DYNAMIC RESPONSE OF A BURIED PIPE IN A SEISMIC ENVIRONMENT

by

S.K. Datta, A.H. Shah and N. E1-Aki1y

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S.K. Datta, Professor Department of Mechanical Engineering University of Colorado, Boulder Member, ASME

A.H. Shah, Associate Professor Department of Civil Engineering University of Manitoba, Winnipeg

N. El-Akily, Assistant Professor Department of Mechanical Engineering University of Colorado, Boulder

Abstract

Axisymmetric dynamic response of a buried pipe due to an incident compressional wave is the subject of this investigation. The pipe has been modelled as a thin cylindrical shell of linear homogeneous isotropic elastic material embedded in a linear isotropic homogeneous elastic medium of infinite extent. The response characteristics of the pipe due to changes in the material properties of the surrounding medium have been carefully studied. It is found that even at long wavelengths and low frequencies the dynamic response is significantly altered by the changes in the Poisson's ratio and the rigidity modulus of the surrounding medium. Further it is found that there are real resonant frequencies of the pipe which are also significantly dependent on these quantities as well as on the wavelength.

1. Introduction

Dynamic response of buried pipelines to seismic excitation has been a subject of considerable interest in recent years. The interest in the subject originates from the desire to design lifelines like gas and water/sewer pipelines against severe damage from earthquakes. The damage to buried pipelines is caused by landslides, liquefaction of soil, faulting and also due to shaking by traveling seismic waves.

Several investigators have recently studied the dynamic response of underground pipelines to seismic excitation. References to these works can be found in the review articles $[1, 2, 3, 4]$. Most of these works have treated the pipeline as a continuous or segmented beam and pave not taken into account the dynamic interaction between the pipe and its surrounding medium.

The departure from the beam model is in the works of Ariman, et al. $[2, 5]$ and in those of Novak and his co-workers $[6, 7, 8]$, where the authors have considered the shell model of the pipe. However, the interaction of the pipe and the soil has not been considered.

In our recent studies [9, 10, 11] it has been shown that the interaction between the pipe and its environment significantly influences the displacements of the pipe wall and the stresses arising in it. It has also been found [9, 10] that the depth of the embedment of the pipe also affects the pipe response.

In this paper we have examined in detail the interaction effects on the dynamic response. In an earlier paper [11] we presented a quasi-static analysis of the problem in which the inertia effects were neglected. It was shown that except for very long wavelengths the pipe does not in general follow the motion of the ground. This implies that the inertia effects should have considerable influence on the response of the pipe. Thus the object of this paper is to analyze the full dynamic problem.

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The related problems of the free and forced vibrations of a pipe in an acoustic medium have received considerable attention in the past [12, l3J. The vibration of a pipe in an elastic medium is complicated by the coupling of the longitudinal and shear waves generated in the surrounding medium. As in the case of ^a freely vibrating pipe in an acoustic medium it is found that there may exist resonant frequencies for a pipe freely vibrating in an elastic medium. However, in the latter case the existence of the resonant frequencies depends on the rigidity ratio of the pipe and its surrounding elastic medium.

We have also considered the forced vibration of the pipe due to an incident longitudinal wave. It is shown that the displacement of the pipe wall I and the axial stress in it depend critically on the Poisson's ratio and the rigidity modulus of the surrounding medium.

 $-2 -$

2. Equations and Solution

As shown in [llJ the equations governing the axisymmetric motion of a shell element are given by

$$
[\frac{1}{R^2} (\mathbb{E}_p + D/R^2) + k_x^2 \cosh \frac{\partial^2}{\partial x^2} - \rho_s h \frac{\partial^2}{\partial t^2}] w + k_x^2 \cosh \frac{\partial^2}{\partial x} - \frac{\mathbb{E}_p v}{R} \frac{\partial u}{\partial x} + p_x^* = 0
$$
\n
$$
- k_x^2 \cosh \frac{\partial w}{\partial x} + [D \frac{\partial^2}{\partial x^2} - k_x^2 \cosh \frac{\partial u}{\partial t} - \rho_x I \frac{\partial^2}{\partial t^2}] \Psi_x +
$$
\n
$$
[\frac{D}{R} \frac{\partial^2}{\partial x^2} - \frac{\rho_s I}{R} \frac{\partial^2}{\partial t^2}] u + p_x^* = 0
$$
\n
$$
[\frac{D}{R} \frac{\partial^2}{\partial x^2} - \frac{\rho_s I}{R} \frac{\partial^2}{\partial t^2}] u + p_x^* = 0
$$
\n
$$
\frac{\mathbb{E}_p v}{R} \frac{\partial w}{\partial x} + [\frac{D}{R} \frac{\partial^2}{\partial x^2} - \frac{\rho_s I}{R} \frac{\partial^2}{\partial t^2}] \Psi_x + [\frac{E}{P} \frac{\partial^2}{\partial x^2} - \rho_s h \frac{\partial^2}{\partial t^2}] u +
$$
\n
$$
p_3^* = 0
$$
\n(3)

Here u, w are the axial and radial displacements of a point on the shellmiddle surface (see Figure 1) and $\mathbb{Y}_{\mathbf{x}}$ is the rotation of the normal to this surface in the meridianal plane. The other shell parameters appearing above are defined as follows.

 $R \equiv$ Mean Radius

 $h \equiv$ Thickness

$$
E_p = \frac{Eh}{1 - v^2}, \qquad E = \text{Young's Modulus}, \quad v = \text{Poisson's Ratio}
$$

$$
D = \frac{E_p h^2}{12}
$$

 $G \equiv$ Shear Modulus

 ρ = Density $I = \frac{h^3}{12}$ $\frac{1}{2}$ $k_{\mathbf{x}}$ = Shear correction factor, taken as $\sqrt{12}$

The vector p^* represents the force and moment exerted per unit area by the surrounding medium on the shell and has the components

$$
p_1^* = (1 + \frac{h}{2R}) \tau_{rr}^*
$$
, $p_2^* = \frac{h}{2} p_3^*$, $p_3^* = (1 + \frac{h}{2R}) \tau_{rx}^*$ (4)

The stress components τ_{rr}^* and τ_{rx}^* arise from the motion of the surrounding medium and are evaluated at $r = R + \frac{h}{2}$.

The displacement $u(r, \theta, x, t)$ of a point of the outer medium satisfies the equations of motion

$$
\tau^2 \mathbb{Z} \cdot \mathbb{Z} \cdot \mathbb{Z} - \mathbb{Z} \wedge \mathbb{Z} \wedge \mathbb{Z} = \frac{1}{c_2^2} \frac{\partial^2 \mathbb{Z}}{\partial t^2}
$$
 (5)

where

$$
\tau = \sqrt{\frac{2(1-\sigma)}{1-2\sigma}} \quad , \qquad c_2 = \sqrt{\frac{\mu}{\rho}}
$$

 σ is the Poisson's ratio and c_2 is the shear wave speed.

Since the object of the present investigation is to analyze the motion of the shell excited by an incoming traveling seismic wave, it will be assumed that ^u is composed of two parts. The part due to the incident disturbance $\ddot{}$ will be denoted by $u^{(1)}$, whose components may be represented for axisymmetric motion as

$$
u_{r}^{(i)} = -u_{0}R\sqrt{\xi^{2} - k_{1}^{2}} I_{1} (\sqrt{\xi^{2} - k_{1}^{2}} r) \cos \xi (x - ct)
$$

$$
u_{x}^{(i)} = u_{0} \xi R I_{0} (\sqrt{\xi^{2} - k_{1}^{2}} r) \sin \xi (x - ct)
$$
 (6)

where I_n is the modified Bessel function of the first kind. In writing (6)

it has been assumed that the disturbance is in the form of a longitudinal wave. Clearly $\mu^{(i)}$ given by (6) with $k_1 = \omega/c_1 = c_5/c_1$ satisfies (5) and is a traveling wave of wavelength $\lambda = 2\pi/5$, moving with speed c along the axis of the pipe. For the special case of a plane longitudinal wave moving along the axis of the pipe (6) will reduce to

$$
u_{r}^{(i)} = 0 \t, \t u_{x}^{(i)} = u_{o} \xi R \sin \xi (x - c_{1} t)
$$
 (7)

The other part of μ , denoted by μ ^(s), may then be written as (see [11]),

$$
u_r^{(s)} = -[A \frac{\gamma}{R} K_1 (\gamma \frac{r}{R}) + B \xi K_1 (\delta \frac{r}{R})] \cos \xi (x - ct)
$$

$$
u_x^{(s)} = [A \xi K_0 (\gamma \frac{r}{R}) + B \frac{\delta}{R} K_0 (\delta \frac{r}{R})] \sin \xi (x - ct)
$$
 (8)

Here K_n is the modified Bessel function of the second kind and

$$
\gamma = \sqrt{\ell^2 - \varepsilon^2} , \quad \delta = \sqrt{\ell^2 - \tau^2 \varepsilon^2} , \quad \varepsilon = \omega R/c_1 , \quad \ell = \xi R
$$
 (9)

The constants A and B are chosen so that the displacement is continuous at the shell outer surface, i.e.,

$$
u_r^{(s)} = w - u_r^{(i)}, \quad u_x^{(s)} = u + \frac{h}{2} \Psi_x - u_x^{(i)}
$$
(10)

 $\tau_{rr}^{(s)*}$, $\tau_{rx}^{(s)*}$ arising from them at $r = R + h/2$. Assuming that Once $u_r^{(s)}$, $u_x^{(s)}$ are known, they can be used to calculate the stresses

$$
w = \overline{w} \cos \xi (x - ct) , \quad \Psi_x = \overline{\Psi}_x \sin \xi (x - ct) , \quad u = \overline{u} \sin \xi (x - ct)
$$
 (11)

it was shown in [llJ that

$$
p_1^* = \frac{\mu}{R} [\ T_{11} w_0 + \frac{h}{2} T_{12} \overline{Y}_x + T_{13} u_0] \cos \xi (x - ct) + (1 + \frac{m}{2}) \tau_{rr}^{(1)} *
$$

\n
$$
p_3^* = \frac{\mu}{R} [\ T_{13} w_0 + \frac{h}{2} T_{23} \overline{Y}_x + T_{33} u_0] \sin \xi (x - ct) + (1 + \frac{m}{2}) \tau_{rx}^{(1)} *
$$

\n
$$
p_2^* = \frac{h}{2} p_3^*
$$
 (12)

where the elements $T_{i,j}$ of the matrix T are given by

$$
T_{11} = -(1 + m/2) \left[\delta \frac{\tau^2 \epsilon^2 K_0 (\text{D K}_0 (\text{A}))}{\text{A}} + \frac{2}{1 + m/2} \right]
$$

\n
$$
T_{12} = T_{21} = T_{13} = T_{31} = -\ell (1 + m/2) \left[2 - \frac{\tau^2 \epsilon^2 K_1 (\text{D K}_0 (\text{A}))}{\text{A}} \right]
$$

\n
$$
T_{22} = T_{33} = T_{23} = T_{32} = -\frac{(1 + m/2) \tau^2 \epsilon^2 \gamma K_1 (\text{A}) K_1 (\text{B})}{\text{A}}
$$

\n
$$
\text{A} = \ell^2 K_0 (\text{A}) K_1 (\text{B}) - \gamma \delta K_1 (\text{A}) K_0 (\text{B})
$$

\nand

$$
m = h/R, \quad w_0 = \overline{w} + u_0 \gamma I_1(\sqrt{4}), \quad u_0 = \overline{u} - u_0 \ell I_0(\sqrt{4})
$$

$$
\sqrt{4} = (1 + m/2) \gamma, \quad \sqrt{6} = (1 + m/2) \delta
$$

Finally equations for the determination of \overline{w} , $\overline{\Psi}_{x}$ and \overline{u} are obtained by substituting (11) and (12) in (1) - (3). In matrix notation these can be written as

$$
[A - MT - \Omega^{2} B]U = -MTU^{(1)} + MT^{(1)}
$$
 (14)

Here A and B are symmetrix 3×3 matrices having the elements

$$
A_{11} = 2Nm (1 + m^{2}/12) + k_{x}^{2} mL^{2}, B_{11} = 1
$$
\n
$$
A_{12} = -2k_{x}^{2} \ell, B_{12} = 0
$$
\n
$$
A_{13} = 2N\nu \ell m, B_{13} = 0
$$
\n
$$
A_{22} = \frac{4}{m} (\frac{N}{6} m^{2} \ell^{2} + k_{x}^{2}), B_{22} = \frac{1}{3}
$$
\n
$$
A_{23} = \frac{1}{3} Nm^{2} \ell^{2}, B_{22} = \frac{m}{6}
$$
\n
$$
A_{33} = 2Nm\ell^{2}, B_{33} = 1
$$
\n(15)

Also,

$$
M = \mu/G, \qquad N = \frac{1}{1 - \nu}, \qquad U_1^{(1)} = -\gamma I_1(Q)
$$

$$
U_2^{(1)} = 0, \qquad U_3^{(1)} = \ell I_0(Q)
$$

$$
T_1^{(1)} = (T^2 \epsilon^2 - 2\ell^2) I_0(Q) + \frac{2\gamma}{1 + m/2} I_1(Q)
$$

$$
T_2^{(1)} = T_3^{(1)} = 2\gamma \ell I_1(Q)
$$

$$
U_1 = \overline{w}/u_0, \qquad U_2 = \frac{h}{2} \overline{Y}_x/u_0, \qquad U_3 = \overline{u}/u_0
$$

$$
\Omega^2 = \frac{\rho_s h R}{G} \omega^2 = T^2 \epsilon^2 M m (\frac{\rho_s}{\rho})
$$

It should be pointed out here that Eq. (14) represents the equation for the determination of the displacement of, and the rotation normal to, the middle surface when the pipe is excited by an axially propagating longitudinal wave. The excitation is given by the right hand side of Eq. (14). Clearly the matrix T is independent of the nature of the excitation and depends solely on the geometry of the pipe, the material properties of the surrounding medium and, of course, on the wavelength and wave-speed of the excitation. So for a different

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excitation only $U^{(i)}$ and $T^{(i)}$ will be changed. Further, the equation determining the frequency of free vibration of the pipe in an elastic medium is given by

$$
\det \left[A - MT - \Omega^2 \right] = 0 \tag{16}
$$

In general these frequencies are complex and depend on M as well as on other material and geometrical parameters of the shell and its surrounding medium. The real frequencies can exist only if

$$
\epsilon < \ell/\tau \tag{17}
$$

These are discussed in the following section. Note that (17) implies that

 $c < c_j$.

3. Numerical Results and Discussion

Eq. (16) was solved for real frequencies for different values of M and the Poisson's ratio, σ , of the outer medium. These are shown in Tables 1 and 2. The following parameters were chosen for the shell.

$$
h/R = .05
$$
, $v = 0.3$, $\rho_s/\rho = 2.9266$

pipe in vacuum. It is further noted that if $\ell \leq \frac{\pi}{3}$ then there are no real It was found that there were no real frequencies for small ^M and for ^M between 0.1 and ¹ there is only one real frequency, which lies between the frequencies of the first and second flexural modes of free vibration of the frequencies for any value of M. This is to be contrasted with the case of a shell vibrating in an acoustic medium in which case it was found [12] that there was always one real frequency for all ℓ . Also, the real frequency exists for smaller values of ℓ as M increases. For example, when $M = 0.1$ the real frequency occurs first for $\lambda = 5\pi/3$ whereas, when $M = 0.3$ it occurs first for $\ell = 5\pi/6$ and when $M = 1$ it is for $\ell = 2\pi/3$. It is observed that the frequency of free vibration decreases with the increase in the rigidity ratio of the soil and. the pipe. Increasing the Poisson's ratio also decreases the frequency.

As an example of forced vibration Eq. (14) was then solved for ^U for different ℓ , ε , M and σ with the same shell parameters indicated above. Knowing U the axial stress $N_{\rm xx}$ is then calculated from the equation

$$
N_{XX} = \frac{E_p u_o}{R} [\& U_3 + v U_1 + \frac{1}{6} mU_2]
$$
 (18)

In order to exemplify the dynamic effect \mid W/w $^{\texttt{static}}\mid$ (= W) and \mid N $_{\texttt{XX}}$ /N $_{\texttt{XX}}^{\texttt{static}}\mid$ (\equiv N) have been plotted in Figures 2 - 13 against ϵ for different ℓ and M. Here w^{static} and $N_{\text{xx}}^{\text{static}}$ are obtained by solving Eq. (14) when $\varepsilon = 0$.

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These static solutions are discussed in [11].

Figures 2 - 5 show the variations of W and N with ϵ and M for $\ell = \pi/6$ and $\sigma = 0.25$ and 0.45, respectively. It is seen from Fig. 2 that for small M, W increases slightly from 1.0 for small ϵ , then decreases to a minimum at about $\epsilon = 0.4$ and continually increases with ϵ thereafter. Both figures 2 and 3 show that for all M, W drops below 1.0 for small ϵ , goes through minima and then increases with ε . Comparing Figs. 2 and 3 it is observed that changing the Poisson's ratio has pronounced effect on the dynamic response of the shell. This is to be contrasted with the observation made in [11] that in the static limit the response of the shell is not very sensitive to changes in the Poisson's ratio. This is clearly not so if the inertia effects are taken into account. It is found then that increasing \circ (softer soil) results in very large displacements of the shell wall and, as Figs. 4 and 5 show, in large axial stresses in the shell. For small ^M the axial stress first decreases with increasing ϵ , but then increases very rapidly with ϵ , the increase being sharper for large σ . It may be noted that for $M \geq 0.1$, N attains asymptotically a constant value for large ε , this value being smaller the larger σ is. It is of particular interest to note that if the soil is soft and $M \geq 0.1$, N increases very rapidly with ϵ reaching a maximum value several times larger than 1 and then drops rapidly to an almost constant value that is not much larger than one. In harder soil, however, N does not reach a sharp maximum. Except for small ϵ it steadily increases to a constant value. These observations seem to be consistent with the evidence that pipes suffer greater damage in soft soils.

The variations of W and N with M and ε are shown in Figs. 6 and 7 for $\ell = \pi/3$ and $\sigma = 0.25$. It is seen that both W and N behave quite differently as the wavelength is increased. Generally, however, it may be observed that for

small M the behavior of W and N with changes in ϵ is the same as for $\ell = \pi/6$, the difference being that the dynamic axial stress is smaller than its statical value for a large range of ϵ . It may be seen that for small ϵ , W first increases sharply when M is small and then drops rapidly, goes through a minimum. The sharp increase for small M and ϵ becomes steeper as ℓ increases (see Figs. 8 and 10). N also behaves in a similar manner.

The variations of W and N with M and ϵ for large ℓ are shown in Figs. 8 - 11. The most important feature to be noticed in these figures is that the pipe begins resonating at a particular frequency that depends on M and ℓ . This has been discussed earlier. Figs. 12 and 13 show the dependence of the resonant behavior when $\sigma = 0.45$. It may be noted that increasing the Poisson's ratio for the same $\texttt{M} \Subset \texttt{0.1}$) generally resul[']ts in larger displacements and axial stresses.

^A general behavior that may be observed from these figures is that for small M and ε both W and N decrease with increasing σ . However, this behavior is reversed for large M. Also to be observed is the fact that for small M and large € both Wand ^N increase with increasing *a.*

In order to see whether the pipe generally follows the motion of the ground for long wavelengths, the values of the radial and axial displacements normalized with respect to the corresponding ground displacements are plotted against *e* for different M when $\ell = \pi/6$. It may be noted that for very small ϵ only U is close to unity if M is large, but \overline{W} is not. It is also interesting to note that for large M, W is close to unity when ϵ is large. For small M an increase in the Poisson's ratio is seen to cause smaller discrepancies between the radial and axial displacements of the pipe \\lall and the corresponding ambient ground displacements.

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Thus the following conclusions may be drawn from the above observations. (1) For long wavelengths when the rigidity ratio of the ground and the pipe is kept at the same small value, larger axial stresses in the pipe are caused in a softer ground. For large rigidity ratios, on the other hand, just the reverse occurs, except when frequency is low.

(2) Pipe resonance occurs only when the rigidity ratio is large and the wavelength is small. No resonance occurs in a soft soil.

(3) For short wavelengths, it appears that the axial stress in the pipe is usually larger in a softer ground except at some small ranges of frequencies. (4) It is important to note that except for very long wavelengths larger axial stresses are caused in the pipeline in a rocky environment than in a soil-like one.

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 $\sqrt{3}$

Table 1

---_ ..__ ._~ ...__.-.- -_.- --..----

 \mathfrak{p}

= wR/G) of the Shell in an Elastic Medium Real Vibrational Frequencies ($e = \omega R/Q$) of the Shell in an Elastic Medium n ies (ϵ Real Vibrational Fr

 $\overline{}$ 0.8699774 1.444238
1.865283 3.141592 None 0.9339887 0.8997447 0.8699774 6.283185 1.845795 1. 741516 1.646022 1. 44L:-238 -1.865283 None G.523599 None None None None $\overline{10}$ 0.10 0.30 1.0 10 0.8997447 1.646022 None 1.0 $(m = 0.05, v = 0.3, \sigma = 0.45, \rho^* = 2.9266)$ $(m = 0.05, v = 0.3, \sigma = 0.45, \rho^* = 2.9266)$ $M = \mu/G$ 0.9339887 1.741516 None 0.30 •, \pmb{i} 1.845795 0.10 None None 0.523599
1.047197 3.141592 6.283185 \circ t

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Table 2

Real Vibrational Frequencies ($\varepsilon = \omega R/G$) of the Shell in an Elastic Medium

Real Vibrational Frequencies ($\varepsilon = \omega R/G$) of the Shell in an Elastic Medium

List of Figures

 \mathcal{L}

Figure 6. Radial displacement of the shell for different M

Figure 7. Axial stress in the shell for different M

Figure 9. Axial stress in the shell at short wavelengths

 $a b$

Figure 12. Axial stress in the shell at short wavelengths ($\sigma = 0.25$)

 $\label{eq:2.1} \mathcal{L}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}))\leq \mathcal{L}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}))\leq \mathcal{L}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}))$ $\label{eq:2.1} \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2.$ \mathcal{L}_{max} $\epsilon_{\rm{eff}}$

 $\label{eq:2} \begin{split} \mathcal{L}_{\text{max}}(\mathbf{r}) = \mathcal{L}_{\text{max}}(\mathbf{r}) \mathcal{L}_{\text{max}}(\mathbf{r}) \end{split}$

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