SEISMIC RESPONSE OF LIGHT ATTACHMENTS TO BUILDINGS

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An approximate simple method for predicting the response of secondary systems attached to buildings subjected to earthquakes is presented. Secondary systems comprise a variety of attachments to the floors and walls of large and complex buildings which, because of their different characteristics and functions, are not considered part of the structures which support them. Current methods to predict the response of secondary systems are either inaccurate or impractical. The method developed in this project overcomes these weaknesses. It may be applied for the analysis of multi-degree-of-freedom secondary systems connected to arbitrary points of a multi-degree-of-freedom primary structure. The method is based on the premise that interaction between a primary and secondary system can be accounted for by analyzing the interconnected system constructed by such primary and secondary systems. The response spectrum method is used to determine the maximum response of the assembled system, and analytical expressions are derived for each step. These expressions are simplified and integrated into a single relationship. Comparative studies indicate that this approximate procedure provides a convenient alternative method for the rational seismic design of secondary systems.

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SUMMARY

A simple approximate method is proposed to compute the maximum response of light secondary systems attached to buildings subjected to earthquakes. The method is derived by considering that a secondary system and its supporting primary structure form a single assembled system, and by applying a modified version of the response spectrum technique to such an assembled system. It is formulated in terms of the dynamic properties of independent primary and secondary systems and of the response spectra of a specified ground motion, is developed for the analysis of any multi-degree of freedom secondary system attached to one or two arbitrary points of a multi-degree of freedom primary structure, and may be applied for secondary systems in resonance with their supporting systems. It is restricted, however, to those cases in which the independent primary and secondary systems are linear elastic systems with classical modes of vibration and the masses of the secondary system are small in comparison with the masses of its primary structure.

The accuracy of the method is verified by means of a comparative study with time-history solutions. In this comparative study, the proposed approximate procedure yields, on the average, errors of no more than about 7%.

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TABLE OF CONTENTS

LIST OF TABLES

Page

LIST OF FIGURES

Page

CHAPTER 1

INTRODUCTION

1.1 Background

Ordinarily, there are a variety of attachments to the floors and walls of large and complex buildings which, because of their different characteristics and functions, may not be considered as part of the structures which support them. Piping systems, electrical equipment, pressure vessels, motors, generators, pumps, tanks, stacks, furnaces, bins, conveyor systems, mixers, precipitators, cranes, antennas, elevator penthouses, and parapets are just a few examples of the attachments which may be found in multi-story buildings, industrial plants or nuclear power facilities. To distinguish them from the structural systems whose main function is to resist forces, these attachments are often referred to as secondary systems.

The experiences from past earthquakes have demonstrated that these secondary systems are particularly vulnerable to the effects of earthquakes, such that in many instances their total failure has been observed in spite of the fact that their supporting structures have shown only moderate damage. Understandably, their low damping values and the amplified motions of the parts of the structures to which they are attached make them undergo accelerations greater than the ones that normally act on their own supporting systems. And often those magnified accelerations are extremely large because of the resonant effect produced by the closeness of their natural frequencies to any of the natural frequencies of their supporting structures, a closeness that is likely to occur because the masses and stiffnesses of a secondary system are usually small compared to those of a primary system. At the same time, past earthquakes have also demonstrated that the survival of some of such secondary systems during the occurrence of an earthquake may be vital to provide emergency services (equipment in power stations and communication facilities, for example) and that their failure may produce loss of human life and property. Thus, it is apparent that these building attachments or secondary systems should be the object of a reliable seismic analysis.

In principle, the analysis of a secondary system may be carried out in conjunction with the analysis of the primary system to which it is connected. That is, the earthquake response of a secondary system may be obtained by considering this secondary system and its supporting structure as a single combined system and by analyzing this combined system by any conventional method of analysis. This procedure, however, presents the following inconveniences:

1) Since a piece of equipment or any other secondary system is customarily designed after the completion of the design of the building where such a piece of equipment or secondary system is housed, a second analysis of this building to include its attachments introduces a problem of schedule and efficiency.

2) The number of degrees of freedom required for the modeling of large and complex facilities makes the analysis of a combined system cumbersome, costly and impractical.

3) The conventional methods of analysis become inaccurate and inefficient when they are applied to a system where there exists a large difference between the values of its various masses, stiffnesses and damping coefficients.

Among all these inconveniences, perhaps the mose serious is the last one. The response spectrum method shows difficulties in computing the natural frequencies, mode shapes and damping ratios of such a combined system and in the combination of its maximum modal responses. The time-history approach becomes prohibitively expensive because: (a) the excessive number of degrees of freedom involved, (b) the necessity of carrying out several analyses to cover the possible variations in the calculated characteristics of the aforementioned combined system (such as natural frequencies and damping ratios) and in the characteristics of the earthquake input, and (c) the different order of magnitude of the values of the masses, stiffnesses and damping coefficients of a primary and a secondary system makes a step-by-step integration extraordinarily sensitive to the selected integration time step. In like manner, a random vibration solution turns out to be particularly susceptible to the spectral density used to represent the ground motions expected in a given area and to the assumptions made about the characteristics of the probabilistic model adopted, such as stationarity and earthquake duration.

1.2 Previous Studies

Several methods have been suggested to simplify the analysis of secondary systems. Initially, a common method of analysis was the so-called floor response spectrum method. With this method, the motion of the supporting point of a secondary system is calculated by the respone-history analysis of its primary structure. Then, a response spectrum, the floor response spectrum, is determined with the time-history of this motion, and the secondary system is analyzed by the response spectrum method in the manner that the analysis of a primary system is usually carried out.

Realizing that this approach is lengthy and impractical, several authors have proposed simple approximate procedures to construct such a floor response spectrum. Biggs and Roesset (1970), Amin et al. (1971), and Kapur and Shao (1973) give empirical rules to predict the response of a secondary system using the information provided by the modal analysis of its supporting building. Atalik (1978) suggests an interesting technique to obtain the floor spectra of a building for a prescribed ground motion by calculating the response spectrum of this ground motion after it has been filtered through simple oscillators and by performing an ordinary modal analysis of the building with this response spectrum. Peters, Schmitz and Wagner (1977) derive approximate analytical expressions to evaluate floor response spectra at any point of a building directly from the response spectrum of the ground motion specified for the building. Singh (1972), Chakravorty and Vanmarcke (1973), and Vanmarcke (1977) develop analogous procedures based on random vibration methods.

All these simplified methods have been proved to give a reasonable accuracy for secondary systems with small masses and with natural frequencies which are not close to or coincide with the natural frequencies of their supporting structures. They have, however, consistently failed for the analysis of secondary systems which are in resonance with the primary systems to which they are connected. The problem is that these methods neglect the interaction between primary and secondary systems and that, as pointed out by Crandall and Mark (1963), Singh (1972), and Kapur and Shao (1973), a significant error is introduced in the analysis of secondary systems under resonant conditions if this interaction is neglected. Based on a comparative study of the mean square response of a two-degree-of-freedom system subjected to an ideal white noise, Crandall and Mark (1963) indicate that when a secondary system is near resonance with its supporting system even a secondary to primary mass ratio of 0.001 is too large for a useful approximation if the existent interaction between primary and secondary systems is neglected. In a similar study, Singh (1972) finds that by neglecting this interaction the response of a secondary system in resonance may be in error by a factor of as much as 7.9.

Upon recognition of the importance of the interaction between primary and secondary systems, several authors have suggested methods in which this interaction is taken into account. Penzien and Chopra (1965) introduce an innovative approach to calculate the response of a single-degree-of-freedom system mounted on the top of a multi-degree of freedom building. They consider that each mode of the building and the secondary system form a coupled two-degree-of-freedom system and obtain the response of the secondary system by analyzing each of these two-degree-of-freedom systems by the conventional response spectrum method. Although in this way they account for the interaction between the primary and secondary systems, the method is nonetheless inaccurate for secondary systems in resonance. This is because the authors suggest the square root of the sum of the squares to combine all the involved modal responses and because this rule is inadequate to combine the modes with similar natural frequencies of a system with resonant components (see Chapter 8). In like manner, Newmark (1971) develops a simple approximate procedure to estimate the maximum response of a multi-degree-of-freedom secondary system connected to an arbitrary point of a primary structure. Based on the modal analyses of the separate secondary system and primary structure, he derives simplified expressions to compute the maximum amplification factors of such a secondary system in each of the modes of its supporting structure. In any case, however, his procedure gives only an upper bound to the true value of the secondary system maximum response since this maximum response is estimated by adding the absolute values of such modal amplification factors.

More recently, Sackman and Kelly (1978) propose a method to determine the maximum response of an attachment to a building ingeniously derived from the frequency response analysis of the composite system formed by the building and the attachment. In many respects, their method is, surprisingly enough, parallel to the one described in this work, which is developed in the time domain. In the derivation of their method, they recognize the importance of the interaction between primary and secondary systems and the deficiencies of the conventional rules to combine modes. In addition, they estimate the maximum response of the attachment or secondary system using the information furnished by the modal analyses of the separate primary and secondary systems and the response spectrum of a specified ground motion. However, they disregard the coupling elements of the damping matrix of the aforementioned composite system. Since, as it will be shown later on, these coupling elements may sometimes be an important aspect of the interaction beween primary and secondary systems, their method is only valid for the few cases in which such a composite system has proportional damping. Another disadvantage of their method is that it models the secondary system as a single-degree-of-freedom system. With such a model, the method may overlook significant contributions of the higher modes of a certain secondary system in the computation of its maximum response, particularly when the frequencies of these higher modes are close or equal to any of the frequencies of the structure to which the secondary system is attached. Besides, it is not possible to consider secondary systems connected to their supporting systems at more than one point. A final objection to their method is that the expression proposed to compute the maximum response of tuned or nearly tuned secondary systems furnishes only an upper bound, for the approximations introduced in its derivation are equivalent to adding the absolute values of the maximum responses in the two modes with nearly equal natural frequencies of the associated composite systems. Since this upper bound may grossly overestimate such a maximum response (sometimes with an error of as much as 4000 %, according to the comparative study in Chapter 8), the method clearly needs refinements in this respect.

1.3 Object and Scope

It is evident from the above discussion that the current methods to predict the response of secondary systems attached to buildings subjected to earthquakes are either inaccurate or impractical, and that there is still a need for a simple and reliable procedure to facilitate the seismic analysis of such secondary systems. Thus, in this work is presented an alternative, approximate method that overcomes the weaknesses of the procedures described above and accurately estimates the maximum response of secondary systems. This method may be applied for the analysis of multi-degree-of-freedom secondary systems connected to arbitrary points of a multi-degree-of-freedom primary structure and exhibits the following characteristics:

1) It is simple enough to carry out the necessary computations by hand.

2) It fully takes into account the interaction between a secondary system and its primary structure, including the damping effect that each

system exerts upon each other.

3) It is formulated in terms of the natural frequencies, mode shapes and damping ratios of independent primary and secondary systems.

4) It uses the ground motion prescribed for the analysis of a primary system to define the earthquake input to its secondary systems.

5) It may be used to analyze secondary systems which are near or in resonance with their supporting structures.

The method, however, is limited to those cases in which the separate primary and secondary systems are linear elastic systems with classical modes of vibration. In addition, it is restricted to the analysis of secondary systems which are connected to a primary system at no more than two points and which have small masses in comparison with the masses of their supporting structures.

1.4 Basic Approach

In the belief that the only possible way that the interaction between a primary and a secondary system may actually be accounted for is by analyzing the interconnected system built up by such primary and secondary systems and that in spite of its difficulties the response spectrum method is not only the most reasonable method of analysis but certainly the most convenient to derive a simple approximate procedure, the development of the approximate method proposed in this study is based on the following basic approach:

1) A primary and a secondary system are considered to form a single assembled system.

2) The response spectrum method is used to determine the maximum response of this assembled system.

3) Simple approximate analytical expressions are derived for each of the steps which constitute the modal analysis of such an assembled system.

4) Considering only the response of the secondary system, these expressions are simplified and integrated into a single relationship.

Obviously, the use of the response spectrum method in the analysis of such an assembled system brings up the inconveniencies mentioned in Sec. 1.1. For the practical application of this approach, therefore, these inconveniences are circumvented as follows:

1) To avoid the computational difficulties involved in the determination of the natural frequencies and mode shapes of an assembled system whose components have masses and stiffnesses of different order of magnitude, a method is derived to calculate these natural frequencies and mode shapes in terms of the natural frequencies and mode shapes of its independent components, i.e., in terms of parameters which normally are of the same order of magnitude.

2) To take into account the complete interaction between given primary and secondary systems, the modal analysis of the corresponding assembled system is carried out in the complex plane; that is, the complex natural frequencies and complex mode shapes of this assembled system are considered.

3) To simplify such a complex modal analysis, an approximate procedure is introduced to estimate the maximum earthquake response of systems with nonproportional damping by the conventional response spectrum method.

4) To accurately predict the maximum response of any assembled system, a rule is established to combine the modal responses of systems with closely-spaced natural frequencies.

1.5 Organization

In Chapter 2 is presented the general procedure by which the response of a secondary system may be obtained through the modal analysis of the assembled system formed by this secondary system and its supporting structure. The aforementioned method to determine the natural frequencies and mode shapes of such an assembled system in terms of the dynamic properties of its separate components is developed, and a general rule to combine its maximum modal responses is introduced. For the sake of clarity, the presentation in Chapter 2 is limited to secondary systems which are connected to only one point of their supporting systems and which together with these supporting systems build up assembled systems with proportional damping.

The derivation of a simplified method to predict the maximum response of secondary systems based on the developments in Chapter 2 is described in Chapter 3. In Chapter 4, this simplified method is extended for the analysis of secondary systems with up to two points of attachment.

Chapter 5 is devoted to the analysis of systems with nonproportional damping. In this chapter, a brief review of the theory of a complex modal analysis is made, a criterion is suggested to define the modal damping ratios and natural frequencies of systems with nonproportional damping, and an approximate procedure is derived to calculate the maximum earthquake response of these systems with nonproportional damping by the conventional response spectrum method. In addition, the rule to combine modes presented in Chapter 2 is generalized for its application to systems with nonproportional damping.

On the basis of the concepts introduced in Chapter 5, the approximate methods developed in Chapters 2 and 3 are generalized in Chapter 6 for the secondary systems which, in combination with their supporting structures, give rise to assembled systems with nonproportional damping. The obtained general approximate method for the analysis of secondary systems is then summarized and illustrated by means of numerical examples in Chapter 7.

The accuracy of the proposed approximate methods is tested by performing a comparative study between the approximate and exact solutions of various different systems. Chapter 8 contains the details and results of this comparative study.

The overall conclusions of the investigation are stated in Chapter 9.

CHAPTER 2

MODAL ANALYSIS

2.1 Introduction

The response of a secondary system attached to a supporting primary structure subjected to a specified earthquake motion may be determined from the separate dynamic properties of the structure and the attachment and the response spectra of the specified earthquake motion if: (a) a modal analysis is carried out for the assembled system formed by the interconnected primary and secondary systems and (b) this modal analysis is performed in terms of the above mentioned dynamic properties of the independent primary and secondary systems. In this chapter, then, such a modal analysis is formulated, and the general procedure by which the maximum response of a secondary system may be obtained through this modal analysis is established. For this purpose, methods are herein developed to compute the natural frequencies, mode shapes, and participation factors of such an assembled system in terms of the mode shapes, natural frequencies, and mass values of its independent components; and a rule is introduced to combine its maximum modal responses when some of its natural frequencies lie close to one another.

In order to introduce the basic concepts in a simple and clear manner, the formulation of the aforementioned general procedure is here limited to the analysis of secondary systems which have only one point of attachment and which, in combination with their supporting structures, give rise to assembled systems with proportional damping (that is, assembled systems whose damping matrices are proportional to their mass or stiffness matrices). Its generalization for systems with two points of attachment and nonproportional damping is left for subsequent chapters.

In accordance with the limitation of assembled systems with proportional damping, it is assumed throughout this chapter that the damping matrices of given primary and secondary systems are proportional to their respective stiffness matrices and that the constant that relates this proportionality for the primary system is always equal to the corresponding one for the secondary system. In other words, it is assumed that the assembled systems studied in this chapter are always systems with classical modes of vibration.

For the sake of clarity, too, the expressions developed hereafter are obtained first for the model depicted in Fig. 2.1 and then generalized for systems with any number of degrees of freedom and different locations of the point of attachment by simple induction.

2.2 Mode Shapes of Assembled System

As mentioned in Sec. 2.1, the development of the sought procedure to determine the maximum response of a secondary system attached to a supporting structure requires the formulation of a method to compute the mode shapes of a compound system using the information furnished by the analysis of its separate components. By following a procedure similar to the component mode synthesis technique introduced by Hurty (1965) and described in Ref. 15, such a method is then formulated in this section as follows:

Consider the assembled system of Fig. 2.1. Each of its components may be isolated and considered independently if the reaction that each subsystem exerts upon each other is taken into account. In this manner, the primary system may be treated as a three-degree-of-freedom system with one fixed end and an applied force on its first mass. Similarly, the secondary system may be viewed as a three-degree-of-freedom system with free ends and a force acting at the point of attachment (see Fig. 2.2).

Let then the independent primary system be defined by its frequency matrix

$$[\omega_p] = \begin{bmatrix} \omega_{p_1} & 0 & 0 \\ 0 & \omega_{p_2} & 0 \\ 0 & 0 & \omega_{p_3} \end{bmatrix}, \tag{2.1}$$

its modal matrix (mode shapes with unit participation factors)

$$[\Phi] = \begin{bmatrix} \Phi_1(1) & \Phi_1(2) & \Phi_1(3) \\ \Phi_2(1) & \Phi_2(2) & \Phi_2(3) \\ \Phi_3(1) & \Phi_3(2) & \Phi_3(3) \end{bmatrix}, \tag{2.2}$$

and its generalized masses M_i^* defined by

$$M_i^* = \sum_{n=1}^{3} M_n \, \Phi_n^2(i), \quad i = 1, 2, 3 . \tag{2.3}$$

In the same fashion, let the independent secondary system be characterized by

$$[\omega_s] = \begin{bmatrix} \omega_{s_0} & 0 & 0 \\ 0 & \omega_{s_1} & 0 \\ 0 & 0 & \omega_{s_2} \end{bmatrix}, \tag{2.4}$$

$$[\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \phi_1(1) & \phi_1(2) \\ 1 & \phi_2(1) & \phi_2(2) \end{bmatrix}, \tag{2.5}$$

and

$${}^{*}_{j} = \sum_{n=1}^{2} m_j \; \phi_n(j), \; j = 1, 2; \tag{2.6}$$

i.e., its frequency matrix, modal matrix (mode shapes with unit participation factors) and generalized masses, respectively. Notice that an extra degree of freedom is added to the independent secondary system to account for the rigid body motion of the system. In the above equations, this extra degree of freedom is identified by the frequency $\omega_{s_0} = 0$ and the mode shape

$$\{\phi\}^{(0)} = \{J\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{2.7}$$

The mode shapes $\{\phi\}^{(j)}$, j = 1, 2, in Eq. 2.5 are selected to be the normal modes of the one-end-fixed secondary system.* As a result, these

*Any set of independent modes may be used to represent the fixed modes, but it is convenient that these modes be the normal or natural modes of vibration [15].

modes constitute a set of orthogonal modes. It should be observed, however, that since such modes are not orthogonal with respect to the rigid body mode, the modal matrix $[\phi]$ is not, as a whole, an orthogonal matrix.

Thus, primary and secondary systems may be considered as two independent conventional systems subjected to external forces, and hence conventional modal analyses may be performed to determine their displacement response. Since the response of these independent systems represents the desired mode shapes of the assembled system (multiplied by a function of time), the mode shapes of this assembled system may be then found from the modal analysis of such independent primary and secondary systems as follows:

Primary System

With reference to Fig. 2.2(a) the equation of motion for the primary system is given by

$$[M]\,\{\ddot{x}_p\} + [K]\,\{x_p\} = \{R(t)\}, \tag{2.8}$$

where $\{x_p\}$ is the vector of displacements, relative to the ground, of the primary masses, $\{R(t)\}$ is the vector of the external forces applied to the system, given by

$$\{R(t)\} = \begin{Bmatrix} R(t) \\ 0 \\ 0 \end{Bmatrix}, \tag{2.9}$$

and $[M]$ and $[K]$ are, respectively, the mass and stiffness matrices of the system.

Under the transformation

$$\{x_p\} = [\Phi]\ \{Y'\}, \tag{2.10}$$

which explicitly may be expressed as

$$\begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} = \begin{Bmatrix} \Phi_1(1) \\ \Phi_2(1) \\ \Phi_3(1) \end{Bmatrix} Y_1' + \begin{Bmatrix} \Phi_1(2) \\ \Phi_2(2) \\ \Phi_3(3) \end{Bmatrix} Y_2' + \begin{Bmatrix} \Phi_1(3) \\ \Phi_2(3) \\ \Phi_3(3) \end{Bmatrix} Y_3' , \tag{2.11}$$

where Y_i', i = 1, 2, 3, are unknown functions of time, this equation of motion may be then written in normal coordinates as

$$\begin{bmatrix} M_1^* & 0 & 0 \\ 0 & M_2^* & 0 \\ 0 & 0 & M_3^* \end{bmatrix} \begin{Bmatrix} \ddot{Y}_1' \\ \ddot{Y}_2' \\ \ddot{Y}_3' \end{Bmatrix} + \begin{bmatrix} K_1^* & & \\ 0 & K_2^* & 0 \\ 0 & 0 & K_3^* \end{bmatrix} \begin{Bmatrix} Y_1' \\ Y_2' \\ Y_3' \end{Bmatrix} = \begin{Bmatrix} \Phi_1(1) \\ \Phi_1(2) \\ \Phi_1(3) \end{Bmatrix} R(t) \tag{2.12}$$

in which M_i^*, i = 1, 2, 3, are the generalized masses of the primary system (see Eq. 2.3) and K_i^*, i = 1, 2, 3, the corresponding generalized stiffnesses ($K_i^* = \omega_{p_i}^2\ M_i^*$).

Now, since $\{x_p\}$ represents the displacements of the primary masses in one of the modes of the assembled system (displacements in a free vibration motion), this vector may then be expressed as

$$\{x_p\} = \{u_p\} \cos(\omega - \theta) \tag{2.13}$$

where $\{u_p\}$ is the part corresponding to the primary system of such a mode shape of the assembled system, ω is the natural frequency in this mode, and θ is a constant phase angle. In the light of Eq. 2.10, the vector $\{Y'\}$ may be therefore written as

$$\{Y'\} = \{Y\} \cos (\omega - \theta) \tag{2.14}$$

in which $\{Y\}$ is simply a vector of unknown amplitudes.

Thus, by substitution of Eqs. 2.13 and 2.14 into Eq. 2.10 one has that

$$\{u_p\} = [\Phi] \{Y\} \quad . \tag{2.15}$$

Similarly, if: (a) Eq. 2.14 is substituted into Eq. 2.12, (b) R(t) is solved from the first and substituted in the second and third component equations of this Eq. 2.12, (c) Y_1 is set equal to unity* and (d) Y_2 and Y_3 are solved from these last two component equations, one obtains

$$Y_2 = \frac{\omega^2 - \omega_{p_1}^2}{\omega^2 - \omega_{p_2}^2} \frac{M_1^*}{M_2^*} \frac{\Phi_1(2)}{\Phi_1(1)} \tag{2.16}$$

$$Y_3 = \frac{\omega^2 - \omega_{p_1}^2}{\omega^2 - \omega_{p_3}^2} \frac{M_1^*}{M_3^*} \frac{\Phi_1(3)}{\Phi_1(1)} \quad . \tag{2.17}$$

It may be inferred, therefore, that for the general case the primary system part of the rth mode shape of an assembled system may be expressed as

$$\{u_p\}^{(r)} = [\Phi] \{Y\}^{(r)} \tag{2.18}$$

where the $Y_i^{(r)}$ factors are of the form

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*Notice by inspection of Eq. 2.15 that because the mode shapes are only relative in value, the factors Y_i, i=1,2,3, are also relative in value.

$$\gamma_i^{(r)} = \frac{\omega_r^2 - \omega_{p_1}^2}{\omega_r^2 - \omega_{p_i}^2} \frac{M_1^*/\Phi_k(1)}{M_i^*/\Phi_k(i)} , \quad i = 1,2, \ldots, N_p \tag{2.19}$$

in which the subscript k indicates the primary mass to which the secondary system is attached, ω_r is the natural frequency corresponding to that rth mode shape, and N_p is the number of degrees of freedom of the primary system.

Secondary System

A similar procedure may be followed for the secondary system. In this case, the equation of motion is

$$[m] \{\ddot{x}_s\} + [k] \{x_s\} = -\{R(t)\} \tag{2.20}$$

where $\{x_s\}$ represents the displacement vector of the secondary masses, also relative to the ground [see Fig. 2.2(b)]; [m] and [k] are the secondary system mass and stiffness matrices, respectively; and {R(t)} is as defined before.

Equation 2.20 may also be transformed into normal coordinates although, because of the rigid body mode of the system, such transformation does not uncouple the equations of motion. Accordingly, if $\{x_s\}$ is written as

$$\{x_s\} = [\phi] \{y'\} , \tag{2.21}$$

which in its expanded form results as

$$\begin{Bmatrix} x_{s_0} \\ x_{s_1} \\ x_{s_2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} y'_0 + \begin{Bmatrix} 0 \\ \phi_1(1) \\ \phi_2(1) \end{Bmatrix} y'_1 + \begin{Bmatrix} 0 \\ \phi_1(2) \\ \phi_2(2) \end{Bmatrix} y'_2 , \tag{2.22}$$

Eq. 2.20 becomes

$$\begin{bmatrix} \sum_n m_n & \sum_n m_n\phi_n(1) & \sum_n m_n\phi_n(2) \\ \sum_n m_n\phi_n(1) & m_1^* & 0 \\ \sum_n m_n\phi_n(2) & 0 & m_2^* \end{bmatrix} \begin{Bmatrix} \ddot{y}'_0 \\ \ddot{y}'_1 \\ \ddot{y}'_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_1^* & 0 \\ 0 & 0 & k_2^* \end{bmatrix} \begin{Bmatrix} y'_0 \\ y'_1 \\ y'_2 \end{Bmatrix} = \begin{Bmatrix} -R(t) \\ 0 \\ 0 \end{Bmatrix} \tag{2.23}$$

where $\{y'\}$ is, again, a vector of unknown functions of time, and m_j^* and k_j^*, j=1,2, are the generalized masses and stiffnesses, respectively, of the secondary system ($k_j^* = \omega_{s_j}^2 m_j^*$).

If by the same argument presented for the primary system the vector $\{x_s\}$ is expressed as

$$\{x_s\} = \{u_s\} \cos(\omega - \theta), \tag{2.24}$$

where $\{u_s\}$ is the secondary system part of the mode shape of the assembled system whose natural frequency is ω, then $\{y'\}$ may be put into the form

$$\{y'\} = \{y\} \cos(\omega - \theta) \tag{2.25}$$

in which, similarly to the vector $\{Y\}$ for the primary system, $\{y\}$ is a vector of unknown amplitudes.

Thus, by virtue of Eqs. 2.21, 2.24 and 2.25 $\{u_s\}$ may be written as

$$\{u_s\} = [\phi]\ \{y\}\ . \tag{2.26}$$

In like manner, if Eq. 2.25 is substituted into Eq. 2.23 and if the following relationship applicable to mode shapes with unit participation factors is employed:

$$\sum_n m_n \phi_n(j) = \sum_n m_n \phi_n^2(j) = m_j^*,\ j = 1,\ 2, \tag{2.27}$$

the last two component equations of Eq. 2.23 lead to

$$y_1 = \frac{\omega^2}{\omega_{s_1}^2 - \omega^2}\ y_0 \tag{2.28}$$

$$y_2 = \frac{\omega^2}{\omega_{s_2}^2 - \omega^2}\ y_0\ . \tag{2.29}$$

In the general case, therefore, the secondary system part of the rth mode shape of an assembled system may be expressed as

$$\{u_s\}^{(r)} = [\phi]\ \{y\}^{(r)} \tag{2.30}$$

where the associated $y_j^{(r)}$ factors result of the form

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2}\ y_0^{(r)},\ j=1,2,\ldots,N_s \tag{2.31}$$

in which N_s represents the number of degrees of freedom of the constrained (no rigid body motion) secondary system.

It should be noticed that in this case the unknown factor y_0 cannot be given an arbitrary value because $\{u_p\}$ and $\{u_s\}$ together represent a mode shape of the assembled system under consideration and because an arbitrary value has been already selected to define this mode shape (i.e., $Y_1 = 1.0$). Consequently, y_0 should be solved from the equations of motion of such an assembled system (Eqs. 2.12 and 2.23) or, more conveniently, from the compatibility conditions. Here, the latter approach is utilized as follows:

Compatibility Conditions

By compatibility, it is known that the displacements of the point of attachment of the secondary system and its supporting primary mass are the same. That is,

$$x_{s_0} = x_{p_1} \quad . \tag{2.32}$$

Therefore, if this compatibility relation is written in normal coordinates by applying the transformations given by Eqs. 2.11 and 2.22, after using Eqs. 2.14 and 2.25 one obtains

$$y_0 = \Phi_1(1)\ Y_1 + \Phi_1(2)\ Y_2 + \Phi_1(3)\ Y_3 \quad . \tag{2.33}$$

The general expression for the factor $y_0^{(r)}$ of Eq. 2.31, which may be called the compatibility factor inasmuch as it depends on the compatibility conditions, results thus as

$$y_0^{(r)} = u_{p_k}(r) = \sum_{i=1}^{N_p} \Phi_k(i)\ Y_i^{(r)} \tag{2.34}$$

where subindex k is, again, the number of the primary mass to which the secondary system is attached and, as before, N_p is the number of degrees of freedom of the primary system.

Summary

Summing up the above results, one has thus that the rth mode shape of a system formed by its assembled primary and secondary systems is given by the following two equations:

$$\{u_p\}^{(r)} = \sum_{i=1}^{N_p} \gamma_i^{(r)} \{\Phi\}^{(i)} \tag{2.35}$$

$$\{u_s\}^{(r)} = \sum_{j=0}^{N_s} y_j^{(r)} \{\phi\}^{(j)} \tag{2.36}$$

where $\{u_p\}^{(r)}$ and $\{u_s\}^{(r)}$ are the parts of this rth mode shape corresponding respectively to the primary and secondary systems,

$$\gamma_i^{(r)} = \frac{\omega_r^2 - \omega_{p_1}^2}{\omega_r^2 - \omega_{p_i}^2} \; \frac{M_1^*/\Phi_k(1)}{M_i^*/\Phi_k(i)}, \quad i=1,2,\ldots,N_p \tag{2.37}$$

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2} \; y_0^{(r)}, \qquad j=1,2,\ldots,N_s \tag{2.38}$$

$$y_{0_c}^{(r)} = u_{p_k}(r) = \sum_{i=1}^{N_p} \Phi_k(i) \; \gamma_i^{(r)} \tag{2.39}$$

and

ω_r = assembled system's rth natural frequency

k = number of the primary mass supporting the secondary system

N_p = number of degrees of freedom of the primary system

N_s = number of degrees of freedom of the secondary system

It may be observed from the inspection of Eqs. 2.37 and 2.38 that whenever one of the frequencies of the assembled system matches one of the frequencies of the independent primary or secondary system, $Y_i^{(r)}$ or $y_j^{(r)}$ may acquire infinite values. As a result, Eq. 2.37 is not valid when $\omega_r = \omega_{p_i}$, and Eq. 2.38 is not valid when $\omega_r = \omega_{s_j}$. It should be noticed, however, that these equations have been derived for closely-coupled systems and that for this kind of systems such cases can never occur.

It is also important to note that the above equations have been derived without having introduced any approximation. Hence, Eqs. 2.35 through 2.39 lead, provided the natural frequencies of the assembled system are known, to the exact mode shapes. In view that Eq. 2.35 and 2.36 are expressed as combinations of the mode shapes of the independent subsystems, by neglecting the insignificant modes of each of these subsystems Eqs. 2.35 through 2.39 lend themselves, nevertheless, for obtaining simple approximate relations for such mode shapes. An approximation used, in fact, in the simplified approach proposed in Chapter 3.

2.3 Natural Frequencies: Resonant Modes

It may be observed that in order to compute the mode shapes of an assembled system with the procedure formulated in the previous section it is necessary to determine first its natural frequencies. To obtain these natural frequencies, then, one might continue that procedure and also solve the associated eigenvalue problem from the transformed equations of motion. This approach, however, becomes to involved and does not lead to explicit relationships. An approximate alternative may be utilized instead by making use of the fact that the natural frequencies of a system are always stationary in value in the neighborhood of its exact mode shapes* (that is, small variations from the true mode shapes only produce higher order variations in the frequency values) [8]. A fact that in combination with Eqs. 2.35 through 2.39 may be used advantageously to obtain accurate estimates of the sought natural frequencies of assembled systems from simple approximations of their mode shapes. In this section, this latter approach is accordingly used to derive an approximate formula for the natural frequencies of assembled systems whose primary and secondary components are under resonant conditions.

It has been observed by Nakhata, Newmark and Hall (1973) that whenever one of the frequencies of a secondary system matches one of the frequencies of its primary system (resonant case) the assembled system has to modes whose frequencies are very close to the frequency in resonance (the closeness depending on the mass values and interconnection of the subsystems in question). From this observation and from the analysis of Eqs. 2.37 and 2.38, it may be seen that the modes of the independent components which most significantly contribute to the summations of Eqs. 2.35 and 2.36, and therefore to the values of a mode shape of an assembled system, are those whose frequencies are the closest to the frequency of the assembled system in such a mode. Consequently, if only such closest component modes are taken into account, the resonant modes of such an assembled system (i.e., the modes whose frequencies are close to the resonant frequency) may be approximated as

$$\{u_p\}^{(r)} = Y_I^{(r)}\{\Phi\}^{(I)} \tag{2.40}$$

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*This property of the natural frequencies of a system is known as Rayleigh's principle [14]

$$\{u_s\}^{(r)} = y_J^{(r)}\{\phi\}^{(J)} \tag{2.41}$$

where subscripts I and J identify respectively the primary and secondary modes whose frequencies are in resonance.

Since by knowing $Y_I^{(r)}$ and $y_J^{(r)}$ one may know $\{u_p\}^{(r)}$ and $\{u_s\}^{(r)}$, Eqs. 2.40 and 2.41 suggest thus that in the resonant modes the assembled system may be reduced to an approximate equivalent system with only two degrees of freedom. Accordingly, if the equation of motion for the system of Fig. 2.1 is written in a partitioned form as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{p_1} \\ \ddot{x}_{p_2} \\ \ddot{x}_{p_3} \end{Bmatrix}^{(r)} + \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix}^{(r)} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix}^{(r)}$$

$$- \begin{bmatrix} k_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{s_1} \\ x_{s_2} \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.42}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{s_1} \\ \ddot{x}_{s_2} \end{Bmatrix}^{(r)} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix}^{(r)} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_{s_1} \\ x_{s_2} \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{2.43}$$

by substitution of Eqs. 2.13, 2.24, 2.40 and 2.41 and premultiplication of Eq. 2.42 by $\{\Phi\}^{(I)^T}$ and Eq. 2.43 by $\{\phi\}^{(J)^T}$ such an equation of motion may be reduced to the following matrix equation:

$$-\omega_r^2 \begin{bmatrix} M_I^* & 0 \\ 0 & m_J^* \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} + \begin{bmatrix} K_I^*+k_1\Phi_1^2(I) & -k_1\Phi_1(I)\phi_1(J) \\ -k_1\Phi_1(I)\phi_1(J) & k_J^* \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \quad (2.44)$$

This equation is the free vibration equation of a two degree of freedom system; from the solution to its characteristic equation the natural frequencies of the assembled system results then approximately as

$$\omega_r^2 = \frac{1}{2}\left[\frac{K_I^*+k_1\Phi_1(I)}{M_I^*} + \frac{k_J^*}{m_J^*}\right] \pm \frac{1}{2}\left[\left(\frac{K_I^*+k_1\Phi_1(I)}{M_I^*} - \frac{k_J^*}{m_J^*}\right)^2 + \frac{4k_1^2\Phi_1^2(I)\phi_1^2(J)}{M_I^* m_J^*}\right]^{1/2} \quad (2.45)$$

which, if it is considered that: (a) $\omega_{p_I}^2 = K_I^*/M_I^*$ and $\omega_{s_J}^2 = k_J^*/m_J^*$, (b) by assumption the jth secondary mode is in resonance with the Ith primary mode and hence

$$\frac{k_J^*}{m_J^*} = \frac{K_I^*}{M_I^*} = \omega_o^2 \quad (2.46)$$

where ω_o is the resonant frequency, and (c) for mode shapes with unit participation factors ω_J^2 may be written as (See Appendix A)

$$\omega_{s_J}^2 = \omega_o^2 = \frac{k_1\phi_1(J)}{m_J^*} \quad , \quad (2.47)$$

may also be expressed as

$$\omega_r^2 = \omega_o^2 + \frac{1}{2}\omega_o^2 \frac{\Phi_1^2(I)}{\phi_1(J)}\gamma_{IJ} \pm \frac{1}{2}\omega_o^2 \frac{\Phi_1^2(I)}{\phi_1(J)}\gamma_{IJ}\left[1 + 4\frac{\phi_1^2(J)}{\Phi_1^2(I)}\frac{1}{\gamma_{IJ}}\right]^{1/2} \quad (2.48)$$

where γ_{IJ} is the mass ratio for the Ith primary and Jth secondary modes defined as

$$\gamma_{IJ} = \frac{m_J^*}{M_I^*} \quad . \tag{2.49}$$

But for small mass ratios ($\gamma_{IJ} << 1.0$), the second term in the right-hand side of the above equation is small when compared to ω_o^2 whereas the second term within the square root is much greater than unity. Therefore, for small mass ratios ω_r^2 may be approximated as

$$\omega_r^2 = \omega_o^2 \left(1 \pm \Phi_1(I) \sqrt{\gamma_{IJ}}\right) \quad . \tag{2.50}$$

For systems with the secondary system attached to the kth primary mass, this expression may be thus generalized as

$$\omega_r^2 = \omega_o^2 \left(1 \pm \Phi_k(I) \sqrt{\gamma_{IJ}}\right) \quad . \tag{2.51}$$

Hence, since for small mass ratios the second term within the parenthesis is less than unity, ω_r results as

$$\omega_r = \omega_o \left(1 \pm \frac{1}{2} \Phi_k(I) \sqrt{\gamma_{IJ}}\right) \quad . \tag{2.52}$$

Equation 2.52 provides thus the simple approximate formula sought to compute the natural frequencies of the resonant modes. Notice that this equation verifies the observation made in Ref.19 and stated at the beginning of this section. That is, it verifies that indeed the interconnection of primary and secondary systems with a common frequency gives rise to an assembled system with two modes whose frequencies are very close to each other and close to the common resonant frequency.

Notice also that Eq. 2.51 indicates that such a closeness increases as the mass ratio decreases. Therefore, the statement made at the end of the last section about the impossibility of having an assembled system with frequencies equal to the frequencies of its separate subsystems is also corroborated by this equation, because it shows that in order to have such a case a mass ratio with a zero value is necessary. A value that is only possible, obviously, for nonexistent secondary systems.

2.4 Natural Frequencies: Nonresonant Modes

It is also noted by Nakhata, Newmark and Hall (1973) that the frequencies of an assembled system which are not close to a resonant frequency (frequencies of nonresonant modes) only depart slightly from the original frequencies of its independent primary and secondary systems. Therefore, a procedure similar to the one used for the resonant case may be followed to derive the natural frequencies of such nonresonant modes.

If, accordingly, it is assumed that each nonresonant mode at the assembled system of Fig. 2.1 is composed by only those modes of the independent components whose frequencies are the closest to the frequency of the nonresonant mode in question, assumption that is tantamount to set in Eqs. 2.11 and 2.22

$$Y_i^{'(r)} = y_j^{'(r)} = 0 \text{ for } \begin{cases} i \neq I \\ j \neq 0 \\ j \neq J \end{cases} , \qquad (2.53)$$

where I and J are, respectively, the subscripts corresponding to the mentioned primary and secondary closest modes, Eqs. 2.12 and 2.23, the equations of motion of this assembled system, may be similarly reduced to the following system of equations:

$$M_I^* \ddot{Y}_I^{'} + K_I^* Y_I^{'} = \Phi_k(1)\ R(t) \qquad (2.54)$$

$$(\sum_{n} m_n)\, \ddot{y}_0' + m_J^* \ddot{y}_J' = -R(t) \tag{2.55}$$

$$m_J^* \ddot{y}_0' + m_J^* \ddot{y}_J' + k_J^* y_J' = 0 \tag{2.56}$$

in which $\Phi_1(1)$ has been changed to $\Phi_k(I)$ in order to generalize this derivation for any support conditions. By the same token, the compatibility factor given by Eq. 2.39 may be approximately written as

$$y_0^{(r)} = \Phi_k(I)\, Y_I^{(r)} \quad . \tag{2.57}$$

Substitution of Eqs. 2.14, 2.25 and 2.57 into the above system of equations and elimination of the reaction R(t) from Eqs. 2.54 and 2.55 lead then to

$$- \omega_r^2 \begin{bmatrix} M_I^* + \Phi_k^2(I)\, (\sum_{n} m_n) & \Phi_k(I)\, m_J^* \\ \Phi_k(I)\, m_J^* & m_J^* \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} + \begin{bmatrix} K_I^* & 0 \\ 0 & k_J^* \end{bmatrix} \begin{Bmatrix} Y_I \\ y_j \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.58}$$

which after neglecting the term $\Phi_k^2(I)\, (\sum_{n} m_n)$ leads in turn to the following characteristic equation:

$$\left(\frac{\omega_{p_I}^2 - \omega_r^2}{\omega_r^2}\right) \left(\frac{\omega_{s_J}^2 - \omega_r^2}{\omega_r^2}\right) = \Phi_k^2(I)\gamma_{IJ} \quad . \tag{2.59}$$

It may be observed thus that for small mass ratios the frequencies of nonresonant modes are essentially the same original frequencies of the independent primary and secondary systems. Therefore, these frequencies may be approximated without much error as

$$\omega_{r_1} = \omega_{p_I} \tag{2.60}$$

$$\omega_{r_2} = \omega_{s_J} \quad . \tag{2.61}$$

The adoption of this approximation gives rise, however, to some difficulties since in these cases the factors $Y_I^{(r)}$ and $y_J^{(r)}$ as given by Eqs. 2.37 and 2.38 reach infinite values. To overcome these difficulties, then, alternative formulations for those particular cases are presented in the following sections.

2.5 $Y_i^{(r)}$ and $y_j^{(r)}$ Factors When $\omega_r = \omega_{p_I}$

When a frequency of an assembled system is very close to one of the frequencies of its primary system, Eq. 2.37 demands great accuracy in the value of the frequency of the assembled system in order to obtain a reliable value of the corresponding $Y_i^{(r)}$ factor. In such a case, a more convenient alternative expression for $Y_I^{(r)}$ may be developed if in the derivation that led to Eq. 2.19, instead of $Y_1^{(r)}$, the $Y_i^{(r)}$ factor corresponding to ω_{p_I}, the closest primary frequency to the frequency ω_r, is now set equal to unity. In this manner, the following equation is obtained:

$$Y_i^{(r)} = \frac{\omega_r^2 - \omega_{p_I}^2}{\omega_r^2 - \omega_{p_i}^2} \, \frac{M_I^*/\Phi_k(I)}{M_i^*/\Phi_k(i)} \quad . \tag{2.62}$$

Thus, if ω_r is approximated as indicated by Eq. 2.60, the $Y_i^{(r)}$ factors when $\omega_r = \omega_{p_I}$ result as

$$Y_i^{(r)} = \begin{cases} 1 \text{ if } i = I \\ 0 \text{ if } i \neq I \end{cases} \tag{2.63}$$

Notice that this result may be verified by analyzing Eq. 2.19 itself. For, if ω_r is very close to ω_{p_I}, then $Y_I^{(r)}$ becomes a very large quantity while all the other $Y_i^{(r)}$ remain comparatively small. Hence, if all these values are normalized with respect to $Y_I^{(r)}$, the same conclusion (i.e., Eq. 2.63) approximately follows.

The $y_j^{(r)}$ factors may be computed directly by Eq. 2.38, even if ω_r is approximated by ω_{p_I}. It should be noted, however, that this approximation is not valid when ω_{p_I} is close to any of the frequencies ω_{s_j} because in such a case the frequency ω_r should approach the value of the frequency of a resonant mode. To establish, then, the separation between ω_{p_I} and ω_{s_j} for which such an approximation is valid, one may observe that in the limiting case when $\omega_{p_I} = \omega_{s_j}$, ω_r^2 is given by Eq. 2.51 and consequently for this limiting case one has that

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2}\, y_0^{(r)} = \frac{1 \mp \Phi_k(I)\sqrt{\gamma_{Ij}}}{\pm\, \Phi_k(I)\sqrt{\gamma_{Ij}}}\, y_0^{(r)} \doteq \frac{y_0^{(r)}}{\Phi_k(I)\sqrt{\gamma_{Ij}}} \quad . \qquad (2.64)$$

Hence, since the $y_j^{(r)}$ factors should be less than or equal to this value, for all cases the following condition need be satisfied:

$$\left| \frac{\omega_{s_j}^2 - \omega_r^2}{\omega_r^2} \right| \geq \left| \Phi_k(I)\sqrt{\gamma_{Ij}} \right| \quad . \qquad (2.65)$$

Thus, when ω_r is approximated by ω_{p_I}, the $y_j^{(r)}$ factors may be calculated by

$$y_j^{(r)} = \frac{\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2}\, y_0^{(r)} \qquad (2.66)$$

if

$$\left|\frac{\omega_{s_j}^2 - \omega_{p_I}^2}{\omega_{p_I}^2}\right| \geq \left|\Phi_k(I)\sqrt{\gamma_{Ij}}\right| \; ; \qquad (2.67)$$

otherwise, $y_j^{(r)}$ should be computed by considering ω_{s_j} and ω_{p_I} as resonant frequencies.

2.6 $Y_i^{(r)}$ and $y_j^{(r)}$ Factors When $\omega_r = \omega_{s_J}$

When $\omega_r = \omega_{s_J}$, the $Y_i^{(r)}$ factors may be calculated directly by Eq. 2.37. But, as in the previous case, the calculation of these $Y_i^{(r)}$ factors should be limited to those values of ω_{p_i} and ω_{s_J} for which such substitution of ω_r by ω_{s_J} remains valid. By following the procedure in that previous case, the valid relation between ω_{p_i} and ω_{s_J} may be therefore established as follows:

According to Eq. 2.37 and 2.51 when $\omega_{p_i} = \omega_{s_J}$ (that is, when ω_{p_i} and ω_{s_J} are in resonance), the corresponding $Y_i^{(r)}$ factor results of the form

$$Y_i^{(r)} = \frac{\omega_r^2 - \omega_{p_1}^2}{\omega_r^2 - \omega_{p_i}^2} \; \frac{M_1^*/\Phi_k(1)}{M_i^*/\Phi_k(i)} = \frac{\omega_{s_J}^2 - \omega_{p_1}^2}{\mp\omega_{s_J}^2 \; \Phi_k(i)\sqrt{\gamma_{iJ}}} \; \frac{M_1^*/\Phi_k(1)}{M_i^*/\Phi_k(i)} \; . \qquad (2.68)$$

Therefore, for all values of ω_{p_i} one has that

$$\left|\frac{\omega_r^2 - \omega_{p_i}^2}{\omega_r^2 - \omega_{p_1}^2} \; \frac{\omega_{s_J}^2 - \omega_{p_1}^2}{\omega_{s_J}^2}\right| \geq \left|\Phi_k(i)\;\sqrt{\gamma_{iJ}}\;\right| \; . \qquad (2.69)$$

Thus, when ω_r is approximated by ω_{s_J}, the $Y_i^{(r)}$ factors may be expressed as

$$Y_i^{(r)} = \frac{\omega_{s_J}^2 - \omega_{p_1}^2}{\omega_{s_J}^2 - \omega_{p_i}^2} \; \frac{M_1^*/\Phi_k(1)}{M_i^*/\Phi_k(i)} \tag{2.70}$$

if

$$\left| \frac{\omega_{s_J}^2 - \omega_{p_i}^2}{\omega_{s_J}^2} \right| \geq |\Phi_k(i) \sqrt{\gamma_{ij}}\,| . \tag{2.71}$$

If this condition is not satisfied, the $Y_i^{(r)}$ factors should then be calculated as if ω_{s_J} and ω_{p_i} were resonant frequencies.

Contrary to the $Y_i^{(r)}$'s, the $y_j^{(r)}$ factors cannot be determined by Eq. 2.38 when the frequency ω_r is assumed equal to ω_{s_J}. In order to be able to approximate the natural frequencies of nonresonant modes by Eq. 2.61, an alternative expression is thus necessary to calculate the $y_j^{(r)}$ factors in such a case. Although not as straightforward as for the $Y_i^{(r)}$ factors in the previous section, this alternative expression may still be developed by recurring to the original formulation that led to Eq. 2.31 as follows:

With reference to the model of Eq. 2.1, let ω be the frequency of the assembled system that is close to ω_{s_1}, the first frequency of the secondary system, and assume that except for y_1, the one that corresponds to ω_{s_1}, all the Y_i and y_j factors in Eqs. 2.12 and 2.23 have been previously determined. Thus, if R(t) is solved from any component equation, say the ith, of Eq. 2.12 and the result is substituted into the first of Eqs. 2.23, the following equation is obtained:

$$m_0^* \ddot{y}_0' + m_1 \ddot{y}_1' + m_2 \ddot{y}_2' + \frac{1}{\Phi_1(i)} [M_i^* \ddot{Y}_i' + K_i Y_i'] = 0 \tag{2.72}$$

in which

$$m_0^* = \{\phi\}^{(0)^T} [m] \{\phi\}^{(0)} = \sum_n m_n \quad . \tag{2.73}$$

By substituting Eqs. 2.14 and 2.25 and solving for y_1, one then obtains

$$y_1 = \frac{(\omega_{p_i}^2 - \omega^2) Y_i - \Phi_1(i) \omega^2 [\gamma_{i0} y_0 + \gamma_{i2} y_2]}{\Phi_1(i) \omega^2 \gamma_{i1}} \tag{2.74}$$

where

$$\gamma_{ij} = \frac{m_j^*}{M_i^*}, \quad j = 0, 1, 2 \; . \tag{2.75}$$

If Eq. 2.31 is used to disclose the relative magnitude of all the y_j factors in Eq. 2.74 and if it is considered that by assumption ω is very close to ω_{s_1}, it is easy to see that the value of y_1 is considerably larger than the value of all the other y_j's are. If in addition it is considered that the values of the mass ratios γ_{I0} and γ_{I1} are small, then it may be deduced that the terms between brackets in Eq. 2.74 are negligibly small. Consequently, y_1 may be approximated as

$$y_1 = \frac{(\omega_{p_i}^2 - \omega^2) Y_i}{\Phi_1(i) \omega^2 \gamma_{i1}} \quad . \tag{2.76}$$

In general, when the rth frequency of an assembled system is close to the Jth one of its secondary system, its $y_J^{(r)}$ factor may be therefore expressed as

$$y_J^{(r)} = \frac{\omega_{p_i}^2 - \omega_r^2}{\Phi_k(i)\omega_r^2\gamma_{iJ}} \gamma_i^{(r)} \quad . \tag{2.77}$$

To complete the derivation, ω_r may now be substituted by ω_{s_J}. As in the previous cases, however, it is necessary to determine first the relation between the values of ω_{p_i} and ω_{s_J} for which such a substitution is applicable. Again, this relation may be obtained by noticing that when ω_{s_J} and ω_{p_i} are equal, ω_r is the frequency of a resonant mode. Accordingly, if Eq. 2.51 is substituted into Eq. 2.77, in such a case $y_J^{(r)}$ results as

$$y_J^{(r)} = \frac{\omega_{p_i}^2 - \omega_r^2}{\Phi_k(i)\ \omega_r^2\ \gamma_{iJ}} \gamma_i^{(r)} = \pm \frac{\gamma_i^{(r)}}{(1 \pm \Phi_k(i)\ \sqrt{\gamma_{iJ}})\ \sqrt{\gamma_{iJ}}} \doteq \pm \frac{\gamma_i^{(r)}}{\sqrt{\gamma_{iJ}}} \ . \tag{2.78}$$

Then, since for this limiting case ω_{p_i} and ω_r get the closest and therefore $y_J^{(r)}$ reaches its minimum value, in all cases the following relationship should be satisfied:

$$\left| \frac{\omega_{p_i}^2 - \omega_r^2}{\omega_r^2} \right| \geq \left| \Phi_k(i)\ \sqrt{\gamma_{iJ}}\ \right| \quad . \tag{2.79}$$

By replacing ω_r by ω_{s_J} in Eqs. 2.77 and 2.79, one has thus that when $\omega_r = \omega_{s_J}$ the $y_J^{(r)}$ factor may be alternatively expressed as

$$y_J^{(r)} = \frac{\omega_{p_i}^2 - \omega_{s_J}^2}{\Phi_k(i)\omega_{s_J}^2\gamma_{iJ}} \gamma_i^{(r)} \tag{2.80}$$

if

$$\left| \frac{\omega_{p_i}^2 - \omega_{s_J}^2}{\omega_{s_J}^2} \right| \geq |\Phi_k(i) \sqrt{\gamma_{iJ}}| \quad . \tag{2.81}$$

Similarly to the previous cases, if ω_{s_J} and ω_{p_i} do not satisfy this relation, $y_J^{(r)}$ should then be computed as for resonant modes.

2.7 Participation Factors

Although the participation factors for an assembled system may be computed directly once its mode shapes are known, it is convenient, nevertheless, to derive an analytic expression for these participation factors in order to study their variability with different values of the different parameters defining that system and to develop hencefrom simple approximate relationships. If, as indicated in Sec. 2.2, the mode shapes of such an assembled system are expressed in normal coordinates, a simplified analytic expression for its participation factors may be then obtained as follows:

By definition, the rth participation factor of an assembled system may be expressed as

$$\alpha_r = \frac{\sum_{n=1}^{N_p} M_n \, u_{p_n}(r) + \sum_{n=1}^{N_s} m_n \, u_{s_n}(r)}{\sum_{n=1}^{N_p} M_n \, u_{p_n}^2(r) + \sum_{n=1}^{N_s} m_n \, u_{s_n}^2(r)} \tag{2.82}$$

where, as before, $u_{p_n}(r)$ and $u_{s_n}(r)$ represent respectively the amplitudes of the primary and secondary masses in the rth mode of this assembled system and N_p and N_s are their respective number of degrees of freedom.

If by virtue of Eq. 2.35 and 2.36 $u_{p_n}(r)$ and $u_{s_n}(r)$ are expressed as

$$u_{p_n}(r) = \sum_{i=1}^{N_p} \gamma_i^{(r)} \, \phi_n(i) \qquad (2.83)$$

$$u_{s_n}(r) = \sum_{j=0}^{N_s} y_j^{(r)} \, \phi_n(j) \quad , \qquad (2.84)$$

then α_r may be written as

$$\alpha_r = \{ \sum_{i=1}^{N_p} \gamma_i^{(r)} [\sum_n M_n \phi_n(i)] + \sum_{j=0}^{N_s} y_j^{(r)} [\sum_n m_n \phi_n(j)] \} / \{ \sum_{i=1}^{N_p} \gamma_i^{(r)^2} [\sum_n M_n \phi_n^2(i)] +$$

$$+ \sum_{\substack{s=1 \\ s \neq t}}^{N_p} \sum_{t=1}^{N_p} \gamma_s^{(r)} \gamma_t^{(r)} [\sum_n M_n \phi_n(s) \phi_n(t)] + \sum_{j=0}^{N_s} y_j^{(r)^2} [\sum_n m_n \phi_n^2(j)] +$$

$$+ \sum_{\substack{s=0 \\ s \neq t}}^{N_s} \sum_{t=0}^{N_s} y_s^{(r)} y_t^{(r)} [\sum_n m_n \phi_n(s) \phi_n(t)] \} \; , \qquad (2.85)$$

where $\sum_n$ simply indicates the sum for all n. However, since $\{\phi\}^{(i)}$, $i = 1, 2, \ldots, N_p$, and $\{\phi\}^{(j)}$, $j = 1, 2, \ldots, N_s$, are mode shapes with unit participation factors and $\{\phi\}^{(0)}$ is a vector of unit elements (see Eq. 2.7), one has that

$$\sum_{n=1}^{N_p} M_n \phi_n(i) = \sum_{n=1}^{N_p} M_n \phi_n^2(i) = M_i^*, \; i = 1, 2, \ldots, N_p \qquad (2.86)$$

$$\sum_{n=1}^{N_s} m_n \phi_n(j) = \sum_{n=1}^{N_s} m_n \phi_n^2(j) = m_j^*, \; j = 1, 2, \ldots, N_s \qquad (2.87)$$

$$\sum_{n=1}^{N_s} m_n \, \phi_n(0) = \sum_{n=1}^{N_s} m_n \, \phi_n{}^2(0) = \sum_n m_n = m_0^* \; . \qquad (2.88)$$

Similarly, in view of the orthogonality conditions it may be observed that

$$\sum_{n=1}^{N_p} M_n \, \Phi_n(s) \, \Phi_n(t) = 0, \; t \neq s, \; t \text{ or } s \neq 0 \qquad (2.89)$$

$$\sum_{n=1}^{N_s} m_n \, \phi_n(s) \, \phi_n(t) = 0, \; t \neq s, \; t \text{ or } s \neq 0 \qquad (2.90)$$

Consequently, Eq. 2.85 may be put into the form

$$\alpha_r = \frac{\sum_{i=1}^{N_p} M_i Y_i^{(r)^2} + y_0^{(r)}(m_0^* - \sum_{j=1}^{N_s} m_j^*) + \sum_{j=1}^{N_s} m_j^*(y_0^{(r)}+y_j^{(r)})}{\sum_{i=1}^{N_p} M_i^* \, Y_i^{(r)^2} + y_0^{(r)^2} \; (m_0^* - \sum_{j=1}^{N_s} m_j^*) + \sum_{j=1}^{N_s} m_j^*(y_0^{(r)}+y_j^{(r)})^2} \, , \qquad (2.91)$$

and thus, if it is considered that the values of m_j^*, $j = 0, 1, \ldots, N_s$, and of the difference $m_0^* - \sum_{j=1}^{N_s} m_j^*$ are small and, hence, that the terms multiplied by this difference may be neglected, α_r may be approximated as

$$\alpha_r = \frac{\sum_{i=1}^{N_p} M_i^* \; Y_i^{(r)} + \sum_{j=1}^{N_s} m_j^* \, [y_0^{(r)} + y_j^{(r)}]}{\sum_{i=1}^{N_p} M_i^* \; Y_i^{(r)2} + \sum_{j=1}^{N_s} m_j^* \, [y_0^{(r)} + y_j^{(r)}]^2} \; . \qquad (2.92)$$

A simpler approximate formula for α_r may be obtained from this equation by considering only the $Y_i^{(r)}$ and $y_j^{(r)}$ factors which significantly contribute to the indicated summations. Accordingly, since by inspection of Eqs. 2.37, 2.38, 2.51, 2.60 and 2.61 one may conclude that of all such factors the largest in value are those corresponding to the closest natural frequencies of the primary and secondary components to the rth natural frequency of the assembled system, by denoting these largest factors by $Y_I^{(r)}$ and $y_J^{(r)}$, Eq. 2.92 may be written approximately as

$$\alpha_r = \frac{B_r Y_I^{(r)} + [y_0^{(r)} + y_J^{(r)}]\gamma_{IJ}}{{Y_I^{(r)}}^2 + [y_0^{(r)} + y_J^{(r)}]^2 \gamma_{IJ}} \tag{2.93}$$

where it is recalled that $\gamma_{IJ} = m_J^*/M_I^*$ and B_r is defined as

$$B_r = \frac{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)}}{M_I^* Y_I^{(r)}} \quad . \tag{2.94}$$

Notice that $\sum_{i=1}^{N_p} M_i Y_i^{(r)}$ in this last expression cannot be approximated by $M_I^* Y_I^{(r)}$ because its different terms may not be very much different in value. Notice, however, that when $M_I^* Y_I^{(r)}$ is indeed larger than the rest of the terms in the summation B_r results very close to unity.

Equation 2.93 is the desired simplified expression to compute the participation factors of an assembled system and the basis to derive with further simplifications the less accurate but simpler relationships in Chapter 3.

2.8 Maximum Modal Responses

In this study, the structural response of a secondary system to any given ground disturbance will be measured by the maximum distortions of the elements between its masses. Therefore, this response will be henceforth identified by what will be called "secondary element distortions"*. Such maximum distortions are of interest because they are directly related to the maximum stresses that earthquake ground motions induce into a system. It should be noticed, however, that the procedure presented in this section is not limited to this kind of response; it may be applied as well to predict any other response, as long as the expressions derived below be adjusted according to the definition of the response under consideration.

In accordance to the response spectrum method, the rth vector of maximum modal distortions of a system is determined by multiplying its rth vector of modal distortions (i.e. the difference in modal amplitudes between adjacent masses) by its rth participation factor and by the ordinate in the displacement response spectrum of the specified earthquake excitation corresponding to the frequency and damping ratio of its rth mode. For an assembled system such an rth vector of maximum modal distortions may be then expressed as

$$\{X\}^{(r)} = \alpha_r\{du\}^{(r)}\ SD\ (\omega_r,\ \xi_r) = \{du'\}\ SD\ (\omega_r,\ \xi_r) \qquad (2.95)$$

where $\{du\}^{(r)}$, the rth vector of modal distortions, is of the form

*Spring distortions or story drifts may be used as alternative names, but given the diverse nature of secondary systems these alternative names might not sound appropriate.

$$\{du\}^{(r)} = \begin{Bmatrix} u_{p_1}^{(r)} \\ u_{p_2}^{(r)} - u_{p_1}^{(r)} \\ \vdots \\ u_{s_1}^{(r)} - u_{p_k}^{(r)} \\ \vdots \\ u_{s_{N_s}}^{(r)} - u_{s_{N_s-1}}^{(r)} \end{Bmatrix}, \qquad (2.96)$$

$\{du'\}^{(r)}$ is the rth vector of unit-participation-factor modal distortions defined as

$$\{du'\}^{(r)} = \alpha_r\{du\}^{(r)}, \qquad (2.97)$$

and $SD(\omega_r, \xi_r)$ is the aforementioned spectral displacement corresponding to the rth natural frequency and the rth damping ratio of the system.

Presumably, the rth modal response of the secondary system alone may be written as

$$\{X_s\}^{(r)} = \alpha_r\{du_s\}^{(r)}SD\ (\omega_r, \xi_r) = \{du'_s\}^{(r)}SD\ (\omega_r, \xi_r) \qquad (2.98)$$

where, correspondingly,

$$\{du'_s\}^{(r)} = \alpha_r\{du_s\}^{(r)} \qquad (2.99)$$

and

$$\{du_s\}^{(r)} = \begin{Bmatrix} u_{s_1}^{(r)} - u_{p_k}^{(r)} \\ \vdots \\ u_{s_{N_s}}^{(r)} - u_{s_{N_s-1}}^{(r)} \end{Bmatrix}. \qquad (2.100)$$

2.9 Maximum Response: Combination of Modal Maxima

It has been recognized by several authors [1, 12, 28, 29,30] that the most commonly used rules to combine modal responses may become greatly inaccurate when they are applied to systems in which two or more of their frequencies lie very close to one another. For instance, the absolute sum of the maxima, an upper bound, may grossly overestimate their true maximum responses and the square root of the sum of the squares (SRSS), although it gives fairly good results for systems with well-separated frequencies, may give values far off their exact solutions. This fact may be explained as follows:

When all the frequencies of a system are well separated from one another the system usually has a dominant mode; therefore, its maximum response may be expected to be close to the maximum response in such dominant mode. For these systems, then, the rule used to estimate that maximum response is of little importance since no rule can deviate very much from the exact solution. In contrast, if a system has two or more natural frequencies close to one another, then it will have two or more modal responses with the same order of magnitude. As a result, since the contribution of each of these modal responses is equally important, the estimate of its maximum response become very sensitive to the rule adopted to combine those modal responses.

Since the assembled systems under study may have closely-spaced natural frequencies (see section 2.3), it may be seen, thus, that the accuracy achieved in the prediction of their maximum response may depend strongly on the rule selected for combining their modal responses. To estimate, then, with a reasonable accuracy the maximum response of these assembled systems, a general criterion to combine modes applicable to any of such systems is next described, discussed, and after a few simplifications, established for the systems treated in this chapter. In Chapter 8 this criterion is evaluated by comparing solutions obtained with conventional rules and with exact methods.

From random vibration theory [12], it has been established that the general expression for the maximum response of a N-degree-of-freedom system is of the form

$$\{X\}_{max} = \sqrt{\sum_{r=1}^{N} \{X\}^{(r)^2} + \sum_{\substack{m=1 \\ m\neq n}}^{N} \sum_{n=1}^{N} \alpha_{mn} \{X\}^{(m)} \{X\}^{(n)}} \qquad (2.101)$$

where $\{X\}_{max}$ is the vector of maximum responses to a given ground disturbance, i.e., the vector of maximum element distortions discussed in Sec. 2.8; $\{X\}^{(r)}$ is the vector of maximum modal responses corresponding to the rth mode; and α_{mn} is a factor, called modal correlation factor, that weights the coupling between the mth and nth modes. The absolute value of α_{mn} varies between 0 and 1.

In view of the fact that formula 2.101 is approximately equivalent to the solution formulated by random vibration methods, it may be assumed that this formula gives the "exact" maximum response. In this context, therefore, the problem of predicting the maximum response of a system is reduced to one of determining its modal correlation factors α_{mn} since, once these factors are known, the calculation of the maximum response readily follows. Unfortunately, such modal correlation factors cannot be determined in a precise manner. The chaotic nature of earthquakes makes extremely difficult the derivation of exact analytical expressions to relate those factors with the characteristics of structures and earthquakes. As a result, the maximum response of a system may only be approximated by introducing assumptions concerning its modal correlation factors. Thus, for example, if the correlation between its modes is assumed small, every one of its correlation factors may be set equal to zero. Then, the SRSS rule is obtained. Similarly, if it is assumed that every one of its modes is perfectly and positively correlated with one another, then each of its correlation factors may be set equal to unity. This assumption is thus tantamount to the absolute sum of the maxima.

It is apparent, therefore, that an accurate rule to combine the modal responses of a system with various modes of similar importance may be obtained if valid assumptions regarding its modal correlation factors may be established. In terms of the separation between its natural frequencies, Rosenblueth (1968) has derived an approximate expression for the modal correlation factors of such a system. Based on a model in which seismic disturbances are idealized as a segment of a stationary white noise process, he proposes

$$\alpha_{mn} = \frac{1}{1 + \left(\frac{\omega_n - \omega_m}{\xi'_m\omega_m + \xi'_n\omega_n}\right)^2} \tag{2.102}$$

where ξ'_r, r = m, n, a corrected damping ratio to account for the transitory nature of actual earthquakes, is given by

$$\xi'_r = \xi_r + \frac{2}{\omega_r s}\ , \quad r = m,\ n \tag{2.103}$$

and ω_r, r = m, n, is the rth natural circular frequency of the system. In this latter formula, ξ_r, r = m, n, is the corresponding rth modal damping ratio and s is the duration of the above mentioned white noise process which most closely represents the earthquake excitation under consideration. In general, this equivalent duration does not coincide with the actual duration of such an earthquake excitation and is different for earthquakes with different characteristics. For design purposes, therefore, s should be determined from the characteristics of the average earthquakes expected in an area of interest and from given site conditions. Thus, for example, Rosenblueth and Bustamante (1962) found that for a group of earthquakes recorded on relatively firm ground along the West Coast such an equivalent duration may be taken as 12.5 sec. In Sec. 2.10, a procedure is suggested to calculate the equivalent duration of a group of earthquakes from its average response spectra.

Equation 2.101 in combination with Eq. 2.102 satisfies the following limiting conditions:

1. When for all m and n ω_m and ω_n are far apart from each other, every α_{mn} approaches zero and hence Eq. 2.101 results as

$$\{X\}_{max} = \sqrt{\sum_{r=1}^{N} \{X\}^{(r)^2}} \qquad (2.104)$$

2. For a two degree of freedom system with $\omega_1 = \omega_2$ and $\xi_1 = \xi_2$ Eq. 2.102 gives $\alpha_{12} = 1.0$ and as a result Eq. 2.101 becomes

$$\{X\}_{max} = \{X\}^{(1)} + \{X\}^{(2)} \qquad (2.105)$$

3. For every value of α_{mn},

$$\{X\}_{max} \leq \sum_{r=1}^{N} |\{X\}^{(r)}| . \qquad (2.106)$$

Observe also that when s, the earthquake equivalent duration, approaches infinity, the modal correlation factor for undamped systems approaches zero; hence, for large s Eq. 2.101 is tantamount to the SRSS rule. On the other hand, when s approaches zero, α_{mn} approaches unity; then Eq. 2.101 turns out to be the algebraic sum of the modal maxima.

Rosenblueth's method remains valid as long as the assumptions on which the derivation of Eq. 2.102 is based are approximately satisfied. According to Newmark and Rosenblueth (1971), this equation is valid if for a given structure and an actual earthquake:

a) The dominant natural periods of the structure are not excessively short,

b) The velocity response spectra as a function of the natural circular frequency do not have too pronounced a curvature in the neighborhood of the natural frequencies of the structure, and

c) The fundamental period of the structure is shorter, or at least not much longer, than the duration of the earthquake.

Accordingly, Rosenblueth's rule will be accurate for most practical structures when they are founded in firm ground at moderate distances from focal points and when the shorter periods of the structure do not significantly contribute to the response.

Notice, however, that even though Rosenblueth's rule may be applied to a broad variety of structures, it may not be considered as a general rule. In this respect, therefore, Eq. 2.101 should be thought as that general rule in which the specification of the required modal correlation factors may ultimately be left to the designer's judgement, who, in any particular case, may consider appropriate to choose, for the sake of simplicity, conservative values. Notwithstanding, the derivations in this work will be limited to those systems for which Rosenblueth's modal correlation factors are applicable.

Thus, if it is taken into account that for an assembled system of the kind studied in this chapter the natural frequencies of its separate primary and secondary components are, by assumption, far apart from one another and that the resulting natural frequencies of its resonant and nonresonant modes are very close to those of such separate components, all its correlation factors other than those between two adjacent resonant modes may be neglected and, as a consequence, for such a system Eq. 2.101 may be simplified as

$$\{X\}_{max} = \sqrt{\sum_{r=1}^{N_p + N_s} \{X\}^{(r)^2} + 2 \sum_{R/2} \alpha_{n(n+1)} \{X\}^{(n)} \{X\}^{(n+1)}} \qquad (2.107)$$

where R is the number of resonant modes in the system, $\{X_s\}^{(n)}$ and $\{X_s\}^{(n+1)}$ are two of such adjacent resonant modes, and $\alpha_{n(n+1)}$ is their associated correlation factor. In turn, this correlation factor may be simplified as follows:

In the light of Eq. 2.102 $\alpha_{n(n+1)}$ may be written as

$$\alpha_{n(n+1)} = \frac{1}{1 + \left(\dfrac{\omega_{n+1} - \omega_n}{\xi'_n \omega_n + \xi'_{n+1} \omega_{n+1}}\right)^2} \quad . \qquad (2.108)$$

But since the systems treated in this chapter are systems with proportional damping and since ω_n and ω_{n+1} are by hypothesis very close to each other, it may be assumed that

$$\xi'_n = \xi'_{n+1} = \xi'_o = \xi_o + \frac{2}{\omega_o s} \quad , \tag{2.109}$$

in which ω_o and ξ_o are respectively a common natural frequency and damping ratio of the above mentioned separate primary and secondary components. Hence, Eq. 2.108 may be expressed as

$$\alpha_{n(n+1)} = \frac{1}{1 + \frac{1}{\xi'_o}\left(\frac{\omega_{n+1}-\omega_n}{\omega_n + \omega_{n+1}}\right)^2} \quad . \tag{2.110}$$

Therefore, if the expression for resonant frequencies given by Eq. 2.52 is substituted, $\alpha_{n(n+1)}$ may be approximated as

$$\alpha_{n(n+1)} = \frac{1}{1 + \frac{\Phi_k^2(I)\ \gamma_{IJ}}{4\xi'^2_o}} \quad . \tag{2.111}$$

Equation 2.107 in combination with Eq. 2.111 and 2.109 will constitute the rule adopted in this chapter to combine the modes of the assembled systems under study.

2.10 Earthquake Duration for Equivalent Ground Motion Excitations

As mentioned in the foregoing section, the derivation of Rosenblueth's rule is based on the idealization of an earthquake excitation as a segment of a white noise process, i.e., a series of random impulses with constant intensity per unit time; and hence, in order to apply this rule, it is necessary to determine an equivalent duration by which specified earthquake excitations may be represented by such an ideal segment of white noise. A procedure by which such an equivalent duration may be obtained is then established in this section as follows:

It is stated by Newmark and Rosenblueth (1971) that for a segment of white noise the ratio of the expected values of its damped to undamped pseudovelocities may be approximated by

$$\beta_E = \frac{E(SV)}{E(SV_0)} = (1 + 0.5\xi\omega s)^{-0.5} \qquad (2.112)$$

where E(SV) denotes the expected pseudovelocity for a damping ratio ξ, ω represents a natural circular frequency, and s is the duration of the process; subscript 0 stands for 0% damping. In theory, then, if an average earthquake motion is equivalent to a white noise process, the β_E ratios calculated from its response spectrum should be equal to those obtained by Eq. 2.112. Thus, the equivalent duration for a group of earthquakes representing the earthquakes expected in a given area may be determined by choosing the duration s that gives the best fit between the β_E values calculated from the average response spectrum for that group of earthquakes and those computed by means of Eq. 2.112. Since the "best fit" is not necessarily the same for different damping values, notice that, in general, different durations will be obtained for different percentages of damping.

Equation 2.112 is useful to adjust the duration of earthquakes for any percentage of damping except zero percent. Therefore, it is necessary to adopt a separate criterion for this particular case. In this work, the duration for zero percent damping will be calculated by assuming that the relation between this duration and that for a small percentage of damping is directly proportional to the relation between the expected values of their corresponding pseudovelocity spectral ordinates. That is, if s_0 denotes the duration for zero percent damping, s_0 will be calculated as

$$s_0 = \frac{E(SV_0)}{E(SV)} s \tag{2.113}$$

where s is the duration for such a small percentage of damping determined by Eq. 2.112 and the procedure introduced at the beginning of this section, and E(SV) represents the expected value of an ordinate in a pseudovelocity response spectrum.

Equation 2.113 may be justified if it is considered that: (1) the ordinates of a response spectrum change only slightly with a small variation in the value of the considered percentage of damping, (2) the duration obtained for a small percentage of damping should consequently be very close to the one for zero damping, and (3) because of their closeness, a linear variation suffices to relate these two durations and their pseudovelocity ordinates.

One should observe that although for a white noise process the expected undamped pseudovelocity is constant for all frequencies (see Rosenblueth and Bustamante, 1962), the average response spectra for a finite sample of earthquakes will be, no doubt, frequency dependent. Therefore, s_0 should be determined by selecting the duration that gives the best fit between the observed ordinates in the pseudovelocity portion of an average zero percent damping response spectrum and those computed by: (a) Eq. 2.113, (b) the observed spectral ordinates in the corresponding response spectrum for a small percentage of damping, and (c) the equivalent duration for this small percentage of damping.

The above criteria are applied in Chapter 8 to find the equivalent durations of three recorded earthquakes.

2.11 Illustrative Example

In order to illustrate and summarize the procedure developed in the foregoing sections, the maximum distortions of a two-degree-of-freedom secondary system connected to the first floor of a three-degree-of-freedom primary structure are here calculated for the case when the base of this primary structure is subjected to a portion of El Centro (May 18, 1940) earthquake ground acceleration record. The primary and secondary systems are depicted in Fig. 2.3, and the response spectrum of the considered portion of the mentioned acceleration record is shown in Fig. 8.3(a). These primary and secondary systems are assumed to be linear elastic structures whose damping matrices are proportional to their respective stiffness ones and to form an assembled system also with proportional damping whose damping ratio in the fundamental mode is of 2 percent. The following are the dynamic properties of such independent primary and secondary systems:

Primary System:

$$[\Phi] = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 1.0 & 0.2 & -0.2 \\ 1.5 & -0.6 & 0.1 \end{bmatrix} \quad \begin{matrix} f_{p_1} = 1.0 \text{ c.p.s} & \xi_{p_1} = 0.02 & M_1^* = 4.5 \\ f_{p_2} = 2.0 \text{ c.p.s} & \xi_{p_2} = 0.04 & M_2^* = 0.9 \\ f_{p_3} = 3.0 \text{ c.p.s} & \xi_{p_3} = 0.06 & M_3^* = 0.1 \end{matrix}$$

Secondary system:

$$[\phi] = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix} \quad \begin{matrix} f_{s_1} = 2.0 \text{ c.p.s} & \xi_{s_1} = 0.040 & m_1^* = 0.009 \\ f_{s_2} = 2.0\sqrt{3} \text{ c.p.s} & \xi_{s_2} = 0.069 & m_2^* = 0.003 \end{matrix}$$

Thus, the described primary and secondary systems give rise to a five-degree-of-freedom assembled system (see Fig. 2.3) whose five mode shapes and secondary modal distortions may be computed, on the basis of these dynamic properties and according to the procedure established in this chapter, as follows.

Mode Shapes and Secondary Distortions

First Mode. In this case, the first mode of the assembled system is a nonresonant mode with a frequency close to the fundamental frequency of the primary system. According to the discussion in Sec. 2.4, the frequency of this first mode may be therefore approximated as

$$f_1 = f_{p_1} = 1.0 \text{ c.p.s.}$$

Thus, Eqs. 2.63, 2.38 and 2.39 lead to the following $Y_i^{(1)}$ and $y_j^{(1)}$ factors:

$$Y_1^{(1)} = 1.0$$

$$Y_2^{(1)} = 0$$

$$Y_3^{(1)} = 0$$

$$y_0^{(1)} = 0.5$$

$$y_1^{(1)} = \frac{1.0}{4.0 - 1.0} (0.5) = 0.16667$$

$$y_2^{(2)} = \frac{1.0}{12.0 - 1.0} (0.5) = 0.04545.$$

Equations 2.35 and 2.36 yield then the following first mode amplitudes of the primary and secondary masses:

$$\{u_p\}^{(1)} = \begin{Bmatrix} 0.5 \\ 1.0 \\ 1.5 \end{Bmatrix}$$

$$\{u_s\}^{(1)} = 0.5\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + 0.16667\begin{Bmatrix} 0.5 \\ 1.5 \end{Bmatrix} + 0.04545\begin{Bmatrix} 0.5 \\ -0.5 \end{Bmatrix} = \begin{Bmatrix} 0.60606 \\ 0.72728 \end{Bmatrix} .$$

In like manner, since in this case

$$B_1 = 1.0,$$

Eq. 2.93 gives

$$\alpha_1 = \frac{1.0 + (0.5 + 0.16667)\ 0.002}{1.0 + (0.5 + 0.16667)^2\ 0.002} = 1.00044.$$

Consequently, the approximate first normalized mode shape of the assembled system results as

$$\{u'\}^{(1)} = 1.00044 \begin{Bmatrix} 0.50000 \\ 1.00000 \\ 1.50000 \\ 0.60606 \\ 0.72728 \end{Bmatrix} = \begin{Bmatrix} 0.50022 \\ 1.00044 \\ 1.50066 \\ 0.60633 \\ 0.72760 \end{Bmatrix}$$

from which one arrives to the following normalized secondary distortions:

$$\{du'_s\}^{(1)} = \begin{Bmatrix} 0.60633 - 0.50022 \\ 0.72760 - 0.60633 \end{Bmatrix} = \begin{Bmatrix} 0.10611 \\ 0.12127 \end{Bmatrix}.$$

Second and Third Modes. Since the second frequency of the primary system is in resonance with the first of the secondary system (i.e., $f_{s_1} = f_{p_2}$), the assembled system results with two modes, resonant modes, whose frequencies are close to the resonant frequency $f_o = 2.0$ c.p.s. By virtue of Eq. 2.51, then, the squares of the frequencies of these two resonant modes are

$$f_2^2 = 4.0\ (1 - 0.4\sqrt{0.01}) = 3.84$$

$$f_3^2 = 4.0\ (1 + 0.4\sqrt{0.01}) = 4.16.$$

Thus, in the light of Eqs. 2.37 through 2.39 the $Y_i^{(2)}$ and $y_j^{(2)}$ factors result as

$$Y_1^{(2)} = 1.0$$

$$Y_2^{(2)} = \frac{3.84 - 1.0}{3.84 - 4.0}\ \frac{4.5/0.5}{0.9/0.4} = -71.00000$$

$$Y_2^{(2)} = \frac{3.84 - 1.0}{3.84 - 9.0}\ \frac{4.5/0.5}{0.1/0.1} = -4.95349$$

$$y_0^{(2)} = 0.5(1.0) + 0.4(-71.00000) + 0.1(-4.95349) = -28.39535$$

$$y_1^{(2)} = \frac{3.84}{4.0 - 3.84}\ (-28.39535) = -681.48838$$

$$y_2^{(2)} = \frac{3.84}{12.0 - 3.84}(-28.39535) = -13.36252$$

from which Eqs. 2.35 and 2.36 lead to

$$\{u_p\}^{(2)} = 1.0\begin{Bmatrix}0.5\\1.0\\1.5\end{Bmatrix} - 71.00000\begin{Bmatrix}0.4\\0.2\\-0.6\end{Bmatrix} - 4.95349\begin{Bmatrix}0.1\\-0.2\\0.1\end{Bmatrix} = \begin{Bmatrix}-28.39535\\-12.20930\\43.60465\end{Bmatrix}$$

$$\{u_s\}^{(2)} = -28.39535\begin{Bmatrix}1\\1\end{Bmatrix} - 681.48838\begin{Bmatrix}0.5\\1.5\end{Bmatrix} - 13.36252\begin{Bmatrix}0.5\\-0.5\end{Bmatrix} = \begin{Bmatrix}-375.82079\\-1043.94665\end{Bmatrix}$$

Similarly, by substitution of the above $Y_i^{(2)}$ and $y_j^{(2)}$ values into Eqs. 2.93 and 2.94 one obtains

$$B_2 = \frac{(4.5 \times 1.0 - 0.9 \times 71.00000 - 0.1 \times 4.95349)}{-0.9 \times 71.00000} = 0.93733$$

$$\alpha_2 = \frac{0.93733\ (-71.00000) + (-28.39535 - 681.48838)\ 0.01}{(-71.00000)^2 + [-28.39535 - 681.48838\ ]^2\ 0.01} = -0.007306 \ .$$

As a result, the approximate second normalized mode is given by

$$\{u'\}^{(2)} = -0.007306 \begin{Bmatrix} -28.39535 \\ -12.20930 \\ 43.60465 \\ -375.82079 \\ -1043.94665 \end{Bmatrix} = \begin{Bmatrix} 0.20746 \\ 0.08920 \\ -0.31858 \\ 2.74583 \\ 7.62730 \end{Bmatrix}$$

whence it may be seen that

$$\{du_s'\}^{(2)} = \begin{Bmatrix} 2.74583 - 0.20746 \\ 7.62730 - 2.74583 \end{Bmatrix} = \begin{Bmatrix} 2.53837 \\ 4.88147 \end{Bmatrix} .$$

With a similar procedure for the third mode, the following values are obtained:

$$Y_1^{(3)} = 1.0$$

$$Y_2^{(3)} = \frac{4.16 - 1.0}{4.16 - 4.0}\ \frac{4.5/0.5}{0.9/0.4} = 79.00000$$

$$Y_3^{(3)} = \frac{4.16 - 1.0}{4.16 - 9.0}\ \frac{4.5/0.5}{0.1/0.1} = -5.87603$$

$$y_0^{(3)} = 0.5(1.0) + 0.4\ (79.00000) + 0.1\ (-5.87603) = 31.51240$$

$$y_1^{(3)} = \frac{4.16\ (31.51240)}{4.0 - 4.16} = -819.32232$$

$$y_2^{(3)} = \frac{4.16\ (31.51240)}{12.0 - 4.16} = 16.72086$$

$$\{u_p\}^{(3)} = 1.0 \begin{Bmatrix} 0.5 \\ 1.0 \\ 1.5 \end{Bmatrix} + 79.00000 \begin{Bmatrix} 0.4 \\ 0.2 \\ -0.6 \end{Bmatrix} - 5.87603 \begin{Bmatrix} 0.1 \\ -0.2 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} 31.51240 \\ 17.97521 \\ -46.48760 \end{Bmatrix}$$

$$\{u_s\}^{(3)} = 31.51240 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - 819.32232 \begin{Bmatrix} 0.5 \\ 1.5 \end{Bmatrix} + 16.72086 \begin{Bmatrix} 0.5 \\ -0.5 \end{Bmatrix} = \begin{Bmatrix} -369.78837 \\ -1205.83163 \end{Bmatrix}$$

$$B_3 = \frac{(4.5 \times 1.0 + 0.9 \times 79.00000 - 0.1 \times 5.87603)}{0.9 \times 79.00000} = 1.05503$$

$$\alpha_3 = \frac{(1.05503)(79.00000) + (31.51240 - 819.32232)\ 0.01}{(79.00000)^2 + [31.51240 - 819.32232)]^2\ 0.01} = 0.006063$$

$$\{u'\}^{(3)} = 0.006063 \begin{Bmatrix} 31.51240 \\ 17.97521 \\ -46.48760 \\ -369.78837 \\ -1205.83163 \end{Bmatrix} = \begin{Bmatrix} 0.19106 \\ 0.10898 \\ -0.28185 \\ -2.24203 \\ -7.31096 \end{Bmatrix}$$

$$\{du_s'\}^{(3)} = \begin{Bmatrix} -2.24203 - 0.19106 \\ -7.31096 + 2.24203 \end{Bmatrix} = \begin{Bmatrix} -2.43309 \\ -5.06893 \end{Bmatrix}.$$

Fourth Mode. The fourth mode is also a nonresonant mode with frequency close to a primary frequency, the third primary one. Hence, f_4 may be approximated as

$$f_4 = f_{p_3} = 3.0 \text{ c.p.s.}$$

According to Eqs. 2.63 and 2.38 one has thus that

$$Y_1^{(4)} = 0$$

$$Y_2^{(4)} = 0$$

$$Y_3^{(4)} = 1.0$$

$$y_0^{(4)} = 0.1$$

$$y_1^{(4)} = \frac{9.0}{4.0 - 9.0}(0.1) = -0.18$$

$$y_2^{(4)} = \frac{9.0}{12.0 - 9.0}(0.1) = 0.30\ .$$

Therefore, $\{u_p\}^{(4)}$, $\{u_s\}^{(4)}$, B_4 and α_4 result as

$$\{u_p\}^{(4)} = \begin{Bmatrix} 0.1 \\ -0.2 \\ 0.1 \end{Bmatrix}$$

$$\{u_s\}^{(4)} = 0.1\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - 0.18\begin{Bmatrix} 0.5 \\ 1.5 \end{Bmatrix} + 0.30\begin{Bmatrix} 0.5 \\ -0.5 \end{Bmatrix} = \begin{Bmatrix} 0.16000 \\ -0.32000 \end{Bmatrix}$$

$$B_4 = 1.0$$

$$\alpha_4 = \frac{1 + (0.1 + 0.30)\ 0.03}{1 + (0.1 + 0.30)^2\ 0.03} = 1.00717\ .$$

Consequently,

$$\{u'\}^{(4)} = 1.00717\begin{Bmatrix} 0.10000 \\ -0.20000 \\ 0.10000 \\ 0.16000 \\ -0.32000 \end{Bmatrix} = \begin{Bmatrix} 0.10072 \\ -0.20143 \\ 0.10072 \\ 0.16115 \\ -0.32229 \end{Bmatrix}$$

and

$$\{du_s'\}^{(4)} = \begin{Bmatrix} 0.16115 - 0.10072 \\ -0.32229 - 0.16115 \end{Bmatrix} = \begin{Bmatrix} 0.06043 \\ -0.48344 \end{Bmatrix}.$$

Fifth Mode. The only one in this example, the fifth mode is a nonresonant mode with its frequency close to one of the frequencies of the secondary system. Accordingly, it is valid to approximate this fifth natural frequency as

$$f_5 = f_{s_2} = 2.0\sqrt{3} \text{ c.p.s.}$$

As a result, Eq. 2.32 and 2.33 give

$$Y_1^{(5)} = 1.0$$

$$Y_2^{(5)} = \frac{12.0 - 1.0}{12.0 - 4.0} \; \frac{4.5/0.5}{0.9/0.4} = 5.5$$

$$Y_3^{(5)} = \frac{12.0 - 1.0}{12.0 - 9.0} \; \frac{4.5/0.5}{0.1/0.1} = 33.0$$

$$y_0^{(5)} = 0.5(1.0) + 0.4(5.5) + 0.1(33.0) = 6.0$$

$$y_1^{(5)} = \frac{12.0}{4.0 - 12.0} (6.0) = 9.0 \cdot$$

In order to find the value of the second $y_j^{(5)}$ factor, it is necessary to resort to the alternative expression given by Eq. 2.80 since in this case $\omega_r = \omega_{s_j}$. Thus, if in this equation i is chosen arbitrarily as 3, one obtains

$$y_2^{(5)} = \frac{9.0 - 12.0}{(0.1)(12.0)(0.03)}(33.0) = -2750.0.$$

The above $Y_i^{(5)}$ and $y_j^{(5)}$ factors and Eqs. 2.35, 2.36, 2.93 and 2.94 lead therefore to

$$\{u_p\}^{(5)} = 1.0\begin{Bmatrix}0.5\\1.0\\1.5\end{Bmatrix} + 5.5\begin{Bmatrix}0.4\\0.2\\-0.6\end{Bmatrix} + 33.0\begin{Bmatrix}0.1\\-0.2\\0.1\end{Bmatrix} = \begin{Bmatrix}6.00000\\-4.50000\\1.50000\end{Bmatrix}$$

$$\{u_s\}^{(5)} = 6.0\begin{Bmatrix}1\\1\end{Bmatrix} + 9.0\begin{Bmatrix}0.5\\1.5\end{Bmatrix} - 2750.0\begin{Bmatrix}0.5\\-0.5\end{Bmatrix} = \begin{Bmatrix}-1364.5\\1394.5\end{Bmatrix}$$

$$B_5 = \frac{(4.5 \times 1.0 + 0.9 \times 5.5 + 0.1 \times 33.0)}{0.1 \times 33.0} = 3.86364$$

$$\alpha_5 = \frac{3.86364(33.0) + (6.0 - 2750.0)\ 0.03}{(33.0)^2 + (6.0 - 2750.0)^2\ 0.03} = 0.000199\ .$$

Hence,

$$\{u'\}^{(5)} = 0.000199 \begin{Bmatrix} 6.00000 \\ -4.50000 \\ 1.50000 \\ -1364.50000 \\ 1394.50000 \end{Bmatrix} = \begin{Bmatrix} 0.00119 \\ -0.00090 \\ 0.00030 \\ -0.27340 \\ 0.27221 \end{Bmatrix}$$

and

$$\{du_s'\}^{(5)} = \begin{Bmatrix} -0.27340 - 0.00119 \\ 0.27221 + 0.27340 \end{Bmatrix} = \begin{Bmatrix} -0.27459 \\ 0.54561 \end{Bmatrix}\ .$$

Observe that even though this is a fifth mode (the highest mode of the system) the modal distortions represent about 10 percent of the distortions in a resonant mode. Contrary to the belief that the resonant modes are the only modes of any importance, this example shows that modes with frequencies equal to secondary frequencies may be of some importance whenever they are among the first modes of a system.

Maximum Modal Responses

From Fig. 8.3(a), the spectral displacements SD (f_r, ξ_r) corresponding to the frequencies and damping ratios of each of the above modes are:

SD (1.0, 0.020) = 0.168 m

SD (2.0, 0.040) = 0.055 m

SD (3.0, 0.060) = 0.017 m

SD $(2\sqrt{3}$, 0.069) = 0.013 m .

In the light of Eq. 2.98 and the foregoing secondary distortions the maximum secondary modal responses result then as

$$\{X_s\}^{(1)} = 0.168\begin{Bmatrix}0.10611\\0.12127\end{Bmatrix} = \begin{Bmatrix}0.018\\0.020\end{Bmatrix}\text{m}$$

$$\{X_s\}^{(2)} = 0.055\begin{Bmatrix}2.53837\\4.88147\end{Bmatrix} = \begin{Bmatrix}0.140\\0.268\end{Bmatrix}\text{m}$$

$$\{X_s\}^{(3)} = 0.055\begin{Bmatrix}-2.43309\\-5.06893\end{Bmatrix} = \begin{Bmatrix}-0.134\\-0.279\end{Bmatrix}\text{m}$$

$$\{X_s\}^{(4)} = 0.017\begin{Bmatrix}0.06043\\-0.48344\end{Bmatrix} = \begin{Bmatrix}0.001\\-0.008\end{Bmatrix}\text{m}$$

$$\{X_s\}^{(5)} = 0.013\begin{Bmatrix}-0.27459\\0.54561\end{Bmatrix} = \begin{Bmatrix}-0.004\\0.007\end{Bmatrix}\text{m}\ .$$

Maximum Secondary Distortions

Since in this case

$$\xi_o = \xi_{p_2} = \xi_{s_1} = 0.040$$

and since from Fig. 8.8 it may be seen that for the excitation under consideration, this damping value , and a frequency of 2.0 c.p.s

$$s = 9.7 \text{ sec},$$

by substitution into Eq. 2.103 the following corrected damping ratio for the resonant modes is obtained:

$$\xi_o' = 0.040 + \frac{2/9.7}{2\pi \times 2.0} = 0.056;$$

hence, their correlation factor (see Eq. 2.111) is

$$\alpha_{23} = \frac{1}{1 + \frac{(0.4)^2\ 0.01}{4\ (0.056)^2}} = 0.887\ .$$

By virtue of Eq. 2.107, the maximum distortions of the secondary system result thus as

$$\left\{X_s\right\}_{max} = \sqrt{\begin{Bmatrix}0.018\\0.020\end{Bmatrix}^2 + \begin{Bmatrix}0.140\\0.268\end{Bmatrix}^2 + \begin{Bmatrix}-0.134\\-0.279\end{Bmatrix}^2 + \begin{Bmatrix}0.001\\-0.008\end{Bmatrix}^2 + \begin{Bmatrix}-0.004\\0.007\end{Bmatrix}^2 - 2(0.887)\begin{Bmatrix}0.140\\0.268\end{Bmatrix}\begin{Bmatrix}0.134\\0.279\end{Bmatrix}} = \begin{Bmatrix}0.068\\0.133\end{Bmatrix} m$$

The approximate results herein obtained may be compared with their corresponding exact solutions in Tables 8.9 and 8.30. For reference, the example just solved corresponds among the systems solved in Chapter 8 to the system B2 with a mass ratio of 1 percent. The exact five modes and natural frequencies are shown in Table 8.9 whereas the exact maximum response is shown, corresponding to El Centro earthquake and 2 percent damping, in Table 8.30.

CHAPTER 3

APPROXIMATE METHOD: PROPORTIONAL DAMPING AND A SINGLE POINT OF ATTACHMENT

3.1 Introduction

In the previous chapter, approximate expressions have been derived to compute the natural frequencies, mode shapes, and participation factors of the system formed by a structure and its attached secondary system. A rule to combine the maximum modal responses of such an assembled system has also been established. With such expressions and this rule, a procedure is then suggested to calculate through the modal analysis of this assembled system the seismic response of the secondary system. In this chapter, these approximate expressions and rule to combine modes are further simplified and incorporated into a single expression to develop a simple formula by which one may obtain, with a reasonable accuracy, quick estimates of the expected maximum responses of secondary systems to any specified ground disturbances.

As in the preceding chapter, the derivation of this simplified formula will be here limited to secondary systems which have only one point of attachment and which in combination with their supporting structures form assembled systems with proportional damping. In Chapters 4 and 6, it will be extended for systems with two points of attachment and nonproportional damping.

3.2 Maximum Modal Responses: Resonant Modes

According to the discussions in Sec. 2.2 and 2.3, the natural frequencies and the secondary system part of the mode shapes of the resonant modes of an assembled system are given respectively by Eqs. 2.51 and 2.36. By the same argument used in Sec. 2.3 and 2.4 to approximate the natural frequencies of such an assembled system, it may be seen thus that if all the insignificant $Y_i^{(r)}$ and $y_j^{(r)}$ factors in Eq. 2.36 are neglected, the amplitudes of the secondary system in these mode shapes may be approximated as

$$\{u_s\}^{(r)} = y_0^{(r)}\{J\} + y_J^{(r)}\{\phi\}^{(J)} \tag{3.1}$$

where by the same token $y_0^{(r)}$ may be written approximately as

$$y_0^{(r)} = u_{p_k}(r) = \Phi_k(I)Y_I^{(r)} \tag{3.2}$$

and where, as before, subscripts I and J identify the modes of the primary and secondary systems whose frequencies are in resonance.

In the light of Eq. 2.99, the vector of secondary modal distortions may be therefore expressed as

$$\{du_s'\}^{(r)} = \alpha_r\, y_J^{(r)}\, \{d\phi\}^{(J)} \tag{3.3}$$

in which $\{d\phi\}^{(J)}$ is of the form

$$\{d\phi\}^{(J)} = \begin{Bmatrix} \phi_1(J) \\ \phi_2(J) - \phi_1(J) \\ \vdots \\ \phi_{N_s}(J) - \phi_{N_s-1}(J) \end{Bmatrix} \tag{3.4}$$

However, by substitution of Eq. 2.51 and 3.2 into Eq. 2.38, $y_J^{(r)}$ may be written as

$$y_J^{(r)} = [\pm \frac{1}{\sqrt{\gamma_{IJ}}} - \Phi_k(I)]\, Y_I^{(r)} \tag{3.5}$$

which for small mass ratios may be approximated as

$$\gamma_J^{(r)} = \pm \frac{1}{\sqrt{\gamma_{IJ}}} \gamma_I^{(r)} . \qquad (3.6)$$

Similarly, if Eqs. 3.2 and 3.6 are substituted into Eq. 2.93 and if, again, all insignificant $\gamma_i^{(r)}$ factors are neglected, the participation factor α_r may be written as

$$\alpha_r = \frac{1}{\gamma_I^{(r)}} \left(\frac{1}{2} + \frac{1}{2} \sqrt{\gamma_{IJ}} \right) \qquad (3.7)$$

from which it may be seen that for small mass ratios a good approximation for this participation factor is

$$\alpha_r = \frac{1}{2} \frac{1}{\gamma_I^{(r)}} . \qquad (3.8)$$

By virtue of Eqs. 3.3, 3.6, 3.8, and 2.98, the maximum secondary distortions in the resonant modes result thus approximately as

$$\{X_s\}^{(r)} = \pm \frac{1}{2} \frac{1}{\sqrt{\gamma_{IJ}}} \{d\phi\}^{(J)} SD(\omega_0, \xi_0) \qquad (3.9)$$

where it has been assumed that the spectral ordinates for two adjacent resonant modes are the same and equal to the one for ω_0 and ξ_0, the natural frequency and damping ratio of the corresponding modes in resonance of the primary and secondary systems.

3.3 Maximum Modal Responses: Nonresonant Modes

The natural frequencies and secondary system part of the mode shapes of the nonresonant modes of an assembled system may be determined by Eqs. 2.60, 2.61 and 2.36. Using the procedure in the preceding section, then, it is also possible to derive approximate expressions for the maximum responses of the secondary system in these nonresonant modes. However, since the expressions derived in the last chapter for the $y_j^{(r)}$ factors of Eq. 2.36 are different for those nonresonant modes with a frequency close to any of the frequencies of the primary system and those with a frequency close to any of the secondary system's, these approximate expressions are here derived separately for each of these cases.

Case I: $\omega_r = \omega_{p_I}$

In view of the discussion in Sec. 2.5, the $y_0^{(r)}$ and $y_j^{(r)}$ factors of Eq. 2.36 result in this case as

$$y_0^{(r)} = u_{p_k}(r) = \phi_k(I) \tag{3.10}$$

$$y_j^{(r)} = \phi_k(I)\,\frac{\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2}\quad . \tag{3.11}$$

By virtue of Eq. 2.35 and by noticing that all these $y_j^{(r)}$ factors may be of the same order of magnitude, the rth vector of secondary modal distortions may be then expressed as

$$\{du_s'\}^{(r)} = \alpha_r \sum_{j=1}^{N_s} y_j^{(r)}\,\{d\phi\}^{(j)} \tag{3.12}$$

in which by substitution of Eqs. 2.63, 3.10 and 3.11 into Eqs. 2.93 and 2.94 the participation factor α_r results of the form

$$\alpha_r = \frac{1 + [\phi_k(I) + \phi_k(I)\,\dfrac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}]\,\gamma_{IJ}}{1 + [\phi_k(I) + \phi_k(I)\,\dfrac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}]^2\,\gamma_{IJ}}\quad . \tag{3.13}$$

If it is observed, however, that the maximum value of $\Phi_k(I)\omega_{p_I}^2/(\omega_{s_J}^2 - \omega_{p_I}^2)$ is $\sqrt{\gamma_{IJ}}$ (see Sec. 2.5), then for small mass ratios the numerator of this equation may be approximated by unity. Similarly, it may be noticed that when ω_{s_J} and ω_{p_I} are well separated from each other the second term in the denominator results negligibly small if compared with unity. On the other hand, when these two frequencies are very close, the first term in the expression between brackets in the same denominator becomes relatively small and may be neglected. Therefore, in all cases it is justified to approximate α_r as

$$\alpha_r = \frac{1}{1 + \Phi_k^2(I)\left(\frac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}\right)^2 \gamma_{IJ}} \ . \tag{3.14}$$

Thus, if one denotes

$$A_o(j) = \Phi_k(I)\,\frac{\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2} \ , \tag{3.15}$$

by which $y_j^{(r)}$ and α_r may be alternatively expressed as

$$y_j^{(r)} = A_o(j) \tag{3.16}$$

$$\alpha_r = \frac{1}{1 + A_o^2(J)\gamma_{IJ}} \ , \tag{3.17}$$

$\{du_s'\}$ may be written as

$$\{du_s'\} = \frac{1}{1 + A_o^2(J)\gamma_{IJ}} \sum_{j=1}^{N_s} A_o(j)\ \{d\phi\}^{(j)} \tag{3.18}$$

or as

$$\{du_s'\} = \frac{A_o(J)}{1 + A_o^2\,(J)\gamma_{IJ}} \sum_{j=1}^{N_s} r_j\,\{d\phi\}^{(j)} \tag{3.19}$$

where r_j, defined as

$$r_j = \frac{A_o(j)}{A_o(J)}\ , \tag{3.20}$$

is a factor that indicates the participation of the jth secondary mode in the formation of the vector $\{du_s'\}$. Notice that r_j varies between -1 and 1 and that it is always equal to 1 for the closest secondary mode (i.e., j = J).

According to Eq. 2.98, the rth vector of maximum secondary distortions in these nonresonant modes may be therefore approximated as

$$\{X_s\}^{(r)} = \frac{A_o(J)}{1 + A_o^2(J)\gamma_{IJ}} \left[\sum_{j=1}^{N_s} r_j\,\{d\phi\}^{(j)}\right] SD(\omega_{p_I},\xi_{p_I}) \tag{3.21}$$

where $SD(\omega_{p_I},\xi_{p_I})$ is the ordinate in the specified displacement response spectrum corresponding to the Ith natural frequency and damping ratio of the primary system.

Notice that since Eq. 3.11 is only valid for the interval (see Sec. 2.5)

$$\left|\frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{p_I}^2}\right| \geq \left|\ \Phi_k(I)\sqrt{\gamma_{IJ}}\ \right|\ , \tag{3.22}$$

Eq. 3.21 is also only valid for this interval.

Case II: $\omega_r \simeq \omega_{s_J}$

Because the closeness between ω_r and ω_{s_J} and hence the large values of $y_j^{(r)}$, the secondary modal distortions in this kind of nonresonant modes may also be approximated by Eq. 3.3, except that in this case the associated subscripts I and J do not refer to the primary and secondary modes in resonance but to those whose frequencies are the closest to the frequency of the nonresonant mode under consideration, and that the indicated $y_j^{(r)}$ factor is now given, according to Eq. 2.80, by

$$y_j^{(r)} = \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\Phi_k(I)\omega_{s_J}^2 \gamma_{IJ}} \, Y_I^{(r)} \quad . \tag{3.23}$$

Therefore, if this equation and Eq. 3.2 are substituted into Eq. 2.93, the corresponding participation factor results as

$$\alpha_r = \frac{1}{Y_I^{(r)}} \frac{B_r + \left[\Phi_k(I) + \dfrac{\omega_{p_I}^2 - \omega_{s_J}^2}{\Phi_k(I)\omega_{s_J}^2 \gamma_{IJ}}\right] \gamma_{IJ}}{1 + \left[\Phi_k(I) + \dfrac{\omega_{p_I}^2 - \omega_{s_J}^2}{\Phi_k(I)\omega_{s_J}^2 \gamma_{IJ}}\right]^2 \gamma_{IJ}} \tag{3.24}$$

which, if it is considered that the minimum value that the second terms between brackets may assume is $1/\sqrt{\gamma_{IJ}}$ (see Sec. 2.6), for small mass ratios may be approximated as

$$\alpha_r = \frac{1}{Y_I^{(r)}} \frac{B_r + \dfrac{1}{\Phi_k(I)} \left(\dfrac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2} \right)}{1 + \dfrac{1}{\Phi_k^2(I)} \left(\dfrac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2} \right)^2 \dfrac{1}{\gamma_{IJ}}} \quad . \tag{3.25}$$

By introducing a new variable $B_o(i)$ defined as

$$B_o(i) = \Phi_k(i) \frac{\omega_{s_J}^2}{\omega_{p_i}^2 - \omega_{s_J}^2} \tag{3.26}$$

α_r may be thus written as

$$\alpha_r = \frac{1}{\gamma_I^{(r)}} \frac{B_r + \frac{1}{B_o(I)}}{1 + \frac{1}{B_o^2(I)\gamma_{IJ}}} \quad . \tag{3.27}$$

But by the definition of the parameter B_r (Eq. 2.94) and by means of Eq. 2.37 one may express this parameter as

$$B_r = \frac{\omega_r^2 - \omega_{p_I}^2}{\Phi_k(I)} \sum_{i=1}^{N_p} \frac{\Phi_k(i)}{\omega_r^2 - \omega_{p_i}^2} \tag{3.28}$$

which, by considering that, by hypothesis, for the case under consideration $\omega_r = \omega_{s_J}$, and after substitution of Eq. 3.26, may also be put into the form

$$B_r = \frac{1}{B_o(I)} \sum_{i=1}^{N_p} B_o(i) \quad . \tag{3.29}$$

Consequently, one may write α_r as

$$\alpha_r = \frac{1}{\gamma_I^{(r)}} \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{B_o(I) + \frac{1}{B_o(I)\gamma_{IJ}}} \quad . \tag{3.30}$$

Thus, since by means of Eq. 3.26 $y_J^{(r)}$ may also be expressed as

$$y_J^{(r)} = \frac{1}{B_o(I)\gamma_{IJ}} \gamma_I^{(r)} \quad , \tag{3.31}$$

by virtue of Eqs. 2.98 and 3.3 the maximum secondary distortions in the nonresonant modes herein being considered result as

$$\{X_s\}^{(r)} = \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{1 + B_o^2(I)\,\gamma_{IJ}} \{d\phi\}^{(J)} SD(\omega_{s_J}, \xi_{s_J}) \tag{3.32}$$

where $SD(\omega_{s_J}, \xi_{s_J})$ represents the spectral displacement corresponding to the

Jth natural frequency and damping ratio of the secondary system.

As in Case I, it should be noted that because Eq. 3.23 is limited to those values of ω_{p_I} and ω_{s_J} for which (see Sec. 2.6)

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2} \right| \geq |\Phi_k(I)\sqrt{\gamma_{IJ}}| , \tag{3.33}$$

Eq. 3.32 is also limited to such values. Notice also that when ω_{p_I} and ω_{s_J} are well separated from each other (i.e., when $B_o^2(I)\gamma_{IJ} << 1.0$) $\{X_s\}^{(r)}$ may be approximated as

$$\{X_s\}^{(r)} = [\ 1 + \sum_{i=1}^{N_p} B_o(i)]\{d\phi\}^{(J)}\ SD(\omega_{s_J},\xi_{s_J}) . \tag{3.34}$$

3.4 <u>Approximate Maximum Response</u>

By using the rule established in Sec. 2.9 for combining the foregoing maximum modal distortions, the approximate maximum distortions of a secondary system may be then expressed as

$$\{X_s\}_{max} = \sqrt{\sum_{r=1}^{N_p+N_s} \{X_s\}^{(r)^2} + 2 \sum_{R/2} \alpha_{n(n+1)} \{X_s\}^{(n)}\{X_s\}^{(n+1)}} \tag{3.35}$$

where $\{X_s\}^{(r)}$ is the rth vector of such maximum modal distortions given by Eq. 3.9, 3.21 or 3.32; R is, as before, the number of resonant modes; and $\alpha_{n(n+1)}$ is as indicated by Eq. 2.111. If, however, the combined response of two adjacent resonant modes is written in a single expression as

$$\{X_s\}^{(s)} = [\{X_s\}^{(n)^2} + \{X_s\}^{(n+1)^2} + 2\alpha_{n(n+1)}\{X_s\}^{(n)}\{X_s\}^{(n+1)}]^{1/2} , \tag{3.36}$$

which by substitution of Eq. 3.9 results of the form

$$\{X_s\}^{(s)} = \sqrt{\frac{1 - \alpha_{n(n+1)}}{2\gamma_{IJ}}}\, \{d\phi\}^{(J)}\ SD(\omega_o,\ \xi_o), \tag{3.37}$$

then Eq. 3.35 may be simplified as

$$\{X_s\}_{max} = \sqrt{\sum_{s=1}^{R/2} \{X_s\}^{(s)^2} + \sum_{r=1}^{N_p+N_s-R} \{X_s\}^{(r)^2}} \quad . \tag{3.38}$$

In like manner, if each modal response is viewed as the product of an amplification factor, a modal configuration, and a spectral ordinate, by virtue of Eqs. 3.37, 3.21 and 3.32 $\{X_s\}^{(s)}$ and $\{X_s\}^{(r)}$ may be then conveniently expressed as follows:

Resonant Modes

$$\{X_s\}^{(s)} = \Psi_R^{(s)}\ \{d\phi\}^{(J)}\ SD(\omega_o,\xi_o) \tag{3.39}$$

where

$$\Psi_R^{(s)} = \sqrt{\frac{1 - \alpha_{IJ}}{2\gamma_{IJ}}} \tag{3.40}$$

and

$$\alpha_{IJ} = \frac{1}{1 + \dfrac{\Phi_k^2(I)\gamma_{IJ}}{4\xi_o'^2}} \quad . \tag{3.41}$$

Nonresonant Modes

Case I: $\omega_r = \omega_{p_I}$

$$\{X_s\}^{(r)} = \Psi_p^{(r)}\ [\sum_{j=1}^{N_s} r_j\ \{d\phi\}^{(j)}]\ SD(\omega_{p_I},\xi_{p_I}) \tag{3.42}$$

in which

$$\Psi_p^{(r)} = \frac{A_o(J)}{1 + A_o^2\,(J)\gamma_{IJ}} \quad . \tag{3.43}$$

Case II: $\omega_r = \omega_{s_J}$

$$\{X_s\}^{(r)} = \Psi_s^{(r)}\ \{d\phi\}^{(J)}\ SD(\omega_{s_J}, \xi_{s_J}) \tag{3.44}$$

where

$$\Psi_s^{(r)} = \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{1 + B_o^2(I)\gamma_{IJ}} \quad . \tag{3.45}$$

Equation 3.38 in combination with Eqs. 3.39 through 3.45 represents thus the desired approximate formula to compute the maximum distortions of a secondary system. Notice the sensitivity of the amplification factors $\Psi_R^{(s)}$ to the variation of the modal correlation factors α_{IJ}: they may vary from zero for $\alpha_{IJ} = 1.0$ to $1/\sqrt{\gamma_{IJ}}$ for $\alpha_{IJ} = -1.0$. Since for small mass ratios the difference between these two extreme values may be considerably large, notice therefore the influence that these modal correlation factors may have in the accuracy of the prediction of such maximum distortions. Observe also that in view that the response of a secondary system resting directly on the ground would be of the form

$$\{X_s\}_{max} = \sqrt{\sum_{j=1}^{N_s} \{d\phi\}^{(j)}\ SD(\omega_{s_j}, \xi_{s_j})} \tag{3.46}$$

the effect of mounting this secondary system on a supporting structure is indicated by the extra terms added to the summation of this equation and the amplification factors multiplying each of the terms of the augmented summation.

CHAPTER 4

EXTENSION OF APPROXIMATE METHOD FOR TWO POINTS OF ATTACHMENT

4.1 Introduction

By following the approach used in the last chapter and introducing the necessary modifications to account for an extra point of attachment, the approximate procedure therein developed is extended in this chapter for secondary systems with up to two points of attachment. To clearly show the basic difference between the systems with one and two points of attachment, the assumption of assembled systems with proportional damping is, however, also kept throughout this chapter.

As in Chapter 2, the expressions derived below will be first obtained for a particular system, the one shown in Fig. 4.1 in this case, and then, by induction, generalized for any other systems. The notation used here will also be that introduced in the previous chapters.

4.2 Mode Shapes of Assembled System

By considering the primary and secondary components of the assembled system of Fig. 4.1 as two independent conventional systems subjected to external forces and by following the procedure utilized in Sec. 2.2, the mode shapes of this assembled system may be determined as follows:

Primary System

With reference to Fig. 4.2(a), the equation of motion of the primary component of the mentioned assembled system may be expressed as

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{p_1} \\ \ddot{x}_{p_2} \\ \ddot{x}_{p_3} \end{Bmatrix} \cdot \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} \begin{Bmatrix} R_1(t) \\ 0 \\ R_3(t) \end{Bmatrix} . \qquad (4.1)$$

Then, if the frequency matrix, modal matrix and generalized masses of this primary component are defined again as indicated by Eqs. 2.1, 2.2 and 2.3, respectively, and if the displacement vector $\{x_p\}$ is transformed into normal coordinates as

$$\begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} = \begin{Bmatrix} \Phi_1(1) \\ \Phi_2(1) \\ \Phi_3(1) \end{Bmatrix} Y_1^{'} + \begin{Bmatrix} \Phi_1(2) \\ \Phi_2(2) \\ \Phi_3(2) \end{Bmatrix} Y_2^{'} + \begin{Bmatrix} \Phi_1(3) \\ \Phi_2(3) \\ \Phi_3(3) \end{Bmatrix} Y_3^{'}, \qquad (4.2)$$

this equation of motion may be written as

$$\begin{bmatrix} M_1^* & 0 & 0 \\ 0 & M_2^* & 0 \\ 0 & 0 & M_3^* \end{bmatrix} \begin{Bmatrix} \ddot{Y}_1^{'} \\ \ddot{Y}_2^{'} \\ \ddot{Y}_3^{'} \end{Bmatrix} + \begin{bmatrix} K_1^* & 0 & 0 \\ 0 & K_2^* & 0 \\ 0 & 0 & K_3^* \end{bmatrix} \begin{Bmatrix} Y_1^{'} \\ Y_2^{'} \\ Y_3^{'} \end{Bmatrix} = \begin{Bmatrix} \Phi_1(1) \\ \Phi_2(2) \\ \Phi_3(3) \end{Bmatrix} R_1(t) + \begin{Bmatrix} \Phi_3(1) \\ \Phi_3(2) \\ \Phi_3(3) \end{Bmatrix} R_3(t) . \qquad (4.3)$$

In the light of Eqs. 2.13 and 2.14, the ith component equation of this matrix equation may be therefore put into the form

$$(\omega^2 - \omega_{p_i}^2)\ Y_i = \frac{1}{M_i^*} [\Phi_1(i)R_1(t) + \Phi_3(i)R_3(t)] \qquad (4.4)$$

which by introducing the parameter

$$\eta = \frac{R_3(t)}{R_1(t)} \tag{4.5}$$

may be alternatively expressed as

$$(\omega^2 - \omega_{p_i}^2)\, Y_i = \frac{1}{M_i^*}\,[\Phi_1(i) + \eta\Phi_3(i)]R_1(t)\,. \tag{4.6}$$

Hence, by: (a) setting $Y_1 = 1.0$, (b) solving $R_1(t)$ from the first of these component equations, (c) substituting the resulting expression for $R_1(t)$ into the ith one, and (d) solving afterwards for Y_i, one obtains

$$Y_i = \frac{\omega^2 - \omega_{p_1}^2}{\omega^2 - \omega_{p_i}^2}\,\frac{M_1^*}{M_i^*}\,\frac{\Phi_1(i)+\eta\Phi_3(i)}{\Phi_1(1)+\eta\Phi_3(1)}\;. \tag{4.7}$$

It may be seen, thus, that the primary system part of the rth mode shape of an assembled system whose secondary system is attached to the kth and ℓth primary masses may be written as

$$\{u_p\}^{(r)} = [\Phi]\{Y\}^{(r)} \tag{4.8}$$

where

$$Y_i^{(r)} = \frac{\omega_r^2-\omega_{p_1}^2}{\omega_r^2-\omega_{p_i}^2}\,\frac{M_1^*}{M_i^*}\,\frac{\hat{\Phi}_r(i)}{\hat{\Phi}_r(1)}\,,\; i=1,2,\ldots,N_p \tag{4.9}$$

$$\hat{\Phi}_r(i) = \Phi_k(i) + \eta_r\,\Phi_\ell(i),\; i=1,2,\ldots,N_p \tag{4.10}$$

and

$$\eta_r = \left[\frac{R_\ell(t)}{R_k(t)}\right]_r\;. \tag{4.11}$$

Secondary System

As shown in Fig. 4.2(b), the secondary system may be considered as an unrestrained four-degree-of-freedom system whose equation of motion is of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{s_0} \\ \ddot{x}_{s_1} \\ \ddot{x}_{s_2} \\ \ddot{x}_{s_c} \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_{s_0} \\ x_{s_1} \\ x_{s_2} \\ x_{s_c} \end{Bmatrix} = -\begin{Bmatrix} R_1(t) \\ 0 \\ 0 \\ R_3(t) \end{Bmatrix} \quad (4.12)$$

But according to Hurty (1965), the displacement vector of this secondary system may be expressed as the combination of: (1) a rigid-body mode, (2) a constraint mode, and (3) the two normal modes of the system when both of its ends are fixed. Then, since the rigid-body mode may be written as

$$\{\Phi\}^{(0)} = \{J\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \; ; \quad (4.13)$$

the constraint mode may be selected to be the mode produced by a unit displacement at the point where the third primary mass is connected while the point where the first is connected is kept fixed, i.e., the following vector of flexibility coefficients:

$$\{\Phi\}^{(c)} = \begin{Bmatrix} 0 \\ f_{1c} \\ f_{2c} \\ f_{cc} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1/k_1 \\ 1/k_1+1/k_2 \\ 1/k_1+1/k_2+1/k_3 \end{Bmatrix} \; ; \quad (4.14)$$

and the normal modes of the two-end-fixed secondary system (see Fig. 4.3) are of the form

$$\{\phi\}^{(j)} = \begin{Bmatrix} 0 \\ \phi_1(j) \\ \phi_2(j) \\ 0 \end{Bmatrix}, \ j=1,2; \tag{4.15}$$

one may express $\{x_s\}$ as

$$\{x_s\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} y_0' + \begin{Bmatrix} 0 \\ \phi_1(1) \\ \phi_2(1) \\ 0 \end{Bmatrix} y_1' + \begin{Bmatrix} 0 \\ \phi_1(2) \\ \phi_2(2) \\ 0 \end{Bmatrix} y_2' + \begin{Bmatrix} 0 \\ \phi_1(c) \\ \phi_2(c) \\ \phi_c(c) \end{Bmatrix} y_c' \tag{4.16}$$

or as

$$\{x_s\} = [\phi]\,\{y'\} \tag{4.17}$$

where

$$[\phi] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \phi_1(1) & \phi_1(2) & \phi_1(c) \\ 1 & \phi_2(1) & \phi_2(2) & \phi_2(c) \\ 1 & 0 & 0 & \phi_c(c) \end{bmatrix} \tag{4.18}$$

and y_j', i=0,1,2,c, is a set of independent generalized coordinates. Consequently, in terms of these generalized coordinates the equation of motion of the system (Eq. 4.12) may be written as

$$\begin{bmatrix} m_0^* & m_1^* & m_2^* & m_{c0}^* \\ m_1^* & m_1^* & 0 & m_{c1}^* \\ m_2^* & 0 & m_2^* & m_{c2}^* \\ m_{c0}^* & m_{c1}^* & m_{c2}^* & m_c^* \end{bmatrix} \begin{Bmatrix} \ddot{y}_0' \\ \ddot{y}_1' \\ \ddot{y}_2' \\ \ddot{y}_c' \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1^* & 0 & 0 \\ 0 & 0 & k_2^* & 0 \\ 0 & 0 & 0 & \phi_c(c) \end{bmatrix} \begin{Bmatrix} y_0' \\ y_1' \\ y_2' \\ y_c' \end{Bmatrix} = -\begin{Bmatrix} R_1(t)+R_3(t) \\ 0 \\ 0 \\ \phi_c(c)\ R_3(t) \end{Bmatrix} \tag{4.19}$$

where

$$m_j^* = \{\phi\}^{(j)^T}[m]\{\phi\}^{(j)} = \sum_n m_n\phi_n^2(j),\ j=0,1,2,c \qquad (4.20)$$

and

$$m_{cj}^* = \{\phi\}^{(c)^T}[m]\{\phi\}^{(j)} = \sum_n m_n\phi_n(c)\phi_n(j),\ j=0,1,2. \qquad (4.21)$$

Thus, since for this secondary system $\{x_s\}$ and $\{y'\}$ may also be expressed as indicated by Eqs. 2.24 and 2.25, after substitution of these two equations into Eq. 4.17 the secondary system part of the mode shape with frequency ω of the assembled system under consideration may be expressed as

$$\{u_s\} = [\phi]\{y\} \qquad (4.22)$$

in which, from the second and third component equations of Eq. 4.19 and by virtue of Eq. 2.25, the y_j factors are of the form

$$y_j = \frac{\omega^2}{\omega_{s_j}^2 - \omega^2}\left(y_0 + \frac{m_{cj}^*}{m_j^*}y_c\right),\ j=1,\ 2\ . \qquad (4.23)$$

In general, then, the secondary system part of the rth mode shape of an assembled system may be written as

$$\{u_s\}^{(r)} = [\phi]\ \{y\}^{(r)} \qquad (4.24)$$

where

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2}\left(y_0^{(r)} + \frac{m_{cj}^*}{m_j^*}y_c^{(r)}\right),\ j=1,2,\ldots,N_s \qquad (4.25)$$

in which N_s is the number of degrees of freedom of such a secondary system when both of its ends are fixed and $y_0^{(r)}$ and $y_c^{(r)}$ are determined from the compatibility conditions as follows.

Compatibility Conditions

In view of the continuity between its primary and secondary components, the assembled system of Fig. 4.1 should satisfy the following compatibility conditions:

$$x_{s_0} = x_{p_1} \tag{4.26}$$

$$x_{s_c} = x_{p_3} \quad . \tag{4.27}$$

If by means of Eqs. 4.2 and 4.16 these compatibility relations are written in generalized coordinates as

$$y_0' = \Phi_1(1)Y_1' + \Phi_1(2)Y_2' + \Phi_1(3)Y_3' \tag{4.28}$$

$$y_0' + \phi_c(c)y_c' = \Phi_3(1)Y_1' + \Phi_3(2)Y_2' + \Phi_3(3)Y_3' \; , \tag{4.29}$$

after introducing Eqs. 2.14 and 2.25 one has therefore that in general for the rth mode of an assembled system with its secondary system attached to the kth and ℓth primary masses the $y_0^{(r)}$ and $y_c^{(r)}$ factors of Eq. 4.25 result of the form

$$y_0^{(r)} = \sum_{i=1}^{N_p} \Phi_k(i)Y_i^{(r)} \tag{4.30}$$

$$y_c^{(r)} = \frac{1}{\phi_c(c)} \sum_{i=1}^{N_p} [\Phi_\ell(i) - \Phi_k(i)]Y_i^{(r)} \quad . \tag{4.31}$$

Alternative Expression for $y_j^{(r)}$ Factors

By substitution of Eqs. 4.30 and 4.31 into Eq. 4.25, one obtains, thus, the following alternative expression for the $y_j^{(r)}$ factors of Eq. 4.24:

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2} \hat{y}_0^{(r)} \tag{4.32}$$

where

$$\hat{y}_0^{(r)} = \sum_{i=1}^{N_p} \Phi_o (i,j) \, Y_i^{(r)} \tag{4.33}$$

in which

$$\Phi_o(i,j) = \Phi_k(i) + \beta_j \, d\Phi(i) \ . \tag{4.34}$$

In this last equation, $d\Phi(i)$ represents the difference between the ith mode shape amplitudes of the two primary masses to which the secondary system is attached, i.e.,

$$d\,\Phi(i) = \Phi_\ell(i) - \Phi_k(i) \tag{4.35}$$

and β_j is defined as

$$\beta_j = \frac{1}{f_{cc}} \frac{m_{cj}^*}{m_j^*} \ . \tag{4.36}$$

A relationship for this parameter β_j in terms of the dynamic parameters of the independent secondary system (assumed with both ends fixed) may be obtained as follows:

Consider the free vibration equation of motion of the secondary system of Fig. 4.3 and let it be conveniently expressed as

$$\omega_{s_j}^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \phi_1(j) \\ \phi_2(j) \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} \phi_1(j) \\ \phi_2(j) \end{Bmatrix} . \qquad (4.37)$$

Then, if both sides of this equation are premultiplied by $\{\phi_1(c)\ \phi_2(c)\}$, where $\phi_1(c)$ and $\phi_2(c)$ are as defined by Eq. 4.14, one is led to

$$\omega_{s_j}^2\ [m_1\phi_1(c)\phi_1(j)+m_2\phi_2(c)\phi_2(j)] = \phi_2(j)\ \frac{k_1k_2+k_1k_3+k_2k_3}{k_1k_2} \qquad (4.38)$$

which in the light of Eq. 4.21 and since by virtue of Eq. 4.14 one has that

$$\frac{k_1k_2 + k_1k_3 + k_2k_3}{k_1k_2} = f_{cc}\ k_3 \qquad (4.39)$$

may also be expressed as

$$\omega_{s_j}^2\ m^*_{cj} = f_{cc}\ k_3\ \phi_2(j) . \qquad (4.40)$$

Thus, from the definition of β_j (Eq. 4.36) it may be seen that this parameter may also be written as

$$\beta_j = \frac{k_3\ \phi_2(j)}{\omega_{s_j}^2\ m^*_j} \qquad (4.41)$$

or, if as shown in Appendix A it is considered that

$$\omega_{s_j}^2\ m^*_j = k^*_j = k_1\phi_1(j) + k_3\phi_2\phi(j), \qquad (4.42)$$

as

$$\beta_j = \frac{k_3\phi_2(j)}{k_1\phi_1(j) + k_3\phi_2(j)} . \qquad (4.43)$$

In general for a secondary system with N_s degrees of freedom, β_j results then of the form

$$\beta_j = \frac{k_{N_s+1}\ \Phi_{N_s}(j)}{\omega_{s_j}^2\ m_j^*} \tag{4.44}$$

or

$$\beta_j = \frac{k_{N_s+1}\ \ \phi_{N_s}(j)}{k_1\phi_1(j) + k_{N_s+1}\ \ \phi_{N_s}(j)}\ . \tag{4.45}$$

Summary

Summarizing the above results, one has therefore that the rth mode shape of an assembled system whose secondary component is attached to the kth and ℓth masses of its primary one may be expressed as

$$\{u_p\}^{(r)} = \sum_{i=1}^{N_p} Y_i^{(r)}\ \{\Phi\}^{(i)} \tag{4.46}$$

$$\{u_s\}^{(r)} = y_0^{(r)}\ \{J\} + \sum_{j=1}^{N_s} y_j^{(r)}\ \{\phi\}^{(j)} + y_c^{(r)}\{f\} \tag{4.47}$$

where $\{u_p\}^{(r)}$ and $\{u_s\}^{(r)}$ are respectively the parts corresponding to the primary and secondary masses of this rth mode shape, $\{J\}$ is a vector of unit elements, $\{f\}$ represents a vector of flexibility coefficients of the form

$$\{f\} = \begin{Bmatrix} 1/k_1 \\ 1/k_1 + 1/k_2 \\ \vdots \\ 1/k_1 + 1/k_2 + \ldots + 1/k_{N_s} \end{Bmatrix}, \tag{4.48}$$

and $\{\phi\}^{(j)}$ is the jth normal mode shape of the secondary system when it has its both ends fixed. In addition,

$$Y_i^{(r)} = \frac{\omega_r^2 - \omega_{p_1}^2}{\omega_r^2 - \omega_{p_i}^2} \frac{M_1^*}{M_i^*} \frac{\hat{\Phi}_r(i)}{\hat{\Phi}_r(1)} , \; i=1,2,\ldots,N_p \tag{4.49}$$

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2} \; \hat{y}_0^{(r)} , \; j=1,2,\ldots,N_s \tag{4.50}$$

$$y_0^{(r)} = \sum_{i=1}^{N_p} \Phi_k(i) \; Y_i^{(r)} \tag{4.51}$$

$$y_c^{(r)} = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i) \; Y_i^{(r)} \tag{4.52}$$

in which

$$f_{cc} = \sum_{j=1}^{N_s+1} \frac{1}{k_j} \; . \tag{4.53}$$

The parameters $\hat{\Phi}_r(i)$ in Eq. 4.49 are defined as

$$\hat{\Phi}_r(i) = \Phi_k(i) + \eta_r \; \Phi_\ell(i), \tag{4.54}$$

and the factor $\hat{y}_0^{(r)}$ of Eq. 4.50 is given by

$$\hat{y}_0^{(r)} = \sum_{i=1}^{N_p} \Phi_0(i,j) \; Y_i^{(r)} \tag{4.55}$$

where

$$\Phi_o(i,j) = \Phi_k(i) + \beta_j d\Phi(i) \quad . \tag{4.56}$$

In these equations, η_r is defined by Eq. 4.5, and β_j and $d\Phi(i)$ are as indicated by Eqs. 4.44 (or 4.45) and 4.35, respectively.

Convergence to the Case of One Point of Attachment

Formulas 4.46 through 4.56 are the generalization of the expressions presented in Sec. 2.2 for systems with only one point of attachment. Therefore, if the conditions which convert a system from two to one point of attachment are introduced, it is obvious that these formulas should converge to the corresponding ones for one point of attachment. To demonstrate, then, that they indeed converge to those in Sec. 2.2, consider once again the assembled system of Fig. 4.1 and assume first that its secondary system is attached to the first primary mass only. In this case, once has that $R_3(t)$ and k_3 are zero and hence

$$\eta_r = \frac{R_3(t)}{R_1(t)} = 0$$

$$\beta_j = \frac{\phi_2(j)\ k_3}{k_1\phi_1(j) + k_3\phi_2(j)} = 0$$

$$f_{cc} = \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_1k_2k_3} = \infty$$

$$y_c^{(r)} = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i)\ Y_I^{(r)} = 0 \ .$$

It may be observed, therefore, that upon substitution of these values Eq. 4.46 through 4.56 lead to Eqs. 2.35 through 2.39. Similarly, assume now that the secondary system is connected to the third primary mass alone. $R_1(t)$ and k_1 are then zero and as a result

$$\eta_r = \frac{R_3(t)}{R_1(t)} = \infty$$

$$\beta_j = \frac{\phi_2(j)k_3}{k_1\phi_1(j) + k_3\phi_2(j)} = 1.0$$

$$f_{cc} = \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_1k_2k_3} = \infty$$

$$y_c^{(r)} = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i)Y_i^{(r)} = 0 .$$

Thus, it may be seen that in this case too the general equations of this chapter converge to the particular ones of Sec. 2.2.

4.3 Natural Frequencies of Resonant Modes

By following the procedure used in Sec. 2.3, the natural frequencies of the resonant modes of an assembled system with two points of attachment may be obtained as follows:

Consider the assembled system of Fig. 4.1. If partitioned to separate the displacements of the primary and secondary components, the equation of motion of this assembled system may be expressed by the following two matrix equations:

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{p_1} \\ \ddot{x}_{p_2} \\ \ddot{x}_{p_3} \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 & 0 \\ -K_2 & K_2+K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} -$$

$$- \begin{bmatrix} k_1 & 0 \\ 0 & 0 \\ 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_{s_1} \\ x_{s_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (4.57)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_{s_1} \\ \ddot{x}_{s_2} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} x_{s_1} \\ x_{s_2} \end{Bmatrix} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} x_{p_1} \\ x_{p_2} \\ x_{p_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \qquad (4.58)$$

Then, if the displacement vectors $\{x_p\}$ and $\{x_s\}$ are approximated as

$$\{x_p\} = Y_I^{(r)} \{\Phi\}^{(I)} \cos(\omega_r - \theta_r) \qquad (4.59)$$

$$\{x_s\} = y_J^{(r)} \{\phi\}^{(J)} \cos(\omega_r - \theta_r) \qquad (4.60)$$

where, as before, subscripts I and J respectively identify the primary and secondary modes whose frequencies are in resonance, after premultiplication of Eq. 4.57 by $\{\Phi\}^{(I)^T}$ and Eq. 4.58 by $\{\phi\}^{(J)^T}$ these two equations lead to the following simplified equation of motion:

$$-\omega_r^2 \begin{bmatrix} M_I^* & 0 \\ 0 & m_J^* \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} + \begin{bmatrix} K_I^*+k_1\Phi_1^2(I)+k_3\Phi_3^2(I) & -k_1\Phi_1(I)\phi_1(J)-k_3\Phi_3(I)\phi_2(J) \\ -k_1\Phi_1(I)\phi_1(J)-k_3\Phi_3(I)\phi_2(J) & k_J^* \end{bmatrix}$$

$$\begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \qquad (4.61)$$

Thus, from the solution of the characteristic equation of this simplified equation of motion one obtains

$$\omega_r^2 = \omega_o^2 + \frac{1}{2}\omega_o^2 \frac{k_1\Phi_1^2(I) + k_3\Phi_3^2(I)}{k_J^*}\gamma_{IJ} \pm$$

$$\pm \frac{1}{2}\omega_o^2 \sqrt{\left(\frac{k_1\Phi_1^2(I) + k_3\Phi_3^2(I)}{k_J^*}\right)^2 \gamma_{IJ}^2 + 4\left(\frac{k_1\Phi_1(I)\phi_1(J) + k_3\Phi_3(I)\phi_2(J)}{k_J^*}\right)^2 \gamma_{IJ}}$$

(4.62)

where according to the notation in the preceding chapters ω_o is a frequency in resonance of the primary and secondary systems, and γ_{IJ} is the mass ratio in the modes of these primary and secondary systems whose frequencies are equal to such a resonant frequency. Hence, if it is observed that for small mass ratios the terms $[k_1\Phi_1^2(I) + k_3\Phi_3^2(I)]\gamma_{IJ}/k_J^*$ in Eq. 4.62 are negligibly small, ω_r^2 may be approximated as

$$\omega_r^2 = \omega_o^2 \pm \omega_o^2 \left[\frac{k_1\Phi_1(I)\phi_1(J) + k_2\Phi_3(I)\phi_2(J)}{k_J^*}\right]\sqrt{\gamma_{IJ}} \quad . \qquad (4.63)$$

It may be noticed, however, that in view of Eq. 4.42 the term between brackets may be written as

$$\frac{k_1\Phi_1(I)\phi_1(J) + k_3\Phi_3(I)\phi_2(J)}{k_J^*} = \Phi_1(I) + \frac{k_3\phi_2(J)}{\omega_{s_J}^2 m_J^*} d\Phi(I) \qquad (4.64)$$

or in the light of Eqs. 4.41 and 4.34 as

$$\frac{k_1\Phi_1(I)\phi_1(J) + k_3\Phi_3(I)\phi_2(J)}{k_J^*} = \Phi_1(I) + \beta_J d\Phi(I) = \Phi_o(I,J). \qquad (4.65)$$

Consequently, Eq. 4.63 may be expressed as

$$\omega_r^2 = \omega_o^2 \; (1 \pm \Phi_o(I,J)\sqrt{\gamma_{IJ}}) \qquad (4.66)$$

and hence,

$$\omega_r \doteq \omega_o \; (1 \pm \frac{1}{2} \; \Phi_o(I,J)\sqrt{\gamma_{IJ}}) \quad . \qquad (4.67)$$

Observe, thus, that the only difference between this expression and the corresponding one for one point of attachment (Eq. 2.52) is that $\Phi_k(I)$ is now replaced by the parameter $\Phi_o(I,J)$, which according to the above equations in its general form results as

$$\Phi_o(I,J) = \Phi_k(I)+\beta_J d\Phi(I) = \frac{k_1\Phi_k(I)\phi_1(J)+k_{N_s+1} \; \Phi_\ell(I)\phi_{N_s}(J)}{k_1\phi_1(J) + k_{N_s+1}\phi_{N_s}(J)} \quad . \qquad (4.68)$$

Observe, also, that when k_1 or k_{N_s+1} are zero, $\Phi_o(I,J)$ turns out to be $\Phi_k(I)$ or $\Phi_\ell(I)$ and that in such cases Eq. 4.66 leads consequently to the corresponding expression proposed for systems with a single point of attachment (see Eq. 2.51). Therefore, this parameter represents a weighted average or central value of the amplitudes of the primary masses to which a secondary system is connected.

4.4 Natural Frequencies of Nonresonant Modes

The natural frequencies of nonresonant modes may also be determined by following the procedure in Chapter 2. Accordingly, if in Eqs. 4.3 and 4.19, the equations of motion of the primary and secondary systems in Fig. 4.2, it is assumed that

$$Y_i' = y_j' = 0 \quad \text{for} \begin{cases} i \neq I \\ j \neq 0 \\ j \neq J \\ j \neq c \end{cases} \tag{4.69}$$

where, again, subscripts I and J refer to the primary and secondary modes whose frequencies are the closest to the frequency of the non-resonant mode under consideration, then these equations of motion may be reduced to the following set of equations:

$$M_I^* \ddot{Y}_I' + K_I^* Y_I' = \Phi_1(I)R_1(t) + \Phi_3(I)R_3(t) \tag{4.70}$$

$$m_0^* \ddot{y}_0' + m_J^* \ddot{y}_J' + m_{c0}^* \ddot{y}_c' = -[R_1(t) + R_3(t)] \tag{4.71}$$

$$m_J^* \ddot{y}_0' + m_J^* \ddot{y}_J' + m_{cJ}^* \ddot{y}_c' + k_J^* y_J' = 0 \tag{4.72}$$

$$m_{c0}^* \ddot{y}_0' + m_{cJ}^* \ddot{y}_J' + m_c^* \ddot{y}_c' + \phi_c(c)y_c' = -\phi_c(c)\ R_3(t)\ . \tag{4.73}$$

Similarly, by virtue of Eqs. 4.69, 4.28, and 4.29 y_0' and y_c' may be approximated as

$$y_0' = \Phi_1(I)\ Y_I' \tag{4.74}$$

$$y_c' = \frac{d\Phi(I)}{\phi_c(c)}\ Y_I'\ . \tag{4.75}$$

After substituting these two equations, Eqs. 2.14 and 2.25, and the expressions for $R_1(t)$ and $R_3(t)$ obtained from Eqs. 4.71 and 4.73, into Eqs. 4.70 and 4.72, and after considering that $\Phi_1(I) + \beta_J\, d\Phi(I) = \Phi_o(I,J)$ and $\phi_c(c) = f_{cc}$, one obtains thus the following simplified equation of motion:

$$-\omega_r^2 \begin{bmatrix} 1+\Phi_1^2(I)\dfrac{m_0^*}{M_I^*} + 2\Phi_1(I)\dfrac{m_c^*}{M_I^*}\dfrac{d\Phi(I)}{f_{cc}} + \dfrac{m_c^*}{M_I^*}\dfrac{d^2\Phi(I)}{f_{cc}^2} & \Phi_o(I,J)\gamma_{IJ} \\ \Phi_o(I,J)\gamma_{IJ} & \gamma_{IJ} \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} +$$

$$+ \begin{bmatrix} \omega_{p_I}^2 + \dfrac{d^2\Phi(I)}{M_I^* f_{cc}} & 0 \\ 0 & \omega_{s_J}^2\,\gamma_{IJ} \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \qquad (4.76)$$

Hence, since for small mass ratios this equation may be written approximately as

$$-\omega_r^2 \begin{bmatrix} 1 & \Phi_o(I,J)\gamma_{IJ} \\ \Phi_o(I,J)\gamma_{IJ} & \gamma_{IJ} \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} + \begin{bmatrix} \omega_{p_I}^2 & 0 \\ 0 & \omega_{s_J}^2\gamma_{IJ} \end{bmatrix} \begin{Bmatrix} Y_I \\ y_J \end{Bmatrix}^{(r)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad (4.77)$$

one is led to the following characteristic equation:

$$\left(\frac{\omega_{p_I}^2 - \omega_r^2}{\omega_r^2}\right)\left(\frac{\omega_{s_J}^2 - \omega_r^2}{\omega_r^2}\right) = \Phi_o^2(I,J)\,\gamma_{IJ}^2 \doteq 0 . \qquad (4.78)$$

It may be seen, therefore, that as in the case of one point of attachment, the natural frequencies of nonresonant modes may also be approximated as

$$\omega_{r_1} = \omega_{p_I} \tag{4.79}$$

$$\omega_{r_2} = \omega_{s_J} \quad . \tag{4.80}$$

4.5 Participation Factors

Since by definition the participation factor of the rth mode of an assembled system with two points of attachment may also be expressed as in Eq. 2.82, and since according to Eqs. 4.46 and 4.47 the modal amplitudes $u_{p_n}(r)$ and $u_{s_n}(r)$ may be written as

$$u_{p_n}(r) = \sum_{i=1}^{N_p} \Phi_n(i)\, Y_i^{(r)} \tag{4.81}$$

$$u_{s_n}(r) = y_0^{(r)} + \sum_{j=1}^{N_s} y_j^{(r)}\, \phi_n(j) + y_c^{(r)} \phi_n(c), \tag{4.82}$$

then in generalized coordinates this rth participation factor results as

$$\alpha_r = [\sum_{i=1}^{N_p} M_i^* Y_i^{(r)} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)} + y_c^{(r)} + y_j^{(r)}) + (m_0^* - \sum_{i=1}^{N_s} m_j^*) y_0^{(r)}$$
$$+ (m_c^* - \sum_{j=1}^{N_s} m_j^*) y_c^{(r)}] / [\sum_{i=1}^{N_p} M_i^* Y_i^{(r)^2} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)} + y_c^{(r)} + y_j^{(r)})^2$$
$$+ (m_0^* - \sum_{j=1}^{N_s} m_j^*) y_0^{(r)^2} + (m_c^* - \sum_{j=1}^{N_s} m_j^*) y_c^{(r)^2} + 2(m_c^* - \sum_{j=1}^{N_s} m_j^*) y_0^{(r)} y_c^{(r)}$$
$$+ 2 \sum_{j=1}^{N_s} (m_{cj}^* - m_j^*) y_c^{(r)} y_j^{(r)}] \quad . \tag{4.83}$$

Therefore, for small mass ratios α_r may be approximated as

$$\alpha_r = \frac{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)}+y_c^{(r)}+y_j^{(r)})}{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)^2} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)}+y_c^{(r)}+y_j^{(r)})^2} \qquad (4.84)$$

or, if irrelevant component modes are neglected, as

$$\alpha_r = \frac{B_r Y_I^{(r)} + (y_0^{(r)}+y_c^{(r)}+y_J^{(r)})\gamma_{IJ}}{Y_I^{(r)^2} + (y_0^{(r)}+y_c^{(r)}+y_J^{(r)})^2\gamma_{IJ}} \qquad (4.85)$$

where, as before,

$$B_r = \frac{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)}}{M_I^* Y_I^{(r)}} \, . \qquad (4.86)$$

4.6 Maximum Response in Resonant Modes

It may be observed from the derivations in the foregoing sections that the form of the equations to determine the natural frequencies and mode shapes of assembled systems with two points of attachment are very similar to the ones for those with just one of these points of attachment. Hence, simplified expressions for the mode shapes and modal distortions in the resonant modes of these assembled systems with two points of attachment, and therefore for the corresponding maximum modal distortions of their secondary systems, may be also obtained by following the approach used in Sec. 3.2.

Thus, if in Eqs. 4.46, 4.47, 4.51, 4.52, and 4.55 all insignificant component modes are neglected, the rth mode shape of such an assembled system may be approximated as

$$\{u_p\}^{(r)} = \gamma_I^{(r)}\{\phi\}^{(r)} \tag{4.87}$$

$$\{u_s\}^{(r)} = y_0^{(r)}\{J\} + y_J^{(r)}\{\phi\}^{(J)} + y_c^{(r)}\{f\} \tag{4.88}$$

where

$$y_0^{(r)} = \phi_k(I)\ \gamma_I^{(r)} \tag{4.89}$$

$$y_c^{(r)} = \frac{d\phi(I)}{f_{cc}}\ \gamma_I^{(r)} \tag{4.90}$$

$$y_J^{(r)} = \phi_o(I,J)\ \frac{\omega_r^2}{\omega_{s_J}^2-\omega_r^2}\ \gamma_I^{(r)}\quad . \tag{4.91}$$

Consequently, since in this rth mode the vector of element distortions of the secondary system may be expressed as

$$\{du_s\}^{(r)} = \begin{Bmatrix} u_{s_1}(r)-u_{p_k}(r) \\ u_{s_2}(r)-u_{s_1}(r) \\ \vdots \\ u_{s_{N_s}}(r)-u_{s_{N_s-1}}(r) \\ u_{p_\ell}(r)-u_{s_{N_s}}(r) \end{Bmatrix}, \tag{4.92}$$

the maximum secondary modal distortions may be written approximately as

$$\{X_s\}^{(r)} = \alpha_r[d\phi(I)\{\frac{df}{f_{cc}}\}\gamma_I^{(r)} + y_J^{(r)}\{d\phi\}^{(J)}]\ SD(\omega_r,\xi_r) \tag{4.93}$$

in which $\{d\phi\}^{(J)}$ is now of the form

$$\{d\phi\}^{(J)} = \begin{Bmatrix} \phi_1(J) \\ \phi_2(J)-\phi_1(J) \\ \cdot \\ \cdot \\ \cdot \\ \phi_{N_s}(J)-\phi_{N_s-1}(J) \\ -\phi_{N_s}(J) \end{Bmatrix} \tag{4.94}$$

and $\{\frac{df}{f_{cc}}\}$ is defined as

$$\{\frac{df}{f_{cc}}\} = \frac{1}{f_{cc}}\begin{Bmatrix} f_{1c} \\ f_{2c}-f_{1c} \\ \cdot \\ \cdot \\ \cdot \\ f_{(N_s+1)c}-f_{N_s c} \end{Bmatrix} = \frac{1}{\frac{1}{k_1}+\frac{1}{k_2}+\cdots\frac{1}{k_{N_s+1}}}\begin{Bmatrix} 1/k_1 \\ 1/k_2 \\ \cdot \\ \cdot \\ \cdot \\ 1/k_{N_s+1} \end{Bmatrix} \tag{4.95}$$

Notice, however, that since for small stiffness constants

$$\{\frac{df}{f_{cc}}\} < \{J\} \tag{4.96}$$

and since for resonant modes $y_J^{(r)}$ is usually large (i.e., $y_J^{(r)}>1.0$), ordinarily one has that

$$d\Phi(I)\{\frac{df}{f_{cc}}\}\ Y_I^{(r)} << y_J^{(r)}\{d\phi\}^{(J)} \quad , \tag{4.97}$$

and, thus, for resonant modes Eq. 4.93 may be simplified as

$$\{X_s\}^{(r)} = \alpha_r \, y_J^{(r)} \{d\phi\}^{(J)} \, SD(\omega_r,\xi_r) \ . \tag{4.98}$$

But by substitution of Eq. 4.66 into Eq. 4.91 $y_J^{(r)}$ may be expressed as

$$y_J^{(r)} = [\pm \frac{1}{\sqrt{\gamma_{IJ}}} - \Phi_0(I,J)] \, Y_I^{(r)} \doteq \pm \frac{1}{\sqrt{\gamma_{IJ}}} Y_I^{(r)} \ . \tag{4.99}$$

Similarly, if in the light of Eqs. 4.89 and 4.90 and this last formula the sum $y_0^{(r)}+y_c^{(r)}+y_J^{(r)}$ is written as

$$y_0^{(r)}+y_c^{(r)}+y_J^{(r)} = [\Phi_k(I) + \frac{d\Phi(I)}{f_{cc}} \pm \frac{1}{\sqrt{\gamma_{IJ}}}] \, Y_I^{(r)} \doteq \pm \frac{1}{\sqrt{\gamma_{IJ}}} Y_I^{(r)} \ , \tag{4.100}$$

then by virtue of Eq. 4.85 and considering that for resonant modes the parameter B_r is very close to unity the participation factor α_r may be approximated as

$$\alpha_r = \frac{1}{Y_I^{(r)}} \frac{1 \pm \sqrt{\gamma_{IJ}}}{2} \doteq \frac{1}{2Y_I^{(r)}} \ . \tag{4.101}$$

Therefore, if Eqs. 4.99 and 4.101 are substituted into Eq. 4.98 and if, as in Chapter 3, it is assumed that the spectral ordinates of two adjacent resonant modes are the same and equal to $SD(\omega_0,\xi_0)$, $\{X_s\}^{(r)}$ may be expressed as

$$\{X_s\}^{(r)} = \pm \frac{1}{2} \frac{1}{\sqrt{\gamma_{IJ}}} \{d\phi\}^{(J)} \, SD(\omega_0,\xi_0) \ . \tag{4.102}$$

Although this expression and the corresponding one for systems with a single point of attachment are identical in form (see Eq. 3.9), it should be observed that these maximum modal distortions are not independent of the number of points of attachment because the mass ratio γ_{IJ} depends on it. (Recall that depending on the number of points of attachment the normal mode shapes of a secondary system are calculated considering that the system has either one or both of its ends fixed.)

4.7 Maximum Response in Nonresonant Modes

Case I: $\omega_r=\omega_{p_I}$

In view of Eqs. 4.79 and 4.46 through 4.52 and proceeding as in Sec. 3.3, one has that for this kind of nonresonant modes,

$$\gamma_i^{(r)} = \begin{cases} 1 \text{ if } i=I \\ 0 \text{ if } i\neq I \end{cases} \qquad (4.103)$$

$$y_j^{(r)} = \Phi_o(I,J)\,\frac{\omega_{p_I}^2}{\omega_{s_j}^2-\omega_{p_I}^2} \qquad (4.104)$$

$$y_0^{(r)} = \Phi_k(I) \qquad (4.105)$$

$$y_c^{(r)} = \frac{d\Phi(I)}{f_{cc}} \qquad (4.106)$$

$$\{u_p\}^{(r)} = \{\Phi\}^{(I)} \qquad (4.107)$$

$$\{u_s\}^{(r)} = y_0^{(r)}\{J\} + \sum_{j=1}^{N_s} y_j^{(r)}\,\{\phi\}^{(j)} + y_c^{(r)}\{f\}\ . \qquad (4.108)$$

Therefore, it if is considered that the corresponding vector of element distortions of the secondary system may be expressed as in

Eq. 4.92, the maximum secondary distortions in these nonresonant modes may be written as

$$\{X_s\}^{(r)} = \alpha_r[d\Phi(I)\{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} y_j^{(r)}\{d\phi\}^{(j)}]\ SD(\omega_{p_I}, \xi_{p_I}) \qquad (4.109)$$

where $\{\frac{df}{f_{cc}}\}$ and $\{d\phi\}^{(j)}$ are as defined by Eqs. 4.94 and 4.95, respectively.

However, by substitution of Eqs. 4.103 through 4.106 into Eqs. 4.85 and 4.86 the participation factor α_r may be approximated as

$$\alpha_r = \frac{1 + [\Phi_k(I) + \frac{d\Phi(I)}{f_{cc}} + \Phi_o(I,J)\frac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}]\gamma_{IJ}}{1 + [\Phi_k(I) + \frac{d\Phi(I)}{f_{cc}} + \Phi_o(I,J)\frac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}]^2\gamma_{IJ}} \qquad (4.110)$$

which, by the same argument used in Sec. 3.3 to simplify Eq. 3.13, may be reduced to

$$\alpha_r = \frac{1}{1 + \Phi_o^2(I,J)\left(\frac{\omega_{p_I}^2}{\omega_{s_J}^2 - \omega_{p_I}^2}\right)^2 \gamma_{IJ}} \ . \qquad (4.111)$$

Thus, if the parameter $A_o(j)$ introduced by Eq. 3.15 is generalized for systems with two points of attachment as

$$A_o(j) = \Phi_o(I,j)\frac{\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2} \ , \qquad (4.112)$$

by which $y_j^{(r)}$ and α_r may also be written as

$$y_j^{(r)} = A_o(j) \tag{4.113}$$

$$\alpha_r = \frac{1}{1 + A_o^2(J)\gamma_{IJ}} \quad , \tag{4.114}$$

$\{X_s\}^{(r)}$ may be expressed as

$$\{X_s\}^{(r)} = \frac{A_o(J)}{1+A_o^2(J)\gamma_{IJ}} [r_c\{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} r_j\{d\phi\}^{(j)}]SD(\omega_{p_I},\xi_{p_I}) \tag{4.115}$$

where

$$r_c = \frac{d\Phi(I)}{A_o(J)} \tag{4.116}$$

and, as before,

$$r_j = \frac{A_o(j)}{A_o(J)} \quad . \tag{4.117}$$

In similarity with the corresponding expression for systems with one point of attachment, notice that Eq. 4.115 is only valid when

$$\left|\frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{p_I}^2}\right| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}| \tag{4.118}$$

and that for other cases ω_{s_J} and ω_{p_I} should be considered as resonant frequencies.

<u>Case II: $\omega_r=\omega_{s_J}$</u>

In virtue of Eqs. 4.80 and 4.50, it may be seen that for these nonresonant modes $y_J^{(r)}$ is large and hence, as for systems with one point of attachment, the maximum secondary distortions may be approximated as

$$\{X_s\}^{(r)} = \alpha_r\, y_J^{(r)} \{d\phi\}^{(J)}\ SD(\omega_{s_J}, \xi_{s_J}) \quad . \tag{4.119}$$

Notice, however, that as in the case of one point of attachment, too, $y_J^{(r)}$ cannot be determined directly from Eq. 4.50 and that as a consequence it is also necessary to derive an alternative expression for this factor $y_J^{(r)}$. With reference to the system in Fig. 4.1 and by following procedure in Sec. 2.6, this alternative expression may be then developed as follows:

Consider Eqs. 4.3 and 4.19 and assume that all the Y_i and y_j factors in these two equations have been, with the exception of y_j, already determined. Thus, if $R_1(t)$ and $R_3(t)$ are solved from the first and last component equations of Eq. 4.19 and substituted into the Ith one of Eq. 4.3, one is led to

$$M_I^* \ddot{Y}_I' + K_I^* Y_I' + \Phi_1(I)[m_0^* \ddot{y}_0' + m_1^* \ddot{y}_1' + m_2^* \ddot{y}_2' + m_{c0}^* \ddot{y}_c'] +$$
$$+ \frac{d\Phi(I)}{f_{cc}} [m_{c0}^* \ddot{y}_0' + m_{c1}^* \ddot{y}_1' + m_{c2}^* \ddot{y}_2' + m_c^* \ddot{y}_c' + f_{cc} \ddot{y}_c'] = 0 \quad . \tag{4.120}$$

By introducing Eqs. 2.14 and 2.25, by considering that because y_J is considerably larger all other y_j factors may be neglected, and since $K_I^* = \omega_{p_I}^2 M_I^*$, this equation may then be simplified as

$$(\omega_{p_I}^2 - \omega^2)\, Y_I + \omega^2 \gamma_{IJ}\, y_J[\Phi_1(I) + \frac{1}{f_{cc}} \frac{m_{cj}^*}{m_J^*}\, d\Phi(I)] = 0 \qquad (4.121)$$

Therefore, solving for y_J, taking into account Eqs. 4.36 and 4.34, and generalizing for the rth mode with frequency ω_r, one arrives to

$$y_J^{(r)} = \frac{\omega_{p_I}^2 - \omega_r^2}{\Phi_o(I,J)\omega_r^2 \gamma_{IJ}}\, Y_I^{(r)} \ . \qquad (4.122)$$

By replacing ω_r by ω_{s_J}, the sought alternative expression for $y_J^{(r)}$ results thus as

$$y_J^{(r)} = \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\Phi_o(I,J)\omega_{s_J}^2 \gamma_{IJ}}\, Y_I^{(r)} \qquad (4.123)$$

which, by analogy with the corresponding expression for systems with one point of attachment, is valid only if

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2} \right| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}| \ . \qquad (4.124)$$

A simplified expression for the participation factor α_r may be obtained as follows:

Observe, first, that by substitution of Eq. 4.123 into Eq. 4.85 and by considering that because the lower bound for $y_J^{(r)}$ is $1/\sqrt{\gamma_{IJ}}$ the factors $y_0^{(r)}$ and $y_c^{(r)}$ are always relatively small and hence negligible, this participation factor may be written as

$$\alpha_r = \frac{1}{\gamma_I^{(r)}} \frac{B_r + \frac{1}{\Phi_o(I,J)} \left(\frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2}\right)}{1 + \frac{1}{\Phi_o^2(I,J)} \left(\frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{s_J}^2}\right)^2 \frac{1}{\gamma_{IJ}}} \tag{4.125}$$

which, if the variable $B_o(i)$ defined in Sec. 3.3 by Eq. 3.26 is now generalized for systems with two points of attachment as

$$B_o(i) = \Phi_o(i,J) \frac{\omega_{s_J}^2}{\omega_{p_i}^2 - \omega_{s_J}^2} \quad , \tag{4.126}$$

may be put into the form

$$\alpha_r = \frac{1}{\gamma_I^{(r)}} \frac{B_r + \frac{1}{B_o(I)}}{1 + \frac{1}{B_o^2(I)\gamma_{IJ}}} \quad . \tag{4.127}$$

Observe, then, that by means of Eqs. 4.86 and 4.49 and substitution of ω_r by ω_{s_J} the parameter B_r may be expressed as

$$B_r = \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\hat{\Phi}_r(I)} \sum_{i=1}^{N_p} \frac{\hat{\Phi}_r(i)}{\omega_{s_J}^2 - \omega_{p_i}^2} \tag{4.128}$$

in which $\hat{\Phi}_r(i)$ may be approximated as follows:

Consider Eq. 4.19 and solve $R_1(t)$ and $R_3(t)$ from the first and last of its component equations. Thus, one obtains

$$R_1(t) = -R_3(t) - [m_0^* \ddot{y}_0' + m_1^* \ddot{y}_1' + m_2^* \ddot{y}_2' + m_c^* \ddot{y}_c'] \tag{4.129}$$

$$R_3(t) = -\frac{1}{f_{cc}} [m_{c0}^* \ddot{y}_0' + m_{c1}^* \ddot{y}_1' + m_{c2}^* \ddot{y}_2' + m_c^* \ddot{y}_c' + f_{cc}\, y_c'] \, . \tag{4.130}$$

From these two equations, the ratio $R_1(t)/R_3(t)$ may be therefore written as

$$\frac{R_1(t)}{R_3(t)} = -1 + f_{cc} \frac{m_0^* \ddot{y}_0' + m_1^* \ddot{y}_1' + m_2^* \ddot{y}_2' + m_c^* \ddot{y}_c'}{m_{c0}^* \ddot{y}_0' + m_{c1}^* \ddot{y}_1' + m^*_{c2} \ddot{y}_2' + m_c^* \ddot{y}_c' + f_{cc}\, y_c'} \tag{4.131}$$

which by neglecting, again, all the y_i factors other than y_J results approximately as

$$\frac{R_1(t)}{R_3(t)} = -1 + f_{cc} \frac{m_J^*}{m_{cJ}^*} \tag{4.132}$$

or in the light of Eq. 4.36 as

$$\frac{R_1(t)}{R_3(t)} = -1 + \frac{1}{\beta_J} \quad . \tag{4.133}$$

Consequently, one may write

$$\eta_r = \frac{R_3(t)}{R_1(t)} = \frac{\beta_J}{1-\beta_J} \tag{4.134}$$

and hence by substitution of this relation into Eq. 4.54 $\hat{\Phi}_r(i)$ may be put into the form

$$\hat{\Phi}_r(i) = \frac{\Phi_k(i) + \beta_J[\Phi_\ell(i) - \Phi_k(i)]}{1 - \beta_J} = \frac{\Phi_o(i,J)}{1 - \beta_J} \quad . \tag{4.135}$$

Thus, Eqs. 4.135 and Eq. 4.128 lead to

$$B_r = \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\Phi_o(I,J)} \sum_{i=1}^{N_p} \frac{\Phi_o(i,J)}{\omega_{s_J}^2 - \omega_{p_i}^2} \tag{4.136}$$

which in terms of the variable $B_o(i)$ results as

$$B_r = \frac{1}{B_o(I)} \sum_{i=1}^{N_p} B_o(i) \quad . \tag{4.137}$$

Accordingly, α_r may be expressed as

$$\alpha_r = \frac{1}{Y_I^{(r)}} \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{B_o(I) + \frac{1}{B_o(I)\gamma_{IJ}}} \quad . \tag{4.138}$$

Since by virtue of Eq. 4.123 and 4.126 $y_J^{(r)}$ may be written as

$$y_J^{(r)} = \frac{1}{B_o(I)\gamma_{IJ}} \; Y_I^{(r)} \quad , \tag{4.139}$$

by substitution of this equation and Eq. 4.138 into Eq. 4.119 it may be seen, then, that the maximum secondary distortions in the kind of nonresonant modes under consideration may be approximated as

$$\{X_s\}^{(r)} = \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{1 + B_o^2(I)\gamma_{IJ}} \{d\phi\}^{(J)} \; SD(\omega_{s_J}, \xi_{s_J}) \; . \qquad (4.140)$$

As in the case of one point of attachment, notice that when ω_{p_I} and ω_{s_J} are well separated from each other (that is, when $B_o^2(I)$ γ_{IJ} <<1.0), one may simplify Eq. 4.140 as

$$\{X_s\}^{(r)} = [1 + \sum_{i=1}^{N_p} B_o(i)] \; \{d\phi\}^{(J)} \; SD(\omega_{s_J}, \xi_{s_J}) \; . \qquad (4.141)$$

Since $y_J^{(r)}$ as given by Eq. 4.123 is limited to the interval indicated by Eq. 4.124, notice too that Eq. 4.140 is also limited to such an interval.

In comparing Eqs. 4.115 and 4.140 with Eqs. 3.21 and 3.32, one may observe that the basic difference between the expressions herein derived and those derived in Chapter 3 for systems with one point of attachment lies, once again, in the substitution of $\Phi_k(I)$ by the parameter $\Phi_o(I,J)$.

4.8 Approximate Maximum Response

In the light of the relationships developed above and in view of the similarity between these relationships and the corresponding ones for the case of one point of attachment, the maximum distortions of a secondary system with two points of attachment may be thus approximated by

$$\{X_s\}_{max} = \sqrt{\sum_{s=1}^{R/2} \{X_s\}^{(s)^2} + \sum_{r=1}^{N_p+N_s-R} \{X_s\}^{(r)^2}} \tag{4.142}$$

where $\{X_s\}^{(s)}$ and $\{X_s\}^{(r)}$ are as indicated below:

Resonant Modes

$$\{X_s\}^{(s)} = \Psi_R^{(s)} \{d\phi\}^{(J)} SD(\omega_o, \xi_o) \tag{4.143}$$

in which

$$\Psi_R = \sqrt{\frac{1-\alpha_{IJ}}{2\gamma_{IJ}}} \tag{4.144}$$

where by substitution of Eq. 4.67 into Eq. 2.108 and by considering again that because of the closeness between the natural frequencies of two adjacent resonant modes one may assume that

$$\xi_n' = \xi_{n+1}' = \xi_o' = \xi_o + \frac{2}{\omega_o s} \ , \tag{4.145}$$

α_{IJ} results of the form

$$\alpha_{IJ} = \alpha_{n(n+1)} = \frac{1}{1 + \frac{\Phi_o^2(I,J)\gamma_{IJ}}{4\ \xi_o'^2}} \ . \tag{4.146}$$

Nonresonant Modes

Case I: $\omega_r = \omega_{p_I}$

$$\{X_s\}^{(r)} = \Psi_p^{(r)} [r_c \{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} r_j \{d\phi\}^{(j)}] SD(\omega_{p_I}, \xi_{p_I}) \tag{4.147}$$

in which

$$\Psi_p^{(r)} = \frac{A_o(J)}{1+A_o^2(J)\gamma_{IJ}} \quad . \tag{4.148}$$

<u>Case II: $\omega_r=\omega_{s_J}$</u>

$$\{X_s\}^{(r)} = \Psi_s^{(r)} \{d\phi\}^{(J)} \text{ SD } (\omega_{s_J}, \xi_{s_J}) \tag{4.149}$$

where

$$\Psi_s^{(r)} = \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{1 + B_o^2(I)\gamma_{IJ}} \quad . \tag{4.150}$$

In these equations, $\Phi_o(I,J)$, $\{\frac{df}{f_{cc}}\}$, r_c, r_j, $A_o(j)$, and $B_o(i)$ are as given by Eqs. 4.56, 4.95, 4.116, 4.117, 4.112, and 4.126, respectively.

CHAPTER 5

EARTHQUAKE RESPONSE OF SYSTEMS WITH NONPROPORTIONAL DAMPING

5.1 Introduction

The approximate methods developed in the preceding chapters are not applicable when a primary and a secondary system form an assembled system without classical modes of vibration, i.e., an assembled system whose damping matrix is not proportional to its mass or stiffness matrices or to any linear combination of them (Caughey, 1960). An extension of these approximate methods is therefore necessary to evaluate the response of secondary systems in such a case.

To analyze an assembled system with nonproportional damping, it would seem natural, at first sight at least, to follow the approximate approach used in the analysis of a conventional structure: a modal analysis in which in order to uncouple the equation of motion of a system the off-diagonal elements of its generalized damping matrix are disregarded. A more careful examination of the problem would indicate, however, that this procedure cannot be used for the systems studied in this work. The great difference in value between the parameters of the primary and secondary systems under consideration makes the off-diagonal elements of the generalized damping matrices of their associated assembled systems to be of the order of magnitude of some of the elements along the main diagonal. By neglecting such off-diagonal elements, one may consequently introduce errors of considerable importance.

Since the main purpose of this study is the derivation of simple approximate methods and since the framework of the response spectrum method is particularly suitable to derive them, it is thus evident that the only viable alternative for the solution of assembled systems with nonproportional damping is a complex modal analysis. (For the description of a complex modal analysis, see Foss, 1958, and Hurty, 1964.)

In this chapter, then, the theory of such a complex analysis is briefly reviewed and extended for the case of earthquakes excitations. Also, an approximate scheme is introduced by which this complex modal analysis for earthquake excitations may be reduced to the form of the conventional response spectrum method. And since the systems of interest in this investigation may have closely-spaced natural frequencies, the rule presented in Chapter 2 for the combination of the modes of such systems is generalized for the case when they have nonproportional damping.

The analysis of response of a secondary system based on the complex analysis of the assembled system that it forms with its supporting structure will be discussed in the next chapter.

5.2 Complex Modal Solution

Reduced Equation of Motion

The equation of motion of a n-degree-of-freedom system described by its mass, damping and stiffness matrices is of the form

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P(t)\} \qquad (5.1)$$

where [M], [C] and [K] are respectively such mass, damping and stiffness matrices, {x} represents the displacement vector of the system, and {P(t)} is the vector of external forces applied to the system. In order to find a modal solution of Eq. 5.1, this equation need be written as

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{P(t)\} \end{Bmatrix} \tag{5.2}$$

or as

$$[A]\{\dot{q}\} + [B]\{q\} = \{Q(t)\} \tag{5.3}$$

where

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \tag{5.4}$$

$$[B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \tag{5.5}$$

$$\{q\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \tag{5.6}$$

and

$$\{Q(t)\} = \begin{Bmatrix} \{0\} \\ \{P(t)\} \end{Bmatrix}. \tag{5.7}$$

Equation 5.3 is a $2n \times 2n$ matrix equation called the reduced equation of motion of the system [14]. Since both [A] and [B] are symmetric and positive definite, it is possible to find a transformation that may simultaneously diagonalize them [13]. It is shown by Foss (1958) that as in the undamped case the matrix of the eignvectors of the system - the solutions to the homogeneous equation of Eq. 5.3 - is such a transformation.

Solution to the Homogeneous Reduced Equation of Motion

The homogeneous reduced equation of motion is given by

$$[A]\{\dot{q}\} + [B]\{q\} = \{0\}, \tag{5.8}$$

and its solution is of the form

$$\{q\} = \{s\}\, e^{\lambda t} . \tag{5.9}$$

Substitution of Eq. 5.9 into Eq. 5.8 leads therefore to the characteristic equation

$$|\lambda[A] + [B]| = 0 \tag{5.10}$$

whose solution leads in turn to a set of 2n eigenvalues λ_r, r = 1,2,..., 2n, and a set of 2n eigenvectors $\{s\}^{(r)}$, r = 1, 2, ..., 2n. When the damping matrix is such that an oscillatory motion occurs, these eigenvalues and eigenvectors result in pairs of complex conjugates [14].

Thus, there are 2n solutions to Eq. 5.8, and they are of the form

$$\{q\}^{(r)} = \{s\}^{(r)}\, e^{\lambda_r t}, \quad r = 1, 2, \ldots, 2n \quad . \tag{5.11}$$

Orthogonality of Eigenvectors $\{s\}^{(r)}$

For the rth mode, Eq. 5.8 results as

$$\lambda_r[A]\,\{s\}^{(r)} + [B]\,\{s\}^{(r)} = \{0\} \; ; \tag{5.12}$$

then, if premultiplied by $\{s\}^{(s)^T}$, the transpose of the sth complex mode shape, this equation may be written as

$$\lambda_r\{s\}^{(s)^T}[A]\,\{s\}^{(r)} + \{s\}^{(s)^T}[B]\,\{s\}^{(r)} = 0. \tag{5.13}$$

Similarly, Eq. 5.8 for the sth mode and premultiplication by $\{s\}^{(r)^T}$ lead to

$$\lambda_s \{s\}^{(r)^T} [A] \{s\}^{(s)} + \{s\}^{(r)^T} [B] \{s\}^{(s)} = 0 \tag{5.14}$$

which in view of the symmetry of [A] and [B] may also be expressed as

$$\lambda_s \{s\}^{(s)^T} [A] \{s\}^{(r)} + \{s\}^{(s)^T} [B] \{s\}^{(r)} = 0. \tag{5.15}$$

Therefore, by substracting Eq. 5.15 to Eq. 5.13 one obtains

$$(\lambda_r - \lambda_s) \{s\}^{(s)^T} [A] \{s\}^{(r)} = 0 \tag{5.16}$$

and hence for any two different modes

$$\{s\}^{(s)^T} [A] \{s\}^{(r)} = 0, \quad r \neq s. \tag{5.17}$$

By substituting this equation into either Eq. 5.13 or Eq. 5.15 one also has that

$$\{s\}^{(s)} [B] \{s\}^{(r)} = 0, \quad r \neq s. \tag{5.18}$$

It may be seen, thus, that the eigenvectors $\{s\}^{(r)}$ are orthogonal with respect to the matrices [A] and [B]. Notice that for a mode and its complex conjugate the difference of frequencies in Eq. 5.16 is also different from zero and that as a consequence for complex conjugates one may write

$$\{\bar{s}\}^{(r)} [A] \{s\}^{(r)} = 0, \qquad r = 1, 2, \ldots n \tag{5.19}$$

$$\{\bar{s}\}^{(r)} [B] \{s\}^{(r)} = 0 \qquad r = 1, 2, \ldots, n \tag{5.20}$$

where $\{\bar{s}\}^{(r)}$ is the complex conjugate of $\{s\}^{(r)}$.*

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*Throughout this study, the complex conjugate of a complex variable will be indicated by a bar above the variable.

Uncoupled Equation of Motion

By substitution of the transformation

$$\{q\} = [s]\ \{z\}\ , \tag{5.21}$$

where [s] is the 2n x 2n matrix of the eigenvectors $\{s\}^{(r)}$ and $\{z\}$ is a vector of unknown normal coordinates, and by premultiplication by $\{s\}^{(r)^T}$ Eq. 5.3 may be written as

$$\{s\}^{(r)^T}[A]\ [s]\ \{\dot{z}\} + \{s\}^{(r)^T}[B][s]\{z\} = \{s\}^{(r)^T}\{Q(t)\} \tag{5.22}$$

which in view of the orthogonality conditions given by Eqs. 5.17 and 5.18 may be reduced to

$$\{s\}^{(r)^T}[A]\ \{s\}^{(r)}\ \dot{z}_r + \{s\}^{(r)^T}[B]\{s\}^{(r)}z_r = \{s\}^{(r)^T}\{Q(t)\} \tag{5.23}$$

where z_r is the rth element of $\{z\}$. Thus, if the following variables are introduced:

$$A_r^* = \{s\}^{(r)^T}[A]\ \{s\}^{(r)} \tag{5.24}$$

$$B_r^* = \{s\}^{(r)^T}[B]\ \{s\}^{(r)} \tag{5.25}$$

$$Q_r^* = \{s\}^{(r)^T}\{Q(t)\}\ , \tag{5.26}$$

where A_r^*, B_r^* and Q_r^* are complex scalars, and if it is observed that an equivalent equation to Eq. 5.23 may be derived for each of the 2n modes of the system under consideration the reduced equation of motion of this system may be transformed to the following set of independent equations:

$$A_r^*\ \dot{z}_r + B_r^*\ z_r = Q_r^*\ , \quad r = 1,\ 2,\ \ldots,\ 2n\ . \tag{5.27}$$

Notice, however, that if Eq. 5.22 is written explicitly for the rth complex conjugate mode shape one is led to

$$\{\bar{s}\}^{(r)^T}[A]\ \{\bar{s}\}^{(r)}\ \dot{z}_{\bar{r}} + \{\bar{s}\}^{(r)^T}[B]\ \{\bar{s}\}^{(r)}\ z_{\bar{r}} = \{\bar{s}\}^{(r)^T}\{Q(t)\}, \qquad (5.28)$$

where $z_{\bar{r}}$ is the normal coordinate corresponding to this rth complex conjugate mode shape, or to

$$\bar{A}^*_r\ \dot{z}_{\bar{r}} + \bar{B}^*_r\ z_{\bar{r}} = \bar{Q}^*_r \qquad (5.29)$$

in view that

$$\bar{A}^*_r = \overline{[\{s\}^{(r)^T}[A]\ \{s\}^{(r)}]} = \{\bar{s}\}^{(r)^T}[A]\ \{\bar{s}\} \qquad (5.30)$$

$$\bar{B}^*_r = \overline{[\{s\}^{(r)^T}[B]\ \{s\}^{(r)}]} = \{\bar{s}\}^{(r)^T}[B]\ \{\bar{s}\}^{(r)} \qquad (5.31)$$

$$\bar{Q}^*_r = \overline{[\{s\}^{(r)^T}\{Q(t)\}]} = \{\bar{s}\}^{(r)^T}\ \{Q(t)\}. \qquad (5.32)$$

Therefore, instead of Eq. 5.27 the equation of motion of the system may be represented by

$$A^*_r\ \dot{z}_r + B^*_r\ z_r = Q^*_r\ , \quad r = 1, 2, \ldots, n \qquad (5.33)$$

and

$$\bar{A}^*_r\ \dot{z}_{\bar{r}} + \bar{B}^*_r\ z_{\bar{r}} = \bar{Q}^*_r\ , \quad r = 1, 2, \ldots, n. \qquad (5.34)$$

Relation Between A^*_r and B^*_r

If Eq. 5.11 is substituted back into Eq. 5.8 and if this equation is premultiplied by $\{s\}^{(r)^T}$, then the homogeneous reduced equation of motion may be expressed as

$$\lambda_r \{s\}^{(r)^T}[A]\{s\}^{(r)} + \{s\}^{(r)^T}[B]\{s\}^{(r)} = 0. \tag{5.35}$$

By virtue of Eqs. 5.24 and 5.25 one has thus that

$$\lambda_r A_r^* + B_r^* = 0 \tag{5.36}$$

and hence

$$\lambda_r = -\frac{B_r^*}{A_r^*} . \tag{5.37}$$

If the above operation is made explicitly for the rth complex conjugate mode shape, one then arrives to

$$\bar{\lambda}_r \{\bar{s}\}^{(r)^T}[A]\{\bar{s}\}^{(r)} + \{\bar{s}\}^{(r)^T}[B]\{\bar{s}\}^{(r)} = 0. \tag{5.38}$$

Therefore, using Eqs. 5.30 and 5.31 one obtains the following similar relation between $\bar{A}_r^*$ and $\bar{B}_r^*$:

$$\bar{\lambda}_r = -\frac{\bar{B}_r^*}{\bar{A}_r^*} . \tag{5.39}$$

Solution of the rth Uncoupled Equation

Equations 5.33 and 5.34 constitute a set of independent ordinary differential equations which may be solved separately by means of either the Laplace transform or the unit impulse function (i.e., Dirac's delta function). Here, this latter approach is used as follows:

According to Eq. 5.33, the rth uncoupled equation of motion of a system when Q_r^* is equal to the unit impulse $\delta(t)$ may be expressed as

$$A_r^* \dot{z}_r + B_r^* z_r = \delta(t). \tag{5.40}$$

Integrating each of its terms from 0 to t this equation may be then written as

$$A_r^* \int_0^t \dot{z}_r \, dt + B_r^* \int_0^t z_r \, dt = \int_0^t \delta(\tau) \, d\tau \tag{5.41}$$

or as

$$A_r^* z_r(t) + B_r^* \int_0^t z_r \, dt = 1. \tag{5.42}$$

Hence, by making t equal to zero one obtains

$$A_r^* z_r(t=0) = 1. \tag{5.43}$$

It may be seen thus that at the end of the unit impulse the system undergoes free vibrations with the initial condition

$$z_r \; (t = 0) = 1/A_r^* \; . \tag{5.44}$$

Since the solution of the homogeneous equation of Eq. 5.40 is of the form

$$z_r = C_r \, e^{\lambda_r t}, \tag{5.45}$$

where C_r is a constant, then the solution of Eq. 5.40 result as

$$z_r = \frac{1}{A_r^*} e^{\lambda_r t}. \tag{5.46}$$

Consequently, by dividing the external force $Q_r^*(t)$ into a series of impulses of magnitude $Q_r^*(\tau) \, d\tau$ and by applying the superposition principle the solution of Eq. 5.33 may be written as

$$z_r(t) = \frac{1}{A_r^*}\int_0^t e^{\lambda_r(t-\tau)}\, Q_r^*(\tau)\, d\tau. \tag{5.47}$$

By following a similar procedure, it is easy to show that the solution of Eq. 5.34 is

$$z_{\bar{r}}(t) = \frac{1}{\bar{A}_r^*}\int_0^t e^{\bar{\lambda}_r(t-\tau)}\, \bar{Q}_r^*(\tau)\, d\tau. \tag{5.48}$$

Observe that since the complex conjugate of a sum is equal to the sum of the conjugates of the terms of the sum and the complex conjugate of a product is equal to the product of the conjugates of the terms of the product, $\bar{z}_r(t)$ may be expressed as

$$\bar{z}_r(t) = \frac{1}{\bar{A}_r^*}\int_0^t [\overline{e^{\lambda_r(t-\tau)}}]\, \bar{Q}_r^*(\tau)\, d\tau \tag{5.49}$$

or as

$$\bar{z}_r(t) = \frac{1}{\bar{A}_r^*}\int_0^t e^{\bar{\lambda}_r(t-\tau)}\, \bar{Q}_r^*(\tau)\, d\tau = z_{\bar{r}}(t). \tag{5.50}$$

Hence, $z_r(t)$ and $z_{\bar{r}}(t)$ are complex conjugates.

Response to Earthquake Excitations

Once the complex eigenvectors $\{s\}^{(r)}$ and the solutions of Eqs. 5.33 and 5.34 are known, the solution of Eq. 5.3 is given directly by Eq. 5.21. This solution, however, may be conveniently expressed as

$$\{q\} = \sum_{r=1}^{n} \{s\}^{(r)}\, z_r + \sum_{r=1}^{n} \{\bar{s}\}^{(r)}\, z_{\bar{r}} \tag{5.51}$$

which by virtue of Eq. 5.50 may also be written as

$$\{q\} = \sum_{r=1}^{n} \{s\}^{(r)} z_r + \sum_{r=1}^{n} \{\bar{s}\}^{(r)} \bar{z}_r \tag{5.52}$$

which in turn may be put into the form

$$\{q\} = 2 \sum_{r=1}^{n} \mathrm{Re}[\{s\}^{(r)} z_r] \tag{5.53}$$

where "Re" stands for "the real part of".

Thus, the solution of Eq. 5.1 may be obtained as follows:

Observe, first, that as indicated by Eq. 5.11 the rth solution of the homogeneous reduced equation of motion is given by

$$\{q\}^{(r)} = \{s\}^{(r)} e^{\lambda_r t} \tag{5.54}$$

and therefore since in the light of Eq. 5.6 $\{q\}^{(r)}$ may be written as

$$\{q\}^{(r)} = \begin{Bmatrix} \{\dot{x}\}^{(r)} \\ \{x\}^{(r)} \end{Bmatrix} \tag{5.55}$$

one has that

$$\begin{Bmatrix} \{\dot{x}\}^{(r)} \\ \{x\}^{(r)} \end{Bmatrix} = \{s\}^{(r)} e^{\lambda_r t}. \tag{5.56}$$

Observe, then, that $\{x\}^{(r)}$ describes the rth mode free vibration displacements of the system defined by Eq. 5.1 and hence this vector may be expressed as the product of a mode shape and a harmonic function of time, i.e.,

$$\{x\}^{(r)} = \{w\}^{(r)} e^{\lambda_r t}. \tag{5.57}$$

Consequently, one may write

$$\begin{Bmatrix} \{\dot{x}\}^{(r)} \\ \\ \{x\}^{(r)} \end{Bmatrix} = \begin{Bmatrix} \frac{d}{dt}(\{w\}^{(r)}e^{\lambda_r t}) \\ \\ \{w\}^{(r)}e^{\lambda_r t} \end{Bmatrix} = \begin{Bmatrix} \lambda_r\{w\}^{(r)} \\ \\ \{w\}^{(r)} \end{Bmatrix} e^{\lambda_r t} \tag{5.58}$$

which together with Eq. 5.56 permits one to conclude that the eigenvector $\{s\}^{(r)}$ may be expressed as

$$\{s\}^{(r)} = \begin{Bmatrix} \lambda_r\{w\}^{(r)} \\ \{w\}^{(r)} \end{Bmatrix} . \tag{5.59}$$

Notice that because $\{s\}^{(r)}$ is complex $\{w\}^{(r)}$ will also be a complex vector.

Conceivably, by substitution of Eqs. 5.6 and 5.59 into Eq. 5.53 one is led to

$$\begin{Bmatrix} \{\dot{x}\} \\ \\ \{x\} \end{Bmatrix} = 2 \sum_{r=1}^{n} \text{Re} \left[\begin{Bmatrix} \lambda_r\{w\}^{(r)} \\ \\ \{w\}^{(r)} \end{Bmatrix} z_r \right] \tag{5.60}$$

whose lower half indicates that

$$\{x\} = 2 \sum_{r=1}^{n} \text{Re}\, [\{w\}^{(r)} z_r] \quad . \tag{5.61}$$

In like manner, the substitution of Eqs. 5.7 and 5.59 into Eq. 5.26 yields

$$Q_r^* = \{w\}^{(r)^T} \{P(t)\} \tag{5.62}$$

while Eq. 5.24 in combination with Eqs. 5.59 and 5.4 leads to

$$A_r^* = \{w\}^{(r)^T}[2\lambda_r[M] + [C]]\{w\}^{(r)} . \quad (5.63)$$

According to Eq. 5.47, $z_r(t)$ may be therefore written as

$$z_r(t) = \frac{\int_o^t e^{\lambda_r(t-\tau)} \{w\}^{(r)^T}\{P(\tau)\}d\tau}{\{w\}^{(r)^T}[2\lambda_r[M]+[C]]\{w\}^{(r)}} . \quad (5.64)$$

But for the case of an earthquake excitation the vector of external forces $\{P(t)\}$ is given by

$$\{P(t)\} = -[M]\{J\}\ddot{q}_g(t) \quad (5.65)$$

in which $\ddot{q}_g(t)$ is the earthquake ground acceleration and $\{J\}$ is a vector of unit elements. Then, for earthquake ground motions $z_r(t)$ may be expressed as

$$z_r(t) = -\gamma_r \int_o^t e^{\lambda_r(t-\lambda)}\ddot{q}_g(t)d\tau \quad (5.66)$$

where γ_r is defined as

$$\gamma_r = \frac{\{w\}^{(r)^T}[M]\{J\}}{\{w\}^{(r)^T}[2\lambda_r [M] + [C]]\{w\}^{(r)}} . \quad (5.67)$$

Thus, by substitution of Eq. 5.66 into Eq. 5.61 the earthquake response of the system described by Eq. 5.1 results as

$$\{x(t)\} = -2 \sum_{r=1}^{n} \text{Re}\,[\gamma_r \{w\}^{(r)} \int_0^t e^{\lambda_r (t-\tau)} \ddot{q}_g(\tau) d\tau] \ . \qquad (5.68)$$

Notice that like the participation factors of a system with proportional damping the parameters γ_r in the above equation indicate the degree of participation of each of the modes of the system herein being considered in this sytem's total response. Therefore, these parameters will be henceforth identified as the complex participation factors of the system.

5.3 Definition of Modal Damping Ratios and Natural Frequencies of Vibration

Motivation

It has been shown in the last section that the earthquake response of a system with nonproportional damping may be expressed as the sum of the individual responses in each of its modes. Then, if a way can be found to obtain the maximum values of these individual responses from a response spectrum, the maximum response of such a system can be conveniently calculated by the response spectrum method. In this regard, it should be noted that in order to use the reponse spectrum method it is necessary to determine first if the concept of the modal damping ratios and natural frequencies of vibration of a system with proportional damping may be extended for the systems with nonproportional damping. In this section, therefore, the significance of the complex natural frequencies, modal damping ratios, and natural frequencies of vibration of a system with proportional damping is reviewed, and then, based on this review, the meaning of the same parameters for a system with non-proportional damping is established.

Damping Ratios and Natural Frequencies of Systems with Proportional Damping

It is well known that the matrix of the undamped mode shapes of a system with proportional damping is the transformation matrix that uncouples its equation of motion [8]. Thus,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}, \tag{5.69}$$

the damped free vibration equation of motion of such a system, is satisfied by

$$\{x\}^{(r)} = \{u\}^{(r)} e^{\lambda_r t}, \quad r=1,\ 2,\ \ldots,\ n \tag{5.70}$$

where $\{u\}^{(r)}$ is the rth undamped mode shape of the system, n denotes the number of its degrees of freedom, and λ_r is an unknown constant.

To determine λ_r, one may observe that by substitution of Eq. 5.70 into Eq. 5.69 and by premultiplication of this latter equation by $\{u\}^{(r)^T}$ one may write the above equation of motion as

$$\lambda_r^2 M_r^* + \lambda_r C_r^* + K_r^* = 0, \quad r=1,\ 2,\ \ldots,\ n, \tag{5.71}$$

where

$$M_r^* = \{u\}^{(r)^T}[M]\{u\}^{(r)} \tag{5.72}$$

$$C_r^* = \{u\}^{(r)^T}[C]\{u\}^{(r)} \tag{5.73}$$

$$K_r^* = \{u\}^{(r)^T}[K]\{u\}^{(r)} \quad , \tag{5.74}$$

and hence after solving for λ_r from Eq. 5.71 one obtains

$$\lambda_r = -\frac{1}{2}\frac{C_r^*}{M_r^*} \pm \frac{1}{2}\sqrt{\left(\frac{C_r^*}{M_r^*}\right)^2 - 4\frac{K_r^*}{M_r^*}} \quad . \tag{5.75}$$

However, since M_r^*, C_r^*, and K_r^* are real, in similarity with a single degree-of-freedom system one has that: (1) the condition for having an oscillatory motion in the rth mode of the system is (see Eq. 5.70)

$$\left(\frac{C_r^*}{M_r^*}\right)^2 < 4\frac{K_r^*}{M_r^*} , \tag{5.76}$$

(2) there exists a critical value of C_r^* given by

$$(C_r^*)_{cr} = 2\sqrt{K_r^* M_r^*} \tag{5.77}$$

with which such an oscillatory motion stops, and (3) C_r^* may be defined in terms of a percentage ξ_r of this critical damping value as

$$C_r^* = 2\,\xi_r\sqrt{K_r^* M_r^*} \quad . \tag{5.78}$$

Consequently, since

$$\omega_r^2 = \frac{K_r^*}{M_r^*} \tag{5.79}$$

C_r^* may be written as

$$C_r^* = 2\xi_r M_r^* \omega_r \tag{5.80}$$

and hence by substitution of these two equations into Eq. 5.75 λ_r may be put into the form

$$\lambda_r = -\xi_r\omega_r \pm i\,\omega_r' \tag{5.81}$$

where

$$\omega_r' = \omega_r\sqrt{1-\xi_r^2} \tag{5.82}$$

is called the rth damped natural frequency.

On the basis of Eq. 5.81, the solution of Eq. 5.69 may be therefore expressed as (see Eq. 5.70)

$$\{x\}^{(r)} = c_1\{u\}^{(r)}e^{-\xi_r\omega_r t+i\omega_r' t} + c_2\{u\}^{(r)}e^{-\xi_r\omega_r t-i\omega_r' t} \tag{5.83}$$

or as

$$\{x\}^{(r)} = \{u\}^{(r)}e^{-\xi_r\omega_r t}[c_1'\cos\omega_r' t + c_2'\sin\omega_r' t] \tag{5.84}$$

where c_1, c_2, c_1' and c_2' are constants.

Notice thus that in this case of proportional damping:

a) The parameter λ_r is in general a complex constant, and it is given in terms of the rth modal damping ratio and rth natural frequency of vibration of the system.

b) λ_r defines the vibrational characteristics of the system in its rth mode; the imaginary part of λ_r describes the frequency with which the system vibrates in that mode while its real part indicates the rate by which such vibrations are damped out with time.

c) The rth modal damping ratio and the rth natural frequency of vibration are defined by means of Eqs. 5.80 and 5.79, respectively.

Damping Ratios and Natural Frequencies of Systems with Nonproportional Damping

It is shown in Sec. 5.2 that the solution of

$$[A]\{\dot{q}\} + [B]\{q\} = \{0\}, \qquad (5.85)$$

the homogeneous reduced equation of motion of a system with nonproportional damping, is of the form

$$\{q\}^{(r)} = \{s\}^{(r)} e^{\lambda_r t}, \quad r=1,2,\ldots,2n \qquad (5.86)$$

where $\{s\}^{(r)}$ and λ_r are respectively the rth complex eigenvector and complex natural frequency of the system and n is the number of its degrees of freedom. By substitution of Eq. 5.86 and by virtue of Eqs. 5.4, 5.5 and 5.59, such an equation of motion may be therefore written as

$$\lambda_r \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \lambda_r\{w\}^{(r)} \\ \{w\}^{(r)} \end{Bmatrix} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \lambda_r\{w\}^{(r)} \\ \{w\}^{(r)} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \qquad (5.87)$$

which in algebraic form results in the following two equations:

$$\lambda_r [M]\{w\}^{(r)} - \lambda_r [M]\{w\}^{(r)} = \{0\} \qquad (5.88)$$

$$\lambda_r^2 [M]\{w\}^{(r)} + \lambda_r[C]\{w\}^{(r)} + [K]\{w\}^{(r)} = \{0\}. \qquad (5.89)$$

From this last equation, then, it may be observed that the homogeneous equation of motion of a system with nonproportional damping given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{5.90}$$

is satisfied by

$$\{x\}^{(r)} = \{w\}^{(r)} e^{\lambda_r t}, \quad r=1,2, \ldots, 2n. \tag{5.91}$$

By noticing the similarity between these two equations and Eqs. 5.69 and 5.70, one may thus follow the approach used for systems with proportional damping to interpret the complex natural frequencies of a system with nonproportional damping. Accordingly, if Eq. 5.89 is premultiplied by $\{\overline{w}\}^{(r)^T}$, the transpose of the rth complex conjugate mode shape, the free vibration equation of motion of the system under consideration may be expressed as

$$\lambda_r^2 M_r^* + \lambda_r C_r^* + K_r^* = 0, \quad r=1,2,\ldots, n \tag{5.92}$$

where M_r^*, C_r^* and K_r^* are defined as

$$M_r^* = \{\overline{w}\}^{(r)^T} [M]\{w\}^{(r)} \tag{5.93}$$

$$C_r = \{\overline{w}\}^{(r)^T} [C]\{w\}^{(r)} \tag{5.94}$$

$$K_r^* = \{\overline{w}\}^{(r)^T} [K]\{w\}^{(r)}. \tag{5.95}$$

(Notice that because of the symmetry of the matrices [M], [C] and [K], the above generalized parameters are the same for a given mode and its complex conjugate, and hence there are only n equations of the kind of Eq. 5.92.)

In order to go further, it is necessary to analyze first the nature of these generalized parameters. For this purpose, consider the complex mode shape $\{w\}^{(r)}$ written explicitly in terms of its real and imaginary parts as follows:

$$\{w\}^{(r)} = \{u\}^{(r)} + i\,\{v\}^{(r)}. \tag{5.96}$$

Observe, then, that after substitution of this equation into Eqs. 5.93 through 5.95 M_r^*, C_r^* and K_r^* may be written as

$$M_r^* = \{u\}^{(r)^T}[M]\{u\}^{(r)} + \{v\}^{(r)^T}[M]\{v\}^{(r)} \tag{5.97}$$

$$C_r^* = \{u\}^{(r)^T}[C]\{u\}^{(r)} + \{v\}^{(r)^T}[C]\{v\}^{(r)} \tag{5.98}$$

$$K_r^* = \{u\}^{(r)^T}[K]\{u\}^{(r)} + \{v\}^{(r)^T}[K]\{v\}^{(r)} \tag{5.99}$$

or as

$$M_r^* = M_{R_r}^* + M_{I_r}^* \tag{5.100}$$

$$C_r^* = C_{R_r}^* + C_{I_r}^* \tag{5.101}$$

$$K_r^* = K_{R_r}^* + K_{I_r}^* \tag{5.102}$$

where

$$M_{R_r}^* = \{u\}^{(r)^T}[M]\{u\}^{(r)} \tag{5.103}$$

$$M^*_{I_r} = \{v\}^{(r)^T}[M]\{v\}^{(r)} \tag{5.104}$$

$$C^*_{R_r} = \{u\}^{(r)^T}[C]\{u\}^{(r)} \tag{5.105}$$

$$C^*_{I_r} = \{v\}^{(r)^T}[C]\{v\}^{(r)} \tag{5.106}$$

$$K^*_{R_r} = \{u\}^{(r)^T}[K]\{u\}^{(r)} \tag{5.107}$$

$$K^*_{I_r} = \{v\}^{(r)^T}[K]\{v\}^{(r)} \quad . \tag{5.108}$$

Notice thus that each of the generalized parameters of a system with nonproportional damping consists of two terms: one corresponding to the real part of the eigenvector $\{w\}^{(r)}$ and the other corresponding to its imaginary part. Notice also that these generalized parameters are always real and that each of their terms is defined like any of the generalized parameters of a system with proportional damping.

Back to Eq. 5.92, it may be seen, then, that this equation is of the form of the corresponding one for a system with proportional damping (see Eq. 5.71) and that consequently it is also possible to define from it a critical damping value, a modal damping ratio, and a natural frequency of vibration. In fact, if λ_r is solved from Eq. 5.92 one is led to

$$\lambda_r = -\frac{1}{2}\frac{C^*_r}{M^*_r} \pm \sqrt{\frac{1}{2}\left(\frac{C^*_r}{M^*_r}\right)^2 - 4\frac{K^*_r}{M^*_r}} \tag{5.109}$$

which by denoting

$$\omega_r = \sqrt{\frac{K_r^*}{M_r^*}} \tag{5.110}$$

and expressing C_r^* in terms of a percentage ξ_r of its critical damping value as

$$C_r^* = \xi_r (C_r)_{cr} = 2\ \xi_r \sqrt{K_r^* M_r^*} = 2\ \xi_r \omega_r M_r^* \tag{5.111}$$

may also be written as

$$\lambda_r = -\xi_r \omega_r \pm i\omega_r' \tag{5.112}$$

where as before

$$\omega_r' = \omega_r \sqrt{1 - \xi_r^2} \tag{5.113}$$

Similarly, if Eq. 5.112 is substituted into Eq. 5.91 one may express $\{x\}^{(r)}$ as*

$$\{x\}^{(r)} = \{w\}^{(r)} e^{-\xi_r \omega_r t} [\cos\omega_r' t + i \sin \omega_r' t], \ r = 1,2,\ldots,2n. \tag{5.114}$$

Evidently, the rth complex frequency of a system with nonproportional damping also describes the vibrational characteristics of the system in its rth mode. As in the case of proprotional damping, its imaginary and real parts indicates respectively the frequency of vibration of the system in the rth mode and the way the associated oscillatory motion dies out

*Notice that the negative sign in Eq. 5.112 corresponds to the complex conjugate of λ_r and therefore the substitution of Eq. 5.112 with this negative sign into Eq. 5.91 would lead to $\{\bar{x}\}^{(r)}$.

with time. Also as in the case of proportional damping, this complex frequency may be written in terms of a modal damping ratio and a natural frequency of vibration. In the light of Eqs. 5.100 through 5.102 and according to Eqs. 5.110 and 5.111, in the case of nonproportional damping such a damping ratio and such a natural frequency of vibration are, however, defined by means of the following two equations:

$$\omega_r = \sqrt{\frac{K^*_{R_r} + K^*_{I_r}}{M^*_{R_r} + M^*_{I_r}}} \tag{5.115}$$

$$\xi_r \omega_r = \frac{1}{2} \frac{C^*_{R_r} + C^*_{I_r}}{M^*_{R_r} + M^*_{I_r}} . \tag{5.116}$$

Observe that since the eigenvectors of a system with proportional damping are always real, for the particular case of proportional damping one has that

$$\{w\}^{(r)} = \{u\}^{(r)}, \tag{5.117}$$

$$K^*_{I_r} = C^*_{I_r} = M^*_{I_r} = 0 \tag{5.118}$$

and as a consequence Eqs. 5.115 and 5.116 lead to

$$\omega_r = \sqrt{\frac{K^*_{R_r}}{M^*_{R_r}}} \tag{5.119}$$

$$\xi_r \omega_r = \frac{1}{2} \frac{C^*_{R_r}}{M^*_{R_r}} . \tag{5.120}$$

Observe also that in view of the discussion in this section, the damping characteristics of a system with nonproportional damping may also be specified in terms of modal damping ratios.

5.4 Earthquake Response by the Conventional Response Spectrum Method

Maximum Earthquake Response

If $\{w\}^{(r)}$ and λ_r in Eq. 5.68 are written explicitly in terms of their real and imaginary parts and if $\{w'\}^{(r)}$ denotes a complex mode shape with unit participation factor, i.e.,

$$\{w'\}^{(r)} = \{u'\}^{(r)} + i\ \{v'\}^{(r)} = \gamma_r \{w\}^{(r)}, \qquad (5.121)$$

the earthquake response of a system with nonproportional damping may be expressed as

$$\{x(t)\} = -2 \sum_{r=1}^{n} \mathrm{Re}\ \{\ [\{u'\}^{(r)} + i\{v'\}^{(r)}] \int_0^t e^{-\xi_r \omega_r (t-\tau)} \ddot{q}_g(\tau) \cdot$$

$$\cdot [\cos \omega_r'(t-\tau) + i \sin \omega_r'(t-\tau)]\ d\tau\ \} \qquad (5.122)$$

or as

$$\{x(t)\} = -2 \sum_{r=1}^{n} \ [\{u'\}^{(r)} \int_0^t e^{-\xi_r \omega_r (t-\tau)} \ddot{q}_g(\tau) \cos\omega_r'(t-\tau) d\tau\ -$$

$$-\ \{v'\}^{(r)} \int_0^t e^{-\xi_r \omega_r (t-\tau)} \ddot{q}_g(\tau) \sin\omega_r'\ (t-\tau) d\tau]. \qquad (5.123)$$

But the second integral in this last equation may be identified as the product of ω_r' and the displacement response of a damped single-degree-of-freedom system with natural frequency ω_r and damping ratio ξ_r to the ground motion $\ddot{q}_g(t)$. Similarly, for small damping ratios the first integral may be considered as the corresponding velocity response. That is

$$-\int_o^t e^{-\xi_r\omega_r(t-\tau)}\ddot{q}_g(\tau)\cos\omega_r'(t-\tau)d\tau \doteq V(\omega_r,\xi_r,t) \tag{5.124}$$

$$-\int_o^t e^{-\xi_r\omega_r(t-\tau)}\ddot{q}_g(\tau)\sin\omega_r'(t-\tau)d\tau = \omega_r'D(\omega_r,\xi_r,t), \tag{5.125}$$

in which $V(\omega_r,\xi_r,t)$ and $D(\omega_r,\xi_r,t)$ stand respectively for the aforementioned velocity and displacement responses at a given time t. Therefore, $\{x(t)\}$ may be alternatively written as

$$\{x(t)\} = 2\sum_{r=1}^{n}[\{u'\}^{(r)}V(\omega_r,\xi_r,t) - \{v'\}^{(r)}\omega_r' D(\omega_r,\xi_r,t)], \tag{5.126}$$

and as a consequence the vector of maximum displacements results of the form

$$\{x_{max}\} = 2\sum_{r=1}^{n}\{u'(r)V(\omega_r,\xi_r,t_{max}) - v'(r)\omega_r' D(\omega_r,\xi_r,t_{max})\} \tag{5.127}$$

where t_{max} signifies the time at which the maximum value of the displacement of a particular mass of the system under consideration is attained.

Notice, thus, that as in the case of proportional damping the maximum earthquake response of a system with nonproportional damping is given by the sum of the individual responses in each of its modes, and hence this maximum response may also be estimated from the maximum values of those individual modal responses. It should be noted, however, that since the maximum values of the velocity and displacement functions in Eq. 5.126

cannot occur at the same time (the displacement function reaches its maximum when the value of the velocity one is zero), such maximum modal responses cannot be evaluated directly from a response spectrum. To determine, then, such a maximum earthquake response by the conventional response spectrum technique, the following approximate formulation for the aforementioned maximum modal responses is introduced.

Approximate Maximum Modal Responses

It may be observed from Eq. 5.127 that an upper bound to the displacement response in the rth mode of a system with nonproportional damping is (notice that u'(r) and v'(r) as well as $V(\omega_r, \xi_r, t_{max})$ and $D(\omega_r, \xi_r, t_{max})$ may be of opposite signs)

$$\{x\}^{(r)} \leq 2 \ \{|u'(r)V(\omega_r, \xi_r, t_{max}) + v'(r)\omega_r' D(\omega_r, \xi_r, t_{max})|\} \quad , \qquad (5.128)$$

and thus since $V(\omega_r, \xi_r, t_{max})$ and $D(\omega_r, \xi_r, t_{max})$ are always less than or equal to their corresponding spectral values the following inequality holds:

$$\{x\}^{(r)} \leq 2 \ \{|u'(r)SV_r + v'(r)\omega_r' SD_r|\} \qquad (5.129)$$

where SV_r and SD_r are respectively the velocity and displacement corresponding to a frequency ω_r and a damping ratio ξ_r in the response spectrum of the ground motion $\ddot{q}_g(t)$. The upper limit in this equation may be evaluated from a response spectrum, and it may therefore be adopted to approximate the maximum modal responses in concern. Less conservative values may be obtained, however, if the two terms in Eq. 5.129 are combined, instead, on the basis of the square root of the sum of their squares.

That is,

$$\{x\}^{(r)} = 2 \quad \{ \sqrt{u'^2(r)SV_r^2 + v'^2(r)\omega_r'^2SD_r^2} \}. \tag{5.130}$$

Since this approximation does not consider the relative sign between the various modal responses of a system and since in some instances this sign may be an important factor in the computation of this sytem's maximum response (when the cross terms in the rule to combine modes established in Chapter 2 need to be considered, for example), it may be assumed that the sign of the argument between the absolute value bars in Eq. 5.129 is also the sign of Eq. 5.130. In this manner, the maximum modal responses $\{x\}^{(r)}$ may be estimated by

$$\{x\}^{(r)} = 2 \{ \text{sgn}[u'(r)SV_r + v'(r)\omega_r' \; SD_r] \sqrt{u'^2(r)SV_r^2 + v'^2(r)\omega_r'^2SD_r^2} \} \tag{5.131}$$

where sgn is a function which reads as "the sign of." Furthermore, if the known approximate relationship between spectral velocities and displacement is used, i.e.,

$$SV_r \doteq \omega_r SD_r, \tag{5.132}$$

and if it is considered that for small damping ratios $\omega_r' \doteq \omega_r,$ $\{x\}^{(r)}$ may be approximated by

$$\{x\}^{(r)} = 2\{\text{sgn}[(u'(r) + v'(r))\omega_r'SD_r] \; \sqrt{u'^2(r) + v'^2(r)} \; \omega_r'SD_r\} \tag{5.133}$$

or by

$$\{x\}^{(r)} = 2 \; \{\text{sgn}(u' + v')|w'|\}^{(r)}\omega_r'SD_r \tag{5.134}$$

where $|w'|$ denotes the absolute value of w'.

Equation 5.134 is the desired expression to determine the maximum modal displacements of a system with nonproportional damping from a specified response spectrum.

Convergence to the Case of Proportional Damping

To demonstrate that Eqs. 5.126 and 5.134 converge to the corresponding equations for systems with proportional damping when the damping of a system is indeed so, one may proceed as follows:

It is well known that when the damping matrix of a system is proportional to either its mass or its stiffness matrix or to any linear combination of these two matrices, all its mode shapes are real. Therefore, for such a system one may write

$$\{w\}^{(r)} = \{u\}^{(r)}. \tag{5.135}$$

Similarly, it has been shown in Sec. 5.3 that when the rth mode shape of a system is real its rth natural frequency and rth damping ratio may be expressed as

$$\omega_r = \frac{K_r^*}{M_r^*} \tag{5.136}$$

$$\xi_r\omega_r = \frac{1}{2}\,\frac{C_r^*}{M_r^*} \tag{5.137}$$

Then, for a system with proportional damping Eq. 5.67 yields

$$\gamma_r = \frac{\{u\}^{(r)^T}[M]\{J\}}{\{u\}^{(r)^T}[2(-\,\xi_r\omega_r + i\omega_r')[M] + [C]]\{u\}^{(r)}} =$$

$$= \frac{\{u\}^{(r)^T}[M]\{J\}}{-2\xi_r\omega_r M_r^* + i\,2\omega_r' M_r^* + C_r^*} \tag{5.138}$$

which by virtue of Eq. 5.137 may also be written as

$$\gamma_r = \frac{\{u\}^{(r)^T}[M]\{J\}}{i\,2\,\omega_r'\,M_r^*} \tag{5.139}$$

or as

$$\gamma_r = \frac{\alpha_r}{2i\omega_r'} , \tag{5.140}$$

where α_r is the conventional participation factor. In such a case, Eq. 5.121 in combination with Eq. 5.135 leads therefore to

$$\{w'\}^{(r)} = \{u'\}^{(r)} + i\{v'\}^{(r)} = \frac{\alpha_r}{2i\omega_r'}\{u\}^{(r)} \tag{5.141}$$

from which it is concluded that

$$\{u'\}^{(r)} = \{0\} \tag{5.142}$$

$$\{v'\}^{(r)} = -\frac{\alpha_r}{2\omega_r'}\{u\}^{(r)} . \tag{5.143}$$

Thus, by substitution of these two equations into Eq. 5.126 one obtains

$$\{x(t)\} = 2\sum_{r=1}^{n} [-(-\frac{\alpha_r}{2\omega_r'})\{u\}^{(r)}\omega_r'\; D(\omega_r,\xi_r,t)] \tag{5.144}$$

or

$$\{x(t)\} = \sum_{r=1}^{n} \alpha_r \{u\}^{(r)} D(\omega_r, \xi_r, t). \tag{5.145}$$

In like manner by substitution of Eqs. 5.141 through 5.143 into Eq. 5.134 one arrives to

$$\{x\}^{(r)} = 2 \{ \text{sng}[(- \frac{\alpha_r}{2\omega_r'}) \, u(r)] \, |\frac{\alpha_r}{2i\omega_r'} \, u(r)| \} \, \omega_r' \, SD_r \tag{5.146}$$

or to

$$\{x\} = \alpha_r \{u\}^{(r)} SD_r \tag{5.147}$$

Equations 5.145 and 5.147 are identical to the expressions used in the modal analysis of a system with proportional damping; the convergence of Eqs. 5.126 and 5.134 to the particular ones for proportional damping is thus proved.

Maximum Element Distortions

In the derivation of Eq. 5.134, the vector of maximum displacements has been considered as the desired response. If the response of interest is instead the vector of maximum element distortions, an expression similar to Eq. 5.134 may be developed as follows:

According to Eq. 5.127 the rth mode displacement of the ith mass of a system at the time the maximum displacement of this ith mass occurs may be written as

$$x_i(r) = 2 \, [u_i'(r) V(\omega_r, \xi_r, t_{max}) - v_i'(r) \omega_r' \, D(\omega_r, \xi_r, t_{max})]. \tag{5.148}$$

Therefore, the distortion of the ith element of the same system in its rth mode may be put into the form

$$X_i(r) = x_i(r) - x_{i-1}(r) =$$

$$= 2 \{[u_i'(r) - u_{i-1}'(r)] \, V(\omega_r, \xi_r, t_{max}) - [v_i'(r) - v_{i-1}'(r)] \, \omega_r' D(\omega_r, \xi_r, t_{max})\}, \tag{5.149}$$

and hence un upper bound to the maximum value of such a distortion is

$$X_i(r) \leq 2 \{|[u_i'(r) - u_{i-1}'(r)] \, SV_r + [v_i'(r) - v_{i-1}'(r)] \, \omega_r' \, SD_r|\}. \tag{5.150}$$

Taking the square root of the sum of the squares of the two terms of this equation while keeping the sign of its argument between the absolute value bars, one then may approximate $X_i(r)$ by

$$X_i(r) = 2 \, \mathrm{sgn} \{ [u_i'(r) - u_{i-1}'(r)] \, SV_r + [v_i'(r) - v_{i-1}'(r)] \omega_r' SD_r \} \cdot$$

$$\cdot \sqrt{[u_i'(r) - u_{i-1}'(r)]^2 \, SV_r^2 + [v_i'(r) - v_{i-1}'(r)]^2 \, \omega_r'^2 \, SD_r^2} \tag{5.151}$$

which, after considering Eq. 5.132 and that for small damping ratios $\omega_r' \doteq \omega_r$, results as

$$X_i(r) = 2 \, \mathrm{sgn} \{[u_i'(r) - u_{i-1}'(r)] + [v_i'(r) - v_{i-1}'(r)]\} \cdot$$

$$\cdot \sqrt{[u_i'(r) - u_{i-1}'(r)]^2 + [v_i'(r) - v_{i-1}'(r)]^2} \;\; \omega_r' \, SD_r \tag{5.152}$$

or as

$$X_i(r) = 2\ \mathrm{sgn}\{[u_i'(r) + v_i'(r)] - [u_{i-1}'(r) + v_{i-1}'(r)]\}|w_i'(r) - w_{i-1}'(r)|\omega_r' SD_r .$$

(5.153)

Thus, the corresponding rth mode vector of maximum distortions may be expressed as

$$\{x\}^{(r)} = 2\{\mathrm{sgn}[(u_i' + v_i') - (u_{i-1}' + v_{i-1}')]|w_i' - w_{i-1}'|\}^{(r)}\omega_r' SD_r .$$

(5.154)

Equations 5.134 and 5.154 reduce the solution of a system with non-proportional damping to one very similar to the conventional modal solution of a system with proportional damping. Consequently, the maximum response of such a system with nonproportional damping may also be estimated by computing its maximum modal responses from a specified response spectrum and by combining these modal maxima in the way established by the rule selected for such a purpose. To complete, then, the procedure by which the systems under study may be analyzed by the response spectrum method, the rules by which their maximum modal responses may be combined are examined next.

5.5 Combinations of Modal Maxima: Generalization of Rosenblueth's Rule

Applicable Rules

By the inspection of Eqs. 5.127 and 5.134, it is easy to see that un upper bound to the maximum response of a system with nonproportional damping may be obtained if the absolute values of its maximum modal responses are considered; hence, the combination of the modes of such a system may also be conservatively made by "the absolute sum of the maxima."

Similarly, if among all the above mentioned maximum modal responses there is one that is significatively greater than the rest of them, it may be seen that a less conservative estimate of such a maximum response may be determined by "the square root of the sum of the squares". In contrast, since the rule suggested by Rosenblueth and presented in Chapter 2 has been derived specifically for systems with proportional damping (see Ref. 26), this rule is not applicable for the systems with nonproportional damping.

In view that the chief interest of this work is in the analysis of systems with closely-spaced natural frequencies and that Rosenblueth's rule is particularly appropriate to combine their modal responses, it is here convenient to generalize this rule for its application in the cases in which these systems have nonproportional damping. Based on the theory developed in this chapter and on the original derivation of Rosenblueth's rule as described in Ref. 26, this generalization may then be accomplished as follows:

Maximum Response in Terms of Modal Maxima

According to Eq. 5.68, the displacement response* of a linear multi-degree-of-freedom system with nonproportional damping is given by

$$\{x(t)\} = -2 \sum_{r=1}^{N} \text{Re} \left[\{w'\}^{(r)} \int_0^t e^{\lambda_r (t-\tau)} \ddot{q}_g(\tau) d\tau \right] \qquad (5.155)$$

where $\{w'\}^{(r)}$ denotes the rth complex mode shape with unit participation factor of the system, N is the number of its degrees of freedom, and

*The generalization of Rosenblueth's rule is made here in terms of the displacement response; notice, however, that this generalization may be obtained as well in terms of any other response, such as the element distortion, velocity or acceleration response.

all other symbols are as denoted before. For any particular mass, say the ith, such displacement response may be then written as

$$x_i(t) = -2 \sum_{r=1}^{N} \text{Re} \; [w_i'(r) \int_o^t e^{\lambda_r(t-\tau)} \ddot{q}_g(\tau) d\tau] \; . \qquad (5.156)$$

But, if a new dummy variable $\theta = t-\tau$ is introduced and if it is considered that for $t < \theta$ (that is, $\tau < o$) $x_i(t)$ vanishes and hence the upper limit of the above integral may be replaced by infinity without changing the value of the integral, this equation may be alternatively expressed as

$$x_i(t) = -2 \sum_{r=1}^{N} \text{Re} \; [w_i'(r) \int_o^\infty e^{\lambda_r \theta} \ddot{q}_g(t-\theta) d\theta] \qquad (5.157)$$

which in turn, since the real part of an integral is equal to the integral of the real part of its argument, may also be put into the form

$$x_i(t) = -2 \sum_{r=1}^{N} \int_o^\infty \text{Re} \; [w_i'(r) \; e^{\lambda_r \theta}] \; \ddot{q}_g(t-\theta) d\theta \; . \qquad (5.158)$$

By denoting

$$\psi_{x_r}(t) = -2 \; \text{Re} \; [w_i'(r) \; e^{\lambda_r t}], \qquad (5.159)$$

where $\psi_{x_r}(t)$ represents the rth transfer function of the system, one may therefore write $x_i(t)$ as

$$x_i(t) = \sum_{r=1}^{N} \int_o^\infty \psi_{x_r}(\tau) \; \ddot{q}_g(t-\tau) d\tau, \qquad (5.160)$$

and hence, since by definition the transfer function is the response to a unit impulse $\delta(t)$[9], where $\delta(t)$ is Dirac's delta function, by substitution of the function $\ddot{q}_g(t-\tau)$ in Eq. 5.160 by $\delta(t-\tau)$ one obtains

$$\psi_x(t) = \sum_{r=1}^{N} \psi_{x_r}(t) \tag{5.161}$$

which in words simply means that the transfer function of a system with nonproportional damping is, as in the case of proportional damping, equal to the sum of the individual transfer functions in each of its modes.

Then, since under the assumption of a stationary white noise excitation the mean square of any response is of the form

$$E[x_i^2(t)] = 2\pi S_o \int_o^\infty \psi_x^2(t)dt \tag{5.162}$$

(see Crandall and Mark, 1963), where S_o is the constant spectral density of such a white noise excitation, by substitution of Eq. 5.161 into the above equation the mean square of the total response $x_i(t)$ may be written as

$$E[x_i^2(t)] = 2\pi S_o \int_o^\infty [\sum_{r=1}^{N} \psi_{x_r}(t)]^2 \, dt =$$

$$= \sum_{r=1}^{N} 2\pi S_o \int_o^\infty \psi_{x_r}^2(t)dt + \sum_{m=1}^{N} \sum_{\substack{n=1 \\ m\neq n}}^{N} 2\pi \, S_o \int_o^\infty \psi_{x_m}(t)\psi_{x_n}(t)dt. \tag{5.163}$$

However, if the argument of the double summation in this last equation is expressed as

$$2\pi S_o \int_o^\infty \psi_{x_m}(t)\psi_{x_n}(t)dt = \alpha_{mn}\sqrt{[2\pi S_o \int_o^\infty \psi_{x_m}(t)dt][2\pi S_o \int_o^\infty \psi_{x_n}(t)dt]} \tag{5.164}$$

and if in the light of Eq. 5.162 the mean squares of the modal responses $x_{i_r}(t)$ are written as

$$E[x_{i_r}^2(t)] = 2\pi S_o \int_o^\infty \psi_{x_r}^2(t)\ dt \tag{5.165}$$

$E[x_i^2(t)]$ may be put into the form

$$E[x_i^2(t)] = \sum_{r=1}^{N} E[x_{i_r}^2(t)] + \sum_{\substack{m=1 \\ m\neq n}}^{N}\sum_{n=1}^{N} \alpha_{mn}\sqrt{E[x_{i_m}^2(t)]\ E\ [x_{i_n}^2(t)]} \tag{5.166}$$

where α_{mn} is a factor, evaluated later on, introduced merely to correlate the double product terms of Eq. 5.163 with their associated mean squares.

Notice thus that if in accordance with the theory of the first passage problem (Ang, 1974) and with the equivalent assumption made in the case of proportional damping (Rosenblueth, 1968) it is now assumed that the absolute maximum value of the response of a system for any given probability of exceedance is proportional to the root mean square of such a response, i.e.,

$$X_{i_{max}} = c\sqrt{E(x_i^2)} \quad , \tag{5.167}$$

where $X_{i_{max}}$ is such a maximum value and c is a proportionality constant, the relation between the total maximum response of a system with nonproportional damping and its maximum modal responses results as

$$X_{i_{max}} = \sqrt{\sum_{r=1}^{N} X_{i_r}^2 + \sum_{m=1}^{N} \sum_{\substack{n=1 \\ m \neq n}}^{N} \alpha_{mn} X_{i_m} X_{i_n}} \tag{5.168}$$

in which X_{i_r} represents such modal maxima.

Modal Correlation Factors

The modal correlation factors α_{mn} may be evaluated as follows*:

According to Eq. 5.164 the modal correlation factor of a system with nonproportional damping is defined as

$$\alpha_{mn} = \frac{\int_0^\infty \psi_{x_m}(t)\, \psi_{x_n}(t)\, dt}{\sqrt{[\int_0^\infty \psi_{x_m}^2(t)dt]\,[\int_0^\infty \psi_{x_n}^2(t)dt]}} \tag{5.169}$$

where ψ_{x_r}, $r = m,n$, is the transfer function defined by Eq. 5.159. Since this transfer function may be written as

$$\psi_{x_r}(t) = -\,[w_i'(r)e^{\lambda_r t} + \overline{w}_i'(r)e^{\overline{\lambda}_r t}], \tag{5.170}$$

then the integral in the numerator of Eq. 5.169 may be expressed as

$$\int_0^\infty \psi_{x_m}(t)\, \psi_{x_n}(t)dt =$$

$$= 2\,\mathrm{Re}\,[w_i'(m)w_i'(n)\int_0^\infty e^{(\lambda_m + \lambda_n)t} dt + w_i'(m)\overline{w}_i'(n)\int_0^\infty e^{(\lambda_m + \overline{\lambda}_n)t} dt] \tag{5.171}$$

*Observe that the modal correlation factors of a system with nonproportional damping differ from those of a similar system with proportional damping because their transfer functions are different (see Eq. 5.159).

and thus after solving these last two integrals one arrives to

$$\int_0^\infty \psi_{x_m}(t)\psi_{x_n}(t)dt = -\ 2\mathrm{Re}\left[\frac{w_i'(m)w_i'(n)}{\lambda_m + \lambda_n} + \frac{w_i'(m)\overline{w}_i'(n)}{\lambda_m + \overline{\lambda}_n}\right] . \qquad (5.172)$$

Similarly, by setting m=n=r in this last equation one has that the integrals in the denominator of the same Eq. 5.169 are of the form

$$\int_0^\infty \psi_{x_r}^2 (t)dt = -\ 2\mathrm{Re}\left[\frac{w_i'^2(r)}{2\lambda_r} + \frac{|w_i'(r)|^2}{2\mathrm{Re}\lambda_r}\right]$$

$$= -\ \frac{\mathrm{Re}[w_i'^2(r)\overline{\lambda}_r]}{\omega_r^2} + \frac{|w_i'(r)|^2}{\xi_r\omega_r} \qquad (5.173)$$

which for small damping ratios may be approximated as

$$\int_0^\infty \psi_{x_r}^2(t)dt \doteq \frac{|w_i'(r)|^2}{\xi_r\omega_r} . \qquad (5.174)$$

In the light of Eqs. 5.169, 5.172 and 5.174, the modal correlation factors α_{mn} result therefore as

$$\alpha_{mn} = 2\mathrm{Re}\left[\frac{w_i'(m)w_i'(n)}{\lambda_m + \lambda_n} + \frac{w_i'(m)\overline{w}_i'(n)}{\lambda_m + \overline{\lambda}_n}\right] \frac{\sqrt{\xi_m\omega_m\xi_n\omega_n}}{|w_i'(m)|\,|w_i'(n)|} . \qquad (5.175)$$

Equivalent Damping Ratios

It may be observed that when the damping ratio ξ_r approaches zero, the value of the integral in Eq. 5.174 approaches infinity. Consequently,

$E[x_{i_r}^2(t)]$ as given by Eq. 5.165 and X_{i_r} under the assumption indicated by Eq. 5.167 also become infinite. Since for real earthquakes the maximum responses X_{i_r} are always bounded, the hypothesis of a stationary process in the above derivation leads thus to an inconsistency that for the accurate application of Eqs. 5.168 and 5.175 needs to be corrected.

Using the concept of equivalent damping ratios introduced in the analysis of systems with proportional damping [26,21], this inconsistency may then be corrected by substituting the damping ratios of the system under analysis (appearing in Eq. 5.175) by the damping ratios of an equivalent system whose maximum response when determined with the model described above (that is, with Eq. 5.167 and the hypothesis of a white noise excitation of infinite duration) is equal to the maximum response obtained when the original system is subjected to a finite segment of white noise (a nonstationary model that accounts for the transient nature of real earthquakes).

In determining such equivalent damping ratios, therefore, one may note that in the light of Eqs. 5.174, 5.165 and 5.167 the maximum response X_{i_r} on the basis of a stationary white noise may be written as

$$X_{i_r} = c\sqrt{2\pi S_o \frac{|w_i'(r)|^2}{\xi_r\omega_r}} = k_1 \frac{|w_i'(r)|}{\sqrt{\xi_r\omega_r}} \qquad (5.176)$$

in which k_1 is a constant. If it is observed, however, that the sought equivalent damping ratios are not employed to compute the mode shapes of the system and that for this reason $w_i'(r)$ in the above expression may be considered as a constant in spite that it varies with ξ_r, one may express

X_{i_r} as

$$X_{i_r} = \frac{k_2}{\sqrt{\xi_r \omega_r}} \tag{5.177}$$

where k_2 is just another constant. In like manner, if it is assumed that the maximum response of a system for a given probability of exceedance is proportional to the expected value of such a response*, and if it is considered that according to Newmark and Rosenblueth (1971) the ratio between the expected values of the damped and undamped maximum responses of such a system to a segment of white noise of duration s_r is of the form

$$\beta_E = (1 + 0.5 \; \xi_r \omega_r s_r)^{-0.5} \quad , \tag{5.178}$$

then when the excitation is such a limited segment of white noise the aforementioned maximum response may be expressed as

$$X_{i_r} = k_3 \; E(X_{i_r}) = \frac{k_4}{\sqrt{1 + 0.5 \; \xi_r \omega_r s_r}} \tag{5.179}$$

in which k_3 and k_4 are other constants. Thus, if ξ_r' represents the rth equivalent damping ratio and if ξ_r in Eq. 5.177 is replaced by this equivalent damping ratio, after equating Eqs. 5.177 and 5.179 one is led to the following relation between ξ_r' and ξ_r:

$$\xi_r' \omega_r = k \; (1 + 0.5 \; \xi_r \omega_r s_r). \tag{5.180}$$

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*Assumption introduced in the original derivation by Rosenblueth [26,21].

By noticing, then, that when s_r approaches infinity, the above mentioned finite segment of white noise becomes a stationary one of infinite duration and that in such a case the equivalent and real damping ratios in Eq. 5.180 coincide, it is easy to show that $k = 2/s_r$ and that the sought expression to compute equivalent damping ratios results consequently as

$$\xi_r' = \xi_r + \frac{2}{\omega_r s_r} \quad . \tag{5.181}$$

Conclusions

If in following the above established criterion the damping ratios ξ_m and ξ_n in Eq. 5.175 are substituted by their equivalent ones ξ_m' and ξ_n', the corrected modal correlation factor α_{mn} is therefore given by

$$\alpha_{mn} = 2\mathrm{Re}\left[\frac{w_i'(m)w_i'(n)}{\lambda_m' + \lambda_n'} + \frac{w_i'(m)\overline{w}_i'(n)}{\lambda_m' + \overline{\lambda}_n'}\right] \frac{\sqrt{\xi_m'\omega_m\xi_n'\omega_n}}{|w_i'(m)||w_i'(n)|} \tag{5.182}$$

where according to the definition of a complex natural frequency the corrected frequencies λ_m' and λ_n' are of the form

$$\lambda_r' = -\xi_r'\omega_r + i\omega_r\sqrt{1 - \xi_r'^2} \ , \ r = m, n. \tag{5.183}$$

Equation 5.168 in combination with Eqs. 5.181 and 5.182 constitutes thus the sought general rule to combine the maximum modal responses of systems with nonportional damping. In examining this rule, one may note that in this case of nonporportional damping:

1) the expression to compute equivalent damping ratios is identical to the one employed for systems with porportional damping.

2) The duration s_r in Eq. 5.181 needs to be adjusted to fit the average characteristics of specified ground disturbances in much the same way as that for systems with proportional damping (see Sec. 2.10).

3) Since Eq. 5.182 is a function of the ratios $w_i'(r)/|w_i'(r)|$, $r = m,n$, and $\overline{w}_i(n)/|w_i'(n)|$, which are nothing else but unit magnitude complex numbers with the arguments of $w_i'(r)$, $r = m,n$, and $\overline{w}_i'(n)$, respectively, the modal correlation factors of a system depend on the phase angles of its mode shapes.

4) Differently from the ones for proportional damping which are always positive (see Eq. 2.102), the modal correlation factors may fluctuate between positive and negative values.

5) Because the various masses of a system vibrate with different phase angles and α_{mn} depends on these phase angles, there is a different modal correlation factor for each of these masses.

CHAPTER 6

GENERALIZATION OF APPROXIMATE METHOD: NONPROPORTIONAL DAMPING AND UP TO TWO POINTS OF ATTACHMENT

6.1 Introduction

Using the concepts developed in the foregoing chapter, it is now possible to derive an approximate procedure based on the response spectrum method to determine the maximum response of those secondary systems which in combination with their supporting structures give rise to assembled systems with nonproportional damping. For this purpose, it may be observed that, as in the case of proportional damping, the maximum response of a system with nonproportional damping may also be obtained by determining the system's mode shapes, natural frequencies, participation factors, and maximum modal responses and by combining its maximum modal responses according to an established rule. Thus, since the determination of mode shapes, natural frequencies, and participation factors and the rule used to combine modes in the cases of proportional and nonproportional damping are very similar in structure, the desired approximate procedure may be derived by a logical extension of the procedure developed in the preceding chapters.

In this chapter, then, the methods introduced in Chapters 2 and 4 are generalized to derive approximate expressions for the computation of the complex mode shapes, natural frequencies and participation factors of an assembled system with nonproportional damping; the rule to combine modes established in Sec. 5.5 is simplified for its application to the systems studied in this chapter; and, on the basis of such approximate expressions and this simplified rule, an approximate procedure--the generalization of the one proposed in Chapter 4--is derived to estimate the maximum response of the secondary systems herein under consideration.

As in the case of proportional damping, this general procedure is here developed on the assumption that any given independent primary and secondary systems are systems whose damping matrices are proportional to their respective stiffness matrices, although in this case of nonproportional damping the associated proportionality constants of such primary and secondary systems obviously need not be the same. Also as in the case of proportional damping, the expressions developed in this chapter are first derived for a particular model and thereafter generalized for systems with any number of degrees of freedom and other configurations by simple induction. The model used in this case is shown in Fig. 6.1.

6.2 Complex Mode Shapes of Assembled System

By following the procedure employed for systems with proportional damping and by considering the reduced equations of motion of the primary and secondary components of an assembled system with nonproportional damping, an expression to obtain the complex mode shapes of this assembled system in terms of the dynamic properties of its independent components may be derived as follows:

Primary System Part of Complex Eigenvectors

Consider the assembled system in Fig. 6.1 and its primary subsystem as depicted in Fig. 6.2(a). The reduced equation of motion of this primary subsystem is given by

$$[A]\{\dot{q}_p\} + [B]\{q_p\} = \{F(t)\} \tag{6.1}$$

where

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \tag{6.2}$$

$$[B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \tag{6.3}$$

$$\{F(t)\} = \begin{Bmatrix} \{0\} \\ \{R(t)\}_p \end{Bmatrix} \tag{6.4}$$

$$\{q_p\} = \begin{Bmatrix} \{\dot{x}_p\} \\ \{x_p\} \end{Bmatrix} \tag{6.5}$$

in which [C] is the damping matrix of the system, $\{R(t)\}_p$ is the vector of applied external forces, and all other symbols are as defined in Chapters 2 and 4.

According to the discussion in Sec. 5.2, the solution to this reduced equation of motion is of the form

$$\{q_p\} = [S]\ \{Z'\} \tag{6.6}$$

where [S] is the $2N_p \times 2N_p$ matrix of the complex eigenvectors of the above mentioned primary subsystem and $\{Z'\}$ is the vector of its normal coordinates. By substitution of Eq. 6.6 into Eq. 6.1 and by premultiplication of this latter equation by $[S]^T$, the reduced equation of motion of the system under consideration may be therefore expressed as

$$[S]^T[A]\,[S]\,\{\dot{Z}'\} + [S]^T[B]\,[S]\,\{Z'\} = [S]^T\{F(t)\} \tag{6.7}$$

which in view of the orthogonality properties of the matrices [A] and [B] (see Sec. 5.2) results in the following set of independent equations:

$$A_i^* \dot{Z}_i' + B_i^* Z_i' = \{S\}^{(i)^T}\{F(t)\}, \quad i = 1, 2, \ldots, 2N_p \tag{6.8}$$

where the generalized parameters A_i^* and B_i^* are of the form

$$A_i^* = \{S\}^{(i)^T}[A]\,\{S\}^{(i)} \tag{6.9}$$

$$B_i^* = \{S\}^{(i)^T}[B]\,\{S\}^{(i)}. \tag{6.10}$$

However, according to Eq. 5.59 $\{S\}^{(i)}$ may be written as

$$\{S\}^{(i)} = \begin{Bmatrix} \lambda_{p_i}\{\Phi\}^{(i)} \\ \{\Phi\}^{(i)} \end{Bmatrix} \tag{6.11}$$

where λ_{p_i} and $\{\Phi\}^{(i)}$ are, respectively, the ith complex natural frequency and ith complex mode shape of the primary system under study. Using Eqs. 6.4 and 6.11 the product $\{S\}^{(i)^T}\{F(t)\}$ may be therefore expressed as

$$\{S\}^{(i)^T}\{F(t)\} = \{\Phi\}^{(i)^T}\{R(t)\}_p \tag{6.12}$$

or if it is considered that

$$\{R(t)\}_p = \begin{Bmatrix} R_1(t) \\ 0 \\ R_3(t) \end{Bmatrix} \tag{6.13}$$

as

$$\{S\}^{(i)^T}\{F(t)\} = \Phi_1(i)\ R_1(t) + \Phi_3(i)\ R_3(t). \tag{6.14}$$

Similarly, if it is considered that $\{q_p\}$ also represents the primary system part of the solution to the homogeneous reduced equation of motion of the assembled system of Fig. 6.1 and that, as a result, this vector may be alternatively written as

$$\{q_p\} = \{\sigma_p\}\ e^{\lambda t}, \tag{6.15}$$

where $\{\sigma_p\}$ is the primary part of one of the complex eigenvectors of the aforementioned assembled system and λ is the corresponding eigenvalue, then in view of Eq. 6.6 the vector $\{Z'\}$ may be expressed as

$$\{Z'\} = \{Z\}\ e^{\lambda t} \tag{6.16}$$

where $\{Z\}$ is a vector of unknown amplitudes. Thus, in the light of Eqs. 6.14 and 6.16 and since according to the discussion in Sec. 5.2 B_i^* may be written as

$$B_i^* = -\lambda_{p_i}\ A_i^* \tag{6.17}$$

Eq. 6.8 may be put into the form

$$(\lambda - \lambda_{p_i})\, A_i^* Z_i\, e^{\lambda t} = \Phi_1(i)R_1(t) + \Phi_3(i)R_3(t), \quad i=1, 2, \ldots, 2Np. \qquad (6.18)$$

Consequently, if Eqs. 6.15 and 6.16 are substituted into Eq. 6.6, one has that the eigenvector $\{\sigma_p\}$ may be expressed as

$$\{\sigma_p\} = [S]\,\{Z\} = \sum_{i=1}^{2N_p} \{S\}^{(i)}\, Z_i\ . \qquad (6.19)$$

In the same fashion, if $R_1(t)$ is solved from the Ith component equation of Eqs. 6.18, and if $R_3(t)$ is expressed in terms of the relation

$$\eta = \frac{R_3(t)}{R_1(t)}\ , \qquad (6.20)$$

one obtains

$$R_1(t) = \frac{(\lambda - \lambda_{p_I})\, A_I^*}{\Phi_1(I) + \eta\Phi_3(I)}\, Z_i\, e^{\lambda t}, \qquad (6.21)$$

and hence by substituting Eqs. 6.20 and 6.21 into Eq. 6.18 and by solving for Z_i from this latter equation one arrives to the following relation for the Z_i factors in Eq. 6.19:

$$Z_i = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\lambda - \lambda_{p_I}}{\lambda - \lambda_{p_i}} \frac{A_I^*}{A_i^*} Z_I, \; i=1, 2, \ldots, 2N_p \tag{6.22}$$

where as before

$$\hat{\Phi}(i) = \Phi_1(i) + \eta\Phi_3(i), \; i=1, 2, \ldots, 2N_p. \tag{6.23}$$

Notice thus that in general the primary system part of the rth complex eigenvector of an assembled system with nonproportional damping is given by

$$\{\sigma_p\}^{(r)} = \sum_{i=1}^{2N_p} Z_i^{(r)} \{S\}^{(i)} \tag{6.24}$$

where

$$Z_i^{(r)} = \frac{\hat{\Phi}_r(i)}{\hat{\Phi}_r(I)} \frac{\lambda_r - \lambda_{p_I}}{\lambda_r - \lambda_{p_i}} \frac{A_I^*}{A_i^*} \; Z_I^{(r)}, \; i=1,2,\ldots,2N_p \tag{6.25}$$

in which λ_r is the rth natural frequency of such an assembled system and N_p and the general expression for $\hat{\Phi}_r(i)$ are as defined in Chapters 2 and 4.

<u>Primary System Part of Complex Mode Shapes</u>

Since according to Eq. 5.59 the eigenvector $\{\sigma_p\}$ is of the form

$$\{\sigma_p\} = \begin{Bmatrix} \lambda\{w_p\} \\ \{w_p\} \end{Bmatrix} \tag{6.26}$$

where $\{w_p\}$ represents the primary system part of a complex mode shape of the assembled system described in Fig. 6.1 (i.e., the amplitudes and phase angles of its primary masses in one of its modes), this vector $\{w_p\}$ may be obtained directly from the lower half of Eq. 6.19. If it is considered, however, that the mode shapes and natural frequencies of a system with nonproportional damping always occur, when underdamped, in pairs of complex conjugates and that, by assumption, the primary system herein being considered has by itself proportional damping, a simplified expression for $\{w_p\}$ may be developed as follows:

Explicitly in terms of the complex mode shapes $\{S\}^{(i)}$ and the corresponding complex conjugates $\{\bar{S}\}^{(i)}$, Eq. 6.19 may be expressed as

$$\{\sigma_p\} = \sum_{i=1}^{N_p} [\{S\}^{(i)} Z_i + \{\bar{S}\} Z_{\bar{i}}] \tag{6.27}$$

where $Z_{\bar{i}}$ is the coordinate corresponding to $\bar{\lambda}_{p_i}$, the complex conjugate of λ_{p_i}. Then, if Eq. 6.11 is substituted into Eq. 6.27, one obtains

$$\{\sigma_p\} = \sum_{i=1}^{N_p} \left[\begin{Bmatrix} \lambda_{p_i} \{\Phi\}^{(i)} \\ \{\Phi\}^{(i)} \end{Bmatrix} Z_i + \begin{Bmatrix} \bar{\lambda}_{p_i} \{\bar{\Phi}\}^{(i)} \\ \{\bar{\Phi}\}^{(i)} \end{Bmatrix} Z_{\bar{i}} \right] \tag{6.28}$$

which, after rearranging terms and taking into consideration that in this case $\{\Phi\}^{(i)}$ is real and hence $\{\bar{\Phi}\}^{(i)} = \{\Phi\}^{(i)}$, may also be written as

$$\{\sigma_p\} = \sum_{i=1}^{N_p} \begin{Bmatrix} \{\Phi\}^{(i)} [\lambda_{p_i} Z_i + \bar{\lambda}_{p_i} Z_{\bar{i}}] \\ \{\Phi\}^{(i)} [Z_i + Z_{\bar{i}}] \end{Bmatrix} . \tag{6.29}$$

But in view of Eq. 6.22 the sum $Z_i + Z_{\bar{i}}$ may be put into the form

$$Z_i + Z_{\bar{i}} = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\lambda - \lambda_{p_I}}{\lambda - \lambda_{p_i}} \frac{A_I^*}{A_i^*} Z_I + \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\lambda - \lambda_{p_I}}{\lambda - \bar{\lambda}_{p_i}} \frac{A_I^*}{\bar{A}_i^*} Z_I \tag{6.30}$$

which, if it is considered that for a system with proportional damping A_i^* and $\overline{A}_i^*$ result as

$$A_i^* = 2\lambda_{p_i} M_i^* + C_i^* = 2\,(-\xi_{p_i}\omega_{p_i} + i\omega'_{p_i}) M_i^* + 2\,\xi_{p_i}\omega_{p_i} M_i^* = 2i\omega'_{p_i} M_i^* \tag{6.31}$$

$$\overline{A}_i^* = 2\bar{\lambda}_{p_i} M_i^* + C_i^* = 2(-\xi_{p_i}\omega_{p_i} - i\,\omega'_{p_i})\, M_i^* + 2\,\xi_{p_i}\omega_{p_i} M_i^* = -\,2i\omega'_{p_i} M_i^*\,, \tag{6.32}$$

may also be expressed as

$$Z_i + Z_{\bar{i}} = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\omega'_{p_I}}{\omega'_{p_i}} \frac{M_I^*}{M_i^*} \left[\frac{1}{\lambda - \lambda_{p_i}} - \frac{1}{\lambda - \bar{\lambda}_{p_i}} \right] (\lambda - \lambda_{p_I})\, Z_I \tag{6.33}$$

or as

$$Z_i + Z_{\bar{i}} = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{(\lambda_{p_I} - \bar{\lambda}_{p_I})(\lambda - \lambda_{p_I})}{(\lambda - \lambda_{p_i})(\lambda - \bar{\lambda}_{p_i})} \frac{M_I^*}{M_i^*} Z_I \;. \tag{6.34}$$

Similarly, by means of Eq. 6.22 the sum $\lambda_{p_i} Z_i + \bar{\lambda}_{p_i} Z_{\bar{i}}$ may be written as

$$\lambda_{p_i} Z_i + \overline{\lambda}_{p_i} Z_{\bar{i}} = \lambda_{p_i} \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\lambda - \lambda_{p_I}}{\lambda - \lambda_{p_i}} \frac{A_I^*}{A_i^*} Z_I + \overline{\lambda}_{p_i} \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{\lambda - \lambda_{p_I}}{\lambda - \overline{\lambda}_{p_i}} \frac{A_I^*}{\overline{A}_i^*} Z_I \tag{6.35}$$

which after substituting Eqs. 6.31 and 6.32 becomes

$$\lambda_{p_i} Z_i + \overline{\lambda}_{p_i} Z_{\bar{i}} = \lambda \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{(\lambda_{p_I} - \overline{\lambda}_{p_I})(\lambda - \lambda_{p_I})}{(\lambda - \lambda_{p_i})(\lambda - \overline{\lambda}_{p_i})} \frac{M_I^*}{M_i^*} Z_I \tag{6.36}$$

and hence by virtue of Eq. 6.34 it results as

$$\lambda_{p_i} Z_i + \overline{\lambda}_{p_i} Z_{\bar{i}} = \lambda (Z_i + Z_{\bar{i}}) \quad . \tag{6.37}$$

In the light of Eqs. 6.26, 6.29 and 6.37, $\{\sigma_p\}$ may be therefore expressed as

$$\{\sigma_p\} = \begin{Bmatrix} \lambda\{\omega_p\} \\ \{\omega_p\} \end{Bmatrix} = \sum_{i=1}^{N_p} \begin{Bmatrix} \lambda\{\Phi\}^{(i)}(Z_i + Z_{\bar{i}}) \\ \{\Phi\}^{(i)}(Z_i + Z_{\bar{i}}) \end{Bmatrix} \tag{6.38}$$

and thus from either the upper or lower half of this equation one may conclude that

$$\{\omega_p\} = \sum_{i=1}^{N_p} \{\Phi\}^{(i)}(Z_i + Z_{\bar{i}}) \ . \tag{6.39}$$

Then, if a new variable Y_i is defined as

$$Y_i = Z_i + Z_{\bar{i}} \tag{6.40}$$

and if it is observed that in accordance with this definition and Eq. 6.22 Y_I may be written as

$$Y_I = \frac{\lambda_{p_I} - \bar{\lambda}_{p_I}}{\lambda - \bar{\lambda}_{p_I}} Z_I \tag{6.41}$$

from which Z_I results as

$$Z_I = \frac{\lambda - \bar{\lambda}_{p_I}}{\lambda_{p_I} - \bar{\lambda}_{p_I}} Y_I , \tag{6.42}$$

$\{w_p\}$ may be written as

$$\{w_p\} = \sum_{i=1}^{N_p} \{\Phi\}^{(i)} Y_I \tag{6.43}$$

where

$$Y_i = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{(\lambda - \lambda_{p_I})(\lambda - \bar{\lambda}_{p_I})}{(\lambda - \lambda_{p_i})(\lambda - \bar{\lambda}_{p_i})} \frac{M_I^*}{M_i^*} Y_I . \tag{6.44}$$

In general, therefore, the primary system part of the rth mode shape of an assembled system with nonproportional damping is given by

$$\{w_p\}^{(r)} = \sum_{i=1}^{N_p} \{\Phi\}^{(i)} Y_i^{(r)} \tag{6.45}$$

where the $Y_i^{(r)}$ factors are of the form

$$Y_i^{(r)} = \frac{\hat{\Phi}_r(i)}{\hat{\Phi}_r(I)} \; \frac{(\lambda_r - \lambda_{p_I})(\lambda_r - \bar{\lambda}_{p_I})}{(\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i})} \; \frac{M_I^*}{M_i^*} \; Y_I^{(r)} \tag{6.46}$$

in which λ_r is the rth complex natural frequency of such an assembled system and $\hat{\Phi}_r(i)$ is given by Eq. 4.10.

Secondary System Part of Complex Eigenvectors

Consider now the independent secondary system shown in Fig. 6.2(b). This system is an unrestrained four-degree-of-freedom system whose reduced equation of motion is

$$[a]\{\dot{q}_s\} + [b]\{q_s\} = -\{f(t)\} \tag{6.47}$$

where

$$[a] = \begin{bmatrix} [o] & [m] \\ [m] & [c] \end{bmatrix} \tag{6.48}$$

$$[b] = \begin{bmatrix} -[m] & [o] \\ [o] & [k] \end{bmatrix} \tag{6.49}$$

$$\{f(t)\} = \begin{Bmatrix} \{o\} \\ \{R(t)\}_s \end{Bmatrix} \tag{6.50}$$

$$\{q_s\} = \begin{Bmatrix} \{\dot{x}_s\} \\ \{x_s\} \end{Bmatrix} \tag{6.51}$$

in which [c] is the damping matrix of the system, $\{R(t)\}_s$ represents the vector of external forces applied to the system, and all other symbols are as denoted before.

Now, according to the component mode synthesis technique (Hurty, 1965) the response of any linear system to given external forces may be represented by a linear combination of its rigid-body modes, a constraint mode for each of the redundancies of the system, and a set of fixed modes whose number is equal to the number of degrees of freedom of the system when its supports are held fixed. By extending this concept to the solution of the reduced equation of motion of a system with nonproportional damping and by considering that, if the system is underdamped, this solution may always be written in terms of the complex eigenvectors of the system and their respective complex conjugates, such a solution may then be expressed as a linear combination of the system's complex rigid-body, constraint and fixed eigenvectors and their corresponding complex conjugates.

Thus, if the complex rigid-body, constraint and fixed eigenvectors of the secondary system under consideration are defined respectively as

$$\{s\}^{(0)} = \begin{Bmatrix} \lambda_{s_0} \{\phi\}^{(0)} \\ \{\phi\}^{(0)} \end{Bmatrix} \tag{6.52}$$

$$\{s\}^{(c)} = \begin{Bmatrix} \lambda_{s_c} \{\phi\}^{(c)} \\ \{\phi\}^{(c)} \end{Bmatrix} \tag{6.53}$$

$$\{s\}^{(j)} = \begin{Bmatrix} \lambda_{s_j} \{\phi\}^{(j)} \\ \{\phi\}^{(j)} \end{Bmatrix} \tag{6.54}$$

where $\{\phi\}^{(0)}$, $\{\phi\}^{(c)}$ and $\{\phi\}^{(j)}$ are the rigid-body, constraint and fixed modes described by Eqs. 4.13, 4.14 and 4.15, respectively, and λs_0, λs_c and λs_j are the corresponding complex natural frequencies,* the solution of Eq. 6.47 may be written as

$$\{q_s\} = \{s\}^{(0)} z_0' + \{s\}^{(1)} z_1' + \{s\}^{(2)} z_2' + \{s\}^{(c)} z_c' +$$

$$+ \{\bar{s}\}^{(0)} z_{\bar{0}}' + \{\bar{s}\}^{(1)} z_{\bar{1}}' + \{\bar{s}\} z_{\bar{2}}' + \{\bar{s}\}^{(c)} z_{\bar{c}}' \qquad (6.55)$$

or as

$$\{q_s\} = [s]\,\{z'\} \qquad (6.56)$$

where [s] is the $2(N_s + 2) \times 2(N_s + 2)$ matrix of the complex eigenvectors of the system and $\{z'\}$ is a vector of unknown independent generalized coordinates.

Upon substitution of Eq. 6.56 and premultiplication by $[s]^T$, Eq. 6.47 may be therefore expressed as

$$[s]^T[a][s]\{\dot{z}'\} + [s]^T[b][s]\{z'\} = -\,[s]^T\{f(t)\}. \qquad (6.57)$$

One may note, however, that the fixed complex eigenvectors of [s] are the normal complex eigenvectors of the secondary system herein being considered and consequently the following orthogonality relations are applicable (see Eqs. 5.17 through 5.20):

---

*Note that because the matrices [a] and [b] in Eq. 6.47 are positive definite, the complex natural frequencies corresponding to the complex rigid-body and constraint modes (λ_0 and λ_c) are different from zero.

$$\{s\}^{(i)^T}[a]\ \{s\}^{(j)} = 0,\ i \neq j;\ i,\ j \neq 0,\ c \tag{6.58}$$

$$\{s\}^{(i)^T}[b]\{s\}^{(j)} = 0,\ i \neq j;\ i,\ j \neq 0,\ c \quad . \tag{6.59}$$

Additionally, it may be observed that since: (a) $[s]^T\{f(t)\}$ may be written as

$$[s]^T\{f(t)\} = [\{s\}^{(0)}\{s\}^{(1)}\{s\}^{(2)}\{s\}^{(c)}\{\bar{s}\}^{(0)}\{\bar{s}\}^{(1)}\{\bar{s}\}^{(2)}\{s\}^{(c)}]^T\{f(t)\}, \tag{6.60}$$

(b) $\{R(t)\}_s$ is given by [see Fig. 6.2(b)]

$$\{R(t)\}_s = \begin{Bmatrix} R_1(t) \\ 0 \\ 0 \\ R_3(t) \end{Bmatrix} \quad , \tag{6.61}$$

and (c) in the light of Eqs. 6.50 and 6.52 through 6.54 each of the products $\{s\}^{(j)^T}\{f(t)\}$ in Eq. 6.60 may be expressed as

$$\{s\}^{(j)^T}\{f(t)\} = \{\phi\}^{(j)^T}\{R(t)\}_s = \begin{cases} R_1(t) + R_3(t) \text{ if } j = 0 \\ 0 \text{ if } \ j \neq 0,\ c \\ \phi_c(c)R_3(t) \text{ if } j = c \end{cases} \tag{6.62}$$

then $[s]^T\{f(t)\}$ may be written as

$$[s]^T\{f(t)\} = \begin{Bmatrix} R_1(t) + R_3(t) \\ 0 \\ 0 \\ \phi_c(c)R_3(t) \\ 0 \\ 0 \\ \phi_c(c)R_3(t) \end{Bmatrix} . \tag{6.63}$$

Hence, under the transformation indicated by Eq. 6.56 the reduced equation of motion of the system in Fig. 6.2(b) results as

$$
\left[\begin{array}{cccc|cccc}
a_{00} & a_{01} & a_{02} & a_{0c} & a_{0\bar{0}} & a_{0\bar{1}} & a_{0\bar{2}} & a_{0\bar{c}} \\
a_{10} & a_{1}^{*} & 0 & a_{1c} & a_{1\bar{0}} & 0 & 0 & a_{1\bar{c}} \\
a_{20} & 0 & a_{2}^{*} & a_{2c} & a_{2\bar{0}} & 0 & 0 & a_{2\bar{c}} \\
a_{c0} & a_{c1} & a_{c2} & a_{cc} & a_{c\bar{0}} & a_{c\bar{1}} & a_{c\bar{2}} & a_{c\bar{c}} \\
\hline
a_{\bar{0}0} & a_{\bar{0}1} & a_{\bar{0}2} & a_{\bar{0}c} & a_{\bar{0}\bar{0}} & a_{\bar{0}\bar{1}} & a_{\bar{0}\bar{2}} & a_{\bar{0}\bar{c}} \\
a_{\bar{1}0} & 0 & 0 & a_{\bar{1}c} & a_{\bar{1}\bar{0}} & \bar{a}_{1}^{*} & 0 & a_{\bar{1}\bar{c}} \\
a_{\bar{2}0} & 0 & 0 & a_{\bar{2}c} & a_{\bar{2}\bar{0}} & 0 & \bar{a}_{2}^{*} & a_{\bar{2}\bar{c}} \\
a_{\bar{c}0} & a_{\bar{c}1} & a_{\bar{c}2} & a_{\bar{c}c} & a_{\bar{c}\bar{0}} & a_{\bar{c}\bar{1}} & a_{\bar{c}\bar{2}} & a_{\bar{c}\bar{c}}
\end{array}\right]
\left\{\begin{array}{c}
\dot{z}'_{0} \\ \dot{z}'_{1} \\ \dot{z}'_{2} \\ \dot{z}'_{c} \\ \hline \dot{z}'_{\bar{0}} \\ \dot{z}'_{\bar{1}} \\ \dot{z}'_{\bar{2}} \\ \dot{z}'_{\bar{c}}
\end{array}\right\} +
$$

$$
+ \left[\begin{array}{cccc|cccc}
b_{00} & b_{01} & b_{02} & b_{0c} & b_{0\bar{0}} & b_{0\bar{1}} & b_{0\bar{2}} & b_{0\bar{c}} \\
b_{10} & b_{1}^{*} & 0 & b_{1c} & b_{1\bar{0}} & 0 & 0 & b_{1\bar{c}} \\
b_{20} & 0 & b_{2}^{*} & b_{2c} & b_{2\bar{0}} & 0 & 0 & b_{2\bar{c}} \\
b_{c0} & b_{c1} & b_{c2} & b_{cc} & b_{c\bar{0}} & b_{c\bar{1}} & b_{c\bar{2}} & b_{c\bar{c}} \\
\hline
b_{\bar{0}0} & b_{\bar{0}1} & b_{\bar{0}2} & b_{\bar{0}c} & b_{\bar{0}\bar{0}} & b_{\bar{0}\bar{1}} & b_{\bar{0}\bar{2}} & b_{\bar{0}\bar{c}} \\
b_{\bar{1}0} & 0 & 0 & b_{\bar{1}c} & b_{\bar{1}\bar{0}} & \bar{b}_{1}^{*} & 0 & b_{\bar{1}\bar{c}} \\
b_{\bar{2}0} & 0 & 0 & b_{\bar{2}c} & b_{\bar{2}\bar{0}} & 0 & \bar{b}_{2}^{*} & b_{\bar{2}\bar{c}} \\
b_{\bar{c}0} & b_{\bar{c}1} & b_{\bar{c}2} & b_{\bar{c}c} & b_{\bar{c}\bar{0}} & b_{\bar{c}\bar{1}} & b_{\bar{c}\bar{2}} & b_{\bar{c}\bar{c}}
\end{array}\right]
\left\{\begin{array}{c}
z'_{0} \\ z'_{1} \\ z'_{2} \\ z'_{c} \\ \hline z'_{\bar{0}} \\ z'_{\bar{1}} \\ z'_{\bar{2}} \\ z'_{\bar{c}}
\end{array}\right\}
= - \left\{\begin{array}{c}
R_{1}(t) + R_{3}(t) \\ 0 \\ 0 \\ \phi_{c}(c)R_{3}(t) \\ \hline R_{1}(t) + R_{3}(t) \\ 0 \\ 0 \\ \phi_{c}(c)\,R_{3}(t)
\end{array}\right\}
\qquad (6.64)
$$

where for $i = 0, c$, and $j = 0, 1, \ldots, N_s, c$,

$$a_{ij} = a_{ji} = \{s\}^{(i)^T}[a]\,\{s\}^{(j)} \tag{6.65}$$

$$a_{\bar{i}j} = a_{j\bar{i}} = \{\bar{s}\}^{(i)^T}[a]\,\{s\}^{(j)} \tag{6.66}$$

$$a_{\bar{i}\bar{j}} = a_{\bar{j}\bar{i}} = \{\bar{s}\}^{(i)^T}[a]\,\{\bar{s}\}^{(j)} \tag{6.67}$$

$$b_{ij} = b_{ji} = \{s\}^{(i)^T}[b]\,\{s\}^{(j)} \tag{6.68}$$

$$b_{\bar{i}j} = b_{j\bar{i}} = \{\bar{s}\}^{(i)^T}[b]\,\{s\}^{(j)} \tag{6.69}$$

$$b_{\bar{i}\bar{j}} = b_{\bar{j}\bar{i}} = \{\bar{s}\}^{(i)^T}[b]\,\{\bar{s}\}^{(j)} \tag{6.70}$$

and where according to the discussion in Sec. 5.2 a_j^* and b_j^*, $j=1, 2, \ldots, 2N_s$, are of the form

$$a_j^* = \{s\}^{(j)^T}[a]\{s\}^{(j)} \tag{6.71}$$

$$b_j^* = \{s\}^{(j)^T}[b]\{s\}^{(j)}. \tag{6.72}$$

Thus, if it is considered that:

a) by the same argument used for the primary system $\{q_s\}$ may be expressed as

$$\{q_s\} = \{\sigma_s\}\, e^{\lambda t} \tag{6.73}$$

where $\{\sigma_s\}$ is the secondary system part of the complex eigenvector with frequency λ of the assembled system in Fig. 6.1;

b) in the light of Eqs. 6.56 and 6.73 the vector $\{z'\}$ may be put into the form

$$\{z'\} = \{z\}\, e^{\lambda t} \tag{6.74}$$

where $\{z\}$ is a vector of unknown amplitudes; and

c) by virtue of Eq. 6.74 and since

$$b_j^* = -\lambda_{s_j} a_j^*,\ j = 1, 2, \ldots, 2N_s \tag{6.75}$$

the jth ($j \neq 0, \bar{0}, c, \bar{c}$) component equation of Eq. 6.64 may be written as

$$(\lambda - \lambda_{s_j})\, a_j^* z_j + (\lambda a_{j0} + b_{j0})\, z_0 + (\lambda a_{j\bar{0}} + b_{j\bar{0}})\, z_{\bar{0}} +$$

$$+ (\lambda a_{jc} + b_{jc})\, z_c + (\lambda a_{j\bar{c}} + b_{j\bar{c}})\, z_{\bar{c}} = 0, \tag{6.76}$$

one may conclude that the secondary system part of a complex mode shape of an assembled system may be expressed as

$$\{\sigma_p\} = [s]\{z\} = \{s\}^{(0)} z_0 + \{\bar{s}\}^{(0)} z_{\bar{0}} + \sum_{j=1}^{2N_s} \{s\}^{(j)} z_j + \{s\}^{(c)} z_c + \{\bar{s}\}^{(c)} z_{\bar{c}} \tag{6.77}$$

where according to Eq. 6.76 the z_j factors ($j=1, 2, \ldots, 2N_s$) are given by

$$z_j = - [(\lambda a_{j0} + b_{j0}) z_0 + (\lambda a_{j\bar{0}} + b_{j\bar{0}})\, z_{\bar{0}} + (\lambda a_{jc} + b_{jc})\, z_c +$$

$$+ (\, a_{j\bar{c}} + b_{j\bar{c}})\, z_{\bar{c}}] \;/\; (\lambda - \lambda_{s_j})\; a_j^* \qquad (6.78)$$

in which z_i, a_{ji} and b_{ji}, i=0, $\bar{0}$, $\bar{c}$, $\bar{c}$, are factors that may be determined from compatibility requirements as follows.

Compatibility Conditions

When the primary and secondary subsystems shown in Figs. 6.2(a) and (b) are interconnected to form the assembled system in Fig. 6.1, one has that

$$x_{s_0} = x_{P_1} \qquad (6.79)$$

$$x_{s_c} = x_{P_3} \qquad (6.80)$$

$$\dot{x}_{s_0} = \dot{x}_{P_1} \qquad (6.81)$$

$$\dot{x}_{s_c} = \dot{x}_{P_3} \quad . \qquad (6.82)$$

Therefore, if in the light of Eqs. 6.6, 6.11, and 6.52 through 6.55 $\{q_p\}$ and $\{q_s\}$ are expressed as

$$\{q_p\} = \sum_{i=1}^{2N_p} \{S\}^{(i)} Z_i' = \sum_{i=1}^{2N_p} \begin{Bmatrix} \lambda_{p_i} \{\Phi\}^{(i)} \\ \{\Phi\}^{(i)} \end{Bmatrix} Z_i' \qquad (6.83)$$

$$\{q_s\} = \sum_{j=1}^{2(N_s+2)} \{s\}^{(j)} z_j' = \begin{Bmatrix} \lambda_{s_0}\{J\} \\ \{J\} \end{Bmatrix} z_0' + \begin{Bmatrix} \overline{\lambda}_{s_0}\{J\} \\ \{J\} \end{Bmatrix} z_{\bar{0}}' +$$

$$+ \begin{Bmatrix} \lambda_{s_c}\{f\} \\ \{f\} \end{Bmatrix} z_c' + \begin{Bmatrix} \overline{\lambda}_{s_c}\{f\} \\ \{f\} \end{Bmatrix} z_{\bar{c}}' + \sum_{j=1}^{2N_s} \begin{Bmatrix} \lambda_{s_j}\{\phi\}^{(j)} \\ \{\phi\}^{(j)} \end{Bmatrix} z_j' , \qquad (6.84)$$

and if it is considered that by virtue of these two equations, Eqs.6.5 and 6.51, and the lower halves of Eqs. 6.15 and 6.73 one may write $\{\dot{x}_p\}$, $\{x_p\}$, $\{\dot{x}_s\}$ and $\{x_s\}$ as

$$\{\dot{x}_p\} = \lambda\{x_p\} = \sum_{i=1}^{2N_p} \lambda_{p_i} \{\Phi\}^{(i)} Z_i' \qquad (6.85)$$

$$\{x_p\} = \sum_{i=1}^{2N_p} \{\Phi\}^{(i)} Z_i' \qquad (6.86)$$

$$\{\dot{x}_s\} = \lambda\{x_s\} = (\lambda_{s_0} z_0' + \overline{\lambda}_{s_0} z_{\bar{0}}')\{J\} + (\lambda_{s_c} z_c' + \overline{\lambda}_{s_c} z_{\bar{c}}')\{f\} +$$

$$+ \sum_{j=1}^{2N_s} \lambda_{s_j} \{\phi\}^{(j)} z_j' \qquad (6.87)$$

$$\{x_s\} = (z_0' + z_{\bar{0}}')\{J\} + (z_c' + z_{\bar{c}}') \{f\} + \sum_{j=1}^{2N_s} \{\phi\}^{(j)} z_j' , \qquad (6.88)$$

in terms of the coordinates $Z_i^{'}$ and $z_j^{'}$ the above compatibility equations may be expressed as (see Eqs. 4.13, 4.14 and 4.15 to recall the definitions of {J}, {f} and $\{\phi\}^{(j)}$)

$$z_0^{'} + z_{\bar{0}}^{'} = \sum_{i=1}^{2N_p} \Phi_1(i)\, Z_i^{'} \tag{6.89}$$

$$z_0^{'} + z_{\bar{0}}^{'} + f_{cc}(z_c^{'} + z_{\bar{c}}^{'}) = \sum_{i=1}^{2N_p} \Phi_3(i)\, Z_i^{'} \tag{6.90}$$

$$\lambda_{s_0} z_0^{'} + \bar{\lambda}_{s_0} z_{\bar{0}}^{'} = \lambda \sum_{i=1}^{2N_p} \Phi_1(i) Z_i^{'} \tag{6.91}$$

$$\lambda_{s_0} z_0^{'} + \bar{\lambda}_{s_0} z_{\bar{0}}^{'} + f_{cc}(\lambda_{s_c} z_c^{'} + \bar{\lambda}_{s_c} z_{\bar{c}}^{'}) = \lambda \sum_{i=1}^{2N_p} \Phi_3(i) Z_i \tag{6.92}$$

which lead, after introducing Eqs. 6.16 and 6.74, to the following compatibility relations:

$$z_0 + z_{\bar{0}} = \sum_{i=1}^{2N_p} \Phi_1(i) Z_i \tag{6.93}$$

$$z_c + z_{\bar{c}} = \frac{1}{f_{cc}} \sum_{i=1}^{2N_p} d\Phi(i) Z_i \tag{6.94}$$

$$\lambda_{s_0} Z_0 + \bar{\lambda}_{s_0} z_{\bar{0}} = \lambda(z_0 + z_{\bar{0}}) \tag{6.95}$$

$$\lambda_{s_c} z_c + \overline{\lambda}_{s_c} z_{\bar{c}} = \lambda(z_c + z_{\bar{c}}) \ . \tag{6.96}$$

Simplified Expressions for $\{s\}^{(0)} z_0 + \{\bar{s}\}^{(0)} z_{\bar{0}}$, $\{s\}^{(c)} z_c + \{\bar{s}\}^{(c)} z_{\bar{c}}$, and z_j

If in the light of Eqs. 6.52 and 6.53 the sums $\{s\}^{(0)} z_0 + \{\bar{s}\}^{(0)} z_{\bar{0}}$ and $\{s\}^{(c)} z_c + \{\bar{s}\}^{(c)} z_{\bar{c}}$ in Eq. 6.77 are expressed as

$$\{s\}^{(0)} z_0 + \{\bar{s}\}^{(0)} z_{\bar{0}} = \begin{Bmatrix} (\lambda_{s_0} z_0 + \overline{\lambda}_{s_0} z_0)\{J\} \\ (z_0 + z_{\bar{0}})\ \{J\} \end{Bmatrix} \tag{6.97}$$

$$\{s\}^{(c)} z_c + \{\bar{s}\}^{(c)} z_{\bar{c}} = \begin{Bmatrix} (\lambda_{s_c} z_{\hat{c}} + \overline{\lambda}_{s_c} z_c)\{f\} \\ (z_c + z_{\bar{c}})\ \{f\} \end{Bmatrix} \ , \tag{6.98}$$

then it may be seen that by virtue of the compatibility relations indicated by Eqs. 6.95 and 6.96 these two sums may be written as

$$\{s\}^{(0)} z_0 + \{\bar{s}\}^{(0)} z_{\bar{0}} = \begin{Bmatrix} \lambda\{J\} \\ \{J\} \end{Bmatrix} (z_{\bar{0}} + z_{\bar{0}}) \tag{6.99}$$

$$\{s\}^{(c)} z_c + \{\bar{s}\}^{(c)} z_{\bar{c}} = \begin{Bmatrix} \lambda\{f\} \\ \{f\} \end{Bmatrix} (z_c + z_{\bar{c}}) \tag{6.100}$$

where the factors $(z_0 + z_{\bar{0}})$ and $(z_c + z_{\bar{c}})$ are given explicitly by Eqs. 6.93 and 6.94.

Similarly, using Eq. 6.95 and since according to Eqs. 6.52 through 6.54 and 6.65 through 6.70 one has that

$$a_{j0} = (\lambda_{s_j} + \lambda_{s_0})\, m_{0j} \tag{6.101}$$

$$a_{j\bar{0}} = (\lambda_{s_j} + \bar{\lambda}_{s_0})\, m_{0j} \tag{6.102}$$

$$b_{j0} = -\lambda_{s_j} \lambda_{s_0} m_{0j} \tag{6.103}$$

$$b_{j\bar{0}} = -\lambda_{s_j} \bar{\lambda}_{s_0} m_{0j} \tag{6.104}$$

where

$$m_{0j} = \{\phi\}^{(j)^T} [m] \{J\}, \tag{6.105}$$

the sum $(\lambda a_{j0} + b_{j0})\, z_0 + (\lambda a_{j\bar{0}} + b_{j\bar{0}})\, z_{\bar{0}}$ in Eq. 6.78 may be expressed as

$$(\lambda a_{j0} + b_{j0})\, z_0 + (\lambda a_{j\bar{0}} + b_{j\bar{0}})\, z_{\bar{0}} =$$

$$= m_{0j} [\lambda \lambda_{s_j} (z_0 + z_{\bar{0}}) + \lambda(\lambda_{s_0} z_0 + \bar{\lambda}_{s_0} z_{\bar{0}}) - \lambda_{s_j} (\lambda_{s_0} z_0 + \bar{\lambda}_{s_0} z_{\bar{0}})\,] =$$

$$= \lambda^2 m_{0j} (z_0 + z_{\bar{0}}) \tag{6.106}$$

whereas by means of Eq. 6.96 and because

$$a_{jc} = (\lambda_{s_j} + \lambda_{s_c})\, m_{cj} + c_{cj} \tag{6.107}$$

$$a_{j\bar{c}} = (\lambda_{s_j} + \bar{\lambda}_{s_c})\, m_{cj} + c_{cj} \tag{6.108}$$

$$b_{jc} = -\, \lambda_{s_j}\, \lambda_{s_c}\, m_{cj} \tag{6.109}$$

$$b_{j\bar{c}} = -\, \lambda_{s_j}\, \bar{\lambda}_{s_c}\, m_{cj} \tag{6.110}$$

where

$$m_{cj} = \{\phi\}^{(j)^T} [m]\, \{f\} \tag{6.111}$$

and

$$c_{cj} = \{\phi\}^{(j)^T} [c]\, \{f\} \quad , \tag{6.112}$$

the sum $(\lambda a_{jc} + b_{jc})\, z_c + (\lambda a_{j\bar{c}} + b_{j\bar{c}})\, z_{\bar{c}}$ in the same Eq. 6.78 results of the form

$$(\lambda a_{jc} + b_{jc})\, z_c + (\lambda a_{j\bar{c}} + b_{j\bar{c}})\, z_{\bar{c}} = \lambda c_{cj}\, (z_c + z_{\bar{c}}) +$$

$$+ m_{cj}\, [\lambda\lambda_{s_j}(z_c + z_{\bar{c}}) + \lambda(\lambda_{s_c} z_c + \bar{\lambda}_{s_c} z_{\bar{c}}) - \lambda_{s_j}(\lambda_{s_c} z_c + \bar{\lambda}_{s_c} z_{\bar{c}})] =$$

$$= (\lambda^2 m_{cj} + \lambda c_{cj})\, (z_c + z_{\bar{c}}) \; . \tag{6.113}$$

Consequently, by substitution of Eqs. 6.106 and 6.113 into Eq. 6.78, the factors z_j of Eq. 6.77 may be expressed as

$$z_j = - \frac{\lambda^2 m_{0j}(z_0 + z_{\bar{0}}) + (\lambda^2 m_{cj} + \lambda c_{cj})\ (z_c + z_{\bar{c}})}{a_j^*\ (\lambda - \lambda_{s_j})} \quad . \tag{6.114}$$

One may consider, however, that

(a) The damping matrix of the secondary system under study is proportional to its own stiffness matrix and thus, according to Eqs. 6.112 and 4.14, c_{cj} results as

$$c_{cj} = a_s\ \{\phi\}^{(j)^T} [k]\ \{f\} = 0 \tag{6.115}$$

in which a_s is simply a proportionality constant.

(b) In view of Eqs. 5.63, 5.67, 6.71, 6.52, 6.54 and 6.65 the parameter a_j^* may be expressed as

$$a_j^* = \gamma_{s_j}\ m_{0j} \tag{6.116}$$

where γ_{s_j} is the jth complex participation factor of the secondary system in Fig. 6.2(b) when both of its ends are fixed.

(c) The factors $(z_0 + z_{\bar{0}})$ and $(z_c + z_{\bar{c}})$ are given directly by Eqs. 6.93 and 6.94.

Therefore, such z_j factors may be alternatively written as

$$z_j = - \gamma_{s_j}\ \lambda^2\ \frac{m_{0j}[\sum_{i=1}^{2N_p} \phi_1(i)\ Z_i] + \frac{m_{cj}}{f_{cc}}\ [\sum_{i=1}^{2N_p}\ d\phi(i)\ Z_i]}{m_{0j}(\lambda-\lambda_{s_j})} \tag{6.117}$$

or as

$$z_j = -\gamma_{s_j} \frac{\lambda^2}{\lambda-\lambda_{s_j}} \hat{z}_0 \qquad (6.118)$$

where $\hat{z}_0$ is defined as

$$\hat{z}_0 = \sum_{i=1}^{2N_p} \Phi_o(i,j)\, Z_i \qquad (6.119)$$

in which $\Phi_o(i,j)$ is given by Eq. 4.34.

In the general case, therefore, the secondary system part of the rth complex eigenvector of an assembled system with nonproportional damping may be expressed as

$$\{\sigma_s\}^{(r)} = z_0\{s\}^{(0)} + z_{\bar{0}}\{\bar{s}\}^{(0)} + \sum_{i=1}^{2N_s} z_j^{(r)}\{s\}^{(j)} + z_c\{s\}^{(c)} + z_c\{\bar{s}\}^{(c)} \qquad (6.120)$$

where

$$z_0^{(r)}\{s\}^{(0)} + z_{\bar{0}}^{(r)}\{\bar{s}\}^{(0)} = \begin{Bmatrix} \lambda_r\{J\} \\ \{J\} \end{Bmatrix} (z_0^{(r)} + z_{\bar{0}}^{(r)}) \qquad (6.121)$$

$$z_c^{(r)}\{s\}^{(c)} + z_{\bar{c}}\{\bar{s}\}^{(c)} = \begin{Bmatrix} \lambda_r\{f\} \\ \{f\} \end{Bmatrix} (z_c^{(r)} + z_{\bar{c}}^{(r)}) \qquad (6.122)$$

$$z_0^{(r)} + z_{\bar{0}}^{(r)} = \sum_{i=1}^{2N_p} \Phi_k(i)\, Z_i^{(r)} \qquad (6.123)$$

$$z_c^{(r)} + z_{\bar{c}}^{(r)} = \sum_{i=1}^{2N_p} \frac{d\Phi(i)}{f_{cc}}\, Z_i^{(r)} \qquad (6.124)$$

and

$$z_j^{(r)} = -\gamma_{s_j} \frac{\lambda_r^2}{\lambda_r - \lambda_{s_j}} \hat{z}_0^{(r)} \tag{6.125}$$

in which

$$\hat{z}_0^{(r)} = \sum_{i=1}^{2N_p} \Phi_o(i,j)\, Z_i^{(r)}. \tag{6.126}$$

Secondary System Part of Complex Mode Shapes

The secondary system part of a complex mode shape of the assembled system in Fig. 6.1 may be determined directly from the lower half of Eq. 6.77 since according to the discussion in Sec. 5.2 the eigenvector $\{\sigma_s\}$ may be written as

$$\{\sigma_s\} = \begin{Bmatrix} \lambda\{w_s\} \\ \{w_s\} \end{Bmatrix}, \tag{6.127}$$

where $\{w_s\}$ represents the secondary system part of the complex mode shape with frequency λ of such an assembled system. However, for a secondary system with proportional damping, and by following the procedure used for the primary system, a simplified expression for this vector $\{w_s\}$ may be obtained as follows:

In the light of Eqs. 6.55, 6.52 through 6.54, 6.73, and 6.74 $\{\sigma_s\}$ may be expressed as

$$\{\sigma_s\} = \begin{Bmatrix} \lambda_{s_0}\{J\} \\ \{J\} \end{Bmatrix} z_0 + \begin{Bmatrix} \bar{\lambda}_{s_0}\{J\} \\ \{J\} \end{Bmatrix} z_{\bar{0}} + \sum_{j=1}^{N_s} \left[\begin{Bmatrix} \lambda_{s_j}\{\phi\}^{(j)} \\ \{\phi\}^{(j)} \end{Bmatrix} z_j + \begin{Bmatrix} \bar{\lambda}_{s_j}\{\bar{\phi}\}^{(j)} \\ \{\bar{\phi}\}^{(j)} \end{Bmatrix} z_{\bar{j}} \right] +$$

$$+ \begin{Bmatrix} \lambda_{s_c}\{f\} \\ \{f\} \end{Bmatrix} z_c + \begin{Bmatrix} \bar{\lambda}_{s_c}\{f\} \\ \{f\} \end{Bmatrix} z_{\bar{c}} \tag{6.128}$$

which, in view of the fact that the mode shapes $\{\phi\}^{(j)}$ are real, and thus $\{\bar{\phi}\}^{(j)} = \{\phi\}^{(j)}$, may also be written as

$$\{\sigma_s\} = \begin{Bmatrix} [\lambda_{s_0} z_0 + \bar{\lambda}_{s_0} z_{\bar{0}}]\{J\} \\ [z_0 + z_{\bar{0}}]\{J\} \end{Bmatrix} + \sum_{j=1}^{N_s} \begin{Bmatrix} [\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}}]\{\phi\}^{(j)} \\ [z_j + z_{\bar{j}}]\{\phi\}^{(j)} \end{Bmatrix} +$$

$$+ \begin{Bmatrix} [\lambda_{s_c} z_c + \bar{\lambda}_{s_c} z_{\bar{c}}]\{f\} \\ [z_c + z_{\bar{c}}]\{f\} \end{Bmatrix} . \tag{6.129}$$

But by virtue of Eq. 6.118, and since according to Eq. 5.140 and to the assumption that each $\{\phi\}^{(j)}$ is a mode shape with a unit participation factor the complex participation factors γ_j and $\overline{\gamma_j}$ result in this case as

$$\gamma_{s_j} = \frac{1}{2i\omega'_{s_j}} \tag{6.130}$$

$$\overline{\gamma}_{s_j} = \frac{-1}{2i\omega'_{s_j}} , \tag{6.131}$$

one has that the sum $z_j + z_{\bar{j}}$ may be expressed as

$$z_j + z_{\bar{j}} = - \frac{\lambda^2}{2i\omega_{s_j}'^2} \left[\frac{1}{\lambda - \lambda_{s_j}} - \frac{1}{\lambda - \bar{\lambda}_{s_j}} \right] \hat{z}_0 \quad , \tag{6.132}$$

or as

$$z_j + z_{\bar{j}} = \frac{- \lambda^2}{(\lambda - \lambda_{s_j})(\lambda - \bar{\lambda}_{s_j})} \hat{z}_0 \quad , \tag{6.133}$$

and that, by the same arguments, the sum $\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}}$ may be written as

$$\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}} = \frac{- \lambda^2}{2i\omega_{s_j}} \left[\frac{\lambda_{s_j}}{\lambda - \lambda_{s_j}} + \frac{\bar{\lambda}_{s_j}}{\lambda - \bar{\lambda}_{s_j}} \right] \hat{z}_0 \quad , \tag{6.134}$$

or as

$$\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}} = \frac{- \lambda^3}{(\lambda - \lambda_{s_j})(\lambda - \bar{\lambda}_{s_j})} \hat{z}_0 \quad . \tag{6.135}$$

In like manner, the sums $[z_0 + z_{\bar{0}}]$, $[\lambda_{s_0} z_0 + \bar{\lambda}_{s_0} z_{\bar{0}}]$, $[z_c + z_{\bar{c}}]$, and $[\lambda_{s_c} z_c + \bar{\lambda}_{s_c} z_{\bar{c}}]$ are given by the compatibility relations indicated by Eqs. 6.93 through 6.96. Therefore, Eq. 6.129 may be put into the form

$$\{\sigma_s\} = \begin{Bmatrix} \lambda\{J\} \\ \\ \{J\} \end{Bmatrix} \sum_{i=1}^{2N_p} \Phi_1(i) Z_i + \sum_{j=1}^{N_s} \frac{-\lambda^2}{(\lambda - \lambda_{s_j})(\lambda - \bar{\lambda}_{s_j})} \begin{Bmatrix} \lambda\{\phi\}^{(j)} \\ \\ \{\phi\}^{(j)} \end{Bmatrix} \hat{z}_0 +$$

$$+ \begin{Bmatrix} \lambda\{f\} \\ \\ \{f\} \end{Bmatrix} \frac{1}{f_{cc}} \sum_{i=1}^{2N_p} d\Phi(i) Z_i \qquad (6.136)$$

which in combination with Eq. 6.127 leads one to conclude that

$$\{w_s\} = \{J\} \sum_{i=1}^{2N_p} \Phi_1(i) Z_i + \sum_{j=1}^{N_s} \frac{-\lambda^2}{(\lambda - \lambda_{s_j})(\lambda - \bar{\lambda}_{s_j})} \hat{z}_0 \{\phi\}^{(j)} +$$

$$+ \{f\} \sum_{i=1}^{2N_p} \frac{d\Phi(i)}{f_{cc}} Z_i \, . \qquad (6.137)$$

Then, since by introducing Eq. 6.40 one has that

$$\sum_{i=1}^{2N_p} \Phi_1(i) Z_i = \sum_{i=1}^{N_p} \Phi_1(i)[Z_i + Z_{\bar{i}}] = \sum_{i=1}^{N_p} \Phi_1(i) Y_i \qquad (6.138)$$

$$\sum_{i=1}^{2N_p} \frac{d\Phi(i)}{f_{cc}} Z_i = \sum_{i=1}^{N_p} \frac{d\Phi(i)}{f_{cc}} [Z_i + Z_{\bar{i}}] = \sum_{i=1}^{N_p} \frac{d\Phi(i)}{f_{cc}} Y_i \qquad (6.139)$$

and since by means of Eq. 6.119 and the same Eq. 6.40 one similarly obtains that

$$\hat{z}_0 = \sum_{i=1}^{N_p} \Phi_o(i,j)[Z_i + Z_i^-] = \sum_{i=1}^{N_p} \Phi_o(i,j)\, Y_i \,, \tag{6.140}$$

by defining the following new variables as

$$y_0 = \sum_{i=1}^{N_p} \Phi_1(i)Y_i \tag{6.141}$$

$$y_c = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i)Y_i \tag{6.142}$$

$$\hat{y}_0 = \sum_{i=1}^{N_p} \Phi_o(i,\, j)Y_i \tag{6.143}$$

$$y_j = \frac{-\lambda^2}{(\lambda - \lambda_{s_j})(\lambda - \bar{\lambda}_{s_j})}\,\hat{y}_0 \tag{6.144}$$

$\{w_s\}$ may be written as

$$\{w_s\} = \{J\}y_0 + \sum_{j=1}^{N_s} \{\phi\}^{(j)} y_j + \{f\}y_c \ . \tag{6.145}$$

It may be inferred, thus, that the secondary system part of the rth complex mode shape of an assembled system with nonproportional damping may be expressed as

$$\{w_s\}^{(r)} = \{J\}y_0^{(r)} + \sum_{j=1}^{N_s}\{\phi\}^{(j)}y_j^{(r)} + \{f\}y_c^{(r)} \tag{6.146}$$

where

$$y_0^{(r)} = \sum_{i=1}^{N_p} \Phi_k(i)Y_i^{(r)} \tag{6.147}$$

$$y_c^{(r)} = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i)Y_i^{(r)} \tag{6.148}$$

$$\hat{y}_0^{(r)} = \sum_{i=1}^{N_p} \Phi_0(i, j)Y_i^{(r)} \tag{6.149}$$

$$y_j^{(r)} = \frac{-\lambda_r^2}{(\lambda_r - \lambda_{s_j})(\lambda_r - \bar{\lambda}_{s_j})} \hat{y}_0^{(r)} \tag{6.150}$$

and where all other symbols are as defined before.

Summary

In summary, the rth complex mode shape of an assembled system with nonproportional damping and whose secondary system is attached to the kth and ℓth masses of its primary system may be written as

$$\{w_p\}^{(r)} = \sum_{i=1}^{N_p} Y_i\{\Phi\}^{(i)} \tag{6.151}$$

$$\{w_s\}^{(r)} = y_0^{(r)}\{J\} + \sum_{j=1}^{N_s} y_j^{(r)}\{\phi\}^{(j)} + y_c^{(r)}\{f\} \tag{6.152}$$

where $\{w_p\}$ and $\{w_s\}$ are the parts of such a complex mode shape corresponding respectively to the primary and secondary systems, and where the $y_i^{(r)}$ and $y_j^{(r)}$ factors are given by

$$Y_i^{(r)} = \frac{\hat{\Phi}_r(i)(\lambda_r - \lambda_{p_I})(\lambda_r - \bar{\lambda}_{p_I})}{\hat{\Phi}_r(I)\ \ (\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i})} \frac{M_I^*}{M_i^*} Y_I^{(r)}, \quad i=1,\ 2,\ \ldots,\ N_p \tag{6.153}$$

$$y_0^{(r)} = \sum_{i=1}^{N_p} \Phi_k(i) Y_i^{(r)} \tag{6.154}$$

$$y_c^{(r)} = \frac{1}{f_{cc}} d\Phi(i) Y_i^{(r)} \tag{6.155}$$

$$y_j^{(r)} = \frac{-\lambda_r^2}{(\lambda_r - \lambda_{s_j})(\lambda_r - \bar{\lambda}_{s_j})} \hat{y}_0^{(r)}, \quad j=1,\ 2,\ \ldots,\ N_s \tag{6.156}$$

in which

$$\hat{y}_0^{(r)} = \sum_{i=1}^{N_p} \Phi_o(i,\ j) Y_i^{(r)}, \tag{6.157}$$

λ_r is the rth complex natural frequency of the system, and {f}, f_{cc}, $\hat{\Phi}_r(i)$, $\Phi_o(i, j)$, and $d\Phi(i)$ are as indicated by Eqs. 4.48, 4.53, 4.54, 4.56, and 4.35, respectively. λ_{p_i}, i=1, 2, ..., N_p, and λ_{s_j}, j=1, 2, ..., N_s, stand for the complex natural frequencies of the independent primary and secondary systems, i.e.,

$$\lambda_{p_i} = - \xi_{p_i}\omega_{p_i} + i\ \omega'_{p_i} \tag{6.158}$$

$$\lambda_{s_j} = - \xi_{s_j}\omega_{s_j} + i\ \omega'_{s_j}, \tag{6.159}$$

and N_p and N_s denote their respective number of degrees of freedom.

Notice thus that the major difference between these expressions and those found for systems with proportional damping is that in the expressions for systems with nonproportional damping it is necessary to consider the complex natural frequencies of the independent primary and secondary systems instead of just their circular natural frequencies. It is important to note, however, that since λ_r, λ_{p_i} and λ_{s_j} are complex parameters, in the case of nonproportional damping the $Y_i^{(r)}$ and $y_j^{(r)}$ factors are complex scalars, and as a consequence the vectors $\{w_p\}$ and $\{w_s\}$ are complex vectors.

Convergence to the Case of Proportional Damping

The equations derived aboved represent the generalization of those developed in Chapter 4 for systems with proportional damping, and hence, if the conditions that transform a system with nonproportional damping to one with proportional damping are introduced, these general expressions should converge to the particular ones for proportional damping. To prove, then, that this is indeed so, one may proceed as follows:

The condition for obtaining an assembled system with a damping matrix proportional to its stiffness matrix when the damping matrices of its independent primary and secondary systems are proportional to their respective stiffness matrices is that the constants that relate the proportionality between the damping and stiffness matrices of these two independent systems be the same. In other words, an assembled system and its primary and secondary components have proportional damping if the damping ratios of these systems may be written as

$$\xi_{p_i} = \frac{1}{2} a\omega_{p_i} \tag{6.160}$$

$$\xi_{s_j} = \frac{1}{2} a\omega_{s_j} \tag{6.161}$$

$$\xi_r = \frac{1}{2} a\omega_r \tag{6.162}$$

where a is a constant. For the particular case of proportional damping, the $Y_i^{(r)}$ and $y_j^{(r)}$ factors given by Eqs. 6.153 and 6.156 result therefore as follows:

$\underline{Y_i^{(r)}\text{ factors}}$. Since in view of Eq. 5.112 the complex frequency λ_r may be written out as

$$\lambda_r = -\xi_r\omega_r + i\,\omega_r' \tag{6.163}$$

and, similarly, λ_{p_i} may be expressed as indicated by Eq. 6.158, the differences $(\lambda_r - \lambda_{p_i})$ and $(\lambda_r - \bar{\lambda}_{p_i})$ in Eq. 6.153 may be put into the form

$$\lambda_r - \lambda_{p_i} = -(\xi_r\omega_r - \xi_{p_i}\omega_{p_i}) + i\,(\omega_r' - \omega_{p_i}') \tag{6.164}$$

$$\lambda_r - \bar{\lambda}_{p_i} = -(\xi_r\omega_r - \xi_{p_i}\omega_{p_i}) + i\,(\omega_r' + \omega_{p_i}')\ , \tag{6.165}$$

and as a consequence the product of these two differences may be expressed as

$$(\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i}) = (\xi_r\omega_r - \xi_{p_i}\omega_{p_i})^2 - (\omega_r'^2 - \omega_{p_i}'^2) + $$

$$+\ i\,2\omega_r'\,(\xi_{p_i}\omega_{p_i} - \xi_r\omega_r) \tag{6.166}$$

which after considering that $\omega_r' = \omega_r\sqrt{1-\xi_r^2}$ and $\omega_{p_i}' = \omega_{p_i}\sqrt{1-\xi_{p_i}^2}$ may also be written as

$$(\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i}) = \omega_{p_i}^2\left(1 - 2\xi_r\omega_r\frac{\xi_{p_i}}{\omega_{p_i}}\right) - \omega_r^2\,(1 - 2\xi_r^2) + $$

$$+\ 2i\omega_r\sqrt{1-\xi_r^2}\ (\xi_{p_i}\omega_{p_i} - \xi_r\omega_r)\ . \tag{6.167}$$

But, if the constant a is eliminated from Eqs. 6.160 and 6.162, in the case of proportional damping ξ_{p_i} may be expressed in terms of ξ_r and and ω_r as

$$\xi_{p_i} = \xi_r \frac{\omega_{p_i}}{\omega_r} \quad . \tag{6.168}$$

In such a case, therefore, Eq. 6.167 may be alternatively expressed as

$$(\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i}) = \omega_{p_i}^2 (1 - 2\xi_r^2) - \omega_r^2 (1 - 2\xi_r^2) + + 2i\xi_r \sqrt{1 - \xi_r^2} \; (\omega_{p_i}^2 - \omega_r^2) \tag{6.169}$$

or as

$$(\lambda_r - \lambda_{p_i})(\lambda_r - \bar{\lambda}_{p_i}) = (\omega_{p_i}^2 - \omega_r^2) \; [1 - 2\xi_r^2 + 2i\xi_r\sqrt{1 - \xi_r^2}] \tag{6.170}$$

which by substitution into Eq. 6.153 leads to

$$Y_i^{(r)} = \frac{\hat{\Phi}(i)}{\hat{\Phi}(I)} \frac{(\omega_{p_I}^2 - \omega_r^2)}{(\omega_{p_i}^2 - \omega_r^2)} \frac{[1 - 2\xi_r^2 + 2i\xi_r\sqrt{1 - \xi_r^2}]}{[1 - 2\xi_r^2 + 2i\xi_r\sqrt{1 - \xi_r^2}]} \frac{M_I^*}{M_i^*} \; Y_I^{(r)} \quad , \tag{6.171}$$

and thus, after setting I=1 and selecting Y_1 to be equal to unity, one has that

$$Y_i^{(r)} = \frac{\hat{\Phi}(i)}{\hat{\Phi}(1)} \frac{\omega_{p_1}^2 - \omega_r^2}{\omega_{p_i}^2 - \omega_r^2} \frac{M_1^*}{M_i^*} \quad . \tag{6.172}$$

<u>$y_j^{(r)}$ factors</u>. By simply replacing the subscripts p_i in Eq. 6.170 by subscripts s_j, it is easy to show that for an assembled system with proportional damping the product $(\lambda_r - \lambda_{s_j})(\lambda_r - \bar{\lambda}_{s_j})$ in Eq. 6.156 results of the form

$$(\lambda_r - \lambda_{s_j})(\lambda_r - \bar{\lambda}_{s_j}) = (\omega_{s_j}^2 - \omega_r^2)(1 - 2\xi_r^2 + 2i\xi_r \sqrt{1 - \xi_r^2}\ ). \tag{6.173}$$

Then, since according to Eq. 6.163 λ_r^2 may be expressed as

$$\lambda_r^2 = \xi_r^2 \omega_r^2 - \omega_r^2 (1 - \xi_r^2) - 2i\xi_r\omega_r^2 \sqrt{1 - \xi_r^2}$$

$$= - \omega_r^2 [(1 - 2\xi_r^2 \omega_r^2) + 2i\xi_r\sqrt{1 - \xi_r^2}\ ], \tag{6.174}$$

in the case of proportional damping the factor $y_j^{(r)}$ given by Eq. 6.156 becomes

$$y_j^{(r)} = \frac{\omega_r [1 - 2\xi_r^2 + 2i\xi_r \sqrt{1 - \xi_r^2}\ ]}{(\omega_{s_j}^2 - \omega_r^2) [1 - 2\xi_r^2 + 2i\xi_r \sqrt{1 - \xi_r^2}\ ]} \hat{y}_0^{(r)} \tag{6.175}$$

or

$$y_j^{(r)} = \frac{\omega_r^2}{\omega_{s_j}^2 - \omega_r^2} \hat{y}_0^{(r)} \quad . \tag{6.176}$$

It may be seen, thus, that since the $Y_j^{(r)}$ and $y_j^{(r)}$ factors given by Eqs. 6.153 and 6.156 converge to the particular ones for proportional damping given by Eqs. 4.49 and 4.50, and since Eqs. 6.151 and 6.152 are identical to the corresponding ones for proportional damping, in the case of an assembled system with proportional damping the general formulas introduced in this section converge to the particular ones derived in Sec. 4.2.

6.3 Complex Natural Frequencies: Resonant Modes

It is shown in Appendix B that Rayleigh's principle may be extended for the case of a system with nonproportional damping. This means, therefore, that the complex natural frequencies of a system with nonproportional damping are also stationary in the neighborhood of their corresponding complex mode shapes and that, as a consequence, it is also possible to derive approximate expressions for these complex frequencies by following the procedure used to derive approximate expressions for the natural frequencies of systems with proportional damping. In this section, then, such a procedure is employed to develop an approximate formula for determining the complex natural frequencies of an assembled system with nonproportional damping whose primary and secondary components are in resonance.*

*In this chapter, it will be understood that two systems are in resonance when they have a common natural frequency, not a common complex natural frequency.

Consider the reduced equation of motion of the assembled system in Fig. 6.1. In terms of this assembled system's mass, damping, and stiffness matrices, such an equation of motion may be expressed as

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\dot{x}\} \end{Bmatrix} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{6.177}$$

where $[M]$, $[C]$, and $[K]$ are respectively such mass, damping, and stiffness matrices and $\{x\}$ is, as before, the displacement vector of the system. But since $[M]$, $[C]$, and $[K]$ are of the form

$$[M] = \left[\begin{array}{ccc|cc} M_1 & & & & \\ & M_2 & & & 0 \\ & & M_3 & & \\ \hline & & & m_1 & \\ & 0 & & & m_2 \end{array}\right] \tag{6.178}$$

$$[C] = \left[\begin{array}{ccc|cc} C_1 + C_2 + c_1 & -C_2 & 0 & -c_1 & 0 \\ -C_2 & C_2 + C_3 & -C_3 & 0 & 0 \\ 0 & -C_3 & C_3 + c_3 & 0 & -c_3 \\ \hline -c_1 & 0 & 0 & c_1 + c_2 & -c_2 \\ 0 & 0 & -c_3 & -c_2 & c_2 + c_3 \end{array}\right] \tag{6.179}$$

$$[K] = \left[\begin{array}{ccc|cc} K_1 + K_2 + k_1 & -K_2 & 0 & -k_1 & 0 \\ -K_2 & K_2 + K_3 & -K_3 & 0 & 0 \\ 0 & -K_3 & K_3 + k_3 & 0 & -k_3 \\ \hline -k_1 & 0 & 0 & k_1 + k_2 & -k_2 \\ 0 & 0 & -k_3 & -k_2 & k_2 + k_3 \end{array}\right], \qquad (6.180)$$

these matrices may be written in terms of the corresponding ones of the independent primary and secondary components of the assembled system under consideration as

$$[M] = \begin{bmatrix} [M] & [0] \\ [0] & [m'] \end{bmatrix} \qquad (6.181)$$

$$[C] = \begin{bmatrix} [C] & [0] \\ [0] & [c'] \end{bmatrix} + \begin{bmatrix} [F] & [D] \\ [D]^T & [0] \end{bmatrix} \qquad (6.182)$$

$$[K] = \begin{bmatrix} [K] & [0] \\ [0] & [k'] \end{bmatrix} + \begin{bmatrix} [H] & [G] \\ [G]^T & [0] \end{bmatrix} \qquad (6.183)$$

where [M], [C], and [K] are the mass, damping and stiffness matrices of the independent primary system; [m'], [c'], and [k'] denote the mass, damping and stiffness matrices of the independent secondary system when both of its ends are fixed; and [D], [F], [G], and [H] are defined as follows:

$$[D] = \begin{bmatrix} -c_1 & 0 \\ 0 & 0 \\ 0 & -c_3 \end{bmatrix} \qquad (6.184)$$

$$[F] = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \qquad (6.185)$$

$$[G] = \begin{bmatrix} -k_1 & 0 \\ 0 & 0 \\ 0 & -k_3 \end{bmatrix} \qquad (6.186)$$

$$[H] = \begin{bmatrix} -k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{bmatrix} . \qquad (6.187)$$

Consequently, in terms of the parameters of the independent primary and secondary systems Eq. 6.177 may be expressed as

$$
\begin{bmatrix}
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[M][0]\\ [0][m']\end{bmatrix}\\
\begin{bmatrix}[M][0]\\ [0][m']\end{bmatrix} & \begin{bmatrix}[C][0]\\ [0][c']\end{bmatrix}
\end{bmatrix}
\begin{Bmatrix}\begin{Bmatrix}\{\ddot{x}_p\}\\ \{\ddot{x}_s\}\end{Bmatrix}\\ \begin{Bmatrix}\{\dot{x}_p\}\\ \{\dot{x}_s\}\end{Bmatrix}\end{Bmatrix}
+
\begin{bmatrix}
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix}\\
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[F][D]\\ [D]^T[0]\end{bmatrix}
\end{bmatrix}
\begin{Bmatrix}\begin{Bmatrix}\{\ddot{x}_p\}\\ \{\ddot{x}_s\}\end{Bmatrix}\\ \begin{Bmatrix}\{\dot{x}_p\}\\ \{\dot{x}_s\}\end{Bmatrix}\end{Bmatrix}
+
$$

$$
+\begin{bmatrix}
-\begin{bmatrix}[M][0]\\ [0][m']\end{bmatrix} & \begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix}\\
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[K][0]\\ [0][k']\end{bmatrix}
\end{bmatrix}
\begin{Bmatrix}\begin{Bmatrix}\{\dot{x}_p\}\\ \{\dot{x}_s\}\end{Bmatrix}\\ \begin{Bmatrix}\{x_p\}\\ \{x_s\}\end{Bmatrix}\end{Bmatrix}
+
\begin{bmatrix}
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix}\\
\begin{bmatrix}[0][0]\\ [0][0]\end{bmatrix} & \begin{bmatrix}[H][G]\\ [G]^T[0]\end{bmatrix}
\end{bmatrix}
\begin{Bmatrix}\begin{Bmatrix}\{\dot{x}_p\}\\ \{\dot{x}_s\}\end{Bmatrix}\\ \begin{Bmatrix}\{x_p\}\\ \{x_s\}\end{Bmatrix}\end{Bmatrix}
=
$$

$$
= \begin{Bmatrix}\begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix}\\ \begin{Bmatrix}\{0\}\\ \{0\}\end{Bmatrix}\end{Bmatrix} \qquad (6.188)
$$

which after rearranging rows and columns may also be written as

$$
\left[\begin{array}{c|c}
[0][M] & [0][0]\\
[M][C] & [0][0]\\ \hline
[0][0] & [0][m']\\
[0][0] & [m'][c']
\end{array}\right]
\left\{\begin{array}{c}
\{\ddot{x}_p\}\\ \{\dot{x}_p\}\\ \hline \{\ddot{x}_s\}\\ \{\dot{x}_s\}
\end{array}\right\}
+
\left[\begin{array}{c|c}
[0][0] & [0][0]\\
[0][F] & [0][D]\\ \hline
[0][0] & [0][0]\\
[0][D]^T & [0][0]
\end{array}\right]
\left\{\begin{array}{c}
\{\ddot{x}_p\}\\ \{\dot{x}_p\}\\ \hline \{\ddot{x}_s\}\\ \{\dot{x}_s\}
\end{array}\right\}
+
$$

$$
+\left[\begin{array}{c|c}
-[M][0] & [0][0]\\
[0][K] & [0][0]\\ \hline
[0][0] & -[m'][0]\\
[0][0] & [0][k']
\end{array}\right]
\left\{\begin{array}{c}
\{\dot{x}_p\}\\ \{x_p\}\\ \hline \{\dot{x}_s\}\\ \{x_s\}
\end{array}\right\}
+
\left[\begin{array}{c|c}
[0][0] & [0][0]\\
[0][H] & [0][G]\\ \hline
[0][0] & [0][0]\\
[0][G]^T & [0][0]
\end{array}\right]
\left\{\begin{array}{c}
\{\dot{x}_p\}\\ \{x_p\}\\ \hline \{\dot{x}_s\}\\ \{x_s\}
\end{array}\right\}
=
$$

$$
= \begin{Bmatrix}\{0\}\\ \{0\}\\ \{0\}\\ \{0\}\end{Bmatrix}, \qquad (6.189)
$$

and thus, by virtue of Eqs. 6.2, 6.3, and 6.5 and the relations analogous to Eqs. 6.48, 6.49, and 6.51 that correspond to the secondary system in Fig. 6.2(b) when both of its ends are considered fixed, the reduced equation of motion of the assembled system in Fig. 6.1 may be put in terms of the parameters of the reduced equations of motion of its independent subsystems as

$$\begin{bmatrix} [A][0] \\ [0][a'] \end{bmatrix} \begin{Bmatrix} \{\dot{q}_p\} \\ \{\dot{q}_s\} \end{Bmatrix} + \begin{bmatrix} [B][0] \\ [0][b'] \end{bmatrix} \begin{Bmatrix} \{q_p\} \\ \{q_s\} \end{Bmatrix} + \begin{bmatrix} [Q][P] \\ [P]^T[0] \end{bmatrix} \begin{Bmatrix} \{\dot{q}_p\} \\ \{\dot{q}_s\} \end{Bmatrix} +$$

$$\begin{bmatrix} [V][T] \\ [T]^T[0] \end{bmatrix} \begin{Bmatrix} \{q_p\} \\ \{q_s\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{6.190}$$

where

$$[P] = \begin{bmatrix} [0][0] \\ [0][D] \end{bmatrix} \tag{6.191}$$

$$[Q] = \begin{bmatrix} [0][0] \\ [0][F] \end{bmatrix} \tag{6.192}$$

$$[V] = \begin{bmatrix} [0][0] \\ [0][H] \end{bmatrix} \tag{6.193}$$

$$[T] = \begin{bmatrix} [0][0] \\ \\ [0][G] \end{bmatrix} \quad . \tag{6.194}$$

Now, it may be seen from the inspection of Eqs. 6.19, 6.22, 6.77, and 6.118 that, similarly to the case of proportional damping, the complex eigenvector of an assembled system associated to a complex frequency λ may be estimated by considering that the only significant component eigenvectors in the summations in Eqs. 6.19 and 6.77 are those whose complex natural frequencies are the closest (in the absolute value sense) to the complex frequency λ. Accordingly, the vectors $\{q_p\}$ and $\{q_s\}$ in Eq. 6.190 may be approximated as

$$\{q_p\} = \{S\}^{(I)} Z_I \; e^{\lambda t} \tag{6.195}$$

$$\{q_s\} = \{s\}^{(J)} z_J \; e^{\lambda t} \tag{6.196}$$

where, using the notation of the preceding chapters, the subscripts I and J identify, respectively, the parameters that correspond to the eigenvectors of the primary and secondary systems herein under consideration whose complex natural frequencies are the closest to the complex frequency λ of their associated assembled system. Then, if Eqs. 6.195 and 6.196 are substituted into Eq. 6.190, and if the first and second component equations of this Eq. 6.190 are premultiplied respectively by $\{S\}^{(I)T}$ and $\{s\}^{(J)T}$, the reduced equation of motion of the assembled system in Fig. 6.1 may be approximated by the following two equations:

$$(\lambda A_I^* + B_I^* + \lambda Q_I^* + V_I^*)\ Z_I + (\lambda P_{IJ}^* + T_{IJ}^*)\ z_J = 0 \tag{6.197}$$

$$(\lambda a_J^* + b_J^*)\ z_J + (\lambda P_{IJ}^* + T_{IJ}^*)\ Z_I = 0 \tag{6.198}$$

where A_I^*, B_I^*, a_J^*, and b_J^* are complex generalized parameters of the primary and secondary systems defined as indicated by Eqs. 6.9, 6.10, 6.71, and 6.72, and where

$$Q_I^* = \{S\}^{(I)^T}[Q]\{S\}^{(I)} \tag{6.199}$$

$$V_I^* = \{S\}^{(I)^T}[V]\{S\}^{(I)} \tag{6.200}$$

$$P_{IJ}^* = \{S\}^{(I)^T}[P]\{s\}^{(J)} \tag{6.201}$$

$$T_{IJ}^* = \{S\}^{(I)^T}[T]\{s\}^{(J)} \quad . \tag{6.202}$$

Thus, since in matrix form Eqs. 6.197 and 6.198 may be written as

$$\begin{bmatrix} \lambda A_I^* + B_I^* + \lambda Q_I^* + V_I^* & \lambda P_{IJ}^* + T_{IJ}^* \\ \lambda P_{IJ}^* + T_{IJ}^* & \lambda a_J^* + b_J^* \end{bmatrix} \begin{Bmatrix} Z_I \\ z_J \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{6.203}$$

after considering that

$$B_I^* = -\lambda_{p_I} A_I^* \tag{6.204}$$

$$b_J^* = -\lambda_{s_J} a_J^* \tag{6.205}$$

those two equations lead to the following simplified eigenvalue problem:

$$\begin{vmatrix} A_I^* (\lambda - \lambda_{p_I}) + (\lambda Q_I^* + V_I^*) & \lambda P_{IJ}^* + T_{IJ}^* \\ \lambda P_{IJ}^* + T_{IJ}^* & a_J^* (\lambda - \lambda_{s_J}) \end{vmatrix} = 0 \tag{6.206}$$

which after expanding the determinant results as

$$\lambda^2 [A_I^* a_J^* + Q_I^* a_J^* - P_{IJ}^{*\,2}] -$$

$$- \lambda[A_I^* a_J^* (\lambda_{p_I} + \lambda_{s_J}) + a_J^* (\lambda_{s_J} Q_I^* - V_I^*) + 2 P_{IJ}^* T_{IJ}^*] +$$

$$+ [A_I^* a_J^* \lambda_{p_I} \lambda_{s_J} - \lambda_{s_J} a_J^* V_I^* - T_{IJ}^{*\,2}] = 0 \quad . \tag{6.207}$$

To express this equation explicitly in terms of the dynamic properties of the independent primary and secondary systems, one may then observe the following:

1) From the definition of Q_I^*, V_I^*, P_{IJ}^*, and T_{IJ}^* one has that

$$Q_I^* = c_1 \Phi_1^2(I) + c_3 \Phi_3^2(I) \tag{6.208}$$

$$V_I^* = k_1 \Phi_1^2(I) + k_3 \Phi_3^2(I) \tag{6.209}$$

$$P_{IJ}^* = -[c_1\Phi_1(I)\phi_1(J) + c_3\Phi_3(I)\phi_2(J)] \tag{6.210}$$

$$T_{IJ}^* = -[k_1\Phi_1(I)\phi_1(J) + k_3\Phi_3(I)\phi_2(J)] \ . \tag{6.211}$$

2) Since by assumption the independent secondary system has proportional damping, the damping constants c_1 and c_3 in the above equations may be expressed as

$$c_j = a_s k_j, \quad j = 1, 2, 3, \tag{6.212}$$

where the proportionality constant a_s is of the form

$$a_s = \frac{2\xi_{s_J}}{\omega_{s_J}} \ , \tag{6.213}$$

and hence Q_I^* and P_{IJ}^* may be written as

$$Q_I^* = a_s V_I^* \tag{6.214}$$

$$P_{IJ}^* = a_s T_{IJ}^* \ . \tag{6.215}$$

3) According to Eqs. 6.211 and 4.65 and since $k_J^* = \omega_{s_J}^2 m_J^*$, T_{IJ}^* results of the form

$$T_{IJ}^* = - \Phi_o(I, J)\omega_{s_J}^2 m_J^* \; . \tag{6.216}$$

4) The complex frequencies λ_{p_I} and λ_{s_J} are given by

$$\lambda_{p_I} = - \xi_{p_I}\omega_{p_I} + i\ \omega'_{p_I} \tag{6.217}$$

$$\lambda_{s_J} = - \xi_{s_J}\omega_{s_J} + i\ \omega'_{s_J} \; . \tag{6.218}$$

5) For a primary and a secondary system with proportional damping the generalized parameters A_I^* and a_J^* may be expressed as

$$A_I^* = 2\ \lambda_{p_I} M_I^* + C_I^* = 2i\omega'_{p_I} M_I^* \tag{6.219}$$

$$a_J^* = 2\ \lambda_{s_J} m_J^* + c_J^* = 2i\omega'_{s_J} m_J^* \, . \tag{6.220}$$

6) For resonant modes

$$\omega_{p_I} = \omega_{s_J} = \omega_o \; . \tag{6.221}$$

7) For small damping ratios $\omega_o' \doteq \omega_o$.

Accordingly, Eq. 6.207 may be written as

$$\lambda^2 \left[1 + \Phi_o^2(I,J)\xi_{s_J}^2 \gamma_{IJ} - i\, \xi_{s_J}\gamma_{IJ} \frac{V_I^*}{k_J^*}\right] -$$

$$- \lambda\left[-(\xi_{p_I} + \xi_{s_J}) + \xi_{s_J}\gamma_{IJ} \frac{V_I^*}{k_J^*} - \Phi_o^2(I,J)\xi_{s_J}\gamma_{IJ} + 2i + \frac{i}{2}(1 + 2\xi_{s_J}^2)\gamma_{IJ}\frac{V_I^*}{k_J^*}\right] \omega_o +$$

$$+ \left[\xi_{p_I}\xi_{s_J} - 1 - \frac{1}{2}\gamma_{IJ}\frac{V_I^*}{k_J^*} + \frac{1}{4}\Phi_o^2(I,J)\gamma_{IJ} - i(\xi_{p_I} + \xi_{s_J}) + \frac{i}{2}\xi_{s_J}\gamma_{IJ}\frac{V_I^*}{k_J^*}\right] \omega_o^2 = 0$$

(6.222)

where

$$\frac{V_I^*}{k_J^*} = \frac{k_1\Phi_1^2(I) + k_3\Phi_3^2(I)}{k_J^*} \qquad (6.223)$$

and, as before,

$$\gamma_{IJ} = \frac{m_J^*}{M_I^*}\,. \qquad (6.224)$$

But since for small damping and mass ratios its second order terms and those multiplied by the ratio V_I^*/k_J^* are comparatively small and may be neglected, Eq. 6.222 may be approximated as

$$\lambda^2 - \lambda[-(\xi_{p_I}+\xi_{s_J}) + 2i]\omega_o - [1 - \xi_{p_I}\xi_{s_J} - \frac{1}{4}\Phi_o^2(I,J)\gamma_{IJ} + i(\xi_{p_I}+\xi_{s_J})]\omega_o^2 = 0. \tag{6.225}$$

Hence, after solving for λ one obtains that

$$\lambda = \frac{\omega_o}{2}[-(\xi_{p_I} + \xi_{s_J}) + 2i] \pm$$

$$\pm \frac{\omega_o}{2}\sqrt{[-(\xi_{p_I}+\xi_{s_J}) + 2i]^2 + 4[1 - \xi_{p_I}\xi_{s_J} - \frac{1}{4}\Phi_o^2(I,J)\gamma_{IJ} + i(\xi_{p_I}+\xi_{s_J})]} \tag{6.226}$$

or

$$\lambda = -\frac{1}{2}(\xi_{p_I}+\xi_{s_J})\omega_o + i\omega_o \pm \frac{\omega_o}{2}\sqrt{(\xi_{p_I}-\xi_{s_J})^2 - \Phi_o(I,J)\gamma_{IJ}} \quad . \tag{6.227}$$

Equation 6.227 is the sought approximate formula to determine the complex natural frequencies of the resonant modes of the assembled system under study. Its generalization for the resonant modes of an assembled system with any number of degrees of freedom and an arbitrary configuration results simply as

$$\lambda_r = - \frac{1}{2} (\xi_{p_I} + \xi_{s_J})\omega_o + i\omega_o \pm \frac{\omega_o}{2} \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \qquad (6.228)$$

where ω_o is the circular natural frequency that is common to the primary and secondary components of such an assembled system and $\Phi_o(I,J)$ is as indicated by Eq. 4.34.

From the analysis of Eq. 6.228 one may observe that when $(\xi_{p_I} - \xi_{s_J})^2$ is much greater than $\Phi_o^2(I,J)\gamma_{IJ}$, λ_r results approximately as

$$\lambda_r \doteq - \frac{1}{2} (\xi_{p_I} + \xi_{s_J}) \omega_o + i\omega_o \pm \frac{\omega_o}{2} (\xi_{p_I} - \xi_{s_J}) , \qquad (6.229)$$

and therefore in such a case the complex frequencies of an assembled system in the two resonant modes that correspond to the resonant frequency ω_o are

$$\lambda_{r_1} = - \xi_{p_I}\omega_o + i\omega_o \qquad (6.230)$$

$$\lambda_{r_2} = - \xi_{s_J}\omega_o + i\omega_o \quad . \qquad (6.231)$$

This means that in such resonant modes the assembled system vibrates with the same frequency, the resonant frequency ω_o, but in one of them it is damped with the damping ratio of its primary system while in the other it is damped with the one of its secondary system. On the other hand, if $\Phi_o^2(I,J)\gamma_{IJ}$ is much greater than $(\xi_{p_I} - \xi_{s_J})^2$, λ_r is given by

$$\lambda_r \doteq -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + i\omega_o \pm i\,\frac{\omega_o}{2}\,\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}\ ; \tag{6.232}$$

hence, the resonant complex frequencies result as

$$\lambda_{r_1} = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + i\omega_o\,[1 - \frac{1}{2}\,\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}\,] \tag{6.233}$$

$$\lambda_{r_2} = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + i\,{}_o[1 + \frac{1}{2}\,\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}\,]\ . \tag{6.234}$$

In this instance, therefore, the assembled system in the corresponding resonant modes is damped approximately with the average of the damping ratios of its primary and secondary systems and vibrates with the circular natural frequencies of the resonant modes of a similar assembled system with proportional damping (see Eq. 4.67).

Thus, between these two extreme cases a system with nonproportional damping in its resonant modes vibrates with a frequency that may not be close to the frequency that the system would have if it were proportionally damped, and is damped with damping ratios that may not be, neither, in the proximity of the average of the damping ratios of its primary and secondary systems.

6.4 Complex Natural Frequencies: Nonresonant Modes

The complex natural frequencies of nonresonant modes may be determined by following the procedure employed in Chapters 2 and 4 to obtain the natural frequencies of the nonresonant modes of systems with proportional damping. That is, it may be assumed that each nonresonant eigenvector of an assembled system with nonproportional damping is made up by only those eigenvectors of the independent components of this assembled system whose complex natural frequencies are the closest to its associated complex frequency.

Thus, since for the sssembled system in Fig. 6.1 such an assumption is equivalent to set in Eqs. 6.6 and 6.55

$$Z_i' = z_j' = 0 \quad \text{for} \begin{cases} i \neq I \\ j \neq 0,\ c,\ J,\ \bar{0},\ \bar{c} \end{cases} \tag{6.235}$$

where, as before, I and J are respectively the subscripts of the closest complex natural frequencies of its primary and secondary systems to its complex frequency λ, the system of equations given by Eqs. 6.8 and 6.64 may be reduced to an approximate system of six equations. If it is observed, however, that by equating the first of Eqs. 6.64 to the negative of the fifth and, similarly, the fourth to the negative of the eight one may conclude that

$$a_{ij} = a_{\bar{i}j}\ ,\quad i = 0,\ c;\quad j = 0,\ J,\ c,\ \bar{0},\ \bar{c} \tag{6.236}$$

$$b_{ij} = b_{\bar{i}j}\ ,\quad i = 0,\ c;\quad j = 0,\ J,\ c,\ \bar{0},\ \bar{c}\ , \tag{6.237}$$

such a system of equations may be reduced further to the following set:

$$A_I^* \dot{Z}_I' + B_I^* Z_I' = \Phi_1(I)R_1(t) + \Phi_3(I)R_3(t) \tag{6.238}$$

$$a_{00}(\dot{z}_0' + \dot{z}_{\bar{0}}') + a_{0J}\dot{z}_J' + a_{0c}(\dot{z}_c' + \dot{z}_{\bar{c}}') + b_{00}(z_0' + z_{\bar{0}}') +$$

$$+ b_{0J}z_J' + b_{0c}(z_c' + z_{\bar{c}}') = -[R_1(t) + R_3(t)] \tag{6.239}$$

$$a_{J0}(\dot{z}_0' + \dot{z}_{\bar{0}}') + a_J^* \dot{z}_J' + a_{Jc}(\dot{z}_c' + \dot{z}_{\bar{c}}') + b_{J0}(z_0' + z_{\bar{0}}') +$$

$$+ b_J^* z_J' + b_{Jc}(z_c' + z_{\bar{c}}') = 0 \tag{6.240}$$

$$a_{c0}(\dot{z}_0' + \dot{z}_{\bar{0}}') + a_{cJ}\dot{z}_J' + a_{cc}(\dot{z}_c' + \dot{z}_{\bar{c}}') + b_{c0}(z_0' + z_{\bar{0}}') +$$

$$+ b_{cJ} z_J' + b_{cc}(z_c' + z_{\bar{c}}') = -\phi_c(c)R_3(t). \tag{6.241}$$

As a result, if it is considered that in the light of Eqs. 6.101 through 6.104 and 6.107 through 6.110 the relations indicated by Eqs. 6.236 and 6.237 are tantamount to the following two equalities:

$$\lambda_{s_0} = \bar{\lambda}_{s_0} \tag{6.242}$$

$$\lambda_{s_c} = \bar{\lambda}_{s_c} \quad , \tag{6.243}$$

and that in virtue of these two equalities and the assumption described by Eq. 6.235 the compatibility relations expressed by Eqs. 6.93 through 6.96 may be simplified as

$$Z_0 + Z_{\bar{0}} = \Phi_1(I) Z_I' \tag{6.244}$$

$$Z_c + Z_c = \frac{d\Phi(I)}{f_{cc}} Z_I' \tag{6.245}$$

$$\lambda_{s_0} = \bar{\lambda}_{s_0} = \lambda \tag{6.246}$$

$$\lambda_{s_c} = \bar{\lambda}_{s_c} = \lambda, \tag{6.247}$$

after eliminating the reactions $R_1(t)$ and $R_3(t)$ from Eqs. 6.238 through 6.241 (by substituting into Eqs. 6.238 and 6.240 the expressions for $R_1(t)$ and $R_2(t)$ obtained from Eqs. 6.239 and 6.241) and introducing Eqs. 6.16, 6.17, 6.74, and 6.75 the reduced equation of motion of the assembled system in Fig. 6.1 may be approximated by

$$[A_I^* (\lambda - \lambda_{p_I}) + \Phi_1^2(I)(\lambda a_{00} + b_{00}) + 2\Phi_1(I) \frac{d\Phi(I)}{f_{cc}} (\lambda a_{0c} + b_{0c}) +$$

$$+ (\frac{d\Phi(I)}{f_{cc}})^2 (\lambda a_{cc} + b_{cc})] Z_I + [\Phi_1(I)(\lambda a_{0J} + b_{0J}) +$$

$$+ \frac{d\Phi(I)}{f_{cc}} (\lambda a_{cJ} + b_{cJ})] z_J = 0 \tag{6.248}$$

$$[\Phi_1(I)(\lambda a_{J0} + b_{J0}) + \frac{d\Phi(I)}{f_{cc}} (\lambda a_{Jc} + b_{Jc})] Z_I + a_J^* (\lambda - \lambda_{s_J}) z_J = 0 \qquad (6.249)$$

which by virtue of Eqs. 6.219, 6.220, 6.101 through 6.104, and 6.107 through 6.110 may also be expressed as

$$M_I^* \lambda^2 \left[2i\omega'_{p_I} \frac{\lambda - \lambda_{p_I}}{\lambda^2} + \Phi_1^2(I) \frac{m_{00}}{M_I^*} + 2\Phi_1(I) \frac{d\Phi(I)}{f_{cc}} \frac{m_{0c}}{M_I^*} + \right.$$

$$\left. + \left(\frac{d\Phi(I)}{f_{cc}}\right)^2 \frac{m_{cc}}{M_I^*} \right] Z_I + \lambda^2 m_J^* \Phi_o(I,J) z_J = 0 \qquad (6.250)$$

$$\lambda^2 m_J^* \Phi_o(I,J) Z_I + 2i\omega'_{s_J} m_J^* (\lambda - \lambda_{s_J}) z_J = 0 . \qquad (6.251)$$

For small mass ratios, therefore, the eigenvalue problem of the assembled system under study may be written approximately as

$$\begin{vmatrix} 2i\omega'_{p_I} \left(\dfrac{\lambda - \lambda_{p_I}}{\lambda^2}\right) & \Phi_o(I,J)\gamma_{IJ} \\ \Phi_o(I,J)\gamma_{IJ} & 2i\omega'_{s_J} \left(\dfrac{\lambda - \lambda_{s_J}}{\lambda^2}\right)\gamma_{IJ} \end{vmatrix} = 0 \qquad (6.252)$$

or as

$$\left(\frac{\lambda - \lambda_{p_I}}{\lambda^2}\right)\left(\frac{\lambda - \lambda_{s_J}}{\lambda^2}\right) = -\frac{\Phi_o^2(I,J)\gamma_{IJ}}{4\omega'_{p_I}\,\omega'_{s_J}} \doteq 0 \quad , \tag{6.253}$$

from which it may be concluded that for small mass ratios the complex natural frequencies of an assembled system with nonproportional damping in its nonresonant modes are approximately given by

$$\lambda_{r_1} = \lambda_{p_I} \tag{6.254}$$

$$\lambda_{r_2} = \lambda_{s_J} \ . \tag{6.255}$$

6.5 Complex Participation Factors

Although the complex participation factors of an assembled system with nonproportional damping may be computed directly from the definition of a complex participation factor introduced in Sec. 5.2, for convenience these complex participation factors are here expressed in terms of the parameters of the primary and secondary components of such an assembled system.

According to Eq. 5.67, the rth complex participation factor of the assembled system in Fig. 6.1 is given by

$$\gamma_r = \frac{\{w\}^{(r)^T}[M]\{J\}}{2\lambda_r\{w\}^{(r)^T}[M]\{w\}^{(r)} + \{w\}^{(r)^T}[C]\{w\}^{(r)}} \tag{6.256}$$

where [M] and [C] are its mass and damping matrices, respectively, and $\{w\}^{(r)}$ represents its rth complex mode shape. Then, its rth complex participation factor may be written as a function of the parameters of its primary and secondary components if in Eq. 6.256 [M] and [C] are expressed in terms of the mass and damping matrices of these primary and secondary components using Eqs. 6.181 and 6.182, and if the mode shape $\{w\}^{(r)}$ is transformed into generalized coordinates by means of Eqs. 6.151 and 6.152. Observe, however, that since the transformation indicated by these two equations is very similar to the corresponding one for systems with proportional damping (the only difference, indeed, is the complex nature of $Y_i^{(r)}$ and $Y_j^{(r)}$), from the results in Sec. 4.5 one may easily infer that

$$\{w\}^{(r)^T}[M]\{J\} = \sum_{i=1}^{N_p} M_i^* Y_i^{(r)} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)} + y_c^{(r)} + y_j^{(r)}) \tag{6.257}$$

$$\{w\}^{(r)^T}[M]\{w\}^{(r)} = \sum_{i=1}^{N_p} M_i^* Y_i^{(r)^2} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)} + y_c^{(r)} + y_j^{(r)})^2 . \tag{6.258}$$

In addition, observe that by substitution of Eq. 6.182 and by the partitioning of $\{w\}^{(r)}$ into its primary and secondary parts $\{w\}^{(r)^T}[C]\{w\}^{(r)}$ may be expressed as

$$\{w\}^{(r)^T}[C]\{w\}^{(r)} = \{w_p\}^{(r)^T}[C]\{w_p\}^{(r)} + \{w_s\}^{(r)^T}[c']\{w_s\}^{(r)} +$$

$$\{w_p\}^{(r)^T}[F]\{w_p\}^{(r)} + 2\{w_p\}^{(r)^T}[D]\{w_s\}^{(r)} \quad , \tag{6.259}$$

and hence, if it is considered that by means of the transformations given by Eqs. 6.151 and 6.152 one may write the following two equalities:

$$\{w_p\}^{(r)^T}[C]\{w_p\}^{(r)} = \{Y\}^{(r)^T}[\Phi]^T[C][\Phi]\{Y\}^{(r)} = \sum_{i=1}^{N_p} C_i^* \, Y_i^{(r)^2} \qquad (6.260)$$

$$\{w_s\}^{(r)^T}[c']\{w_s\}^{(r)} = \{y\}^{(r)^T}[\phi]^T[c'][\phi]\{y\}^{(r)} = \sum_{j=1}^{N_s} c_j^* \, (y_0^{(r)} + y_j^{(r)} + y_c^{(r)})^2 +$$

$$+ (c_{00} - c_1^* - c_2^*)y_0^{(r)^2} + (c_{cc} - c_1^* - c_2^*)y_c^{(r)^2} + 2\,(c_{0c} - c_1^* - c_2^*)y_0^{(r)}\,y_c^{(r)} +$$

$$+ 2\,(c_{c1} - c_1^*)y_1^{(r)}\,y_c^{(r)} + 2\,(c_{c2} - c_2^*)y_2^{(r)}\,y_c^{(r)} \qquad (6.261)$$

where

$$C_i^* = \{\Phi\}^{(i)^T}[C]\{\Phi\}^{(i)}, \quad i = 1, 2, \ldots, N_p \qquad (6.262)$$

$$c_{ij} = \{\phi\}^{(i)^T}[c]\{\phi\}^{(j)}, \; i=0,\, c; \quad j=0,\, 1,\, 2,\, c \qquad (6.263)$$

$$c_j^* = \{\phi\}^{(j)^T}[c']\{J\} = \{\phi\}^{(j)^T}[c']\{\phi\}^{(j)}, \; j=1, 2, \ldots, N_s \qquad (6.264)$$

(for the proof of this last identity, see Apendix C), and that according to the definitions of [F] and [D] (Eqs. 6.184 and 6.185) one has that

$$\{w_p\}^{(r)^T}[F]\{w_p\}^{(r)} = c_1 w_{p_1}^2(r) + c_3\, w_{p_3}^2(r) \tag{6.265}$$

$$\{w_p\}^{(r)^T}[D]\{w_s\}^{(r)} = -\,[c_1 w_{p_1}(r) w_{s_1}(r) + c_3 w_{p_3}(r) w_{s_2}(r)] \tag{6.266}$$

in which

$$w_{p_1}(r) = \sum_{i=1}^{N_p} \Phi_1(i) Y_i^{(r)} = y_0^{(r)} \tag{6.267}$$

$$w_{p_3}(r) = \sum_{i=1}^{N_p} \Phi_3(i)\; Y_i^{(r)} = f_{cc}(y_0^{(r)} + y_c^{(r)}) \tag{6.268}$$

$$w_{s_i}(r) = y_0^{(r)} + \sum_{j=1}^{N_s} \phi_i(j)\; y_j^{(r)} + f_{ci}\; y_c^{(r)} \quad , \; i=1,\, 2, \tag{6.269}$$

after discarding second order terms one obtains that

$$\{w\}^{(r)^T}[C]\{w\}^{(r)} = \sum_{i=1}^{N_p} C_i^*\; Y_i^{(r)^2} + \sum_{j=1}^{N_s} c_j^*\; (y_0^{(r)} + y_j^{(r)} + y_c^{(r)})^2 \;. \tag{6.270}$$

Thus, it is easy to see that in terms of the parameters of its independent primary and secondary systems the rth complex participation factor of an assembled system may be expressed as

$$\gamma_r = \frac{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)} + \sum_{j=1}^{N_s} m_j^* (y_0^{(r)} + y_j^{(r)} + y_c^{(r)})}{\sum_{i=1}^{N_p} [2\lambda_r M_i^* + C_i^*] Y_i^{(r)^2} + \sum_{j=1}^{N_s} [2\lambda_r m_j^* + c_j^*] (y_0^{(r)} + y_j^{(r)} + y_c^{(r)})^2} \tag{6.271}$$

On the basis of this equation and by (a) writing λ_r explicitly in terms of its real and imaginary parts, (b) considering that for a primary and a secondary system with proportional damping C_i^* and c_j^* result of the form (see Eq. 5.80)

$$C_i^* = 2\xi_{p_i} \omega_{p_i} M_i^* \tag{6.272}$$

$$c_j^* = 2\xi_{s_j} \omega_{s_j} m_j^* , \tag{6.273}$$

and (c) neglecting insignificant component modes one may therefore write λ_r approximately as

$$\gamma_r = \frac{1}{2} \frac{B_r Y_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)}) \gamma_{IJ}}{[-(\xi_r \omega_r - \xi_{p_I} \omega_{p_I}) + i\omega_r] Y_I^{(r)^2} + [-(\xi_r \omega_r - \xi_{s_J} \omega_{s_J}) + i\omega_r](y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2 \gamma_{IJ}} \tag{6.274}$$

where, as before, B_r is defined as

$$B_r = \frac{\sum_{i=1}^{N_p} M_i^* Y_i^{(r)}}{M_I^* Y_I^{(r)}} . \tag{6.275}$$

6.6 Maximum Modal Responses: Resonant Modes

It has been shown in Sec. 5.4 that the rth mode maximum response of an assembled system with nonproportional damping may be calculated by means of Eq. 5.154, which when applied to its secondary system alone results of the form

$$\{X_s\}^{(r)} = 2\{\text{sgn}[(u'_{s_i} + v'_{s_i}) - (u'_{s_{i-1}} + v'_{s_{i-1}})]|dw'_{s_i}|\}^{(r)} \omega_r SD(\omega_r, \xi_r) \tag{6.276}$$

where

$$\{|dw'_s|\}^{(r)} = |\gamma_r|\{|dw_s|\}^{(r)} \tag{6.277}$$

and $\{dw_s\}^{(r)}$ is of the form

$$\{dw_s\}(r) = \begin{Bmatrix} w_{s_1}(r) - w_{p_k}(r) \\ w_{s_2}(r) - w_{s_1}(r) \\ \vdots \\ w_{s_{N_s}}(r) - w_{s_{N_s-1}}(r) \\ w_{p_\ell}(r) - w_{s_{N_s}}(r) \end{Bmatrix} . \tag{6.278}$$

Thus, it may be seen that a simplified formula for the maximum modal responses of a secondary system may be obtained if a simple approximate expression for its vector of complex modal distortions may be derived. Also, since the formulation presented in Sec. 6.2 to compute the mode shapes of assembled systems with nonproportional damping is very similar in form to the one introduced in Chapters 2 and 4 to determine the mode shapes of those with proportional damping, it is apparent that such an approximate expression may be developed by applying the criteria employed in the derivation of the corresponding one for systems with proportional damping.

Accordingly, if it is assumed again that the rth mode of an assembled system with nonproportional damping is composed by only those component modes whose complex frequencies are, among all, the closest to the complex frequency of this rth mode, and if $\{\Phi\}^{(I)}$ and $\{\phi\}^{(J)}$ are such closest component modes, Eqs. 6.151 through 6.157 lead to the following approximate expressions for $\{w_p\}^{(r)}$ and $\{w_s\}^{(r)}$:

$$\{w_p\}^{(r)} = \{\Phi\}^{(I)} \, Y_I^{(r)} \tag{6.279}$$

$$\{w_s\}^{(r)} = \{J\}y_0^{(r)} + \{\phi\}^{(J)}y_J^{(r)} + \{f\}y_c^{(r)} \tag{6.280}$$

where

$$y_0^{(r)} = \Phi_k(I) \, Y_I^{(r)} \tag{6.281}$$

$$y_c^{(r)} = \frac{d\Phi(I)}{f_{cc}} \gamma_I^{(r)} \tag{6.282}$$

$$y_j^{(r)} = \Phi_o(I,J) \frac{-\lambda_r^2}{(\lambda_r - \lambda_{s_J})(\lambda_r - \bar{\lambda}_{s_J})} \gamma_I^{(r)} . \tag{6.283}$$

On the basis of these equations and in similarity with the corresponding derivation for systems with proportional damping presented in Sec. 4.6, the rth vector of secondary element distortions may be therefore written as

$$\{dw_s\}^{(r)} = d\Phi(I) \{\frac{df}{f_{cc}}\} \gamma_I^{(r)} + y_J^{(r)} \{d\phi\}^{(J)} \tag{6.284}$$

which, considering that when λ_r is close to λ_{s_J} the first term in the right-hand side of this equation is small when compared to the second one, may be approximated as

$$\{dw_s\}^{(r)} = y_J^{(r)} \{d\phi\}^{(J)} , \tag{6.285}$$

and hence the vector of the absolute values of such secondary distortions results of the form

$$\{|dw_s|\}^{(r)} = |y_J^{(r)}| \{|d\phi|\}^{(J)} . \tag{6.286}$$

Equations 6.276, 6.277, and 6.286 indicate thus that the desired simplified expression for $\{x_s\}^{(r)}$ may be obtained by deriving approximate relationships for the absolute values of the factor $y_J^{(r)}$, the complex participation factor γ_r, and the vector of secondary modal distortions $\{dw_s\}^{(r)}$, and by evaluating the sign function sgn $[(u'_{s_i} + v'_{s_i}) - (u'_{s_{i-1}} + v'_{s_{i-1}})]$. In what follows, then, such approximate relationships are derived, and this sign function is evaluated.

$\underline{y_J^{(r)}\text{ factors}}$

According to Eq. 6.228, the resonant complex natural frequencies of an assembled system are given by

$$\lambda_r = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_0 + i\omega_0 \pm \frac{\omega_0}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_0^2(I,J)\,\gamma_{IJ}}\ . \qquad (6.287)$$

Consequently, since λ_{s_J}, the Jth complex natural frequency of the independent secondary system of such an assembled system, may be written as

$$\lambda_{s_J} = -\xi_{s_J}\omega_{s_J} + i\omega'_{s_J} \doteq -\xi_{s_J}\omega_0 + i\omega_0\ , \qquad (6.288)$$

the difference $\lambda_r - \lambda_{s_J}$ in Eq. 6.283 results of the form

$$\lambda_r - \lambda_{s_J} = -\frac{1}{2}(\xi_{p_I} - \xi_{s_J})\omega_0 \pm \frac{\omega_0}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_0^2(I,J)\gamma_{IJ}}\ . \qquad (6.289)$$

Similarly, the difference $\lambda_r - \bar{\lambda}_{s_J}$ may be expressed as

$$\lambda_r - \bar{\lambda}_{s_J} = -\frac{1}{2}(\xi_{p_I} - \xi_{s_J})\omega_o + 2i\omega_o \pm \frac{\omega_o}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \tag{6.290}$$

from which it may be seen that a close approximation for $(\lambda_r - \bar{\lambda}_{s_J})$ is

$$\lambda_r - \bar{\lambda}_{s_J} \doteq 2i\omega_o \ . \tag{6.291}$$

In addition, if λ_r is written in its rectangular form as

$$\lambda_r = -\xi_r\omega_r + i\omega_r' \ , \tag{6.292}$$

it may be seen that $i\lambda_r$ may be put into the form

$$i\lambda_r = -\omega_r'\left[1 + i\sqrt{\frac{\xi_r^2}{1 - \xi_r^2}}\right] \tag{6.293}$$

which in polar form may be written as

$$i\lambda_r = -\omega_r e^{i\theta_r} \tag{6.294}$$

where

$$\theta_r = \tan^{-1} \frac{\xi_r}{\sqrt{1 - \xi_r^2}} , \tag{6.295}$$

and thus, since for small damping ratios one may approximate $i\lambda_r$ as

$$i\lambda_r = - \omega_r e^{i\xi_r}, \tag{6.296}$$

λ_r^2 may be expressed as

$$\lambda_r^2 = - \omega_r^2 e^{i2\xi_r} . \tag{6.297}$$

Upon substitution of Eqs. 6.289, 6.291, and 6.297, and by considering that for resonant modes ω_r is approximately equal to ω_0, Eq. 6.283 may be therefore written as

$$y_J^{(r)} = i \frac{\Phi_0(I,J)\, e^{i2\xi_r}}{(\xi_{p_I} - \xi_{s_J}) \mp \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_0^2(I,J)\, \gamma_{IJ}}} \gamma_I^{(r)} . \tag{6.298}$$

In expressing this approximate expression in its polar form to find an approximate relationship for $|y_J^{(r)}|$, one should note that the argument of its square root may be positive or negative and hence its denominator may be real or complex. Thus, the following two cases need to be considered separately:

<u>Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$</u> . In this case the denominator of Eq. 6.298 is real. Consequently, $y_J^{(r)}$ may be expressed as

$$y_J^{(r)} = i\,\frac{\Phi_o(I,J)e^{i2\xi_r}}{(\xi_{p_I} - \xi_{s_J}) \mp (\xi_{p_I} - \xi_{s_J})\sqrt{1 - \dfrac{\Phi_o^2(I,J)\gamma_{IJ}}{(\xi_{p_I} - \xi_{s_J})^2}}}\,Y_I^{(r)} \qquad (6.299)$$

which by introducing the transformation

$$\frac{\Phi_o(I,J)\sqrt{\gamma_{IJ}}}{\xi_{p_I} - \xi_{s_J}} = \sin\Psi_r \qquad (6.300)$$

may also be written as

$$y_J^{(r)} = i\,\frac{\Phi_o(I,J)e^{i2\xi_r}}{(\xi_{p_I} - \xi_{s_J})[1 \pm \cos\Psi_r]}\,Y_I^{(r)} \;. \qquad (6.301)$$

In such a case, then, $|y_J^{(r)}|$ results as

$$|y_J^{(r)}| = \frac{|\Phi_o(I,J)|}{(\xi_{p_I} - \xi_{s_J})[1 \mp \cos\Psi_r]}\,|Y_I^{(r)}|. \qquad (6.302)$$

Case II: $|\xi_{p_I} - \xi_{s_J}| \le |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$. When $|\xi_{p_I} - \xi_{s_J}|$ is smaller than or equal to $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, Eq. 6.298 may be put into the form

$$y_J^{(r)} = \frac{\Phi_o(I,J)e^{i2\xi_r}}{\mp\sqrt{\Phi_o^2(I,J)\,\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} - i(\xi_{p_I} - \xi_{s_J})}\, Y_I^{(r)} \tag{6.303}$$

which in polar form results as

$$y_J^{(r)} = \frac{e^{-i(\psi_r - 2\xi_r)}}{\sqrt{\gamma_{IJ}}}\, Y_I^{(r)} \tag{6.304}$$

where

$$\psi_r = \pm\tan^{-1}\frac{\xi_{p_I} - \xi_{s_J}}{\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}}\ . \tag{6.305}$$

Thus, it is easy to see that for the case under consideration $|y_J^{(r)}|$ is of the form

$$|y_J^{(r)}| = \frac{1}{\sqrt{\gamma_{IJ}}}\,|Y_I^{(r)}|. \tag{6.306}$$

Participation Factors

In the light of the above expressions for $y_J^{(r)}$ and Eqs. 6.274 and 6.275, simplified expressions for the complex participation factors may be derived as follows:

Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$. In this instance, if $y_0^{(r)}$, $y_c^{(r)}$, and $y_J^{(r)}$ are approximated as indicated by Eqs. 6.281, 6.282 and 6.301, and if it is considered that $\Phi_o(I,J) = \Phi_k(I) + \beta_J\, d\Phi(I)$, the sum $y_0^{(r)} + y_c^{(r)} + y_J^{(r)}$ in Eq. 6.274 may be expressed as

$$y_0^{(r)} + y_c^{(r)} + y_J^{(r)} = \left[1 + i\,\frac{e^{i2\xi_r}}{(\xi_{p_I} - \xi_{s_J})(1 \mp \cos\Psi_r)}\right]\Phi_k(I)Y_I^{(r)} +$$

$$+\left[\frac{1}{f_{cc}} + i\,\frac{\beta_J\, e^{i2\xi_r}}{(\xi_{p_I} - \xi_{s_J})(1 \mp \cos\Psi_r)}\right] d\Phi(I)Y_I^{(r)} \qquad (6.307)$$

from which it may be seen that for small damping ratios

$$y_0^{(r)} + y_c^{(r)} + y_J^{(r)} \doteq i\,\frac{\Phi_0(I,J)}{(\xi_{p_I} - \xi_{s_J})(1 \mp \cos\Psi_r)}\,Y_I^{(r)}\ . \qquad (6.308)$$

Understandably, since for resonant modes the parameter B_r given by Eq. 6.275 is very close to unity, the numerator of the right-hand side of Eq. 6.274 may be written as

$$B_rY_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\gamma_{IJ} = \left[1 + i\,\frac{\Phi_0(I,J)\,\gamma_{IJ}}{(\xi_{p_I} - \xi_{s_J})(1 \mp \cos\Psi_r)}\right]Y_I^{(r)} \qquad (6.309)$$

or, if it is expressed in polar form and Eq. 6.300 is introduced, as

$$B_r Y_I^{(r)} + (y_0^{(r)} + y_C^{(r)} + y_J^{(r)})\gamma_{IJ} = Y_I^{(r)} \sqrt{1 + \left(\frac{\sin\Psi_r}{1 \mp \cos\Psi_r}\right)^2 \gamma_{IJ}} \; e^{i\zeta_r} \tag{6.310}$$

where ζ_r is such that

$$\tan\zeta_r = \frac{\sin\Psi_r \sqrt{\gamma_{IJ}}}{1 \mp \cos\Psi_r} \quad . \tag{6.311}$$

But the analysis of Eq. 6.311 with its plus sign shows that for small mass ratios $\tan\zeta_r$ is always much smaller than unity because $|\sin\Psi_r|$ and $|\cos\zeta_r|$ are always less than or equal to unity. In like manner, if it is considered that by substituting into Eq. 6.311 the value of $\sqrt{\gamma_{IJ}}$ solved from Eq. 6.300 $\tan\zeta_r$ may be alternatively expressed as

$$\tan\zeta_r = \frac{\xi_{p_I} - \xi_{s_J}}{\Phi_0(I,J)} \, \frac{\sin^2\Psi_r}{1 \mp \cos\Psi_r} \, , \tag{6.312}$$

it may be observed that in the limiting case when Ψ_r approaches zero (that is, when the denominator of the right-hand side of Eq. 6.312 approaches its minimum) this equation with its negative sign yields

$$\tan\zeta_r \Big|_{\Psi_r \to 0} = \frac{\xi_{p_I} - \xi_{s_J}}{\Phi_o(I,J)} \frac{2\sin\Psi_r \cos\Psi_r}{\sin\Psi_r} \Bigg|_{\Psi_r=0} = 2\,\frac{\xi_{p_I} - \xi_{s_J}}{\Phi_o(I,J)} \tag{6.313}$$

which indicates that for small damping ratios $\tan\zeta_r$ is also always small when the negative sign of Eq. 6.312 is considered. For small mass and damping ratios, therefore, the second term within the radical of Eq. 6.310 may be neglected, $e^{i\zeta_r}$ may be set equal to unity, and, as a consequence, the numerator of the right-hand side of Eq. 6.274 may be written approximately as

$$B_r\, Y_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\, \gamma_{IJ} \doteq Y_I^{(r)}. \tag{6.314}$$

To write a simplified expression for the denominator of the same right-hand side of Eq. 6.274, it may be observed that when $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o\,(I,J)\gamma_{IJ}|$ Eq. 6.287 leads to

$$-\,(\xi_r\omega_r - \xi_{p_I}\omega_{p_I}) = \frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})\omega_o \pm \frac{\omega_o}{2}\sqrt{\,(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2\,(I,J)\gamma_{IJ}} \tag{6.315}$$

$$-\,(\xi_r\omega_r - \xi_{s_J}\omega_{s_J}) = -\frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})\omega_o \pm \frac{\omega_o}{2}\sqrt{\,(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2\,(I,J)\gamma_{IJ}} \tag{6.316}$$

$$\omega_r = \omega_0 \; , \tag{6.317}$$

and therefore by virtue of Eqs. 6.308 and 6.300 one has that

$$-(\xi_r\omega_r - \xi_{p_I}\omega_{p_I})Y_I^{(r)^2} - (\xi_r\omega_r - \xi_{s_J}\omega_{s_J})(y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\gamma_{IJ} =$$

$$= \frac{\omega_r}{2}(\xi_{p_I} - \xi_{s_J})\left[(1 \pm \cos\Psi_r) + (1 \mp \cos\Psi_r)\frac{\sin^2\Psi_r}{(1 \mp \cos\Psi_r)^2}\right]Y_I^{(r)^2} =$$

$$= \frac{\omega_r}{2}(\xi_{p_I} - \xi_{s_J})\left(\frac{2\sin^2\Psi_r}{1 \mp \cos\Psi_r}\right)Y_I^{(r)^2} \tag{6.318}$$

and

$$i\omega_r\left[Y_I^{(r)^2} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\gamma_{IJ}\right] = i\omega_r\left[1 - \frac{\Phi_0^2(I,J)\gamma_{IJ}}{(\xi_{p_I} - \xi_{s_J})^2(1 \mp \cos\Psi_r)^2}\right]Y_I^{(r)^2} =$$

$$= \mp\, i\omega_r\left[\frac{2\cos\Psi_r}{1 \mp \cos\Psi_r}\right]Y_I^{(r)^2} \; . \tag{6.319}$$

Thus, by substitution of Eqs. 6.314, 6.318, and 6.319 into Eq. 6.274, the complex participation factors in the case under study may be expressed as

$$\gamma_r = \frac{1}{2\omega_r}\,\frac{1}{2Y_I^{(r)}}\,\frac{1 \mp \cos\Psi_r}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})\sin^2\Psi_r \mp i\cos\Psi_r} \tag{6.320}$$

and hence

$$|\gamma_r| = \frac{1}{2\omega_r} \frac{1}{2|Y_I^{(r)}|} \frac{1 \mp \cos\Psi_r}{\sqrt{\frac{1}{4}(\xi_{p_I} - \xi_{s_J})^2 \sin^4\Psi_r + \cos^2\Psi_r}} \quad . \tag{6.321}$$

Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_0(I,J)\sqrt{\gamma_{IJ}}|$. In this case, by virtue of Eqs. 6.281, 6.282, and 6.304 the sum $y_0^{(r)} + y_c^{(r)} + y_J^{(r)}$ in Eq. 6.274 may be expressed as

$$y_0^{(r)} + y_c^{(r)} + y_J^{(r)} = [\Phi_k(I) + \frac{d\Phi(I)}{f_{cc}} + \frac{e^{-i(\psi_r - 2\xi_r)}}{\sqrt{\gamma_{IJ}}}]\, Y_I^{(r)} \tag{6.322}$$

which for small mass and damping ratios may be approximated as

$$y_0^{(r)} + y_c^{(r)} + y_J^{(r)} \doteq \frac{e^{-i\psi_r}}{\sqrt{\gamma_{IJ}}} Y_I^{(r)} \quad . \tag{6.323}$$

Therefore, by substituting Eq. 6.323 and taking into account that as in the previous case $B_r \doteq 1.0$, the numerator of the right-hand side of Eq. 6.274 may be written as

$$B_r Y_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\gamma_{IJ} = [1 + \sqrt{\gamma_{IJ}}\, e^{-i\psi_r}]\, Y_I^{(r)} \doteq Y_I^{(r)} \quad . \tag{6.324}$$

Similarly, since for the case herein being considered Eq. 6.287 yields

$$-(\xi_r\omega_r - \xi_{p_I}\omega_{p_I}) = \frac{1}{2}(\xi_{p_I} - \xi_{s_J})\omega_o \tag{6.325}$$

$$-(\xi_r\omega_r - \xi_{s_J}\omega_{s_J}) = -\frac{1}{2}(\xi_{p_I} - \xi_{s_J})\,\omega_o \tag{6.326}$$

$$\omega_r = \omega_o \pm \frac{\omega_o}{2}\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\quad , \tag{6.327}$$

the terms of the denominator of the right-hand side of Eq. 6.274 result as

$$-(\xi_r\omega_r - \xi_{p_I}\omega_{p_I})Y_I^{(r)^2} - (\xi_r\omega_r - \xi_{s_J}\omega_{s_J})(y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\gamma_{IJ} =$$

$$= \frac{1}{2}(\xi_{p_I} - \xi_{s_J})(1 - e^{-i2\psi_r})\,Y_I^{(r)^2} \tag{6.328}$$

$$i\omega_r[Y_I^{(r)^2} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\,\gamma_{IJ}] = i\omega_r\,(1 + e^{-i2\psi_r})\,Y_I^{(r)^2}\quad . \tag{6.329}$$

Thus, if Eqs. 6.324, 6.328, and 6.329 are substituted into Eq. 6.274, and if it is considered that the factors $(1-e^{-i2\psi_r})$ and $(1 + e^{-i2\psi_r})$ in Eqs. 6.328 and 6.329 may be written as

$$1 - e^{-i2\psi_r} = i2\sin\psi_r\, e^{-i\psi_r} \tag{6.330}$$

$$1 + e^{-i2\psi_r} = 2\cos\psi_r \, e^{-i\psi_r} \quad , \tag{6.331}$$

the complex participation factors in the case under consideration may be expressed as

$$\gamma_r = \frac{1}{2i\omega_r} \frac{1}{2Y_I^{(r)}} \frac{e^{i\psi_r}}{\frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})\,\frac{\omega_o}{\omega_r}\sin\psi_r + \cos\psi_r} \tag{6.332}$$

which, in view of the fact that ω_r is approximately equal to ω_o and since according to Eq. 6.305 $\sin\psi_r$ and $\cos\psi_r$ are given by

$$\sin\psi_r = \frac{-\,(\xi_{p_I} - \xi_{s_J})}{\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}} \tag{6.333}$$

$$\cos\psi_r = \frac{\mp\,\sqrt{\Phi_o(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}}{\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}} \quad , \tag{6.334}$$

may also be put into the form

$$\gamma_r = \frac{1}{2i\omega_r} \frac{1}{2Y_I^{(r)}} \frac{\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}\; e^{i\psi_r}}{-\,\frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2 \mp \sqrt{\Phi_o^2\,(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}} \quad . \tag{6.335}$$

Consequently, in this Case II $|\gamma_r|$ may be written as

$$|\gamma_r| = \frac{1}{2\omega_r}\frac{1}{2\,|\gamma_I^{(r)}|}\frac{|\Phi_o(I,J)|\sqrt{\gamma_{IJ}}}{\left|\frac{1}{2}(\xi_{p_I}-\xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ}-(\xi_{p_I}-\xi_{s_J})^2}\right|} \cdot \quad (6.336)$$

Secondary Modal Distortions

Case I: $|\xi_{p_I}-\xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$. By virtue of Eqs. 6.277, 6.286, and 6.302 one has that

$$\{|d\omega_s'|\}^{(r)} = |\gamma_r| \frac{|\Phi_o(I,J)|\;|\gamma_I^{(r)}|}{(\xi_{p_I}-\xi_{s_J})[1 \mp \cos\Psi_r]}\{|d\phi|\}^{(J)} \quad (6.337)$$

which in combination with Eq. 6.321 leads to

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\frac{1}{2}\frac{|\Phi_o(I,J)|}{\sqrt{\frac{1}{4}(\xi_{p_I}-\xi_{s_J})^4\sin^2\Psi_r + \cos^2\Psi_r(\xi_{p_I}-\xi_{s_J})}}\{|d\phi|\}^{(J)}. \quad (6.338)$$

Thus, if Eq. 6.300 is considered, the vector of secondary modal distortions when $|\xi_{p_I}-\xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ may be expressed as

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\frac{1}{2}\frac{|\Phi_o(I,J)|}{\sqrt{(\xi_{p_I}-\xi_{s_J})^2-\Phi_o^2(I,J)\gamma_{IJ}[1-\frac{1}{4}\Phi_o^2(I,J)\gamma_{IJ}]}}\{|d\phi|\}^{(J)}. \tag{6.339}$$

Notice that this equation is not valid when

$$(\xi_{p_I}-\xi_{s_J})^2-\Phi_o^2(I,J)\gamma_{IJ}-[\frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ}]^2 = 0. \tag{6.340}$$

However, if this equality is rewritten as

$$\left[\frac{\xi_{p_I}-\xi_{s_J}}{\Phi_o(I,J)\sqrt{\gamma_{IJ}}}\right]^2 = 1-\frac{1}{4}\Phi_o^2(I,J)\gamma_{IJ}\ , \tag{6.341}$$

it is evident that for the case under consideration Eq. 6.340 can never be satisfied because $\Phi_o(I,J)$ and γ_{IJ} are always positive and because by hypothesis

$$\left[\frac{\xi_{p_I}-\xi_{s_J}}{\Phi_o^2(I,J)\sqrt{\gamma_{IJ}}}\right]^2 \geq 1.0. \tag{6.342}$$

Therefore, Eq. 6.339 is defined for all the possible relations between $|\xi_{p_I}-\xi_{s_J}|$ and $|\Phi_o(I,J)\gamma_{IJ}|$.

Notice also that since for small mass ratios the term $[1 - \frac{1}{4}\,\Phi_o^2(I,J)\gamma_{IJ}]$ in Eq. 6.339 is very close to unity, for the cases in which $|(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}|$ is not very small $\{|d\omega_s'|\}^{(r)}$ may be approximated as

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\,\frac{1}{2}\,\frac{|\Phi_o(I,J)|}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}}}\,\{|d\phi|\}^{(J)}. \tag{6.343}$$

Case II: $|\xi_{p_I} - \xi_{s_J}| \le |\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}\,|$. When this condition is satisfied, $|y_J^{(r)}|$ is given by Eq. 6.306. In this case, therefore, Eqs. 6.277 and 6.286 yield

$$\{|d\omega_s'|\}^{(r)} = \frac{|\gamma_r|}{\sqrt{\gamma_{IJ}}}\,|Y_I^{(r)}|\,\{|d\phi|\}^{(J)}, \tag{6.344}$$

and hence, since $|\gamma_r|$ is given by Eq. 6.336, the vector of secondary modal distortions results of the form

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\,\frac{1}{2}\,\frac{|\Phi_o(I,J)|}{\left|\frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\,\right|}\{|d\phi|\}^{(J)}. \tag{6.345}$$

Evidently, for small damping ratios and when $|\Phi_o^2\,(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|$ is not very small these modal distortions may be approximated as

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\frac{1}{2}\frac{|\Phi_o(I,J)|}{\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}}\{|d\phi|\}^{(J)}. \tag{6.346}$$

It is interesting to note that in both cases the maximum values of the modal distortions of a secondary system are obtained when $|\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}\ | = |\xi_{p_I} - \xi_{s_J}|$ and that in both cases, too, such maximum values are

$$\{|d\omega_s'|\}_{max}^{(r)} = \frac{1}{2\omega_r}\frac{1}{|\Phi_o(I,J)\gamma_{IJ}|}\{|d\phi|\}^{(J)} = \frac{1}{2\omega_r}\frac{|\Phi_o(I,J)|}{(\xi_{p_I} - \xi_{s_J})^2}\{|d\phi|\}^{(J)} . \tag{6.347}$$

It is also interesting to note that when the values of $|\xi_{p_I} - \xi_{s_J}|$ and $|\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}\ |$ are not very close to each other, the vector of secondary modal distortions is given, independently of the relation between those two values, by the following single expression:

$$\{|d\omega_s'|\}^{(r)} = \frac{1}{2\omega_r}\frac{1}{2}\frac{|\Phi_o(I,J)|}{\sqrt{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2\ |}}\{|d\phi|\}^{(J)}. \tag{6.348}$$

Evaluation of sgn $[(u_{s_i}' + v_{s_i}') - (u_{s_{i-1}}' + v_{s_{i-1}}')]$

As pointed out earlier, the need for the evaluation of the signs of the resonant modal responses of a system is because such signs indicate if the cross-product terms in the established rule to combine modes (see Eq. 5.168) are to be added or subtracted to the squared terms in it. It may be observed, however, that what actually determines the positive or negative nature of any of such cross-product terms is the relative sign between its two associated resonant modal responses. On these premises, therefore, the evaluation of the sign function in Eq. 6.276 will be here limited to the determination of such a relative sign.

One may note that because u_s' and v_s' are respectively the real and imaginary parts of w_s', the argument of the sign function herein being evaluated may be put into the form

$$(u_{s_i}' + v_{s_i}') - (u_{s_{i-1}}' + v_{s_{i-1}}') = \mathrm{Re}[\gamma_r dw_{s_i}] + \mathrm{Im}[\gamma_r dw_{s_i}] \; . \tag{6.349}$$

In view of Eq. 6.285, it may therefore be written approximately as

$$(u_{s_i}' + v_{s_i}') - (u_{s_{i-1}}' + v_{s_{i-1}}') = [\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}(\gamma_r y_J^{(r)})] d\phi_i(J) \; , \tag{6.350}$$

and thus, if it is considered that $d\phi_i(J)$ is a parameter common to two adjacent resonant modes (i.e., two modes whose natural frequencies lie close to the same resonant frequency), the sign function in Eq. 6.276 may be expressed as

$$\mathrm{sgn}[(u_{s_i}' + v_{s_i}') - (u_{s_{i-1}}' + v_{s_{i-1}}')] = \mathrm{sgn}[\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}(\gamma_r y_J^{(r)})]. \tag{6.351}$$

On the basis of this equality and the approximate expressions for γ_r and $y_J^{(r)}$ found above, such a sign function may then be evaluated as follows:

Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$. In this case, Eqs. 6.301 and 6.320 permit one to write the product $\gamma_r y_J^{(r)}$ as

$$\gamma_r y_J^{(r)} = \frac{1}{4\omega_r} \frac{i\Phi_o(I,J)\, e^{i2\xi_r}}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \sin^2\Psi_r \mp i\cos\Psi_r (\xi_{p_I} - \xi_{s_J})} \tag{6.352}$$

which after introducing Eq. 6.300 becomes

$$\gamma_r y_J^{(r)} = \frac{1}{4\omega_r} \frac{i\Phi_o(I,J)\, e^{i2\xi_r}}{\frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ} \mp i\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}}} . \tag{6.353}$$

Hence, since ω_r and ξ_r are always positive and $\Phi_o(I,J)$ is a parameter common to two adjacent resonant modes, one has that

$$\text{sgn}[\text{Re}(\gamma_r y_J^{(r)}) + \text{Im}(\gamma_r y_J^{(r)})] =$$

$$= \text{sgn}[\mp \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} + \frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ}] . \tag{6.354}$$

Thus, it may be seen that when

$$\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2\,(I,J)\gamma_{IJ}} > \frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ} \tag{6.355}$$

or. what is tantamount, when

$$\left(\frac{\xi_{p_I} - \xi_{s_J}}{\Phi_o^2\,(I,J)\,\sqrt{\gamma_{IJ}}}\right)^2 > 1 + \frac{1}{4}\,\Phi_o^2(I,J)\gamma_{IJ}, \tag{6.356}$$

two adjacent resonant modes are always of opposite signs; otherwise, they are of the same sign. That is,

$$\mathrm{sgn}\,[\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}\,(\gamma_r y_J^{(r)})] =$$

$$= \begin{cases} \mathrm{sgn}(\mp\,1.0) \text{ when } (\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} > [\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}]^2 \\ \\ \mathrm{sgn}\,(+\,1.0) \text{ otherwise} \end{cases} \tag{6.357}$$

<u>Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}|$</u> . According to Eqs. 6.304 and 6.335, the product $\gamma_r y_J^{(r)}$ for the systems within this Case II may be expressed as

$$\gamma_r y_J^{(r)} = \frac{i}{4\omega_r} \frac{\Phi_o(I,J)\, e^{i2\xi_r}}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm\sqrt{\Phi_o^2\,(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}} \cdot \tag{6.358}$$

Hence, in this case $\mathrm{sgn}[\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}(\gamma_r y_J^{(r)})]$ results as

$$\mathrm{sgn}[\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}(\gamma_r y_J^{(r)})] =$$

$$= \mathrm{sgn}\ \frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J) - (\xi_{p_I} - \xi_{s_J})^2} \tag{6.359}$$

which indicates that two adjacent resonant modes always have opposite signs when

$$\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} > \frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2 \tag{6.360}$$

or

$$\left(\frac{\Phi_o(I,J)\,\sqrt{\gamma_{IJ}}}{\xi_{p_I} - \xi_{s_J}}\right)^2 > 1 + \frac{1}{4}(\xi_{p_I} - \xi_{s_J})^2 \tag{6.361}$$

and the same sign in all other cases. In other words,

$$\mathrm{sgn}[\mathrm{Re}(\gamma_r y_J^{(r)}) + \mathrm{Im}(\gamma_r y_J^{(r)})] =$$

$$= \begin{cases} \mathrm{sgn}(\pm 1.0) \text{ when } (\xi_{p_I} - \xi_{s_J})^2 - \Phi_0^2(I,J) > [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2 \\ \mathrm{sgn}(+ 1.0) \quad \text{otherwise.} \end{cases} \tag{6.362}$$

The results indicated by Eqs. 6.357 and 6.362 are somewhat expected, because if it is considered that when $|\xi_{p_I} - \xi_{s_J}| = |\Phi_0(I,J)\gamma_{IJ}|$ a system has two modes with the same complex natural frequency (see Eq. 6.287) and therefore identical mode shapes, it is logical to expect that in the neighborhood of that equality the same system have two modes with similar mode shapes and, consequently, the same sign.

<u>Maximum Modal Responses</u>

In the light of Eqs. 6.276, 6.339, 6.345, 6.351, 6.354 and 6.359, the maximum distortions of a secondary system in its resonant modes may be thus expressed as follows:

<u>Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_0(I,J)\sqrt{\gamma_{IJ}}|$</u>

$$\{X_s\}^{(r)} = \frac{1}{2}\frac{\mathrm{sgn}(\Delta_{IJ})\ \Phi_0(I,J)}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_0^2(I,J)\gamma_{IJ}[1 - \frac{1}{4}\Phi_0^2(I,J)\gamma_{IJ}]}}\{d\phi\}^{(J)} SD(\omega_r, \xi_r) \tag{6.363}$$

where

$$\Delta_{IJ} = \frac{1}{2} \Phi_o^2(I,J)\gamma_{IJ} \mp \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \tag{6.364}$$

and, according to Eq. 6.287,

$$\xi_r = \frac{1}{2}(\xi_{p_I} + \xi_{s_J}) \pm \frac{1}{2} \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \tag{6.365}$$

and

$$\omega_r = \omega_{p_I} = \omega_{s_J} = \omega_o \quad . \tag{6.366}$$

<u>Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$</u>

$$\{X_s\}^{(r)} = \frac{1}{2} \frac{\text{sgn}(\varepsilon_{IJ})\ \Phi_o(I,J)}{\left|\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\right|} \{d\phi\}^{(J)} SD(\omega_r,\xi_r) \tag{6.367}$$

where

$$\varepsilon_{IJ} = \frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} \tag{6.368}$$

and by virtue of Eq. 6.287

$$\xi_r = \frac{1}{2}\,(\xi_{p_I} + \xi_{s_J}) \tag{6.369}$$

$$\omega_r = \omega_o \pm \frac{\omega_o}{2}\,\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\;. \tag{6.370}$$

Observe that in this case $\{X_s\}^{(r)}$ may be written simply as

$$\{X_s\}^{(r)} = \frac{1}{2}\,\frac{\Phi_o(I,J)}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}}\{d\phi\}^{(J)}SD(\omega_r,\xi_r)\;. \tag{6.371}$$

By the inspection of these relationships it is easy to show that for an assembled system with proportional damping Eqs. 6.363 and 6.367 converge to the corresponding one derived in Chapter 4 for systems with proportional damping. In fact, since for such an assembled system $\xi_{p_I} - \xi_{s_J}$ is zero (when the damping matrix of the system is proportional to its stiffness matrix, for example, $\xi_{p_I} = \xi_{s_J} = \frac{1}{2}\,a\omega_o$, where a is the proportionality constant common to its primary and secondary components), it is apparent that for systems with proportional damping Case II always applies and that Eq. 6.371 is reduced to

$$\{X_s\}^{(r)} = \pm \frac{1}{2} \frac{\{d\phi\}^{(J)}}{\sqrt{\gamma_{IJ}}} SD\ (\xi_r, \omega_r) \tag{6.372}$$

which, as it may be seen by comparison with Eq. 4.102, is the expression for $\{X_s\}^{(r)}$ derived in Chapter 4 for systems with proportional damping.

Similarly, it may be observed that in accordance with Eqs. 6.348, 6.357, and 6.362 when $|\xi_{p_I} - \xi_{s_J}|$ and $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ are not very close to each other (i.e., when $|(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}|$ is greater than $[\frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ}]^2$ and $[\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2$, the maximum distortions of a secondary system in its resonant modes may be approximated, independently of the relation between $|\xi_{p_I} - \xi_{s_J}|$ and $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, by

$$\{X_s\}^{(r)} = \pm \frac{1}{2} \frac{\Phi_o(I,J)}{\sqrt{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|}} \{d\phi\}^{(J)} SD(\omega_r, \xi_r). \tag{6.373}$$

For most practical purposes, this formula may be considered the sought simplified expression to determine the maximum response of secondary systems in resonant modes.

6.7 Maximum Modal Responses: Nonresonant Modes

Simplified relationships for the maximum distortions of a secondary system in the nonresonant modes of its assembled system may also be obtained by following the corresponding approach utilized in Chapters 2 and 4.

These simplified relationships, therefore, are here derived separately for those nonresonant modes in which the assembled system has natural frequencies close to the natural frequencies of its primary system and those in which it has natural frequencies close to the natural frequencies of its secondary system.

Case I: $\lambda_r \doteq \lambda_{p_I}$

According to the discussion in Sec. 6.4, the complex natural frequencies of an assembled system in some of its nonresonant modes may be approximated as

$$\lambda_r = \lambda_{p_I} \quad . \tag{6.374}$$

For such nonresonant modes, then, the $Y_i^{(r)}$ and $y_j^{(r)}$ factors of Eqs. 6.151 and 6.152 result as

$$Y_i^{(r)} = \begin{cases} Y_I^{(r)} & \text{if } i = I \\ 0 & \text{if } i \neq I \end{cases} \tag{6.375}$$

$$y_0^{(r)} = \Phi_k(I) Y_I^{(r)} \tag{6.376}$$

$$y_c = \frac{d\Phi(I)}{f_{cc}} \; Y_I^{(r)} \tag{6.377}$$

$$y_j^{(r)} = \Phi_0(I,j) \frac{-\lambda_{p_I}^2}{(\lambda_{p_I} - \lambda_{s_j})(\lambda_{p_I} - \bar{\lambda}_{s_j})} \gamma_I^{(r)} \quad , \tag{6.378}$$

and consequently the primary and secondary parts of their associated mode shapes may be expressed as

$$\{w_{p_I}\}^{(r)} = \{\Phi\}^{(I)}\gamma_I^{(r)} \tag{6.379}$$

$$\{w_s\}^{(r)} = \Phi_k(I)\gamma_I^{(r)}\{J\} + \sum_{j=1}^{N_s} y_j^{(r)}\{\phi\}^{(j)} + \frac{d\Phi(I)}{f_{cc}} \gamma_I^{(r)}\{f\} \tag{6.380}$$

from which it may be seen that the corresponding unit-participation-factor secondary modal distortions may be written as

$$\{dw_s'\}^{(r)} = \gamma_r \left[d\Phi(I)\gamma_I^{(r)} \{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} y_j^{(r)}\{d\phi\}^{(j)}\right] , \tag{6.381}$$

where according to Eq. 6.274 γ_r is of the form

$$\gamma_r = \frac{1}{2} \frac{\gamma_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\gamma_{IJ}}{i\omega_{p_I}\gamma_I^{(r)^2} + [-(\xi_{p_I}\omega_{p_I} - \xi_{s_J}\omega_{s_J}) + i\omega_{p_I}](y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2 \, \gamma_{IJ}} \; . \tag{6.382}$$

Thus, the maximum response of a secondary system in the nonresonant modes under consideration may be easily calculated by means of Eqs. 5.154 and 6.381. It is convenient, however, to derive, based directly on the concepts of Sec. 5.4, an alternative simplified expression as follows:

According to Eq. 5.150 and by considering that $SV_r \doteq \omega_r SD_r$ and $\omega'_r \doteq \omega_r$, un upper bound to the maximum modal distortions of a secondary system is given by

$$\{X_s\}^{(r)} \leq 2|Re\{dw'_s\}^{(r)} + Im\{dw'_s\}^{(r)}|\omega_r SD(\omega_r, \xi_r) \tag{6.383}$$

which, by virtue of Eqs. 6.374 and 6.381 and assuming that $Y_I^{(r)}$ is real, for the nonresonant modes herein being studies results as

$$\{X_s\}^{(r)} \leq 2|(Re\gamma_r + Im\gamma_r)\ d\Phi(I)Y_I^{(r)}\{\frac{df}{f_{cc}}\} +$$

$$+ \sum_{j=1}^{N_s} [Re(\gamma_r y_j^{(r)}) + Im(\gamma_r y_j^{(r)})]\{d\phi\}^{(j)}|\omega_r SD(\omega_{p_I}, \xi_{p_I}). \tag{6.384}$$

Hence, since

$$Re(\gamma_r y_j^{(r)}) + Im(\gamma_r y_j^{(r)}) \leq (Re\gamma_r + Im\gamma_r)(Re y_j^{(r)} + Im y_j^{(r)}) \quad , \tag{6.385}$$

one may consider that

$$\{X_s\}^{(r)} \leq 2|\mathrm{Re}\gamma_r + \mathrm{Im}\gamma_r| \Big| d\Phi(I)Y_I^{(r)} \left\{\frac{df}{f_{cc}}\right\} +$$

$$+ \sum_{j=1}^{N_s} (\mathrm{Re}\; y_j^{(r)} + \mathrm{Im} y_j^{(r)})\{d\phi\}^{(j)} \Big|\omega_r \; SD(\omega_{p_I}, \xi_{p_I}) \qquad (6.386)$$

and thus, if by adopting the less conservative approach introduced in Sec. 5.4 the sums $\mathrm{Re}\gamma_r + \mathrm{Im}\gamma_r$ and $\mathrm{Re}\; y_j^{(r)} + \mathrm{Im}\; y_j^{(r)}$ are replaced by the square root of the sum of the square of their terms times their original signs, the vector $\{X_s\}^{(r)}$ may be written approximately as

$$\{X_s\}^{(r)} = 2|\gamma_r| \; \Big| d\Phi(I)Y_I^{(r)} \left\{\frac{df}{f_{cc}}\right\} +$$

$$+ \sum_{j=1}^{N_s} \mathrm{sgn}(\mathrm{Re}\; y_j^{(r)} + \mathrm{Im} y_j^{(r)}) \; |y_j^{(r)}|\{d\phi\}^{(j)} \Big| \omega_{p_I} SD(\omega_{p_I}, \omega_{p_I}) \; . \qquad (6.387)$$

To obtain a simplified formula for $\{X_s\}^{(r)}$, then, approximate expressions for $|y_j^{(r)}|, |\gamma_r|$, and $\mathrm{sgn}\; (\mathrm{Re}\; y_j^{(r)} + \mathrm{Im}\; y_j^{(r)})$ are derived next.

<u>$y_j^{(r)}$ factors</u>. If λ_{p_I} and λ_{s_j} are written explicitly in terms of their real and imaginary parts, and if $\lambda_{p_I}^2$ is put into the form indicated by Eq. 6.297, Eq. 6.378 may be alternatively expressed as

$$y_j^{(r)} = \frac{\Phi_o(I,j)\omega_{p_I}^2 e^{i2\xi_{p_I}} \gamma_I^{(r)}}{[-(\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}) + i(\omega'_{p_I} - \omega'_{s_j})][-(\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}) + i(\omega'_{p_I} + \omega'_{s_j})]} \tag{6.388}$$

which for small damping ratios may be written as

$$y_j^{(r)} = \frac{-\Phi_o(I,j)\omega_{p_I}^2 \gamma_I^{(r)}}{(\omega_{p_I} + \omega_{s_j})[(\omega_{p_I} - \omega_{s_j}) + i(\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j})]} . \tag{6.389}$$

In polar form, then, $y_j^{(r)}$ may be expressed as

$$y_j^{(r)} = \frac{-\Phi_o(I,j)\omega_{p_I}^2 e^{-i\theta_j} \gamma_I^{(r)}}{(\omega_{p_I} + \omega_{s_j})[(\omega_{p_I} - \omega_{s_j})^2 + (\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j})^2]^{1/2}} , \tag{6.390}$$

or as

$$y_j^{(r)} = \frac{-\Phi_o(I,j)\omega_{p_I}^2}{\omega_{p_I}^2 - \omega_{s_j}^2} \cos\theta_j \, e^{-i\theta_j} \gamma_I^{(r)} \tag{6.391}$$

where θ_j is such that

$$\tan\theta_j = \frac{\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}}{\omega_{p_I} - \omega_{s_j}} , \tag{6.392}$$

or as

$$y_j^{(r)} = A_o(j)\ \cos\theta_j\ e^{-i\theta_j}\ \gamma_I^{(r)} \qquad (6.393)$$

where according to Eq. 4.112 $A_o(j)$ is of the form

$$A_o(j) = \frac{\Phi_o(I,j)\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2} \quad . \qquad (6.394)$$

Thus, $y_j^{(r)}$ may be written as

$$|y_j^{(r)}| = |A_o(j)\ \cos\ \theta_j||\gamma_I^{(r)}| \ . \qquad (6.395)$$

By examining Eq. 6.378, one may observe that $y_j^{(r)}$ is undefined when $\lambda_{p_I} = \lambda_{s_j}$. Observe, however, that for such a case Eq. 6.393 is not valid because whenever the frequencies ω_{p_I} and ω_{s_j} get very close to each other the complex frequency λ_r approaches that of a resonant mode and consequently the hypothesis $\lambda_r \doteq \lambda_{p_I}$ used in its derivation is no longer valid. To establish, then, the range of validity of Eq. 6.393, it may be considered that there exists an upper bound for $|y_j^{(r)}|$ when $\omega_{p_I} = \omega_{s_j}$ and $\xi_{p_I} = \xi_{s_j}$ (see Eq. 6.298) and that thus for all other relations between ω_{p_I} and ω_{s_j}, and ξ_{p_I} and ξ_{s_j}, $|y_j^{(r)}|$ should always be less than or equal to that upper bound. Accordingly, since for $\omega_{p_I} = \omega_{s_j}$ and $\xi_{p_I} = \xi_{s_j}$ Eq. 6.298 yields

$$|y_j^{(r)}|_{max} = \frac{1}{\sqrt{\gamma_{Ij}}}|\ \gamma_I^{(r)}|\ , \qquad (6.396)$$

from Eq. 6.395 one has that

$$|A_0(j)\cos\theta_j| \le \frac{1}{\sqrt{\gamma_{Ij}}} \tag{6.397}$$

and therefore Eq. 6.393 is valid when

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_j}^2}{\omega_{p_I}^2} \sec\theta_j \right| \ge |\Phi_0(I,j)\sqrt{\gamma_{Ij}}| \ . \tag{6.398}$$

When given primary and secondary nonresonant modes do not satisfy this condition, they should be considered as resonant modes.

Participation Factors. Based on the proof that for resonant modes the term $(y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\,\gamma_{IJ}$ in Eq. 6.274 is negligibly small, and by considering that for nonresonant modes the factors $y_j^{(r)}$ are always smaller than the corresponding ones for resonant modes, one may infer that the numerator of the right-hand side of Eq. 6.382 may be approximated as

$$Y_I^{(r)} + (y_0^{(r)} + y_c^{(r)} + y_J^{(r)})\,\gamma_{IJ} \doteq Y_I^{(r)} \ . \tag{6.399}$$

Similarly, it may be observed that the term $(y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\,\gamma_{IJ}$ in the denominator of the right-hand side of the same Eq. 6.382 may always be written approximately as

$$(y_0^{(r)} + y_c^{(r)} + y_J^{(r)})^2\,\gamma_{IJ} \doteq y_J^{(r)^2}\,\gamma_{IJ} \tag{6.400}$$

because if $y_J^{(r)}$ is large, $y_0^{(r)} + y_c^{(r)}$ are comparatively small and if, on the other hand, $y_J^{(r)}$ is small, the term in its totality is negligibly small. Consequently, for the nonresonant modes under study the participation factors may be expressed as

$$\gamma_r = \frac{1}{2} \frac{\gamma_I^{(r)}}{i\omega_{p_I}\gamma_I^{(r)^2} + [-(\xi_{p_I}\omega_{p_I} - \xi_{s_J}\omega_{s_J}) + i\omega_{p_I}]\, y_J^{(r)^2}\, \gamma_{IJ}} \qquad (6.401)$$

or as

$$\gamma_r = \frac{1}{2i\omega_{p_I}} \frac{1}{y_J^{(r)2}\gamma_{IJ}} \frac{\gamma_I^{(r)}}{(1+\gamma_I^{(r)2}/y_J^{(r)2}\,\gamma_{IJ})+i(\xi_{p_I}\omega_{p_I}-\xi_{s_J}\omega_{s_J})/\omega_{p_I}} \cdot \qquad (6.402)$$

Notice, however, that by introducing Eq. 6.393 one has that

$$1 + \frac{\gamma_I^{(r)^2}}{y_J^{(r)^2}\gamma_{IJ}} = 1 + \frac{\cos 2\theta_J + i \sin 2\theta_J}{A_o^2(J)\cos^2\theta_J\,\gamma_{IJ}} \qquad (6.403)$$

which by considering that

$$\cos 2\theta_j = (1-\tan^2\theta_j)\cos^2\theta_j \qquad (6.404)$$

$$\sin 2\theta_j = 2\tan\theta_j\cos^2\theta_j \qquad (6.405)$$

may also be written as

$$1 + \frac{\gamma_I^{(r)^2}}{y_J^{(r)^2}\gamma_{IJ}} = 1 + \frac{(1-\tan^2\theta_J) + 2i\tan\theta_J}{A_o^2(J)\gamma_{IJ}} \quad . \qquad (6.406)$$

Therefore, by means of Eqs. 6.393 and 6.406 γ_r may be alternatively expressed as

$$\gamma_r = \frac{1}{2i\omega_{p_I}} \frac{1}{\gamma_I^{(r)}} \frac{(1+\delta_J^2)\, e^{i2\theta_J}}{[1+A_o^2(J)\gamma_{IJ}-\delta_J^2] + i\,[2+\frac{\omega_{p_I}-\omega_{s_J}}{\omega_{p_I}}A_o^2(J)\gamma_{IJ}]\,\delta_J} , \qquad (6.407)$$

where, if j = J, δ_J is given by

$$\delta_j = \tan \theta_j = \frac{\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}}{\omega_{p_I} - \omega_{s_j}} \quad , \tag{6.408}$$

from which one obtains that

$$|\gamma_r| = \frac{1}{2\omega_{p_I}} \frac{1}{|\gamma_I^{(r)}|} \frac{1 + \delta_J^2}{\left\{[1 + A_o^2(J)\gamma_{IJ} - \delta_J^2]^2 + [2 + \frac{\omega p_I - \omega s_J}{\omega_{p_I}} A_o^2(J)\gamma_{IJ}]^2 \delta_J^2\right\}^{1/2}} . \tag{6.409}$$

<u>sgn(Re $y_j^{(r)}$ + Im $y_j^{(r)}$)</u>, Since Eq. 6.389 may be put into the form

$$y_j^{(r)} = \frac{-\Phi_o(I,j)\omega_{p_I}^2 [(\omega_{p_I} - \omega_{s_j}) - i(\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j})]}{(\omega_{p_I} + \omega_{s_j})[(\omega_{p_I} - \omega_{s_j})^2 + (\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j})^2]} \gamma_I^{(r)} , \tag{6.410}$$

it is easy to see that for the case under consideration the sign function sgn (Re $y_j^{(r)}$ + Im $y_j^{(r)}$) results as

$$\mathrm{sgn}(\mathrm{Re}\, y_j^{(r)} + \mathrm{Im}\, y_j^{(r)}) = -\mathrm{sgn}\{\Phi_o(I,j)[(\omega_{p_I} - \omega_{s_j}) - (\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j})]\} \tag{6.411}$$

which after introducing the parameter δ_j defined by Eq. 6.408 becomes

$$\text{sgn}(\text{Re } y_j^{(r)} + \text{Im } y_j^{(r)}) = -\text{sgn}[\Phi_o(I,j)(\omega_{p_I} - \omega_{s_j})(1 - \delta_j)]$$

$$= -\text{sgn }[\Phi_o(I,j)(\omega_{p_I} - \omega_{s_j})] \text{ sgn}(1 - \delta_j). \tag{6.412}$$

Maximum Modal Secondary Distortions. By virtue of Eqs. 6.395 and 6.412, the product $\text{sgn}(\text{Re } y_j^{(r)} + \text{Im } y_j^{(r)})|y_j^{(r)}|$ in Eq. 6.387 may be expressed as

$$\text{sgn }(\text{Re } y_j^{(r)} + \text{Im } y_j^{(r)})|y_j^{(r)}| =$$

$$= \text{sgn }[\Phi_o(I,j)(\omega_{p_I} - \omega_{s_j})] \text{ sgn}(1 - \delta_j)|A_o(j)||\cos \theta_j||Y_I^{(r)}|. \tag{6.413}$$

But since in the light of Eq. 6.394 and by considering that ω_{p_I} and ω_{s_j} are always positive one has that

$$- \text{sgn}[\Phi_o(I,j)(\omega_{p_I} - \omega_{s_j})]|A_o(j)| = A_o(j), \tag{6.414}$$

after expressing $\cos \theta_j$ in terms of δ_j such a product may also be written as

$$\text{sgn}(\text{Re } y_j^{(r)} + \text{Im } y_j^{(r)})|y_j^{(r)}| = \text{sgn}(1 - \delta_j) \frac{A_o(j)}{\sqrt{1 + \delta_j^2}} |Y_I^{(r)}|. \tag{6.415}$$

By substitution of this equation into Eq. 6.387, the vector of maximum modal secondary distortions may be therefore expressed as

$$\{X_s\}^{(r)} = 2|\gamma_r||Y_I^{(r)}| \left[d\Phi(I)\{\frac{df}{f_{cc}}\} + \right.$$

$$\left. + \sum_{j=1}^{N_s} \text{sgn}(1-\delta_j) \frac{A_o(j)}{\sqrt{1+\delta_j^2}} \{d\phi\}^{(j)} \right] \omega_{p_I} \, SD(\omega_{p_I}, \xi_{p_I}) \qquad (6.416)$$

or as

$$\{X_s\}^{(r)} = A.F. \left[r_c \{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} r_j \{d\phi\}^{(j)} \right] SD(\omega_{p_I}, \xi_{p_I}) \qquad (6.417)$$

where

$$r_c = \frac{d\Phi(I)}{A_o(J)} \sqrt{1+\delta_J^2} \qquad (6.418)$$

$$r_j = \text{sgn}\,(1-\delta_j) \frac{A_o(j)}{A_o(J)} \sqrt{\frac{1+\delta_J^2}{1+\delta_j^2}} \; , \qquad (6.419)$$

where by virtue of Eq. 6.409

$$A.F. = \frac{A_o(J)\sqrt{1+\delta_J^2}}{\left\{ [1 + A_o^2(J)\gamma_{IJ} - \delta_J^2]^2 + [2 + \frac{\omega_{p_I} - \omega_{s_J}}{\omega_{p_I}} A_o^2(J)\gamma_{IJ}]^2 \, \delta_J^2 \right\}^{1/2}} \; , \qquad (6.420)$$

and where the outer absolute value bars indicated in Eq. 6.387 have been ignored because the sign of $\{X_s\}^{(r)}$ is of no importance.

Equation 6.417 in combination with Eqs. 6.418 through 6.420 represents thus the desired simplified expression to calculate the maximum response of a secondary system in the nonresonant modes of its associated assembled system whose complex natural frequencies are nearly equal to those of its supporting primary system. Notice that since $|y_j^{(r)}|$ is only valid for the range indicated by Eq. 6.398, Eqs. 6.416 and 6.420 are also only valid if

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_j}^2}{\omega_{p_I}^2} \right| \sqrt{1 + \delta_j^2} \geq \left| \Phi_o(I,j) \sqrt{\gamma_{Ij}} \right| . \qquad (6.421)$$

By the inspection of Eq. 6.417, one may also note that the maximum response of a secondary system in the nonresonant modes under consideration is not, in general, proportional to its response when it is mounted directly on the ground. Rather, it is given by the product of an amplification factor, a distortion configuration, and a response spectrum ordinate, where the distortion configuration is a linear combination of the most significant modal distortions of the independent secondary system (the significance measured by the ratios r_c and r_j). In the cases, however, in which one of the complex frequencies of the independent secondary system is comparatively close to the complex frequency λ_{p_I} while all others are well separated from it (that is, when $r_J \doteq 1$ and $r_j \ll 1.0$ for $j \neq J$), $\{X_s\}^{(r)}$ may be approximated as

$$\{X_s\}^{(r)} = \text{A.F. } \{d\phi\}^{(J)} \, SD(\omega_{p_I}, \xi_{p_I}) \tag{6.422}$$

where $\{d\phi\}^{(J)}$ is the vector of modal distortions corresponding to the close secondary frequency.

To demonstrate that Eq. 6.417 converges to the corresponding one derived in Chapter 4 for systems with proportional damping, one may observe that for an undamped system*

$$\delta_j = \frac{\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}}{\omega_{p_I} - \omega_{s_j}} = 0 \, , \tag{6.423}$$

and consequently by setting $\delta_j = 0$ in Eqs. 6.418 through 6.420 one obtains

$$r_c = \frac{d\Phi(I)}{A_o(J)} \tag{6.424}$$

$$r_j = \frac{A_o(j)}{A_o(J)} \tag{6.425}$$

---

* Actually, Eq. 6.417 converges to the one for proportional damping whenever the damping matrices of the independent primary and secondary components of an assembled system are proportional, with the same proportionality constants, to any linear combination of their respective mass and stiffness matrices; for convenience, however, the demonstration is here restricted to the undamped case. In particular, notice that Eq. 6.423 also holds for independent components with damping matrices proportional to their respective mass matrices.

$$A.F. = \frac{A_o(J)}{1 + A_o^2(J)\gamma_{IJ}} \quad , \tag{6.426}$$

which after being substituted into Eq. 6.417 lead, precisely, to Eq. 4.115, the expression for $\{X_s\}^{(r)}$ in the case of proportional damping.

Similarly, it may be seen that whenever $\omega_{p_I} = \omega_{s_J}$ and $|\xi_{p_I} - \omega_{s_J}| = |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ the amplification factor given by Eq. 6.420 converges to the maximum of the corresponding amplification factor for resonant modes.* For, in such a case Eq. 6.298 yields

$$y_J^{(r)} \doteq i\,\frac{\gamma_I^{(r)}}{\sqrt{\gamma_{IJ}}} = \frac{e^{i\pi/2}}{\sqrt{\gamma_{IJ}}}\,\gamma_I^{(r)} \tag{6.427}$$

which together with Eq. 6.393 indicates that

$$|A_o(J)\cos\,\theta_J| = \frac{1}{\sqrt{\gamma_{IJ}}} \tag{6.428}$$

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*Observe that the limitation in the closeness between the frequencies ω_{p_I} and ω_{s_J} indicated by Eq. 6.398 is set only for the evaluation of $|y_j^{(r)}|$. The actual condition of resonance is, however, when $\omega_{p_I} = \omega_{s_J}$.

and that

$$\theta_J = -\pi/2 \ . \tag{6.429}$$

Therefore, if Eq. 6.420 is rewritten as

$$A.F. = [A_o(J)\cos\theta_J] \;/\; \{[\cos 2\theta_J + A_o^2(J)\ \cos^2\theta_J\ \gamma_{IJ}]^2 +$$

$$+ [\frac{\xi_{p_I}\omega_{p_I} - \xi_{s_J}\omega_{s_J}}{\omega_{p_I}} A_o^2(J)\cos^2\theta_J\gamma_{IJ} + \sin 2\theta_J]^2\ \}^{1/2} \tag{6.430}$$

and Eqs. 6.428 and 6.429 are substituted, A.F. results as

$$A.F. = \frac{1}{\sqrt{\gamma_{IJ}}}\ \frac{\omega_{p_I}}{\xi_{p_I}\omega_{p_I} - \xi_{s_J}\omega_{s_J}} \tag{6.431}$$

from which, since by assumption $\omega_{p_I} = \omega_{s_J}$ and $|\xi_{p_I} - \omega_{s_J}| = |\Phi_o(I,J)\sqrt{\gamma_{IJ}}\,|$, it may be seen that

$$A.F. = \frac{1}{|\Phi_o(I,J)\ \gamma_{IJ}|} = \frac{|\Phi_o(I,J)|}{(\xi_{p_I} - \xi_{s_J})^2} \ . \tag{6.432}$$

Under the conditions specified above, the amplification factor in Eq. 6.417 converges thus to the one that is implicitly indicated by Eq. 6.347.

Finally, for those systems in which their frequencies ω_{p_I} and ω_{s_J} are well separated from each other (that is, when $A_0^2(J)\ \gamma_{IJ} << 1.0$) the terms multiplied by $A_0^2(J)\ \gamma_{IJ}$ in Eq. 6.420 are relatively small, and hence by neglecting them the amplification factor in Eq. 6.417 may be approximated as

$$A.F. = \frac{A_0(J)}{\sqrt{1 + \delta_J^2}} \quad . \tag{6.433}$$

The maximum modal secondary distortions of such systems may be therefore expressed as

$$\{X_s\}^{(r)} = \frac{A_0(J)}{\sqrt{1 + \delta_J^2}} \left[r_c \{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} r_j \{d\phi\}^{(j)} \right] SD(\omega_{p_I}, \xi_{p_I}) . \tag{6.434}$$

In comparing this equation with Eq. 4.115, note the factor $1/\sqrt{1 + \delta_J^2}$ that differentiates the modal responses of systems with nonproportional damping from those of systems with proportional damping.

<u>Case II: $\lambda_r \doteq \lambda_{s_J}$</u>

As noted in Sec. 6.4, the complex natural frequencies of an assembled system in some of its nonresonant modes are close to the complex natural frequencies of its independent secondary system, and thus those complex natural frequencies may be approximated as

$$\lambda_r = \lambda_{s_J} \quad . \tag{6.435}$$

Under this approximation, therefore, the $Y_i^{(r)}$ and $y_j^{(r)}$ factors of Eqs. 6.151 and 6.152 for such nonresonant modes become

$$Y_i^{(r)} = \frac{\hat{\Phi}_r(i)\ (\lambda_{s_J} - \lambda_{p_I})(\lambda_{s_J} - \bar{\lambda}_{p_I})\ M_I^*}{\hat{\Phi}_r(I)\ (\lambda_{s_J} - \lambda_{p_i})(\lambda_{s_J} - \bar{\lambda}_{p_i})\ M_i^*}\ Y_I^{(r)} \tag{6.436}$$

$$y_0^{(r)} = \sum_{i=1}^{N_p} \Phi_k(i)\ Y_i^{(r)} \tag{6.437}$$

$$y_c^{(r)} = \frac{1}{f_{cc}} \sum_{i=1}^{N_p} d\Phi(i) Y_i^{(r)} \tag{6.438}$$

$$y_j^{(r)} = \frac{-\lambda_{s_J}^2}{(\lambda_{s_J} - \lambda_{s_j})(\lambda_{s_J} - \bar{\lambda}_{s_j})}\ \hat{y}_0^{(r)},\ j \neq J\ . \tag{6.439}$$

From the inpsection of Eq. 6.436, it is thus evident that in this case there may not be a dominant $Y_i^{(r)}$ factor in Eq. 6.151 because there may not be a primary frequency λ_{p_i} distinctively close to the frequency λ_{s_J}. In contrast, although Eq. 6.439 is inappropriate to compute a $y_j^{(r)}$ factor for which $\lambda_{s_j} = \lambda_{s_J}$, it is apparent from the analysis of this equation that such a $y_j^{(r)}$ factor is always so large that all the other $y_j^{(r)}$ factors in Eq. 6.152 become negligibly small. On these premises, then, the complex mode shape of an assembled system whose complex natural frequency is nearly equal to the complex frequency λ_{s_J} of its independent secondary system may be expressed as

$$\{w_p\}^{(r)} = \sum_{i=1}^{N_p} \{\Phi\}^{(i)} \; Y_i^{(r)} \tag{6.440}$$

$$\{w_s\}^{(r)} = \{\phi\}^{(J)} \; y_J^{(r)} \; , \tag{6.441}$$

and hence the corresponding vector of unit-participation-factor modal secondary distortions may be expressed as

$$\{dw_s'\}^{(r)} = \gamma_r \; y_J^{(r)} \; \{d\phi\}^{(J)} \tag{6.442}$$

from which one obtains that

$$\{|dw_s'|\}^{(r)} = |\gamma_r| \, | \, y_J^{(r)}| \{|d\phi|\}^{(J)} \; . \tag{6.443}$$

According to Eq. 5.154 and recalling that the sign of the maximum response of a system in its nonresonant modes is of no importance, the associated vector of maximum modal distortions may be therefore expressed as

$$\{X_s\}^{(r)} = 2|\gamma_r| \, | \, y_J^{(r)}| \{d\phi\}^{(J)} \omega_{s_J} \; SD(\omega_{s_J}, \; \xi_{s_J}) \; . \tag{6.444}$$

Evidently, a simplified formula for the maximum modal response of secondary systems in the nonresonant modes under study may also be obtained if approximate relationships are derived for the absolute values of the participation factors γ_r and the $y_J^{(r)}$ factors. To derive such a simplified formula, then, these approximate relationships are derived next.

$\underline{y_J^{(r)}\text{ factor}}$. As pointed out earlier, Eq. 6.156 is unbounded when λ_r is approximated by λ_{s_J}. Consequently, to develop a simplified expression for $y_J^{(r)}$, it is necessary to derive first an alternative relationship for the calculation of this factor. Hence, by following the procedure employed in Chapter 4 for systems with proportional damping this alternative relationship is here obtained as follows:

If in the original equations that led to Eqs. 6.151 through 6.157, i.e., Eqs. 6.18 and 6.64, it is assumed that all the Z_i' and z_j' factors have been, with the exception of z_J' and $z_{\bar{J}}'$, previously determined, then the system of equations described by Eqs. 6.18 and 6.64 may be reduced to

$$(\lambda - \lambda_{p_I})\, A_I^* \, Z_I e^{\lambda t} = \Phi_1(I)R_1(t) + \Phi_3(I)R_3(t) \tag{6.445}$$

$$\sum_j a_{0j}\dot{z}_j' + \sum_j a_{0\bar{j}}\, \dot{z}_{\bar{j}}' + \sum_j b_{0j} z_j' + \sum_j b_{0\bar{j}}\, z_{\bar{j}}' = -[R_1(t) + R_3(t)] \tag{6.446}$$

$$\sum_j a_{cj}\dot{z}_j' + \sum_j a_{c\bar{j}}\, \dot{z}_{\bar{j}}' + \sum_j b_{cj} z_j' + \sum_j b_{c\bar{j}}\, z_{\bar{j}}' = -\, f_{cc}\, R_3(t) \tag{6.447}$$

where $\sum_j$ indicates a summation for $j = 0, 1, \ldots, N_s, c$, which by means of Eqs. 6.31, 6.74, and 6.75 may also be written as

$$2i\omega_{p_I}'(\lambda - \lambda_{p_I})\, M_I^* \, Z_I \, e^{\lambda t} = \Phi_1\,(I)R_1(t) + \Phi_3(I)R_3(t) \tag{6.448}$$

$$\sum_j [(\lambda a_{0j} + b_{0j}) z_j + (\lambda a_{0\bar{j}} + b_{0\bar{j}} z_{\bar{j}}] e^{\lambda t} = - [R_1(t) + R_3(t)] \tag{6.449}$$

$$\sum_j [(\lambda a_{cj} + b_{cj}) z_j + (\lambda a_{c\bar{j}} + b_{c\bar{j}}) z_{\bar{j}}] e^{\lambda t} = - f_{cc} R_3(t) . \tag{6.450}$$

Observe, however, that by virtue of Eqs. 6.101 through 6.104, 6.107 through 6.110, and 6.115, one has that for $k = 0$, c, and $j = 0, 1, \ldots, N_s$, c

$$(\lambda a_{kj} + b_{kj}) z_j + (\lambda a_{k\bar{j}} + b_{k\bar{j}}) z_{\bar{j}} =$$

$$[(\lambda - \lambda_{s_k})(\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}}) + \lambda\lambda_{s_k}(z_j + z_{\bar{j}})] m_{kj}, \tag{6.451}$$

and thus, since according to Eqs. 6.95 and 6.96. and Eqs. 6.133, 6.135, End 6.140 in combination with Eqs. 6.143 and 6.144, $(z_j + z_{\bar{j}})$ and $(\lambda_{s_j} z_j + \bar{\lambda}_{s_j} z_{\bar{j}})$ may be expressed as

$$z_j + z_{\bar{j}} = y_j \tag{6.452}$$

$$\lambda_{s_j} z_j + \bar{\lambda}_{s_{\bar{j}}} = \lambda y_j , \tag{6.453}$$

one may write

$$(\lambda a_{kj} + b_{kj})\ z_j + (\lambda a_{k\bar{j}} + b_{k\bar{j}})\ z_{\bar{j}} = \lambda^2\ y_j\ m_{kj}\ . \qquad (6.454)$$

Therefore, Eqs. 6.449 and 6.450 may also be put into the form

$$\lambda^2\ e^{\lambda t} \sum_j y_j\ m_{0j} = -\ [R_1(t) + R_3(t)] \qquad (6.455)$$

$$\lambda^2\ e^{\lambda t} \sum_j y_j\ m_{cj} = f_{cc}\ R_3(t)\ . \qquad (6.456)$$

If now $R_1(t)$ and $R_3(t)$ are solved respectively from Eqs. 6.455 and 6.456 and substituted in Eq. 6.448, one arrives to

$$2i\ \omega'_{p_I}\ (\lambda - \lambda_{p_I})\ Z_I + \lambda^2 \sum_j [\Phi_1(I) + \frac{m_{cj}}{m_{0j}}\ \frac{d\Phi(I)}{f_{cc}}]\ \frac{m_{0j}}{M_I^*}\ y_j = 0 \qquad (6.457)$$

which, by considering that $y_J^{(r)}$ is a factor distinctively larger than the other $y_j^{(r)}$ factors and that thus for small ratios $m_{0j}/\ M_I^*$ all its terms multiplied by these other y_j factors are comparatively small, may be approximated as

$$2i\ \omega'_{p_I}\ (\lambda - \lambda_{p_I})\ Z_I + \lambda^2\ [\Phi_1(I) + \frac{m_{cJ}}{m_{0J}}\ \frac{d\Phi(I)}{f_{cc}}]\frac{m_{0J}}{M_I^*}\ y_J = 0\ . \qquad (6.458)$$

Hence, solving for y_J and recalling that $m_{OJ} = m_J^*$, $m_{cJ} = \beta_J f_{cc} m_J^*$, and $\Phi_o(I,J) = \Phi_1(I) + \beta_J d\Phi(I)$, one obtains

$$y_J = - \frac{2i\, \omega'_{p_I} (\lambda - \lambda_{p_I})}{\lambda^2 \Phi_o(I,J)\, \gamma_{IJ}} Z_I \tag{6.459}$$

which by substitution of Eq. 6.42 and generalized for a nonresonant mode with complex frequency λ_r leads to

$$y_J^{(r)} = - \frac{(\lambda_r - \lambda_{p_I})(\lambda_r - \bar{\lambda}_{p_I})}{\lambda_r^2 \Phi_o(I,J)\, \gamma_{IJ}} Y_I^{(r)} \quad . \tag{6.460}$$

Conceivably, if in accordance with Eq. 6.435 λ_r is replaced by λ_{s_J} , an alternative expression for $y_J^{(r)}$ is

$$y_J^{(r)} = - \frac{(\lambda_{s_J} - \lambda_{p_I})(\lambda_{s_J} - \bar{\lambda}_{p_I})}{\lambda_{s_J}^2 \Phi_o(I,J)\gamma_{IJ}} Y_I^{(r)} \ . \tag{6.461}$$

On the basis of this equation, a simplified formula for $|y_J^{(r)}|$ may then be obtained as follows:

If λ_{s_J} and λ_{p_I} are written explicitly in terms of their real and imaginary parts, and if it is considered that for small damping ratios $\lambda_{s_J} - \bar{\lambda}_{p_I} \doteq i\,(\omega_{p_I} + \omega_{s_J})$ and $-\lambda_{s_J}^2 \doteq \omega_{s_J}^2$, Eq. 6.461 may be approximated as

$$y_J^{(r)} = - \frac{\omega_{p_I} + \omega_{s_J}}{\omega_{s_J}^2} \frac{(\omega_{s_J} - \omega_{p_I}) + i\,(\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I})}{\Phi_o(I,J)\gamma_{IJ}} \; Y_I^{(r)} \qquad (6.462)$$

which in polar form may be expressed as

$$y_J^{(r)} = - \frac{\omega_{p_I} + \omega_{s_J}}{\omega_{s_J}^2} \frac{\sqrt{(\omega_{s_J} - \omega_{p_I})^2 + (\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I})^2}}{\Phi_o(I,J)\gamma_{IJ}} \; e^{i\theta_I} \, Y_I^{(r)} \qquad (6.463)$$

or as

$$y_J^{(r)} = - \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{s_J}^2 \, \Phi_o(I,J)\gamma_{IJ}} \; \frac{e^{i\theta_I}}{\cos\theta_I} \; Y_I^{(r)} \qquad (6.464)$$

where θ_I is such that for $i = I$

$$\tan\theta_i = \frac{\xi_{s_J}\omega_{s_J} - \xi_{p_i}\omega_{p_i}}{\omega_{s_J} - \omega_{p_i}} \, . \qquad (6.465)$$

Then, in terms of the parameter $B_o(i)$ defined by Eq. 6.126, i.e., if

$$B_o(i) = \frac{\Phi_o(i, J)\ \omega_{s_J}^2}{\omega_{p_i}^2 - \omega_{s_J}^2}\ , \tag{6.466}$$

$y_J^{(r)}$ may be expressed as

$$y_J^{(r)} = \frac{e^{i\theta_I}}{B_o(I)\cos\ \theta_I\ \gamma_{IJ}}\ Y_I^{(r)} \tag{6.467}$$

and thus in a simplified form $|y_J^{(r)}|$ may be written as

$$|y_J^{(r)}| = \frac{|Y_I^{(r)}|}{|B_o(I)\cos\ \theta_I\ \gamma_{IJ}|}\ . \tag{6.468}$$

As in the previous case, notice that when λ_{s_J} approaches λ_{p_I} , a nonresonant mode becomes a resonant one and consequently the approximation indicated by Eq. 6.435 is not longer valid. For such nonresonant modes, therefore, Eq. 6.467 is not valid, either. To establish its range of validity, then, it may be observed from Eq. 6.461 that $|y_J^{(r)}|$ reaches its minimum when λ_{s_J} gets the closest to λ_{p_I} and that for all other relations between λ_{s_J} and λ_{p_I} $|y_J^{(r)}|$ should be greater than this minimum. Understandably, since by setting $\omega_{s_J} = \omega_{p_I}$ and $\xi_{s_J} = \xi_{p_I}$ in Eq. 6.298 such a minimum is given by

$$|y_J^{(r)}|_{min} = \frac{|Y_I^{(r)}|}{\sqrt{\gamma_{IJ}}} \quad , \tag{6.469}$$

one has that for all cases

$$\frac{1}{|B_o(I)\cos\,\theta_I\;\gamma_{IJ}|} \geq \frac{1}{\sqrt{\gamma_{IJ}}} \tag{6.470}$$

and as a result Eq. 6.467 is applicable if

$$\left| \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{s_J}^2} \sec\,\theta_I \right| \geq \left| \Phi_o(I,J)\,\sqrt{\gamma_{IJ}} \right| . \tag{6.471}$$

Participation Factors. If, as discussed above, it is considered that the frequency λ_r may be approximated by λ_{s_J} and that $y_J^{(r)}$, the $y_j^{(r)}$ factor corresponding to λ_{s_J}, is considerably larger than any other $y_j^{(r)}$ factor, then according to Eq. 6.274 the complex participation factors for the nonresonant modes under consideration may be approximated as

$$\gamma_r = \frac{1}{2}\,\frac{B_r Y_I^{(r)} + y_J^{(r)}\;\gamma_{IJ}}{i\omega_{s_J}\;Y_I^{(r)^2} + [-(\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I}) + i\omega_{s_J}]\;y_J^{(r)^2}\;\gamma_{IJ}} \tag{6.472}$$

where, as defined by Eq. 6.275, B_r is of the form

$$B_r = \frac{\sum_{i=1}^{N_p} M_i^* \, Y_i^{(r)}}{M_I^* \, Y_I^{(r)}} \quad . \tag{6.473}$$

In developing a simplified relationship for γ_r , one may note, thus, that since in this case there may not be a predominant $Y_i^{(r)}$ factor in the summation of this equation, the parameter B_r may not be close to and therefore may not be approximated by unity. Consequently, for an accurate evaluation of the complex participation factors an approximate expression for this parameter B_r is here obtained as follows:

By definition, the parameters $\hat{\Phi}_r(I)$ in the expression for $Y_I^{(r)}$ (Eq. 6.436) are given by (see Eq. 4.54)

$$\hat{\Phi}_r(i) = \Phi_1(i) + \eta_r \, \Phi_3(i) \tag{6.474}$$

where

$$\eta_r = \frac{R_3(t)}{R_1(t)} \quad . \tag{6.475}$$

But if $R_1(t)$ and $R_3(t)$ are solved from Eqs. 6.455 and 6.456 one has that

$$R_1(t) = - R_3(t) - \lambda^2 \, e^{\lambda t} \sum_j y_j \, m_{0j} \tag{6.476}$$

$$R_3(t) = -\frac{1}{f_{cc}} \lambda^2 e^{\lambda t} \sum_j y_j m_{cj} \tag{6.477}$$

from which the ratio $R_1(t) / R_3(t)$ may be written as

$$\frac{R_1(t)}{R_3(t)} = -1 + f_{cc} \frac{\sum_j y_j m_{0j}}{\sum_j y_j m_{cj}} \tag{6.478}$$

and, by neglecting all the y_j factors for which $j \neq J$, approximated as

$$\frac{R_1(t)}{R_3(t)} = -1 + f_{cc} \frac{m_{0J}}{m_{cJ}} . \tag{6.479}$$

Therefore, since by virtue of Eq. 4.36 and recalling that $m_{0J} = m_J^*$ Eq. 6.479 may also be put into the form

$$\frac{R_1(t)}{R_3(t)} = -1 + \frac{1}{\beta_J} , \tag{6.480}$$

η_r may be expressed as

$$\eta_r = \frac{R_3(t)}{R_1(t)} = \frac{\beta_J}{1 - \beta_J} , \tag{6.481}$$

and as a result $\hat{\Phi}_r(i)$ may be written as

$$\hat{\Phi}_r(i) = \frac{\Phi_1(i) + \beta_J\, d\Phi(i)}{1 - \beta_J} = \frac{\Phi_o(i, J)}{1 - \beta_J} . \qquad (6.482)$$

On the basis of this equation and Eqs. 6.473 and 6.436, B_r may be thus expressed as

$$B_r = \frac{(\lambda_{s_J} - \lambda_{p_I})(\lambda_{s_J} - \bar{\lambda}_{p_I})}{\Phi_o(I,J)} \sum_{i=1}^{N_p} \frac{\Phi_o(i,J)}{(\lambda_{s_J} - \lambda_{p_i})(\lambda_{s_J} - \bar{\lambda}_{p_i})} \qquad (6.483)$$

which, by writing λ_{s_J} and λ_{p_i} in terms of their real and imaginary parts and by considering that $\lambda_{s_J} - \bar{\lambda}_{p_i} \doteq i(\omega_{s_J} + \omega_{p_i})$, may be approximated as

$$B_r = \left\{ \frac{(\omega_{s_J} + \omega_{p_I}) \left[(\omega_{s_J} - \omega_{p_I}) + i\, (\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I}) \right]}{\Phi_o(I,J)} \right\} .$$

$$\cdot \sum_{i=1}^{N_p} \frac{\Phi_o(i,J)}{(\omega_{s_J} + \omega_{p_i}) \left[(\omega_{s_J} - \omega_{p_i}) + i\, (\xi_{s_J}\omega_{s_J} - \xi_{p_i}\omega_{p_i}) \right]} , \qquad (6.484)$$

and hence by virtue of Eq. 6.462 and by expressing the terms of the summation in polar form one may write

$$B_r = -\frac{y_J^{(r)}\gamma_{IJ}}{\gamma_I^{(r)}} \sum_{i=1}^{N_p} \frac{\Phi_o(i, J)\,\omega_{s_J}^2}{\omega_{s_J}^2 - \omega_{p_i}^2} \cos\theta_i\, e^{-i\theta_i} \tag{6.485}$$

which in terms of the parameter $B_o(i)$ defined by Eq. 6.466 may also be expressed as

$$B_r = \frac{y_J^{(r)}\,\gamma_{IJ}}{\gamma_I^{(r)}} \sum_{i=1}^{N_p} B_o(i)\cos\,\theta_i\, e^{-i\theta_i}\ . \tag{6.486}$$

In the light of Eqs. 6.472 and 6.486, the complex participation factors γ_r may be consequently written as

$$\gamma_r = \frac{1/y_J^{(r)}}{2i\omega_{s_J}} \frac{1 + \sum_{i=1}^{N_p} B_o(i)\cos\,\theta_i e^{-i\theta_i}}{(1 + \gamma_I^{(r)2}/y_J^{(r)2}\,\gamma_{IJ}) + i\,\dfrac{\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I}}{\omega_{s_J}}} \tag{6.487}$$

or, since by means of Eq. 6.467 one has that

$$1 + \frac{\gamma_I^{(r)2}}{y_J^{(r)2}\ \gamma_{IJ}} = 1 + B_o^2\ (I)\ \cos^2\theta_I\ \gamma_{IJ}\ e^{-i2\theta_I}\ , \tag{6.488}$$

as

$$\gamma_r = \frac{1/y_J^{(r)}}{2i\omega_{s_J}} \frac{1 + \sum_{i=1}^{N_p} B_o(i)\cos\theta_i e^{-i\theta_i}}{1 + B_o^2(I)\gamma_{IJ}\cos^2\theta_I\, e^{-i2\theta_I} + i(\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I})/\omega_{s_J}} , \qquad (6.489)$$

from which it is easy to see that

$$|\gamma_r| = \frac{|1/y_J^{(r)}|}{2\omega_{s_J}} \left\{ \frac{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\,\delta_i]^2}{\left[1 + B_o'^2(I)\gamma_{IJ}(1 - \delta_I^2)\right]^2 + \left[\frac{\omega_{s_J} - \omega_{p_I}}{\omega_{s_J}} + 2B_o'^2(I)\gamma_{IJ}\right]^2 \delta_I^2} \right\}^{1/2}$$

(6.490)

where

$$B_o'(i) = \frac{B_o(i)}{1 + \delta_i^2} \qquad (6.491)$$

$$\delta_i = \tan\theta_i = \frac{\xi_{s_J}\omega_{s_J} - \xi_{p_i}\omega_{p_i}}{\omega_{s_J} - \omega_{p_i}} . \qquad (6.492)$$

Maximum Modal Secondary Distortions. In view of Eqs. 6.444 and 6.490, the maximum distortions of a secondary system in the nonresonant modes herein being considered may be thus expressed as

$$\{X_s\}^{(r)} = \text{A.F. } \{d\phi\}^{(J)} \, SD(\omega_{s_J}, \xi_{s_J}) \quad , \tag{6.493}$$

where

$$\text{A.F.} = \left\{ \frac{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2}{[1 + B_o'^2(I)\gamma_{IJ}(1 - \delta_I^2)]^2 + [\frac{\omega_{s_J} - \omega_{p_I}}{\omega_{s_J}} + 2B_o'^2(I)\gamma_{IJ}]^2 \, \delta_I^2} \right\}^{1/2} \tag{6.494}$$

which in accordance with Eqs. 6.471 and 6.492 is valid if

$$\left| \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{s_J}^2} \right| \sqrt{1 + \delta_I^2} \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}\,| . \tag{6.495}$$

As the corresponding equation for nonresonant modes with a primary frequency, notice that Eq. 6.493 converges to the expressions for the particular cases previously studied. Thus, for example, since for an undamped system δ_I is equal to zero, it is easy to see that for a system with proportional damping Eq. 6.494 yields

$$\text{A.F.} = \frac{1 + \sum_{i=1}^{N_p} B_o(i)}{1 + B_o^2(I)\gamma_{IJ}} \tag{6.496}$$

and that consequently Eq. 6.493 coincides with the corresponding equation derived for systems with proportional damping (see Eq. 4.140). In like manner, by virtue of Eqs. 6.427 and 6.467 one has that whenever $\omega_{p_I} = \omega_{s_J}$ and $|\xi_{p_I} - \xi_{s_J}| = |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ (conditions for the maximum amplification factor of a resonant mode)

$$|B_o(I)\cos\theta_I| = \frac{1}{\sqrt{\gamma_{IJ}}} \tag{6.497}$$

$$\theta_I = \pi/2 \tag{6.498}$$

$$|B_o(I)\cos\theta_I| >> |B_o(i)\cos\theta_i| \text{ for } i \neq I , \tag{6.499}$$

and hence, if Eq. 6.494 is rewritten as

$$\text{A.F.} = \{[1 + \sum_{i=1}^{N_p} B_o(I)\cos\theta_i\ \cos\theta_i]^2 + [\sum_{i=1}^{N_p} B_o(i)\cos\theta_i\ \sin\theta_i]^2\}^{1/2} \Big/$$

$$\{[1 + B_o^2(I)\cos^2\theta_I\gamma_{IJ}\cos 2\theta_I]^2 + [\frac{\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I}}{\omega_{s_J}} + B_o^2(I)\cos^2\theta_I\gamma_{IJ}\sin 2\theta_J]^2\}^{1/2} , \tag{6.500}$$

it may be seen that for such a case A.F. results as

$$\text{A.F.} = \frac{\sqrt{1 + 1/\gamma_{IJ}}}{(\xi_{s_J}\omega_{s_J} - \xi_{p_I}\omega_{p_I})/\omega_{s_J}} \tag{6.501}$$

or, since by assumption $\omega_{p_I} = \omega_{s_J}$ and $|\xi_{p_I} - \xi_{s_J}| = |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, and for small mass ratios $1 + 1/\gamma_{IJ} \doteq 1/\gamma_{IJ}$, as

$$A.F. = \frac{1}{|\Phi_o(I,J)\gamma_{IJ}|} = \frac{|\Phi_o(I,J)|}{(\xi_{p_I} - \xi_{s_J})^2} \tag{6.502}$$

which according to Eq. 6.347 is the maximum amplification factor for resonant modes.

Notice also that for systems whose nonresonant frequencies are well separated from one another (that is, systems for which $B_o^2(I)\gamma_{IJ} << 1.0$) the denominator of the right-hand side of Eq. 6.494 is, for small damping ratios, very close to unity. As a result, their amplification factors may be approximated as

$$A.F. = \sqrt{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2} \quad , \tag{6.503}$$

and consequently their maximum modal distortions may be calculated by

$$\{X_s\}^{(r)} = \sqrt{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2}\ \{d\phi\}^{(J)} SD(\omega_{s_J}, \xi_{s_J}). \tag{6.504}$$

In distinction with the relationship obtained in Case I, observe that this expression is different from the corresponding one for systems with proportional damping (Eq. 4.141) not only because of the factors $(1/1 + \delta_i^2)$ that multiply the parameters $B_0(i)$ (that is, the use of $B_0'(i)$ instead of $B_0(i)$) but also because of the addition of the extra terms indicated by the second summation within its radical.

6.8 Simplified Modal Correlation Factors

It has been shown in the preceding chapter that the combination of modes of systems with nonproportional damping may be attained by a rule of the form

$$X_{i_{max}} = \sqrt{\sum_{r=1}^{N} X_{i_r}^2 + \sum_{m=1}^{N} \sum_{\substack{n=1 \\ m \neq n}}^{N} \alpha_{mn} X_{i_m} X_{i_n}} \tag{6.505}$$

in which according to Eq. 5.182 the modal correlation factors α_{mn} are given by

$$\alpha_{mn} = 2\mathrm{Re}\left[\frac{w_i'(m)w_i'(n)}{\lambda_m' + \lambda_n'} + \frac{w_i'(m)\bar{w}_i'(n)}{\lambda_m' + \bar{\lambda}_n'}\right] \frac{\sqrt{\xi_m'\omega_m \; \xi_n'\omega_n}}{|w_i'(m)|\,|w_i'(n)|} \quad , \tag{6.506}$$

where for r = m, n

$$\lambda_r' = -\xi_r'\omega_r + i\omega_r\sqrt{1 - \xi_r'^2} = -\xi_r'\omega_r + i\omega_r \tag{6.507}$$

and

$$\xi_r' = \xi_r + \frac{2}{\omega_r s_r} \quad , \tag{6.508}$$

or, if the responses X_{i_r} in Eq. 6.505 represent the maximum distortions of a secondary system, the response of interest in this work, by

$$\alpha_{mn} = 2\,\mathrm{Re}\left[\frac{dw'_{s_i}(m)\; dw'_{s_i}(n)}{\lambda'_m + \lambda'_n} + \frac{dw'_{s_i}(m)\overline{dw'_{s_i}}(n)}{\lambda'_m + \overline{\lambda}'_n}\right] \frac{\sqrt{\xi'_m\,\omega_m\,\xi'_n\,\omega_n}}{|dw'_{s_i}(m)|\,|dw'_{s_i}(n)|} \qquad (6.509)$$

where, as before,

$$dw'_{s_i}(r) = w'_{s_i}(r) - w'_{s_{i-1}}(r)\ , \quad r = m,n\ . \qquad (6.510)$$

Thus, the maximum response of a secondary system may be determined by means of Eqs. 6.505 and 6.509 and the relationships derived in the preceding sections to compute its maximum modal distortions. One may note, however, that Eq. 6.509 is rather complicated to be used in the simplified method herein being developed. In this section, therefore, a simple approximate expression is derived to calculate the modal correlation factors in Eq. 6.505 based on Eq. 5.509 and the approximate formulas obtained in Secs. 6.3 and 6.6 for the resonant complex frequencies of assembled systems and the modal distortions of their secondary systems.

In the development of this simple approximate expression, one may then note the following:

1) Since the natural frequencies of an assembled system in two of its resonant modes are close to a resonant frequency ω_o and hence close to each other (that is, $\omega_m = \omega_n = \omega_o$), then in their polar form the sums $\lambda_m' + \lambda_n'$ and $\lambda_m' + \bar{\lambda}_n'$ in Eq. 6.509 for such resonant modes may be written approximately as

$$\lambda_m' + \lambda_n' \doteq \omega_o \sqrt{4 + (\xi_m' + \xi_n')^2}\; e^{i\theta_1} \tag{6.511}$$

$$\lambda_m' + \bar{\lambda}_n' \doteq \omega_o (\xi_m' + \xi_n')\, e^{i\theta_2} \quad , \tag{6.512}$$

where θ_1 and θ_2 are respectively the arguments of $\lambda_m' + \lambda_n'$ and $\lambda_m' + \bar{\lambda}_n'$. For small damping ratios, therefore, $1/|\lambda_m' + \bar{\lambda}_n'|$ is always much greater than $1/|\lambda' + \lambda_n'|$, and consequently α_{mn} may be approximated without much error as

$$\alpha_{mn} = 2 \operatorname{Re}\left| \frac{dw_{s_i}'(m)\, \overline{dw}_{s_i}'(n)}{\lambda_m' + \bar{\lambda}_n'} \right| \frac{\sqrt{\xi_m'\omega_m\, \xi_n'\omega_n}}{|dw_{s_i}'(m)|\,|dw_{s_i}'(n)|} \quad . \tag{6.513}$$

2) The approximate expression for $dw_{s_i}'(r)$ derived in Sec. 6.6 is of the form (see Eq. 6.285)

$$dw_{s_i}'(r) = \gamma_r\; y_J^{(r)}\, d\phi(J) \; . \tag{6.514}$$

Hence, the ratios $dw'_{s_i}(m)/|dw'_{s_i}(m)|$ and $\overline{dw}'_{s_i}(n)/|dw'_{s_i}(n)|$ in Eq. 6.513 may be expressed as

$$\frac{dw'_{s_i}(m)}{|dw'_{s_i}(m)|} = e^{\arg[\gamma_m y_J^{(m)}]} \tag{6.515}$$

$$\frac{\overline{dw}'_{s_i}(n)}{|dw'_{s_i}(n)|} = e^{\arg[\bar{\gamma}_n \bar{y}_J^{(n)}]} \tag{6.516}$$

where "arg" stands for "the argument of".

3) Separate modal correlation factors should be derived for the cases in which $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ and $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ because different expressions for $\gamma_r y_J^{(r)}$ were obtained in Sec. 6.6 for these two cases.

Under the above premises, the sought approximate correlation factors may then be determined for each of the aforementioned cases as follows:

<u>Case I:</u> $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$

According to Eq. 6.228, the complex frequencies λ_m and λ_n of the systems for which this inequality holds may be expressed as

$$\lambda_m = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\,\omega_o + \frac{\omega_o}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} + i\omega_o \tag{6.517}$$

$$\lambda_n = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o - \frac{\omega_o}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} + i\omega_o \qquad (6.518)$$

from which it may be seen that

$$\omega_m = \omega_n = \omega_o \qquad (6.519)$$

$$\xi_m = \frac{1}{2}(\xi_{p_I} + \xi_{s_J}) - \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \qquad (6.520)$$

$$\xi_n = \frac{1}{2}(\xi_{p_I} + \xi_{s_J}) + \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \quad , \qquad (6.521)$$

and that for such systems $\lambda_m' + \bar{\lambda}_n'$ results as

$$\lambda_m' + \bar{\lambda}_n' = -(\xi_{p_I} + \xi_{s_J})\omega_o + \frac{2}{\omega_o s_m} + \frac{2}{\omega_o s_n} = -(\xi_m' + \xi_n')\omega_o \qquad (6.522)$$

where ξ_m' and ξ_n' are given by Eqs. 6.520, 6.251, and 6.508. Similarly, in view of the discussion in Sec. 6.6 the product $\gamma_r\, y_J^{(r)}$ in the case under consideration may be written as (see Eq. 6.353)

$$\gamma_r\, y_J^{(r)} \doteq \frac{1}{4\omega_r}\; \frac{\Phi_o(I,J)}{\mp\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} - \frac{i}{2}\Phi_o^2(I,J)\gamma_{IJ}} \quad , \qquad (6.523)$$

and therefore in their polar form $\gamma_m \; y_J^{(m)}$ and $\bar{\gamma}_n \; \bar{y}_J^{(n)}$ may be expressed as

$$\gamma_m \, y_J^{(m)} = \frac{1}{4\omega_m} \frac{\Phi_o(I,J) \; e^{-i(\nu_{IJ} - \pi)}}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + \frac{1}{4}[\Phi_o^2(I,J)\gamma_{IJ}]^2}} \tag{6.524}$$

$$\bar{\gamma}_n \, \bar{y}_J^{(n)} = \frac{1}{4\omega_n} \frac{\Phi_o(I,J) \; e^{-i\nu_{IJ}}}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + \frac{1}{4}[\Phi_o^2(I,J)\gamma_{IJ}]^2}} \tag{6.525}$$

where ν_{IJ} is such that

$$\tan \nu_{IJ} = \frac{\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}}} \tag{6.526}$$

and whence it is easy to see that the arguments of $\gamma_m \; y_J^{(m)}$ and $\bar{\gamma}_n \; \bar{y}_J^{(n)}$ are

$$\arg\,(\gamma_m \, y_J^{(m)}) = -\,(\nu_{IJ} - \pi) \tag{6.527}$$

$$\arg\,(\bar{\gamma}_n \, \bar{y}_J^{(n)}) = -\,\nu_{IJ} \; . \tag{6.528}$$

Thus, by substitution of Eqs. 6.527 and 6.528 into Eqs. 6.515 and 6.516, and by substitution in turn of these last two equations and Eqs. 6.519 and 6.522 into Eq. 6.513, the modal correlation factors α_{mn} for the systems within this Case I may be written as

$$\alpha_{mn} = 2\,\mathrm{Re}\left[\frac{e^{-i(2\nu_{IJ} - \pi)}}{-(\xi_m' + \xi_n')\omega_o}\right]\sqrt{\xi_m'\,\omega_o\,\xi_n'\,\omega_o}$$

$$= 2 \cos 2\,\nu_{IJ}\,\frac{\sqrt{\xi_m'\xi_n'}}{\xi_m' + \xi_n'} \qquad (6.529)$$

which in view of Eq. 6.526 may also be expressed as

$$\alpha_{mn} = 2\,\tau_{IJ}\,\frac{\sqrt{\xi_m'\,\xi_n'}}{\xi_m' + \xi_n'} \qquad (6.530)$$

where

$$\tau_{IJ} = \cos 2\nu_{IJ} = \frac{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} - \frac{1}{4}[\Phi_o^2(I,J)\gamma_{IJ}]^2}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + \frac{1}{4}[\Phi_o^2(I,J)\gamma_{IJ}]^2} \;. \qquad (6.531)$$

Notice that when $|\xi_{p_I} - \xi_{s_J}|$ is not very close to $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}\,|$, the terms $[\Phi_o^2(I,J)\gamma_{IJ}]^2/4$ in this last equation become negligible and thus in such a case τ_{IJ} is very close to unity. On the other hand, if

$|\xi_{p_I} - \xi_{s_J}| = |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, τ_{IJ} is equal to negative one. It may be observed, therefore, that since ξ_m' and ξ_n' are always positive, α_{mn} may fluctuate between -1.0 and 1.0. Observe, however, that since by assumption $(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}$ is always positive, τ_{IJ}, and hence α_{mn}, is positive when

$$(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} > [\tfrac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}]^2 \,. \tag{6.532}$$

Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$

When $|\xi_{p_I} - \xi_{s_J}|$ is less than or equal to $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, the complex frequencies λ_m and λ_n are given by (see Eq. 6.228)

$$\lambda_m = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + i\,[\omega_o + \frac{\omega_o}{2}\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\,] \tag{6.533}$$

$$\lambda_n = -\frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + i\,[\omega_o - \frac{\omega_o}{2}\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\,] \,. \tag{6.534}$$

Therefore, in this case $\xi_m\omega_m$, $\xi_n\omega_n$, ω_m, and ω_n result as

$$\xi_m\omega_m = \xi_n\omega_n = \frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o \tag{6.535}$$

$$\omega_m = \omega_o + \frac{\omega_o}{2}\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} \tag{6.536}$$

$$\omega_n = \omega_o - \frac{\omega_o}{2} \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} \tag{6.537}$$

from which and by virtue of Eq. 6.508 one may write

$$\xi_m'\omega_m = \xi_n'\omega_n = \frac{1}{2}(\xi_{p_I} + \xi_{s_J})\omega_o + \frac{2}{s_o} = \xi_o'\omega_o \tag{6.538}$$

where

$$\xi_o = \frac{1}{2}(\xi_{p_I} + \xi_{s_J}) \tag{6.539}$$

and s_o is the earthquake duration corresponding to ξ_o, and

$$\lambda_m' + \lambda_n' = -2\,\xi_o'\omega_o + i\omega_o \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} \;. \tag{6.540}$$

In like manner, since according to Eq. 6.358 the product $\gamma_r\, y_J^{(r)}$ is given in this case by

$$\gamma_r\, y_J^{(r)} \doteq \frac{1}{4\omega_r} \frac{i\Phi_o(I,J)}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}} , \tag{6.541}$$

the ratios $\gamma_m\, y_J^{(m)} / |\gamma_m\, y_J^{(m)}|$ and $\bar{\gamma}_n\, \bar{y}_J^{(n)} / |\gamma_n\, y_J^{(n)}|$ may be expressed as

$$\frac{\gamma_m\, y_J^{(m)}}{|\gamma_m\, y_J^{(m)}|} = e^{i\pi/2} \tag{6.542}$$

$$\frac{\bar{\gamma}_n \bar{y}_J^{(n)}}{|\gamma_n y_J^{(n)}|} = -\mathrm{sgn}\ [-\tfrac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 + \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\ \ e^{-i\pi/2}. \tag{6.543}$$

Consequently, by means of Eqs. 6.513, 6.538, 6.540, 6.542 and 6.543 the modal correlation factors α_{mn} in the case under consideration may be written as

$$\alpha_{mn} = 2\ \mathrm{Re}\left[\frac{-\mathrm{sgn}\ [-\frac{1}{2}(\xi_{p_I}-\xi_{s_J})^2 + \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I}-\xi_{s_J})^2}]}{-2\xi_o'\,\omega_o + i\omega_o\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I}-\xi_{s_J})^2}}\right]\xi_o'\omega_o \tag{6.544}$$

or as

$$\alpha_{mn} = \mathrm{sgn}(\kappa_{IJ})\ \frac{1}{1 + \dfrac{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}{4\,\xi_o'^2}} \tag{6.545}$$

where

$$\kappa_{IJ} = -\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 + \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}\ . \tag{6.546}$$

Notice that in this case, too, α_{mn} may fluctuate between positive and negative values. Specifically, α_{mn} is positive when

$$\Phi_o^2\,(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 > [\, \tfrac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2\,]^2 \,. \tag{6.547}$$

Notice, however, that this condition is satisfied when $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}\,|$ is not very close to $|\xi_{p_I} - \xi_{s_J}|$, and hence α_{mn} is positive for most practical cases. Observe, finally, that for a system with proportional damping (i.e., when $\xi_{p_I} = \xi_{s_J}$) Eq. 6.545 coincides with Eq. 4.146.

6.9 Approximate Maximum Response

On the basis of the approximate expressions derived in Secs. 6.6 and 6.7 to calculate the modal responses of secondary systems and the rule established in Sec. 6.8 to combine these modal responses, a simple approximate procedure to compute their maximum response may be then developed as follows:

Consider Eqs. 6.505, 6.530, and 6.545. Since according to these last two equations the modal correlation factors of an assembled system of the kind herein being considered do not depend on the phase angles of their various masses, the maximum distortions of its secondary system may be expressed as

$$\{X_s\}_{max} = \sqrt{\sum_{r=1}^{N_p+N_s} \{X_s\}^{(r)^2} + \sum_{m=1}^{N_p+N_s} \sum_{\substack{n=1 \\ m \neq n}}^{N_p+N_s} \alpha_{mn}\, \{X_s\}^{(m)} \{X_s\}^{(n)}} \,. \tag{6.548}$$

Observe, however, that the modal correlation factors for the nonresonant modes of such an assembled system are negligible, and thus $\{X_s\}_{max}$ may be written in a simplified form as

$$\{X_s\}_{max} = \sqrt{\sum_{s=1}^{R/2} \{X_s\}^{(s)^2} + \sum_{r=1}^{N_p + N_s - R} \{X_s\}^{(r)^2}} \tag{6.549}$$

where $\{X_s\}^{(s)}$ represents the combined response of the secondary system in two resonant modes with adjacent natural frequencies of its assembled system and is given by

$$\{X_s\}^{(s)} = \left[\{X_s\}^{(m)^2} + \{X_s\}^{(n)^2} + 2\,\alpha_{mn}\{X_s\}^{(m)}\{X_s\}^{(n)}\right]^{1/2}, \tag{6.550}$$

$\{X_s\}^{(r)}$ denotes its maximum response in a nonresonant mode of the same assembled system, and R indicates the number of resonant modes in the assembled system in question. Observe, then, that in terms of the relations obtained in the foregoing section to compute the maximum modal distortions of secondary systems these vectors $\{X_s\}^{(s)}$ and $\{X_s\}^{(r)}$ may be expressed as follows.

Resonant Modes

According to Eqs. 6.363 and 6.371, the general expression for the maximum response of a secondary system in the resonant modes of its assembled system is

$$\{X_s\}^{(r)} = (A.F.)_r \ \{d\phi\}^{(J)} SD(\omega_r, \xi_r), \tag{6.551}$$

where $(A.F.)_r$ represents an amplification factor. Consequently, the vector $\{X_s\}^{(s)}$ defined by Eq. 6.550 may be written as

$$\{X_s\}^{(s)} = [(A.F.)_m^2 \, SD^2(\omega_m, \xi_m) + (A.F.)_n^2 \, SD^2(\omega_n, \xi_n) +$$

$$+ \; 2\alpha_{mn}(A.F.)_m \, (A.F.)_n \, SD(\omega_m, \xi_m) \, SD(\omega_n, \xi_n)]^{1/2} \, \{d\phi\}^{(J)} , \qquad (6.552)$$

and thus making the distinction between those cases for which $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J) \sqrt{\gamma_{IJ}}\,|$ and those for which $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J) \sqrt{\gamma_{IJ}}\,|$ such a vector may be expressed as follows:

<u>Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J) \gamma_{IJ}|$</u>. In this case, the amplification factor in Eq. 6.551 is given by (see Eq. 6.363)

$$(A.F.)_r = \frac{1}{2} \frac{sgn(\Delta_{IJ}) \, \Phi_o(I,J)}{\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + [\frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ}]^2}} , \qquad (6.553)$$

where

$$\Delta_{IJ} = \frac{1}{2} \Phi_o^2(I,J)\gamma_{IJ} \mp \sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} , \qquad (6.554)$$

and according to Eq. 6.530 the corresponding modal correlation factor is of the form

$$\alpha_{mn} = 2\,\tau_{IJ}\,\frac{\sqrt{\xi_m'\,\xi_n'}}{\xi_m' + \xi_n'} \tag{6.555}$$

where

$$\tau_{IJ} = \frac{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} - [\,\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}\,]^2}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + [\,\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}\,]^2}\ . \tag{6.556}$$

Observe, then, that the product $(A.F.)_m(A.F.)_n$ is negative and the parameter τ_{IJ} is positive when

$$(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} > [\,\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}]^2 \tag{6.557}$$

and that they are positive and negative, respectively, otherwise.

Since the sign of α_{mn} depends on the sign of τ_{IJ}, it may be seen, therefore, that independently of the signs of $(A.F.)_r$, $r = m,n$, and α_{mn} the product $\alpha_{mn}\ (A.F.)_m(A.F.)_n$ in Eq. 6.552 is always negative, and that consequently in the case under consideration this equation may be written as

$$\{X_s\}^{(s)} = |A.F.|_r\ [SD^2(\omega_m,\xi_m) + SD^2(\omega_n,\xi_n)\ - $$

$$- 2\,\alpha_{mn}\,SD(\omega_m,\xi_m)\,SD(\omega_n,\xi_n)\,]^{1/2}\,\{d\phi\}^{(J)} \tag{6.558}$$

where α_{mn} is redefined as

$$\alpha_{mn} = 2\ |\tau_{IJ}|\ \frac{\sqrt{\xi_m'\xi_n'}}{\xi_m' + \xi_n'} \quad . \tag{6.559}$$

Hence, by introducing a new parameter ρ_{mn} defined as

$$\rho_{mn} = \frac{1}{2}\left[\frac{SD(\omega_m,\ \xi_m)}{SD(\omega_n,\ \xi_n)} + \frac{SD(\omega_n,\xi_n)}{SD(\omega_m,\xi_m)}\right] \tag{6.560}$$

$\{X_s\}^{(s)}$ may be expressed as

$$\{X_s\}^{(s)} = |A.F.|_r \sqrt{2(\rho_{mn} - \alpha_{mn})}\ \ \{d\phi\}^{(J)} \sqrt{SD(\omega_m,\xi_m)\ SD(\omega_n,\xi_n)} \tag{6.561}$$

which in combination with Eq. 6.553 leads to

$$\{X_s\}^{(s)} = \sqrt{\frac{\frac{1}{2}(\rho_{mn} - \alpha_{mn})\ \Phi_o^2(I,J)}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + [\ \frac{1}{2}\ \Phi_o^2(I,J)\gamma_{IJ}]^2}} \quad .$$

$$\cdot\ \{d\phi\}^{(J)}\ \ \sqrt{SD(\omega_m,\ \xi_m)\ SD(\omega_n,\ \xi_n)} \tag{6.562}$$

where according to Eqs. 6.519 through 6.521 ω_m, ω_n, ξ_m, and ξ_n are given by

$$\omega_m = \omega_n = \omega_o \tag{6.563}$$

$$\xi_m = \xi_o - \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \qquad (6.564)$$

$$\xi_n = \xi_o + \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ}} \qquad (6.565)$$

in which

$$\xi_o = \frac{1}{2}(\xi_{p_I} + \xi_{s_J}) \quad . \qquad (6.566)$$

Notice that the parameter ρ_{mn} in Eq. 6.562 is very close to unity when the spectral ordinates $SD(\omega_m, \xi_m)$ and $SD(\omega_n, \xi_n)$ are nearly equal. Observe, however, that it may not be approximated by unity because, as indicated by Eqs. 6.564 and 6.565, ξ_m and ξ_n are usually far apart from each other and hence the spectral ordinates corresponding to those damping ratios may differ significantly.

Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\gamma_{IJ}|$. According to Eqs. 6.371 and 6.545, the amplification and modal correlation factors of the resonant modes of systems for which this condition is satisfied are

$$(A.F.)_r = \frac{1}{2}\,\frac{\Phi_o(I,J)}{\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 \pm \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}} \qquad (6.567)$$

$$\alpha_{mn} = \text{sgn}\,(\kappa_{IJ}) \frac{1}{1 + \dfrac{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}{4\xi_o'^2}} \tag{6.568}$$

where

$$\kappa_{IJ} = -\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2 + \sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} \quad . \tag{6.569}$$

Hence, it may be seen that the product $(A.F.)_m\ (A.F.)_n$ is negative whereas α_{mn} is positive when

$$\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 > [\tfrac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2 \ , \tag{6.570}$$

that otherwise $(A.F.)_m\ (A.F.)_n$ is positive and α_{mn} is negative, and that consequently in this case too the product $\alpha_{mn}\ (A.F.)_m\ (A.F.)_n$ in Eq. 6.552 is always negative. Then, by substituting Eq. 6.567 into Eq. 6.552, and by considering that for the systems within this Case II the damping ratios ξ_m and ξ_n and the natural frequencies ω_m and ω_n are always close to each other (see Eqs. 6.535 through 6.537) and that thus their corresponding spectral ordinates may be approximated as

$$SD(\omega_m,\ \xi_m) = SD(\omega_n,\ \xi_n) \doteq SD(\omega_o,\ \xi_o) \tag{6.571}$$

where as before

$$\omega_o = \omega_{p_I} = \omega_{s_J} \tag{6.572}$$

and

$$\xi_o = \frac{1}{2}\,(\xi_{p_I} + \xi_{s_J}) \quad , \tag{6.573}$$

in the case herein being considered one may express $\{X_s\}^{(s)}$ as

$$\{X_s\}^{(s)} = \left\{\left(\frac{\frac{1}{2}\,\Phi_o(I,J)}{\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} - \frac{1}{2}\,(\xi_{p_I} - \xi_{s_J})^2}\right)^2 + \left(\frac{\frac{1}{2}\,\Phi_o(I,J)}{\sqrt{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2} + \frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2}\right)^2 - 2|\alpha_{mn}|\,\frac{[\frac{1}{2}\,\Phi_o(I,J)]^2}{\left|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2\right|}\right\}^{1/2} \{d\phi\}^{(J)} SD(\omega_o,\xi_o) \tag{6.574}$$

or as

$$\{X_s\}^{(s)} = \sqrt{\frac{\frac{1}{2}\,(\mu_{IJ} - \alpha_{IJ})\;\Phi_o^2(I,J)}{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2|}}\;\{d\phi\}^{(J)} SD(\omega_o,\xi_o) \tag{6.575}$$

where

$$\mu_{IJ} = \left| \frac{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 + [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2}{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2} \right| \tag{6.576}$$

and

$$\alpha_{IJ} = \frac{1}{1 + \dfrac{\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}{4\,\xi_o'^2}} \; . \tag{6.577}$$

Notice that the parameter μ_{IJ} in Eq. 6.575 is very close to unity when $|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|$ is not very close to $[\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2$ and that consequently it may be assumed equal to unity in most practical cases.

Equations 6.562 and 6.575 furnish thus the expressions to compute the combined response of a secondary system in two adjacent resonant modes. If, however, this combined response is viewed as the product of an amplification factor, a modal configuration, and a response spectrum ordinate, those expressions may be conveniently written as follows:

Case I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$

$$\{X_s\}^{(s)} = \Psi_R^{(s)}\,\{d\phi\}^{(J)}\,\sqrt{SD(\omega_m,\xi_m)\,SD(\omega_n,\xi_n)} \tag{6.578}$$

where

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(\rho_{mn} - \alpha_{mn})\,\Phi_o^2(I,J)}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\gamma_{IJ} + [\frac{1}{2}\,\Phi_o^2(I,J)\gamma_{IJ}]^2}} \tag{6.579}$$

<u>Case II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\gamma_{IJ}|$</u>

$$\{X_s\}^{(s)} = \Psi_R^{(s)}\,\{d\phi\}^{(J)}\,SD(\omega_o,\xi_o) \tag{6.580}$$

in which

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(\mu_{IJ} - \alpha_{IJ})\,\Phi_o^2(I,J)}{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2|}} \tag{6.581}$$

In analyzing these amplification factors, it is interesting to note the following:

1) In the particular case when $|\Phi_o^2(I,J)\gamma_{IJ}| = |\xi_{p_I} - \xi_{s_J}|^2$, the modal correlation factors α_{mn} and α_{IJ} given by Eqs. 6.559 and 6.577 as well as the parameters ρ_{mn} and μ_{IJ} defined by Eqs. 6.560 and 6.576 are equal to unity. In such a case, therefore, Eqs. 6.579 and 6.581 yield zero amplification factors and, as a result, $\{X_s\}^{(s)}$ is equal to zero. Thus, although according to the discussion in Sec. 6.6 the maximum of the response of a secondary system in a resonant mode is obtained when

$\left|\Phi_o(I,J)\sqrt{\gamma_{IJ}}\right| = \left|\xi_{p_I} - \xi_{s_J}\right|$, the combined response of the system in two adjacent resonant modes reaches its minimum in such a case.

2) When the values of $|\xi_{p_I} - \xi_{s_J}|^2$ and $|\Phi_o^2(I,J)\gamma_{IJ}|$ are not very close to each other, the amplification factor $\Psi_R^{(s)}$ may be approximated by

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(\rho_{mn} - \alpha_{mn})\ \Phi_o^2(I,J)}{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|}} \tag{6.582}$$

where

$$\alpha_{mn} = 2\,\frac{\sqrt{\xi_m'\,\xi_n'}}{\xi_m' + \xi_n'} \quad , \tag{6.583}$$

if $|\xi_{p_I} - \xi_{s_J}| > |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$, and by

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(1 - \alpha_{IJ})\ \Phi_o^2(I,J)}{|\Phi_o^2(I,J)\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|}} \tag{6.584}$$

if $|\xi_{p_I} - \xi_{s_J}| < |\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}|$.

Nonresonant Modes

The response of a secondary system in a nonresonant mode of its assembled system is given by either Eqs. 6.417 or 6.493. Therefore, the vectors $\{X_s\}^{(r)}$ in Eq. 6.549 may be expressed as follows:

Case I: $\omega_r \doteq \omega_{p_I}$

$$\{X_s\}^{(r)} = \psi_p^{(r)} \left[r_c \{\frac{df}{f_{cc}}\} + \sum_{j=1}^{N_s} r_j \{d\phi\}^{(j)} \right] SD(\omega_{p_I}, \xi_{p_I}) \tag{6.585}$$

where

$$r_c = \frac{d\Phi(I)}{A_o(J)} \sqrt{1 + \delta_J^2} \tag{6.586}$$

$$r_j = \text{sgn}(1 - \delta_j) \frac{A_o(j)}{A_o(J)} \sqrt{\frac{1 + \delta_J^2}{1 + \delta_j^2}} \tag{6.587}$$

and

$$\psi_p^{(r)} = \frac{A_o(J)\sqrt{1 + \delta_J^2}}{\left\{[1 + A_o^2(J)\gamma_{IJ} - \delta_J^2]^2 + [2 + \frac{\omega_{p_I} - \omega_{s_J}}{\omega_{p_I}} A_o^2(J)\gamma_{IJ}]^2 \delta_J^2\right\}^{1/2}} \tag{6.588}$$

which is valid only if

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{p_I}^2} \right| \sqrt{1 + \delta_J^2} \geq \left| \Phi_o(I,J) \sqrt{\gamma_{IJ}} \right| . \tag{6.589}$$

In the above equations,

$$A_o(j) = \frac{\Phi_o(I,j)\,\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2} \tag{6.590}$$

and

$$\delta_j = \frac{\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}}{\omega_{p_I} - \omega_{s_j}} \quad . \tag{6.591}$$

<u>Case II: $\omega_r \doteq \omega_{s_J}$</u>

$$\{X_s\}^{(r)} = \Psi_s^{(r)}\ \{d\phi\}^{(J)}\ SD(\omega_{s_J},\ \xi_{s_J}) \tag{6.592}$$

where

$$\Psi_s(r) = \left\{ \frac{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2}{[1 + B_o'^2(I)\gamma_{IJ}(1 - \delta_I^2)]^2 + [\frac{\omega_{s_J} - \omega_{p_I}}{\omega_{s_J}} + 2B_o'^2(I)\gamma_{IJ}]^2\ \delta_I^2} \right\}^{1/2} \tag{6.593}$$

which is valid only when

$$\left| \frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{s_J}^2} \right| \sqrt{1 + \delta_I^2} \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}| \quad , \tag{6.594}$$

and where

$$B_o'(i) = \frac{B_o(i)}{1 + \delta_i^2} , \tag{6.595}$$

$$B_o(i) = \frac{\Phi_o(i,J)\omega_{s_J}^2}{\omega_{p_i}^2 - \omega_{s_J}^2} \tag{6.596}$$

and

$$\delta_i = \frac{\xi_{s_J}\omega_{s_J} - \xi_{p_i}\omega_{p_i}}{\omega_{s_J} - \omega_{p_i}} . \tag{6.597}$$

For a secondary system that together with its supporting primary structure gives rise to an assembled system with nonresonant frequencies that are well separated from one another (i.e., a secondary system for which $A_o^2(J)\gamma_{IJ}$ and $B_o^2(I)\gamma_{IJ}$ in Eqs. 6.588 and 6.593 are much smaller than unity), the above amplification factors may be approximated as

$$\Psi_p^{(r)} = \frac{A_o(J)}{\sqrt{1 + \delta_J^2}} \tag{6.598}$$

$$\Psi_s^{(r)} = \sqrt{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2} . \tag{6.599}$$

It may be seen, thus, that an estimate of the maximum distortions of a given secondary system may be obtained in a straightforward manner by the application of Eq. 6.549 in combination with Eqs. 6.578, 6.580, and 6.592. These equations constitute therefore the sought approximate procedure to compute the maximum response of secondary systems. In the next chapter, this approximate procedure will be summarized and illustrated by means of numerical examples.

CHAPTER 7

RECOMMENDED APPROXIMATE PROCEDURE

7.1 Introduction

An approximate method has been developed in the last chapters for the computation of the maximum response of secondary systems attached to primary supporting structures when these structures are subjected to specified ground motions. With the idea of providing a recommended procedure that may be applied directly in the analysis of such secondary systems without having to understand the preceding derivations, in this chapter this approximate method is summarized and illustrated by several numerical examples. For completeness, its scope and limitations are also summarized here, and the parameters and variables involved are defined and described again.

7.2 Limitations

The method described below has been derived under certain assumptions and, consequently, it has some limitations. Its application should be therefore restricted to cases within such limitations. Specifically, the approximate method herein being proposed is applicable if:

1) The independent primary and secondary systems are linear elastic structures with proportional damping,and each of these independent systems has all its natural frequencies well separated from one antoher (that is, no resonant frequencies).
2) The secondary system is attached to its supporting primary system at no more than two points.
3) The primary structure is found in firm ground at a moderate distance from the focal points of the ground disturbances under

consideration.

4) The fundamental periods of primary and secondary systems are shorter, or at least not much longer, than the duration of the earthquakes in the analysis, and the periods of their dominant higher modes are not excessively short.

5) The secondary to primary mass ratios of the independent systems are small when compared to unity.

7.3 Scope

This method may be employed to analyze any multi-degree of freedom secondary system connected to one or two arbitrary points of any multi-degree of freedom primary system. These primary and secondary systems may have one or more coinciding natural frequencies (i.e., components under resonant conditions) and may give rise to assembled systems with nonproportional damping. Although formulated in this work to obtain specifically maximum distortions, the method may be used as well to estimate any other response, such as maximum displacement, velocity or acceleration. (In such cases, the vectors $\{d\phi\}^{(J)}$ and $\{df/f_{cc}\}$ in the expressions below are substituted by equivalent vectors for the response of interest, and the spectral displacement $SD(\omega,\xi)$ is replaced by a consistent response spectrum ordinate.)

7.4 Summary of Procedure

Consider a secondary system attached to one or two arbitrary points of a supporting primary structure. Let the independent primary system be described by its matrix of unit-participation-factor mode shapes $[\Phi]$, its natural frequencies ω_{p_i}, $i = 1, 2, \ldots, N_p$, its generalized masses[†] M_i^*, $i = 1, 2, \ldots, N_p$, and its modal damping ratios ξ_{p_i}, $i = 1, 2, \ldots, N_p$, where N_p represents the number of degrees of freedom of the system. Similarly, let the independent secondary system be fixed at its points of attachment with the primary structure, and let it be characterized by its modal matrix $[\phi]$ (mode shapes also with unit participation factors), its natural frequencies ω_{s_j}, $j = 1, 2, \ldots, N_s$, its generalized masses[†] m_j^*, $j = 1, 2, \ldots, N_s$, and its modal damping ratios ξ_{s_j}, $j = 1, 2, \ldots, N_s$, in which N_s signifies the number of degrees of freedom of such a secondary system.

Let then the following variables be defined as follows:

$SD(\omega,\xi)$ = response spectrum displacement ordinate corresponding to a natural frequency ω and a damping ratio ξ,

γ_{ij} = m_j^*/M_i^* = mass ratio in jth secondary and ith primary modes,

$\phi_n(j)$ = amplitude of the jth mode shape of the independent secondary system at the level of its nth mass,

$$\{d\phi\}^{(j)} = \begin{Bmatrix} \phi_1(j) \\ \phi_2(j) - \phi_1(j) \\ \vdots \\ -\phi_{N_s+1}(j) \end{Bmatrix}$$ = vector of element distortions in the jth mode of the independent secondary system,

---

[†]The ith generalized mass of a system with N degrees of freedom is defined as

$$M_i^* = \sum_{n=1}^{N} M_n \, \phi_n^2(i)$$

where $\phi_n(i)$ is the amplitude of the ith mode shape of the system at the level of its nth mass, and M_n represents the value of such a nth mass.

$$\{\frac{df}{f_{cc}}\} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_{N_s+1}}} = \begin{Bmatrix} 1/k_1 \\ 1/k_2 \\ \vdots \\ 1/k_{N_s+1} \end{Bmatrix} =$$

= vector of normalized secondary differential flexibilities,

$$\beta_j = \frac{k_{N_s+1}\ \phi_{N_s}(j)}{\omega_{s_j}^2\ m_j^*},$$

$\Phi_o(i,j) = \Phi_k(i) + \beta_j[\Phi_\ell(i) - \Phi_k(i)]$ = central value of the amplitudes of the points of attachment in the ith primary and jth secondary modes.

In the above expressions, k_j, $j = 1, 2, \ldots, N_s+1$, represents the stiffness constants of the secondary system, and $\Phi_k(i)$ and $\Phi_\ell(i)$ are the amplitudes of the kth and ℓth primary masses, the masses of the primary system to which the secondary system is attached, in the ith mode of the independent primary system. For secondary systems with a single point of attachment, $\{df/f_{cc}\} = \{0\}$, $\beta_j = 0$ and $\Phi_o(i,j) = \Phi_k(i)$.

Assume now that the assembled system (primary and secondary systems together) is a $N_p + N_s$ degree of freedom system whose natural frequencies are the frequencies of its independent primary and secondary components, and classify a mode of this assembled system as a resonant mode if its natural frequency is a frequency common to both independent components and as a nonresonant mode if its frequency is any other. Then, let R denote the number of these resonant modes, and let subscripts I and J respectively identify the parameters in the modes of the separate primary and secondary systems whose frequencies are the closest to or coincide with the frequency of one of such resonant or nonresonant modes.

Thus, if the base of the primary system is subjected to a given ground motion and if this ground motion is specified by its response spectrum, the vector of maximum distortions of the secondary system may be calculated by

$$\{X_s\}_{max} = \sqrt{\sum_{s=1}^{R/2} \{X_s\}^{(s)^2} + \sum_{r=1}^{N_p+N_s-R} \{X_s\}^{(r)^2}} \tag{7.1}$$

where $\{X_s\}^{(s)}$, which represents the combined maximum response of the secondary system in two resonant modes with equal frequency, and $\{X_s\}^{(r)}$, which is the maximum secondary response in the rth nonresonant mode, may be determined as follows:

Resonant Modes

Let ω_0 and ξ_0 be the natural frequency and damping ratio of the resonant modes, and let them be defined as

$$\omega_0 = \omega_{p_I} = \omega_{s_J} \tag{7.2}$$

$$\xi_0 = \frac{1}{2}\,(\xi_{p_I} + \xi_{s_J}) \quad . \tag{7.3}$$

For any ω_r and ξ_r, define also equivalent damping ratios as

$$\xi'_r = \xi_r + \frac{2}{\omega_r\, s(\xi_r)} \tag{7.4}$$

where $s(\xi_r)$, a function of ξ_r, is an equivalent earthquake duration calculated as described in Sec. 2.10.

Depending on the relation between the damping and mass ratios of the separate primary and secondary systems, the vector $\{X_s\}^{(s)}$ for two given resonant modes with equal frequency may be then computed by the following formulas:

CASE I: $|\xi_{p_I} - \xi_{s_J}| \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$

$$\{X_s\}^{(s)} = \Psi_R^{(s)}\{d\phi\}^{(J)} \sqrt{SD(\omega_m, \xi_m)\, SD(\omega_n, \xi_n)} \tag{7.5}$$

where

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(\rho_{mn} - \alpha_{mn})\, \Phi_o^2(I,J)}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\,\gamma_{IJ} + [\frac{1}{2}\Phi_o^2(I,J)\gamma_{IJ}]^2}} \tag{7.6}$$

$$\omega_m = \omega_n = \omega_o \tag{7.7}$$

$$\xi_m = \xi_o - \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\,\gamma_{IJ}} \tag{7.8}$$

$$\xi_n = \xi_o + \frac{1}{2}\sqrt{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\,\gamma_{IJ}} \tag{7.9}$$

$$\rho_{mn} = \frac{1}{2}\left[\frac{SD(\omega_m, \xi_m)}{SD(\omega_n, \xi_n)} + \frac{SD(\omega_n, \xi_n)}{SD(\omega_m, \xi_m)}\right] \tag{7.10}$$

$$\alpha_{mn} = 2\,|\tau_{IJ}|\,\frac{\sqrt{\xi_m'\,\xi_n'}}{\xi_m' + \xi_n'} \tag{7.11}$$

in which

$$\tau_{IJ} = \frac{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\,\gamma_{IJ} - [\frac{1}{2}\Phi_o^2(I,J)\,\gamma_{IJ}]^2}{(\xi_{p_I} - \xi_{s_J})^2 - \Phi_o^2(I,J)\,\gamma_{IJ} + [\frac{1}{2}\Phi_o^2(I,J)\,\gamma_{IJ}]^2} \tag{7.12}$$

For systems in which the values of $|\xi_{p_I} - \xi_{s_J}|$ and $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ are not very close to each other, τ_{IJ}, α_{mn} and $\Psi_R^{(s)}$ may be approximated as follows:

$$\tau_{IJ} = 1.0 \tag{7.13}$$

$$\alpha_{mn} = 2\,\frac{\sqrt{\xi'_m\ \xi'_n}}{\xi'_m + \xi'_n} \tag{7.14}$$

$$\psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(\rho_{mn} - \alpha_{mn})\ \Phi_o^2(I,J)}{|\Phi_o^2(I,J)\ \gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|}}\ . \tag{7.15}$$

CASE II: $|\xi_{p_I} - \xi_{s_J}| \leq |\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}|$

$$\{X_s\}^{(s)} = \psi_R^{(s)}\ \{d\phi\}^{(J)}\ SD(\omega_o,\ \xi_o) \tag{7.16}$$

in which

$$\psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}\ (\mu_{IJ} - \alpha_{IJ})\ \Phi_o^2(I,J)}{\left|\Phi_o^2(I,J)\ \gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\ \frac{1}{2}(\xi_{p_I} - \xi_{s_J})^2]^2\right|}} \tag{7.17}$$

$$\mu_{IJ} = \left|\frac{\Phi_o^2(I,J)\ \gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 + [\frac{1}{2}\ (\xi_{p_I} - \xi_{s_J})^2]^2}{\Phi_o^2(I,J)\ \gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2 - [\frac{1}{2}\ (\xi_{p_I} - \xi_{s_J})^2]^2}\right| \tag{7.18}$$

$$\alpha_{IJ} = \frac{1}{1 + \dfrac{\Phi_o^2(I,J)\ \gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2}{4\xi'^2_o}}\ . \tag{7.19}$$

When the values of $|\Phi_o^2(I,J)\ \gamma_{IJ}|$ and $|\xi_{p_I} - \xi_{s_J}|^2$ are not very similar, μ_{IJ} and $\psi_R^{(s)}$ may be approximated by the following simiplified relationships:

$$\mu_{IJ} = 1.0 \tag{7.20}$$

$$\Psi_R^{(s)} = \sqrt{\frac{\frac{1}{2}(1-\alpha_{IJ})\,\Phi_o^2(I,J)}{|\Phi_o^2(I,J)\,\gamma_{IJ} - (\xi_{p_I} - \xi_{s_J})^2|}} \quad . \tag{7.21}$$

Nonresonant Modes

For the computation of $\{X_s\}^{(r)}$, a distinction is made between those nonresonant modes with a frequency equal to one of the frequencies of the primary system and those with a frequency equal to one of the secondary system's. If ω_r denotes the frequency of the rth nonresonant mode, the vector $\{X_s\}^{(r)}$ in each of these cases is then determined as follows:

CASE I: $\omega_r = \omega_{p_I}$ –

$$\{X_s\}^{(r)} = \Psi_p^{(r)} \left[r_c \left\{\frac{df}{f_{cc}}\right\} + \sum_{j=1}^{N_s} r_j \{d\phi\}^{(j)} \right] SD(\omega_{p_I}, \xi_{p_I}) \tag{7.22}$$

where

$$r_c = \frac{\Phi_\ell(I) - \Phi_k(I)}{A_o(J)} \sqrt{1 + \delta_J^2} \tag{7.23}$$

$$r_j = \text{sgn}\,(1 - \delta_j)\,\frac{A_o(j)}{A_o(J)} \sqrt{\frac{1 + \delta_J^2}{1 + \delta_j^2}} \tag{7.24}$$

and

$$\Psi_p^{(r)} = \frac{A_o(J)\sqrt{1 + \delta_J^2}}{\left\{[1 + A_o^2(J)\,\gamma_{IJ} - \delta_J^2]^2 + [2 + \frac{\omega_{p_I} - \omega_{s_J}}{\omega_{p_I}} A_o^2(J)\,\gamma_{IJ}]^2\,\delta_J^2\right\}^{1/2}} \quad . \tag{7.25}$$

This expression for $\Psi_p^{(r)}$ is valid only if

$$\left| \frac{\omega_{p_I}^2 - \omega_{s_J}^2}{\omega_{p_I}^2} \right| \sqrt{1 + \sigma_J^2} \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}| \quad . \tag{7.26}$$

When the frequencies ω_{p_I} and ω_{s_J} are so close that Eq. 7.26 is not satisfied, consider them as resonant frequencies.

In the above equations,

$$A_o(j) = \frac{\Phi_o(I,j)\omega_{p_I}^2}{\omega_{s_j}^2 - \omega_{p_I}^2} \tag{7.27}$$

$$\delta_j = \frac{\xi_{p_I}\omega_{p_I} - \xi_{s_j}\omega_{s_j}}{\omega_{p_I} - \omega_{s_j}} \tag{7.28}$$

and sgn is a function which reads "the sign of".

If $A_o^2(J)\ \gamma_{IJ} << 1.0$, that is, if ω_{p_I} and ω_{s_J} are well separated from each other, $\Psi_p^{(r)}$ may be approximated as

$$\Psi_p^{(r)} = \frac{A_o(J)}{\sqrt{1 + \delta_J^2}} \tag{7.29}$$

<u>CASE II: $\omega_r = \omega_{s_J}$</u>

$$\{X_s\}^{(r)} = \Psi_s^{(r)}\ \{d\phi\}^{(J)}\ SD(\omega_{s_J},\ \xi_{s_J}) \tag{7.30}$$

where

$$\Psi_s^{(r)} = \left\{ \frac{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2}{[1+B_o'^2(I)\gamma_{IJ}(1-\delta_I^2)]^2 + [\frac{\omega_{s_J} - \omega_{p_I}}{\omega_{s_J}} + 2B_o'^2(I)\gamma_{IJ}]^2\ \delta_I^2} \right\}^{1/2} . \tag{7.31}$$

As in Case I, this expression is valid only when

$$\left|\frac{\omega_{s_J}^2 - \omega_{p_I}^2}{\omega_{s_J}^2}\right| \sqrt{1 + \delta_I^2} \geq |\Phi_o(I,J)\sqrt{\gamma_{IJ}}| \quad . \tag{7.32}$$

If ω_{s_J} and ω_{p_I} are so close that Eq. 7.32 is not satisfied, they should be considered as resonant frequencies.

In the equations above,

$$B_o'(i) = \frac{B_o(i)}{1 + \delta_i^2} \tag{7.33}$$

$$B_o(i) = \frac{\Phi_o(i,J)\,\omega_{s_J}^2}{\omega_{p_i}^2 - \omega_{s_J}^2} \tag{7.34}$$

$$\delta_i = \frac{\xi_{s_J}\omega_{s_J} - \xi_{p_i}\omega_{p_i}}{\omega_{s_J} - \omega_{p_i}} \quad . \tag{7.35}$$

When $B_o^2(I)\gamma_{IJ} << 1.0$, i.e., when ω_{s_J} and ω_{p_I} are far apart from each other, $\Psi_s^{(r)}$ may be approximated as

$$\Psi_s^{(r)} = \sqrt{[1 + \sum_{i=1}^{N_p} B_o'(i)]^2 + [\sum_{i=1}^{N_p} B_o'(i)\delta_i]^2} \quad . \tag{7.36}$$

The procedure presented above is a general method to compute the response of any secondary system of the class considered in this study. Consequently, the introduced equations constitute the most general expressions of such a procedure. It is important to note, however, that this procedure does not always require the use of such general expressions to obtain accurate estimates of the response of a given specific secondary system. Rather, without overlooking that this is an approximate method, one should interpret these expressions and use a simplified version of them to analyze this given specific problem. Thus, for example, although Eq. 7.1 indicates the use of all the modes of the assembled system whose secondary system is to be analyzed and Eqs. 7.22 and 7.36 consider all the modes of this secondary system and its associated primary structure, in similarity with a conventional modal analysis one should take into account only those of such modes which significantly affect the value of the response of such a secondary system.

7.5 Illustrative Examples

To clarify the use of the procedure established above, the maximum distortions of the secondary systems shown in Fig. 7.1(b) are here calculated by this procedure for the cases when these secondary systems are mounted on several locations of the primary system in Fig. 7.1(a), and this latter system is subjected to the portion of El Centro (May 18, 1940) earthquake ground acceleration whose response spectrum is shown in Fig. 8.3(a). Three cases are considered: a) secondary system S1 on the third floor of the primary system, b) secondary system S1 on the first floor of the primary system, and c) secondary system S2 attached to the first and third floors of the primary system (see Fig. 7.2). In every case, the damping ratios of the fundamental modes of the primary and secondary systems are considered to be 2 and 0.1%, respectively. In addition, their damping matrices are assumed proportional to their respective stiffness matrices. The units of the mass and stiffness values indicated in Fig. 7.1 are T-sec^2/m for the masses and T/m for the stiffness constants.

The modal matrices, natural frequencies, modal damping ratios and generalized masses of such independent primary and secondary systems are:

Primary System

$$[\Phi] = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 1.0 & 0.2 & -0.2 \\ 1.5 & -0.6 & 0.1 \end{bmatrix} \quad \begin{matrix} f_{p_1} = 1.0 \text{ c.p.s.} & \xi_{p_1} = 0.02 & M_1^* = 4.5 \\ f_{p_2} = 2.0 \text{ c.p.s.} & \xi_{p_2} = 0.04 & M_2^* = 0.9 \\ f_{p_3} = 3.0 \text{ c.p.s.} & \xi_{p_3} = 0.06 & M_3^* = 0.1 \end{matrix}$$

Secondary System S1

$$[\phi] = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix} \quad \begin{matrix} f_{s_1} = 1.0 \text{ c.p.s.} & \xi_{s_1} = 0.001 & m_1^* = 0.0045 \\ f_{s_2} = \sqrt{3} \text{ c.p.s.} & \xi_{s_2} = 0.00173 & m_2^* = 0.0015 \end{matrix}$$

Secondary System S2

$$[\phi] = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix} \quad \begin{matrix} f_{s_1} = 1.0 \text{ c.p.s.} & \xi_{s_1} = 0.001 & m_1^* = 0.0045 \\ f_{s_2} = \sqrt{2} \text{ c.p.s.} & \xi_{s_2} = 0.00141 & m_2^* = 0.0015 \end{matrix}$$

On the basis of these dynamic properties of the independent primary and secondary systems and the response spectrum of Fig. 8.3(a), the maximum distortions of the secondary systems in each of the above mentioned cases may be then calculated as follows:

Case 1: System S1 on Third floor of Primary System

Contrary to what has been said at the end of the last section, in this example all the assembled system modes as well as all the components modes will be taken into account, even though the value of the calculated response would hardly be affected if some of these modes were neglected. Here, all those modes are considered to illustrate the application of different cases of the proposed procedure and to show that indeed some of the mentioned modes may be negligible. It should be kept in mind, however, that in an ordinary analysis such negligible modes would have been disregarded, as they are in examples

2 and 3, in order to simplify the calculations.

In accordance with the procedure established in the foregoing section, in this case the secondary system and its supporting primary structure give rise to a five degree of freedom assembled system whose natural frequencies are:

$$f_1 = 1.0 \text{ c.p.s.}$$
$$f_2 = 1.0 \text{ c.p.s.}$$
$$f_3 = \sqrt{3} \text{ c.p.s.}$$
$$f_4 = 2.0 \text{ c.p.s.}$$
$$f_5 = 3.0 \text{ c.p.s.}$$

Thus, this assembled system has two resonant modes and three nonresonant modes. The first two modes are the resonant modes; the third mode is a nonresonant mode with a frequency of the secondary system whereas the fourth and fifth are nonresonant modes with frequencies of the primary system. Hence, the maximum secondary distortions in each of these modes may be calculated as follows:

First and Second Modes: Resonant Modes

According to Eqs. 7.2 and 7.3, the natural frequency and damping ratio of the resonant modes are:

$$\omega_0 = 2\pi(1.0) = 2\pi \text{ rad/sec}$$
$$\xi_0 = \frac{1}{2}(0.02 + 0.001) = 0.0105 \text{ .}$$

Because there is only point of attachment, in this example the central value of the modal amplitudes of the points of attachment is simply the amplitude of the third floor in the first mode of the primary system. That is,

$$\Phi_0(I,J) = \Phi_0(1,1) = \Phi_3(1) = 1.5 \text{ .}$$

Then, in this case

$$\xi_{p_I} - \xi_{s_J} = \xi_{p_1} - \xi_{s_1} = 0.02 - 0.001 = 0.019$$

and

$$\Phi_o(I,J)\sqrt{\gamma_{IJ}} = \Phi_o(1,1)\sqrt{\gamma_{11}} = 1.5\sqrt{0.0045/4.5} = 0.04743 .$$

It may be seen, thus, that the value of $|\xi_{p_I} - \xi_{s_J}|$ is less than the one of $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}|$ and, consequently, the computation of $\{X_s\}^{(s)}$ should be made by the formulas for Case II of resonant modes as follows:

As indicated in Fig. 8.8(a), the equivalent earthquake duration $s(\xi_r)$ for El Centro earthquake, a damping ratio of 1.05%, and a natural frequency less than or equal to 1.0 c.p.s. may be taken, if interpolated linearly, as (see Sec. 8.4 for the determination of equivalent earthquake durations)

$$s(0.0105) = 17.2 \text{ sec};$$

hence, the equivalent damping ratio ξ_o' results as (see Eq. 7.4)

$$\xi_o' = 0.0105 + 2/2\pi(17.2) = 0.02901 \cdot$$

If it is observed that the values of $|\xi_{p_I} - \xi_{s_J}|$ and $|\Phi_o(I,J)\sqrt{\gamma_{IJ}}\,|$ are not very close to each other, then Eq. 7.21 may be used to compute the amplification factor $\Psi_R^{(1)}$. Accordingly, by means of Eqs. 7.19 and 7.21 one obtains

$$\alpha_{11} = \frac{1}{1 + \dfrac{(0.04743)^2 - (0.019)^2}{4(0.02901)^2}} = 0.64060$$

$$\Psi_R^{(r)} = \sqrt{\frac{\frac{1}{2}(1 - 0.64060)\ (1.5)^2}{(0.04743)^2 - (0.019)^2}} = 14.632 \quad .$$

Consequently, since for $\xi_o = 0.0105$ and $\omega_o = 2\pi$ the response spectrum displacement ordinate for El Centro is (see Fig. 8.3(a))

$SD(2\pi,\ 0.0105) = 0.201$ m,

the maximum secondary distortions in the resonant modes result as

$$\{X_s\}^{(1)} = 14.632 \left\{ \begin{matrix} 0.5 \\ 1.5 - 0.5 \end{matrix} \right\} 0.201 = \left\{ \begin{matrix} 1.470 \\ 2.941 \end{matrix} \right\} \text{m} .$$

Third Mode: Nonresonant Mode with $\omega_r = \omega_{s_2}$

In this mode, the frequencies and damping ratios of the closest primary and secondary modes are

$$\omega_{p_I} = \omega_{p_2} = 2(2\pi) = 4\pi; \xi_{p_I} = \xi_{p_2} = 0.04$$

$$\omega_{s_J} = \omega_{s_2} = 2\pi\ \sqrt{3.0}; \qquad \xi_{s_J} = \xi_{s_2} = 0.00173$$

and the factors $\Phi_o(i,J)$ result as

$$\Phi_o(1,2) = \Phi_3(1) = 1.5$$

$$\Phi_o(2,2) = \Phi_3(2) = -0.6$$

$$\Phi_o(3,2) = \Phi_3(3) = 0.1.$$

Therefore, for i = 1,2,3, Eqs. 7.33 through 7.35 give

$$\delta_1 = \frac{(0.00141)\sqrt{3.0} - 0.02(1)}{\sqrt{3.0} - 1} = -0.02398$$

$$\delta_2 = \frac{(0.00141)\sqrt{3.0} - 0.04(2)}{\sqrt{3.0} - 2} = 0.28945$$

$$\delta_3 = \frac{(0.00141)\sqrt{3.0} - 0.06(3)}{\sqrt{3.0} - 3} = 0.14004$$

$$B_o(1) = \frac{-1.5(3)}{3 - 1} = -2.25$$

$$B_o(2) = \frac{0.6(3)}{3 - 4} = -1.80$$

$$B_o(3) = \frac{-0.1(3)}{3 - 9} = 0.05$$

$$B_o'(1) = \frac{-2.25}{1 + (-0.02398)^2} = -2.24871$$

$$B_o'(2) = \frac{-1.80}{1 + (0.28945)^2} = -1.66085$$

$$B_o'(3) = \frac{0.05}{1 + (0.14004)^2} = 0.04904 \quad .$$

Based on the fact that ω_{p_2} and ω_{s_2} are well separated from each other $[B_o(2)\ \dot{\gamma}_{22} = 0.0054 << 1.0]$, one may see that in this mode the amplification factor $\Psi_s^{(1)}$ may be computed by Eq. 7.36. With the above values of $B_o'(i)$ and δ_i, this amplification factor results then as

$$\Psi_s^{(1)} = \sqrt{(1 - 3.86052)^2 + (0.41994)^2} = 2.89118 \quad .$$

Thus, since for El Centro

$$SD(2\pi\sqrt{3.0},\ 0.00173) = 0.124\text{m} ,$$

the maximum modal secondary distortions in this first nonresonant mode are

$$\{X_s\}^{(1)} = 2.89118 \begin{Bmatrix} 0.5 \\ -0.5 - 0.5 \end{Bmatrix} 0.124 = \begin{Bmatrix} 0.179 \\ -0.359 \end{Bmatrix}\text{m} \quad .$$

Fourth Mode: Nonresonant Mode with $\omega_r = \omega_{p_2}$

In the fourth mode, the frequencies and damping ratios of the closest component modes are:

$$\omega_{p_I} = \omega_{p_2} = 2\pi(2) = 4\pi \ ; \quad \xi_{p_I} = \xi_{p_2} = 0.04$$

$$\omega_{s_J} = \omega_{s_2} = 2\pi\sqrt{3.0} \ ; \quad \xi_{s_J} = \xi_{s_2} = 0.00173$$

Hence, since in this case

$$\Phi_o(2,1) = \Phi_o(2,2) = \Phi_3(2) = -0.6,$$

Eqs. 7.27 and 7.28 lead to the following values of δ_j and $A_o(j)$ for $j = 1, 2$:

$$\delta_1 = \frac{0.04(2) - 0.001(1.0)}{2-1} = 0.079$$

$$\delta_2 = \frac{0.04(2) - 0.00173(\sqrt{3})}{2-\sqrt{3}} = 0.28738$$

$$A_o(1) = \frac{0.6(4)}{4-1} = 0.8$$

$$A_o(2) = \frac{0.6(4)}{4-3} = 2.4 \quad .$$

Thus, since ω_{p_2} and ω_{s_2} may be considered, once again, well separated from each other $[A_o^2(2)\ \gamma_{22}^{22} = 0.0096 << 1.0]$, and since for a system with a single point of attachment $\{df/f_{cc}\} = \{0\}$, from Eqs. 7.29 and 7.24 one obtains

$$\psi_p^{(2)} = \frac{2.4}{\sqrt{1+(0.28738)^2}} = 2.30664$$

$$r_1 = \frac{0.8}{2.4}\sqrt{\frac{1+(0.28738)^2}{1+(0.079)^2}} = 0.346$$

$$r_2 = 1.0 \quad .$$

Considering then that for this mode the spectral displacement is

$$SD(4\pi,\ 0.04) = 0.058\ \text{m},$$

Eq. 7.22 yields

$$\{X_s\}^{(2)} = 2.30664[0.346\ \begin{Bmatrix}0.5\\1.0\end{Bmatrix} + 1.0\ \begin{Bmatrix}0.5\\-1.0\end{Bmatrix}]\ 0.058 = \begin{Bmatrix}0.090\\-0.087\end{Bmatrix}\ \text{m} \ .$$

<u>Fifth Mode: Nonresonant Mode with $\omega_r = \omega_{p_3}$</u>

In view that this is also a nonresonant mode with a primary frequency, the secondary distortions in this fifth mode may be calculated in the same form as those in the preceding mode. Accordingly, since for this mode the natural frequencies and damping ratios of the closest component modes are

$$\omega_{p_I} = \omega_{p_3} = 2\pi(3) = 6\pi \ ; \qquad \xi_{p_I} = \xi_{p_3} = 0.06$$

$$\omega_{s_J} = \omega_{s_2} = 2\pi\sqrt{3.0} \ ; \qquad \xi_{s_J} = \xi_{s_2} = 0.00173$$

one has that

$$\Phi_o(3,1) = \Phi_o(3,2) = \Phi_3(3) = 0.1$$

$$\delta_1 \approx \frac{0.06(3) - 0.001(1.0)}{3-1} = 0.0895$$

$$\delta_2 = \frac{0.06(3) - 0.00173(\sqrt{3})}{3-\sqrt{3}} = 0.1396$$

$$A_o(1) = \frac{-0.1(9)}{9-1} = -0.1125$$

$$A_o(2) = \frac{-0.1(9)}{9-3} = -0.1500$$

$$\psi_p^{(3)} = \frac{-0.1500}{\sqrt{1+(0.1396)^2}} = -0.14856$$

$$r_1 = \frac{-0.1125}{-0.1500}\sqrt{\frac{1+(0.1396)^2}{1+(0.0895)^2}} = 0.754$$

$$r_2 = 1.0$$

$$SD(6\pi,\ 0.06) = 0.017\ \text{m}$$

$$\{X_s\}^{(3)} = 0.14856\ [0.754 \begin{Bmatrix} 0.5 \\ 1.0 \end{Bmatrix} + 1.0 \begin{Bmatrix} 0.5 \\ -1.0 \end{Bmatrix}]\ 0.017 = \begin{Bmatrix} 0.002 \\ -0.001 \end{Bmatrix}\ \text{m.}$$

Maximum Secondary Distortions

In the light of Eq. 7.1, the maximum distortions of the secondary system result thus as

$$\{X_s\}_{max} = \sqrt{\begin{Bmatrix} 1.470 \\ 2.941 \end{Bmatrix}^2 + \begin{Bmatrix} 0.179 \\ 0.359 \end{Bmatrix}^2 + \begin{Bmatrix} 0.090 \\ 0.087 \end{Bmatrix}^2 + \begin{Bmatrix} 0.002 \\ 0.001 \end{Bmatrix}^2} = \begin{Bmatrix} 1.484 \\ 2.964 \end{Bmatrix}\ \text{m.}$$

Case 2: System S1 on First Floor of Primary System

Because in this and in the former case the independent components are the same, the natural frequencies of the assembled system in this Case 2 (the one that results from the connection of the secondary system S1 to the first floor of the given primary structure) are identical to those of the assembled system in the preceding example. It may be observed, thus, that the only difference between this and the previous case is the location of the point of attachment, and as consequence they only differ in the values of the parameters $\Phi_o(i,j)$. Based on this fact and on the fact that the maximum response in that preceding example is controlled by the response in its resonant modes, one may then infer that in this case too the maximum response will be controlled by the response in the resonant modes. In this example, the maximum response of the secondary system will be therefore estimated by considering only the response in such resonant modes.

Since in this example the central value of the amplitudes of the points of attachment in the mentioned resonant modes results as

$$\Phi_o(I,J) = \Phi_o(1,1) = \Phi_1(1) = 0.5 \ ,$$

one has that

$$\xi_{p_I} - \xi_{s_J} = \xi_{p_1} - \xi_{s_1} = 0.02 - 0.001 = 0.019$$

and

$$\Phi_o(I,J)\ \sqrt{\gamma_{IJ}} = \Phi_o(1,1)\ \sqrt{\gamma_{11}} = 0.5\ \sqrt{0.0045/4.5} = 0.01581 \ .$$

Notice, therefore, that now $|\xi_{p_I} - \xi_{s_J}|$ is greater than $|\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}|$ and, hence, the desired response should be computed by the formulas for Case I of resonant modes.

Accordingly, since as in the previous example ω_o and ξ_o for this case results as

$$\omega_o = 2\pi(1.0) = 2\pi \text{ rad/sec}$$

$$\xi_o = \frac{1}{2}(0.02 + 0.001) = 0.0105 \ ,$$

then Eqs. 7.7 through 7.9 lead to

$$\omega_m = \omega_n = 2\pi \text{ rad/sec}$$

$$\xi_m = 0.0105 - \frac{1}{2}\sqrt{(0.019)^2 - (0.01581)^2} = 0.00523$$

$$\xi_n = 0.0105 + \frac{1}{2}\sqrt{(0.019)^2 - (0.01581)^2} = 0.01577.$$

For these values of ω_m, ω_n, ξ_m and ξ_n, Fig. 8.8(a) yields therefore the following earthquake equivalent durations:

$$s(0.00523) = 18.5 \text{ sec}$$

$$s(0.01577) = 16.0 \text{ sec.}$$

With such damping ratios, equivalent durations and Eq. 7.4, one then obtains the following equivalent damping ratios:

$$\xi'_m = 0.00523 + 2/2\pi(18.5) = 0.02244$$

$$\xi'_n = 0.01577 + 2/2\pi(16.0) = 0.03566 \quad .$$

In like manner, it may be seen from Fig. 8.3(a) that for the above values of ω_m, ω_n, ξ_m and ξ_n,

$$SD(2\pi, 0.00523) = 0.215 \text{ m}$$

$$SD(2\pi, 0.01577) = 0.195 \text{ m} \quad .$$

Thus, Eqs. 7.10 through 7.12 and Eqs. 7.6 and 7.5 yield

$$\rho_{mn} = \frac{1}{2}\left[\frac{0.215}{0.195} + \frac{0.195}{0.215}\right] = 1.00477$$

$$\tau_{11} = \frac{(0.019)^2 - (0.01581)^2 - [\frac{1}{2}(0.01581)^2]^2}{(0.019)^2 - (0.01581)^2 + [\frac{1}{2}(0.01581)^2]^2} = 0.99972$$

$$\alpha_{mn} = 2(0.99972)\frac{\sqrt{(0.02244)\ (0.03566)}}{0.02244 + 0.03566} = 0.97350$$

$$\Psi_R^{(1)} = \sqrt{\frac{\frac{1}{2}(1.00477 - 0.97350)\ (0.5)^2}{(0.019)^2 - (0.01581)^2 + [\frac{1}{2}\ (0.01581)^2]^2}} = 5.93255$$

$$\{X_s\}^{(1)} = 5.93255 \begin{Bmatrix} 0.5 \\ 1.0 \end{Bmatrix} \sqrt{(0.215)(0.195)} = \begin{Bmatrix} 0.607 \\ 1.215 \end{Bmatrix} \text{ m} .$$

If, as established above, the responses in all other modes are neglected, then by means of Eq. 7.1 one may conclude that

$$\{X_s\}_{max} = \begin{Bmatrix} 0.607 \\ 1.215 \end{Bmatrix} \text{ m} .$$

Case 3: System S2 Attached to First and Third Floors of Primary System

In this example, the primary and the secondary system form an assembled system whose natural frequencies are:

f_1 = 1.0 c.p.s.

f_2 = 1.0 c.p.s.

f_3 = $\sqrt{2.0}$ c.p.s.

f_4 = 2.0 c.p.s.

f_5 = 3.0 c.p.s.

Notice thus that besides the different attaching configurations the only difference between the assembled systems of this and the last two examples is the frequency f_3, which in this instance is somewhat more separated from the adjacent frequencies f_2 and f_4 than it is in the mentioned last two examples. Notice, in addition, that the dynamic properties of the independent secondary systems S1 and S2 are almost the same. It may be concluded, then, that in this case too the response in the resonant modes is the only significant response of the system. As in the previous case, therefore, the maximum response of the secondary system in this example will be here approximated by the maximum response in such resonant modes.

Observe, thus, that the frequency and damping ratio of the resonant modes of the assembled system in this example are identical to those in the two previous cases. That is,

$$\omega_0 = 2\pi\ (1.0) = 2\pi \text{ rad/sec}$$

$$\xi_0 = \frac{1}{2}(0.02 + 0.001) = 0.0105.$$

Observe, also, that in this case the central value of the modal amplitudes of the points of attachment is given by

$$\Phi_0(I,J) = \Phi_0(1,1) = \phi_1(1) + \beta_1[\Phi_3(1) - \Phi_1(1)].$$

Then, since

$$\beta_1 = \frac{k_3\phi_2(1)}{\omega_{s_1}^2 m_1^*} = \frac{0.00075(1.5)}{1.0(0.0045)} = 0.25,$$

$\Phi_0(1,1)$ results as

$$\Phi_o(1,1) = 0.5 + 0.25\ [1.5 - 0.5] = 0.75$$

and consequently the values of $\xi_{p_I} - \xi_{s_J}$ and $\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}$ are

$$\xi_{p_I} - \xi_{s_J} = \xi_{p_1} - \xi_{s_1} = 0.02 - 0.001 = 0.019$$

$$\Phi_o(I,J)\ \sqrt{\gamma_{IJ}} = \Phi_o(1,1)\ \sqrt{\gamma_{11}} = 0.75\ \ \sqrt{0.0045/4.5} = 0.02372\ .$$

Evidently, $|\xi_{p_I} - \xi_{s_J}|$ is smaller than $|\Phi_o(I,J)\ \sqrt{\gamma_{IJ}}|$, and hence, as in the first example, the maximum response of the secondary system herein under consideration should be calculated by the formulas for Case II of resonant modes.

As in the first example, therefore, one has that

$$s(0.0105) = 17.2 \text{ sec}$$

$$\xi_o' = 0.02901$$

$$SD(2\pi,\ 0.0105) = 0.201 \text{ m.}$$

As a result, Eqs. 7.16 through 7.19 yield

$$\mu_{11} = \frac{(0.02372)^2 - (0.019)^2 + [\frac{1}{2}(0.019)^2]^2}{(0.02372)^2 - (0.019)^2 - [\frac{1}{2}(0.019)^2]^2} = 1.00032$$

$$\alpha_{11} = \frac{1}{1 + \dfrac{(0.02372)^2 - (0.019)^2}{4(0.02901)^2}} = 0.94349$$

$$\Psi_R^{(1)} = \sqrt{\frac{\frac{1}{2}(1.00032 - 0.94349)(0.75)^2}{(0.02372)^2 - (0.019)^2 - [\frac{1}{2}(0.019)^2]^2}} = 8.90397$$

$$\{X_s\}^{(1)} = 8.90397\left\{\begin{matrix} 0.5 \\ 1.5 - 0.5 \\ 1.5 \end{matrix}\right\} 0.201 = \left\{\begin{matrix} 0.895 \\ 1.790 \\ 2.685 \end{matrix}\right\} m \; .$$

Neglecting the responses in the rest of the modes, the vector of maximum secondary distortions results thus approximately as

$$\{X_s\}_{max} = \left\{\begin{matrix} 0.895 \\ 1.790 \\ 2.685 \end{matrix}\right\} m \quad .$$

CHAPTER 8

COMPARATIVE ANALYSES

8.1 General

The accuracy of the approximate expressions proposed in Chapter 2 to compute the natural frequencies and mode shapes of assembled systems, the approximate formula of Chapter 6 to calculate the complex natural frequencies of systems with nonproportional damping, the rule introduced and adopted in the same Chapter 2 to combine the modes of systems with closely-spaced natural frequencies, and the approximate method summarized in Chapter 7 to estimate the maximum distortions of secondary systems is here evaluated by comparing the approximate and exact solutions of a number of selected idealized systems.

In the development of this comaprative analysis, three main categories are considered separately:

1) Systems with resonant modes and proportional damping.
2) Systems with resonant modes and nonproportional damping.
3) Systems with nonproportional damping but without resonant modes.

With the first category, the essential idea of the proposed approximate procedures is tested without the complication of a complex analysis. The second category examines the applicability of these procedures in their most general form. And since in the first two categories the resonant modes always govern the value of the response, the third category is included to assess the validity of the proposed general formulation for nonresonant modes.

Within each of these categories, the parameters of the systems considered are varied to study the accuracy of the methods being evaluated for systems with different characteristics. Systems with different mass ratios, frequency distributions, damping characteristics, location of the points of attachment, and number of these points of attachment are considered. To test for the variability in the characteristics of the earthquake input, each of these systems is analyzed, in addition, for three different earthquake excitations.

The comparison of approximate and exact responses is made within a statistical framework. That is, accepting the fact that a comparison based on a single earthquake is meaningless and inconsistent with the average response spectra used in the engineering practice (Newmark, 1970), the accuracy evaluation of the proposed approximate methods is here made over the average response to the various earthquake excitations chosen for the comparative analysis.* To adequately consider excitations of different magnitudes, such statistical averages are then taken over the approximate responses normalized with respect to their respective exact values.

In order to disclose in a concise and descriptive manner the accuracy achieved with the evaluated approximated methods, the general conclusions of the performed comparative analysis are also presented in statistical terms; the overall effectiveness of these approximate

*The statistical averages calculated in this study are used to describe the results obtained for the systems and earthquakes here analyzed, not to infer conclusions about other systems and earthquakes.

methods is indicated by the group statistics--the averages obtained for all the analyzed systems within a specified group--of the approximate to exact response ratios in each of the different categories introduced above.

8.2 Systems and Parameters Studied

A three-degree-of-freedom primary system and two two-degree-of-freedom secondary systems are selected for the comparative study. These systems are idealized as shear beams and are defined by their mass and spring constant values. The primary system is depicted in Fig. 8.1(a), and its modal matrix, natural frequencies and generalized masses are as indicated below.

Properties of Primary System

Modal Matrix	Natural Frequencies	Generalized Masses
$[\Phi] = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 1.0 & 0.2 & -0.2 \\ 1.5 & -0.6 & 0.1 \end{bmatrix}$	$f_{p_1} = 1.0$ c.p.s $f_{p_2} = 2.0$ c.p.s $f_{p_3} = 3.0$ c.p.s	$M_1^* = 4.5$ $M_2^* = 0.9$ $M_3^* = 0.1$

The models for the secondary systems are shown in Figs. 8.1(b) and (c), and the different parameters considered are listed in Table 8.1. The values of the mass ratios relative to the primary system of Fig. 8.1(a), the natural frequencies, and the generalized masses in each case are also shown in this table. The following is the modal matrix of the secondary system in all cases.

Modal Matrix of Secondary Systems

$$[\phi] = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix}$$

To test for different locations and number of the points of attachment, the following three cases are considered:

1) Secondary system of Fig. 8.1(b) attached to the third mass of the primary system.
2) Secondary systems of Fig. 8.1(b) attached to the first mass of the primary system.
3) Secondary system of Fig. 8.1(c) attached to the first and third masses of the primary system.

The assembled systems corresponding to each of these cases are shown in Fig. 8.2.

For each system with a given mass ratio, frequency distribution, and location and number of the points of attachment, three different cases of damping are considered. The nominal damping percentages selected and the resulting modal damping ratios are indicated in Table 8.2. In this table, the first group (cases A1 through C3) corresponds to systems with proportional damping; the second and third correspond to those with nonproportional damping. The nominal damping values are those in the first modes of the independent primary and secondary systems. The modal damping ratios are calculated from these nominal values under the assumption that the damping matrices of such independent primary and secondary systems are proportional to their respective stiffness matrices. For the cases in which the assembled systems have proportional damping, the modal damping ratios are those which approximately give the nominal damping values and comply, in addition, with the condition of independent primary and secondary systems with proportional damping and equal proportionality constant (see Sec. 2.2).

Each of the different cases described above will be identified hereafter by its mass ratio, its nominal damping ratios and a label consisting of a letter and a number. The mass ratios will be those in the resonant modes in the case of a system with resonant modes and those in the first modes of the primary and secondary systems when a system has no resonant modes. The nominal damping ratios will correspond to those indicated in Table 8.2. The letters A, B and C in the above mentioned label will identify systems with resonant modes whereas the letters D, E and F will identify those without resonant modes. The secondary system in systems A and D, B and E, and C and F is attached to the primary system as described in the above cases 1, 2 and 3, respectively. The number in the label refers to one of the secondary systems considered in this analysis whose parameters and natural frequencies are defined in Table 8.1.

8.3 Selected Earthquake Records

Because of their different characteristics, the following three earthquake ground acceleration records are selected:

1. El Centro, May 18, 1940, Component S00E
2. Taft, July 21, 1952, Component N21E
3. Pacoima Dam, February 9, 1971, Component S16E

To achieve economy, however, only the approximately first ten seconds of each of these records are used. Within the neighborhood of these ten seconds, then, the last point of each record is chosen to be the one corresponding to a zero ground velocity. In each record, too, extra points are added beyond this last point to be able to detect the maximum of the response of a system when this maximum occurs after the considered ground motion stops.

The response spectra obtained for each of these records with the characteristics just described are shown for 0, 2, 10 and 20 percent damping in Figs. 8.3.

8.4 Adjustment of Earthquake Durations for Equivalent White Noise Excitations

According to the discussion in Sec. 2.10, the use of Rosenblueth's rule for the combination of modes requires the calculation of an equivalent earthquake duration to represent the earthquake excitations employed in the analysis of a system by a finite segment of white noise of duration equal to such an equivalent duration. For the application of Rosenblueth's rule in this comparative analysis, the duration of the above three earthquake records is therefore adjusted in this section by following the criterion established in that Sec. 2.10. In this work, however, the adjustment is made individually for each record rather than for the average response spectrum for those three records.

This procedure is adopted because the accuracy of the approximate methods herein being evaluated is to be measured by average ratios of approximate to exact responses and because, if the desired adjustment were made for the average response spectrum, there would not be a way to compute the exact responses for such an average response spectrum. It offers, in addition, the advantage of a deterministic-like treatment and of avoiding arbitrary normalizations. Statistically speaking there is no reason to believe that such a procedure is not valid because, even though only rough adjustments can be made for each individual record, the aforementioned accuracy is not measured by individual approximate to exact response ratios but, as stated earlier, by the average of such ratios for the total number of records.

Another variation introduced in this section to the procedure established in Sec. 2.10 is the separate adjustment of the earthquake duration for different portions of a given response spectrum.

That is, because in this study such an adjustment for a set of earthquake records, is made individually for each of these records, and because some of these records may be very much different from a white noise excitation, the adjustment of the duration of an earthquake record based on the total frequency range of its response spectrum may lead to a rough representation of that record as a white noise and, consequently, to a not very accurate application of the adopted rule to combine modes. Thus, to obtain a better white noise representation, and hence an improved accuracy of such a rule, the adjustment of the earthquake duration of the records considered in this comparative analysis is here accomplished by following the criterion of Sec. 2.10

but applied separately to different frequency ranges of the specified response spectra.

From the "average" response spectra* drawn with broken lines in Figs. 8.3 and by applying the least square method to obtain the best fit of Eq. 2.112 or 2.113, whichever applies, for each of these average response spectra, the equivalent durations of each of the three earthquake records described in Sec. 8.3 are accordingly determined for 0, 2, 10 and 20 percent damping. Because El Centro and Pacoima Dam earthquakes result very far off an ideal white noise excitation (opposite to Taft that is quite close), it is necessary to perform their fitting for two different frequency ranges. The first fit is made for all frequencies between 0.2 and 1.0 c.p.s. while the second includes only those between 1.0 and 5.0 c.p.s. The fittings and the durations obtained are shown in Figs. 8.4 through 8.7. The variation of these durations with the percentage of damping is sketched in Fig. 8.8, where a linear variation is assumed between the values for 0, 2, 10 and 20 percent.

8.5 Approximate and Exact Natural Frequencies and Mode Shapes

In the development of the approximate method proposed in this study, the derived expressions to approximate the natural frequencies of assembled systems play a fundamental role in the accuracy of the entire method because the accuracy achieved in the determination of mode shapes and, consequently, maximum modal responses depend directly on the exactitude with which such natural frequencies may be determined. To prove, then, that such approximate expressions to compute the natu-

*By average response spectra it is meant here the smooth response spectra obtained after eliminating local peaks and valleys.

ral frequencies of assembled systems are indeed accurate and lead therefore to accurate values of the mode shapes of these systems, the natural frequencies and mode shapes of the systems with proportional damping and a single point of attachment of those described in Sec. 8.2 are here calculated by the approximate method introduced in Chapter 2 and compared with their respective exact values. Similarly, the natural frequencies of those with proportional damping and two points of attachment are computed by the approximate expressions suggested in Chapter 4 and compared with the corresponding exact answers. The mode shapes of the systems with two points of attachment are not calculated because no expression was developed to determine these mode shapes in terms of all their component modes. Since the method to compute such mode shapes would be very similar to the one for a single point of attachment, it may be considered, nevertheless, that in this case too accurate frequencies lead to accurate mode shapes.

The approximate and exact natural frequencies and mode shapes with unit participation factors of the systems with proportional damping and one point of attachment (systems A and B in Fig. 8.2) are presented in Tables 8.3 through 8.14. The approximate frequencies are computed by Eq. 2.51 for the resonant modes and by Eqs. 2.60 and 2.61 for the nonresonant ones. Using the approximate values of the natural frequencies, the approximate mode shapes are determined by Eqs. 2.35 through 2.39 and by multiplying the values thus obtained by approximate participation factors calculated by Eq. 2.93. The $y_j^{(r)}$ factors of Eq. 2.36 are computed by either Eq. 2.38 or 2.80, depending on the case. In table 8.15 are shown the approximate and exact natural frequencies for the systems with two points of attachment (systems C in Fig. 8.2). The frequencies of the resonant modes are calculated from Eq. 4.66 whereas those of the nonresonant ones are obtained by Eqs. 4.79 and 4.80. In all cases, the exact natural frequencies and unit-participation-factor mode shapes are computed using the SAP IV computer program described in Ref. 4. Double precision is used in this program to avoid truncation errors that might occur because the great difference in the values of the parameters of primary and secondary systems.

The accuracy achieved in each case is measured by the approximate to exact ratios included in each of the above mentioned tables.

8.6 Approximate and Exact Complex Natural Frequencies of Systems with Nonproportional Damping

To extend the preceding analysis and verify the accuracy of the expressions derived in Chapter 6 to determine the complex natural frequencies of systems with nonproportional damping, the approximate and exact complex natural frequencies of the assembled systems with nonproportional damping analyzed in this comparative study are also calculated and compared with respect to each other. The approximate complex frequencies of resonant modes are obtained by Eq. 6.251, and the ones for nonresonant modes by Eqs. 6.280 and 6.281. The exact values are computed by the EISPAC control program (eigenproblem subroutine package) of the IBM 360/75 computer system at the University of Illinois [10]. The approximate and exact complex frequencies of the two resonant modes are shown in Tables 8.16

and 8.17. In each case, approximate or exact, the value in the left column represents the real part of the complex frequency under consideration while the value in the right one corresponds to its imaginary part. In all the analyzed cases, the approximate and complex frequencies of nonresonant modes result very close to each other. Since for this reason their presentation would be superfluous, the values obtained for these frequencies are not shown.

8.7 Evaluation of Several Rules to Combine Modal Responses

It has been emphasized throughout this study the importance of the rule by which the modal responses of a secondary system are combined to estimate this secondary system maximum response. In this section, this importance is confirmed by comparing the results obtained by several commonly used approximate rules and those determined from an exact analysis.

The approximate and exact maximum distortions of the secondary systems in each of the assembled systems with proportional damping defined in Sec. 8.2 are calculated for the three earthquake excitations described in Sec. 8.3. The approximate solutions are computed by three rules: (a) the sum of the absolute modal maxima (Abs Sum), (b) the square root of the sum of the squares (SRSS), and (c) the one proposed by Rosenblueth and described in Sec. 2.9. In order to evaluate only the error introduced by these rules, the approximate maximum responses are computed from the exact modal maxima. These exact modal maxima as well as the exact maximum responses are determined by the "Response History Analysis by Mode Superposition" option of

SAP IV [4], modified to account for different damping ratios in the different modes of a system. The exact solutions and the approximate to exact maximum secondary distortion ratios obtained for each of the above specified rules are presented in Tables 8.18 through 8.23. To summarize the information contained in these tables, the mean (μ) and coefficient of variation (c.o.v.) of the three approximate to exact maximum distortion ratios obtained in each case for the three earthquakes used in the analysis are calculated and shown in Tables 8.24 through 8.26. Inasmuch as no major statistical differences are found among the responses of the various secondary elements, these responses are indistinctively considered in the average of these tables. To verify that the conventional rules to combine modes become inaccurate when they are applied to systems whose frequencies are close to one another, the parameter $\Delta\omega/(\omega_I + \omega_J)$, where $\Delta\omega = |\omega_J - \omega_I|$ and ω_I and ω_J are the nearly equal natural frequencies in the resonant modes of the analyzed assembled systems, is calculated for each of these systems and included in Tables 8.24 through 8.26.

Table 8.27 shows the computed group statistics. For each rule and each percentage of damping, four statistics are furnished. The first two are the mean (μ) and coefficient of variation (c.o.v.) of the sample formed by all the approximate to exact ratios of Tables 8.24 through 8.26 corresponding to one of the percentages of damping considered. The last two are the maximum (MAX) and minimum (MIN) values of these approximate to exact ratios found in such a sample. Notice that each of the coefficients of variation shown in Table 8.27

represents the deviation of the mean of one of the above described samples, not the average of the corresponding individual coefficients of variation listed in Tables 8.24 through 8.26. Notice also that the damping ratios indicated in all the tables mentioned in this section are the damping ratios in the first modes of the assembled systems in the analysis (see Sec. 8.2).

8.8 Comparison of Approximate and Exact Maximum Distortions of Secondary Systems

In this final comparative study are examined the accuracy of the approximate method for the computation of the maximum response of secondary systems and the overall effectiveness of the approximate procedures suggested in the development of this approximate method. The approximate maximum distortions of the secondary systems of each of the assembled systems described in Sec. 8.2 are determined for each of the ground motion records selected for this study and compared with the solutions obtained by a more accurate time-history analysis. The approximate responses are calculated by the procedure summarized in Chapter 7. Such more accurate solutions are computed by the "Response History Analysis by Direct Integration" section of SAP IV [4] in the case of systems with proportional damping and a modified version of it in the case of those with nonproportional damping. The obtained approximate and exact maximum responses as well as the corresponding approximate to exact ratios are presented in Tables 8.28 through 8.37. Tables 8.28 through 8.33 show the values for the systems with resonant modes (systems A, B and C) and proportional damping, Tables 8.34 through 8.36 the ones for some of the same systems but with nonproporional damping, and Table 8.37 those with nonproportional damping and no resonant modes (systems D, E and F). To compare, once again, the approximate and exact solutions on the basis of the average response to the three earthquakes considered in the analysis, the last two columns of these tables show the mean and coefficient of variation of the three approximate to exact ratios obtained in each case for these three earthquakes.

The results in Tables 8.28 through 8.37 are statistically summarized in Table 8.38. For each of the three categories herein being studied, this table lists the mean values and coefficients of variation of the average approximate to exact ratios of Tables 8.28 through 8.37 within groups classified by their damping characteristics. To supplement this information, the maximum and minimum values of such ratios within each of these groups are also listed in Table 8.38.

8.9 Discussion of Results and Conclusions

From the results of the comparative analyses presented in the foregoing sections, the following may be concluded:

1. The proposed approximate expressions to compute the natural frequencies of assembled systems furnish, in all cases, an adequate accuracy.
2. The approximate method suggested in Sec. 2.2 accurately predicts the resonant and nonresonant unit-participation-factor mode shapes of such assembled systems.
3. The approximate formulas for the calculation of the complex natural frequencies of systems with nonproportional damping estimate

with an excellent accuracy these complex natural frequencies.

4. The observation made in Sec. 6.3 about the nature of the complex natural frequencies of resonant modes is confirmed. That is, it is verified that depending on the relation between the mass and damping ratios of the primary and secondary components of an assembled system the values of the natural frequencies and damping ratios of two adjacent resonant modes of this assembled system vary between the following two extreme cases:
   a) Both frequencies equal to the resonant frequency of the independent components; one damping ratio equal to the damping ratio of the primary system and the other equal to the damping ratio of the secondary system.
   b) Frequencies equal to the frequencies of a similar assembled system with proportional damping; both damping ratios equal to the average damping ratio of the independent components.
5. The conventional rules to combine modal responses become extraordinarily conservative when they are applied to the analysis of light secondary systems or, more generally, to the analysis of systems whose natural frequencies are very close to one another. On the average for the systems analyzed in this study, the absolute sum overestimates 3.2, 6.1 and 11.2 times the exact responses for 0, 2 and 10 percent damping, respectively. Similarly, for these same percentages of damping the square root of the sum of the squares overestimates the true solutions by factors of 2.1, 4.1 and 7.4, respectively.

6. The rule suggested by Rosenblueth achieves a reasonable accuracy. In the analysis of the same systems mentioned above, this rule yields on the average for 0, 2 and 10 percent damping approximate responses equal respectively to 1.02, 0.93 and 1.01 times the exact answers.
7. The inaccuracy of the absolute sum and the square root of the sum of the squares generally increases with the closeness between natural frequencies, the closeness measured in this study with the parameter $\Delta\omega/(\omega_I + \omega_J)$. In contrast, the accuracy of Rosenblueth's rule remains practically unaltered with the variation of this parameter. These results confirm thus the importance of the cross terms in the general expression to combine the modes of systems with closely-spaced natural frequencies (see Eq. 2.101).
8. The exactitude of this latter rule depends strongly on the value selected for the earthquake duration; consequently, the adjustment of this duration as described in Sec. 2.10 is an important step in the application of the rule.
9. Among the three earthquake records employed in the comparative analysis, the adjusted earthquake duration for Taft is closer to its actual duration than the adjusted durations for El Centro and Pacoima are to their respective actual durations. Therefore, Taft is an earthquake closer to an ideal white noise than El Centro and Pacoima are; a fact that may also be confirmed by the inspection of the form of their pseudovelocity response spectra.
10. Rosenblueth's rule may be applied to excitations with non-smooth pseudovelocity response spectra (Pacoima, for example), provided

their durations are adjusted separately for different portions of their response spectra in the fashion suggested in Sec. 8.4.

11. The proposed approximate method to estimate the maximum distortions of secondary systems predicts with a fairly good accuracy the maximum distortions of all the secondary systems analyzed in this study. For all the categories and damping characteristics considered, this approximate method gives average errors of no more than 7%. Individually in each of the analyzed systems, the error is always less than about 35%, in either the conservative or non-conservative side.

12. In the case of systems with proportional damping, the accuracy obtained by the approximate method is consistent with the accuracy attained in the prediction of maximum responses using the exact modal maxima and Rosenblueth's rule to combine these modal maxima. For this reason, a significant improvement in the accuracy of the proposed approximate method may be achieved only if a substantial improvement in the accuracy of the rule used for the combination of modes may be accomplished.

CHAPTER 9

CONCLUSIONS AND RECOMMENDATIONS

9.1 Summary

A simple approximate procedure has been proposed to predict the maximum response of light secondary systems attached to buildings subjected to earthquakes. Formulated in terms of the separate dynamic properties of a primary and a secondary system and a specified ground response spectrum, this procedure is derived on the basis of the modal analysis of the assembled system formed by the interconnected primary and secondary systems and the development of variations to the conventional response spectrum method. As presented, it may be applied for estimating the response of any multi-degree-of-freedom secondary system attached to one or two arbitrary points of a multi-degree-of-freedom primary structure, but it is restricted to those cases in which the primary and secondary systems are linear elastic systems with classical modes of vibration and the masses of the secondary system are small in comparison with the masses of its primary structure. It may consider a secondary system that is close to or in resonance with its supporting system.

The applicability and accuracy of the proposed approximate procedure and the various methods developed for its derivation have been evaluated by means of a comparative study between the approximate and exact solutions of a number of different systems subjected to diverse earthquakes.

9.2 Conclusions

The analytical developments summarized in Chapter 7 and the numerical results of the comparative study described in Chapter 8 indicate that the proposed approximate procedure is a simple general method of analysis that eliminates the unnecessary complications of other procedures, and that furnishes an accuracy consistent with the uncertainties of the response spectrum method and adequate enough for all practical purposes. Thus, it may be concluded that this approximate procedure provides a convenient alternative method for the rational seismic design of secondary systems.

9.3 Recommendations for Future Studies

The approximate procedure herein developed was restricted to the analysis of secondary systems with up to two points of attachment. Therefore, although in an approximate manner any secondary system with more than two points of attachment can be treated as a series of separate subsystems with one ore two of such points of attachment, additional studies are needed to extend the proposed procedure for the analysis of multiply-connected secondary systems. In the same fashion, since this investigation was limited to the study of linear elastic systems, further research is necessary to consider, when applicable, the inelastic behavior of primary and secondary systems. In this respect, it is believed that the use of the procedure recommended in this work in combination with inelastic response spectra suffices for a practical nonlinear analysis of secondary systems. It is important, however, to find a method to determine the ductility factor of an assembled system in terms of the ductility factors of its separate components.

Finally, it is considered that the approach used here to derive the suggested approximate method for the analysis of secondary systems may be applied to solve other similar engineering problems, such as the problem of the interaction between shear walls and frames, a torsional and a translational motion, or a soil mass and a structure.

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TABLE 8.1 PROPERTIES OF SELECTED SECONDARY SYSTEMS FOR COMPARATIVE ANALYSIS

MASS RATIO	CASE	m_1 (T-sec^2/m)	m_2 (T-sec^2/m)	k_1 (T/M)	k_2 (T/M)	k_3 (T/M)	f_{s_1} (c.p.s.)	f_{s_2} (c.p.s.)	m_1^*	m_2^*
	A1 and B1	0.0450	0.0150	0.0900	0.0225		1.000	1.732	0.045	0.015
	A2 and B2	0.0090	0.0030	0.0720	0.0180		2.000	3.464	0.009	0.003
0.01	A3 and B3	0.1350	0.0450	0.0900	0.0225		0.577	1.000	0.135	0.045
	C1	0.0450	0.0150	0.06750	0.01125	0.00750	1.000	1.414	0.045	0.015
	C2	0.0090	0.0030	0.05400	0.00900	0.00600	2.000	2.828	0.009	0.003
	C3	0.1350	0.0450	0.10125	0.016875	0.01125	0.707	1.000	0.135	0.045
	A1 and B1	0.00450	0.00150	0.00900	0.00225		1.000	1.732	0.0045	0.0015
	A2 and B2	0.00090	0.00030	0.00720	0.00180		2.000	3.464	0.0009	0.0003
0.001	A3 and B3	0.01350	0.00450	0.00900	0.00225		0.577	1.000	0.0135	0.0045
	C1	0.00450	0.00150	0.006750	0.001125	0.000750	1.000	1.414	0.0045	0.0015
	C2	0.00090	0.00030	0.005400	0.000900	0.000600	2.000	2.828	0.0009	0.0003
	C3	0.01350	0.00450	0.010125	0.0016875	0.001125	0.707	1.000	0.0135	0.0045
	D1 and E1	0.0450	0.0150	0.045000	0.011250		0.707	1.225	0.045	0.015
	D2 and E2	0.0450	0.0150	0.180000	0.045000		1.414	2.449	0.045	0.015
0.01	D3 and E3	0.0450	0.0150	0.022500	0.005625		0.500	0.866	0.045	0.015
	F1	0.0450	0.0150	0.050625	0.0084375	0.005625	0.866	1.225	0.045	0.015
	F2	0.0450	0.0150	0.202500	0.0337500	0.022500	1.732	2.449	0.045	0.015
	F3	0.0450	0.0150	0.016875	0.0028125	0.001875	0.500	0.707	0.045	0.015

TABLE 8.2 MODAL DAMPING RATIOS OF INDEPENDENT COMPONENTS

CASE	NOMINAL DAMPING		PRIMARY			SECONDARY	
	ξ_{P_1}	ξ_{S_1}	MODE 1	MODE 2	MODE 3	MODE 1	MODE 2
A1 and B1	0% 2% 10%	0% 2% 10%	0.0 0.022 0.108	0.0 0.043 0.216	0.0 0.065 0.325	0.0 0.022 0.108	0.0 0.037 0.187
C1	0% 2% 10%	0% 2% 10%	0.0 0.021 0.104	0.0 0.042 0.208	0.0 0.062 0.312	0.0 0.021 0.104	0.0 0.029 0.147
A2 and B2	0% 2% 10%	0% 4% 20%	0.0 0.020 0.100	0.0 0.040 0.200	0.0 0.060 0.301	0.0 0.040 0.200	0.0 0.069 0.347
C2	0% 2% 10%	0% 4% 20%	0.0 0.020 0.100	0.0 0.040 0.200	0.0 0.060 0.300	0.0 0.040 0.200	0.0 0.057 0.283
A3 and B3	0% 3.5% 17.5%	0% 2% 10%	0.0 0.035 0.175	0.0 0.070 0.351	0.0 0.105 0.526	0.0 0.020 0.101	0.0 0.035 0.175
C3	0% 2.8% 14.1%	0% 2% 10%	0.0 0.028 0.141	0.0 0.057 0.283	0.0 0.085 0.424	0.0 0.020 0.100	0.0 0.028 0.141
A1 and B1	4% 0% 2%	0% 4% 0.1%	0.040 0.0 0.020	0.080 0.0 0.040	0.120 0.0 0.060	0.0 0.040 0.0010	0.0 0.0693 0.0017
C1	4% 0% 2%	0% 4% 0.1%	0.040 0.0 0.020	0.080 0.0 0.040	0.120 0.0 0.060	0.0 0.040 0.0010	0.0 0.0566 0.0014
A2 and B2	4% 0% 2%	0% 8% 0.1%	0.040 0.0 0.020	0.080 0.0 0.040	0.120 0.0 0.060	0.0 0.080 0.0010	0.0 0.1386 0.0017
C2	4% 0% 2%	0% 8% 0.1%	0.040 0.0 0.020	0.080 0.0 0.040	0.120 0.0 0.060	0.0 0.08 0.0010	0.0 0.1131 0.0014
A3 and B3	7% 0% 2%	0% 4% 0.1%	0.070 0.0 0.020	0.140 0.0 0.040	0.210 0.0 0.060	0.0 0.0404 0.0010	0.0 0.070 0.0017
C3	5.6% 0% 2%	0% 4% 0.1%	0.056 0.0 0.020	0.112 0.0 0.040	0.168 0.0 0.060	0.0 0.040 0.0010	0.0 0.056 0.0014
D1 and E1, F1, D2 and E2, F2, D3 and E3, F3	2%	0%	0.020	0.040	0.060	0.0	0.0

TABLE 8.3 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A1
MASS RATIO = 1%

SYSTEM	MASS	MODE 1			MODE 2			MODE 3			MODE 4			MODE 5		
		FREQUENCIES (c.p.s)														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
		0.92195	0.92405	0.998	1.07238	1.07267	1.000	1.73205	1.72607	1.003	2.0000	2.02341	0.988	3.00000	3.00200	0.999
		MODE SHAPES														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
PRIMARY	1	0.24351	0.23786	1.024	0.25955	0.26023	0.997	0.04199	0.03992	1.052	0.39089	0.36296	1.007	0.09921	0.09904	1.002
PRIMARY	2	0.50528	0.49310	1.025	0.49963	0.50086	0.998	0.04199	0.04033	1.041	0.19545	0.16438	1.189	-0.19843	-0.19868	0.999
PRIMARY	3	0.81409	0.79307	1.027	0.69251	0.69398	0.998	-0.02099	-0.01892	1.109	-0.58634	-0.56927	1.030	0.09921	0.10113	0.981
SECONDARY	1	3.28158	3.26765	1.004	-1.74688	-1.74102	1.003	-1.40485	-1.36755	1.027	0.97723	0.87238	1.120	-0.03100	-0.03154	0.983
SECONDARY	2	7.57289	7.58603	0.998	-7.48661	-7.47454	1.002	1.42584	1.38665	1.028	-0.58633	-0.50442	1.162	0.00620	0.00630	0.984

TABLE 8.4 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A1
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1			MODE 2			MODE 3			MODE 4			MODE 5		
		FREQUENCIES (c.p.s)														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
		0.97599	0.97614	1.000	1.02344	1.02353	1.000	1.73205	1.73141	1.000	2.00000	2.00239	0.999	3.00000	3.00020	1.000
		MODE SHAPES														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
PRIMARY	1	0.24767	0.24623	1.006	0.25263	0.25359	0.996	0.00447	0.00444	1.007	0.39904	0.39384	1.008	0.09992	0.09990	1.000
PRIMARY	2	0.50122	0.49825	1.006	0.49926	0.50115	0.996	0.00447	0.00445	1.004	0.19952	0.19602	1.018	-0.19984	-0.19987	1.000
PRIMARY	3	0.76960	0.76492	1.006	0.73106	0.73375	0.996	-0.00223	-0.00221	1.009	-0.59857	-0.59660	1.003	0.09992	0.10011	0.998
SECONDARY	1	8.67606	8.67230	1.000	-7.14448	-7.14137	1.000	-1.48990	-1.48525	1.003	0.99761	0.98550	1.012	0.03123	-0.03128	0.998
SECONDARY	2	23.77290	23.77499	1.000	-23.67989	-23.67960	1.000	1.49214	1.48744	1.003	-0.59856	-0.58904	1.016	0.00624	0.00625	0.998

TABLE 8.5 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B1
MASS RATIO = 1%

SYSTEM	MASS	MODE 1			MODE 2			MODE 3			MODE 4			MODE 5		
		FREQUENCIES (c.p.s)														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
		0.97468	0.97423	1.000	1.02469	1.02417	1.001	1.73205	1.72582	1.004	2.00000	2.01039	0.995	3.00000	3.00198	0.999
		MODE SHAPES														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
PRIMARY	1	0.27965	0.28472	0.982	0.22242	0.21771	1.022	0.01088	0.01015	1.072	0.38281	0.38661	0.990	0.09921	0.10084	0.984
	2	0.52685	0.53585	0.983	0.47497	0.46421	1.023	-0.00000	-0.00015	1.000	0.19141	0.20036	0.955	-0.19843	-0.20028	0.991
	3	0.77100	0.78385	0.984	0.73073	0.71378	1.024	-0.02176	-0.02052	1.060	-0.57422	-0.57703	0.995	0.09921	0.09995	0.993
SECONDARY	1	3.00110	3.00591	0.998	-2.05314	-2.05789	0.998	0.72814	0.70374	1.035	-0.63802	-0.62028	1.029	-0.03100	-0.03145	0.986
	2	8.18481	8.18501	1.000	-6.84378	-6.84336	1.000	-0.73902	-0.71400	1.035	0.38281	0.36607	1.046	0.00620	0.00628	0.987

TABLE 8.6 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B1
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1			MODE 2			MODE 3			MODE 4			MODE 5		
		FREQUENCIES (c.p.s)														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
		0.99206	0.99201	1.000	1.00787	1.00782	1.000	1.73205	1.73140	1.000	2.00000	2.00106	0.999	3.00000	3.00200	1.000
		MODE SHAPES														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
PRIMARY	1	0.25916	0.26074	0.994	0.24104	0.23951	1.006	0.00112	0.00111	1.009	0.39824	0.39856	0.999	0.09992	0.10009	0.998
PRIMARY	2	0.50832	0.51137	0.994	0.49185	0.48865	1.007	-0.00000	0.00000	1.000	0.19912	0.20003	0.995	-0.19984	-0.20003	0.999
PRIMARY	3	0.75651	0.76101	0.994	0.74365	0.73878	1.007	-0.00224	-0.00223	1.004	-0.59737	-0.59753	1.000	0.09992	0.10000	0.999
SECONDARY	1	8.38823	8.38979	1.000	-7.44033	-7.44197	1.000	0.74774	0.74478	1.004	-0.66374	-0.66164	1.003	-0.03123	-0.03127	0.999
SECONDARY	2	24.39331	24.39262	1.000	-23.04985	-23.05067	1.000	-0.74886	-0.74594	1.004	0.39824	0.39631	1.005	0.00624	0.00625	0.998

TABLE 8.7 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A2
MASS RATIO = 1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
		FREQUENCIES (c.p.s)														
		1.00000	0.99620	1.004	1.93907	1.94285	0.998	2.05913	2.06238	0.998	3.00000	2.99929	1.000	3.46410	3.47214	0.998
		MODE SHAPES														
PRIMARY	1	0.49802	0.49547	1.005	0.25590	0.23283	1.099	0.16723	0.17182	0.973	0.10072	0.09979	1.009	0.00008	0.00007	1.000
PRIMARY	2	0.99603	0.99282	1.003	0.15866	0.14264	1.112	0.06355	0.06414	0.991	-0.20143	-0.19938	1.010	-0.00027	-0.00025	1.080
PRIMARY	3	1.49405	1.49489	0.999	-0.33410	-0.30694	1.088	-0.27854	-0.28763	0.968	0.10072	0.09905	1.017	0.00066	0.00061	1.082
SECONDARY	1	1.81097	1.80892	1.001	-3.02740	-2.94808	1.027	2.10580	2.04715	1.029	0.16115	0.15816	1.019	-0.07065	-0.06615	1.068
SECONDARY	2	2.17316	2.16744	1.003	-8.10908	-7.94912	1.020	7.17887	7.03285	1.021	-0.32229	-0.31676	1.017	0.06999	0.06554	1.068

TABLE 8.8 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A2
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
		FREQUENCIES (c.p.s)														
		1.00000	0.99962	1.000	1.98094	1.98130	1.000	2.01888	2.01923	1.000	3.00000	2.99993	1.000	3.46410	3.46490	1.000
		MODE SHAPES														
PRIMARY	1	0.49980	0.49953	1.001	0.21460	0.20991	1.022	0.18766	0.19056	0.985	0.10007	0.09998	1.001	0.00001	0.00001	1.000
PRIMARY	2	0.99960	0.99927	1.000	0.11545	0.11277	1.024	0.08671	0.08792	0.986	-0.20014	-0.19994	0.001	-0.00003	-0.00003	1.000
PRIMARY	3	1.49940	1.49948	1.000	-0.30938	-0.30285	1.022	-0.29190	-0.29661	0.984	0.10007	0.09991	1.002	0.00006	0.00006	1.000
SECONDARY	1	1.81745	1.81723	1.000	-8.38255	-8.35932	1.003	7.47133	7.45081	1.003	0.16012	0.15986	1.002	-0.06843	-0.06798	1.007
SECONDARY	2	2.18095	2.18033	1.000	-24.22823	-24.17819	1.002	23.29810	23.25007	1.002	-0.32023	-0.31972	1.002	0.06837	0.06791	1.007

TABLE 8.9 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B2
MASS RATIO = 1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
FREQUENCIES (c.p.s)																
		1.00000	0.99958	1.000	1.95959	1.96002	1.000	2.03961	2.03992	1.000	3.00000	2.99929	1.000	3.46410	3.46788	0.999
MODE SHAPES																
PRIMARY	1	0.50022	0.50097	0.999	0.20746	0.20572	1.008	0.19106	0.19297	0.990	0.10072	0.09915	1.016	0.00119	0.00118	1.008
PRIMARY	2	1.00044	1.00088	1.000	0.08920	0.08861	1.007	0.10898	0.11018	0.989	-0.20143	-0.19881	1.013	-0.00090	-0.00088	1.023
PRIMARY	3	1.50066	1.50070	1.000	-0.31858	-0.31582	1.009	-0.28185	-0.28464	0.990	0.10072	0.09948	1.012	0.00030	0.00029	1.034
SECONDARY	1	0.60633	0.60712	0.999	2.74583	2.75035	0.998	-2.24203	-2.24525	0.999	0.16115	0.15834	1.018	-0.27340	-0.27043	1.011
SECONDARY	2	0.72760	0.72844	0.999	7.62730	7.64612	0.998	-7.31096	-7.32654	0.998	-0.32229	-0.31711	1.016	0.27221	0.26926	1.011

TABLE 8.10 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B2
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
		FREQUENCIES (c.p.s)														
		1.00000	0.99996	1.000	1.98731	1.98735	1.000	2.01261	2.01264	1.000	3.00000	2.99993	1.000	3.46410	3.46448	1.000
		MODE SHAPES														
PRIMARY	1	0.50002	0.50011	1.000	0.20255	0.20196	1.003	0.19731	0.19791	0.997	0.10007	0.09791	1.002	0.00012	0.00012	1.000
PRIMARY	2	1.00004	1.00009	1.000	0.09679	0.09653	1.003	0.10303	0.10335	0.997	-0.20014	-0.19987	1.001	-0.00009	-0.00009	1.000
PRIMARY	3	1.50006	1.50008	1.000	-0.30585	-0.30495	1.003	-0.29420	-0.29509	0.997	0.10007	0.09994	1.001	0.00003	0.00003	1.000
SECONDARY	1	0.60609	0.60617	1.000	8.15698	8.15826	1.000	-7.65061	-7.65174	1.000	0.16012	0.15986	1.002	-0.27280	-0.27243	1.001
SECONDARY	2	0.72730	0.72740	1.000	23.86711	23.87238	1.000	-23.54752	-23.55268	1.000	-0.32023	-0.31972	1.002	0.27268	0.27230	1.001

TABLE 8.11 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A3
MASS RATIO = 1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
FREQUENCIES (c.p.s)																
		0.57735	0.56634	1.019	0.92195	0.93533	0.986	1.07238	1.08335	0.990	2.00000	2.01103	0.995	3.00000	3.00164	0.999
MODE SHAPES																
PRIMARY	1	0.01347	0.01352	0.996	0.24351	0.19885	1.225	0.25955	0.29836	0.870	0.40319	0.39004	1.034	0.09943	0.09924	1.002
PRIMARY	2	0.03142	0.03163	0.993	0.50528	0.41015	1.232	0.49963	0.57082	0.875	0.20160	0.18640	1.082	-0.19886	-0.19898	0.999
PRIMARY	3	0.06210	0.06277	0.989	0.81409	0.65333	1.246	0.69251	0.78076	0.887	-0.60479	-0.59781	1.012	0.09943	0.10096	0.985
SECONDARY	1	0.87017	0.87725	0.992	2.45101	2.40889	1.017	-2.44970	-2.40276	1.020	0.12829	0.12504	1.026	-0.00813	-0.00824	0.987
SECONDARY	2	2.42421	2.44687	0.991	-3.50144	-3.21313	1.090	1.88439	1.78335	1.057	-0.01833	-0.01764	1.039	0.00048	0.00048	1.000

TABLE 8.12 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM A3
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
FREQUENCIES (c.p.s)																
		0.57735	0.57622	1.002	0.97599	0.97727	0.999	1.02344	1.02464	0.999	2.00000	2.00109	0.999	3.00000	3.00016	1.000
MODE SHAPES																
PRIMARY	1	0.00143	0.00143	1.000	0.24767	0.23424	1.057	0.25263	0.26541	0.952	0.40032	0.39900	1.003	0.09994	0.09993	1.000
PRIMARY	2	0.00334	0.00335	0.997	0.50122	0.47373	1.058	0.49926	0.52419	0.952	0.20016	0.19863	0.008	-0.19989	-0.19991	1.000
PRIMARY	3	0.00661	0.00662	0.998	0.76960	0.72651	1.059	0.73106	0.76659	0.954	-0.60048	-0.59981	1.001	0.09994	0.10010	0.998
SECONDARY	1	0.85182	0.85269	0.999	7.90511	7.88892	1.002	-7.87672	-7.86052	1.002	0.12737	0.12706	1.002	-0.00817	-0.00818	0.999
SECONDARY	2	2.53564	2.53832	0.999	-8.73365	-8.66787	1.008	7.19422	7.14737	1.007	-0.01820	-0.01813	1.004	0.00048	0.00048	1.000

TABLE 8.13 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B3
MASS RATIO = 1%

SYSTEM	MASS	MODE 1 APP	MODE 1 EX	MODE 1 APP/EX	MODE 2 APP	MODE 2 EX	MODE 2 APP/EX	MODE 3 APP	MODE 3 EX	MODE 3 APP/EX	MODE 4 APP	MODE 4 EX	MODE 4 APP/EX	MODE 5 APP	MODE 5 EX	MODE 5 APP/EX
FREQUENCIES (c.p.s)																
		0.57735	0.57549	1.003	0.97468	0.97557	0.999	1.02469	1.02530	0.999	2.00000	2.00484	0.998	3.00000	3.00163	0.999
MODE SHAPES																
PRIMARY	1	0.00824	0.00827	0.996	0.27965	0.26747	1.046	0.22242	0.22607	0.984	0.39698	0.39749	0.999	0.09943	0.10071	0.987
PRIMARY	2	0.00965	0.00967	0.998	0.52685	0.50491	1.043	0.47497	0.48359	0.982	0.19849	0.20212	0.982	-0.19886	-0.20027	0.993
PRIMARY	3	0.01086	0.01087	0.999	0.77100	0.73952	1.043	0.73073	0.74446	0.982	-0.59547	-0.59483	1.001	0.09943	0.09997	0.995
SECONDARY	1	0.64917	0.65064	0.998	2.72090	2.69932	1.008	-2.27595	-2.25802	1.008	-0.08421	-0.08380	1.005	-0.00813	-0.00822	0.989
SECONDARY	2	1.92277	1.92726	0.998	-3.02322	-2.98765	1.012	2.06905	2.04809	1.010	0.01203	0.01191	1.010	0.00048	0.00048	1.000

TABLE 8.14 APPROXIMATE AND EXACT FREQUENCIES AND MODE SHAPES OF ASSEMBLED SYSTEM B3
MASS RATIO = 0.1%

SYSTEM	MASS	MODE 1			MODE 2			MODE 3			MODE 4			MODE 5		
		FREQUENCIES (c.p.s)														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
		0.57735	0.57716	1.000	0.99206	0.99214	1.000	1.00787	1.00795	1.000	2.00000	2.00048	1.000	3.00000	3.00016	1.000
		MODE SHAPES														
		APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX
PRIMARY	1	0.00083	0.00083	1.000	0.25916	0.25634	1.011	0.24104	0.24302	0.992	0.39970	0.39975	1.000	0.09994	0.10007	0.999
PRIMARY	2	0.00097	0.00097	1.000	0.50832	0.50287	1.011	0.49185	0.49597	0.992	0.19985	0.20021	0.998	-0.19989	-0.20003	0.999
PRIMARY	3	0.00110	0.00110	1.000	0.75651	0.74845	1.011	0.74365	0.74994	0.992	-0.59955	-0.59948	1.000	0.09994	0.10000	0.999
SECONDARY	1	0.64551	0.64572	1.000	8.12902	8.12255	1.001	-7.68141	-7.67532	1.001	-0.08478	-0.08475	1.000	-0.00817	-0.00818	0.999
SECONDARY	2	1.93405	1.93460	1.000	-8.39447	-8.38510	1.001	7.44596	7.43792	1.001	0.01211	0.01210	1.001	0.00048	0.00048	1.000

TABLE 8.15 APPROXIMATE AND EXACT NATURAL FREQUENCIES OF ASSEMBLED SYSTEMS C

MASS RATIO:		0.01			0.001		
CASE	MODE	APP	EX	APP/EX	APP	EX	APP/EX
C1	1	0.96177	0.96299	0.999	0.98807	0.98819	1.000
	2	1.03682	1.03795	0.999	1.01179	1.01190	1.000
	3	1.41421	1.41136	1.002	1.41421	1.41392	1.000
	4	2.00000	2.00610	0.997	2.00000	2.00061	1.000
	5	3.00000	3.00149	1.000	3.00000	3.00015	1.000
C2	1	1.00000	0.99957	1.000	1.00000	0.99996	1.000
	2	1.98494	1.98503	1.000	1.99525	1.99526	1.000
	3	2.01494	2.01496	1.000	2.00474	2.00474	1.000
	4	2.82843	2.82781	1.000	2.82843	2.82836	1.000
	5	3.00000	3.00465	0.998	3.00000	3.00047	1.000
C3	1	0.70711	0.70086	1.009	0.70711	0.70647	1.001
	2	0.98107	0.98411	0.997	0.99405	0.99440	1.000
	3	1.01858	1.02265	0.996	1.00591	1.00629	1.000
	4	2.00000	2.00681	0.997	2.00000	2.00068	1.000
	5	3.00000	3.00204	0.999	3.00000	3.00020	1.000

TABLE 8.16 APPROXIMATE AND EXACT COMPLEX FREQUENCIES OF RESONANT MODES
MASS RATIO = 1%

CASE	DAMPING ξ_p	ξ_s	APPROXIMATE $(\xi_r\omega_r)_1$	$(\omega_r')_1$	EXACT $(\xi_r\omega_r)_1$	$(\omega_r')_1$	APPROXIMATE $(\xi_r\omega_r)_2$	$(\omega_r')_2$	EXACT $(\xi_r\omega_r)_2$	$(\omega_r')_2$
A1			0.0200	0.92772	0.0187	0.92629	0.0200	1.07228	0.0209	1.06972
B1	4%	0%	0.0200	0.98500	0.0191	0.98368	0.0200	1.01500	0.0210	1.01393
C1			0.0200	0.96828	0.0192	0.96834	0.0200	1.03172	0.0208	1.03181
A1			0.0200	0.92772	0.0154	0.92634	0.0200	1.07228	0.0251	1.06963
B1	0%	4%	0.0200	0.98500	0.0186	0.98368	0.0200	1.01500	0.0202	1.01394
C1			0.0200	0.96828	0.0178	0.96835	0.0200	1.03172	0.0224	1.03179
A1			0.0105	0.92560	0.0097	0.92453	0.0105	1.07440	0.0111	1.07200
B1	2%	0.1%	0.0105	0.97688	0.0100	0.97598	0.0105	1.02312	0.0110	1.02221
C1			0.0105	0.96372	0.0100	0.96411	0.0105	1.03628	0.0110	1.03663
A2			0.1329	2.00000	0.1328	1.99203	0.0271	2.00000	0.0273	2.00690
B2	8%	0%	0.1493	2.00000	0.1491	1.99359	0.0107	2.00000	0.0109	2.00003
C2			0.1586	2.00000	0.1580	1.99929	0.0014	2.00000	0.0017	2.00084
A2			0.1329	2.00000	0.1353	2.00769	0.0271	2.00000	0.0258	1.99117
B2	0%	8%	0.1493	2.00000	0.1494	1.99633	0.0107	2.00000	0.0107	1.99729
C2			0.1586	2.00000	0.1548	1.99479	0.0014	2.00000	0.0018	1.99894
A2			0.0410	1.95440	0.0438	1.95542	0.0410	2.04560	0.0403	2.04823
B2	4%	0.1%	0.0410	1.99111	0.0426	1.98652	0.0410	2.00889	0.0414	2.01184
C2			0.0770	2.00000	0.0766	1.99976	0.0050	2.00000	0.0072	2.00084
A3	7%	0%	0.0350	0.93367	0.0272	0.94226	0.0350	1.06633	0.0424	1.07401
B3	7%	0%	0.0595	1.00000	0.0595	0.99764	0.0105	1.00000	0.0595	0.99764
C3	5.6%	0%	0.0646	1.00000	0.0638	1.00773	0.0054	1.00000	0.0056	0.99647
A3	0%	7%	0.0350	0.93367	0.0340	0.94207	0.0350	1.06633	0.0398	1.07407
B3	0%	7%	0.0595	1.00000	0.0591	0.99771	0.0105	1.00000	0.0109	1.00077
C3	0%	5.6%	0.0646	1.00000	0.0638	0.99521	0.0054	1.00000	0.0078	1.00893
A3	2%	0.17%	0.0109	0.92556	0.0087	0.93576	0.0109	1.07444	0.0131	1.08272
B3	2%	0.17%	0.0109	0.97673	0.0099	0.97718	0.0109	1.02328	0.0119	1.02350
C3	2%	0.14%	0.0109	0.98362	0.0076	0.98595	0.0109	1.01638	0.0140	1.02060

TABLE 8.17 APPROXIMATE AND EXACT COMPLEX FREQUENCIES OF RESONANT MODES
MASS RATIO = 0.1%

CASE	DAMPING		APPROXIMATE		EXACT		APPROXIMATE		EXACT	
	ξ_p	ξ_S	$(\xi_r\omega_r)_1$	$(\omega_r')_1$	$(\xi_r\omega_r)_1$	$(\omega_r')_1$	$(\xi_r\omega_r)_2$	$(\omega_r')_2$	$(\xi_r\omega_r)_2$	$(\omega_r')_2$
A1			0.0200	0.98725	0.0199	0.98665	0.0200	1.01275	0.0201	1.01223
B1	4%	0%	0.0016	1.00000	0.0016	0.99983	0.0384	1.00000	0.0384	0.99921
C1			0.0361	1.00000	0.0361	0.99936	0.0039	1.00000	0.0039	0.99994
A1			0.0200	0.98725	0.0178	0.98659	0.0200	1.01275	0.0222	1.01229
B1	0%	4%	0.0384	1.00000	0.0384	0.99924	0.0016	1.00000	0.0016	0.99981
C1			0.0361	1.00000	0.0361	0.99964	0.0039	1.00000	0.0039	0.99965
A1			0.0105	0.97827	0.0103	0.97802	0.0105	1.02173	0.0107	1.02145
B1	2%	0.1%	0.0052	1.00000	0.0052	0.99977	0.0158	1.00000	0.0158	0.99986
C1			0.0105	0.99290	0.0104	0.99285	0.0105	1.00710	0.0106	1.00705
A2			0.1577	2.00000	0.1577	1.99359	0.0023	2.00000	0.0023	2.00054
B2	8%	0%	0.1590	2.00000	0.1590	1.99359	0.0010	2.00000	0.0010	2.00000
C2			0.1599	2.00000	0.1598	1.99352	0.0001	2.00000	0.0002	2.00008
A2			0.1577	2.00000	0.1578	1.99468	0.0023	2.00000	0.0023	1.99944
B2	0%	8%	0.1590	2.00000	0.1590	1.99383	0.0010	2.00000	0.0010	1.99976
C2			0.1599	2.00000	0.1598	1.99371	0.0001	2.00000	0.0002	1.99990
A2			0.0751	2.00000	0.0749	1.99839	0.0069	2.00000	0.0091	2.00054
B2	4%	0.1%	0.0779	2.00000	0.0778	1.99841	0.0041	2.00000	0.0062	1.99998
C2			0.0797	2.00000	0.0797	1.99832	0.0023	2.00000	0.0043	2.00008
A3	7%	0%	0.0607	1.00000	0.0607	1.00019	0.0093	1.00000	0.0093	0.99927
B3	7%	0%	0.0691	1.00000	0.0691	0.99780	0.0009	1.00000	0.0009	0.99984
C3	5.6%	0%	0.05537	1.00000	0.05527	0.99938	0.00063	1.00000	0.00067	0.99973
A3	0%	7%	0.0607	1.00000	0.0608	0.99900	0.0093	1.00000	0.00096	1.00045
B3	0%	7%	0.0691	1.00000	0.0691	0.99758	0.0009	1.00000	0.0009	1.00006
C3	0%	5.6%	0.05537	1.00000	0.05531	0.99827	0.00063	1.00000	0.00083	1.00083
A3	2%	0.17%	0.0109	0.97811	0.0102	0.97899	0.0109	1.02189	0.0116	1.02273
B3	2%	0.17%	0.0154	1.00000	0.0154	1.00027	0.0063	1.00000	0.0063	0.99962
C3	2%	0.14%	0.00355	1.00000	0.00354	0.99955	0.01786	1.00000	0.01786	1.00093

TABLE 8.18 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 1%

E_Q	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
ELCENTRO	A1	0	1	0.910	1.345	0.783	0.766
			2	1.963	1.224	0.753	0.738
		2	1	0.634	1.499	0.881	0.769
			2	1.236	1.506	0.929	0.815
		10	1	0.183	2.744	1.581	0.854
			2	0.359	2.689	1.613	0.849
	A2	0	1	0.336	1.488	0.967	0.909
			2	0.622	1.517	1.005	0.939
		2	1	0.118	2.807	1.735	0.953
			2	0.195	3.189	2.035	1.014
		10	1	0.039	5.043	3.162	1.048
			2	0.053	7.017	4.524	1.034
	A3	0	1	0.948	1.331	0.830	0.816
			2	1.809	1.300	0.806	0.789
		2	1	0.530	1.614	0.965	0.803
			2	0.983	1.677	1.033	0.846
		10	1	0.127	3.224	1.849	1.042
			2	0.228	3.500	2.109	1.180
TAFT	A1	0	1	0.364	1.611	0.962	0.914
			2	0.755	1.502	0.962	0.916
		2	1	0.209	1.822	1.097	0.903
			2	0.476	1.554	0.991	0.821
		10	1	0.066	3.191	1.921	0.977
			2	0.131	3.116	1.931	0.927
	A2	0	1	0.187	1.284	0.837	0.773
			2	0.372	1.278	0.843	0.779
		2	1	0.047	2.913	1.829	0.919
			2	0.084	3.131	2.011	0.964
		10	1	0.013	5.169	3.164	1.078
			2	0.019	6.962	4.390	0.991
	A3	0	1	0.384	1.552	0.991	0.954
			2	0.774	1.393	0.876	0.833
		2	1	0.193	1.727	1.034	0.829
			2	0.398	1.593	0.974	0.755
		10	1	0.073	2.612	1.498	0.856
			2	0.138	2.753	1.653	0.964
PACOIMA	A1	0	1	2.104	1.298	0.826	0.817
			2	4.469	1.183	0.776	0.767
		2	1	1.404	1.438	0.913	0.825
			2	2.874	1.346	0.874	0.783
		10	1	0.486	2.779	1.788	0.953
			2	0.871	3.019	1.966	0.971
	A2	0	1	0.745	1.483	0.931	0.767
			2	1.402	1.497	0.962	0.779
		2	1	0.246	2.651	1.606	0.766
			2	0.451	2.725	1.704	0.714
		10	1	0.071	4.528	2.642	1.178
			2	0.086	6.825	4.100	1.166
	A3	0	1	2.509	1.236	0.721	0.714
			2	5.228	1.186	0.731	0.724
		2	1	1.486	1.371	0.783	0.668
			2	2.787	1.495	0.925	0.810
		10	1	0.538	2.446	1.409	0.795
			2	1.096	2.449	1.487	0.914

TABLE 8.19 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 0.1%

E_Q	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
EL CENTRO	A1	0	1	2.025	1.762	1.152	0.953
			2	4.065	1.730	1.152	0.955
		2	1	0.878	3.224	2.121	1.058
			2	1.798	3.103	2.075	1.034
		10	1	0.240	6.094	3.966	0.835
			2	0.471	6.099	4.035	0.785
	A2	0	1	0.373	3.296	2.211	1.367
			2	0.701	3.438	2.344	1.440
		2	1	0.125	7.442	4.974	0.986
			2	0.208	8.739	5.950	1.059
		10	1	0.040	14.195	9.559	1.047
			2	0.054	20.447	13.984	1.036
	A3	0	1	2.045	1.731	1.150	0.956
			2	4.233	1.644	1.102	0.912
		2	1	0.697	3.538	2.324	0.937
			2	1.402	3.483	2.321	0.942
		10	1	0.139	7.908	5.071	1.013
			2	0.259	8.410	5.486	1.131
TAFT	A1	0	1	0.439	2.317	1.477	1.029
			2	0.825	2.381	1.567	1.084
		2	1	0.237	3.728	2.443	1.051
			2	0.527	3.269	2.188	0.923
		10	1	0.073	8.243	5.454	1.056
			2	0.151	7.861	5.275	0.935
	A2	0	1	0.305	2.281	1.567	0.952
			2	0.625	2.212	1.527	0.929
		2	1	0.056	7.348	4.983	0.887
			2	0.100	8.190	5.611	0.954
		10	1	0.014	14.251	9.499	1.070
			2	0.019	19.946	13.532	0.986
	A3	0	1	0.428	2.392	1.533	1.070
			2	0.846	2.355	1.555	1.087
		2	1	0.223	3.741	2.420	0.867
			2	0.409	4.033	2.663	0.990
		10	1	0.078	6.325	4.015	0.814
			2	0.154	6.387	4.122	0.981
PACOIMA	A1	0	1	6.142	1.356	0.922	0.829
			2	12.339	1.331	0.916	0.823
		2	1	2.588	2.316	1.565	0.871
			2	4.958	2.382	1.632	0.904
		10	1	0.559	7.286	4.956	0.991
			2	1.012	7.978	5.474	1.017
	A2	0	1	0.514	4.717	3.138	1.314
			2	0.894	5.296	3.584	1.455
		2	1	0.257	7.072	4.696	0.775
			2	0.466	7.633	5.164	0.742
		10	1	0.071	11.940	7.711	1.186
			2	0.087	18.915	12.555	1.167
	A3	0	1	6.601	1.323	0.857	0.772
			2	12.722	1.366	0.899	0.811
		2	1	2.086	2.744	1.762	0.777
			2	4.328	2.636	1.722	0.793
		10	1	0.579	6.220	3.984	0.800
			2	1.213	5.935	3.845	0.902

TABLE 8.20 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 1%

E_Q	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
EL CENTRO	B1	0	1	0.667	1.781	1.115	0.942
EL CENTRO	B1	0	2	1.351	1.701	1.093	0.920
EL CENTRO	B1	2	1	0.319	2.956	1.870	1.008
EL CENTRO	B1	2	2	0.583	3.126	2.025	1.062
EL CENTRO	B1	10	1	0.098	5.052	3.150	0.954
EL CENTRO	B1	10	2	0.162	5.832	3.730	0.944
EL CENTRO	B2	0	1	0.188	2.255	1.501	1.291
EL CENTRO	B2	0	2	0.365	2.284	1.527	1.308
EL CENTRO	B2	2	1	0.066	4.519	2.979	1.074
EL CENTRO	B2	2	2	0.121	4.856	3.236	1.107
EL CENTRO	B2	10	1	0.019	9.482	6.303	1.056
EL CENTRO	B2	10	2	0.027	13.295	8.904	1.079
EL CENTRO	B3	0	1	0.744	1.621	0.994	0.841
EL CENTRO	B3	0	2	1.519	1.561	0.992	0.844
EL CENTRO	B3	2	1	0.247	3.442	2.085	0.945
EL CENTRO	B3	2	2	0.545	3.095	1.952	0.962
EL CENTRO	B3	10	1	0.076	5.181	3.028	1.027
EL CENTRO	B3	10	2	0.163	4.751	2.902	1.138
TAFT	B1	0	1	0.157	2.268	1.356	0.998
TAFT	B1	0	2	0.282	2.342	1.472	1.058
TAFT	B1	2	1	0.082	3.637	2.276	1.109
TAFT	B1	2	2	0.156	3.650	2.365	1.085
TAFT	B1	10	1	0.031	6.449	4.119	1.168
TAFT	B1	10	2	0.056	6.909	4.512	1.053
TAFT	B2	0	1	0.176	1.301	0.878	0.746
TAFT	B2	0	2	0.347	1.335	0.896	0.762
TAFT	B2	2	1	0.032	4.083	2.714	0.927
TAFT	B2	2	2	0.065	4.041	2.687	0.915
TAFT	B2	10	1	0.005	11.749	7.679	1.216
TAFT	B2	10	2	0.008	14.857	9.787	1.135
TAFT	B3	0	1	0.176	2.084	1.210	0.887
TAFT	B3	0	2	0.322	2.177	1.353	1.007
TAFT	B3	2	1	0.078	3.761	2.228	0.969
TAFT	B3	2	2	0.197	2.955	1.838	0.885
TAFT	B3	10	1	0.049	3.675	2.134	0.829
TAFT	B3	10	2	0.089	3.991	2.411	1.071
PACOIMA	B1	0	1	2.122	1.293	0.860	0.784
PACOIMA	B1	0	2	4.053	1.310	0.889	0.808
PACOIMA	B1	2	1	0.919	2.155	1.417	0.846
PACOIMA	B1	2	2	1.755	2.179	1.465	0.854
PACOIMA	B1	10	1	0.241	5.550	3.702	1.036
PACOIMA	B1	10	2	0.407	6.407	4.331	1.018
PACOIMA	B2	0	1	0.428	2.108	1.382	0.947
PACOIMA	B2	0	2	0.790	2.254	1.480	1.001
PACOIMA	B2	2	1	0.142	4.116	2.679	0.784
PACOIMA	B2	2	2	0.274	4.229	2.776	0.778
PACOIMA	B2	10	1	0.039	7.051	4.438	0.810
PACOIMA	B2	10	2	0.047	11.453	7.327	0.946
PACOIMA	B3	0	1	2.463	1.252	0.746	0.683
PACOIMA	B3	0	2	4.807	1.280	0.794	0.730
PACOIMA	B3	2	1	0.829	2.466	1.454	0.769
PACOIMA	B3	2	2	1.884	2.168	1.330	0.758
PACOIMA	B3	10	1	0.342	3.794	2.233	0.838
PACOIMA	B3	10	2	0.713	3.653	2.217	0.962

TABLE 8.21 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 0.1%

EQ	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
EL CENTRO	B1	0	1	0.857	3.981	2.685	1.189
			2	1.810	3.721	2.539	1.120
		2	1	0.337	8.187	5.548	1.102
			2	0.628	8.688	5.950	1.146
		10	1	0.101	14.270	9.607	0.965
			2	0.171	16.660	11.351	0.943
	B2	0	1	0.254	4.679	3.227	1.487
			2	0.492	4.814	3.332	1.531
		2	1	0.068	13.247	9.142	1.112
			2	0.124	14.405	9.987	1.148
		10	1	0.019	28.599	19.793	1.059
			2	0.027	40.459	28.097	1.084
	B3	0	1	0.923	3.713	2.491	1.104
			2	1.988	3.417	2.317	1.033
		2	1	0.269	9.103	6.078	0.947
			2	0.567	8.566	5.781	0.992
		10	1	0.076	14.206	9.301	1.033
			2	0.163	13.090	8.678	1.150
TAFT	B1	0	1	0.188	5.201	3.441	1.091
			2	0.338	5.627	3.811	1.176
		2	1	0.088	9.793	6.602	1.145
			2	0.163	10.382	7.110	1.151
		10	1	0.032	18.679	12.695	1.204
			2	0.057	20.462	14.051	1.087
	B2	0	1	0.222	3.071	2.138	0.971
			2	0.433	3.157	2.193	0.998
		2	1	0.034	11.740	8.136	0.929
			2	0.069	11.719	8.122	0.922
		10	1	0.005	35.221	24.240	1.222
			2	0.008	44.645	30.835	1.136
	B3	0	1	0.198	4.986	3.264	1.029
			2	0.356	5.433	3.638	1.178
		2	1	0.081	10.122	6.679	0.993
			2	0.200	8.097	5.416	0.917
		10	1	0.049	9.812	6.373	0.828
			2	0.090	10.647	6.987	1.074
PACOIMA	B1	0	1	2.595	3.070	2.116	1.198
			2	5.064	3.113	2.166	1.223
		2	1	1.046	5.585	3.838	0.873
			2	2.012	5.741	3.985	0.884
		10	1	0.244	16.538	11.427	1.040
			2	0.417	19.181	13.339	1.015
	B2	0	1	0.353	6.405	4.385	1.258
			2	0.645	6.991	4.798	1.349
		2	1	0.147	11.932	8.202	0.783
			2	0.279	12.516	8.634	0.795
		10	1	0.039	20.538	13.936	0.811
			2	0.047	33.665	23.010	0.944
	B3	0	1	2.990	2.789	1.838	1.047
			2	5.517	3.004	2.000	1.146
		2	1	0.906	6.205	4.069	0.764
			2	1.976	5.660	3.748	0.773
		10	1	0.345	10.238	6.693	0.835
			2	0.722	9.752	6.419	0.956

TABLE 8.22 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 1%

E_Q	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
EL CENTRO	C1	0	1	0.719	1.580	1.028	0.945
			2	1.429	1.548	1.036	0.951
			3	2.086	1.547	1.060	0.974
		2	1	0.413	2.184	1.432	0.966
			2	0.781	2.251	1.512	1.010
			3	1.152	2.227	1.528	1.017
		10	1	0.125	3.704	2.400	0.838
			2	0.233	3.848	2.559	0.794
			3	0.338	3.837	2.617	0.738
	C2	0	1	0.122	3.497	2.148	1.174
			2	0.221	3.941	2.389	1.343
			3	0.294	3.916	2.647	1.391
		2	1	0.046	7.052	4.334	1.043
			2	0.110	5.987	3.623	1.053
			3	0.089	9.766	6.591	1.017
		10	1	0.021	9.375	5.828	1.067
			2	0.055	7.193	4.412	1.036
			3	0.027	19.177	13.014	1.053
	C3	0	1	0.594	1.913	1.220	0.943
			2	1.085	1.975	1.311	1.032
			3	0.630	1.881	1.284	1.200
		2	1	0.223	3.808	2.392	0.923
			2	0.472	3.423	2.255	1.063
			3	0.512	1.825	1.230	1.073
		10	1	0.085	5.019	2.954	0.933
			2	0.157	5.171	3.195	1.330
			3	0.236	2.279	1.448	1.302
TAFT	C1	0	1	0.189	2.026	1.267	1.060
			2	0.342	2.138	1.411	1.179
			3	0.525	2.037	1.382	1.159
		2	1	0.109	2.835	1.819	1.089
			2	0.206	2.891	1.922	1.133
			3	0.325	2.670	1.818	1.069
		10	1	0.040	4.910	3.215	1.101
			2	0.078	4.834	3.232	0.976
			3	0.108	5.087	3.484	0.937
	C2	0	1	0.084	2.889	1.791	0.943
			2	0.202	2.410	1.507	0.827
			3	0.241	2.738	1.869	0.966
		2	1	0.019	7.703	4.745	0.958
			2	0.054	5.349	3.289	0.842
			3	0.040	9.684	6.580	0.920
		10	1	0.009	7.968	4.814	1.018
			2	0.022	6.507	3.890	1.074
			3	0.010	18.979	12.756	1.052
	C3	0	1	0.182	2.100	1.172	0.806
			2	0.310	2.293	1.376	0.993
			3	0.351	1.493	0.935	0.894
		2	1	0.092	3.419	1.968	0.896
			2	0.215	2.782	1.691	0.901
			3	0.276	1.522	0.969	0.897
		10	1	0.053	3.741	2.150	0.895
			2	0.092	4.121	2.472	1.236
			3	0.143	1.921	1.246	1.161
PACOIMA	C1	0	1	2.272	1.198	0.809	0.775
			2	4.402	1.205	0.828	0.792
			3	6.511	1.197	0.829	0.793
		2	1	1.094	1.772	1.190	0.877
			2	2.188	1.725	1.182	0.860
			3	3.199	1.734	1.199	0.867
		10	1	0.307	4.259	2.889	1.022
			2	0.573	4.481	3.075	0.974
			3	0.789	4.764	3.309	0.961
	C2	0	1	0.201	4.245	2.527	1.013
			2	0.405	4.272	2.504	1.039
			3	0.369	6.012	4.028	1.370
		2	1	0.092	7.112	4.237	1.017
			2	0.254	5.242	3.098	0.966
			3	0.191	8.955	5.988	0.759
		10	1	0.058	5.418	3.101	0.973
			2	0.139	4.696	2.670	1.086
			3	0.048	16.353	10.705	1.190
	C3	0	1	2.210	1.414	0.827	0.724
			2	3.958	1.520	0.928	0.824
			3	3.427	1.210	0.780	0.763
		2	1	0.928	2.341	1.352	0.721
			2	1.881	2.227	1.345	0.805
			3	2.255	1.328	0.861	0.809
		10	1	0.388	3.660	2.128	0.870
			2	0.749	3.674	2.213	1.101
			3	1.135	1.729	1.131	1.052

TABLE 8.23 APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS AS COMPUTED WITH THREE DIFFERENT RULES, MASS RATIO = 0.1%

E_Q	CASE	ξ (%)	ELEM	EXACT	ABS. SUM / EXACT	SRSS / EXACT	ROSENBLUETH / EXACT
EL CENTRO	C1	0	1	1.282	2.626	1.802	1.068
			2	2.393	2.784	1.931	1.143
			3	3.614	2.740	1.917	1.134
		2	1	0.455	5.954	4.099	1.150
			2	0.918	5.487	4.063	1.125
			3	1.344	5.938	4.157	1.144
		10	1	0.134	10.576	7.242	0.857
			2	0.253	11.049	7.647	0.802
			3	0.368	11.259	7.867	0.747
	C2	0	1	0.131	9.335	6.271	1.252
			2	0.225	10.926	7.288	1.497
			3	0.326	10.790	7.515	1.441
		2	1	0.046	20.043	13.462	1.044
			2	0.111	16.731	11.144	1.048
			3	0.090	29.764	20.720	1.022
		10	1	0.021	27.032	18.249	1.071
			2	0.055	20.512	13.753	1.036
			3	0.028	58.799	41.011	1.054
	C3	0	1	0.688	4.868	3.322	1.158
			2	1.274	5.190	3.583	1.273
			3	0.804	4.330	2.879	1.280
		2	1	0.231	11.001	7.470	0.954
			2	0.502	10.024	6.883	1.086
			3	0.547	4.898	3.218	1.088
		10	1	0.088	13.622	8.957	0.922
			2	0.164	14.419	9.604	1.331
			3	0.250	5.351	3.294	1.292
TAFT	C1	0	1	0.236	4.069	2.740	1.199
			2	0.441	4.246	2.920	1.266
			3	0.672	4.136	2.871	1.239
		2	1	0.121	7.049	4.809	1.151
			2	0.225	7.437	5.144	1.195
			3	0.356	6.984	4.873	1.110
		10	1	0.041	14.362	9.862	1.131
			2	0.081	14.254	9.876	0.997
			3	0.113	15.219	10.645	0.955
	C2	0	1	0.087	8.038	5.424	1.039
			2	0.209	6.660	4.498	0.896
			3	0.249	8.084	5.649	1.060
		2	1	0.019	21.794	14.675	0.957
			2	0.055	15.090	10.134	0.830
			3	0.041	29.359	20.489	0.919
		10	1	0.009	22.437	14.972	1.019
			2	0.022	18.143	12.020	1.076
			3	0.010	57.787	40.174	1.051
	C3	0	1	0.187	5.446	3.462	0.934
			2	0.368	5.418	3.521	1.008
			3	0.375	3.181	1.873	0.962
		2	1	0.095	9.018	5.847	0.900
			2	0.224	7.569	4.979	0.919
			3	0.289	3.412	2.058	0.913
		10	1	0.054	9.951	6.412	0.902
			2	0.094	11.183	7.285	1.263
			3	0.148	4.205	2.512	1.177
PACOIMA	C1	0	1	3.730	2.148	1.490	1.069
			2	7.339	2.164	1.513	1.084
			3	10.932	2.168	1.522	1.089
		2	1	1.478	3.931	2.723	0.881
			2	2.880	3.996	2.793	0.893
			3	4.272	4.020	2.821	0.897
		10	1	0.320	12.501	8.692	1.023
			2	0.598	13.316	9.310	0.983
			3	0.829	14.312	10.056	0.976
	C2	0	1	0.193	11.881	7.848	1.072
			2	0.407	11.348	7.437	1.050
			3	0.352	18.548	12.863	1.465
		2	1	0.092	19.844	13.166	1.020
			2	0.254	14.423	9.502	0.966
			3	0.192	27.174	18.860	0.762
		10	1	0.058	14.493	9.410	0.974
			2	0.139	12.291	7.890	1.086
			3	0.049	48.910	33.682	1.191
	C3	0	1	2.531	3.313	2.159	1.013
			2	4.298	3.861	2.541	1.207
			3	3.927	2.140	1.477	0.877
		2	1	0.963	6.123	3.965	0.733
			2	2.028	5.764	3.771	0.803
			3	2.405	2.846	1.712	0.807
		10	1	0.394	9.806	6.355	0.875
			2	0.766	10.001	6.534	1.117
			3	1.175	3.825	2.300	1.064

TABLE 8.24 MEAN AND COEFFICIENT OF VARIATION FOR THREE EARTHQUAKES OF THE APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS COMPUTED WITH THREE DIFFERENT RULES

DAMPING = 0%

MASS RATIO		0.01							0.001						
CASE	ELEM	$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT		$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT	
			μ	c.o.v.	μ	c.o.v.	μ	c.o.v.		μ	c.o.v.	μ	c.o.v.	μ	c.o.v.
A1	1	0.07443	1.418	0.119	0.857	0.109	0.832	0.090	0.02370	1.812	0.266	1.167	0.217	0.937	0.108
	2		1.303	0.133	0.830	0.138	0.807	0.118		1.814	0.292	1.212	0.272	0.954	0.137
A3	1	0.07333	1.373	0.118	0.847	0.160	0.828	0.145	0.02366	1.815	0.297	1.180	0.287	0.933	0.161
	2		1.293	0.080	0.804	0.090	0.782	0.070		1.788	0.285	1.185	0.283	0.937	0.149
C1	1	0.03748	1.601	0.259	1.035	0.221	0.927	0.155	0.01190	2.948	0.339	2.011	0.324	1.112	0.068
	2		1.630	0.289	1.092	0.271	0.974	0.200		3.065	0.349	2.121	0.341	1.164	0.080
	3		1.594	0.265	1.090	0.255	0.975	0.188		3.015	0.336	2.103	0.330	1.154	0.067
A2	1	0.02984	1.418	0.082	0.912	0.074	0.816	0.098	0.00948	3.431	0.357	2.305	0.343	1.211	0.187
	2		1.431	0.093	0.937	0.090	0.832	0.111		3.649	0.426	2.485	0.417	1.275	0.235
B1	1	0.02499	1.781	0.274	1.110	0.223	0.908	0.122	0.00791	4.084	0.262	2.747	0.242	1.159	0.051
	2		1.784	0.292	1.151	0.257	0.929	0.135		4.154	0.316	2.839	0.304	1.173	0.044
B3	1	0.02485	1.652	0.252	0.983	0.236	0.804	0.133	0.00790	3.829	0.288	2.531	0.282	1.060	0.037
	2		1.673	0.274	1.046	0.271	0.860	0.162		3.951	0.329	2.652	0.328	1.119	0.068
B2	1	0.01998	1.888	0.272	1.254	0.264	0.995	0.277	0.00632	4.718	0.353	3.250	0.346	1.239	0.209
	2		1.950	0.276	1.301	0.270	1.024	0.267		4.987	0.386	3.441	0.380	1.293	0.210
C3	1	0.01938	1.809	0.196	1.073	0.200	0.824	0.134	0.00580	4.542	0.243	2.981	0.240	1.035	0.110
	2		1.929	0.201	1.205	0.201	0.950	0.116		4.823	0.174	3.215	0.182	1.163	0.119
	3		1.528	0.220	1.000	0.258	0.952	0.235		3.317	0.287	2.076	0.348	1.040	0.204
C2	1	0.00750	3.769	0.286	2.155	0.171	1.043	0.114	0.00250	9.751	0.200	6.514	0.189	1.121	0.102
	2		3.541	0.281	2.133	0.256	1.070	0.242		9.645	0.269	6.408	0.258	1.148	0.272
	3		4.222	0.393	2.848	0.384	1.242	0.193		12.474	0.435	8.676	0.432	1.322	0.172

TABLE 8.25 MEAN AND COEFFICIENT OF VARIATION FOR THREE EARTHQUAKES OF THE APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS COMPUTED WITH THREE DIFFERENT RULES

DAMPING = 2%

MASS RATIO		0.01							0.001						
CASE	ELEM	$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT		$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT	
			μ	C.O.V.	μ	C.O.V.	μ	C.O.V.		μ	C.O.V.	μ	C.O.V.	μ	C.O.V.
A1	1	0.07443	1.586	0.130	0.964	0.121	0.832	0.081	0.02370	3.089	0.232	2.043	0.217	0.993	0.107
	2		1.469	0.074	0.931	0.063	0.806	0.025		2.918	0.162	1.965	0.150	0.954	0.074
A3	1	0.07333	1.571	0.116	0.927	0.140	0.785	0.083	0.02366	3.341	0.158	2.169	0.164	0.860	0.093
	2		1.588	0.057	0.977	0.055	0.804	0.057		3.384	0.208	2.235	0.213	0.908	0.113
C1	1	0.03748	2.264	0.237	1.480	0.214	0.977	0.109	0.01190	5.645	0.280	3.877	0.274	1.061	0.147
	2		2.289	0.255	1.539	0.241	1.001	0.137		5.760	0.299	4.000	0.294	1.071	0.148
	3		2.210	0.212	1.515	0.204	0.984	0.107		5.647	0.266	3.950	0.264	1.050	0.127
A2	1	0.02984	2.790	0.047	1.723	0.065	0.879	0.113	0.00948	7.287	0.026	4.884	0.033	0.883	0.120
	2		3.015	0.084	1.917	0.096	0.897	0.179		8.187	0.068	5.575	0.071	0.918	0.176
B1	1	0.02499	2.916	0.254	1.854	0.232	0.988	0.134	0.00791	7.855	0.270	5.329	0.262	1.040	0.141
	2		2.985	0.250	1.952	0.233	1.000	0.127		8.270	0.284	5.682	0.278	1.060	0.144
B3	1	0.02485	3.223	0.209	1.922	0.214	0.894	0.122	0.00790	8.477	0.240	5.609	0.244	0.901	0.134
	2		2.739	0.182	1.707	0.194	0.868	0.119		7.441	0.210	4.982	0.218	0.894	0.124
B2	1	0.01998	4.239	0.057	2.791	0.059	0.928	0.156	0.00632	12.306	0.067	8.493	0.066	0.941	0.175
	2		4.375	0.098	2.900	0.102	0.933	0.177		12.880	0.107	8.914	0.108	0.955	0.187
C3	1	0.01938	3.189	0.238	1.904	0.275	0.847	0.130	0.00580	8.714	0.282	5.761	0.304	0.862	0.134
	2		2.811	0.213	1.764	0.260	0.923	0.141		7.786	0.275	5.211	0.311	0.936	0.152
	3		1.558	0.161	1.020	0.186	0.926	0.145		3.719	0.285	2.329	0.339	0.936	0.152
C2	1	0.00750	7.289	0.049	4.439	0.061	1.006	0.043	0.00250	20.560	0.052	13.768	0.058	1.007	0.045
	2		5.526	0.073	3.337	0.080	0.954	0.111		15.415	0.077	10.260	0.081	0.948	0.116
	3		9.468	0.047	6.386	0.054	0.899	0.145		28.766	0.048	20.023	0.051	0.901	0.145

TABLE 8.26 MEAN AND COEFFICIENT OF VARIATION FOR THREE EARTHQUAKES OF THE APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS COMPUTED WITH THREE DIFFERENT RULES
DAMPING = 10%

MASS RATIO		0.01							0.001						
CASE	ELEM	$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT		$\frac{\Delta\omega}{\omega_I + \omega_J}$	ABS. SUM/EXACT		SRSS/EXACT		ROSENBLUETH/EXACT	
			μ	C.O.V.	μ	C.O.V.	μ	C.O.V.		μ	C.O.V.	μ	C.O.V.	μ	C.O.V.
A1	1	0.07443	2.905	0.086	1.763	0.097	0.928	0.070	0.02370	7.208	0.149	4.792	0.158	0.961	0.118
	2		2.941	0.076	1.837	0.106	0.916	0.067		7.313	0.144	4.928	0.158	0.912	0.129
A3	1	0.07333	2.761	0.148	1.585	0.147	0.898	0.143	0.02366	6.818	0.139	4.357	0.142	0.876	0.136
	2		2.901	0.186	1.750	0.184	1.019	0.139		6.911	0.191	4.484	0.196	1.005	0.116
C1	1	0.03748	4.291	0.141	2.835	0.145	0.987	0.137	0.01190	12.480	0.152	8.599	0.153	1.004	0.138
	2		4.388	0.114	2.955	0.119	0.915	0.114		12.873	0.128	8.944	0.130	0.927	0.117
	3		4.563	0.142	3.137	0.146	0.879	0.139		13.597	0.153	9.523	0.154	0.893	0.142
A2	1	0.02984	4.913	0.069	2.989	0.101	1.101	0.062	0.00948	13.462	0.098	8.923	0.118	1.101	0.068
	2		6.935	0.014	4.338	0.050	1.064	0.086		19.769	0.040	13.357	0.055	1.063	0.088
B1	1	0.02499	5.684	0.125	3.657	0.133	1.053	0.103	0.00791	16.496	0.134	11.243	0.138	1.070	0.114
	2		6.383	0.084	4.191	0.098	1.005	0.055		18.768	0.103	12.914	0.108	1.015	0.071
B3	1	0.02485	4.217	0.199	2.465	0.199	0.898	0.125	0.00790	11.419	0.212	7.456	0.215	0.899	0.130
	2		4.132	0.136	2.510	0.141	1.057	0.084		11.163	0.155	7.361	0.160	1.060	0.092
B2	1	0.01998	9.427	0.249	6.140	0.265	1.027	0.199	0.00632	28.119	0.262	19.323	0.267	1.031	0.201
	2		13.202	0.129	8.673	0.144	1.053	0.092		39.590	0.140	27.314	0.145	1.055	0.094
C3	1	0.01938	4.140	0.184	2.411	0.195	0.899	0.035	0.00580	11.126	0.194	7.241	0.205	0.900	0.026
	2		4.322	0.178	2.627	0.194	1.222	0.094		11.868	0.193	7.808	0.205	1.237	0.088
	3		1.976	0.141	1.275	0.126	1.172	0.107		4.460	0.178	2.702	0.194	1.178	0.097
C2	1	0.00750	7.587	0.264	4.582	0.301	1.019	0.046	0.00250	21.321	0.298	14.210	0.314	1.021	0.048
	2		6.132	0.210	3.657	0.244	1.065	0.025		16.982	0.249	11.221	0.268	1.066	0.025
	3		18.169	0.087	12.158	0.104	1.098	0.072		55.165	0.099	38.289	0.105	1.099	0.073

TABLE 8.27 GROUP AVERAGE STATISTICS OF APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS OF SECONDARY SYSTEMS FOR THREE APPROXIMATE RULES

DAMPING	ABS. SUM/EXACT				SRSS/EXACT				ROSENBLUETH/EXACT			
	μ	c.o.v.	MAX	MIN	μ	c.o.v.	MAX	MIN	μ	c.o.v.	MAX	MIN
0%	3.195	0.755	12.474	1.293	2.113	0.785	8.676	0.804	1.022	0.149	1.322	0.782
2%	6.108	0.890	28.766	1.469	4.072	0.923	20.023	0.927	0.934	0.077	1.071	0.785
10%	11.164	0.923	55.165	1.976	7.441	0.966	38.289	1.275	1.015	0.093	1.237	0.876

TABLE 8.28 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
A1	0	0	1	0.719	0.910	0.790	0.360	0.364	0.989	1.814	2.104	0.862	0.880	0.114
			2	1.427	1.963	0.727	0.710	0.755	0.940	3.608	4.469	0.807	0.825	0.130
	2	2	1	0.497	0.634	0.784	0.209	0.209	1.000	1.168	1.404	0.832	0.872	0.130
			2	0.984	1.236	0.796	0.414	0.476	0.870	2.320	2.874	0.807	0.824	0.048
	10	10	1	0.170	0.183	0.929	0.073	0.066	1.106	0.441	0.486	0.907	0.981	0.111
			2	0.333	0.359	0.928	0.144	0.131	1.099	0.872	0.871	1.001	1.009	0.085
A2	0	0	1	0.249	0.336	0.741	0.159	0.187	0.850	0.496	0.745	0.666	0.752	0.123
			2	0.486	0.622	0.781	0.316	0.372	0.849	0.960	1.402	0.685	0.772	0.107
	2	4	1	0.115	0.118	0.975	0.045	0.047	0.957	0.196	0.246	0.797	0.910	0.108
			2	0.208	0.195	1.067	0.086	0.084	1.024	0.341	0.451	0.756	0.949	0.178
	10	20	1	0.035	0.039	0.897	0.013	0.013	1.000	0.074	0.071	1.042	0.980	0.076
			2	0.049	0.053	0.925	0.018	0.019	0.947	0.096	0.086	1.116	0.996	0.105
A3	0	0	1	0.712	0.948	0.751	0.362	0.384	0.943	1.944	2.509	0.775	0.823	0.127
			2	1.420	1.809	0.785	0.722	0.774	0.933	3.882	5.228	0.743	0.820	0.122
	3.5	2	1	0.420	0.530	0.792	0.177	0.193	0.917	1.106	1.486	0.744	0.818	0.109
			2	0.838	0.983	0.852	0.353	0.398	0.887	2.207	2.787	0.792	0.844	0.057
	17.5	10	1	0.150	0.127	1.181	0.069	0.073	0.945	0.465	0.538	0.864	0.997	0.165
			2	0.300	0.228	1.316	0.138	0.138	1.000	0.928	1.096	0.847	1.054	0.227

TABLE 8.29 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	$\frac{APP}{EX}$	APP	EX	$\frac{APP}{EX}$	APP	EX	$\frac{APP}{EX}$		
A1	0	0	1	2.071	2.025	1.023	0.513	0.439	1.169	5.315	6.142	0.865	1.019	0.149
			2	4.138	4.065	1.018	1.018	0.825	1.234	10.623	12.339	0.861	1.038	0.180
	2	2	1	0.978	0.878	1.114	0.254	0.237	1.072	2.294	2.588	0.886	1.024	0.118
			2	1.950	1.798	1.085	0.503	0.527	0.954	4.578	4.958	0.923	0.987	0.087
	10	10	1	0.196	0.240	0.817	0.076	0.073	1.041	0.483	0.559	0.864	0.907	0.130
			2	0.385	0.471	0.817	0.149	0.151	0.987	0.955	1.012	0.944	0.916	0.096
A2	0	0	1	0.501	0.373	1.343	0.326	0.304	1.072	0.666	0.514	1.296	1.237	0.117
			2	0.994	0.701	1.418	0.651	0.625	1.042	1.307	0.894	1.462	1.307	0.177
	2	4	1	0.125	0.125	1.000	0.050	0.056	0.893	0.206	0.257	0.802	0.898	0.110
			2	0.230	0.208	1.106	0.096	0.100	0.960	0.363	0.466	0.779	0.948	0.173
	10	20	1	0.035	0.040	0.875	0.013	0.014	0.929	0.074	0.071	1.042	0.949	0.090
			2	0.049	0.054	0.907	0.018	0.019	0.947	0.097	0.087	1.115	0.990	0.111
A3	0	0	1	2.068	2.045	1.011	0.514	0.428	1.201	5.360	6.601	0.812	1.008	0.193
			2	4.135	4.233	0.977	1.026	0.846	1.213	10.719	12.722	0.843	1.011	0.185
	3.5	2	1	0.707	0.697	1.014	0.199	0.223	0.892	1.693	2.086	0.812	0.906	0.112
			2	1.413	1.402	1.008	0.397	0.409	0.971	3.383	4.328	0.782	0.920	0.132
	17.5	10	1	0.155	0.139	1.115	0.068	0.078	0.872	0.475	0.579	0.820	0.936	0.168
			2	0.309	0.259	1.193	0.136	0.154	0.883	0.948	1.213	0.782	0.953	0.225

TABLE 8.30 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
B1	0	0	1	0.672	0.667	1.007	0.175	0.157	1.115	1.704	2.122	0.803	0.975	0.163
			2	1.338	1.351	0.990	0.339	0.282	1.202	3.398	4.053	0.838	1.010	0.181
	2	2	1	0.328	0.319	1.028	0.088	0.082	1.073	0.762	0.919	0.829	0.977	0.133
			2	0.648	0.583	1.111	0.170	0.156	1.090	1.510	1.755	0.860	1.020	0.136
	10	10	1	0.073	0.098	0.745	0.028	0.031	0.903	0.170	0.241	0.705	0.784	0.133
			2	0.137	0.162	0.846	0.053	0.056	0.946	0.327	0.407	0.803	0.865	0.085
B2	0	0	1	0.220	0.188	1.170	0.145	0.176	0.824	0.372	0.428	0.869	0.954	0.197
			2	0.439	0.365	1.203	0.290	0.347	0.836	0.739	0.790	0.935	0.991	0.192
	2	4	1	0.073	0.066	1.106	0.030	0.032	0.938	0.116	0.142	0.817	0.954	0.152
			2	0.142	0.121	1.174	0.060	0.065	0.923	0.224	0.274	0.818	0.972	0.188
	10	20	1	0.015	0.019	0.789	0.006	0.005	1.200	0.029	0.039	0.744	0.911	0.276
			2	0.026	0.027	0.963	0.009	0.008	1.125	0.046	0.047	0.979	1.022	0.087
B3	0	0	1	0.678	0.744	0.911	0.180	0.176	1.023	1.787	2.463	0.726	0.887	0.169
			2	1.355	1.519	0.892	0.357	0.322	1.109	3.573	4.807	0.743	0.915	0.201
	3.5	2	1	0.259	0.247	1.049	0.080	0.078	1.026	0.678	0.829	0.818	0.964	0.132
			2	0.516	0.545	0.947	0.160	0.197	0.812	1.354	1.884	0.719	0.826	0.139
	17.5	10	1	0.091	0.076	1.197	0.044	0.049	0.898	0.306	0.342	0.895	0.997	0.174
			2	0.182	0.163	1.117	0.088	0.089	0.989	0.610	0.713	0.856	0.987	0.132

TABLE 8.31 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ε_{p_1} (%)	ε_{s_1} (%)	ELEM	EL CENTRO APP	EL CENTRO EX	EL CENTRO APP/EX	TAFT APP	TAFT EX	TAFT APP/EX	PACOIMA APP	PACOIMA EX	PACOIMA APP/EX	MEAN	C.O.V.
B1	0	0	1	1.123	0.857	1.310	0.209	0.188	1.112	3.136	2.595	1.208	1.210	0.082
			2	2.243	1.810	1.239	0.409	0.338	1.210	6.266	5.064	1.237	1.229	0.013
	2	2	1	0.377	0.337	1.119	0.090	0.088	1.023	0.871	1.046	0.833	0.992	0.147
			2	0.748	0.628	1.191	0.175	0.163	1.074	1.729	2.012	0.859	1.041	0.162
	10	10	1	0.075	0.101	0.743	0.030	0.032	0.938	0.174	0.244	0.713	0.798	0.153
			2	0.140	0.171	0.819	0.056	0.057	0.982	0.334	0.417	0.801	0.867	0.115
B2	0	0	1	0.372	0.254	1.465	0.243	0.222	1.095	0.450	0.353	1.275	1.278	0.145
			2	0.742	0.492	1.508	0.487	0.433	1.125	0.898	0.645	1.392	1.342	0.146
	2	4	1	0.076	0.068	1.118	0.032	0.034	0.941	0.120	0.147	0.816	0.958	0.158
			2	0.150	0.124	1.210	0.064	0.069	0.928	0.232	0.279	0.832	0.990	0.198
	10	20	1	0.015	0.019	0.789	0.006	0.005	1.200	0.029	0.039	0.744	0.911	0.276
			2	0.026	0.027	0.963	0.010	0.008	1.250	0.046	0.047	0.979	1.064	0.152
B3	0	0	1	1.127	0.923	1.221	0.213	0.198	1.076	3.182	2.990	1.064	1.120	0.078
			2	2.253	1.988	1.133	0.423	0.356	1.188	6.362	5.517	1.153	1.158	0.024
	3.5	2	1	0.280	0.269	1.041	0.082	0.081	1.012	0.723	0.906	0.798	0.950	0.140
			2	0.557	0.567	0.982	0.162	0.200	0.810	1.443	1.976	0.730	0.841	0.153
	17.5	10	1	0.091	0.076	1.197	0.044	0.049	0.898	0.306	0.345	0.887	0.994	0.177
			2	0.182	0.163	1.117	0.088	0.090	0.978	0.612	0.722	0.848	0.981	0.137

TABLE 8.32 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO APP	EL CENTRO EX	EL CENTRO APP/EX	TAFT APP	TAFT EX	TAFT APP/EX	PACOIMA APP	PACOIMA EX	PACOIMA APP/EX	MEAN	C.O.V.
C1	0	0	1	0.706	0.719	0.982	0.223	0.189	1.180	1.823	2.272	0.802	0.988	0.191
			2	1.408	1.429	0.985	0.442	0.342	1.292	3.641	4.402	0.827	1.035	0.229
			3	2.107	2.086	1.010	0.662	0.525	1.261	5.458	6.511	0.838	1.036	0.205
	2	2	1	0.416	0.413	1.007	0.122	0.109	1.119	0.990	1.094	0.905	1.010	0.106
			2	0.829	0.781	1.061	0.241	0.206	1.170	1.975	2.188	0.903	1.045	0.129
			3	1.239	1.152	1.076	0.360	0.325	1.108	2.960	3.199	0.925	1.036	0.094
	10	10	1	0.097	0.125	0.776	0.038	0.040	0.950	0.241	0.307	0.785	0.837	0.117
			2	0.190	0.233	0.815	0.075	0.078	0.962	0.478	0.573	0.834	0.870	0.092
			3	0.278	0.338	0.822	0.109	0.108	1.009	0.713	0.789	0.904	0.912	0.103
C2	0	0	1	0.143	0.122	1.172	0.090	0.084	1.071	0.199	0.201	0.990	1.078	0.085
			2	0.298	0.221	1.348	0.181	0.202	0.896	0.434	0.405	1.072	1.105	0.206
			3	0.404	0.294	1.374	0.263	0.241	1.091	0.508	0.369	1.377	1.281	0.128
	2	4	1	0.050	0.046	1.087	0.017	0.019	0.895	0.094	0.092	1.022	1.001	0.098
			2	0.121	0.110	1.100	0.039	0.054	0.722	0.232	0.254	0.913	0.912	0.207
			3	0.092	0.089	1.034	0.037	0.040	0.925	0.150	0.191	0.785	0.915	0.136
	10	20	1	0.022	0.021	1.048	0.008	0.009	0.889	0.050	0.058	0.862	0.933	0.108
			2	0.058	0.055	1.055	0.022	0.022	1.000	0.132	0.139	0.950	1.002	0.052
			3	0.026	0.027	0.963	0.010	0.010	1.000	0.056	0.048	1.167	1.043	0.104
C3	0	0	1	0.610	0.594	1.027	0.162	0.182	0.890	1.730	2.210	0.783	0.900	0.136
			2	1.219	1.085	1.124	0.320	0.310	1.032	3.457	3.958	0.873	1.010	0.126
			3	0.673	0.630	1.068	0.304	0.351	0.866	2.650	3.427	0.773	0.902	0.167
	2.8	2	1	0.236	0.223	1.058	0.093	0.092	1.011	0.732	0.928	0.798	0.956	0.145
			2	0.470	0.472	0.996	0.185	0.215	0.860	1.460	1.881	0.776	0.877	0.127
			3	0.361	0.512	0.705	0.230	0.276	0.833	1.677	2.255	0.744	0.761	0.086
	14.1	10	1	0.089	0.085	1.047	0.052	0.053	0.981	0.367	0.388	0.946	0.991	0.052
			2	0.176	0.157	1.121	0.103	0.092	1.120	0.733	0.749	0.979	1.073	0.076
			3	0.249	0.236	1.055	0.151	0.143	1.056	1.074	1.135	0.946	1.019	0.062

TABLE 8.33 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
C1	0	0	1	1.500	1.282	1.170	0.290	0.236	1.229	4.073	3.730	1.092	1.164	0.059
			2	2.998	2.393	1.253	0.576	0.441	1.306	8.144	7.339	1.110	1.223	0.083
			3	4.494	3.614	1.243	0.863	0.672	1.284	12.214	10.932	1.117	1.215	0.072
	2	2	1	0.560	0.455	1.231	0.132	0.121	1.091	1.316	1.478	0.890	1.071	0.160
			2	1.117	0.918	1.217	0.261	0.225	1.160	2.629	2.880	0.913	1.097	0.147
			3	1.674	1.344	1.246	0.390	0.356	1.096	3.942	4.272	0.923	1.088	0.149
	10	10	1	0.103	0.134	0.769	0.041	0.041	1.000	0.254	0.320	0.794	0.854	0.148
			2	0.203	0.253	0.802	0.080	0.081	0.988	0.504	0.598	0.843	0.878	0.111
			3	0.297	0.368	0.807	0.118	0.113	1.044	0.752	0.829	0.907	0.919	0.129
C2	0	0	1	0.162	0.131	1.237	0.103	0.087	1.184	0.204	0.193	1.057	1.159	0.080
			2	0.334	0.225	1.484	0.206	0.209	0.986	0.444	0.407	1.091	1.187	0.221
			3	0.463	0.326	1.420	0.303	0.249	1.217	0.527	0.352	1.497	1.378	0.105
	2	4	1	0.050	0.046	1.087	0.018	0.019	0.947	0.094	0.092	1.022	1.019	0.069
			2	0.121	0.111	1.090	0.039	0.055	0.709	0.232	0.254	0.913	0.904	0.211
			3	0.093	0.090	1.033	0.037	0.041	0.902	0.151	0.192	0.786	0.907	0.136
	10	20	1	0.022	0.021	1.048	0.008	0.009	0.889	0.050	0.058	0.862	0.933	0.108
			2	0.058	0.055	1.055	0.022	0.022	1.000	0.132	0.139	0.950	1.002	0.052
			3	0.026	0.028	0.929	0.010	0.010	1.000	0.056	0.049	1.143	1.024	0.106
C3	0	0	1	0.882	0.688	1.282	0.181	0.187	0.968	2.605	2.531	1.029	1.093	0.152
			2	1.762	1.274	1.383	0.359	0.368	0.976	5.208	4.298	1.212	1.190	0.172
			3	0.926	0.804	1.152	0.315	0.375	0.840	3.289	3.927	0.838	0.943	0.192
	2.8	2	1	0.251	0.231	1.087	0.093	0.095	0.979	0.758	0.963	0.787	0.951	0.160
			2	0.499	0.502	0.994	0.185	0.224	0.826	1.513	2.028	0.746	0.855	0.148
			3	0.370	0.547	0.676	0.230	0.289	0.796	1.688	2.405	0.702	0.725	0.087
	14.1	10	1	0.090	0.088	1.023	0.052	0.054	0.963	0.367	0.394	0.931	0.972	0.048
			2	0.178	0.164	1.085	0.103	0.094	1.096	0.732	0.766	0.956	1.046	0.074
			3	0.249	0.250	0.996	0.151	0.148	1.020	1.074	1.175	0.914	0.977	0.057

TABLE 8.34 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
A1	4	0	1	1.161	0.953	1.218	0.268	0.223	1.202	2.716	2.956	0.919	1.113	0.151
			2	2.318	2.004	1.157	0.535	0.544	0.983	5.427	5.653	0.960	1.033	0.104
	0	4	1	1.151	0.886	1.299	0.273	0.233	1.172	2.717	3.024	0.898	1.123	0.182
			2	2.297	1.778	1.292	0.536	0.522	1.027	5.425	5.724	0.948	1.089	0.165
	2	0.1	1	1.486	1.272	1.168	0.359	0.280	1.282	3.548	4.047	0.877	1.109	0.188
			2	2.969	2.508	1.184	0.715	0.654	1.093	7.089	7.675	0.924	1.067	0.124
A2	4	0	1	0.115	0.132	0.871	0.072	0.087	0.828	0.215	0.218	0.986	0.895	0.091
			2	0.216	0.235	0.919	0.141	0.165	0.855	0.394	0.418	0.943	0.906	0.050
	0	8	1	0.126	0.147	0.857	0.072	0.084	0.857	0.241	0.323	0.746	0.820	0.078
			2	0.224	0.236	0.949	0.142	0.163	0.871	0.414	0.544	0.761	0.860	0.110
	2	0.1	1	0.177	0.186	0.952	0.108	0.138	0.783	0.310	0.344	0.901	0.879	0.099
			2	0.341	0.342	0.997	0.214	0.268	0.799	0.589	0.657	0.896	0.897	0.110
A3	7	0	1	0.863	0.744	1.160	0.203	0.228	0.890	1.925	2.490	0.773	0.941	0.211
			2	1.726	1.499	1.151	0.406	0.427	0.951	3.849	5.185	0.742	0.948	0.216
	0	4	1	0.852	0.712	1.197	0.199	0.209	0.952	1.847	2.205	0.838	0.996	0.184
			2	1.700	1.484	1.146	0.392	0.402	0.975	3.687	4.503	0.819	0.980	0.167
	2	0.1	1	1.486	1.292	1.150	0.365	0.322	1.134	3.613	4.380	0.825	1.036	0.177
			2	2.971	2.539	1.170	0.729	0.625	1.166	7.224	8.785	0.822	1.053	0.190

TABLE 8.35 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
B1	4	0	1	0.436	0.396	1.101	0.095	0.092	1.033	0.990	1.173	0.844	0.993	0.134
			2	0.867	0.811	1.069	0.186	0.175	1.063	1.975	2.400	0.823	0.985	0.142
	0	4	1	0.432	0.360	1.200	0.104	0.100	1.040	0.997	1.246	0.800	1.013	0.199
			2	0.855	0.681	1.256	0.190	0.165	1.152	1.977	2.327	0.850	1.086	0.194
	2	0.1	1	0.618	0.519	1.191	0.131	0.112	1.170	1.492	1.603	0.931	1.097	0.132
			2	1.231	1.110	1.109	0.257	0.209	1.230	2.978	3.266	0.912	1.084	0.148
B2	4	0	1	0.072	0.076	0.947	0.048	0.059	0.814	0.130	0.145	0.897	0.886	0.076
			2	0.141	0.145	0.972	0.096	0.115	0.835	0.255	0.293	0.870	0.892	0.080
	0	8	1	0.074	0.078	0.949	0.048	0.050	0.960	0.135	0.179	0.754	0.888	0.131
			2	0.143	0.137	1.044	0.097	0.098	0.990	0.258	0.309	0.835	0.956	0.113
	2	0.1	1	0.114	0.109	1.046	0.075	0.086	0.872	0.195	0.228	0.855	0.924	0.114
			2	0.226	0.213	1.061	0.150	0.164	0.915	0.384	0.454	0.846	0.941	0.117
B3	7	0	1	0.331	0.345	0.959	0.091	0.102	0.892	0.865	1.260	0.687	0.846	0.168
			2	0.661	0.710	0.931	0.181	0.218	0.830	1.729	2.576	0.671	0.811	0.162
	0	4	1	0.312	0.309	1.010	0.084	0.082	1.024	0.759	1.034	0.734	0.923	0.177
			2	0.621	0.602	1.032	0.161	0.188	0.856	1.510	1.968	0.767	0.885	0.152
	2	0.1	1	0.627	0.600	1.045	0.141	0.133	1.060	1.584	2.006	0.790	0.965	0.157
			2	1.252	1.236	1.013	0.281	0.267	1.052	3.165	3.965	0.798	0.954	0.143

TABLE 8.36 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 0.1%

CASE	ξ_{p_1} (%)	ξ_{s_1} (%)	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
				APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
C1	4	0	1	0.632	0.523	1.208	0.138	0.124	1.113	1.443	1.691	0.853	1.058	0.174
			2	1.262	1.040	1.213	0.276	0.235	1.174	2.883	3.361	0.858	1.082	0.180
			3	1.886	1.475	1.279	0.411	0.372	1.105	4.322	4.983	0.867	1.084	0.191
	0	4	1	0.631	0.501	1.259	0.141	0.138	1.022	1.444	1.751	0.825	1.035	0.210
			2	1.258	0.970	1.297	0.275	0.232	1.185	2.883	3.278	0.879	1.120	0.193
			3	1.886	1.444	1.306	0.412	0.367	1.123	4.323	4.906	0.881	1.103	0.193
	2	0.1	1	0.913	0.758	1.204	0.191	0.165	1.158	2.261	2.359	0.958	1.107	0.118
			2	1.824	1.489	1.225	0.382	0.305	1.252	4.520	4.680	0.966	1.148	0.138
			3	2.731	2.154	1.268	0.570	0.467	1.221	6.778	6.944	0.976	1.155	0.136
C2	4	0	1	0.044	0.046	0.957	0.027	0.037	0.730	0.088	0.089	0.989	0.892	0.158
			2	0.106	0.117	0.906	0.057	0.081	0.704	0.215	0.225	0.956	0.855	0.156
			3	0.086	0.098	0.878	0.058	0.071	0.817	0.159	0.170	0.935	0.877	0.067
	0	8	1	0.055	0.061	0.902	0.023	0.027	0.852	0.114	0.153	0.745	0.833	0.096
			2	0.141	0.156	0.904	0.053	0.062	0.855	0.295	0.394	0.749	0.836	0.095
			3	0.094	0.109	0.862	0.058	0.056	1.036	0.179	0.238	0.752	0.883	0.162
	2	0.1	1	0.059	0.062	0.952	0.034	0.046	0.739	0.110	0.114	0.965	0.885	0.143
			2	0.138	0.151	0.914	0.071	0.105	0.676	0.260	0.275	0.945	0.845	0.174
			3	0.134	0.136	0.985	0.089	0.098	0.908	0.228	0.248	0.919	0.937	0.044
C3	5.6	0	1	0.276	0.278	0.993	0.106	0.120	0.883	0.916	1.272	0.720	0.865	0.159
			2	0.551	0.539	1.022	0.212	0.253	0.838	1.830	2.714	0.674	0.845	0.206
			3	0.393	0.523	0.751	0.275	0.338	0.814	2.184	2.778	0.786	0.784	0.040
	0	4	1	0.276	0.279	0.989	0.087	0.096	0.906	0.746	0.974	0.766	0.887	0.127
			2	0.547	0.525	1.042	0.167	0.204	0.819	1.484	1.895	0.783	0.881	0.159
			3	0.381	0.588	0.648	0.196	0.293	0.669	1.482	2.390	0.620	0.646	0.038
	2	0.1	1	0.466	0.401	1.162	0.131	0.149	0.879	1.305	1.750	0.746	0.929	0.229
			2	0.930	0.807	1.152	0.261	0.275	0.949	2.608	3.406	0.766	0.956	0.202
			3	0.545	0.610	0.893	0.286	0.353	0.810	2.366	3.237	0.731	0.811	0.100

TABLE 8.37 APPROXIMATE AND EXACT MAXIMUM DISTORTIONS IN SECONDARY SYSTEMS
MASS RATIO = 1%

DAMP	CASE	ELEM	EL CENTRO			TAFT			PACOIMA			MEAN	C.O.V.
			APP	EX	APP/EX	APP	EX	APP/EX	APP	EX	APP/EX		
ξ_{p_1} = 2%	D1	1	0.366	0.389	0.941	0.209	0.226	0.925	1.031	1.065	0.968	0.945	0.023
		2	1.258	1.172	1.073	0.506	0.477	1.061	2.902	3.524	0.823	0.986	0.143
	D2	1	0.248	0.195	1.272	0.073	0.092	0.793	0.393	0.357	1.101	1.055	0.230
ξ_{s_1} = 0%		2	0.450	0.385	1.169	0.141	0.161	0.876	0.663	0.696	0.953	0.999	0.152
	D3	1	0.904	0.818	1.105	0.308	0.258	1.194	1.963	2.422	0.810	1.036	0.194
		2	1.391	1.327	1.048	0.523	0.457	1.144	3.177	3.387	0.938	1.043	0.099
ξ_{p_1} = 2%	E1	1	0.097	0.100	0.970	0.082	0.080	1.025	0.627	0.602	1.042	1.012	0.037
		2	0.412	0.385	1.070	0.192	0.200	0.960	1.449	1.569	0.924	0.985	0.077
	E2	1	0.067	0.064	1.047	0.021	0.021	1.000	0.126	0.134	0.940	0.996	0.054
ξ_{s_1} = 0%		2	0.138	0.138	1.000	0.052	0.048	1.083	0.256	0.278	0.921	1.001	0.081
	E3	1	0.406	0.435	0.933	0.147	0.142	1.035	0.905	1.172	0.772	0.913	0.145
		2	0.690	0.829	0.832	0.267	0.261	1.023	1.596	1.878	0.850	0.902	0.117
	F1	1	0.402	0.368	1.092	0.152	0.127	1.197	0.920	0.900	1.022	1.104	0.080
		2	0.914	0.753	1.214	0.327	0.296	1.105	2.036	2.170	0.938	1.086	0.128
		3	1.422	1.180	1.205	0.501	0.442	1.133	3.147	3.384	0.930	1.089	0.131
ξ_{p_1} = 2%	F2	1	0.066	0.059	1.119	0.021	0.022	0.955	0.128	0.121	1.058	1.044	0.079
		2	0.186	0.187	0.995	0.067	0.055	1.218	0.360	0.409	0.880	1.031	0.167
ξ_{s_1} = 0%		3	0.094	0.082	1.146	0.028	0.023	1.217	0.124	0.121	1.025	1.129	0.086
	F3	1	0.239	0.220	1.086	0.109	0.104	1.048	0.760	0.986	0.771	0.968	0.178
		2	0.370	0.408	0.907	0.200	0.202	0.990	1.394	1.632	0.854	0.917	0.075
		3	0.594	0.552	1.072	0.232	0.211	1.110	1.437	1.949	0.737	0.973	0.211
											GROUP AVERAGE:	1.010	0.062

TABLE 8.38 GROUP AVERAGE STATISTICS OF APPROXIMATE TO EXACT MAXIMUM DISTORTION RATIOS

CATEGORY	DAMPING	MEAN	C.O.V.	MAX	MIN
SYSTEMS WITH PROPORTIONAL DAMPING AND RESONANT MODES	$\xi_1 = 0\%$	1.058	0.153	1.378	0.752
	$\xi_1 = 2\%$	0.941	0.090	1.097	0.725
	$\xi_1 = 10\%$	0.956	0.074	1.073	0.784
SYSTEMS WITH NON-PROPORTIONAL DAMPING AND RESONANT MODES	$\xi_{P_1} \geq 4\%$, $\xi_{S_1} = 0\%$	0.933	0.103	1.113	0.784
	$\xi_{P_1} = 0\%$, $\xi_{S_1} \geq 4\%$	0.945	0.130	1.123	0.646
	$\xi_{P_1} = 2\%$, $\xi_{S_1} = 0.1\%$	0.989	0.104	1.155	0.811
SYSTEMS WITH NON-PROPORTIONAL DAMPING AND NO RESONANT MODES	$\xi_{P_1} = 2\%$, $\xi_{S_1} = 0\%$	1.010	0.062	1.129	0.902

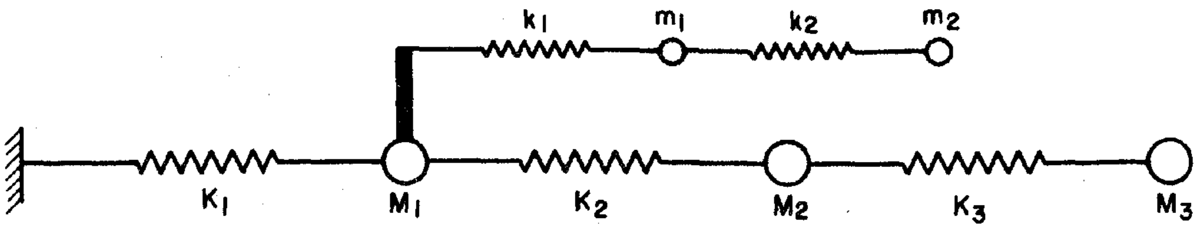


FIG. 2.1 ASSEMBLED SYSTEM

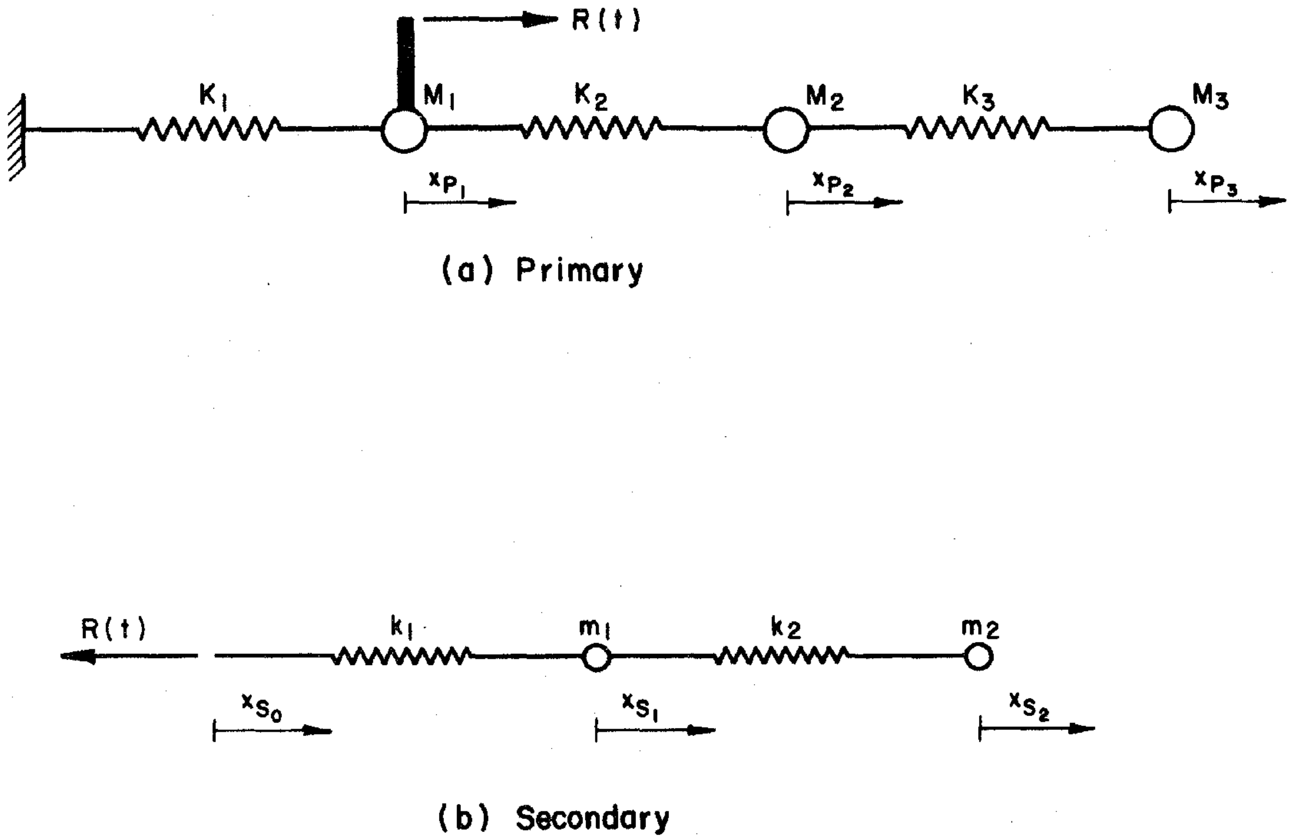


FIG. 2.2 PRIMARY AND SECONDARY SUBSYSTEMS

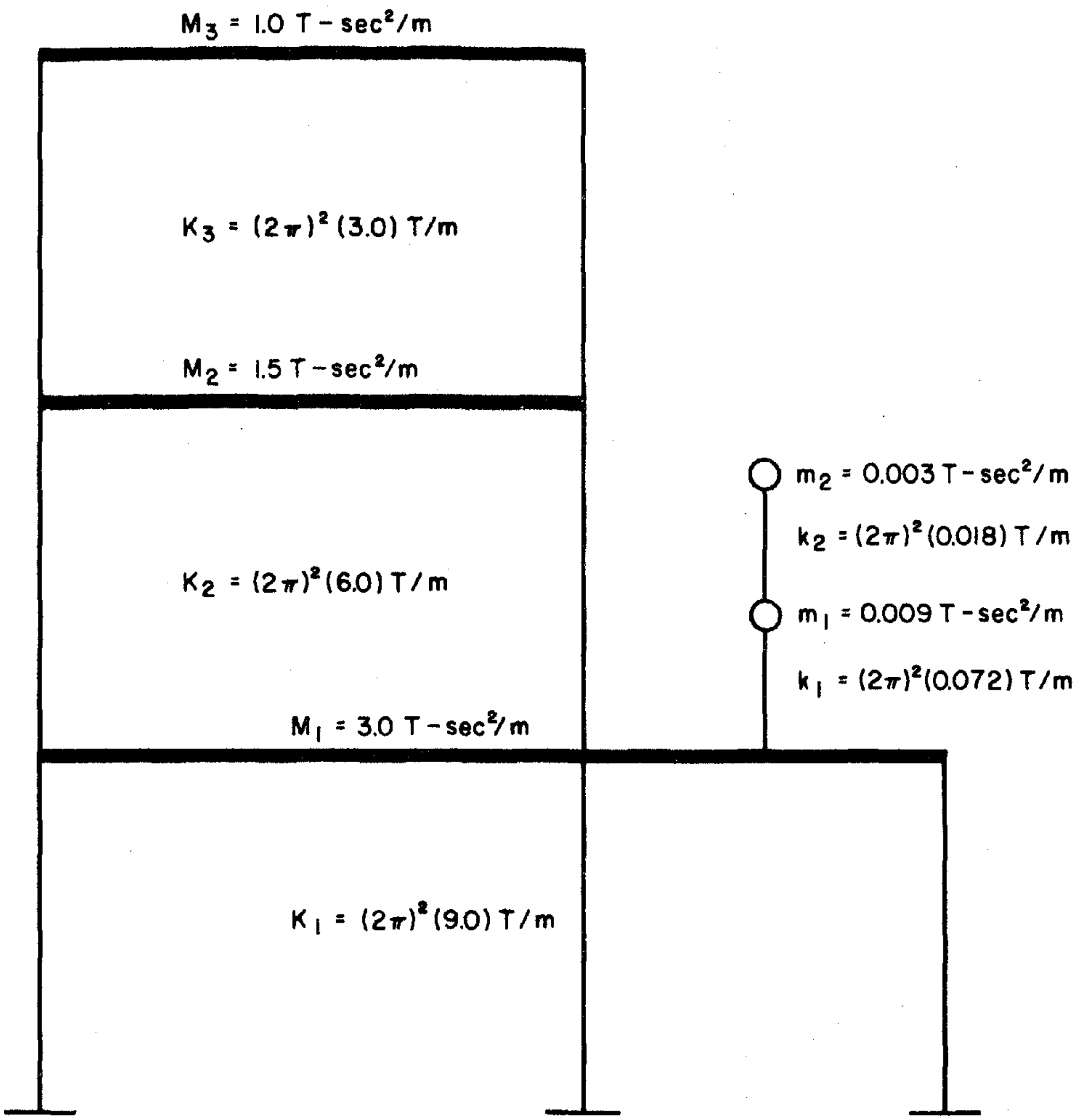


FIG. 2.3 ASSEMBLED SYSTEM IN ILLUSTRATIVE EXAMPLE OF SEC. 2.11

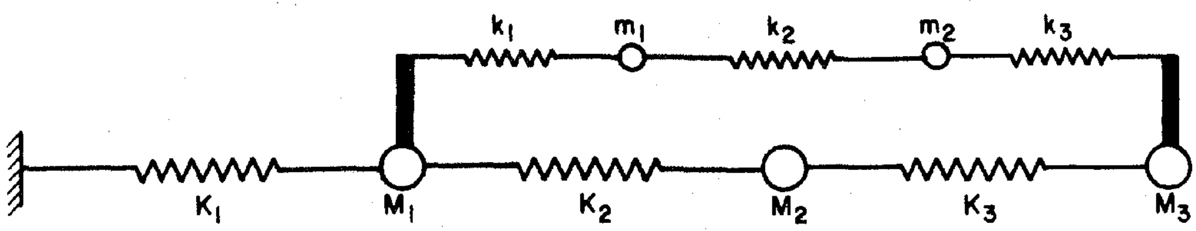


FIG. 4.1 ASSEMBLED SYSTEM WITH TWO POINTS OF ATTACHMENT

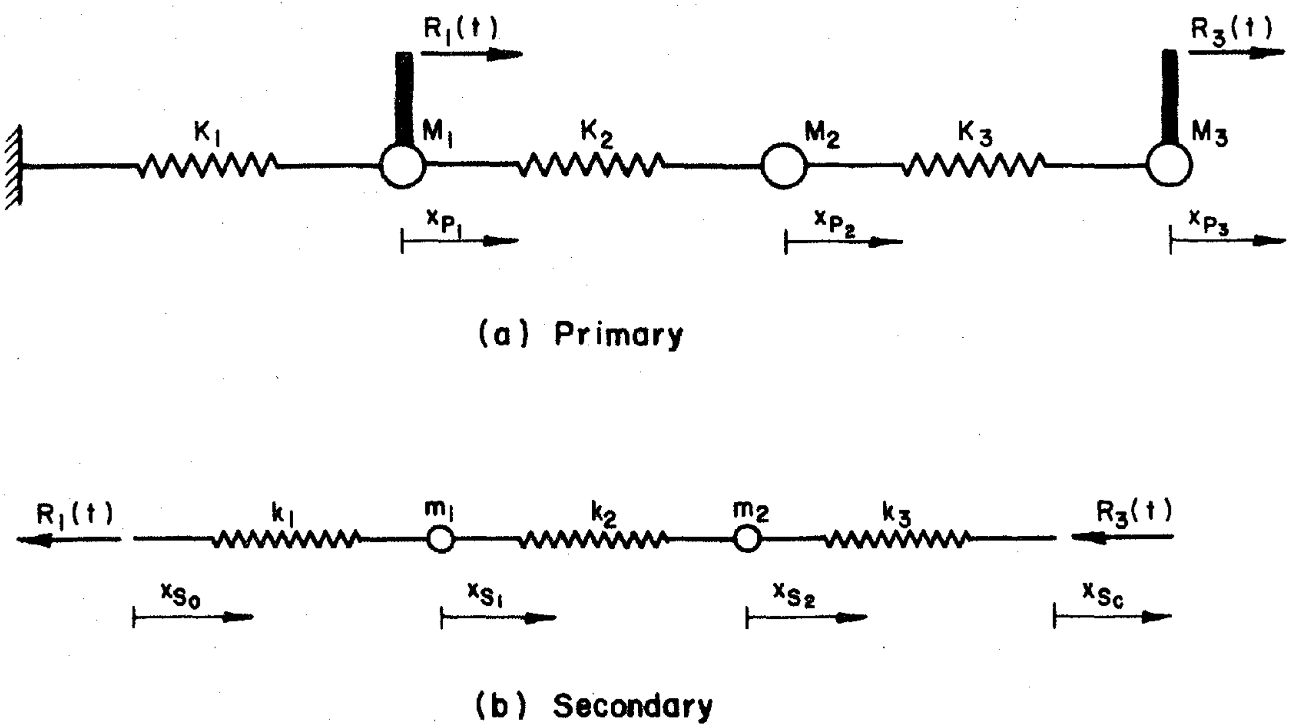


FIG. 4.2 PRIMARY AND SECONDARY SUBSYSTEMS

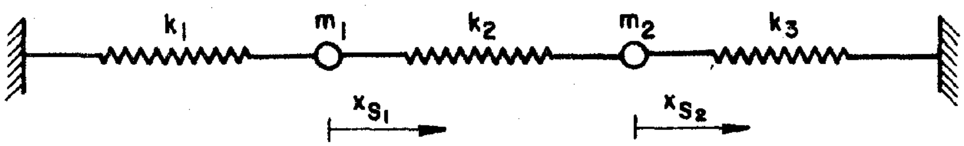


FIG. 4.3 INDEPENDENT SECONDARY SYSTEM

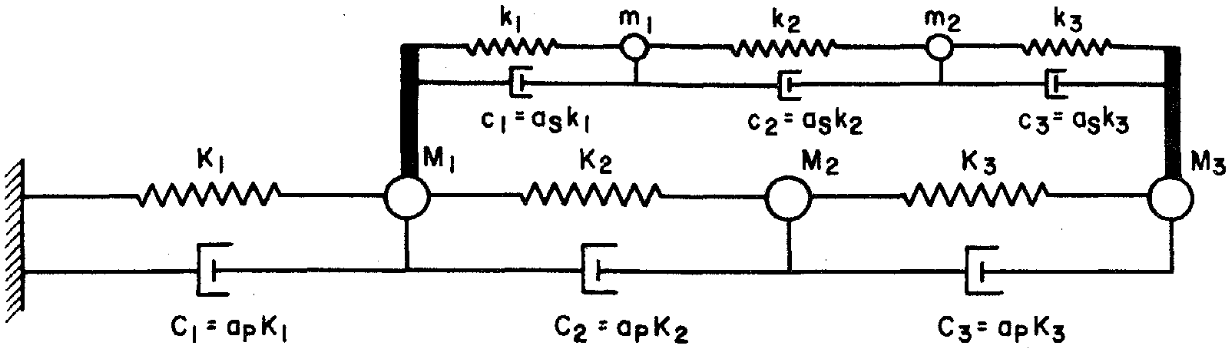


FIG. 6.1 DAMPED ASSEMBLED SYSTEM

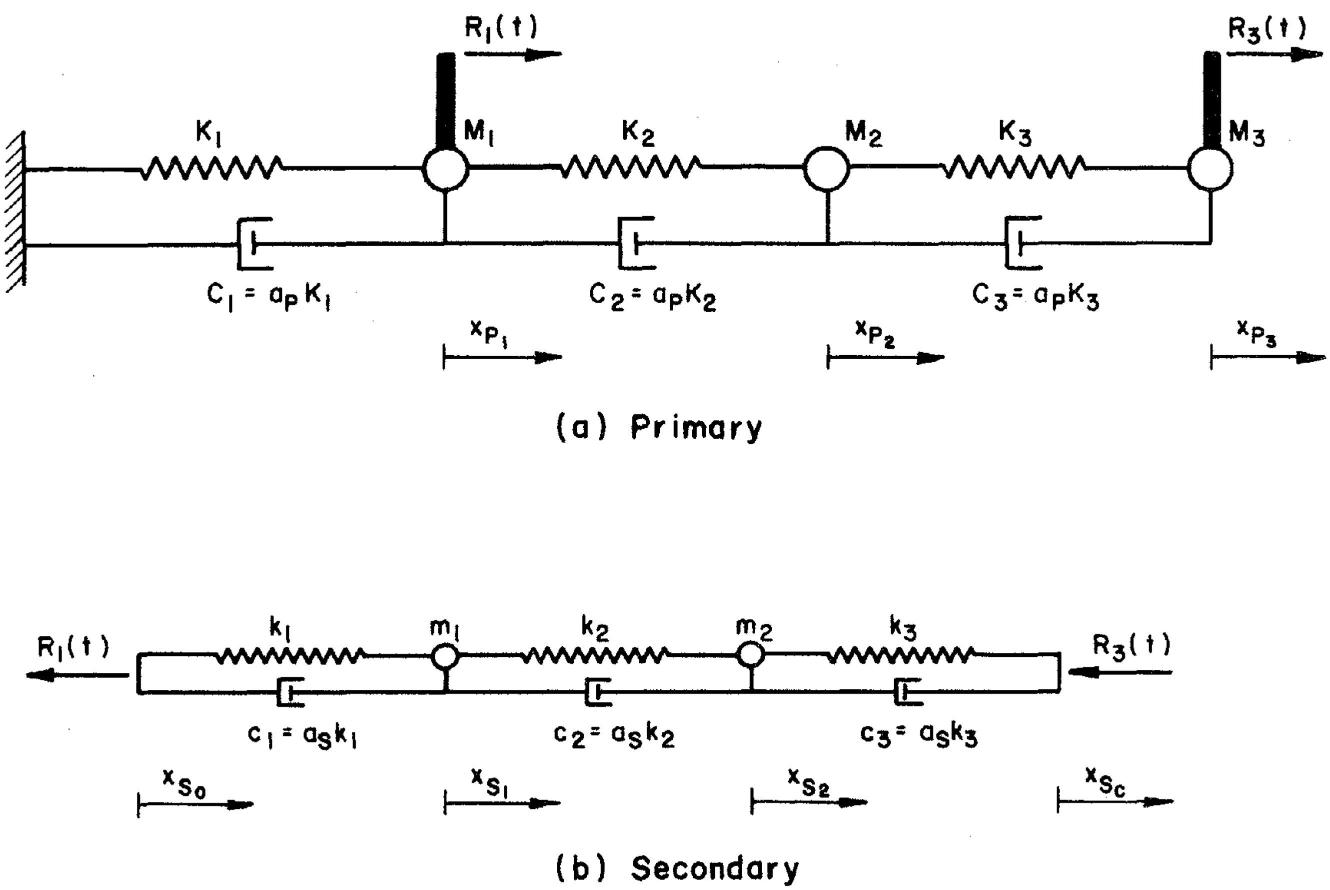


FIG. 6.2 DAMPED PRIMARY AND SECONDARY SUBSYSTEMS

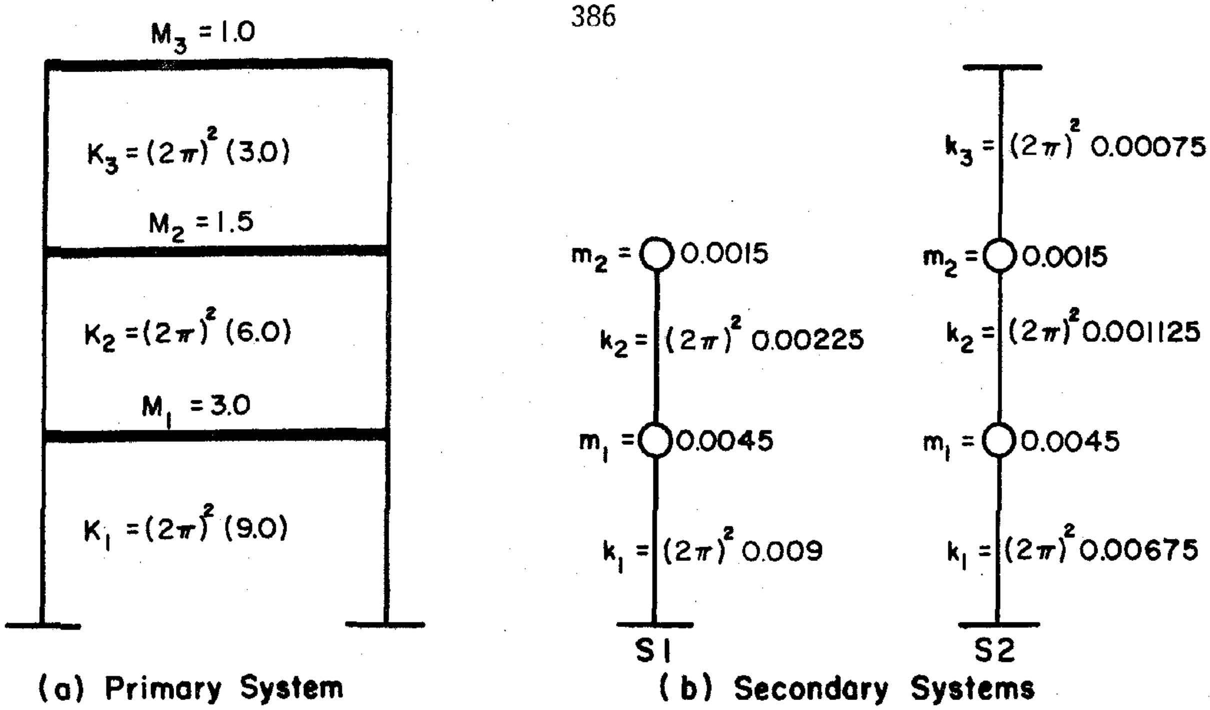


FIG. 7.1 INDEPENDENT PRIMARY AND SECONDARY SYSTEMS IN ILLUSTRATIVE EXAMPLES

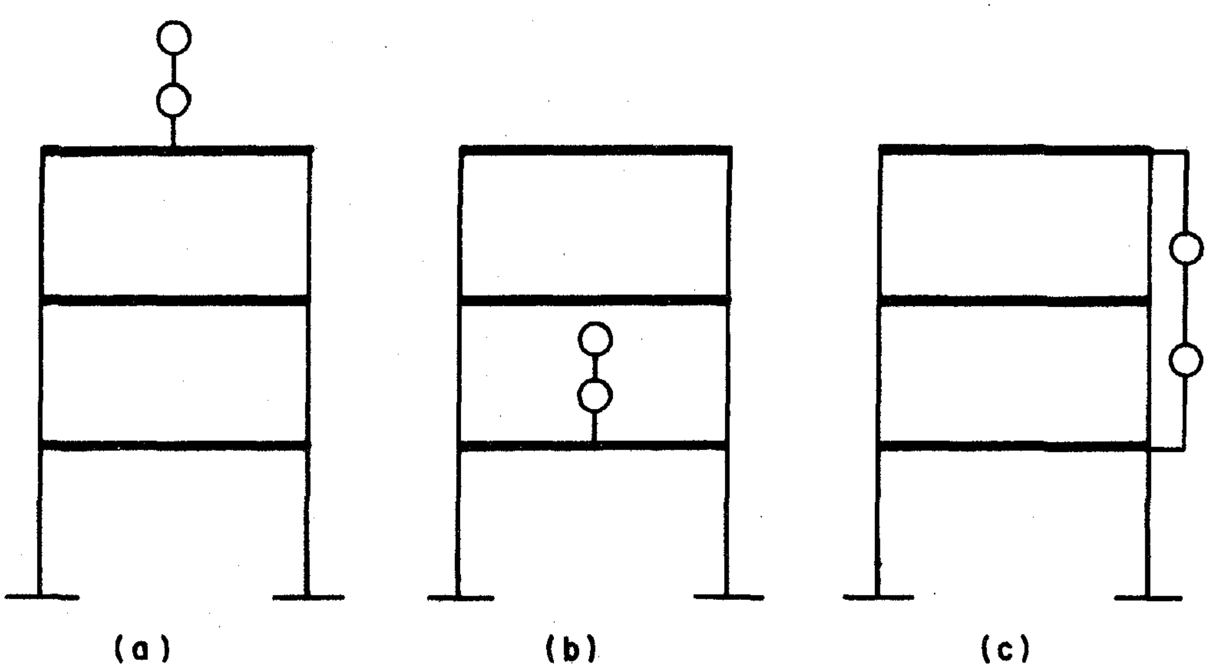
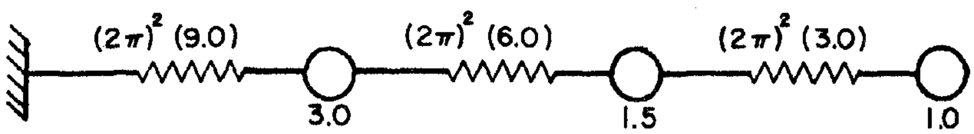
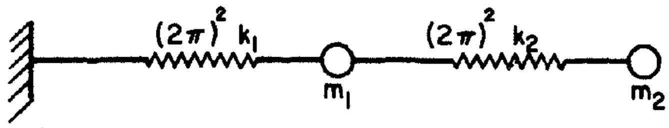


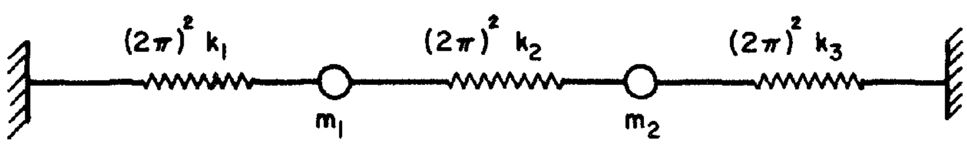
FIG. 7.2 LOCATIONS OF SECONDARY SYSTEMS ON PRIMARY SYSTEM IN ILLUSTRATIVE EXAMPLES



(a) Primary System



(b) Secondary System In Cases A, B, D And E



(c) Secondary System In Cases C And F

FIG. 8.1 INDEPENDENT PRIMARY AND SECONDARY SYSTEMS IN COMPARATIVE ANALYSES

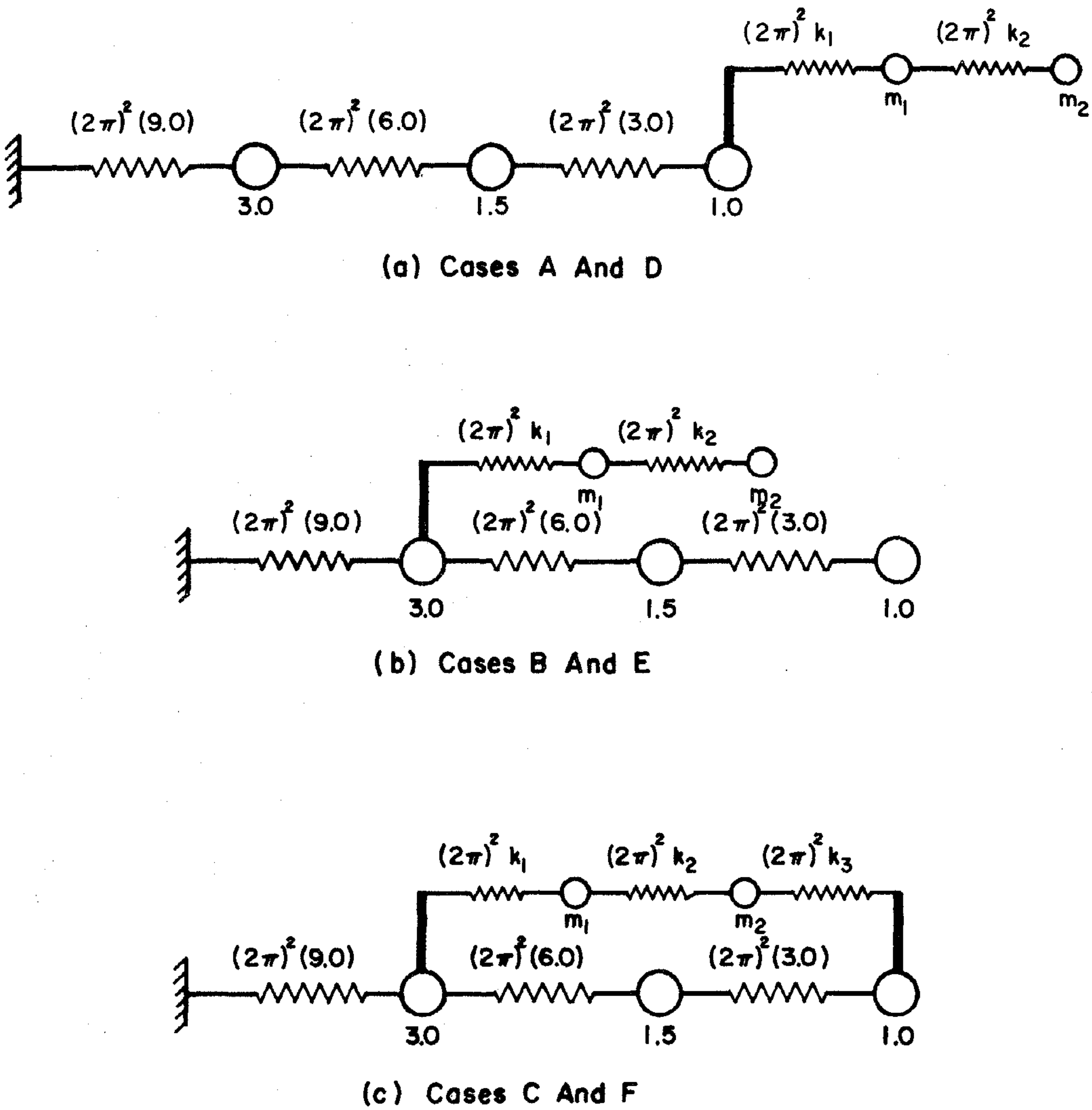


FIG. 8.2 ASSEMBLED SYSTEMS IN COMPARATIVE ANALYSES

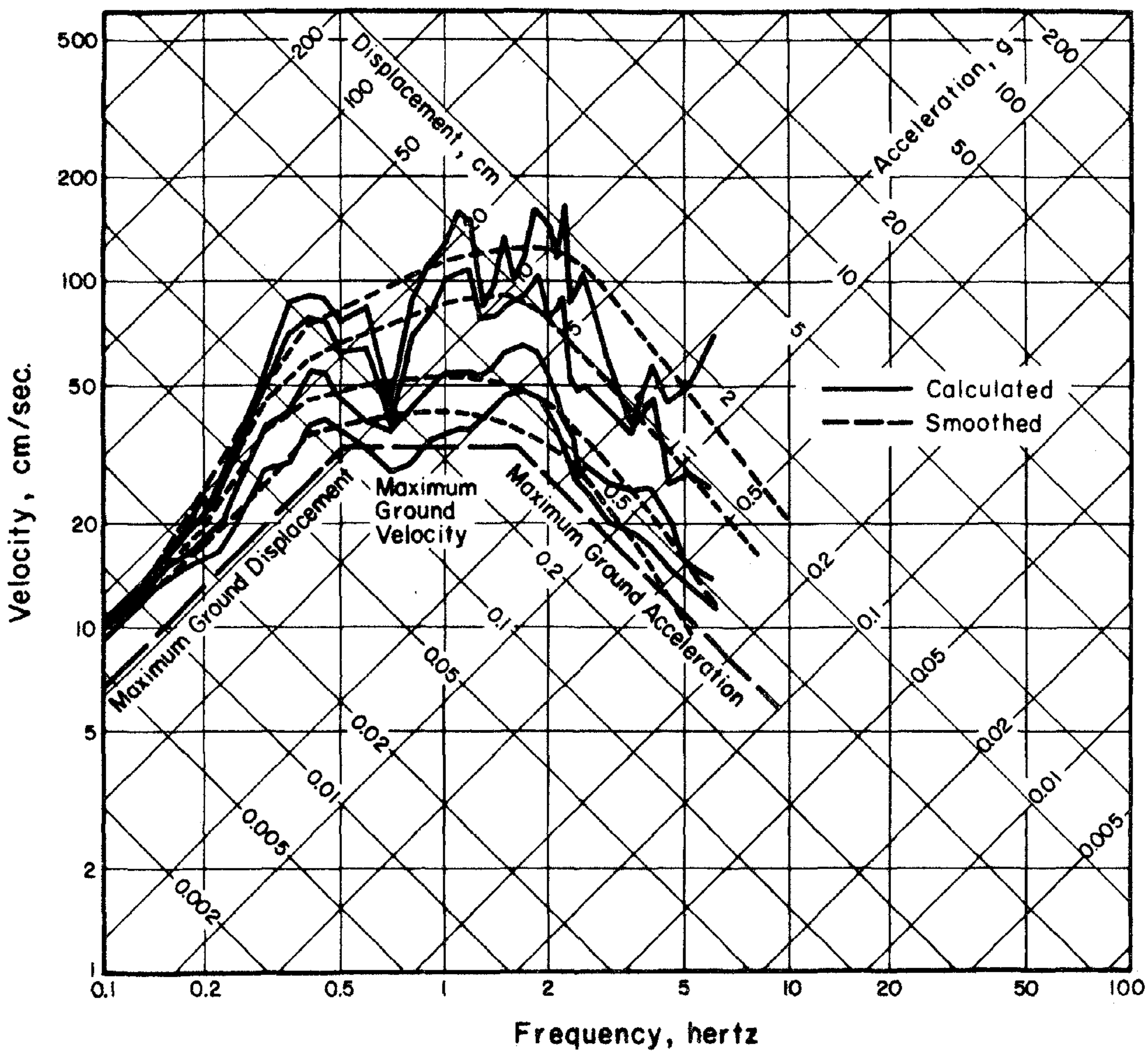


FIG. 8.3(a) RESPONSE SPECTRA, EL CENTRO, MAY 18, 1940, COMP SOOE, DURATION = 10 SEC; 0, 2, 10 AND 20 PERCENT DAMPING

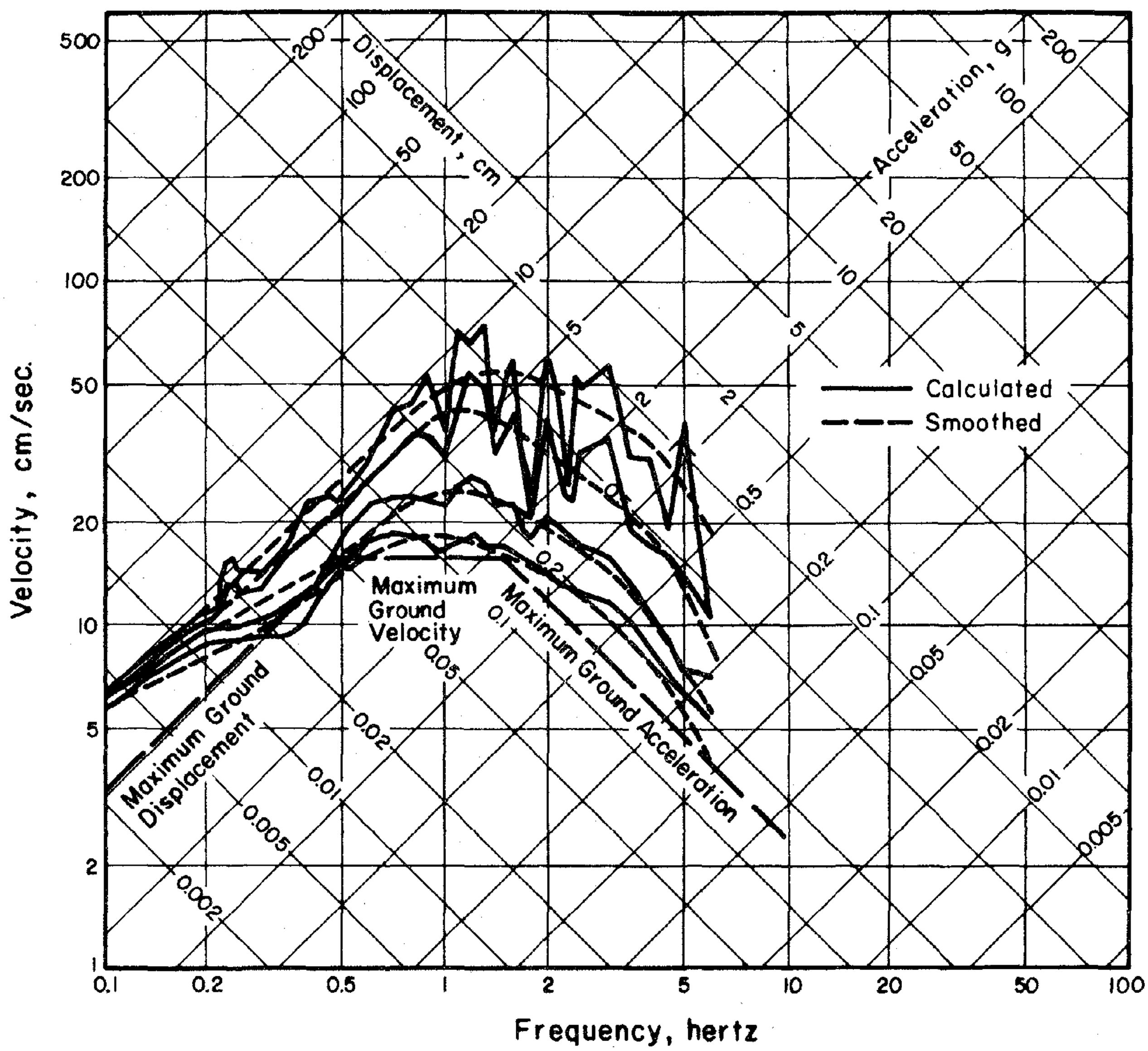


FIG. 8.3(b) RESPONSE SPECTRA, TAFT, JULY 21 1952, COMP N21E, DURATION = 10 SEC; 0, 2, 10 AND 20 PERCENT DAMPING

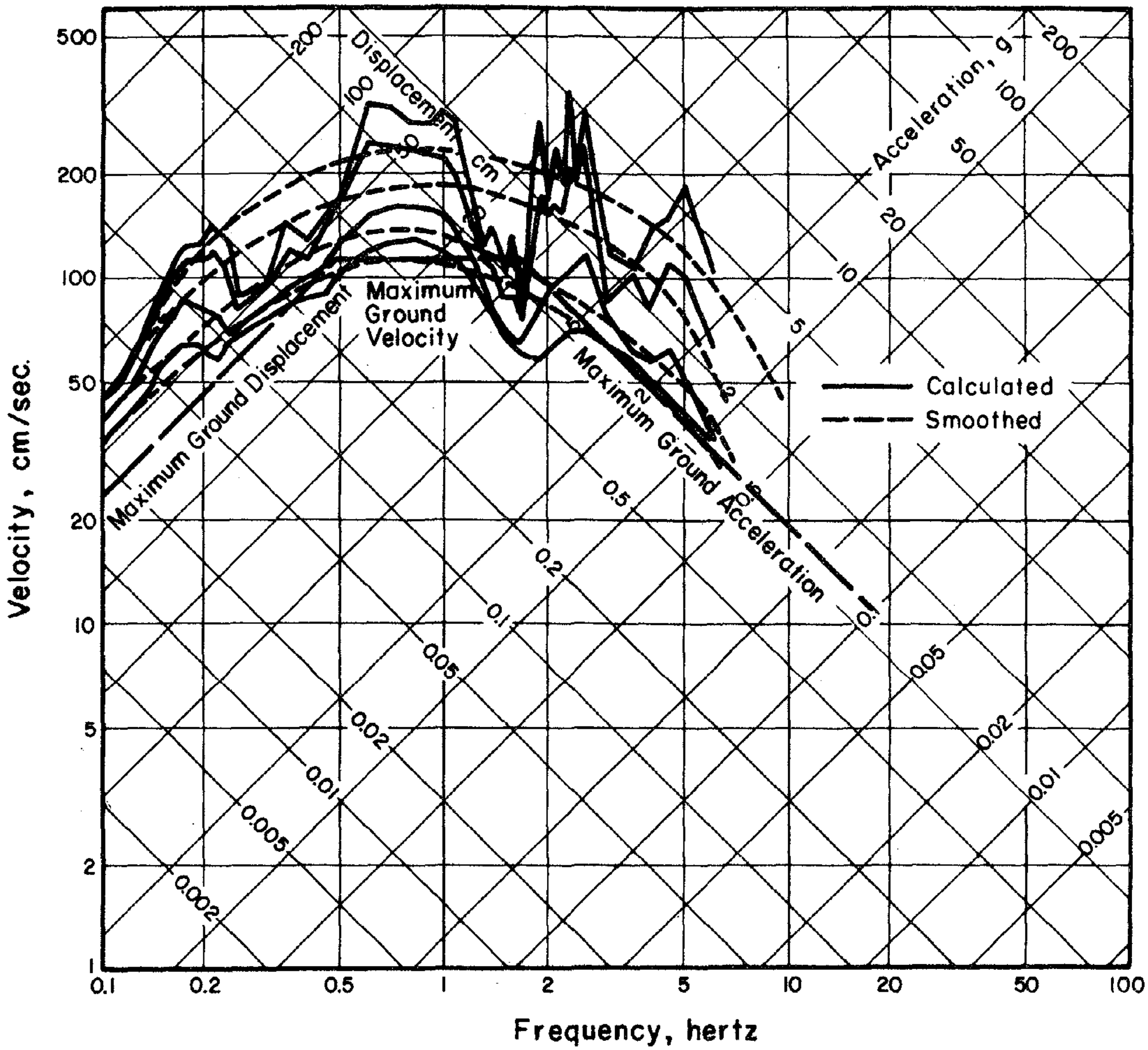


FIG. 8.3(c) RESPONSE SPECTRA, PACOIMA DAM, FEBRUARY 9, 1971, COMP S16E, DURATION 10 SEC; 0, 2, 10 AND 20 PERCENT DAMPING

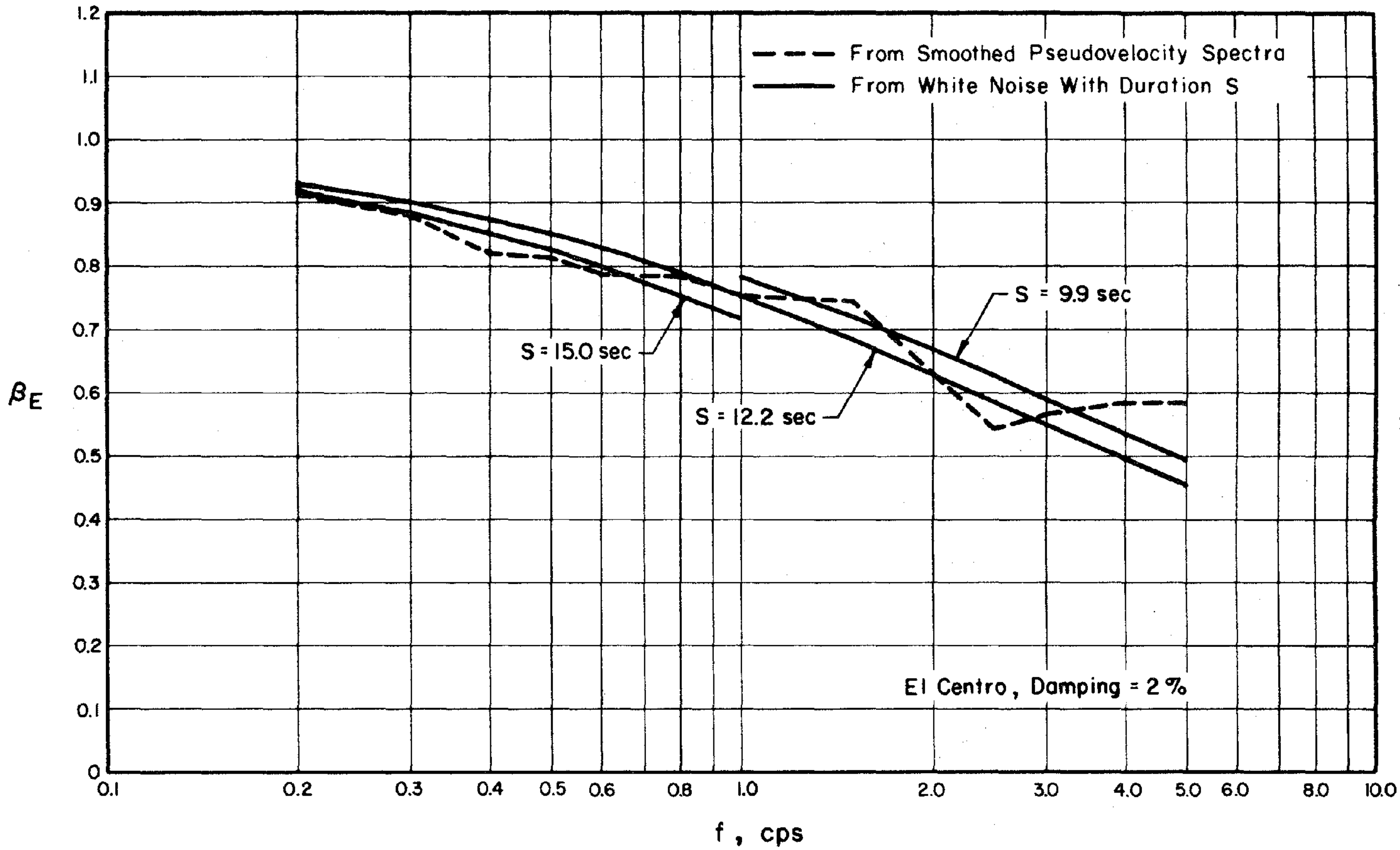


FIG. 8.4(a) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

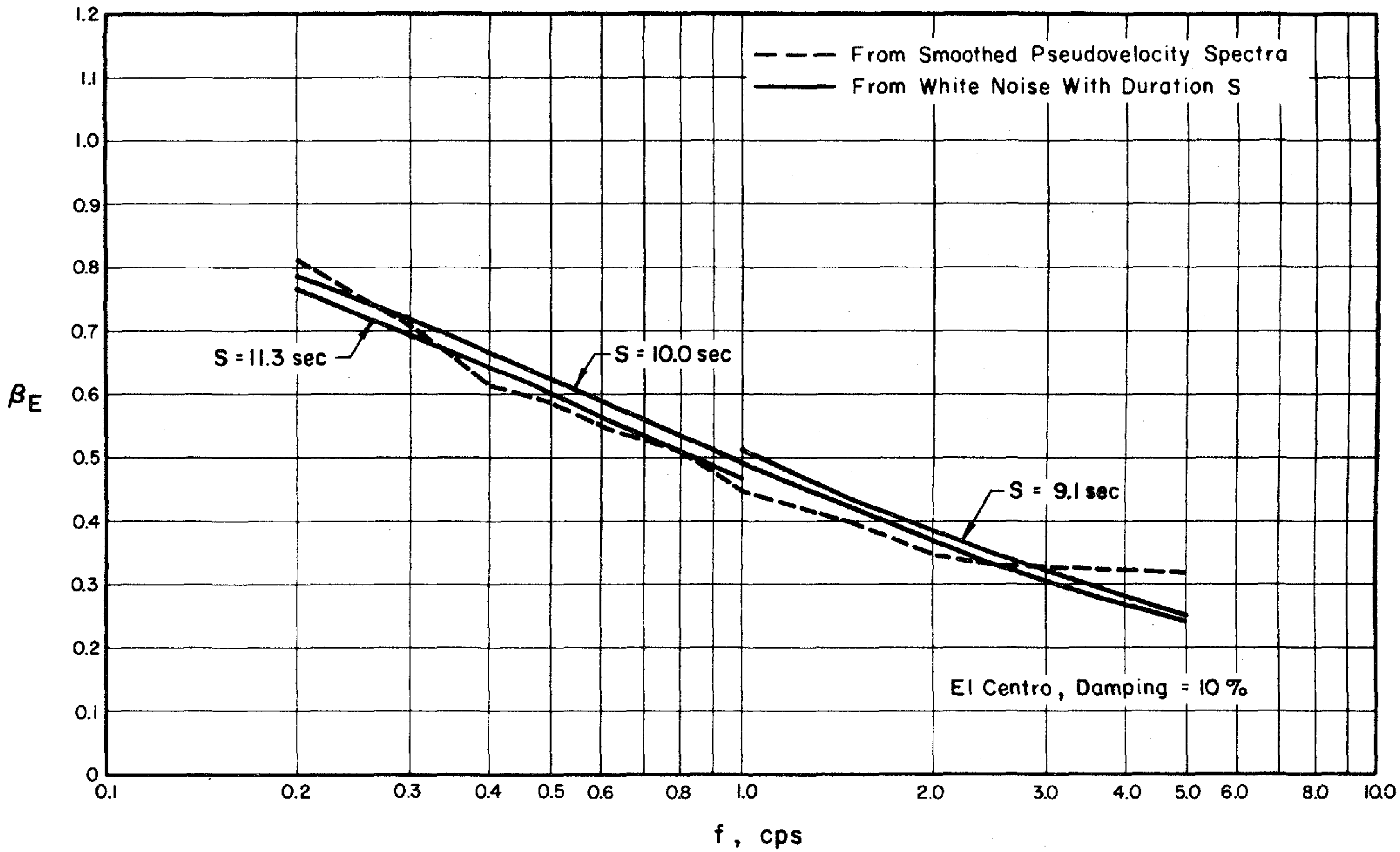


FIG. 8.4(b) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

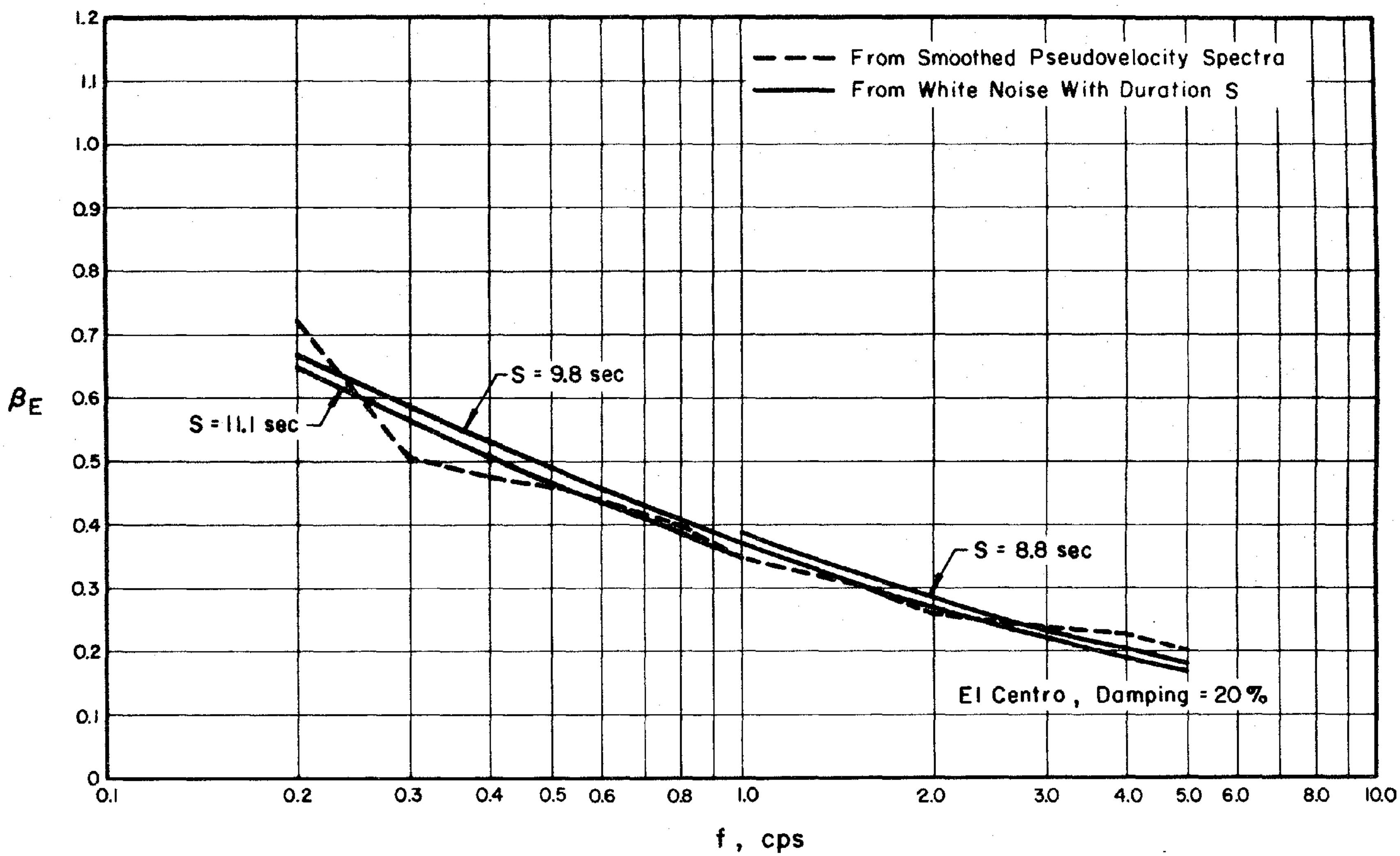


FIG. 8.4(c) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

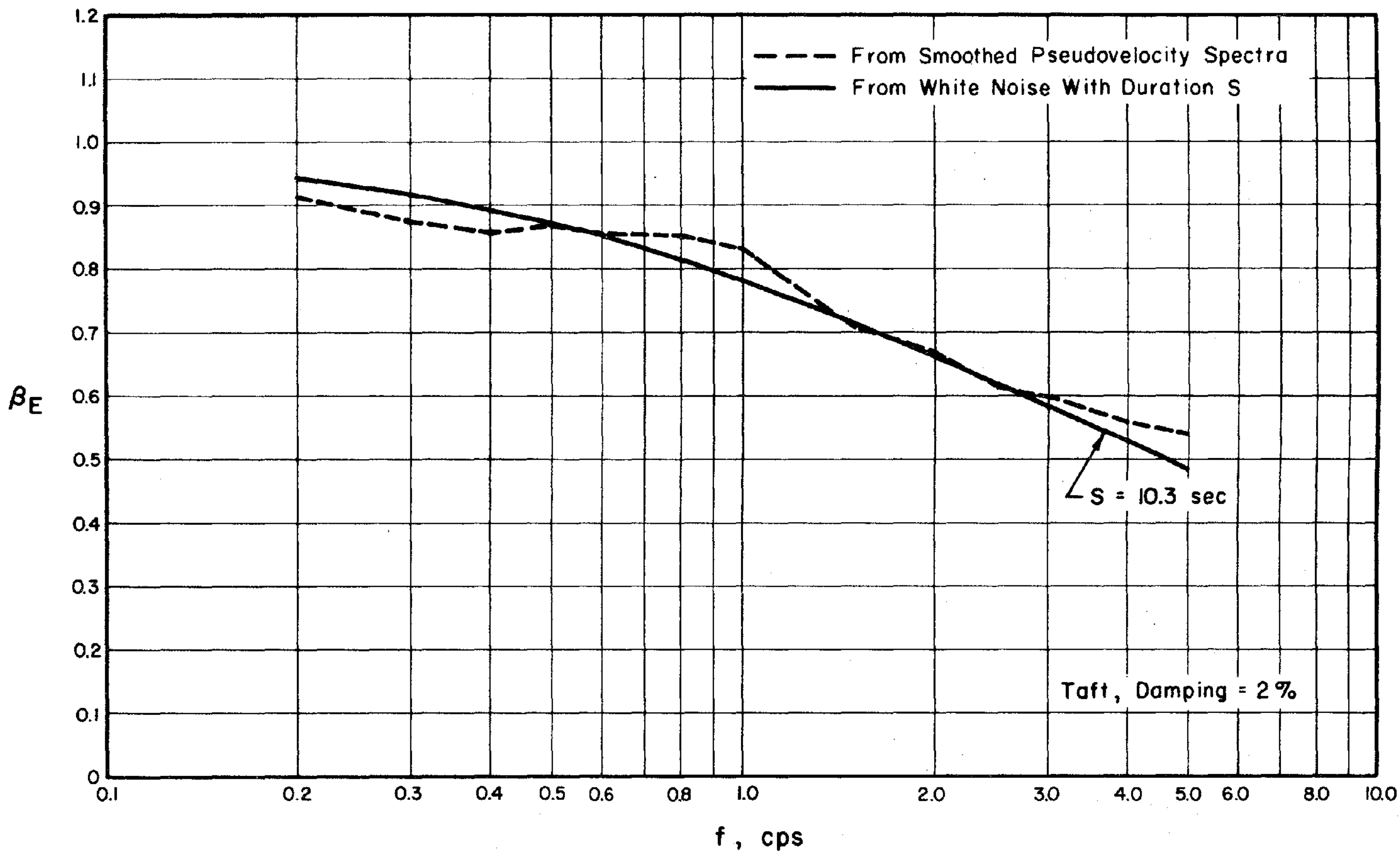


FIG. 8.5(a) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

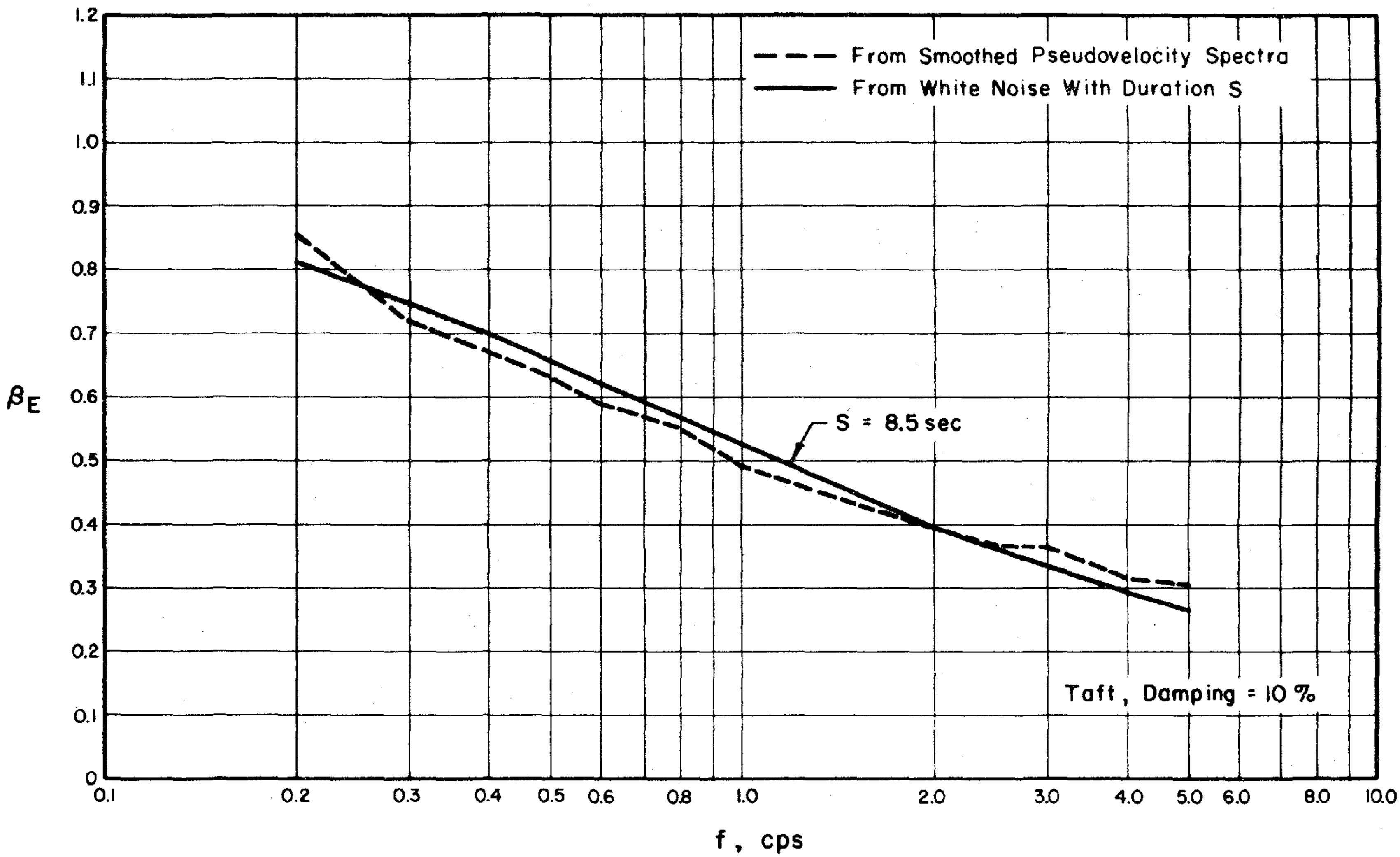


FIG. 8.5(b) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

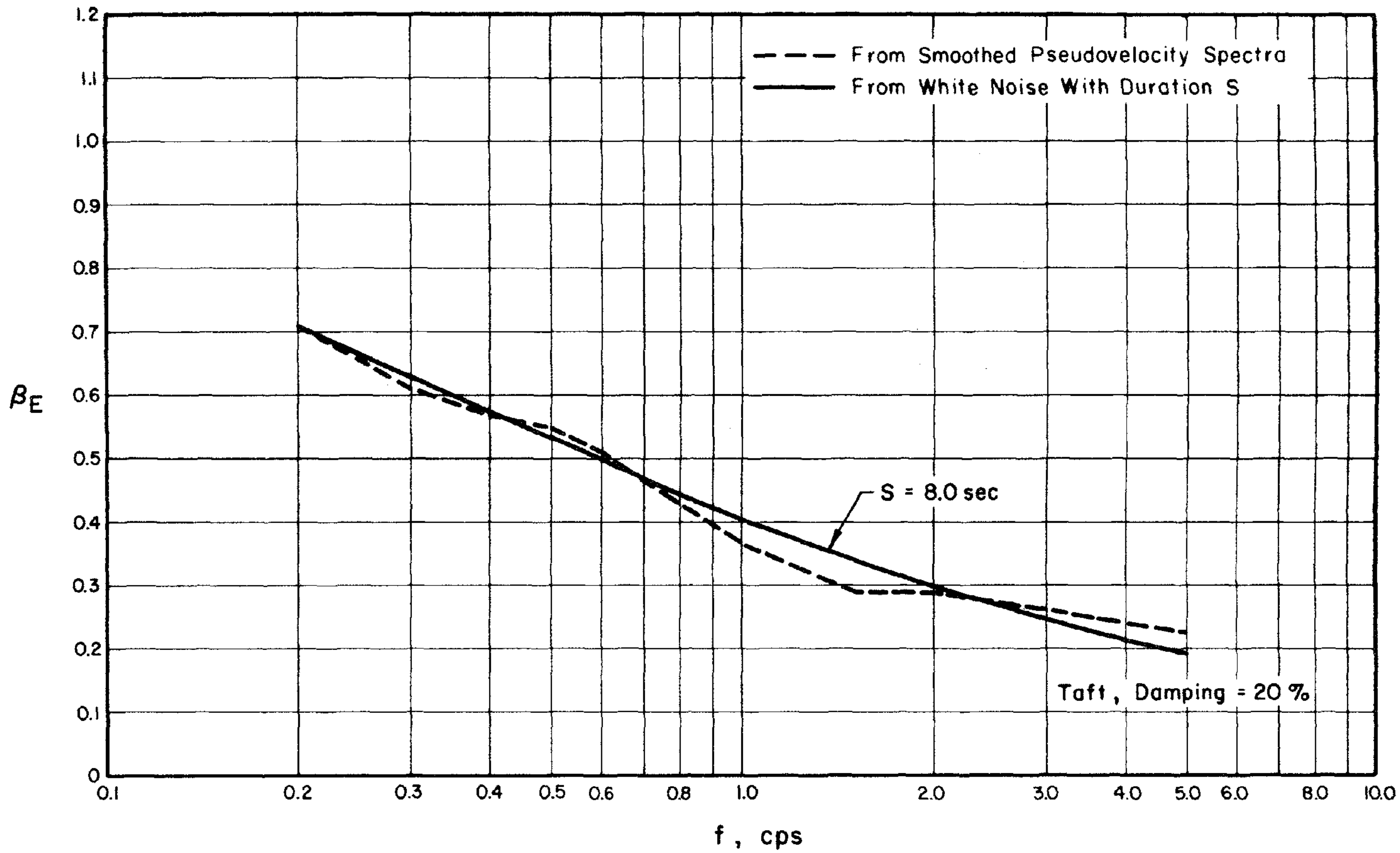


FIG. 8.5(c) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

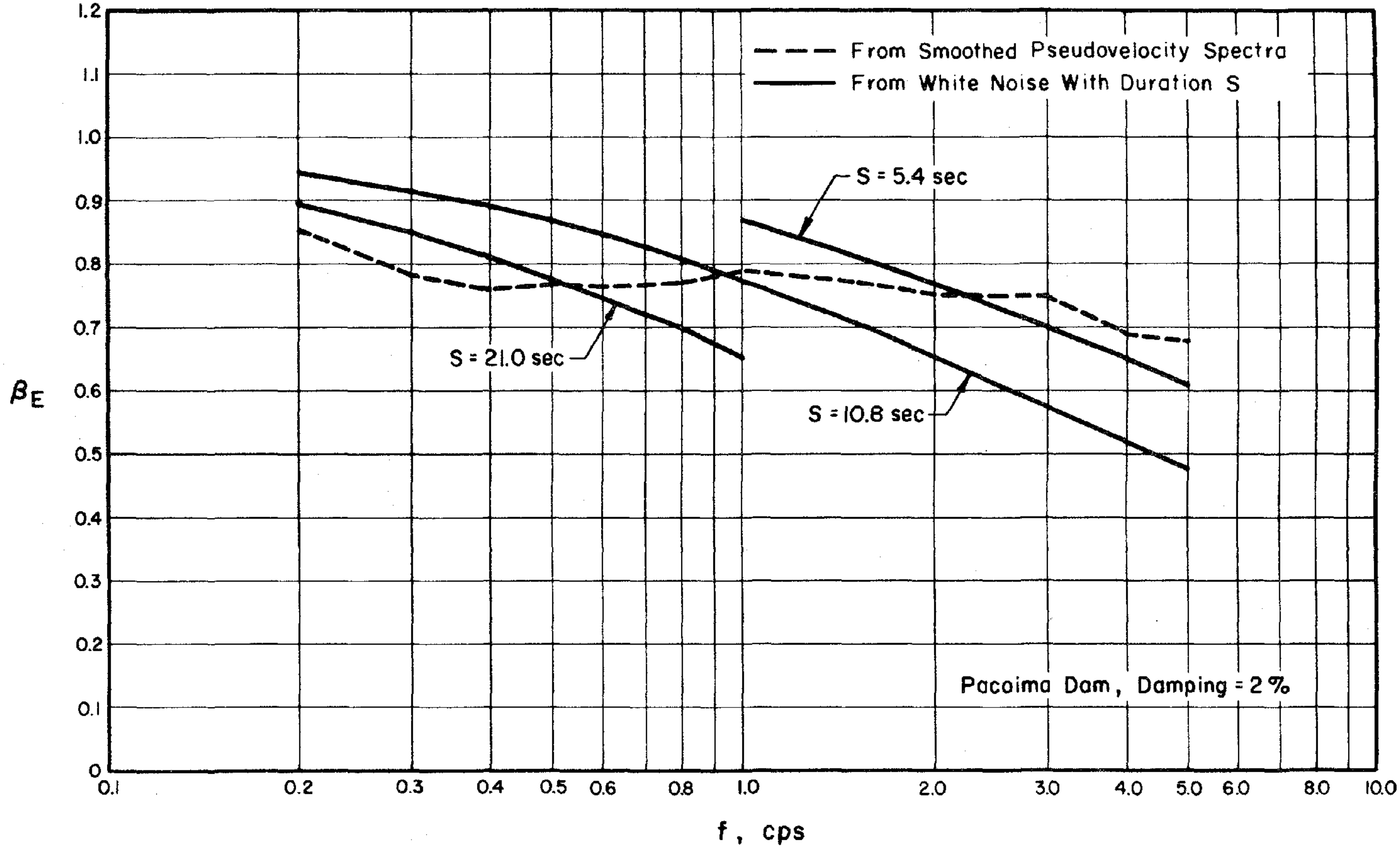


FIG. 8.6(a) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

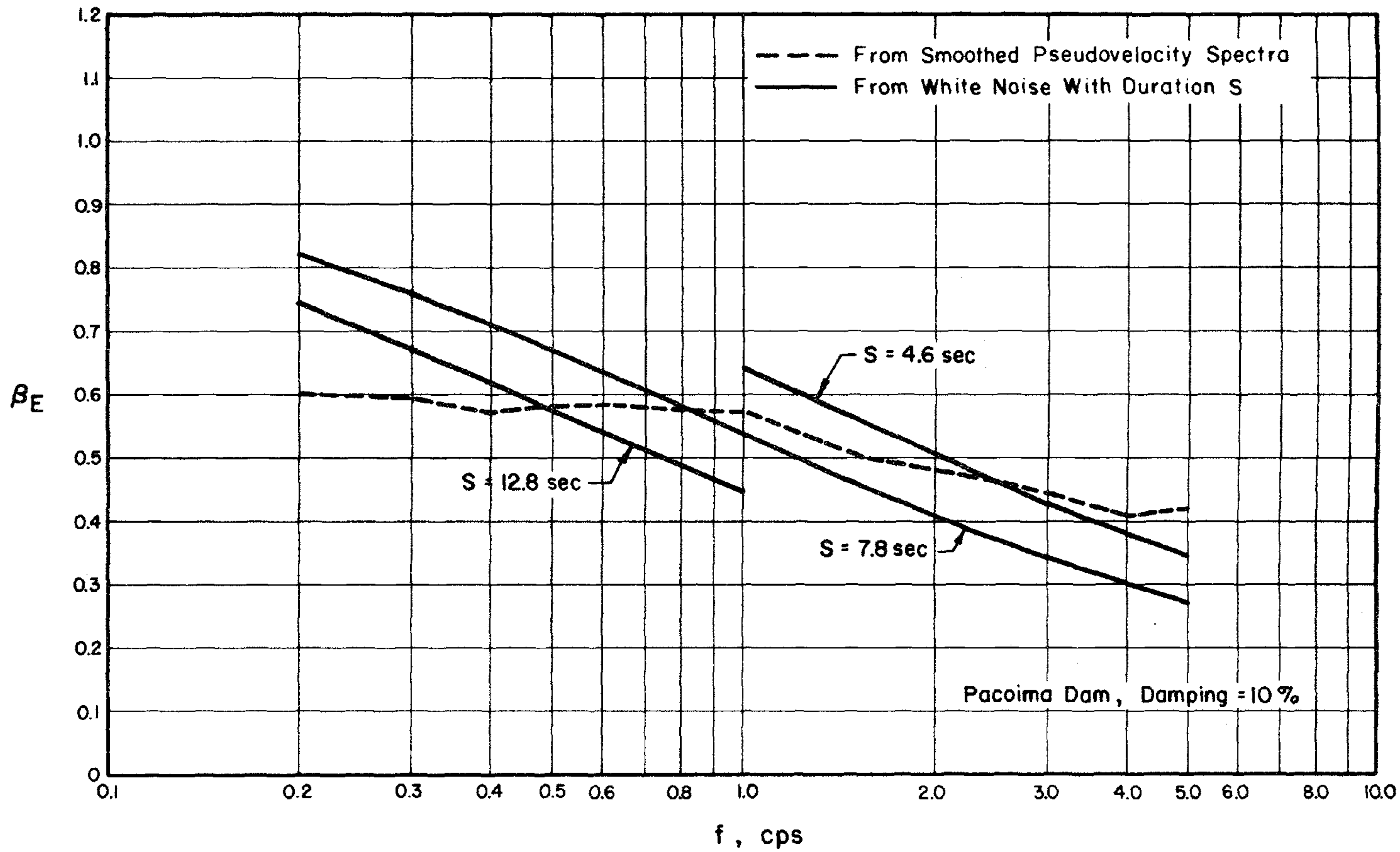


FIG. 8.6(b) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

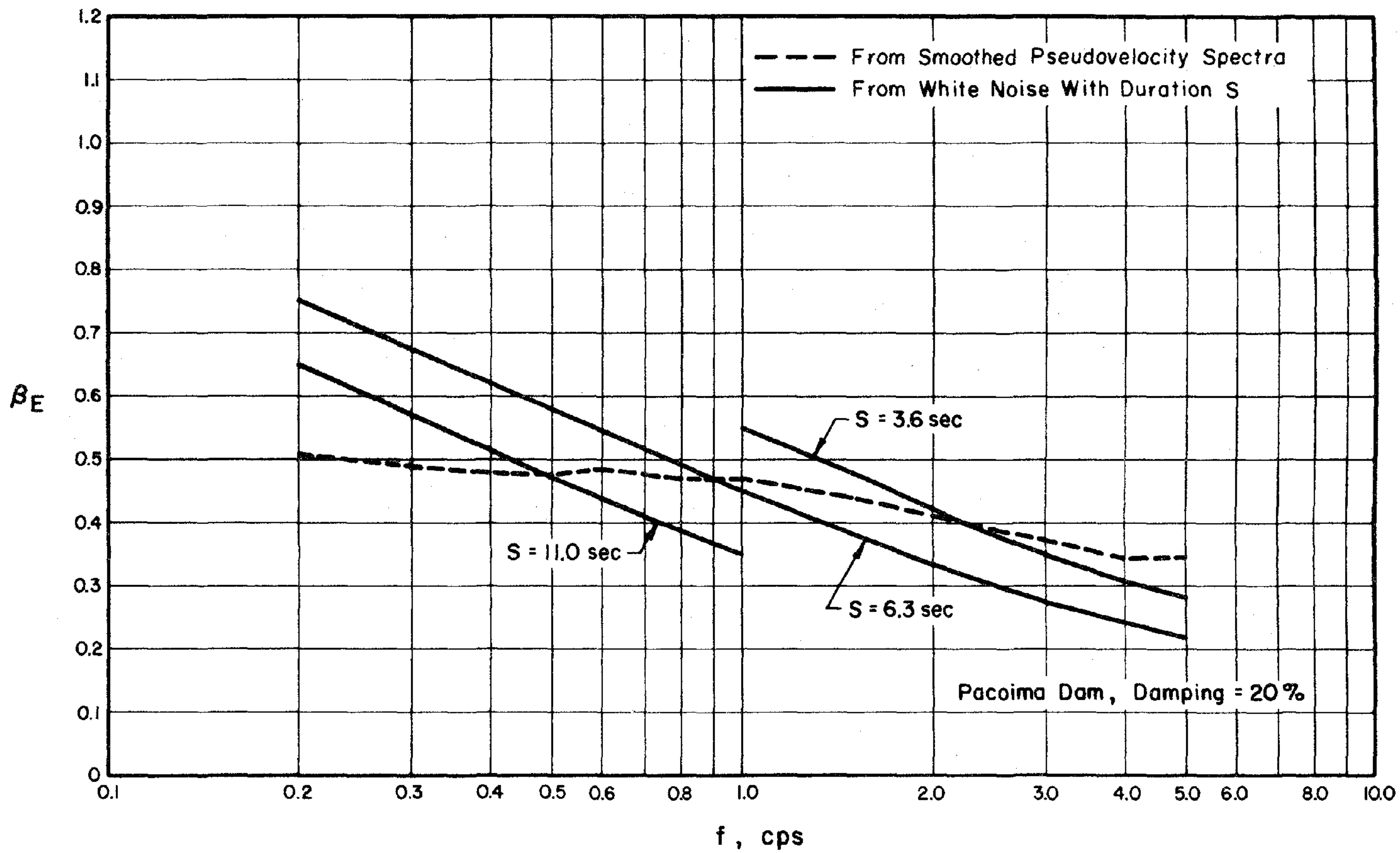


FIG. 8.6(c) ADJUSTMENT OF EARTHQUAKE DURATION FOR AN EQUIVALENT WHITE NOISE EXCITATION

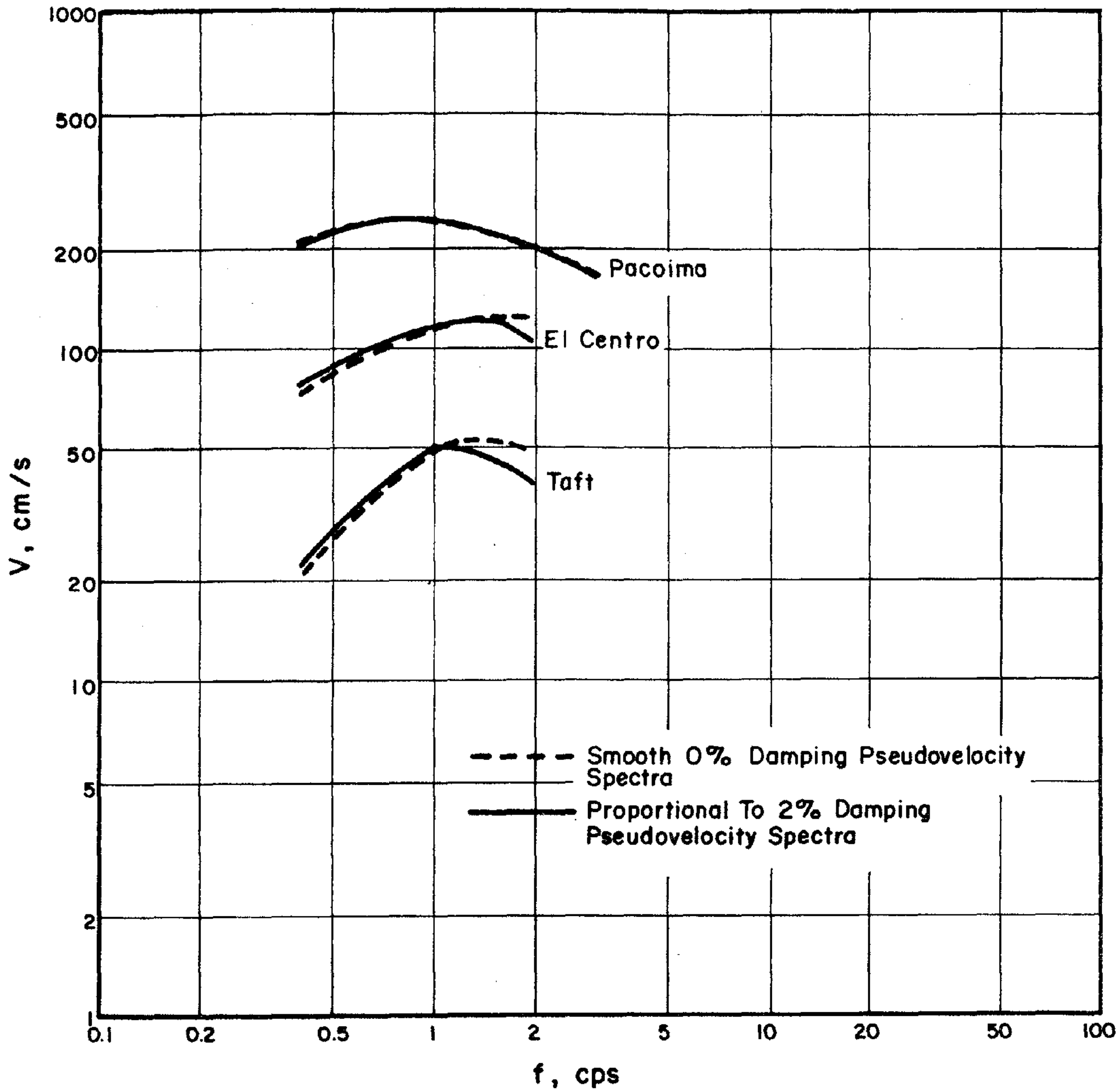


FIG. 8.7 ADJUSTMENT OF EARTHQUAKE DURATION FOR 0% DAMPING

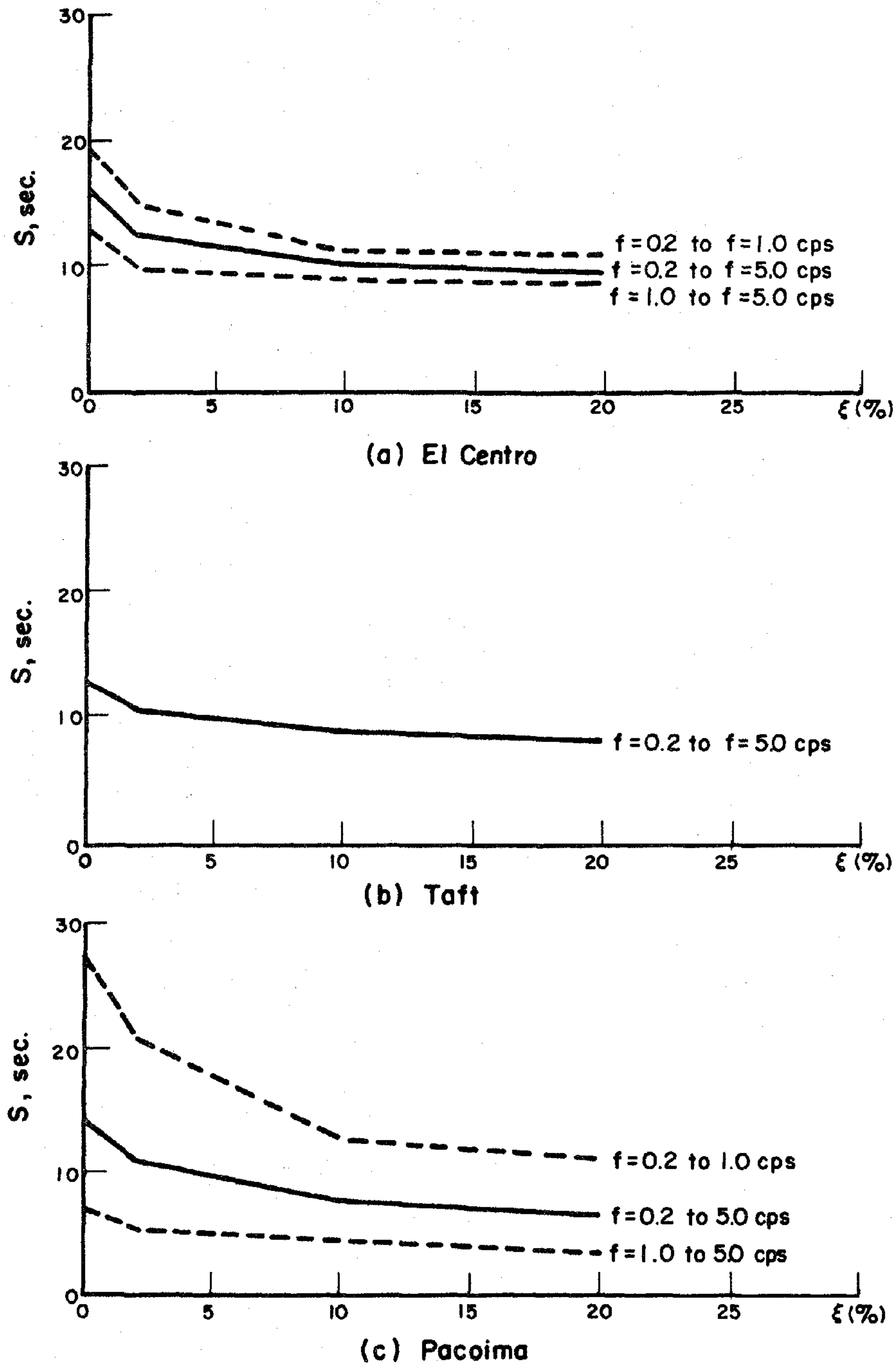


FIG. 8.8 VARIATION OF EQUIVALENT EARTHQUAKE DURATION WITH DAMPING

APPENDIX A

DERIVATION OF THE EQUALITY $\omega_{s_j}^2 = [k_1\phi_1(j) + k_3\phi_2(j)]/m_j^*$

Consider the system shown in Fig. 4.3. Its free vibration equation of motion is of the form

$$[m]\,\{\ddot{x}_s\} + [k]\,\{x_s\} = \{0\} \tag{A.1}$$

where [m] and [k] are respectively its mass and stiffness matrices and $\{x_s\}$ is its displacement vector, and the jth solution to this equation of motion may be written as

$$\{x_s\}^{(j)} = \{\phi\}^{(j)} \cos(\omega_{s_j} - \theta_j) \tag{A.2}$$

in which $\{\phi\}^{(j)}$ and ω_{s_j} represent the system's jth unit-participation-factor mode shape and jth natural frequency and θ_j is a constant phase angle. Then, if Eq. A.2 is substituted into Eq. A.1, one has that

$$-\omega_{s_j}^2\,[m]\{\phi\}^{(j)} + [k]\,\{\phi\}^{(j)} = \{0\} \tag{A.3}$$

which by premultiplication by $\{\phi\}^{(j)^T}$ leads to

$$\omega_{s_j}^2\,m_j^* = k_j^* \tag{A.4}$$

where

$$m_j^* = \{\phi\}^{(j)^T}[m]\{\phi\}^{(j)} \tag{A.5}$$

$$k_j^* = \{\phi\}^{(j)^T}[k]\,\{\phi\}^{(j)}\ . \tag{A.6}$$

In the same fashion, premultiplication of Eq. A.3 by {J}, a vector of unit elements, yields

$$-\omega_{s_j}^2 \{J\}^T [m]\{\phi\}^{(j)} + \{J\}^T [k]\{\phi\}^{(j)} = 0 \tag{A.7}$$

and thus by taking the transpose of both sides of this equation one obtains

$$\omega_{s_j}^2 \{\phi\}^{(j)^T} [m] \{J\} = \{\phi\}^{(j)^T} [k]\{J\}. \tag{A.8}$$

But

$$\{\phi\}^{(j)^T} [k]\{J\} = k_1\phi_1(j) + k_3\phi_2(j), \tag{A.9}$$

where $\phi_i(j)$, i = 1,2, represents the amplitude of the ith mass of the system under consideration in its jth mode, and since by assumption $\{\phi\}^{(j)}$ is a mode shape with a unit-participation factor, one has that

$$\{\phi\}^{(j)} [m]\{J\} = \{\phi\}^{(j)^T} [m]\{\phi\}^{(j)} = m_j^* \quad . \tag{A.10}$$

Therefore, Eq. A.8 may be written as

$$\omega_{s_j}^2 \; m_j^* = k_1\phi_1(j) + k_3\phi_2(j) \tag{A.11}$$

which in combination with Eq. A.4 permits one to conclude that

$$\omega_{s_j}^2 = \frac{k_j^*}{m_j^*} = \frac{k_1\phi_1(j) + k_3\phi_2(j)}{m_j^*} \quad . \tag{A.12}$$

Notice that when the system in Fig. 4.3 has its right end free, k_3 is equal to zero. Hence, for this particular case Eq. A.12 gives

$$\omega_{s_j}^2 = \frac{k_1 \phi_1(j)}{m_j^*} . \tag{A.13}$$

APPENDIX B

EXTENSION OF RAYLEIGH'S PRINCIPLE FOR SYSTEMS WITH NONPROPORTIONAL DAMPING

It is shown elsewhere (Cherry, 1968; Hurty and Rubinstein, 1964) that the natural frequencies of an undamped or proportionally damped system are stationary with respect to their respective mode shapes (Rayleigh's principle). Here, based on the developments introduced in Chapter 5, it is demonstrated that it is possible to extend this principle for the complex natural frequencies of systems with nonproportional damping.

It has been shown in Sec. 5.3 that the damped free-vibration equation of motion of a system with nonproportional damping

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{B.1}$$

is satisfied by

$$\{x\}^{(r)} = \{w\}^{(r)} e^{\lambda_r t} \tag{B.2}$$

where $\{w\}^{(r)}$ is the rth complex mode shape of the system and λ_r represents its rth natural frequency. Then, if Eq. B.2 is substituted into Eq. B.1, such an equation of motion may be written as

$$\lambda_r^2 [M]\{w\}^{(r)} + \lambda_r [C]\{w\}^{(r)} + [K]\{w\}^{(r)} = \{0\} , \tag{B.3}$$

which after premultiplication by $\{\bar{w}\}^{(r)^T}$, the transpose of the conjugate of $\{w\}^{(r)}$, leads to

$$\lambda_r^2 M_r^* + \lambda_r C_r^* + K_r^* = 0 \tag{B.4}$$

where M_r^*, C_r^*, and K_r^* are real parameters defined as

$$M_r^* = \{\bar{w}\}^{(r)^T}[M]\{w\}^{(r)} \tag{B.5}$$

$$C_r^* = \{\bar{w}\}^{(r)^T}[C]\{w\}^{(r)} \tag{B.6}$$

$$K_r^* = \{\bar{w}\}^{(r)^T}[K]\{w\}^{(r)}. \tag{B.7}$$

Accordingly, if the mode shape $\{w\}^{(r)}$ is approximated by a complex vector $\{a\}$ that is close, in the absolute value sense, to $\{w\}^{(r)}$, the equation of motion of the system under consideration may be expressed approximately as

$$\lambda_a^2 M_a + \lambda_a C_a + K_a = 0 \tag{B.8}$$

where λ_a represents the approximate value of the complex frequency λ_r corresponding to the approximate mode shape $\{a\}$, and M_a, C_a, and K_a are given by

$$M_a = \{\bar{a}\}^T [M]\{a\} \tag{B.9}$$

$$C_a = \{\bar{a}\}^T [C]\{a\} \tag{B.10}$$

$$K_a = \{\bar{a}\}^T [K]\{a\} \tag{B.11}$$

in which $\{\bar{a}\}$ denotes the complex conjugate of the vector $\{a\}$. Hence, if Eq. B.8 is derived with respect to a_i, the ith element of $\{a\}$, one obtains

$$2\lambda_a \frac{\partial\lambda_a}{\partial a_i} M_a + \lambda_a^2 \frac{\partial M_a}{\partial a_i} + \lambda_a \frac{\partial C_a}{\partial a_i} + \frac{\partial\lambda_a}{\partial a_i} C_a + \frac{\partial K_a}{\partial a_i} = 0 \tag{B.12}$$

from which it may be seen that

$$\frac{\partial\lambda_a}{\partial a_i} = \frac{1}{2\lambda_a M_a + C_a} [\lambda_a^2 \frac{\partial M_a}{\partial a_i} + \lambda_a \frac{\partial C_a}{\partial a_i} + \frac{\partial K_a}{\partial a_i}] \tag{B.13}$$

and, consequently, by letting n denote the number of degrees of freedom of the system herein being considered, one may write

$$\{\frac{\partial\lambda_a}{\partial a}\}^T = \{\frac{\partial\lambda_a}{\partial a_1} \ \frac{\partial\lambda_a}{\partial a_2} \ . \ . \ . \ \frac{\partial\lambda_a}{\partial a_n}\} =$$

$$= \frac{1}{2\lambda_a M_a + C_a} \{(\lambda_a^2 \frac{\partial M_a}{\partial a_1} + \lambda_a \frac{\partial C_a}{\partial a_1} + \frac{\partial K_a}{\partial a_1})$$

$$(\lambda_a^2 \frac{\partial M_a}{\partial a_2} + \lambda_a \frac{\partial C_a}{\partial a_2} + \frac{\partial K_a}{\partial a_2}) \ldots (\lambda_a^2 \frac{\partial M_a}{\partial a_n} + \lambda_a \frac{\partial C_a}{\partial a_n} + \frac{\partial K_a}{\partial a_n})\} \tag{B.14}$$

which after rearranging terms may also be expressed as

$$\{\frac{\partial\lambda_a}{\partial a}\}^T = \frac{1}{2\lambda_a M_a + C_a} [\lambda_a^2 \{\frac{\partial M_a}{\partial a_1} \ \frac{\partial M_a}{\partial a_2} \ . \ . \ . \ \frac{\partial M_a}{\partial a_n}\} +$$

$$+ \lambda_a \{\frac{\partial C_a}{\partial a_1} \ \frac{\partial C_a}{\partial a_2} \ . \ . \ . \ \frac{\partial C_a}{\partial a_n}\} + \{\frac{\partial K_a}{\partial a_1} \ \frac{\partial K_a}{\partial a_2} \ . \ . \ . \ \frac{\partial K_a}{\partial a_n}\}]. \tag{B.15}$$

But,

$$\{\frac{\partial M_a}{\partial a_1}\ \frac{\partial M_a}{\partial a_2}\ \ldots\ \frac{\partial M_a}{\partial a_n}\} = \{\frac{\partial}{\partial a_1}\{a\}^T [M]\{a\}\ \frac{\partial}{\partial a_2}\{a\}^T [M]\{a\} \ldots$$

$$\frac{\partial}{\partial a_n}\{a\}^T [M]\{a\}\} = \{a\}^T[M] \qquad (B.16)$$

and similarly

$$\{\frac{\partial C_a}{\partial a_1}\ \frac{\partial C_a}{\partial a_2}\ \ldots\ \frac{\partial C_a}{\partial a_n}\} = \{a\}^T[C] \qquad (B.17)$$

$$\{\frac{\partial K_a}{\partial a_1}\ \frac{\partial K_a}{\partial a_2}\ \ldots\ \frac{\partial K_a}{\partial a_n}\} = \{a\}^T [K]. \qquad (B.18)$$

Therefore, $\{\frac{\partial \lambda_a}{\partial a}\}$ is of the form

$$\{\frac{\partial \lambda_a}{\partial a}\} = \frac{1}{2\lambda_a M_a + C_a} (\lambda_a^2 [M] + \lambda_a [C] + [K]) \{a\}. \qquad (B.19)$$

Thus, since when $\{a\}$ approaches $\{w\}^{(r)}$, λ_a approaches λ_r and Eq. B.3 indicates that

$$(\lambda_a^2 [M] + \lambda_a [C] + [K]) \{a\} \rightarrow \{0\}, \qquad (B.20)$$

and since the term $2\lambda_a M_a + C_a$ is always different from zero, one may conclude that

$$\{\frac{\partial \lambda_a}{\partial a}\} = \{0\} \qquad (B.21)$$

when the approximate mode shape $\{a\}$ is in the proximity of the exact mode shape $\{w\}^{(r)}$. Hence, since the first variation of λ_a when the system is given a virtual displacement from the configuration $\{a\}$ is given by

$$\delta\lambda_a = \frac{\partial\lambda_a}{\partial a_1}\ \delta a_1 + \frac{\partial\lambda_a}{\partial a_2}\ \delta a_2 + \ .\ .\ .\ + \frac{\partial\lambda_a}{\partial a_n}\ \delta a_n \tag{B.22}$$

which in matrix form may be expressed as

$$\delta\lambda_a = \{ \frac{\partial\lambda_a}{\partial a} \}^T \{\delta a\}, \tag{B.23}$$

one has that

$$\delta\lambda_a = 0 \tag{B.24}$$

and thus it may be seen that the complex natural frequencies of a system with nonproportional damping are also stationary in the neighborhood of their respective complex mode shapes.

Notice that Rayleigh's principle for an undamped system may be derived directly from the above relationships, for in such a case

$$C_a = 0 \tag{B.25}$$

$$\{w\}^{(r)} = \{u\}^{(r)} \tag{B.26}$$

$$\lambda_a^2 = -\omega_a^2 \tag{B.27}$$

$$2\lambda_a \frac{\partial\lambda_a}{\partial a_i} = - \frac{\partial\omega_a^2}{\partial a_i} \tag{B.28}$$

where $\{u\}^{(r)}$ represents the real rth mode shape of such a system and ω_a denotes its corresponding approximate natural frequency, and as a result Eq. B.12 becomes

$$- \frac{\partial\omega_a^2}{\partial a_i}\ M_a - \omega_a^2 \frac{\partial M_a}{\partial a_i} + \frac{\partial K_a}{\partial a_i} = 0 \tag{B.29}$$

from which one obtains that

$$\frac{\partial \omega_a^2}{\partial a_i} = \frac{1}{M_a} \left(\frac{\partial K_a}{\partial a_i} - \omega_a^2 \frac{\partial M_a}{\partial a_i} \right) . \qquad (B.30)$$

Consequently, one has that

$$\{\frac{\partial \omega_a^2}{\partial a}\} = \frac{1}{M_a} ([K] - \omega_a^2 [M]) \{a\} \qquad (B.31)$$

which in combination with Eq. B.3 for the case when [C] = 0 permits one to conclude that

$$\{\frac{\partial \omega_a^2}{\partial a}\} = \{0\} \qquad (B.32)$$

and

$$\delta \omega_a^2 = \{\frac{\partial \omega_a^2}{\partial a}\}^T \{\delta a\} = 0 \qquad (B.33)$$

when $\{a\} \rightarrow \{u\}^{(r)}$.

APPENDIX C

DERIVATION OF THE EQUALITY

$$\{\phi\}^{(j)^T}[c]\{J\} = \{\phi\}^{(j)^T}[c]\{\phi\}^{(j)}$$

The homogeneous equation of a proportionally damped system with mass, damping, and stiffness matrices [m], [c], and [k] is

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{0\} \quad (C.1)$$

where {x} represents the displacement vector of the system, and the solutions to this equation of motion are of the form

$$\{x\}^{(j)} = \{\phi\}^{(j)} e^{\lambda_{s_j} t} \quad (C.2)$$

in which $\{\phi\}^{(j)}$ signifies the system's jth unit-participation-factor mode shape and λ_{s_j} is its jth complex natural frequency. Therefore, if Eq. C.2 is substituted into Eq. C.1, one arrives to

$$\lambda_{s_j}^2 [m]\{\phi\}^{(j)} + \lambda_{s_j}[c]\{\phi\}^{(j)} + [k]\{\phi\}^{(j)} = 0 \quad (C.3)$$

which after premultiplaction by $\{\phi\}^{(j)T}$ leads to

$$\lambda_{s_j}^2 m_j^* + \lambda_{s_j} c_j^* + k_j^* = 0 \quad (C.4)$$

where

$$m_j^* = \{\phi\}^{(j)T} [m]\{\phi\}^{(j)} \tag{C.5}$$

$$c_j^* = \{\phi\}^{(j)T} [c]\{\phi\}^{(j)} \tag{C.6}$$

$$k_j^* = \{\phi\}^{(j)T} [k]\{\phi\}^{(j)} \ . \tag{C.7}$$

Similarly, if Eq. C.3 is premultiplied by $\{J\}^T$, a unit vector, one obtains

$$\lambda_{s_j}^2 \ \{J\}^T [m]\{\phi\}^{(j)} + \lambda_{s_j} \ \{J\}^T [c]\{\phi\}^{(j)} + \{J\}^T [k]\{\phi\}^{(j)} = 0 \tag{C.8}$$

which in view of the symmetry of the matrices [m], [c], and [k] may also be expressed as

$$\lambda_{s_j}^2 \{\phi\}^{(j)T}[m]\{J\} + \lambda_{s_j} \{\phi\}^{(j)T}[c]\{J\} + \{\phi\}^{(j)T}[k]\{J\} = 0. \tag{C.9}$$

But since $\{\phi\}^{(j)}$ is a mode with a unit participation factor, one has that

$$\{\phi\}^{(j)T}[m]\{J\} = \{\phi\}^{(j)T}[m] \ \{\phi\}^{(j)} \ . \tag{C.10}$$

In like manner, if it is assumed that the damping matrix [c] is proportional to the stiffness matrix [k], one may write

$$c_j^* = a_S \ k_j^* \tag{C.11}$$

and

$$\{\phi\}^{(j)T}[c]\{J\} = a_S \ \{\phi\}^{(j)T}[k]\{J\} \tag{C.12}$$

where a_S is a proportionality constant. Therefore, Eqs. C.4 and C.9 may be written as

$$\lambda_{s_j}^2 \ m_j^* + (a_S\lambda_{s_j} + 1) \ k_j^* = 0 \tag{C.13}$$

$$\lambda_{s_j}^2 \ m_j^* + (a_S\lambda_{s_j} + 1) \ \{\phi\}^{(j)T}[k] \ \{J\} = 0 \ , \tag{C.14}$$

and hence, by equating these two equations, one obtains that

$$k_j^* = \ \{\phi\}^{(j)T}[k]\{J\} \ . \tag{C.15}$$

From Eqs. C.11 and C.15, one may then conclude that

$$c_j^* = a_S \ \{\phi\}^{(j)T} \ [k] \ \{J\} \tag{C.16}$$

which in combination with Eqs. C.6 and C.12 leads to

$$\{\phi\}^{(j)T}[c]\{J\} = \{\phi\}^{(j)T}[c] \ \{\phi\}^{(j)} \ . \tag{C.17}$$

APPENDIX D

NOTATION

A_r^*	generalized parameter of primary system defined by Eq. 5.24 or 6.9
$A_o(j)$	parameter defined by Eq. 3.15 or 4.112
$[A]$	square matrix defined by Eq. 5.4 or 6.2
A.F.	amplification factor
$(A.F.)_r$	amplification factor in rth mode
a	constant
a_p	constant of proportionality for primary system
a_s	constant of proportionality for secondary system
a_{ij}	parameters defined by Eqs. 6.65 through 6.67
a_j^*	generalized parameter of secondary system defined by Eq. 6.71
$\{a\}$	approximate mode shape
$[a]$	square matrix defined by Eq. 6.48
arg	"the argument of"
B_r	parameter defined by Eq. 2.94
B_r^*	generalized parameter of primary system defined by Eq. 5.25 or 6.10
$B_o(i)$	parameter defined by Eq. 3.26 or 4.126
$B_o'(i)$	parameter defined by Eq. 6.491
$[B]$	square matrix defined by Eq. 5.5 or 6.3
b_{ij}	parameters defined by Eqs. 6.68 through 6.70
b_j^*	generalized parameter of secondary system defined by Eq. 6.72
$[b]$	square matrix defined by Eq. 6.49

C_i	ith damping constant of primary system
C_i^*	ith generalized damping constant of independent primary system
$(C_r^*)_{cr}$	critical damping value
$C_{R_r}^*$	real part of rth generalized damping constant
$C_{I_r}^*$	imaginary part of rth generalized damping constant
[C]	damping matrix of primary system
c_j	jth damping constant of secondary system
c_{ij}	parameter defined by Eq. 6.263
c_j^*	jth generalized damping constant of independent secondary system
c_{cj}	parameter defined by Eq. 6.112
[c]	damping matrix of secondary system
[c']	damping matrix of secondary system with both ends fixed
$\mathcal{C}_r^*$	generalized damping constant defined by Eq. 5.94
$\mathcal{C}_a$	generalized damping constant for an approximate mode shape {a}
$[\mathcal{C}]$	damping matrix of assembled system
$D(\omega_r,\xi_r,t)$	earthquake displacement response at time t of a sdof system with frequency ω_r and damping ratio ξ_r
[D]	matrix defined by Eq. 6.184
$d\Phi(i)$	difference between amplitudes of points of attachment in ith mode of independent primary system
$d\phi_i(j)$	ith element of $\{d\phi\}^{(j)}$
$dw_{s_i}(r)$	ith element of $\{dw_s\}^{(r)}$
$dw'_{s_i}(r)$	ith element of $\{dw'_s\}^{(r)}$
$\{dw_s\}^{(r)}$	secondary system part of rth vector of modal distortions of assembled system described by Eq. 6.278
$\{dw'_s\}^{(r)}$	$= \gamma_r\{dw_s\}^{(r)}$
$\{d\phi\}^{(J)}$	vector of modal distortions of independent secondary system in its Jth mode given by Eq. 3.4

$\{du\}^{(r)}$	rth vector of modal distortions of assembled system described by Eq. 2.96
$\{du'\}$	$= \alpha_r\{du\}$
$\{du_s\}^{(r)}$	secondary system part of rth vector of modal distoritons of assembled system described by Eq. 2.100
$\{du_s'\}^{(r)}$	$= \alpha_r\{du_s\}^{(r)}$
$\{\frac{df}{f_{cc}}\}$	vector defined by Eq. 4.95
E[ ]	expected value
$\{F(t)\}$	vector of external forces defined by Eq. 6.4
[F]	square matrix defined by Eq. 6.185
f_o	resonant frequency in cycles per second
f_r	rth natural frequency of assembled system in c.p.s.
f_{p_i}	ith natural frequency in c.p.s. of independent primary system
f_{s_j}	jth natural frequency in c.p.s. of independent secondary system
f_{cc}	$= \phi_c(c)$ = amplitude of second point of attachment in the vector of flexibilities {f}
$\{f\}$	$= \{\phi\}^{(c)}$ = vector of flexibilities defined by Eq. 4.14
$\{f(t)\}$	vector of external forces defined by Eq. 6.50
[G]	matrix defined by Eq. 6.186
[H]	matrix defined by Eq. 6.187
{J}	vector of unit elements
K_i	ith stiffness constant of primary system
K_i^*	ith generalized stiffness of independent primary system
K_r^*	generalized stiffness defined by Eq. 5.95
$K_{R_r}^*$	real part of rth generalized stiffness
$K_{I_r}^*$	imaginary part of rth generalized stiffness

$\mathcal{K}_a$	generalized stiffness for an approximate mode shape {a}
[K]	stiffness matrix of primary system
[$\mathcal{K}$]	stiffness matrix of assembled system
k	number of the first primary mass to which a secondary system is attached
k_j	jth stiffness constant of secondary system
k_{N_s+1}	stiffness constant of last element of secondary system
k_j^*	jth generalized stiffness of independent secondary system
[k]	stiffness matrix of secondary system
[k']	stiffness matrix of secondary system with both ends fixed
ℓ	number of the second primary mass to which a secondary system is attached
M_i	ith mass of primary system
M_i^*	ith generalized mass of independent primary system
$\mathcal{M}_a$	generalized mass for an approximate mode shape {a}
$M_{R_r}^*$	real part of rth generalized mass
$M_{I_r}^*$	imaginary part of rth generalized mass
[M]	mass matrix of primary system
[$\mathcal{M}$]	mass matrix of assembled system
m_j	jth mass of secondary system
m_j^*	jth generalized mass of independent secondary system
m_0^*	parameter defined by Eq. 2.73
m_c^*	parameter defined by Eq. 4.20 for j=c
m_{0j}	parameter defined by Eq. 6.105
m_{cj}	parameter defined by Eq. 6.111
m_{cj}^*	parameter defined by Eq. 4.21

[m]	mass matrix of secondary system
[m']	mass matrix of secondary system with both ends fixed
N_p	number of degrees of freedom of independent primary system
N_s	number of degrees of freedom of independent secondary system
P^*_{IJ}	parameter defined by Eq. 6.201
{P(t)}	vector of external forces
[P]	matrix defined by Eq. 6.191
Q^*_I	parameter defined by Eq. 6.199
Q^*_r	parameter defined by Eq. 5.26
{Q(t)}	vector defined by Eq. 5.7
[Q]	matrix defined by Eq. 6.192
$\ddot{q}_g(t)$	earthquake ground acceleration
{q}	vector defined by Eq. 5.6
$\{q\}^{(r)}$	rth solution to homogeneous reduced equation of motion
$\{q_p\}$	vector defined by Eq. 6.5
$\{q_s\}$	vector defined by Eq. 6.51
R	number of resonant modes
R(t)	reaction force between primary and secondary systems
$R_1(t)$	reaction force acting on first mass of primary system
$R_3(t)$	reaction force acting on third mass of primary system
Re	"the real part of"
{R(t)}	vector of reactions defined by Eq. 2.9
$\{R(t)\}_p$	vector of reactions on primary system
$\{R(t)\}_s$	vector of reactions on secondary system

r_j parameter defined by Eq. 3.20 or 6.419

r_c parameter defined by Eq. 4.116 or 6.418

S_o constant spectral density of white noise excitation

SV pseudovelocity

SV_0 undamped pseudovelocity

$SD(\omega_r,\xi_r)$ = SD_r = ordinate in a displacement response spectrum corresponding to a natural frequency ω_r and a damping ratio ξ_r

$SV(\omega_r,\xi_r)$ = SV_r = ordinate in a pseudovelocity response spectrum corresponding to a natural frequency ω_r and a damping ratio ξ_r

$\{S\}^{(i)}$ ith complex eigenvector of independent primary system

[S] matrix of complex eigenvectors of primary system

s equivalent earthquake duration

s_0 equivalent earthquake duration for 0% damping

s_r equivalent earthquake duration for a damping ratio ξ_r

sgn "the sign of"

$\{s\}^{(0)}$ rigid-body complex eigenvector of secondary system

$\{s\}^{(c)}$ constraint complex eigenvector of secondary system

$\{s\}^{(j)}$ jth complex eigenvector of independent secondary system

[s] matrix of complex eigenvectors of secondary system

T^*_{IJ} parameter defined by Eq. 6.202

[T] matrix defined by Eq. 6.194

t time

$u_{p_i}(r)$ amplitude of ith primary mass in rth mode shape of assembled system

$u_{p_k}(r)$ amplitude of supporting primary mass in rth mode shape of assembled system

$u_{s_j}(r)$ amplitude of jth secondary mass in rth mode shape of assembled system

$u'(r)$ element of $\{u'\}^{(r)}$

$u'_{s_i}(r)$ ith element of $\{u'_s\}^{(r)}$

$\{u\}^{(r)}$ real part of rth complex mode shape

$\{u'\}^{(r)}$ real part of $\{w'\}^{(r)}$

$\{u_p\}$ primary system part of mode shape of assembled system

$\{u_p\}^{(r)}$ primary system part of rth mode shape of assembled system

$\{u_s\}$ secondary system part of mode shape of assembled system

$\{u_s\}^{(r)}$ secondary system part of rth mode shape of assembled system

$\{u'_s\}^{(r)}$ real part of $\{w'_s\}^{(r)}$

$V(\omega_r,\xi_r,t)$ earthquake velocity response at time t of a sdof system with frequency ω_r and damping ratio ξ_r

V_I^* parameter defined by Eq. 6.200

[V] matrix defined by Eq. 6.193

$v'(r)$ element of $\{v'\}^{(r)}$

$v'_{s_i}(r)$ ith element of $\{v'_s\}^{(r)}$

$\{v\}^{(r)}$ imaginary part of rth complex mode shape

$\{v'_s\}^{(r)}$ imaginary part of $\{w'_s\}^{(r)}$

$w'_i(r)$ amplitude of ith mass in rth unit-participation-factor complex mode shape

$w_{p_i}(r)$ ith element of $\{w_p\}^{(r)}$

$w_{s_j}(r)$ jth element of $\{w_s\}^{(r)}$

$\{w\}^{(r)}$ rth complex mode shape

$\{w'\}^{(r)}$ rth complex mode shape with unit complex participation factor

$\{w_p\}$ primary system part of complex mode shape of assembled system

$\{w_s\}$ secondary system part of complex mode shape of assembled system

$\{w_p\}^{(r)}$ primary system part of rth complex mode shape of assembled system

$\{w_s\}^{(r)}$ secondary system part of rth complex mode shape of assembled system

$X_i(r)$ $= X_{i_r}$ = distortion of ith element in rth mode

$\{X\}_{max}$ vector of maximum distortions of assembled system

$\{X\}^{(r)}$ rth vector of maximum modal distortions of assembled system

$\{X_s\}_{max}$ vector of maximum distortions of secondary system

$\{X_s\}^{(r)}$ vector of maximum distortions of secondary system in rth mode of assembled system

$\{X_s\}^{(s)}$ combined response in two adjacent resonant modes given by Eq. 3.36

$x_i(t)$ displacement response function of ith element

$x_{i_{max}}$ maximum value of $x_i(t)$

x_{p_i} displacement of ith primary mass in assembled system

x_{s_j} displacement of jth secondary mass in assembled system

$\{x_{max}\}$ vector of maximum displacements

$\{x_p\}$ displacement vector of primary masses in assembled system

$\{x_s\}$ displacement vector of secondary masses in assembled system

$\{x\}^{(r)}$ rth displacement vector

Y_i factors defined by Eq. 2.16, 2.17, 4.7, or 6.44

Y_i' ith normal coordinate of primary system

$Y_i^{(r)}$ ith primary system factor in rth mode of assembled system defined by Eq. 2.19, 4.9, or 6.46

$\{Y'\}$ vector of normal coordinates Y_i'

$\{Y\}^{(r)}$ vector of $Y_i^{(r)}$ factors

y_j factors defined by Eq. 2.28, 2.29, 4.23 or 6.144

y_j'	jth generalized coordinate of secondary system
$y_j^{(r)}$	jth secondary system factor in rth mode of assembled system defined by Eq. 2.31, 4.25, or 6.150
y_0	factor defined by Eq. 2.33, 4.30, or 6.141
y_0'	generalized coordinate in Eq. 2.22 or 4.16
$\hat{y}_0^{(r)}$	parameter defined by Eq. 4.33 or 6.149
$y_0^{(r)}$	factor defined by Eq. 2.34, 4.30, or 6.147
y_c'	generalized coordinate in Eq. 4.16
$y_c^{(r)}$	factor defined by Eq. 4.31 or 6.148
$\{y\}$	vector of y_j factors
$\{y'\}$	vector of generalized coordinates y_j'
$\{y\}^{(r)}$	vector of $y_j^{(r)}$ factors
Z_r	rth element of $\{Z\}$
$Z_{\bar{i}}$	element of $\{Z\}$ corresponding to the complex conjugate of $\{S\}^{(i)}$ in Eq. 6.27
Z_i	factor defined by Eq. 6.22
Z_i'	ith element of $\{Z'\}$
$Z_i^{(r)}$	ith primary system factor in rth mode of assembled system defined by Eq. 6.25
$\{Z\}$	vector of Z_i factors
$\{Z'\}$	vector of complex normal coordinates of primary system
z_j	factor defined by Eq. 6.78 or 6.118
z_j'	element of $\{z'\}$
$z_{\bar{j}}'$	element of $\{z'\}$ corresponding to the complex conjugate of $\{s\}^{(j)}$ in Eq. 6.55
$\hat{z}_0$	parameter defined by Eq. 6.119
$\hat{z}_0^{(r)}$	parameter defined by Eq. 6.126

$z_j^{(r)}$	jth secondary system factor in rth mode of assembled system defined by Eq. 6.125
$\{z\}$	vector of factors z_j
$\{z'\}$	vector of complex generalized coordinates of secondary system
α_r	rth participation factor of assembled system
α_{mn}	modal correlation factor between modes m and n
$\alpha_{n(n+1)}$	modal correlation factor between two adjacent modes
α_{IJ}	modal correlation factor defined by Eq. 6.577
β_E	ratio of expected values of damped to undamped pseudovelocities
β_j	parameter defined by Eq. 4.36
γ_r	rth complex participation factor
γ_{s_j}	jth complex participation factor of independent secondary system
γ_{ij}	primary to secondary mass ratio in ith primary and jth secondary modes
Δ_{IJ}	parameter defined by Eq. 6.364
$\delta(t)$	Dirac's delta function
δ_i	parameter defined by Eq. 6.492
δ_j	parameter defined by Eq. 6.408
$\delta\lambda_a$	first variation of λ_a
$\delta\omega_a^2$	first varation of ω_a^2
ε_{IJ}	parameter defined by Eq. 6.368
ζ_r	phase angle defined by Eq. 6.311
η	ratio of $R_3(t)$ to $R_1(t)$
η_r	ratio of $R_3(t)$ to $R_1(t)$ in rth mode of assembled system

θ	phase angle or dummy variable
θ_r	phase angle defined by Eq. 6.295
θ_j	phase angle defined by Eq. 6.392
θ_i	phase angle defined by Eq. 6.465
κ_{IJ}	parameter defined by Eq. 6.546
λ	complex natural frequency
λ_r	rth complex natural frequency
λ_r'	rth corrected complex natural frequency defined by Eq. 5.183
λ_{p_i}	ith complex natural frequency of independent primary system
λ_{s_j}	jth complex natural frequency of independent secondary system
λ_{s_0}	complex natural frequency of secondary system in rigid-body mode
λ_{s_c}	complex natural frequency of secondary system in constraint mode
λ_a	approximate value of complex natural frequency corresponding to an approximate mode shape {a}
μ_{IJ}	parameter defined by Eq. 6.576
ν_{IJ}	phase angle defined by Eq. 6.526
ξ	damping ratio
ξ_r	rth damping ratio
ξ_r'	rth corrected damping ratio defined by Eq. 2.103
ξ_o	damping ratio common to two resonant modes or defined by Eq. 6.539
ξ_{p_i}	ith damping ratio of independent primary system
ξ_{s_j}	jth damping ratio of independent secondary system
π	3.14159...

ρ_{mn}	parameter defined by Eq. 6.560
$\sum_n$	summation for all n
$\{\sigma_p\}$	primary system part of complex eigenvector of assembled system
$\{\sigma_s\}$	secondary system part of complex eigenvector of assembled system
$\{\sigma_p\}^{(r)}$	primary system part of rth complex eigenvector of assembled system
$\{\sigma_s\}^{(r)}$	secondary system part of rth complex eigenvector of assembled system
τ	dummy variable
τ_{IJ}	parameter defined by Eq. 6.531
$\Phi_n(i)$	amplitude of nth mass in ith mode of primary system
$\hat{\Phi}_r(i)$	parameter defined by Eq. 4.10
$\Phi_o(i,j)$	central value of the amplitudes of the points of attachment in a primary system in the ith primary and jth secondary modes
$\{\Phi\}^{(i)}$	ith mode shape of independent primary system
$[\Phi]$	modal matrix of independent primary system
$\phi_n(j)$	amplitude of nth mass in jth mode of secondary system
$\phi_c(c)$	$= f_{cc}$ = amplitude of the second point of attachment in the vector of flexibilities $\{\phi\}^{(c)}$
$\{\phi\}^{(j)}$	jth mode shape of independent secondary system
$\{\phi\}^{(0)}$	= {J} = vector of unit elements
$\{\phi\}^{(c)}$	= {f} = vector of flexibilities defined by Eq. 4.14
$[\phi]$	modal matrix of independent secondary system
Ψ_r	parameter defined by Eq. 6.300
$\Psi_R^{(s)}$	sth amplification factor in resonant modes

$\Psi_p^{(r)}$	amplification factor in a nonresonant mode with frequency close to a primary frequency
$\Psi_s^{(r)}$	amplification factor in a nonresonant mode with frequency close to a secondary frequency
ψ_r	phase angle defined by Eq. 6.305
$\psi_x(t)$	transfer function
$\psi_{x_r}(t)$	transfer function in rth mode
ω	natural frequency of assembled system
ω_r	rth natural frequency of assembled system
ω_r'	rth damped natural frequency
ω_o	resonant natural frequency of primary and secondary systems
ω_{p_i}	ith natural frequency of independent primary system
ω_{s_j}	jth natural frequency of independent secondary system
$[\omega_p]$	frequency matrix of independent primary system
$[\omega_s]$	frequency matrix of independent secondary system
$[\ ]$	rectangular or square matrix
$\{\ \}$	column vector
$[\ ]^T$	transpose of a matrix
$\{\ \}^T$	transpose of a column vector
$\dot{}$	differentiation with respect to time
$\bar{}$	complex conjugate
$\|\ \|$	absolute value or determinant of square matrix