where K_{12} has the same expression as in (2.19) and

$$S_1(x) = G(x-x')\ N_{12}\ (x-x')\Big|_{x'=0}$$

$$K_{11}(x-x') = G(x-x')\ N_{11}\ (x-x') \qquad (2.27)$$

Consequently by the substitution of (2.24) and (2.26) in (2.2) the field equation for the averaged displacement becomes:

$$(\frac{1 - N_{22}(o) + i\kappa}{1 + i\kappa})\ \frac{d^2\bar{u}}{dx^2} + q_o^2\ \bar{u} - q_o^2\ S_3\ X_o$$

$$+ (1 + i\kappa)^{-1}\ (\frac{dS_2}{dx} + q_o^2\ (S_1 + S_4)\ )\ Y_o$$

$$+ q_o^2 \int_{x'=0}^{x} (\ (1 + i\kappa)^{-1}\ K_{22}\ \frac{d^2\bar{u}'}{dx'^2} - \frac{2 + i\kappa}{1 + i\kappa}\ K_{12}\ \frac{d\bar{u}'}{dx'} - q_o^2 K_{11}\ \bar{u}')dx' = 0 \qquad (2.28)$$

3. SOLUTION FOR A SINGLE LAYER

One of the aims of the above derivations is to account for random changes in the properties of soils as related to earthquake phenomena. Towards this end, typically for modeling the effect of the local geology, the problem of a single layer which is forced by a harmonic displacement at one face and is traction free at the other is solved. In this case, first the auxiliary initial value problem as defined by the integro-differential equation in (2.28) and the initial conditions $\bar{u}(o) = X_o$, $\frac{d\bar{u}(o)}{dx} = Y_o$ is studied. Once the general solution for the system in (2.28) is obtained for arbitrary X_o and Y_o , the solution to the actual problem is generated by imposing the actual boundary conditions. The Green's function for the problem in (2.11) is:

$$G(x - x') = \sin q_o(x - x')/q_o \tag{3.1}$$

Furhtermore, a common choice for the correlation function is

$$N_{ij}(x - x') = N^o_{ij}\, e^{-|x - x'|/\alpha} \tag{3.2}$$

Such a stochastic process is termed as an "Uhlenbeck-Ornstein Process" [7]. An overall measure of the stochasticity may be taken as $\frac{1}{d} \int_0^d N_{ij}(|x|)dx \approx N^o_{ij}\, \alpha/d$. Since by the construction of the equations in (2.28), $0 \leq x' \leq x$ the absolute value sign in (3.2) may be dropped. In (3.2), although not necessary, the correlation length α is taken to be the same for all (i, j). Thus collecting (2.19), (2.23), (2.27), (3.1) and (3.2) together the kernels and surface correlation terms become:

$$S_1(x) = \frac{1}{q_o} N^o_{12} \sin q_o x \exp(-x/\alpha)$$

$$S_2(x) = N^o_{22} \cos q_o x \exp(-x/\alpha)$$

$$S_3(x) = N^o_{12} \cos q_o x \exp(-x/\alpha)$$

$$S_4(x) = \frac{1}{q_o} N^o_{22} \sin q_o x \exp(-x/\alpha)$$

$$K_{11}(x-x') = \frac{1}{q_o} N^o_{11} \sin q_o(x-x')\exp(-(x-x')/\alpha)$$

$$K_{12}(x-x') = N^o_{12} \cos q_o(x-x')\exp(-(x-x')/\alpha)$$

$$K_{22}(x-x') = \frac{1}{q_o} N^o_{22} \sin q_o(x-x')\exp(-(x-x')/\alpha) \tag{3.3}$$

The form of the integrals in (2.28) being of convolution type, the Laplace transform method is ideally suited for the solution. In fact this is the major motivation for formulating the system as an initial value problem. For the script letters denoting Laplace transforms, e.g.

$$\mathcal{U}(s) = \int_0^\infty \bar{u}(x)\exp(-sx)\, dx \tag{3.4}$$

(2.28) yields:

$$\mathcal{U}(s) = \frac{F(s)X_o + \mathcal{G}(s)Y_o}{\mathcal{D}(s)} \tag{3.5}$$

where

$$F(s) = \left(\frac{1 - N^o_{22} + i\kappa}{1 + i\kappa}\right)s + q_o^2\, S_3 + q_o^2(1 + i\kappa)^{-1} s\, K_{22} - \frac{2 + i\kappa}{1 + i\kappa}\, K_{12} \tag{3.6}$$

$$G(s) = 1 + (1 + i\kappa)^{-1}(-sS_2 - q_o^2(S_1 + S_4 - K_{22})\ )$$

$$D(s) = \left(\frac{1 - N^o_{22} + i\kappa}{1 + i\kappa}\right)s^2 + q_o^2 - q_o^4\, K_{11} + (1 + i\kappa)^{-1} q_o^2 s^2 K_{22} - \frac{2 + i\kappa}{1 + i\kappa}\, s\, K_{12}$$

In view of (3.3), with $\beta = 1/\alpha$,

$$S_1 = N^o_{12}/d(s)$$

$$S_2 = N^o_{22}(s + \beta)/d(s)$$

$$S_3 = N^o_{12}(s + \beta)/d(s)$$

$$S_4 = N^o_{22}/d(s)$$

$$K_{11} = N^o_{11}/d(s)$$

$$K_{12} = N^o_{12}(s + \beta)/d(s)$$

$$K_{22} = N^o_{22}/d(s)$$

$$d(s) = (s + \beta)^2 + q_o^2 \tag{3.7}$$

Substitution of (3.7) into (3.6) yields:

$$F(s) = \left(\frac{1-N^{o}_{22}+i\kappa}{1+i\kappa}\right)s\left(\,(s+\beta)^{2}+q_{o}^{2}\right) - N^{o}_{12}\left(\frac{2+i\kappa}{1+i\kappa} - q_{o}^{2}\right)(s+\beta)$$

$$+\, q_{o}^{2}(1+i\kappa)^{-1}N^{o}_{22}\;s$$

$$G(s) = \left(\,(s+\beta)^{2}+q_{o}^{2}\right) - (1+i\kappa)^{-1}\left(N^{o}_{22}s(s+\beta)+q_{o}^{2}N^{o}_{12}\right)$$

$$\mathcal{D}(s) = \left(\frac{1-N^{o}_{22}+i\kappa}{1+i\kappa}\;s^{2}+q_{o}^{2}\right)\left(\,(s+\beta)^{2}+q_{o}^{2}\right)$$

$$-\,\frac{2+i\kappa}{1+i\kappa}\,N^{o}_{12}\;s(s+\beta) + (1+i\kappa)^{-1}q_{o}^{2}N^{o}_{22}s^{2} - q_{o}^{4}N^{o}_{11} \qquad (3.8)$$

Above it is seen that $F(s)$, $G(s)$ and $\mathcal{D}(s)$ are respectively polynomials of degree 3 , 2 and 4 in s . Thus once the poles s_k ($k = 1,2,3,4$) (i.e. the zeros of $\mathcal{D}(s)$) are determined, the inverse transform of (3.5) is readily obtained [9]. In this case,

$$\bar{u}(x) = X_{o}\sum_{k=1}^{u} A_{k}e^{s_{k}x} + Y_{o}\sum_{k=1}^{4} B_{k}e^{s_{k}x} \equiv X_{o}f(x) + Y_{o}g(x) \qquad (3.9)$$

where

$$A_{k} = \left.\frac{F(s)}{\frac{d}{ds}\mathcal{D}(s)}\right|_{s_{k}} \qquad\qquad B_{k} = \left.\frac{G(s)}{\frac{d}{ds}\mathcal{D}(s)}\right|_{s_{k}}$$

$$f(x) = \sum_{k=1}^{4} A_{k}e^{s_{k}x} \qquad\qquad g(x) = \sum_{k=1}^{4} B_{k}e^{s_{k}x} \qquad (3.10)$$

Due to the construction, a check of the caluclations is obtained by the considerations

$$\bar{u}\Big|_{x=0} = X_o\sum_{k=1}^{4} A_k + Y_o\sum_{k=1}^{4} B_k = X_o f(o) + Y_o g(o) = X_o$$

$$\frac{d\bar{u}}{dx}\Big|_{x=0} = X_o\sum_{k=1}^{4} s_k A_k + Y_o\sum_{k=1}^{4} s_k B_k = X_o \frac{df(o)}{dx} + Y_o \frac{dg(o)}{dx} = Y_o \qquad (3.11)$$

i.e.

$$f(o) = \sum_{k=1}^{4} A_k = 1 \qquad\qquad g(o) = \sum_{k=1}^{4} B_k = 0$$

$$\frac{df(o)}{dx} = \sum_{k=1}^{4} s_k A_k = 0 \qquad\qquad \frac{dg(o)}{dx} = \sum_{k=1}^{4} s_k B_k = 1 \qquad (3.12)$$

For calculating the stress according to (2.18), first note the integrals

$$I_{1k}(x) = e^{-s_k x}\int_{x'=0}^{x} K_{22}(x-x')e^{s_k x'}\, dx'$$

$$= \frac{N_{22}^{o}}{(\ (s_k+\beta)^2+q_o^2)}\{1 - e^{-(s_k+\beta)x}(\frac{1}{q_o}(s_k+\beta)\sin q_o x + \cos q_o x)\}$$

$$I_{2k}(x) = e^{-s_k x}\int_{x'=0}^{x} K_{12}(x-x')e^{s_k x'}\, dx'$$

$$= \frac{N_{12}^{o}}{(\ (s_k+\beta)^2+q_o^2)}\{(s_k+\beta) - e^{-(s_k+\beta)x}(\ (s_k+\beta)\cos q_o x - q_o \sin q_o x\} \qquad (3.13)$$

In the integrals above, obviously

$$I_{1k}(o) = I_{2k}(o) = 0 \tag{3.14}$$

for the upper and lower limits being the same. Then by the use of the above integrals in (2.18), $\bar{\tau}$ is found to be:

$$\bar{\tau} = \mu_0(1+i\kappa)[X_0\ \phi(x) + Y_0\psi(x)] \tag{3.15}$$

where

$$\phi(x) = \left(\frac{1-N_{22}^{o}+i\kappa}{1+i\kappa}\right)\frac{df}{dx}(x) + q_0^2\sum_{k=1}^{4}[(1+i\kappa)^{-1}s_k I_{1k}(x) - I_{2k}(x)]A_k e^{s_k x}$$

$$\psi(x) = \left(\frac{1-N_{22}^{o}+i\kappa}{1+i\kappa}\right)\frac{dg(x)}{dx} + q_0^2\sum_{k=1}^{4}[(1+i\kappa)^{-1}s_k I_{1k} - I_{2k}(x)]B_k e^{s_k x} \tag{3.16}$$

In view of (3.12) and (3.14),

$$\phi(0) = 0 \qquad\qquad \psi(o) = 1 \tag{3.17}$$

and

$$\bar{\tau}\Big|_{x=0} = \mu_0(1 + iK)Y_0 \tag{3.18}$$

Consider now a layer where the displacement is prescribed at one face while the stress vanishes at the other. In this problem the choice of the coordinates as $x = 0$ for the face where zero stress is prescribed proves more convenient for it leads to simpler expressions. (Clearly the physical values are the same in all coordinate systems.) Thus, we take the boundary conditions as:

$$\bar{\tau}\Big|_{x=0} = 0 \qquad\qquad \bar{u}\Big|_{x=d} = 0 \tag{3.19}$$

The arbitrary coefficients X_o and Y_o are determined by the use of (3.19) in the expressions in (3.9) and (3.15) as:

$$Y_o = 0 \qquad\qquad X_o = \frac{a}{f(d)} \tag{3.20}$$

Consequently, the solution satisfying the actual boundary conditions becomes

$$\bar{u} = a\,\frac{f(x)}{f(d)}$$

$$\bar{\tau} = \mu_o(1+i\kappa)a\,\frac{\phi(x)}{f(d)} \tag{3.21}$$

Similarly the amplification coefficient is:

$$A \equiv \frac{u(o)}{u(d)} = \frac{1}{f(d)} \tag{3.22}$$

The preceeding equations take a particularly simple form and the poles s_k can be found explicitly when only the density is stochastic.

Thus for $n_2 = 0$ we have $N_{22} = N_{12} = 0$ and consequently $S_1 = S_2 = S_3 = S_4 = K_{12} = K_{22} = I_{1k} = I_{2k} = 0$. In this case, (3.8) reduces to:

$$F(s) = ((s+\beta)^2 + q_o^2)s$$

$$G(s) = F(s)/s$$

$$\mathcal{D}(s) = (s^2+q_o^2)((s+\beta)^2+q_o^2) - q_o^4 N_{11}^o \tag{3.23}$$

Furthermore it is observed that $\mathcal{D}(s)$ is a biquadratic in $(s + \beta/2)$. Indeed for

$$\mathcal{D}(s) = (s+\beta/2)^4 + 2(q_o^2-\beta^2/4)(s+\beta/2)^2+((q_o^2+\beta^2/4)^2 - q_o^4N_{11}^o) \tag{3.24}$$

the poles in (3.24) are readily obtained by solving $\mathcal{D}(s) = 0$ for $(s+\beta/2)$. The poles therefore are:

$$s_1 = ir_o - \beta/2 \qquad s_2 = ir_o^* - \beta/2$$

$$s_3 = -ir_o - \beta/2 \qquad s_4 = -ir_o^* - \beta/2 \tag{3.25}$$

where

$$r_o = q_o((1-\beta^2/4q_o^2) + i(\beta^2/q_o^2 - N_{11}^o)^{1/2})^{1/2} \tag{3.26}$$

Similarly by the use of (3.23) in (3.10):

$$A_k = \frac{1}{4}\frac{s_k(\ (s_k+\beta)^2+q_o^2}{(s_k+\beta/2)(\ (s_k+\beta/2)^2+(q_o^2-\beta^2/4)\ )} \qquad B_k = A_k/s_k \tag{3.25}$$

Furthermore, for $I_{1k} = I_{2k} = 0$, (3.16) reduces to

$$\phi(x) = \frac{df(x)}{dx} \qquad \psi(x) = \frac{dg(x)}{dx} \tag{3.26}$$

i.e.

$$\bar{\tau} = \mu_o(1+i\kappa)(X_o\ \frac{df(x)}{dx} + Y_o\ \frac{dg(x)}{dx}) \equiv \mu_o(1+i\kappa)\ \frac{d\bar{u}(x)}{dx} \tag{3.27}$$

4. MULTILAYER SYSTEM

The extension of the solution of the previous section to a multilayer system is achieved through the usual transfer matrix formalism [8]. Let $k = 2,3, \ldots n$ denote the layers and $k = 2,3 \ldots n-1$ the interfaces and $k = 1$ and n the lower and upper faces of a multilayer system. Thus, typically for the k^{th} layer, the lower face is $(k-1)$ and the upper face is k. In this case from (3.9) and (3.15) the displacement and stress at the lower face, by construction are:

$$\bar{U}_{k-1} \equiv \bar{u}(o) = X_{k-1}$$

$$\bar{T}_{k-1} \equiv \bar{\tau}(o) = (1+i\kappa)\mu_o \quad Y_{k-1} \tag{4.1}$$

Similarly, on the upper face of the layer

$$\bar{U}_k \equiv \bar{u}(d) = X_{k-1} \; f(d) + Y_{k-1} g(d)$$

$$\bar{T}_k \equiv \bar{\tau}(d) = (1+i\kappa)\mu_o \quad X_{k-1} \; \phi(d)+(1+i\kappa) \; \mu_o \quad Y_{k-1} \; \psi(d) \tag{4.2}$$

The substitution of X_{k-1} and Y_{k-1} in (4.2) in terms of the expressions in (4.1) yields the transfer matrix equation:

$$\begin{bmatrix} \bar{U}_k \\ \\ \bar{T}_k \end{bmatrix} = \begin{bmatrix} f(d) & g(d)/\mu_o(1+i\kappa) \\ \\ \mu_o(1+i\kappa)\phi(d) & \psi(d) \end{bmatrix} \begin{bmatrix} \bar{U}_{k-1} \\ \\ \bar{T}_{k-1} \end{bmatrix} \tag{4.3}$$

By this notation the continuity in the displacements and stresses across the interfaces is achieved automatically. The information from the first face to the n^{th} face can be carried by successive matrix multiplications [8]. Rewriting (4.3) in more compact form with $\underline{Z}_{k-1} = (\bar{U}_{k-1}, \bar{T}_{k-1})$ and $\underline{Z}_k = (\bar{U}_k, \bar{T}_k)$, one has

$$\underline{Z}_k = \underline{\underline{A}}_k \cdot \underline{Z}_{k-1} \tag{4.4}$$

Successive matrix multiplications yield:

$$\underline{Z}_k = \underline{\underline{B}}_k \cdot \underline{Z}_1 \qquad k = 2,3, \ldots \tag{4.5}$$

where $\underline{\underline{B}}_1$ = Identity matrix and

$$\underline{\underline{B}}_k = \underline{\underline{A}}_k \cdot \underline{\underline{B}}_{k-1} \qquad k = 2,3, \ldots \tag{4.6}$$

Consequently on the last interface,

$$\begin{bmatrix} \bar{U}_n \\ \\ \bar{T}_n \end{bmatrix} = \begin{bmatrix} R_{11,n} & R_{12,n} \\ \\ R_{21,n} & R_{22,n} \end{bmatrix} \begin{bmatrix} \bar{U}_1 \\ \\ \bar{T}_1 \end{bmatrix} \tag{4.7}$$

For $\bar{U}_1 = a$, the prescribed bedrock displacement and $\bar{T}_n = 0$, $\bar{T}_1$ is solved from the last equation above and with $\underline{Z}_1 = (\bar{U}_1, \bar{T}_1)$ thus being determined, $\underline{Z}_k = (\bar{U}_k, \bar{T}_k)$ is obtained automatically by the use of (4.5).

For this problem with $(n-1)$ layers, there are normally $2(n-1)$ integration constants which are determined by the $2(n-2)$ continuity conditions at the interfaces and the 2 prescribed boundary conditions one at each of the faces 1 and n. The transfer matrix formalism avoids the solution of the $2(n-1)$ equations for the integration constants and provides the solution from a single equation for a single unknown.

In this case the amplification coefficient is obtained as:

$$A = U_n/U_1 \tag{4.8}$$

Figure (2b) shows the amplification response spectrum for a multilayer system taken as homogeneous and stochastic. The layer thicknesses, average shear modulus and density for each layer are taken from actual data for a site in California, USA [10].

5. ARBITRARY INPUT

The response of a single or multilayer system to an arbitrary forcing at its boundary is readily found by Fourier analysis. Let the forcing at the face $k = 1$ be:

$$F(t) = \sum_{\ell=-L}^{L} a_\ell e^{i\omega_\ell t} \tag{5.1}$$

Here the index ℓ is introduced to differentiate between the various frequencies $\omega = \omega_\ell$. For the particular mode ℓ , the displacement amplification factor A_ℓ is obtained from (3.22) for a single layer and (4.8) for a multilayer system. Thus the response of the system becomes:

$$G(t) = \sum_{\ell=-L}^{L} a_\ell A_\ell e^{i\omega e t} \tag{5.2}$$

Figures (3 -5) show various responses. In Figure 3, the input function $F(t)$ is of Gaussian distribution given by

$$F(t) = Be^{-t^2/2} \tag{5.3}$$

where B is the amplitude of the impulse and the response is given by $G(t)/B$. Figures 4 and 5 show respectively the responses to an actual earthquake (N21E component of the 1952 Taft strong motion record) of a single and multilayer system. The multilayer system is that shown in Figure 2a and the parameters for the top layer in this latter configuration are used for the single layer calculations in Figures 3 and 4.

6. DISCUSSION

In the solution of the differential equations with stochastic coefficients, one of the obvious attempts would be to apply a perturbation analysis, assuming for small deviations from the mean. However such a perturbation is of a singular kind [1] and diverges through the generation of secular terms. Instead, the integro-differential equation of the type in (2.28) needs to be solved. This amounts to the incorporation of the long range interactions and though they may be small are not of perturbative nature. Formally, the equations are of the same structure as in the nonlocal continuum theories. Conceptually also, the correlations are indeed expressions of nonlocal properties. An important consequence of the integro-differential equation describing the mean motion is that this latter is different from the solution of the equation with the mean properties. For the homogeneous elastic case based on the average material coefficients, Eq. (3.5) in the Laplace transform domain has only the poles at $s = \pm ik_o$. These pure imaginary numbers generate trigonometric functions which display resonance infinities at $k_o d = \frac{\pi}{2}(2n + 1)$. In the problem with the integro-differential equation however, the solution in the Laplace transform domain has four complex poles [9]. The real parts of the roots amounts to decays and as a result the solutions are not pure trigonometric functions and the resonances are removed. Physically this is a situation analogous to a viscoelastic medium where the real parts of the poles characterize the damping. However, the equations for the mean field display a decay of a different origin. The wave energy through random mode coupling is distributed into various modes to the expense of the energy in the mean motion. This energy decay in the mean is physically a manifestation of the scattering of the waves through the inhomogeneities in the medium. In Figure 1, it is observed that the effect of the inhomogeneities gets stronger as the frequencies increase and that the uncertainties get large at "resonant" frequencies. Random mode coupling is not confined to the coupling of modes of the same type e.g., shear modes. The three dimensional equations show that there are no pure shear and volumetric waves as in homogeneous media and these are coupled through random mode mixing. In certain measurements indeed, the arrival of the volumetric, shear and Rayleigh surface waves are not detected separately, but rather a combined wave group is observed. From the study of Figures 1a-c it is seen that the stochastic changes in the shear modulus shift the spectra to the left, while those in the density shift the spectra to the right, as compared with the homogeneous models.

The effect of the stochastic changes in increasing the damping characteristics is displayed in Figs.3-5. The figures show the displacement response at the top of a layer subjected to a Gaussian and an earthquake record (N21E component of the 1952 Taft strong motion record) [11] as input at the bottom. The curve with the broken lines is the response of a homogeneous layer with hysteretic damping coefficient κ of 0.02 or 0.05. The curve drawn in solid line corresponds to a stochastic layer with the same average properties of a layer as in the preceding case. The coefficients describing the stochasticity are shown in the figure legends. It is seen that the waves in the stochastic case are damped out more rapidly than those in the homogeneous case. The significance of this observation is in that the value for the damping coefficient obtained in the laboratory corresponds to a macroscapically homogeneous sample. In the field however the stochastic inhomogeneities are always present and are likely to increase the damping characteristics in a significant way. The method presented here provides a tool for accounting for this phenomenon.

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FIGURE CAPTIONS

Figure 1 Effect of the stochastic correlation parameters on the mean displacement amplification spectrum for a single layer.
(a) $N^o_{22} = N^o_{12} = 0$, $\beta = 5$; (b) $N^o_{11} = N^o_{12} = 0$, $\beta = 5$;
(c) $N^o_{11} = N^o_{12} = N^o_{22}$, $\beta = 5$.

Figure 2a Data for a multilayer system

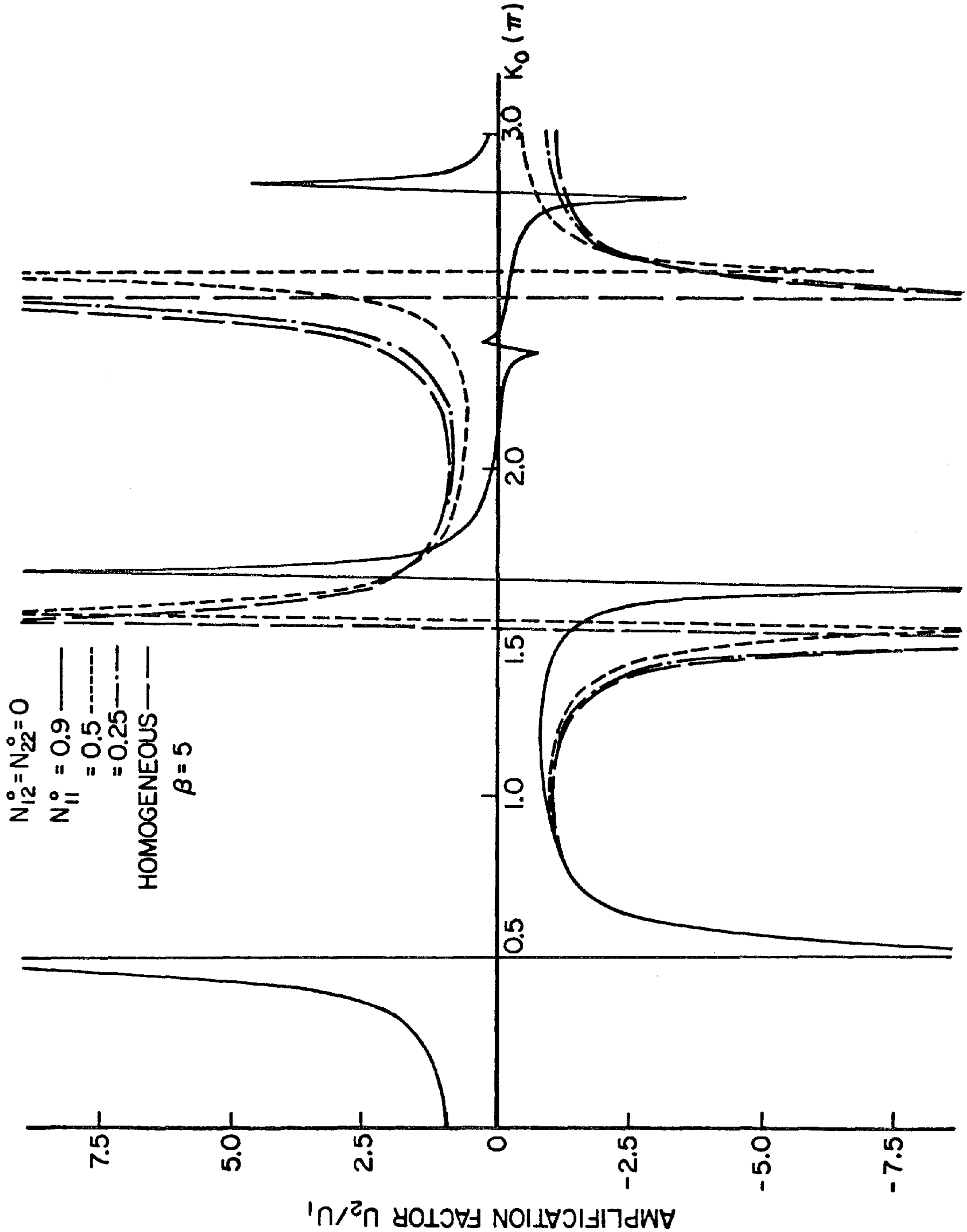
Figure 2b Effect of the stochastic correlation parameters on the mean displacement amplification spectrum for a multilayer system.
$N^o_{11} = N^o_{12} = N^o_{22} = 1/4$, $\beta = 5$.

Figure 3 Response of a single layer to a Gaussian pulse (a) $N^o_{11} = 0.5$, $\beta = 1.0$ $\kappa = 0.02$; (b) $N^o_{11} = 0.5$, $\beta = 5$ $\kappa = 0.05$;
(c) $N^o_{11} = 0.9$, $\beta = 1.0$, $\kappa = 0.02$; (d) $N^o_{11} = 0.9$, $\beta = 5$, $\kappa = 0.02$

Figure 4 Response of a single layer to an earthquake input (a) $N^o_{11} = 0.9$, $\beta = 1.0$, $\kappa = 0.02$; (b) $N^o_{11} = 0.9$, $\beta = 5$, $\kappa = 0.02$
(c) $N^o_{11} = 2.0$, $\beta = 5$, $\kappa = 0.02$

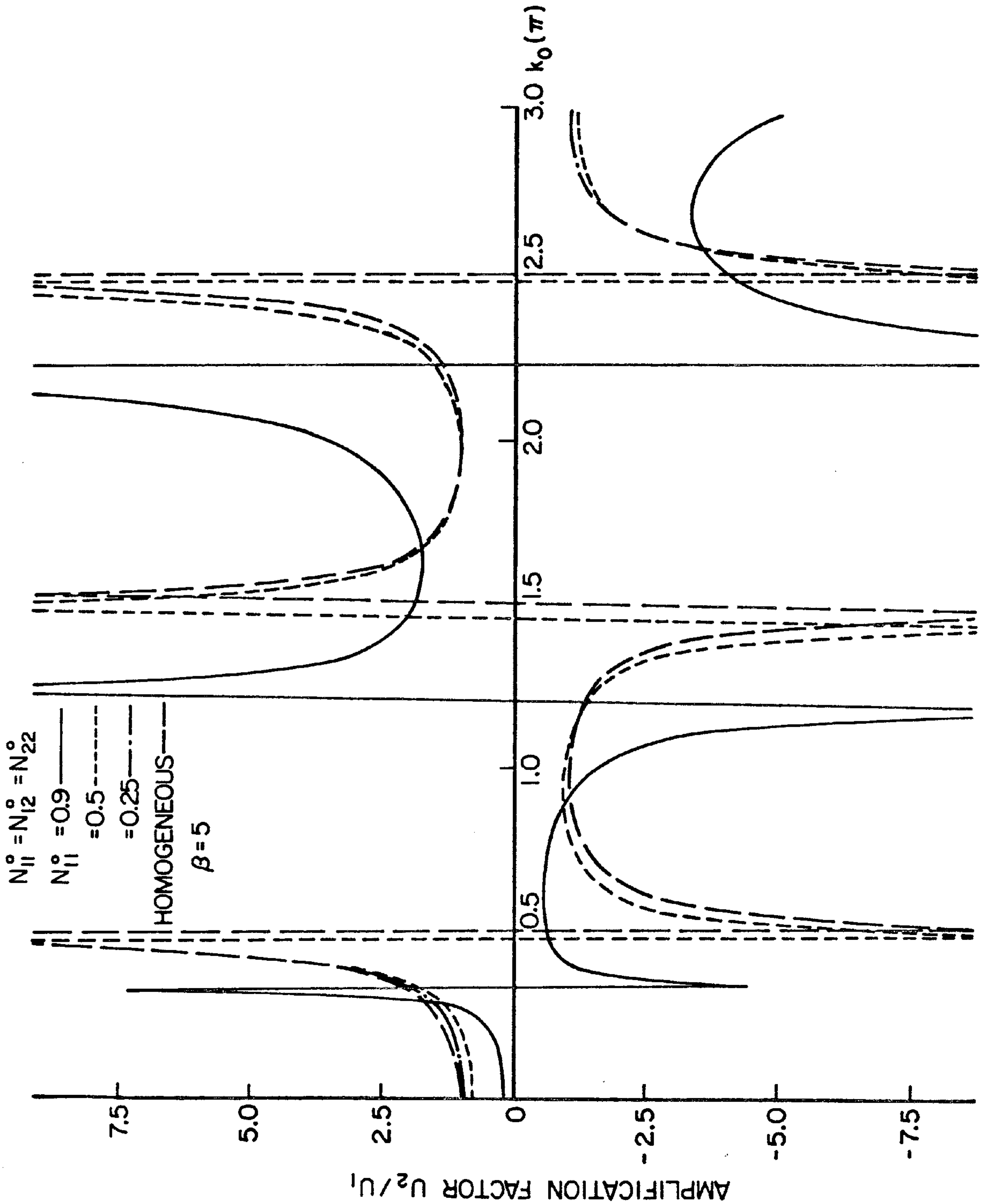
Figure 5 Response of a multilayer system to an earthquake input.
$N^o_{11} = 2.0$, $\kappa = 0.02$, $\beta = 5$

Figure 1a



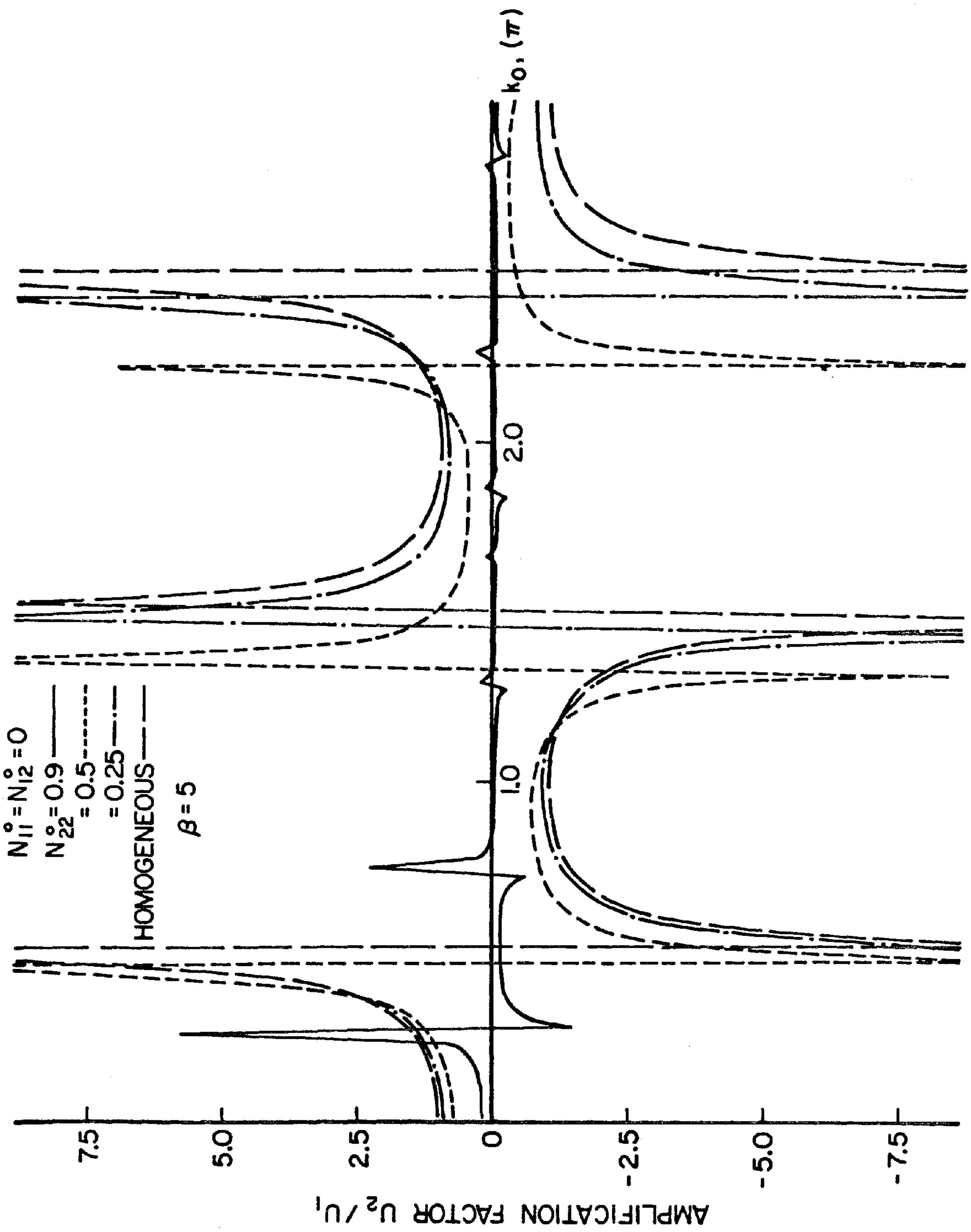
$N^{\circ}_{12} = N^{\circ}_{22} = 0$
$N^{\circ}_{11} = 0.9$
$= 0.5$
$= 0.25$
HOMOGENEOUS
$\beta = 5$
$K_0\ (\pi)$
0.5
1.0
1.5
2.0
3.0
7.5
5.0
2.5
0
-2.5
-5.0
-7.5
AMPLIFICATION FACTOR U_2/U_1

Figure 1c



$N^o_{11} = N^o_{12} = N^o_{22}$
N^o_{11} =0.9
=0.5
=0.25
HOMOGENEOUS
β = 5
0.5
1.0
1.5
2.0
2.5
3.0 $k_o(\pi)$
7.5
5.0
2.5
0
-2.5
-5.0
-7.5
AMPLIFICATION FACTOR U_2/U_1

Figure 1b


$N_{11}^{\circ} = N_{12}^{\circ} = 0$
$N_{22}^{\circ} = 0.9$
$= 0.5$
$= 0.25$
HOMOGENEOUS
$\beta = 5$
$k_0, (\pi)$
1.0
2.0
7.5
5.0
2.5
0
-2.5
-5.0
-7.5
AMPLIFICATION FACTOR U_2/U_1

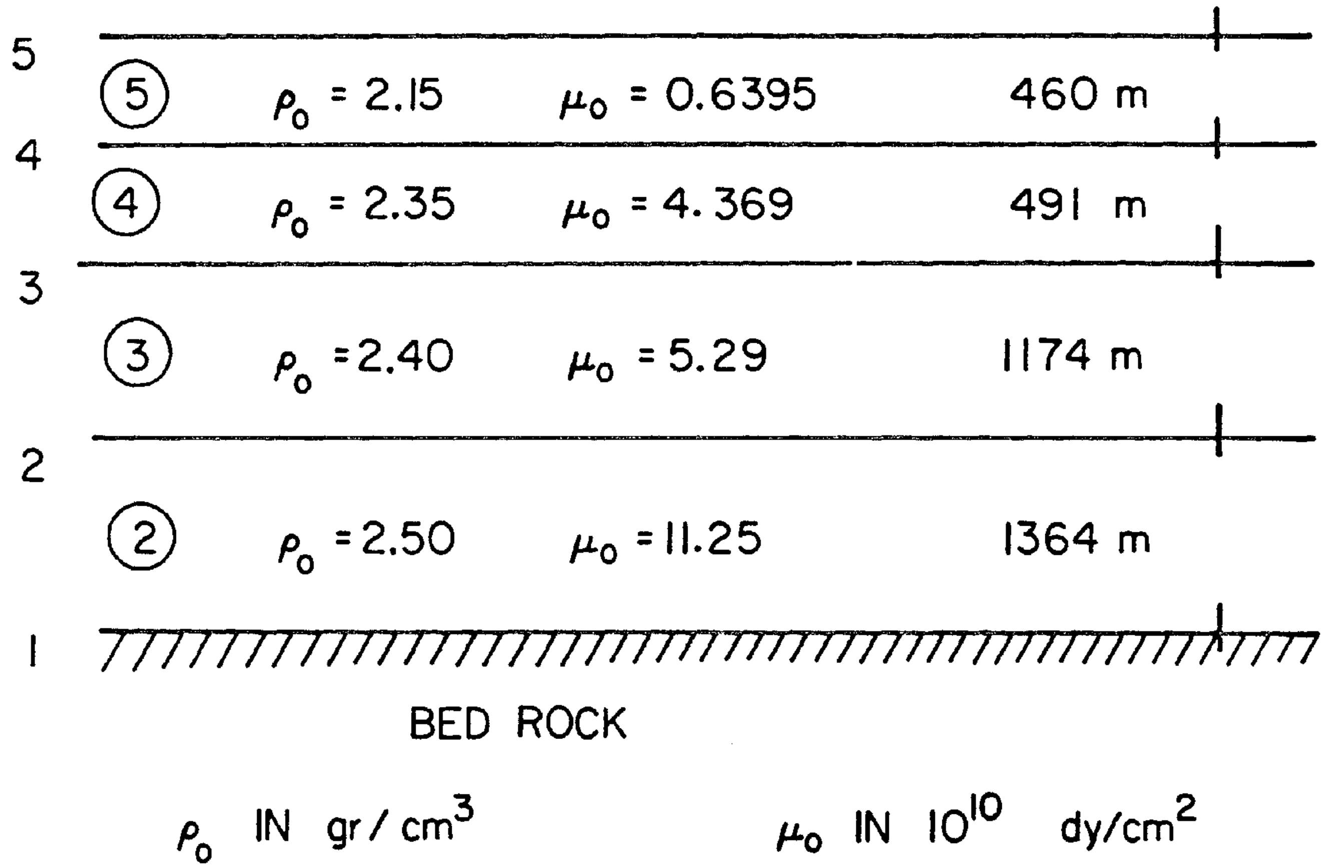
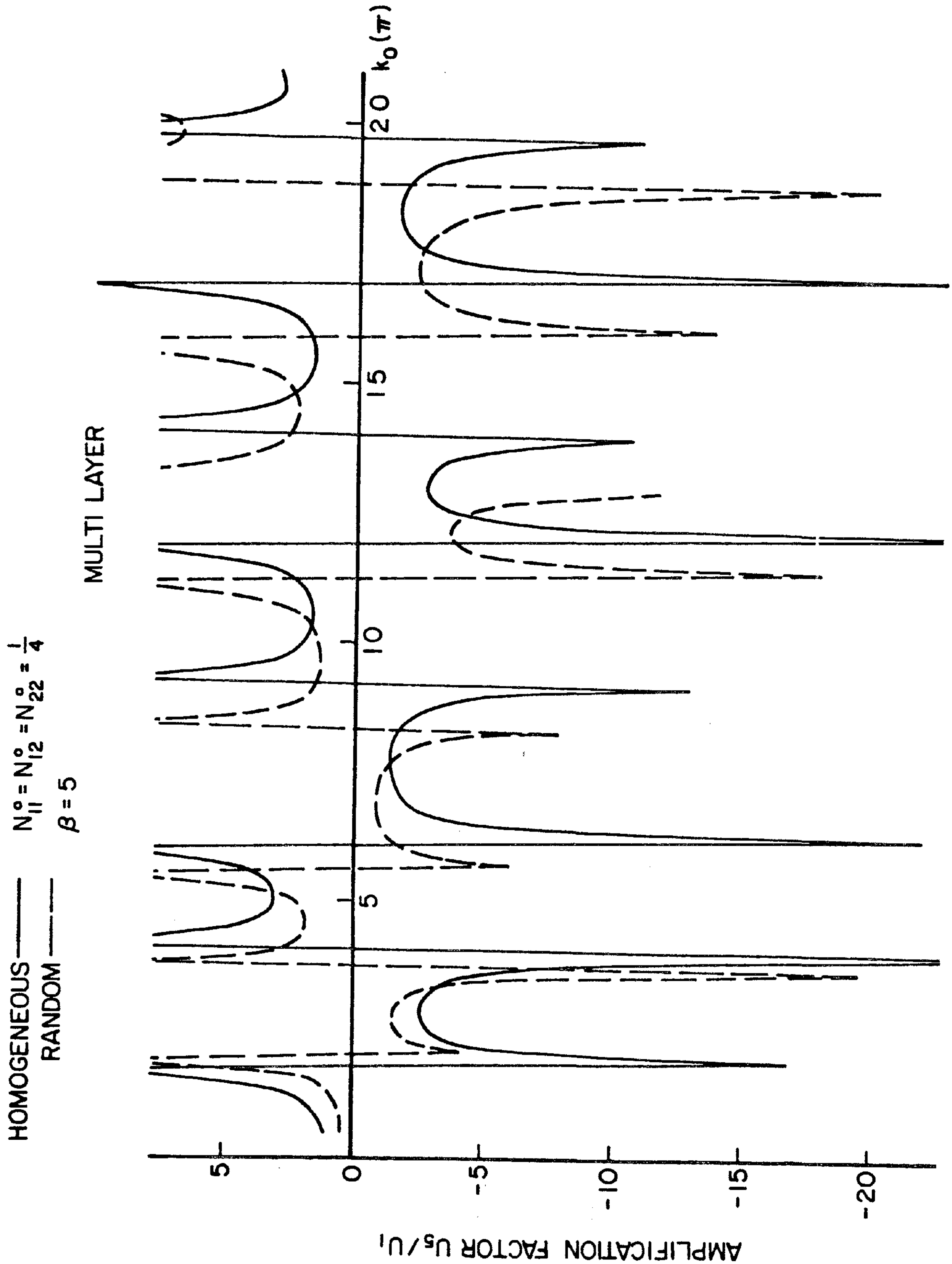

5
(5) ρ_o = 2.15 μ_o = 0.6395 460 m
4
(4) ρ_o = 2.35 μ_o = 4.369 491 m
3
(3) ρ_o =2.40 μ_o = 5.29 1174 m
2
(2) ρ_o =2.50 μ_o =11.25 1364 m
1
BED ROCK
ρ_o IN gr/cm^3
μ_o IN 10^{10} dy/cm^2

FIGURE 2a

Figure 2b



HOMOGENEOUS
RANDOM
$N^{o}_{11} = N^{o}_{12} = N^{o}_{22} = \frac{1}{4}$
$\beta = 5$
MULTI LAYER
$k_o(\pi)$
5
10
15
20
5
0
-5
-10
-15
-20
AMPLIFICATION FACTOR U_s/U_1

Figure 3a

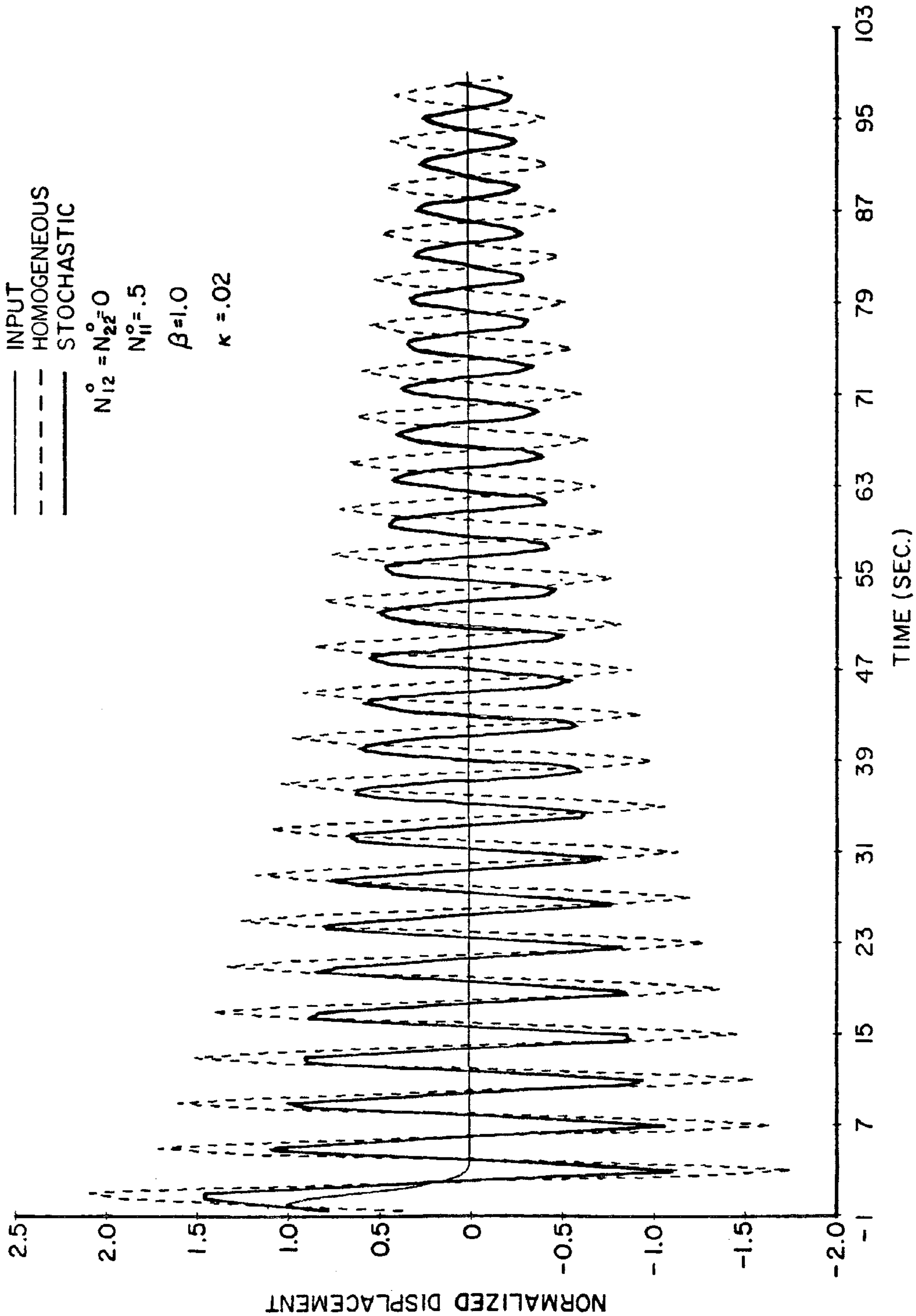

INPUT
HOMOGENEOUS
STOCHASTIC
$N^{o}_{12} = N^{o}_{22} = O$
$N^{o}_{11} = .5$
$\beta = 1.0$
$\kappa = .02$
TIME (SEC.)
NORMALIZED DISPLACEMENT
1
7
15
23
31
39
47
55
63
71
79
87
95
103
2.5
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0

Figure 3b

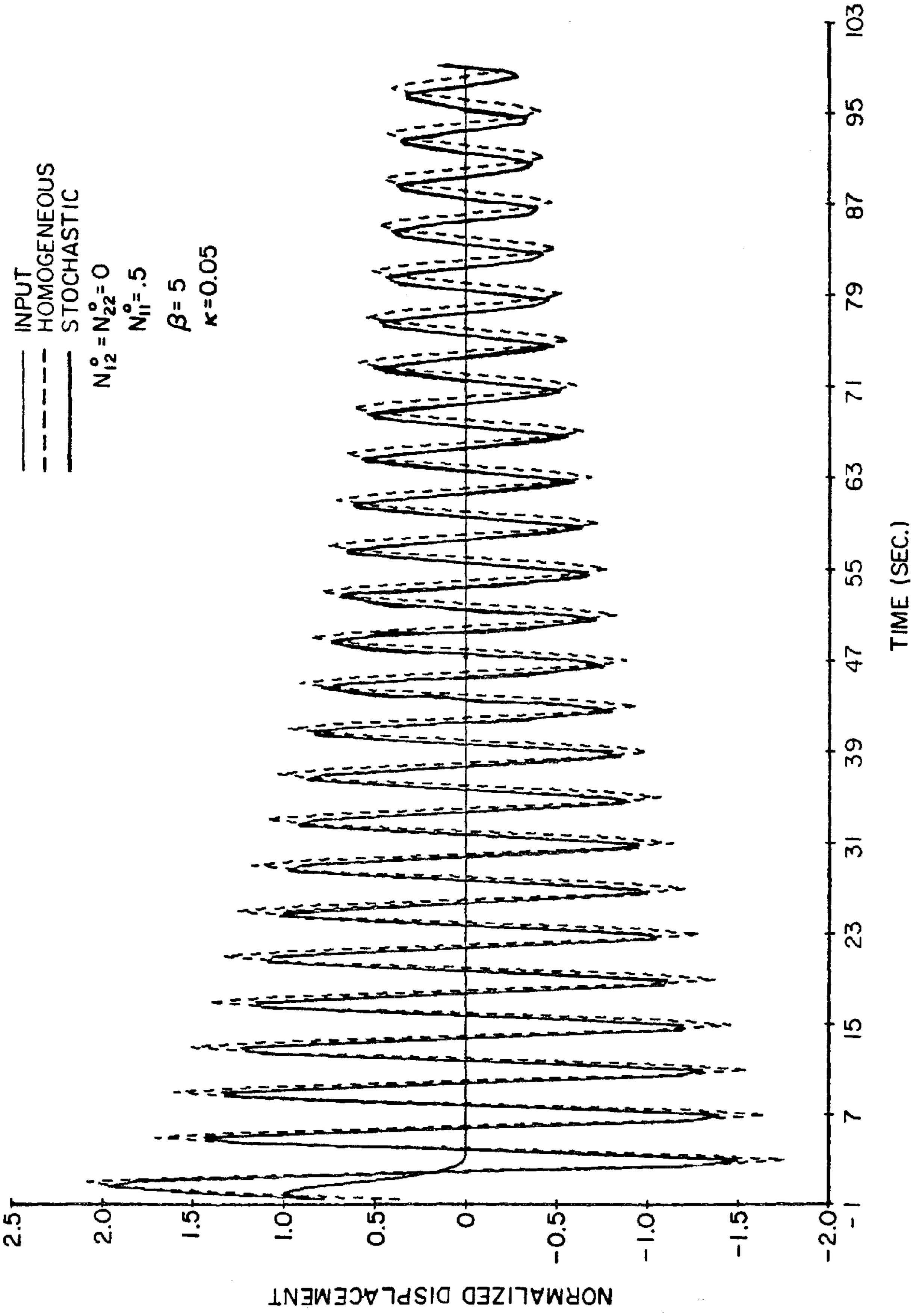

INPUT
HOMOGENEOUS
STOCHASTIC
$N^{o}_{12} = N^{o}_{22} = 0$
$N^{o}_{11} = .5$
$\beta = 5$
$\kappa = 0.05$
TIME (SEC.)
NORMALIZED DISPLACEMENT
1
7
15
23
31
39
47
55
63
71
79
87
95
103
2.5
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0

Figure 3c

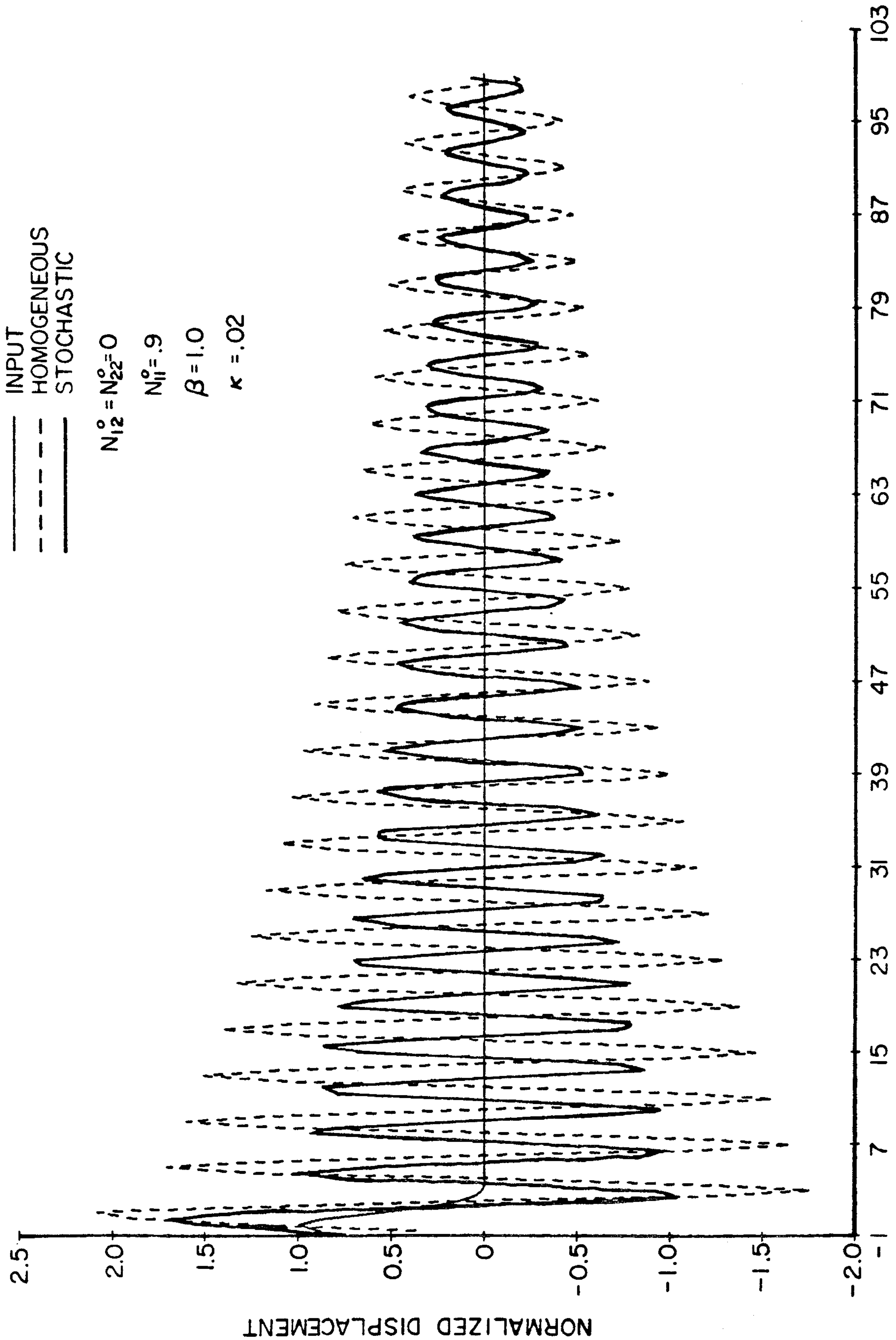
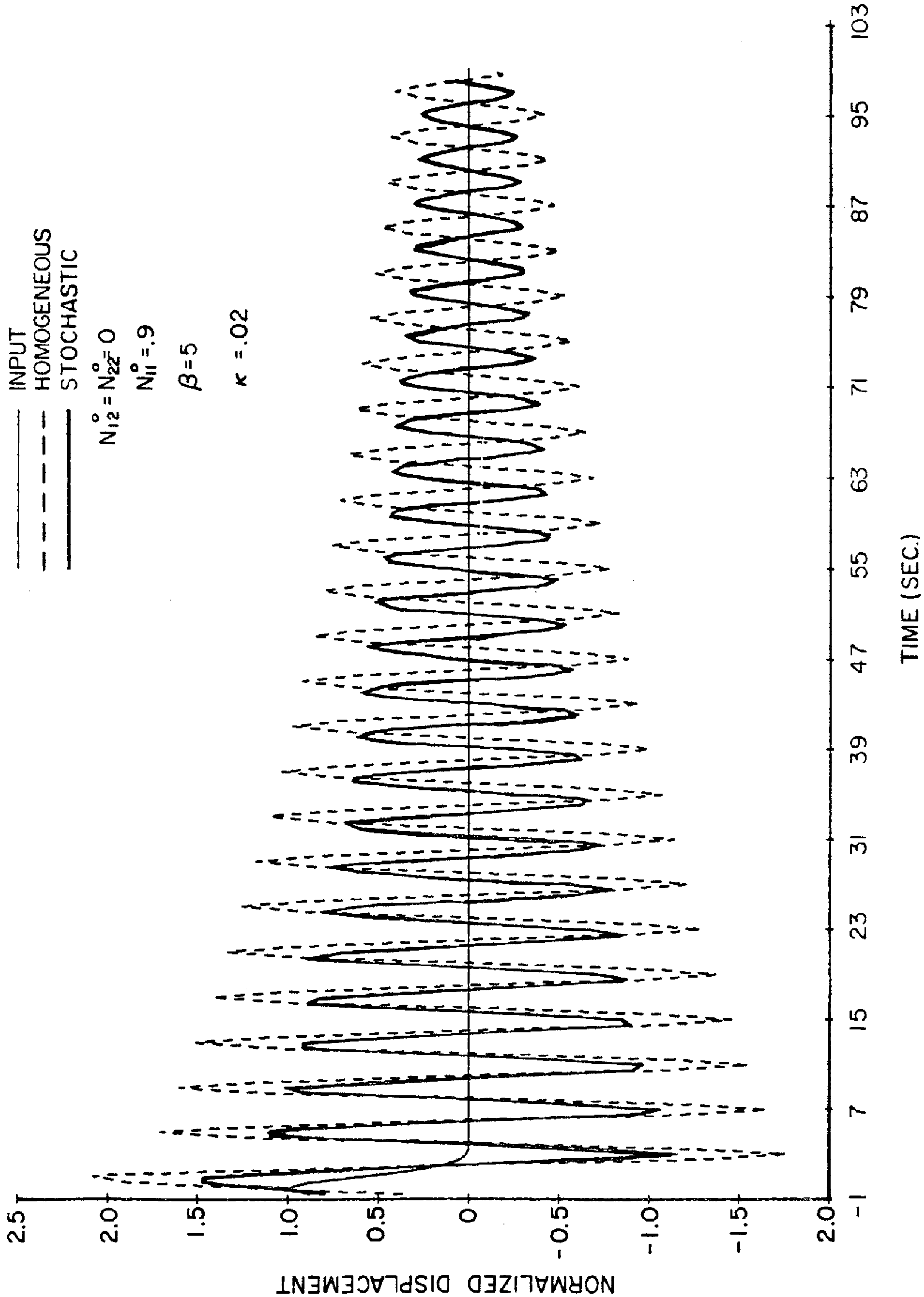

INPUT
HOMOGENEOUS
STOCHASTIC
$N^o_{12} = N^o_{22} = 0$
$N^o_{11} = .9$
$\beta = 1.0$
$\kappa = .02$
TIME (SEC.)
1
7
15
23
31
39
47
55
63
71
79
87
95
103
NORMALIZED DISPLACEMENT
2.5
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0

Figure 3d



INPUT
HOMOGENEOUS
STOCHASTIC
$N^{\circ}_{12} = N^{\circ}_{22} = 0$
$N^{\circ}_{11} = .9$
$\beta = 5$
$\kappa = .02$
NORMALIZED DISPLACEMENT
2.5
2.0
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
2.0
TIME (SEC.)
-1
7
15
23
31
39
47
55
63
71
79
87
95
103

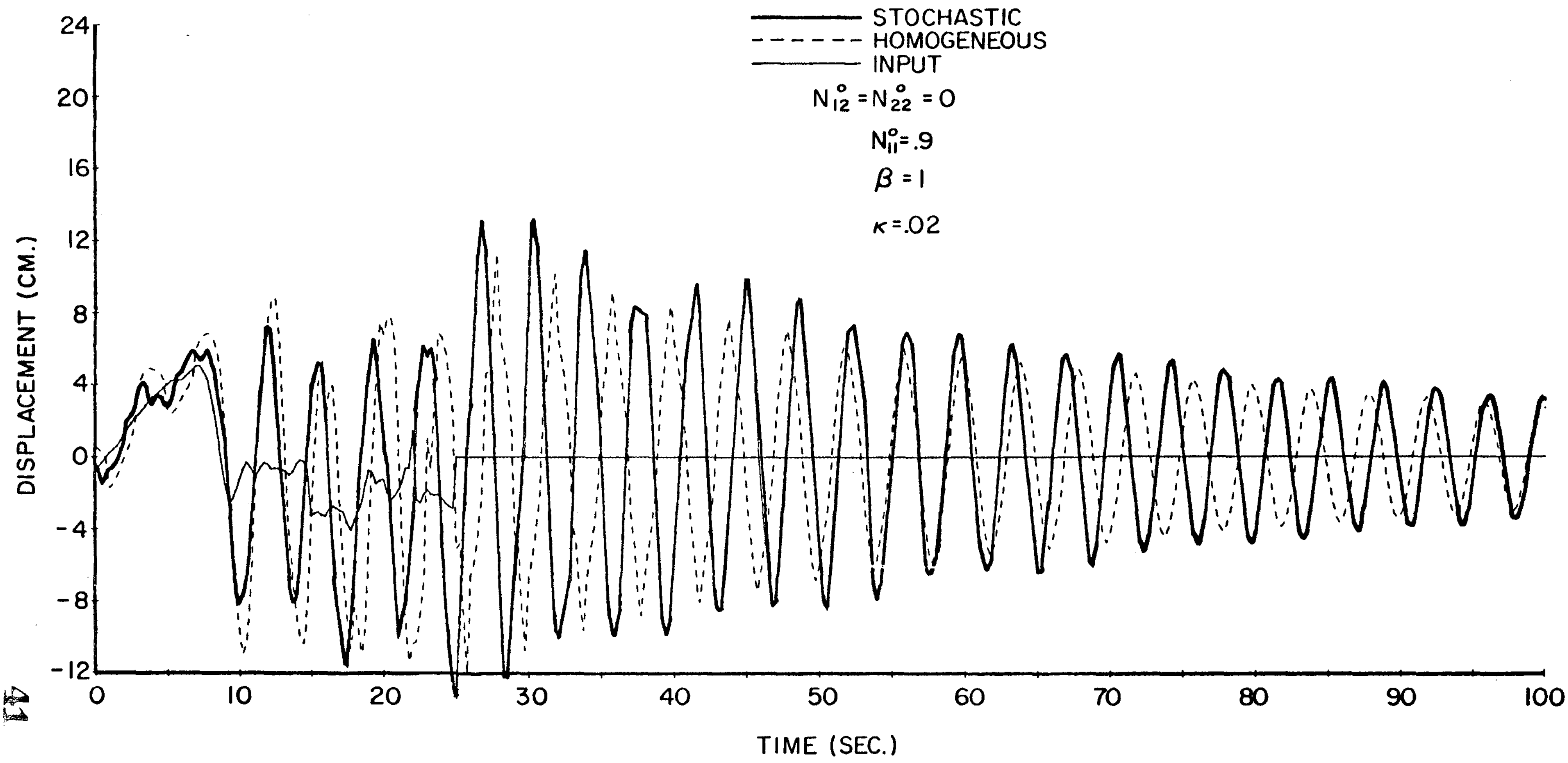
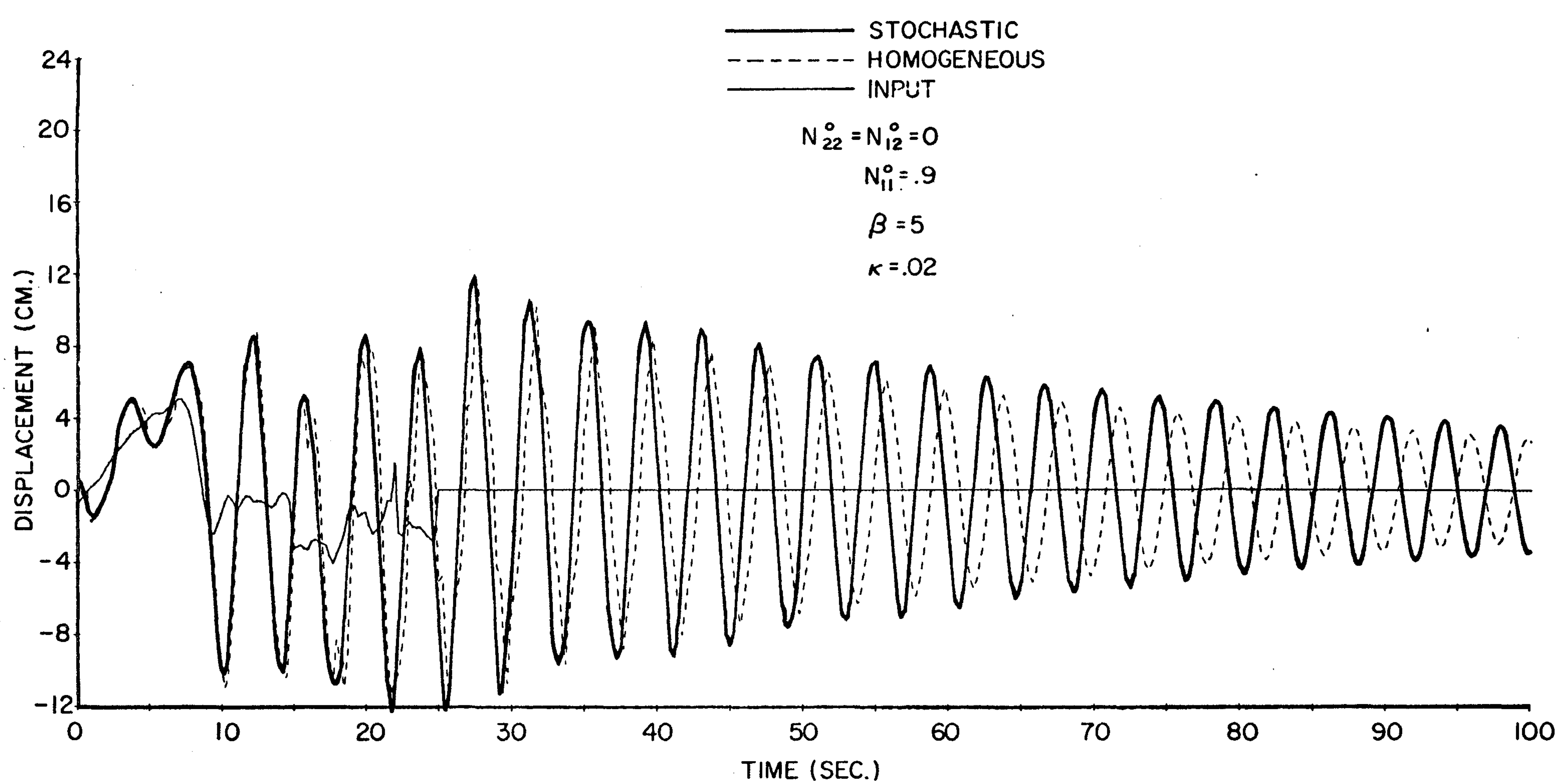

STOCHASTIC
HOMOGENEOUS
INPUT
$N^o_{12} = N^o_{22} = 0$
$N^o_{11} = .9$
$\beta = 1$
$\kappa = .02$
DISPLACEMENT (CM.)
24
20
16
12
8
4
0
-4
-8
-12
0
10
20
30
40
50
60
70
80
90
100
TIME (SEC.)

FIGURE 4a



STOCHASTIC
HOMOGENEOUS
INPUT
$N^{\circ}_{22} = N^{\circ}_{12} = 0$
$N^{\circ}_{11} = .9$
$\beta = 5$
$\kappa = .02$
DISPLACEMENT (CM.)
TIME (SEC.)
24
20
16
12
8
4
0
-4
-8
-12
0
10
20
30
40
50
60
70
80
90
100

FIGURE 4b

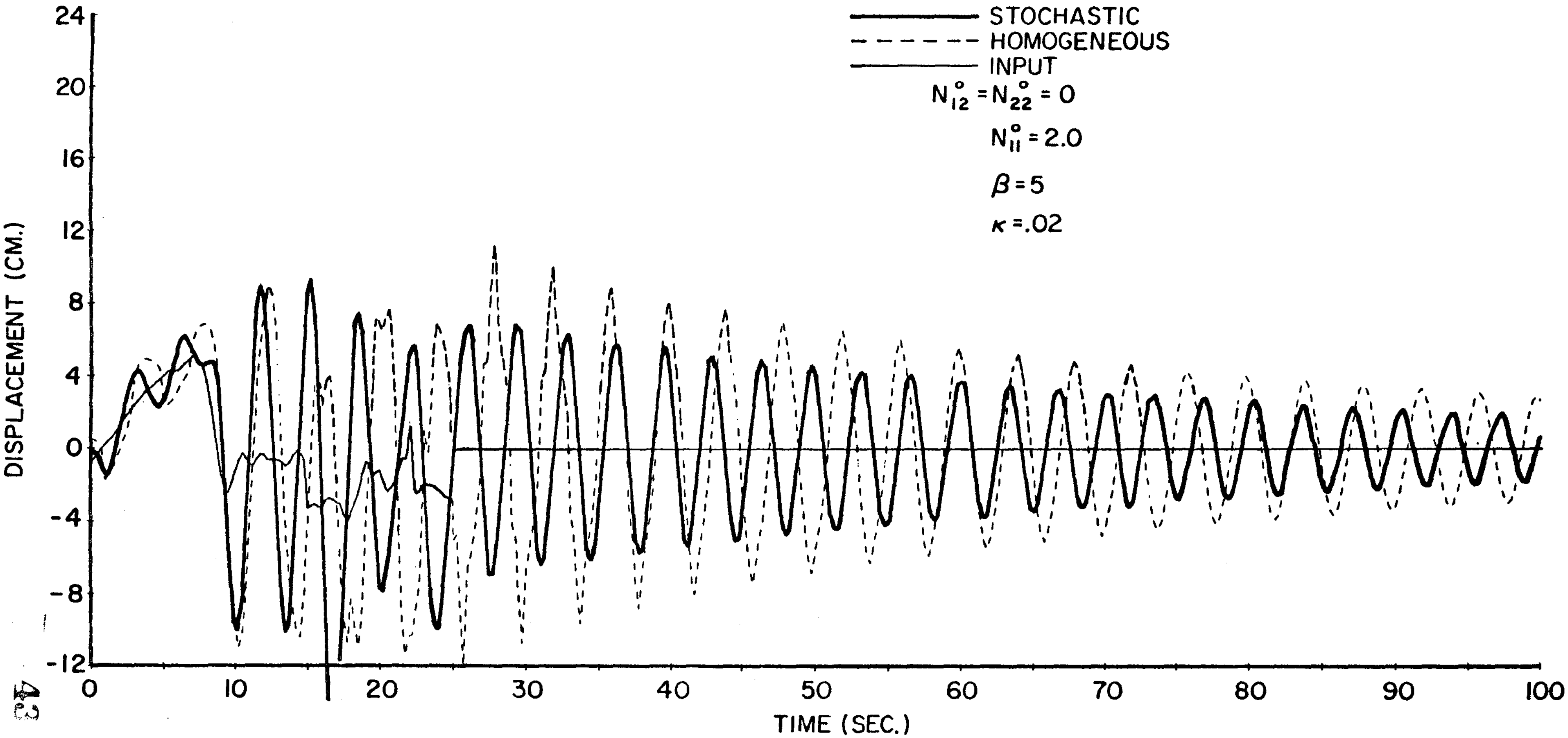
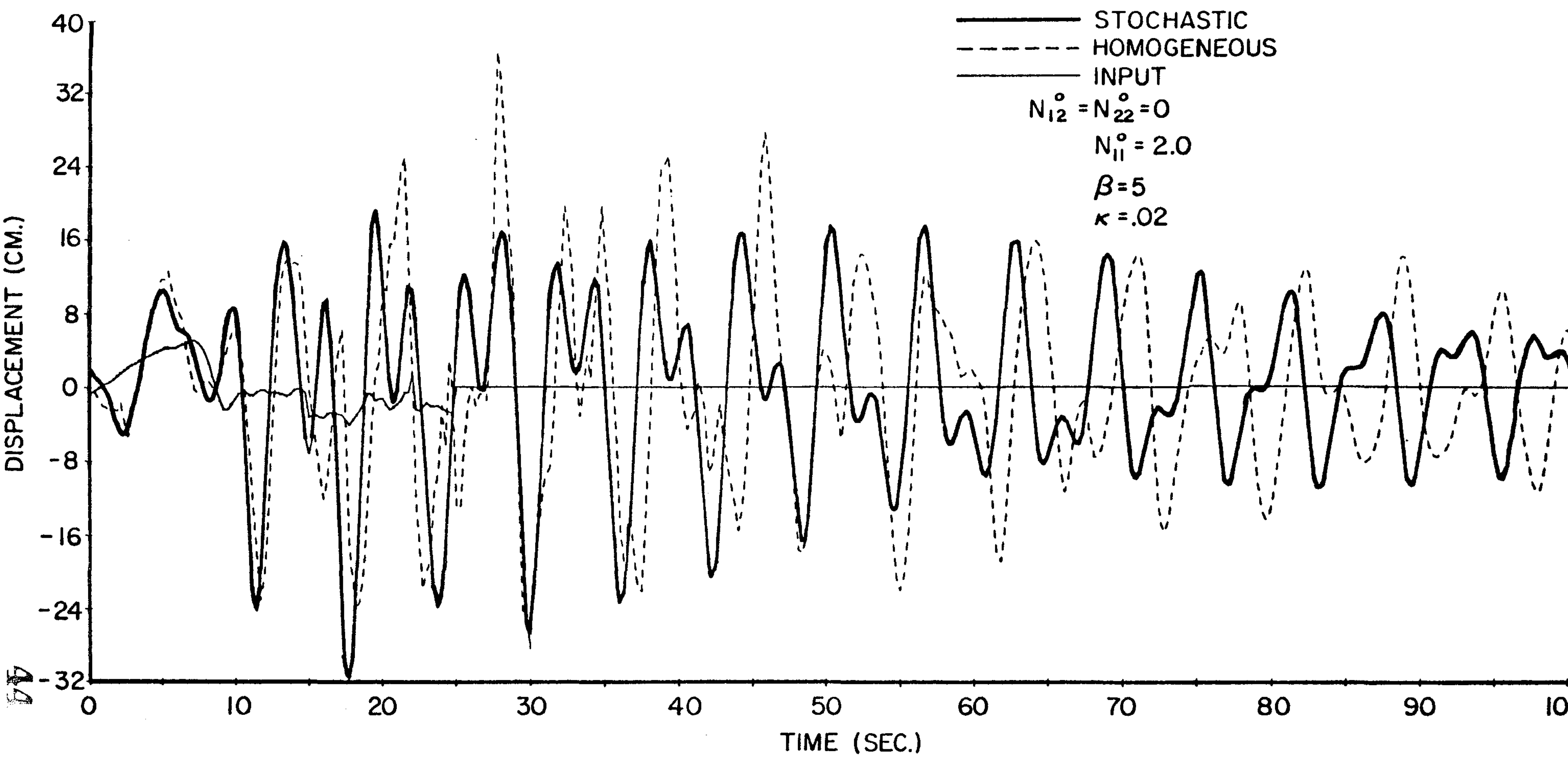

STOCHASTIC
HOMOGENEOUS
INPUT
$N^{\circ}_{12} = N^{\circ}_{22} = 0$
$N^{\circ}_{11} = 2.0$
$\beta = 5$
$\kappa = .02$
DISPLACEMENT (CM.)
TIME (SEC.)
24
20
16
12
8
4
0
-4
-8
-12
0
10
20
30
40
50
60
70
80
90
100

FIGURE 4c



STOCHASTIC
HOMOGENEOUS
INPUT
$N^{\circ}_{12} = N^{\circ}_{22} = 0$
$N^{\circ}_{11} = 2.0$
$\beta = 5$
$\kappa = .02$
DISPLACEMENT (CM.)
40
32
24
16
8
0
-8
-16
-24
-32
0
10
20
30
40
50
60
70
80
90
100
TIME (SEC.)

FIGURE 5