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DISPERSION RELATIONS OF AN ELASTIC ROD EMBEDDED IN AN ELASTIC SOIL

by

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ABSTRACT

The dynamic interaction of pipes buried in soil depends strongly on resonant frequencies of the pipe-soil system. Such resonant frequencies are of interest since a possible cause of failure of buried pipes during earthquakes is believed to be due to resonant or near-resonant behavior under certain prescribed ground motion inputs. For any given wavelength, the resonant frequencies of the system are immediately known once the phase velocities of the system are established. These phase velocities are dependent on the wavelengths of waves propagating in the system and thus reveal the dispersive character of the system.

The model considered is represented by an elastic rod of radius a embedded in a linear elastic medium, and the interaction between the surrounding soil and pipe is assumed to occur through a shear mechanism acting at the interface. The problem then is to consider at what velocities a wave of wavelength λ can propagate under steady state conditions. The dispersive behavior is found to be dependent on several non-dimensional parameters defined by the geometric ratio a/λ, as well as on non-dimensional ratios of the pipe and medium properties. It is shown that the resulting waves which can propagate under steady-state conditions are surface waves which decay with the radial distance and which permit no radiation damping of energy. In addtion, it is shown that such waves may propagate only for systems with $\bar{c} < c_s$, where $\bar{c}$ is the longitudinal velocity of waves in a free pipe and c_s is the shear wave velocity in the medium. If $\bar{c} \geq c_s$, such waves are precluded from propagating in the system.

Results are presented by means of dispersion curves and surfaces. From a study of the analytical results obtained, lower and upper bounds on the phase velocities are established.

LIST OF SYMBOLS

A = constant

a = radius of rod

B = constant; real number, see Eq. (A.3)

b_p, b_s = real positive numbers, see Eq. (A.1)

c = wave propagation velocity (phase velocity)

c_p, c_s = velocity of P- and S-waves in medium

$\bar{c}$ = propagation velocity of longitudinal waves in rod

D = real number, see Eq. (A.19)

d = real number, see Eq. (A.4)

d_p, d_s = real positive numbers, see Eq. (A.15)

$\bar{E}$ = Young's modulus of rod

f, f_s = frequency, frequency of S-waves in medium

G = real positive number, see Eq. (A.21)

$H_o^{(1)}$, $H_1^{(1)}$ = Hankel functions of the first kind of order 0, 1.

h^* = coefficient, see Eq. (5a)

J_o, J_1 = Bessel function (of first kind) of order 0, 1

K_o, K_1 = modified Bessel function of order 0, 1

k^* = coefficient, see Eq. (5b)

$R_c(\nu) = c_p/c_s$

$R_D = \rho/\bar{\rho}$, non-dimensional density parameter

$R_v = c_p/\bar{c}$

r = radial coordinate

t = time

$U_p(z, t)$ = axial displacement of rod

U_r, U_z = radial, axial displacement in medium

$v = 2\pi\eta$

W = Wronskian

Y_o, Y_1 = Bessel function of second kind of order 0, 1

z = axial coordinate

α = variable, see Eq. (A.3)

β_p, β_s = non-dimensional coefficients, see Eqs. (26)

$\Gamma(\eta, R_v, R_D, R_c) = c/\bar{c}$, non-dimensional wave velocity parameter

γ_p, γ_s = non-dimensional arguments, see Eqs. (19)

$\eta = a/\lambda$, non-dimensional geometric parameter

η_{cr} = critical cut-off value of η

λ = wavelength

μ = shear modulus in medium

ν = Poisson's ratio in medium

ρ, $\bar{\rho}$ = density of medium, rod

τ_{rz} = shear stress

$\Phi(r)$ = radial dependence of U_r

$\Psi(r)$ = radial dependence of U_z

1. INTRODUCTION

The effect of earthquakes on pipe systems buried in soil has been the subject of a number of investigations, e.g. Refs. [1]-[2]. Since the dynamic response of such systems is a question of concern, it is of particular importance to determine the parameters of the system under which resonant response can occur under forced ground motions. Such responses are believed to be possibly one of the causes of failure of buried pipes due to earthquakes.

The resonant response depends naturally on the frequency f of the earthquake component as well as on the wavelength λ. Hence the propagation velocity $c = f\lambda$ is a critical factor. If this propagation velocity corresponds to the phase velocity of a given pipe-soil system, resonance can occur under steady-state conditions in the same sense as it occurs when the frequency of a forcing function corresponds to the natural frequency of a vibrating system.

In this report, the phase velocities of a pipe embedded in a soil are determined, i.e. we investigate at what velocity c, a wave of wavelength λ can propagate freely in the system. The system is represented by a model consisting of an elastic rod of radius a embedded in an elastic medium where P- and S-waves propagate with velocites c_p and c_s respectively. Since the radial displacements of the rod are known to be small, for mathematical simplicity, the pipe is assumed to be radially rigid. Such an assumption has been used previously, Ref. [3].

The phase velocity c of the system, which is defined as the apparent propagation velocity in the longitudinal direction, is found to be dependent on the wavelength ratio a/λ, thus revealing the dispersive character of the system. The phase velocities are observed to depend also on several non-dimensional parameters of the system; viz. the ratio $c_p/\bar{c}$ (where $\bar{c}$ is the propagation velocity of longitudinal waves in a free rod), the ratio of mass densities of medium to rod, and finally on the Poisson ratio of the medium (which defines the ratio c_p/c_s).

The resulting dispersion relations, obtained from the roots of a frequency equation, are presented by means of several figures. In addition, upper and lower bounds on the phase velocities are determined.

From the study of the analytic solutions and observations of the dispersion curves and surfaces presented, general conclusions are established which govern possible resonance of the pipe-soil system.

2. GENERAL FORMULATION

The model considered consists of an infinite cylindrical rod of radius a with modulus of elasticity $\bar{E}$ and density $\bar{\rho}$ embedded in a linear isotropic elastic medium [Fig. (1)]. The motion of the rod in the longitudinal z-direction is denoted by $U_p(z, t)$. Following the assumption of radial rigidity, the radial displacements are taken as zero throughout the rod.

The surrounding medium is assumed to behave as a linear elastic material having density ρ and defined by a shear modulus μ and Poisson ratio ν. For the axisymmetric case considered here, the soil medium can undergo time-dependent radial and axial displacements denoted by $U_r(r,z,t)$ and $U_z(r,z,t)$ respectively.

The interaction between the rod and surrounding medium is due to an interactive shear mechanism, acting along the cylindrical interface, which prevents slip between the rod and the medium.

The governing equation of the rod is then written as:

$$\bar{E}\,\frac{\partial^2 U_p(z,t)}{\partial z^2} + \frac{2\tau_{rz}(a,z,t)}{a} - \bar{\rho}\,\frac{\partial^2 U_p(z,t)}{\partial t^2} = 0 \qquad (1)$$

where $\tau_{rz}(a,z,t)$ represents the interactive shear stress at the interface.

With the assumptions stated above, together with the requirements on continuity of displacements at the rod-medium interface, the boundary conditions on the medium displacements become

$$U_r(a,z,t) = 0, \; U_z(a,z,t) = U_p(z,t) \qquad (2a,b)$$

We now consider the propagation of waves having wavelength λ and which propagate in the system with an apparent propagation velocity c in the z-direction.

The subsequent dynamic displacements of the surrounding medium, $r > a$ may be expressed in terms of outgoing wave expressions which decay as $r \to \infty$ (in a revised form from that given in Ref. [3]) as follows:

$$U_r(r,z,t) = \Phi(r)\ e^{[\frac{i2\pi}{\lambda}(z-ct)]} \tag{3a}$$

$$U_z(r,z,t) = \Psi(r) i\ e^{[\frac{i2\pi}{\lambda}(z-ct)]} \tag{3b}$$

where

$$\Phi(r) = A\,\frac{\lambda h^*}{2\pi}\,H_1^{(1)}(h^*r) + BH_1^{(1)}(k^*r) \tag{4a}$$

$$\Psi(r) = AH_o^{(1)}(h^*r) - B\,\frac{\lambda k^*}{2\pi}\,H_o^{(1)}(k^*r) \tag{4b}$$

In the above, A and B are undetermined constants, $H_n^{(1)}(x)$ are Hankel functions of the first kind of order n,

$$h^{*2} = (\frac{2\pi}{\lambda})^2\ [c^2/c_p^2 - 1] \tag{5a}$$

and

$$k^{*2} = (\frac{2\pi}{\lambda})^2\ [c^2/c_s^2 - 1] \tag{5b}$$

where

$$c_s = [\frac{\mu}{\rho}]^{1/2} \quad \text{and} \quad c_p = [\frac{2(1-\nu)}{1-2\nu}\cdot\frac{\mu}{\rho}]^{1/2} \tag{6a,b}$$

are the propagation speeds in the elastic medium of outgoing S- and P-waves respectively. Thus, the terms associated with A represent the P-waves, while the B terms represent the propagation of S-waves. The constants A and B must then satisfy the boundary conditions of Eq. (2). From Eqs. (2a) and (4a)

$$[\frac{\lambda h^*}{2\pi}\,H_1^{(1)}(h^*a)]A + H_1^{(1)}(k^*a)\ B = 0 \tag{7}$$

Furthermore, since $U_r(a,z) = 0$, the shear stress at the interface is given by

$$\tau_{rz}(a,z,t) = \mu \frac{\partial U_z(a,z,t)}{\partial r} \tag{8}$$

Substituting the above and the continuity condition, Eq. (2b), in Eq. (1), we obtain

$$\frac{\partial^2 U_z(a,z,t)}{\partial t^2} - \bar{c}^2 \frac{\partial^2 U_z(a,z,t)}{\partial z^2} - \frac{2\mu}{a\bar{\rho}} \frac{\partial U_z(a,z,t)}{\partial r} = 0 \tag{9}$$

where

$$\bar{c} = \sqrt{\bar{E}/\bar{\rho}} \tag{10}$$

is the propagation velocity of longitudinal waves in a free cylindrical rod.

Using Eq. (3b), Eq. (9) is satisfied under steady-state conditions, if

$$\Psi(a) = \frac{2\mu}{a\bar{\rho}\bar{c}^2(\frac{2\pi}{\lambda})^2[1-(\frac{c}{\bar{c}})^2]} \cdot \frac{\partial \Psi(r)}{\partial r}\Bigg|_{r=a} \tag{11}$$

Substituting Eq. (4b) and noting that (Ref. [4])

$$\frac{dH_o^{(1)}(x)}{dx} = - H_1^{(1)}(x) \tag{12}$$

we obtain the second required condition on the constants A and B; viz.

$$a(\frac{2\pi}{\lambda})^2 [1 - (\frac{c}{\bar{c}})^2][AH_o^{(1)}(h^*a) - B\frac{\lambda k^*}{2\pi} H_o^{(1)}(k^*a)] = -\frac{2\rho c_s^2}{\bar{\rho}\bar{c}^2}[Ah^* H_1^{(1)}(h^*a) - B\frac{\lambda k^{*2}}{2\pi} H_1^{(1)}(k^*a)] \tag{13}$$

where use has been made of Eq. (6).

At this point, it is advantageous to express the solution in terms of non-dimensional quantities, and more specifically in terms of non-dimensional ratios relating the propagation velocities of the P- and S-waves in the medium and the propagation velocity $\bar{c}$ of waves in a free rod. To this end, the following non-dimensional variables are defined:

$$\eta = a/\lambda \ , \tag{14}$$

$$\Gamma = c/\bar{c} \ , \tag{15}$$

and

$$R_c = c_p/c_s = \left[\frac{2(1-\nu)}{1-2\nu}\right]^{1/2} \ , \quad R_v = c_p/\bar{c} \tag{16a,b}$$

Also, let the ratio of the densities of medium to rod be

$$R_D = \rho/\bar{\rho} \tag{17}$$

Using the above definitions, Eqs. (7) and (13) lead, after some manipulation, to the following system of equations on A and B:

$$\begin{bmatrix} \gamma_p H_1^{(1)}(\gamma_p) & 2\pi\eta H_1^{(1)}(\gamma_s) \\ (2\pi\eta)\left[(2\pi\eta)^2(1-\Gamma^2)H_o^{(1)}(\gamma_p) + \dfrac{2R_v^2R_D}{R_c^2}\gamma_p H_1^{(1)}(\gamma_p)\right] & -\gamma_s\left[\dfrac{2R_DR_v^2}{R_c^2}\gamma_s H_1^{(1)}(\gamma_s) + (2\pi\eta)^2(1-\Gamma^2)H_o^{(1)}(\gamma_s)\right] \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18}$$

where

$$\gamma_p = 2\pi\eta \left[\frac{\Gamma^2}{R_v^2} - 1\right]^{1/2}$$

$$\gamma_s = 2\pi\eta \left[\frac{R_c^2\Gamma^2}{R_v^2} - 1\right]^{1/2} \tag{19a,b}$$

For non-trivial solutions, the determinant of Eq. (18) must vanish. Expansion of the determinant then leads to the frequency equation:

$$(1-\Gamma^2)[H_o^{(1)}(\gamma_p)H_1^{(1)}(\gamma_s) + \frac{\gamma_p\gamma_s}{(2\pi\eta)^2} H_o^{(1)}(\gamma_s)H_1^{(1)}(\gamma_p)] + \frac{R_D\Gamma^2}{\pi\eta}[(\Gamma/R_v)^2 - 1]^{1/2}H_1^{(1)}(\gamma_s)H_1^{(1)}(\gamma_p) = 0 \tag{20}$$

whose roots $\Gamma = c/\bar{c}$ yield the appropriate phase velocities.

3. EVALUATION OF PHASE VELOCITIES: DISPERSION RELATIONS, PHYSICAL INTERPRETATION

From the derived frequency equation, we note that the roots $\Gamma = c/\bar{c}$ which define the phase velocities c, are dependent on the non-dimensional parameters previously defined; i.e.

$$\Gamma = \Gamma(\eta, R_v, R_D, R_c) \tag{21}$$

where $R_c \geq 2$ is a parameter dependent explicitly on Poisson's ratio ν.

For real values of η, it is proven (in the Appendix) that no real roots Γ can occur for values

$$\Gamma \geq R_v/R_c \qquad (22)^{*)}$$

Thus the phase velocities $c < c_s$, i.e. the phase velocities are always less than that of S-waves which can propagate in the medium.

It follows that γ_p and γ_s as defined by Eqs. (19) are always complex. Noting that, Ref. [4]

$$H_o^{(1)}(ix) = \frac{2}{\pi i} K_o(x) \quad , \quad H_1^{(1)}(ix) = -\frac{2}{\pi} K_1(x) \qquad (23a,b)$$

where $K_n(x)$ are the modified Bessel functions, the frequency equation is written in a more convenient form, involving only real quantities, as

$$(1-\Gamma^2)[K_o(v\beta_p)K_1(v\beta_s) - \beta_p\beta_s K_o(v\beta_s)K_1(v\beta_p)] + \frac{R_D\beta_p\Gamma^2}{\pi\eta} K_1(v\beta_p)K_1(v\beta_s) = 0 \tag{24}$$

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*) In all expressions and equations, Γ always appears as Γ^2. Results presented here and subsequently, taken from the positive branch of the square root, are equally valid for $-\Gamma$. Thus c may be replaced everywhere by $|c|$ indicating waves travelling in either the positive or negative z-directions with speed $|c|$.

where $$v = 2\pi\eta \tag{25}$$

$$\beta_p = [1 - (\Gamma/R_v)^2]^{1/2} > 0$$

$$\beta_s = [1 - (R_c\Gamma/R_v)^2]^{1/2} > 0 \tag{26a,b}$$

are real.

From an examination of Eq. (24) it is proven (in the Appendix) that for all finite η (with $R_D > 0$), all roots must lie in the region

$$1 < \Gamma < R_v/R_c \tag{27a}$$

i.e. the phase velocity lies in the range

$$\bar{c} < c < c_s \tag{27b}$$

For $\eta \to \infty$ or $R_D = 0$ the only roots of the frequency equation are $\Gamma^2 = 1$, i.e. $c = \bar{c}$.

The range of possible values of phase velocity given by Eq. (27b) may now be interpreted readily.

It is obvious that the condition $c > \bar{c}$ must be satisfied if we recognize that due to increased restraint the effective stiffness of the embedded rod is always stiffer than the free rod, and consequently the propagation velocity must be greater than $\bar{c}$.

To interpret the upper bound on c, let us consider the contrary case if c were greater than c_s. If this were indeed the case, the propagation of wave fronts would be as shown in Fig. (2) since the phase velocity is the apparent velocity in the z-direction. Such a situation would imply a continuous radiation of energy due to outward propagating waves, thus contradicting steady-state conditions. In effect then, the condition $c < c_s$ is required in order to maintain steady-state conditions. Similar conclusions were found by Biot in studying the propagation of waves along an empty cylindrical bore, Ref. [5].

It is worthwhile to consider the form of the displacements for the permissible range $c < c_s$. In this range h^* and k^* appearing in Eqs. (5) are both imaginary and hence, using the relations of Eq. (23), the displacements U_r and U_z appearing in Eqs. (3) become:

$$U_r = -\frac{2}{\pi}[iA\beta_p K_1(v\beta_p \frac{r}{a}) + BK_1(v\beta_s \frac{r}{a})]e^{[\frac{i2\pi}{\lambda}(z-ct)]} \tag{28a}$$

$$U_z = -\frac{2}{\pi}[iAK_o(v\beta_p \frac{r}{a}) + B\beta_s K_o(v\beta_s \frac{r}{a})]e^{[\frac{i2\pi}{\lambda}(z-ct)]} \tag{28b}$$

where v, β_p and β_s are defined in Eqs. (25)-(26).

The displacements, which are observed to decay monotonically with $r(r > a)$, represent essentially surface waves similar to Rayleigh waves [See Fig. (3)]. No energy is radiated out and, in effect, all energy is confined to a region adjacent to the rod, thus permitting steady-state conditions.

Numerical results showing the dispersion curves, $c/\bar{c}$ versus $\eta = a/\lambda$, are presented in Figs. (4a) - (4d) for a medium with $\nu = 0.25$ for several parameter values $R_v = c_p/\bar{c}$ and $R_D = \rho/\bar{\rho}$. The values $R_D = 0.05$ and $R_D = 5.0$ represent lower and upper bounds for all reasonable combinations of soil and rod densities. All curves are seen to tend to unity as η becomes large; the asymptotic behavior approaching unity occurs, in particular, much more rapidly for relatively small values of R_D. Thus for either small ratios $\rho/\bar{\rho}$ or $\eta >> 1$, the phase velocity approaches the velocity of longitudinal waves in a free bar, and hence the interaction between the rod and surrounding medium is seen to be small

It is also observed in Fig. (4) that for sufficiently small η, i.e. for relatively large wavelengths λ, no waves can propagate in the system under steady-state conditions. That is, a critical cut-off value of η, η_{cr}, exists below which no real solutions are possible. (These critical values occur in Fig. (4) at values of η indicated by vertical lines). At these critical values of η, the phase velocity is such that

$$\Gamma \equiv c/\bar{c} = R_V/R_c \tag{29}$$

i.e. $c = c_s$, which was shown to be the upper bound for the roots of the frequency equation. Having observed that values $c \geq c_s$ are physically incompatible with steady-state conditions, values of η less than or equal to η_{cr} (which would perforce propagate with $c \geq c_s$) are thus precluded from propagating in the system.

Upon noting that $c = c_s$ is, in all cases, the upper bound of the solution it is now advantageous to represent the dispersion relation as c/c_s versus $\eta = a/\lambda$. Recognizing also from Eq. (27b), that $R_V/R_c = c_s/\bar{c} \geq 1$ is the permissible range of this second governing parameter, the dispersion relations for c/c_s are represented as functions of η and $c_s/\bar{c}$ for several values of the parameter R_D. These relations are shown in Figs. (5a-d) and appear as dispersion surfaces for a medium with $\nu = 0.25$.

From these figures, the strong variation with R_D is immediately observed. Furthermore, it is noted that for large R_D, c approaches c_s, i.e. the behavior of the system is dominated by the surrounding medium for moderate wavelengths. The values of η_{cr}, recognized to be the curve defined by the intersection of the dispersion surfaces and the plane $c/c_s = 1$ appearing in Figs. (5), are shown in Fig. (6) as a function of $c_s/\bar{c}$ for various values of R_D . From this figure, it is noted, e.g. that for small $R_D = .05$, waves having relatively large wavelengths, $\lambda > 10a$ may propagate in the system, while for a relatively dense material, ($R_D = 2$), waves with length $\lambda > 1.5a$ are precluded from propagating, irrespective of the ratio $c_s/\bar{c}$.

One may also conclude from Figs. (5) and (6) that changes in phase velocity are most sensitive to values of $c_s/\bar{c}$ in the region approaching unity, and are most sensitive to η for relatively small values of η .

4. SUMMARY

It has been shown that for a given pipe-soil system, waves can propagate freely under steady-state conditions only if $\bar{c} < c_s$. The resulting waves are seen to be surface waves of wavelength λ (causing disturbances confined to the region of the medium near the surface of the pipe) which propagate with a phase velocity c, $\bar{c} < c < c_s$. Thus, if such a system is subjected to time-harmonic periodically spaced axial forces with frequency f, resonant behavior can be expected for certain corresponding input frequencies f/f_s (where $f = c/\lambda$, $f_s = c_s/\lambda$). Such ratios are found from the dispersion surfaces presented, which are functions of the density ratio, medium Poisson ratio and aspect ratio a/λ. For systems with $c_s < \bar{c}$, waves cannot propagate freely and hence no resonant behavior can occur.

REFERENCES

[1] "Proceedings of U.S.-Japan Seminar on Earthquake Engineering Research With Emphasis on Lifeline Systems," Tokyo, 1976.

[2] "The Current State of Knowledge of Lifeline Earthquake Engineering," Proceedings, Technical Council on Lifeline Earthquake Engineering Speciality Conference, University of California at Los Angeles, ASCE, 1977.

[3] Toki, K. and Takada, S., "Earthquake Response Analysis of Underground Tubular Structures," Bull. Disas. Prev. Res. Inst., Kyoto University, Vol. 24, No. 221, June 1964.

[4] McLachlan, N.W., "Bessel Functions for Engineers," Oxford, 1955.

[5] Biot, M.A., "Propagation of Elastic Waves in a Cylindrical Bore Containing a Fluid", J. Appl. Phys., Vol. 23, No. 9, Sept. 1952, pp. 997-1005.

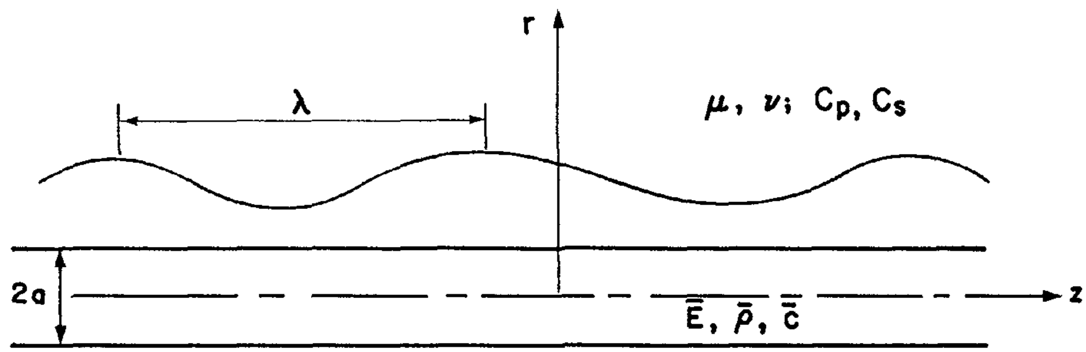

FIG. I GEOMETRY OF PROBLEM

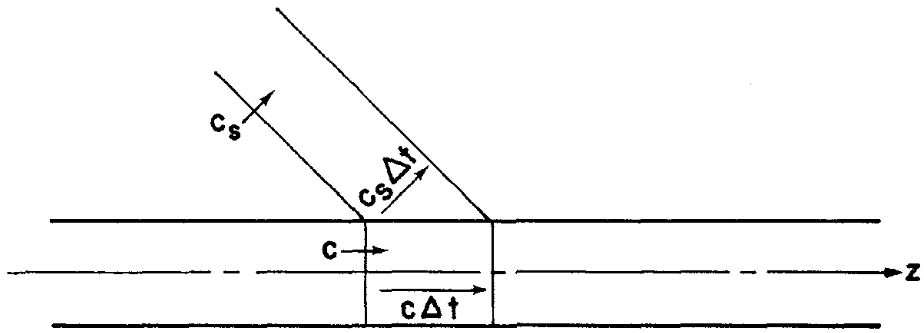

FIG. 2 CONFIGURATION FOR $c > c_s$

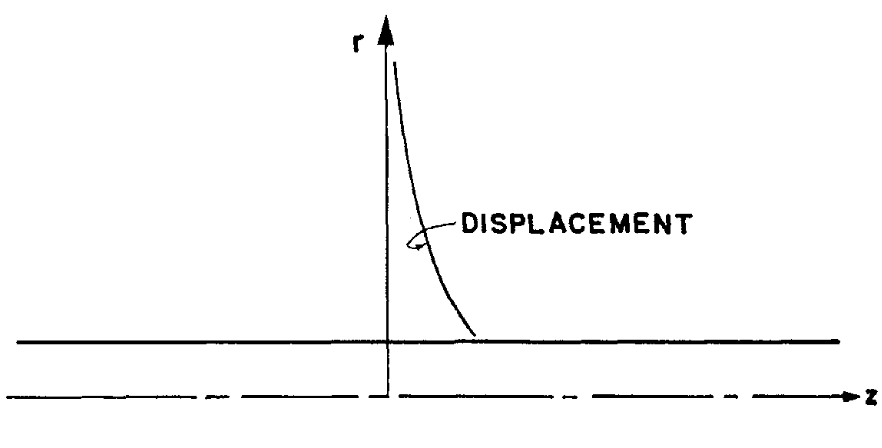

FIG. 3 SURFACE WAVE RESPONSE, $c < c_s$

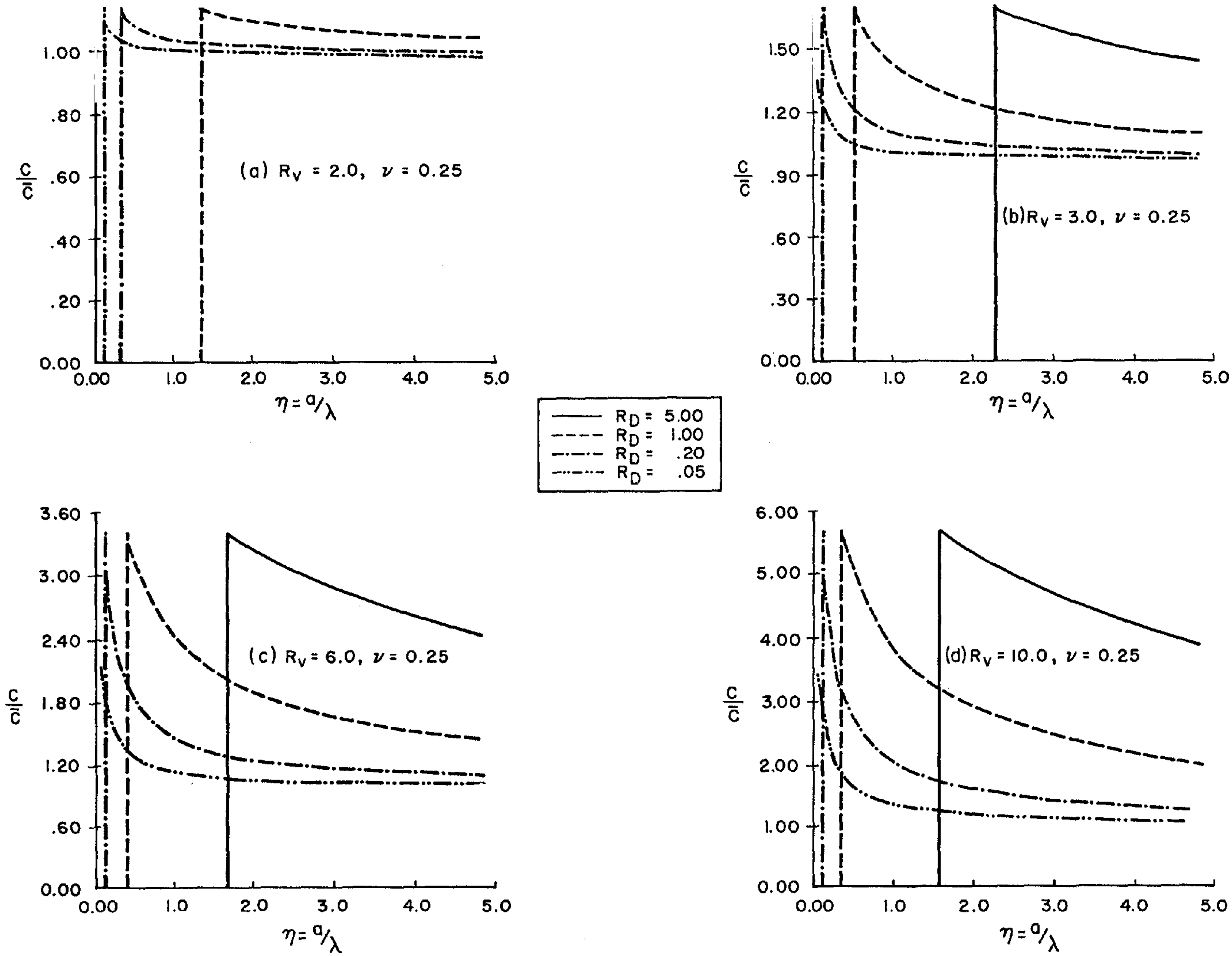

FIG. 4 DISPERSION CURVES ν = 0.25

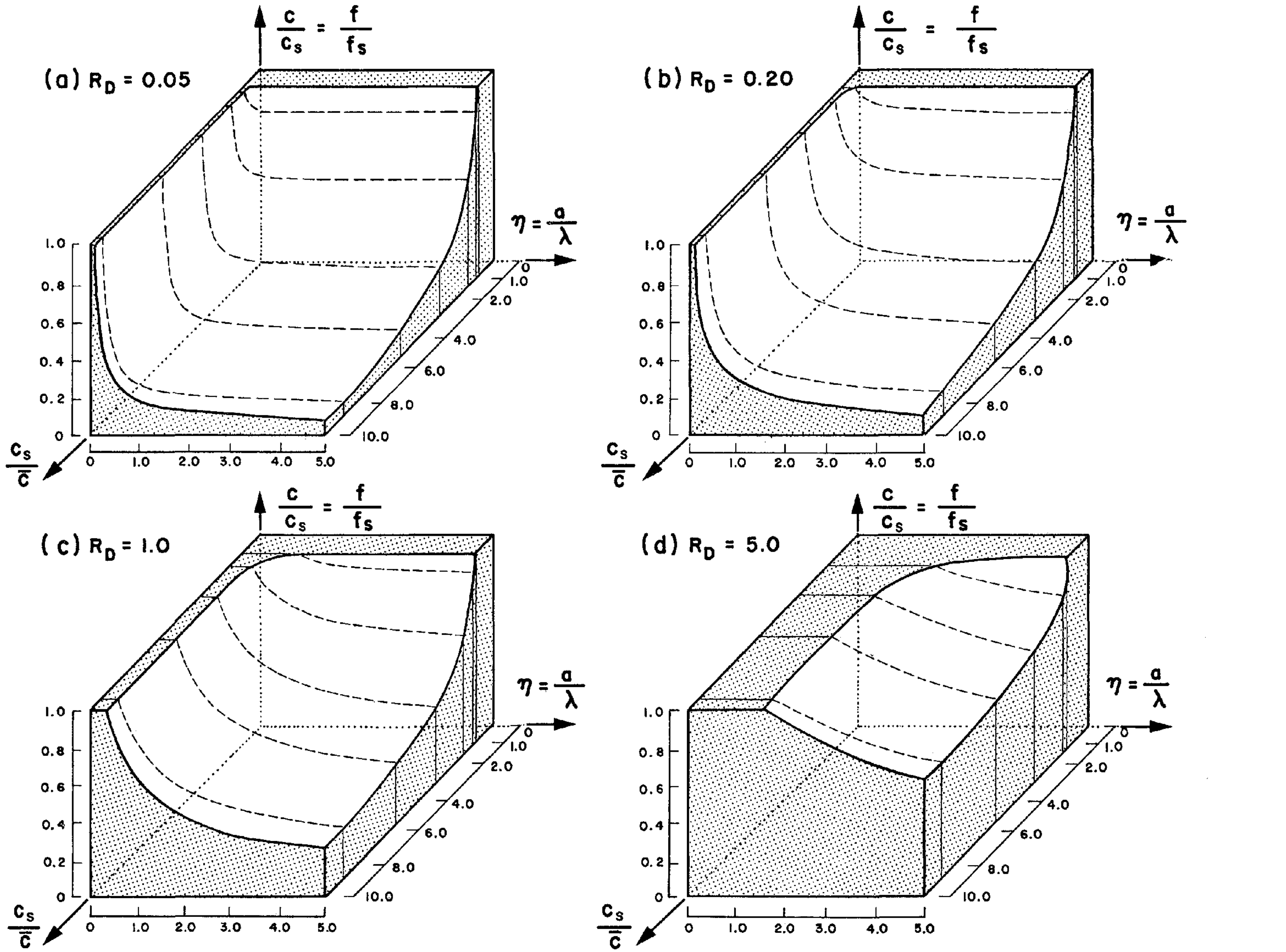

FIG. 5 DISPERSION SURFACES, $\nu = 0.25$

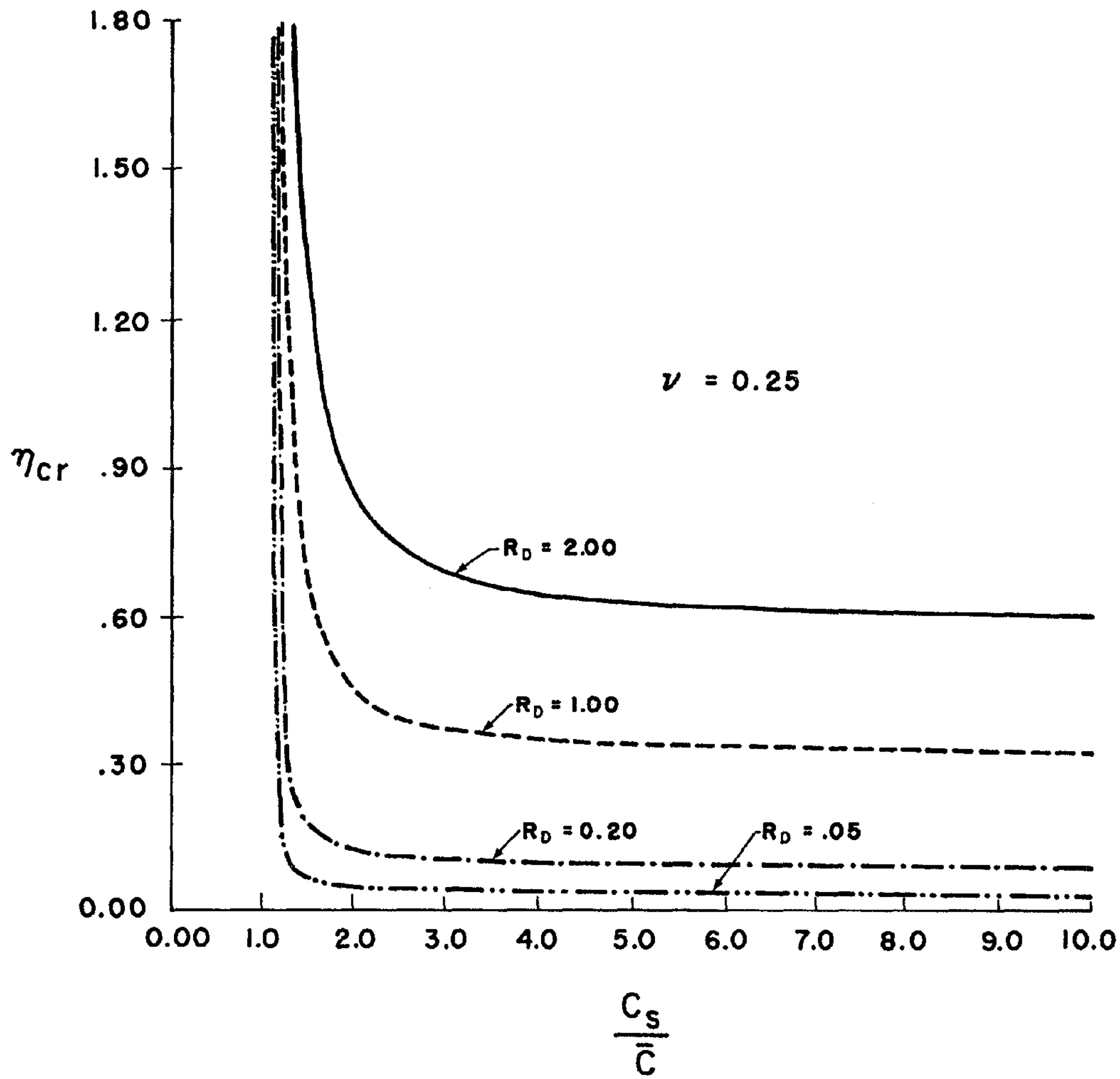

FIG. 6 CUT OFF WAVE LENGTH RATIOS, $\nu = 0.25$

APPENDIX: ESTABLISHMENT OF RANGE OF ROOTS Γ OF FREQUENCY EQUATION

1. Proof that no real roots Γ can exist for $\Gamma > R_v/R_c$

We first prove that no roots Γ exist in the range $R_v^2/R_c^2 < \Gamma^2 < R_v^2$. To prove this, we postulate the existence of roots in this range and show that this assumption leads to a contradiction.

If Γ does exist in this range, then from their definition, Eqs. (19)

$$\gamma_p = ib_p \ , \quad \gamma_s = b_s \tag{A.1}$$

where $b_p > 0$, $b_s > 0$ are real.

Dividing Eq. (20) through by $(1-\Gamma^2)H_1^{(1)}(\gamma_p)H_1^{(1)}(\gamma_s)$

$$\frac{H_o^{(1)}(ib_p)}{H_1^{(1)}(ib_p)} + i\alpha\,\frac{H_o^{(1)}(b_s)}{H_1^{(1)}(b_s)} + Bi = 0 \tag{A.2}$$

where

$$\alpha = \frac{b_p b_s}{(1-\Gamma^2)(2\pi\eta)^2}$$

$$B = \frac{R_D\Gamma^2}{(1-\Gamma^2)\pi\eta}\,[1 - \Gamma^2/R_v^2]^{1/2} \tag{A.3}$$

are real.

Noting that

$$\frac{H_o^{(1)}(ib_p)}{H_1^{(1)}(ib_p)} = i\,\frac{K_o(b_p)}{K_1(b_p)} = id \tag{A.4}$$

where d is real, Eq. (A.2) becomes

$$\frac{H_o^{(1)}(b_s)}{H_1^{(1)}(b_p)} = -\frac{1}{\alpha}\,[B + d] \quad , \quad \text{real.} \tag{A.5}$$

To satisfy this condition, we require, using the basic definition of the Hankel functions

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) \tag{A.6}$$

that

$$\text{Im}\ \{\frac{J_o(b_s) + i\ Y_o(b_s)}{J_1(b_s) + i\ Y_1(b_s)}\} = 0 \tag{A.7}$$

and hence the required condition becomes

$$Y_o(b_s)J_1(b_s) - J_o(b_s)Y_1(b_s) = 0 \tag{A.8}$$

which is recognized as the vanishing of the Wronskian

$$W(b_s) = \begin{vmatrix} J_o(b_s) & Y_o(b_s) \\ J_1(b_s) & Y_1(b_s) \end{vmatrix} = 0 \tag{A.9}$$

Since $W(b_s)$ can never vanish, it follows that no roots $R_v^2/R_c^2 < \Gamma^2 < R_v^2$ can exist.

To prove that no real roots can exist in the range $\Gamma^2/R_v^2 > 1$, we similarly postulate the existence of roots in this range and show that it leads to a contradiction.

If $\Gamma^2/R_v^2 > 1$, then

$$\gamma_p = b_p \quad , \quad \gamma_s = b_s \tag{A.10}$$

are real, with $\qquad 0 < b_p < b_s \qquad$ (A.11)

Dividing the frequency equation, Eq. (20), by $(1-\Gamma^2)\ H_o^{(1)}(\gamma_p)H_1^{(1)}(\gamma_s)$ as before, we obtain

$$\frac{H_o^{(1)}(b_p)}{H_1^{(1)}(b_p)} + \alpha\ \frac{H_o^{(1)}(b_s)}{H_1^{(1)}(b_s)} = -B \quad , \quad \text{real} \tag{A.12}$$

where $\alpha > 0$. Hence

$$\text{Im}\left\{ \frac{H_o^{(1)}(b_p)}{H_1^{(1)}(b_p)} + \alpha \frac{H_o^{(1)}(b_s)}{H_1^{(1)}(b_s)} \right\} = 0 \qquad \text{(A.13a)}$$

or, from Eq. (A.6)

$$\text{Im}\left\{ \frac{J_o(b_p) + i\, Y_o(b_p)}{J_1(b_p) + i\, Y_1(b_p)} + \alpha \left[\frac{J_o(b_s) + i\, Y_o(b_s)}{J_1(b_s) + i\, Y_1(b_s)}\right] \right\} = 0 \qquad \text{(A.13b)}$$

Therefore

$$d_p[Y_o(b_p)J_1(b_p) - J_o(b_p)Y_1(b_p)] + \alpha d_s[Y_o(d_s)J_1(d_s) - J_o(b_s)Y_1(b_s)] = 0 \qquad \text{(A.14)}$$

where

$$d_{p,s} = [J_1^2\,(b_{p,s}) + Y_1^2\,(b_{p,s})]^{-1/2} > 0 \qquad \text{(A.15)}$$

Using the Wronskian relation, Ref. [4],

$$W(x) \equiv Y_o(x)J_1(x) - J_o(x)Y_1(x) = 2/\pi x \qquad \text{(A.16)}$$

the required condition, Eq. (A.14), for the existence of the roots, becomes

$$d_p/b_p + \alpha d_s/b_s = 0 \qquad \text{(A.17)}$$

Since b_p, b_s, α, d_p, d_s are all positive real quantities, we conclude that this condition is violated and therefore no real roots Γ can exist for $\Gamma^2/R_v^2 > 1$.

It remains to be shown that roots $\Gamma = R_v/R_c$ ($c = c_s$) are also excluded for finite η . Upon examining the original equations of the system, Eqs. (3) - (5), it is noted that for this value of Γ, $k^* = 0$. Noting that $\lim_{x\to 0}|H_1^{(1)}(x)| = \infty$, it is readily seen that the explicit boundary conditions, Eqs. (7) and (13) can only be satisfied if A = B = 0 identically.

It follows that possible roots can only exist in the range $\Gamma^2 < R_v^2/R_c^2$.

2. Proof that real roots Γ can exist only in the range $\Gamma^2 \geq 1$.

The frequency equation, given in the form of Eq. (24), is written as

$$(1-\Gamma^2)[K_o(v\beta_p)K_1(v\beta_s) - \beta_p\beta_s K_o(v\beta_p)K_1(v\beta_p)] + D = 0 \tag{A.18}$$

where β_p and β_s have been defined by Eqs. (26) and where

$$D = \frac{R_D\Gamma^2\beta_p}{\pi\eta} K_1(\beta_p) K_1(\beta_s) \tag{A.19}$$

is real.

We note that for $\Gamma^2 < (R_v/R_c)^2$,

$$0 < \beta_s < \beta_p < 1 , \tag{A.20}$$

since $R_c \geq 2$. Hence $D > 0$ for all finite η, provided $R_D > 0$, while $D = 0$ only if $\eta \rightarrow \infty$ or $R_D = 0$.

Now, letting

$$G = K_o(v\beta_p)K_1(v\beta_s) - \beta_p\beta_s K_o(v\beta_s)K_1(v\beta_p) \tag{A.21}$$

we note that

$$G > K_o(v\beta_p)K_1(v\beta_s) - K_o(v\beta_s)K_1(v\beta_p)$$

$$= K_o(v\beta_s)K_1(v\beta_p)\left[\frac{K_o(v\beta_p)}{K_1(v\beta_p)} - \frac{K_o(v\beta_s)}{K_1(v\beta_s)}\right] = A > 0 \tag{A.22}$$

since $\beta_p > \beta_s$. Therefore the frequency equation, Eq. (24), is of the form

$$(1 - \Gamma^2)G + D = 0 , \quad \text{where} \tag{A.23}$$

$G > 0$, $D > 0$. Hence,

$$\Gamma^2 \geq 1$$

are the only possible solutions. Furthermore $\Gamma^2 = 1$ only if $\eta \rightarrow \infty$ or $R_D = 0$.