# SEISMIC STABILITY EVALUATION OF EARTH STRUCTURES

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Final Report of the Research Supported by the
National Science Foundation under Grant
No. PFR-7823095

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November 1980

| BIBLIOGRAPHIC DATA SHEET | 1. Report No. VPI-E-80.30 | 2. | 3. Recipient's Accession No. PB81 123068 |
|---|---|---|---|

**4. Title and Subtitle**
SEISMIC STABILITY EVALUATION OF EARTH STRUCTURES

**5. Report Date**
November 1980

**6.**

**7. Author(s)**
Mahendra P. Singh and M. Ghafory-Ashtiany

**8. Performing Organization Rept. No.**

**9. Performing Organization Name and Address**
Virginia Polytechnic Institute and
State University

**10. Project/Task/Work Unit No.**

**11. Contract/Grant No.**
PFR-7823095

**12. Sponsoring Organization Name and Address**
National Science Foundation
Washington, D.C. 20550

**13. Type of Report & Period Covered**
Final

**14.**

**15. Supplementary Notes**

**16. Abstracts**

An analytical approach is proposed for seismic design evaluation of earthstructures like foundations and slopes. The approach is based on random vibration principles. Seismic Design inputs defined in terms of response spectra curves can be directly used. For verification of the proposed approach a comparative study of the results obtained by a rigorous nonlinear analysis approach and the proposed approach is conducted. It is observed that the proposed approach will provide a safe and conservative evaluation of an earth structure design for earthquake induced ground motions.

**17. Key Words and Document Analysis. 17a. Descriptors**

seismic design
earth systems
foundations
dynamic response
liquefaction
earthquake engineering
random vibration
seismic stability

**17b. Identifiers/Open-Ended Terms**

**17c. COSATI Field Group**

| 18. Availability Statement | 19. Security Class (This Report) | 21. No. of Pages |
|---|---|---|
| Release Unlimited | UNCLASSIFIED | 124 |
| | 20. Security Class (This Page) UNCLASSIFIED | 22. Price |

## Acknowledgements

This investigation was sponsored by the National Science Foundation under Grant No. PFR-7823095. The opinions, findings and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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# 1. INTRODUCTION

## 1.1 General

Earth structures like foundations and slopes behave nonlinearly when subjected to earthquake induced ground motions. Depending upon the type of soil these structures may become unserviceable due to excessive accumulation of strains or due to loss of strength as a result of excessive pore water pressure buildup in a process usually called liquefaction.

For accurate evaluation of a design of an earth structure for seismic loads, dynamic analyses which consider nonlinear soil behavior should be performed for a class of earthquake motions which could be expected on the site. For design purposes, the expected earthquake motions at a site are usually characterized by response spectra curves, such as those proposed by Housner (14), Newmark et al (29,30), Seed et al (40) and the ATC report (2). These spectra are presumably equivalent to an ensemble of earthquake time histories that can occur on a site, and therefore represent a more generalized form of seismic design input than any single earthquake accelerogram. Other generalized form of seismic inputs which are collectively equivalent to an ensemble of earthquake motions are usually defined stochastically, such as by a spectral density function (43,52).

For earthquake inputs defined in the form of acceleration or velocity time histories, two different approaches are available for dynamic analysis of nonlinear soil systems: (1) step-by-step nonlinear approach and (2) iterative equivalent linear approach (28,36,37,41). For a general soil system step-by-step nonlinear approach will give the most accurate answer, but the cost of analysis can be prohibitive.

For simple one dimensional problems, however, an efficient approach (48,49) based on the method of characteristics of partial differential equations has been developed. A computer program called CHARSOIL has been written for analysis by this method. Hysteretic stress-strain behavior of soil is defined by Ramberg-Osgood Curves (33) in this program. Further attempts have also been made to apply these methods to some simple two dimensional problems (3,31). However, it seems that these extensions are still in developmental stage.

In the equivalent linear approach, initially developed by Seed et al (37) for nonlinear soil system and subsequently used in many investigations (15,16,17,26,28,35,36,41), hysteretic soil behavior is characterized by equivalent and strain dependent shear moduli and damping ratios (38); since the soil modulus and damping values are not constant but now depend upon the level of strain, the analysis for a given earthquake motion is still nonlinear. To obtain the solution of equations of motion with such nonlinearity, the equivalent linear approach adopts an iterative scheme in which in each iteration a linear analysis is performed. The solution of the equations of motion in an iteration for a given earthquake time history is either performed in time (16,17) or frequency domain (26).

To evaluate a design of a soil system by one of the above mentioned approaches for a generalized seismic design input like response spectra, an ensemble of earthquake time histories which are equivalent to the generalized seismic design input must be obtained as these approaches accept seismic input only in this form. The analyses must then be performed for individual time histories. Also, the analyses results should be statistically processed to make a meaningful evaluation of the soil system design. As an analysis for a single time history is computationally expensive, the design evaluation for an ensemble of earthquake time histories becomes rather unattractive. Therefore, the methods are being sought and developed (35,44,45) in which generalized seismic inputs like response spectra and spectral density function can be directly used and the time history analyses avoided. The approaches which can use ground response spectra directly in a design evaluation are often called direct approaches.

## 1.2 Objective & Scope

The objective of this investigation is to develop and verify such a direct approach. We have been involved in the development of a direct approach earlier (44,45). This approach is essentially an equivalent linear approach, but it can use a design response spectra directly. In this investigation, this approach has been re-examined and its analytical formulation has been refined further. Herein, it is being proposed to use this approach for design evaluation of earth structures.

For verification of this approach a comprehensive analytical simulation study has been conducted. In this study the dynamic response results of a typical horizontal layered strata obtained by the direct approach are compared with the results obtained by a step-by-step nonlinear approach. The program CHARSOIL (48) has been used in the nonlinear approach. The seismic input in the nonlinear approach consists of an ensemble of earthquake time history, whereas that in the direct approach it consists of a set of averaged response spectra curves generated for the ensemble of time histories.

CHARSOIL program, which has been used in the simulation study, provides a most accurate solution of a one dimensional problem. Algorithms of comparable accuracy are, however, not yet available to solve a two dimensional nonlinear problem. Therefore, in this investigation, the verification of the proposed approach has been limited to only one-dimensional problems. The formulation of the proposed approach is, however, sufficiently general for its use with two and three dimensional problems as well.

## 1.3 Report Organization

Analytical development of the direct approach examined in this study is provided in Chapter 2. Some parts of this formulation are also available elsewhere (44) in a rather condensed form, but they have been repeated here for the sake of continuity and completeness of the rest of the presentation in the report.

The elements of the simulation study planned for the corroboration of the direct approach are described in Chapter 3. This chapter includes the development of consistent seismic inputs used in the two analysis procedures, a discussion of selected soil system parameters and their consistency, a description of the methods used for the construction of system damping matrix used in the direct approach and the procedure for an evaluation of safety indices for a stress response time history.

In chapter 4 the numerical results obtained by the two approaches are presented and critically reviewed. The summary and the conclusions are provided in chapter 5.

Some related analytical developments are given in the three appendices. More specifically, Appendix-I provides an analytical procedure used to estimate the correlation between nodal velocities and displacements; Appendix-II, a newly developed procedure to calculate design response by SRSS for nonproportionally damped systems; and Appendix-III, an analytical procedure to obtain certain expected values required in the implementation of the proposed approach.

## 2. ANALYTICAL FORMULATION OF DIRECT APPROACH

### 2.1 Equivalent Linear Equations of Motion

For a finite element descretization of an earth structure with strain dependent soil modulus and damping properties, the equation of motion at any time instant t can be written in the following form

$$[M]\{\ddot{x}\} + [D(\varepsilon)]\{\dot{x}\} + [S(\varepsilon)]\{x\} = - [M]\{r\}\ddot{X}_g(t) \qquad (1)$$

in which [M], [D] and [S] are the mass, damping and stiffness matrices of the structure; {X} = relative displacement vector; $\ddot{X}_g(t)$ = the base acceleration at time t; {r} = displacement influence coefficient vector (8). The dot over a vector represents its time derivative. Thus $\{\ddot{x}(t)\}$ represents the relative acceleration vector. The matrices [D] and [S] depend upon the levels of strain in various finite elements.

In the proposed direct approach it is desired to obtain the solution of Eq. 1, which has strain dependent parameters, for earthquake excitations characterized by a spectral density function or a set of response spectra curves. For this a quasi-linear and iterative approach is proposed, wherein Eq. 1 is replaced by another equation in which structural matrices do not depend upon strain, i.e.,

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = -[M]\{r\}\ddot{X}_g(t) \qquad (2)$$

in which [C] = damping matrix and [K] = stiffness matrix, which are assumed to be independent of the level of strain. This obviously introduced an error in the equation, and also in the solution as follows:

$$\{e\} = [D(\varepsilon) - C]\{\dot{x}\} + [S(\varepsilon) - K]\{x\} \qquad (3)$$

The minimization of the mean square value of the error, i.e. $E[\{e\}^T e]$, provides a mathematically convenient condition for the selection of elemental damping ratios and shear moduli required for construction of [C] and [K] matrices. In the equivalent linear time history approach (17,37), these elemental parameters are selected corresponding to a level of strain in the elements which usually is equal to 0.65 times the maximum strain in the element. The factor of 0.65 represents a kind of averaging effect which should be included in the selection of these elemental parameters.

The minimization of mean square error as suggested here for linearizing the equations has been used by many researchers earlier (6,18) in one form or another. This process is often called stochastic linearization technique. For minimization of the mean square error

$$\frac{\partial}{\partial \lambda_q^*} E[\{e\}'\{e\}] = 0 \tag{4a}$$

$$\frac{\partial}{\partial G_q^*} E[\{e\}'\{e\}] = 0 \tag{4b}$$

in which $\lambda_q^*$ and $G_q^*$ are the damping ratio and shear modulus, respectively, for the qth element. Eqs. 4 are obtained for all finite elements. It has been shown by Iwan (18) that the solution of Eq. 4 will minimize and not maximize the error.

Substituting for {e} from Eq. 3 and noting that [K] is independent of $\lambda_q^*$'s and [C] is independent of $G_q^*$'s, the following equations are obtained for the minimization of the mean square error:

$$E[(\{\dot{x}\}[D(\varepsilon) - C]' + \{x\}'[S(\varepsilon) - K]') \frac{\partial}{\partial \lambda_q^*} [C]\{\dot{x}\}] = 0 \tag{5a}$$

$$E[(\{\dot{x}\}'[D(\varepsilon) - C]' + \{x\}'[S(\varepsilon) - K]') \frac{\partial}{\partial G_q^*} [K]\{x\}] = 0 \tag{5b}$$

or

$$E[\{\dot{x}\}'[D(\varepsilon) - C]' \frac{\partial}{\partial \lambda^*_q} [C]\{\dot{x}\}] + E[\{x\}'[S(\varepsilon) - K]' \frac{\partial}{\partial \lambda^*_q} [C]\{\dot{x}\}] = 0 \quad (6a)$$

$$E[\{x\}'[S(\varepsilon) - K]' \frac{\partial}{\partial G^*_q} [K]\{x\}] + E[\{\dot{x}\}'[D(\varepsilon) - C]' \frac{\partial}{\partial G^*_q} [K]\{x\}] = 0 \quad (6b)$$

The second expected value terms in Eqs. 6a and b contain terms which represent the cross correlation between the displacement and velocity at the nodal points of a finite element. The correlation between velocity and displacement at the same nodal point is of course zero if the stochastically stationary excitation and responses are being considered. It has also been found that the cross correlation, that is, the expected values of the velocity at one node and displacement at the other nodes of an element are nearly zero. (The expressions to evaluate these correlations are given in Appendix I.) As a result, the second terms in Eqs. 6a and b can be dropped to simplify these equations as follows:

$$E[\{\dot{x}\}'[C]' \frac{\partial}{\partial \lambda^*_q} [C]\{\dot{x}\}] = E[\{\dot{x}\}'[D(\varepsilon)]' \frac{\partial}{\partial \lambda^*_q} [C]\{\dot{x}\}] \quad (7a)$$

$$E[\{x\}'[K]' \frac{\partial}{\partial G^*_q} [K]\{x\}] = E[\{x\}'[S(\varepsilon)]' \frac{\partial}{\partial G^*_q} [K]\{x\}] \quad (7b)$$

The matrices [C], [D], [K] and [S] are made up of elemental damping and stiffness matrices. For an element these matrices can be written as

$$\begin{aligned} [C_q] &= \bar{\lambda}_q[\bar{C}_q] \\ [D_q(\varepsilon)] &= \bar{\lambda}_q(\varepsilon)[\bar{C}_q] \\ [K_q] &= \bar{G}_q[\bar{K}_q] \\ [S_q(\varepsilon)] &= G_q(\varepsilon)[\bar{K}_q] \end{aligned} \quad (8)$$

in which $\bar{\lambda}_q = \lambda^*_q/\lambda_{qm}$ and $\bar{G}_q = G^*_q/G_{qm}$ are normalized damping and shear modulus factors for element q; $[\bar{C}_q]$ = element damping matrix obtained with $\lambda_{qm}$ as damping ratio; $[\bar{K}_q]$ = the stiffness matrix obtained with a shear modulus value of $G_{qm}$($\lambda_{qm}$ and $G_{qm}$, respectively, may be chosen as the values corresponding to high and low shear strains); $\lambda_q(\varepsilon)$ = strain dependent damping ratio curve normalized by $\lambda_{qm}$; and $G_q(\varepsilon)$ = strain dependent shear modulus curve normalized by $G_{qm}$.

For a particular value of q, Eqs. 7a or b will have $\bar{\lambda}_q$ (or $\bar{G}_q$) of all the elements connected with the element q. Thus $\bar{\lambda}_q$ (or $\bar{G}_q$) values are coupled. To obtain $\bar{\lambda}_q$ and $\bar{G}_q$, Eqs. 7a and b will have to be set up for all the finite elements of the system and solved simultaneously. However, if the connectivity of the elements is ignored, Eq. 7a or b become uncoupled which finally lead to a set of simplified expressions to obtain $\bar{\lambda}_q$ and $\bar{G}_q$. For example, Eqs. 7a gives:

$$\bar{\lambda}_q \, E(\{\dot{x}_q\}'[C_q]'[C_q]\{\dot{x}_q\}) = E(\lambda_q(\varepsilon)\{\dot{x}_q\}'[C_q][C_q]\{\dot{x}_q\}) \tag{9}$$

in which $\{x_q\}$ the displacement vector for the qth finite element. Noting that the strain $\varepsilon$ in an element depends only on the nodal displacements and that the nodal velocities and displacements in an element are only mildly correlated, the expected value on the right hand side of Eq. 9 can be written as

$$E[\lambda_q(\varepsilon)\{\dot{x}\}'[C_q]'[C_q]\{\dot{x}\}] = E[\lambda_q(\varepsilon)]E[\{\dot{x}\}'[C_q]'[C_q]\{\dot{x}\}] \tag{10}$$

Using Eq. 10 in Eq. 9 we obtain

$$\bar{\lambda}_q = E[\lambda_q(\varepsilon)] \tag{11}$$

That is, the most appropriate value of the damping ratio in an element which will minimize te mean square error in Eq. 3 is the expected value of the damping ratio random variable.

Proceeding on the same lines as above, and ignoring coupling between various $\bar{G}_q$ values, Eq. 7b gives the following for the shear modulus factor

$$\bar{G}_q = \frac{E[G_q(\varepsilon)\{x_q\}'[\bar{K}_q]'[\bar{K}_q]\{x_q\}]}{E[\{x_q\}'[\bar{K}_q]'[\bar{K}_q]\{x_q\}]} \tag{12}$$

In neglecting the coupling between $\bar{\lambda}_q$ and $\bar{G}_q$ in Eqs. 7a and b, and thus Eqs. 11 and 12, we are in essence neglecting the effect of the nodal forces applied by the adjacent element on the error minimizing process. This, however, does not mean that the forces applied by the adjacent elements are altogether neglected in the solution of the problem. As the complete global matrices [C] and [K] are used in the analysis, the effect of these forces is included in the final solution.

The effect of including and disregarding these forces on $\bar{\lambda}_q$ and $\bar{G}_q$ values and the final results has been investigated herein. For the sample problem considered in this investigation, the response and factor of safety results obtained with the use of Eq. 11 and 12 were only imperceptibly different from the results obtined with the use of complete Eqs. 7a and b. (This, however, may not always be the case for other, more complicated soil systems (44).) Thus, here only Eqs. 11 and 12 are used.

It is noted that the solution of Eq. 2 is required to be known to obtain $\bar{\lambda}_q$ and $\bar{G}_q$ from Eqs. 7a and b or from Eqs. 11 and 12. Specifically, the second order statistics of the response of Eq. 2 is required before these equations can be used. For this an iterative approach is used, in which one starts with some predecided values of $\bar{\lambda}_q$ and $\bar{G}_q$ and solves Eq. 2 to further refine these values in the next iteration. The iterations are continued till a conversion in the final values of $\bar{\lambda}_q$ and $\bar{G}_q$ is obtained. In this respect, therefore, this methodology is similar to that being used in the equivalent linear methods (17,36,37).

In the following section, the method to obtain the solution of Eq. 2 to obtain the response statistics required in Eqs. 7a, b, 11 and 12, and that required in the stability evaluation, for earthquake inputs defined by spectral density function or response spectra is described.

## 2.2 Response Analysis

In any iteration, the system is assumed to behave linearly and the solution of equivalent linear Eq. 2 can be obtained by any suitable technique. For an input earthquake motion defined by an acceleration time history, step-by-step numerical integration scheme can be used, as is done in the finite element equivalent linear techniques (16,17). For the motion characterized by a random process, the statistics of the response quantity can be obtained by standard random vibration procedures (25,43). For earthquake input motions defined by response spectra curves, one has to use the normal modes approach to solve Eq. 2 as the spectra provide the maximum response in each mode directly.

If the ground motion can be assumed to be characterized by a spectral density function, the stationary mean square value of a response quantity can be obtained by using the normal mode procedure as follows (43).

$$E[R_q^2] = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \psi_j(q) \psi_k(q) \int_{-\infty}^{\infty} \Phi_g(\omega) H_j(\omega) H_k^*(\omega) d\omega \qquad (13)$$

in which $R_q$ = response quantity of interest, N = number of significant modes; $\gamma_j$ = jth mode participation factor defined as equal to $\{\phi_j\}^T[M]\{r\}/\{\phi_j\}^T[M]\{\phi_j\}$; $\phi_j$ = relative displacement mode shape of the system; $\psi_j(q)$ = jth mode shape of the response quantity of interest, $\Phi_g(\omega)$ = spectral density function of the acceleration at the base of the system; $\omega$ =

frequency variable; and $H_j(\omega)$ = relative displacement frequency transfer function for mode j, defined as

$$H_j(\omega) = 1/(\omega_j^2 - \omega^2 + 2\beta_j\omega_j\omega) \quad (14)$$

in which $\omega_j$ = jth undamped frequency of the vibration and $\beta_j$ = jth mode damping ratio. An asterisk (*) as superscript on the frequency response function denotes the complex conjugate.

The solution given by Eq. 13 is of course valid only when the equivalent linear matrix [C] can be diagonalized by the undamped normal modes. That is, the matrix [C] is so-called classical type (5). Since this matrix in any iteration is assembled* from individual elemental matrices constructed with different damping ratios in different elements , it may not be proportional, and may have off-diagonal terms when pre and post multiplied by the modal matrix $[\Phi]^+$ of the system. To check what error is introduced if the off-diagonal terms of the matrix $[\Phi]^T[C][\Phi]$ are neglected, a method has been developed to obtain the response in the case in which these terms are retained. The theoretical basis for this new procedure, which can also be used with the earthquake inputs defined by response spectra curves, is given in Appendix II which contains the copy of the paper, Ref. (46).

Using this mathematically exact procedure and the stiffness and damping matrices obtained in the final iteration of the proposed approach for the layered soil system considered in this investigation, Fig. 1, the displacement and elastic force response values have been obtained. These values are shown in Table 2 and 3. Also shown are the values obtained by the normal mode approach in which the off diagonal terms

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* See Sec. 3.4
+ $[\Phi]$ - modal matrix

are neglected. The seismic input used to obtain these results is defined by a spectral density function, Eq. 43, with parameters as given in Table 1. It is seen that the difference between the two results is rather small (in this case not more than 0.35%). Thus the off-diagonal terms of $[\phi]^T[C][\phi]$ can be ignored without introducing any serious error, and conventional modal analysis approach, Eq. 13, can be used. All results in this report are obtained with this assumption with the modal damping ratio $\beta_j$ defined as follows:

$$\beta_j = \frac{1}{2\omega_j} \{\phi_j\}^T[C]\{\phi_j\} \tag{15}$$

in which $\{\phi_j\}$ are mass normalized mode shapes.

Eq. 13 can be used to obtain the mean square response value of a response quantity like displacement, strain, etc. in any iteration. In evaluation of $G_q$ by Eq. 12 we also need to obtain the cross covariance between two nodal displacements and between displacements and strains. These can also be similarly obtained using the following expressions

$$E[x_r x_s] = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \phi_j(r) \phi_k(s) \int_{-\infty}^{\infty} \Phi_g(\omega) H_j(\omega) H_k^*(\omega) d\omega \tag{16}$$

$$E[\varepsilon_q x_r] = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \phi_j(r) \zeta_k(q) \int_{-\infty}^{\infty} \Phi_g(\omega) H_j(\omega) H_k^*(\omega) d\omega \tag{17}$$

in which $x_r$ = relative displacement at node r, $\phi_j(r)$ = jth relative modal displacement at node r, $\zeta_k(q)$ = kth modal strain for element q.

For seismic input defined by response spectra curve, Eq. 13 can be written in terms of given spectra values and peak factor $F_i$ (implicitly built in the response spectra definition) as follows:

$$E[R_q^2] = \sum_{j=1}^{N} \frac{1}{F_j^2} \left(\frac{\gamma_j^2 \psi_j^2(q) R_a^2(\omega_j)}{\omega_j^4}\right) + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \gamma_j \gamma_k \psi_j(q) \psi_k(q) \mathrm{CTXF} \tag{18}$$

in which CTXF is defined as:

$$CTXF = \{A_{jk}R_a^2(\omega_j) + B_{jk}\omega_j^2 r^2 R_v^2(\omega_j)\}/(F_j^2\omega_j^4)$$

$$+ \{C_{jk}R_a^2(\omega_k) + D_{jk}\omega_k^2 R_v^2(\omega_k)\}/(F_k^2\omega_k^4) \quad (19)$$

and $R_a(\omega_j)$, $R_v(\omega_j)$ = pseudo-acceleration and relative velocity spectra value at frequency $\omega_j$ and damping $\beta_j$, respectively; $F_j$ = the peak factor of response spectra value at frequency $\omega_j$ and damping $\beta_j$; $r = \omega_j/\omega_k$; and the factors $A_{jk}$, $B_{jk}$ etc. are obtained from the solution of the following simultaneous equations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & v_k & 1 & v_j r^2 \\ v_k & 1 & v_j r^2 & r^4 \\ 1 & 0 & r^4 & 0 \end{bmatrix} \begin{Bmatrix} A_{jk} \\ B_{jk} \\ C_{jk} \\ D_{jk} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ u \\ r^2 \end{Bmatrix} \quad (22)$$

in which $v_k = -2(1-2\beta_k^2)$, $v_j = -2(1-2\beta_j^2)$ and $u = -(1+r^2-4\beta_j\beta_k r)$.

$F_j$, generally depend upon the frequency and damping ratio of the mode, see Sec. 3.2 and Fig. 12. However, not much error is introduced if only a single value, probably corresponding to the most dominant mode, is used. In this investigation a single value, F, corresponding to the fundamental mode of the system has been used. Also the same peak factor is used for pseudo acceleration, relative velocity and relative acceleration response spectra values. With this, Eqs. 16, 17, 18 can be defined in terms of given spectra values as follows:

$$E[R_q^2] = \frac{1}{F^2}\left(\sum_{j=1}^{N} \gamma_j^2\psi_j^2(q)R_a^2(\omega_j)/\omega_j^4 + 2\sum_{j=1}^{N}\sum_{k=j+1}^{N} \gamma_j\gamma_k\psi_j(q)\psi_k(q)CTX\right) \quad (21)$$

$$E[x_r x_s] = \frac{1}{F^2}\left(\sum_{j=1}^{N} \gamma_j^2\phi_j(r)\phi_j(s)R_a^2(\omega_j)/\omega_j^4 + \sum_{\substack{j=1 \\ j\neq k}}^{N}\sum_{k=1}^{N} \gamma_j\gamma_k\{\phi_j(r)\phi_k(s)\right.$$

$$+ \phi_j(s)\phi_k(r)\}CTX) \tag{22}$$

$$E[\varepsilon_q x_r] = \frac{1}{F^2}[\sum_{j=1}^{N} \gamma_j^2 \phi_j(r)\zeta_j(q)R_a^2(\omega_j)/\omega_j^4$$

$$+ \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k\neq j}}^{N} \gamma_j\gamma_k\{\phi_j(r)\zeta_k(q) + \phi_k(r)\zeta_j(q)\}CTX) \tag{23}$$

in which CTX is defined as

$$CTX = \{A_{jk}R_a^2(\omega_j) + B_{jk}\omega_j^2 r^2 R_v^2(\omega_j)\}/\omega_j^4 + \{C_{jk}R_a^2(\omega_k) + D_{jk}\omega_k^2 R_v^2(\omega_k)\}/\omega_k^4 \tag{24}$$

It is seen that these expressions require relative velocity response spectra for the input motion which is, however, different from the pseudo velocity spectra, especially in the high frequency range. If this spectra is not available, the approximate approach suggested in Ref. (47) can be used to obtain $R_v(\omega_j)$ from the acceleration spectra values. In the simulation study, however, this spectra was generated for its use in these expressions.

Eqs. 7a, b and 12 require the evaluation of the expected values of expressions like $[G_q(\varepsilon)x_r x_s]$ and $[G_q(\varepsilon)x_r^2]$. These can be obtained as explained in Appendix III.

## 2.3 Evaluation of Stability Indices

Seismic stability of a soil system is often defined in terms of sustained cyclic damage and factor of safety (1,24,39). The method to be used with the proposed stochastic approach has been discussed by the writer earlier (44,45) and is further elaborated upon in this section for the sake of completeness of the solution of the problem.

### 2.3.1 Cyclic Damage

Soil systems lose their strength when subjected to cyclic stress due to successive accumulation of deformation (cohesive soils) or pore water pressure (cohesionless soils). A gradual degradation occurs in the soil which finally renders it unserviceable for load carrying purposes. The extent of degradation in soil when subjected to cyclic stresses induced by earthquake ground motion is often measured in terms of so-called cyclic damage (1,24,44). The concept of cyclic damage in soils is analogous to the concept of fatigue damage in metals and often similar procedures are used to quantify such damages for the two materials.

To assess cyclic damage (also called damage potential) the Palmgren-Miner hypothesis is commonly used: the cyclic damage is defined as the ratio of the applied number of cycles at a stress level to the number of cycles the soil can take up to failure. According to this hypothesis, the damages due to the application of different amplitudes of stress cycles can be linearly added to obtain the total damage. Furthermore, the order in which different stress cycles are applied is considered immaterial. This latter assumption makes this hypothesis especially attractive to use for randomly varying stresses such as those applied during an earthquake. A material is assumed to have failed when the total damage approaches 1.0.

It is thus seen that the calculation of damage potential for a soil requires the stress amplitude versus the number of cycles-to-failure plot. Such a plot is often called as S-N curve. It is obtained by experiments on soil specimens, and depends on parameters like confining pressure, relative density, etc. When plotted on log-log scale, an S-N curve for a soil may be a straight line. In such a case, it may be defined analytically as follows:

$$Ns^b = C \tag{25}$$

where b and C are constants which depend upon the type of soil, effective confining pressure, relative density, etc. The S-N curves of various layers of the model as indicated in Table 4 are shown in Fig. 2.

The procedure to obtain cumulative damage for stochastically defined stationary stress time history is given by Crandall (9) and Lin (25). It is shown that the expected value of the cumulative damage is given by

$$E[D] = \frac{\bar{T}M}{C} \int_0^\infty s^b f_s(s)ds \tag{26}$$

in which $\bar{T}$ = duration of time history, M = expected number of stress peaks per unit time, s = stress and $f_s(s)$ = the probability density function of stress peaks. This probability density function is defined as (25)

$$f_s(s) = \frac{\sqrt{1-\bar{\alpha}^2}}{\sigma_s\sqrt{2\pi}} \exp(-s^2/[2\sigma_s^2(1-\bar{\alpha}^2)])$$

$$+ \frac{\bar{\alpha}s}{2\sigma_s^2} [1 + \mathrm{erf}(s\bar{\alpha}/\{\sigma_s\sqrt{2(1-\bar{\alpha}^2)}\})]\exp(-s^2/2\sigma_s^2) \tag{27}$$

in which $\bar{\alpha}$ = the band width parameter defined as = $\nu_0/M$; $\nu_0$ = zero crossing rate of stress response; and $\sigma_s$ = standard deviation of stress.

Substitution of Eq. 27 into Eq. 26 gives the following expression for the expected value of damage

$$E[D] = \frac{\bar{T}M}{C} [\frac{\bar{\alpha}(\sigma_s\sqrt{2})^b}{2} \{\Gamma(\frac{b+2}{2}) + I_1\} + I_2] \tag{28}$$

in which

$$I_1 = \int_0^\infty t^{b/2}\mathrm{erf}\{\bar{\alpha}\sqrt{t}/\sqrt{1-\bar{\alpha}^2}\}\exp(-t)dt \tag{29}$$

and

$$I_2 = \frac{\sqrt{1-\bar{\alpha}^2}}{\sigma_s\sqrt{2\pi}} \frac{\{2\sigma_s^2(1-\bar{\alpha}^2)\}^{\frac{b+1}{2}}}{2} \Gamma(\frac{b+1}{2}) \tag{30}$$

where $\Gamma(\cdot)$ is the complete Gamma function.

The response due to an earthquake excitation will not be stationary. To obtain cyclic damage from Eq. 26, it is, however, suggested that an equivalent stationary duration be used instead of the total duration $\bar{T}$. To obtain a conservative estimate of safety, the procedure suggested by Hou (13), can be used to obtain $\bar{\bar{T}}$.

For earthquake ground excitations defined in stochastic form or by response spectra, one can obtain the standard deviation of stress in an element from Eqs. 13 or 21 with proper substitution for $\psi_i$s. The parameter $\bar{\alpha}$ is defined in terms of the mean square response of stress and its time derivative as follows:

$$\alpha = \frac{\nu_0}{M} = \frac{\sigma_{\dot{s}}^2}{\sigma_s \; \sigma_{\ddot{s}}} \tag{31}$$

To obtain $\sigma_{\dot{s}}$ and $\sigma_{\ddot{s}}$ the following expressions can be used

$$\sigma_{\dot{s}}^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \xi_j(q) \xi_k(q) \int_{-\infty}^{\infty} \omega^2 \Phi_g(\omega) H_j(\omega) H_k^*(\omega) d\omega \tag{32}$$

$$\sigma_{\ddot{s}}^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \xi_j(q) \xi_k(q) \int_{-\infty}^{\infty} \omega^4 \Phi_g(\omega) H_j(\omega) H_k^*(\omega) d\omega \tag{33}$$

To define these quantities in terms of ground response spectra values the following expressions can be used:

$$\sigma_{\dot{s}}^2 = \frac{1}{F^2} \left( \sum_{j=1}^{N} \gamma_j^2 \xi_j^2(q) R_v^2(\omega_j) + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \gamma_j \gamma_k \xi_j(q) \xi_k(q) \cdot \mathrm{CTW} \right) \tag{34}$$

$$\sigma_{\ddot{s}}^2 = \frac{1}{F^2} \left( \sum_{j=1}^{N} \gamma_j^2 \xi_j^2(q) R_r^2(\omega_j) + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \gamma_j \gamma_k \xi_j(q) \xi_k(q) \cdot \mathrm{CTWW} \right) \tag{35}$$

in which $\xi_j(q)$ = jth stress mode shape, $R_r(\omega_j)$ = relative acceleration response spectrum value at frequency $\omega_j$ and damping $\beta_j$, and CTW and CTWW are defined as

$$CTW = \{\omega_k^2 R_a^2(\omega_j)A' + \omega_j^4 R_v^2(\omega_j)B'\}/\omega_j^4 + \{R_a^2(\omega_k)C' + \omega_k^2 R_v^2(\omega_k)D'\}/\omega_k^2 \quad (36)$$

$$CTWW = \omega_k^2 R_v^2(\omega_j)A' + R_r^2(\omega_j)B' + \omega_k^2 R_v^2(\omega_k)C' + R_r^2(\omega_k)D' \quad (37)$$

in which A', B', C' and D' are obtained from the solution of the following simultaneous equations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & v_k & 1 & v_j r^2 \\ v_k & 1 & v_j r^2 & r^4 \\ 1 & 0 & r^4 & 0 \end{bmatrix} \begin{Bmatrix} A' \\ B' \\ C' \\ D' \end{Bmatrix} = \begin{Bmatrix} 1 \\ -u \\ r^2 \\ 0 \end{Bmatrix} \quad (38)$$

where $v_j$, $v_k$ and u are the same as defined in Eq. 20.

The relative acceleration response spectrum value can be obtained if such spectra curves are available; however it can also be obtained using the relative velocity and absolute acceleration spectra values as follows

$$R_r^2(\omega_j) = A_g^2 + 2(1-2\beta_j^2)\omega_j^2 R_v^2(\omega_j) - R_a^2(\omega_j) \quad (39)$$

Evaluation of Eq. 28 requires numerical integration which can be performed using a suitable integration technique.

Eq. 28 provides only the mean value of damage. Mark (9) has established a method to assess the variability of damage potential. As discussed in Reference (9), the variability in damage is less if the damping involved is large. Since soil systems will tend to have a large effective damping ratio at the stress level of any practical consequence the magnitude of damage variability may not be significant. This is substantiated by the numerical results obtained in the simulation study; see Sec. 4.7.

### 2.3.2 Factor of Safety

According to Palmgren-Miner's criterion, a system becomes unserviceable when cumulative damage in the system reaches 1.0. However, for any value less than 1.0, the level of cumulative damage does not provide a direct measure of available safety margin simply because it is not a linear measure of safety. In civil engineering practice, the term factor of safety is commonly used to measure the available safety margin. As a result the use of this term to quantify stability in seismic loading of earth structures has also been suggested (1,44).

To obtain factor of safety, some allowable stress is compared with an "equivalent uniform stress" (1,39). Two stress time histories are considered equivalent, if they induce the same amount of damage in a soil element. This concept of equivalence is used to characterize a irregular stress time history, as induced in an earthquake, by a uniform stress time history applied for a certain number of cycles. Thus by varying the level of stress and corresponding number of cycles, an infinite number of equivalent uniform stress time histories can be obtained for the same amount of damage. For each such equivalent time history with stress level $S_e$ and the number of stress cycles, $N_e$, there is an allowable stress $S_a$ which when applied $N_e$ times cyclically will cause the soil to fail. The ratio

$$F.S. = \frac{S_a}{S_e} \tag{40}$$

is called the factor of safety. For linear log-log S-N curve, this factor of safety is related to the damage as follows (44,45)

$$F.S. = D^{-1/b} \tag{41}$$

It is independent of the chosen parameters, $S_e$ and $N_e$, of the equivalent stress time history for linear log-log S-N curve only. For nonlinear curves, the definition of factor of safety is not as straight forward. See Ref. (44).

# 3. SIMULATION STUDY

## 3.1 General

For a given set of response spectra curves, the approach described in the previous sections can be used to assess the stability of an earth structure. However, various assumptions have been made in formulating the approach in a numerically tractable form. To check the validity of the proposed approach, therefore, a numerical simulation study has been planned. The idea is to compare the results obtained by the proposed direct approach with the results obtained by a truly nonlinear approach which considers the hysteretic behavior of soils under cyclic loads. The approach developed by Streeter, Wylic and Richart (48) is a non-linear approach which uses the method of characteristics to solve the continuum equation of motion. The approach was originally developed for one-dimensional problems and then extended to the two dimensional case by the lattice work approach (31). A finite-element consistent two dimensional approach has also been proposed by Ayala (3). In this investigation, however, only one dimensional problems have been analyzed by the nonlinear approach (using the program CHARSOIL) and the proposed direct approach.

Since the proposed approach and the nonlinear approach are quite different from each other in as much as they use different forms of seismic input, model the soil system differently and even use different soil parameters, it is very essential that the problem parameters be chosen consistently to obtain comparable results. This consistency in various parameters chosen in the two approaches seems to have been achieved as described in the following sections of this chapter.

## 3.2 Seismic Input

The proposed direct approach is meant to be used in design evaluations and therefore has been formulated for its use with seismic input defined by ground response spectra curves. The nonlinear approach on the other hand uses earthquake time histories as seismic input. To make these inputs consistent, one could choose a set of spectra curves and obtain ground acceleration (and thus velocity) time histories by a variety of methods that have been proposed (12,50,52). However, these methods have some constraints such as that of enveloping a given spectrum by the generated time history spectrum. Some problem is also encountered in making the generated time histories consistent with all the spectra curves for various damping. It was, therefore, thought better to artifically generate an ensemble of time histories (with certain frequency and intensity characteristics) for their use in the time history approach, and generate a set of average spectra curves for these time histories for their use in the proposed approach.

For random generation of time histories, the base accelerations are assumed to be represented as follows:

$$\ddot{X}_g(t) = \ddot{X}_b(t)e(t) \tag{42}$$

in which $\ddot{X}_g(t)$ a sample earthquake base acceleration; $\ddot{X}_s(t)$ = a stochastically stationary time history motion; and e(t) = intensity modulation function. $\ddot{X}_s(t)$ is characterized by a broad-band spectral density function of the following form:

$$\Phi_s(\omega) = \sum_{i=1}^{3} S_i \frac{\omega_i^4+4\beta_i^2\omega_i^2\omega^2}{(\omega_i^2-\omega^2)^2+4\beta_i^2\omega_i^2\omega^2} \tag{43}$$

The parameters $S_i$, $\omega_i$ and $\beta_i$ of this spectral density function are given in Table 1. This is a modified Kanai-Tajimi type of spectral density function in which more terms have been added to get a broad-band effect. The sample acceleration time history functions corresponding to this density function are generated using the standard technique, as originally proposed by Rice (34) and subsequently used by many researchers (12, 42). Each stationary function is then further modified by an envelope function as shown in Fig. 5. Such an envelope function represents an earthquake motion somewhere between Class B and C type of motions as classified by Jenning, Housner and Tsai (21). The choice of this function in this investigation has been made rather arbitrarily. Fig. 6 shows a sample acceleration time history record generated by this procedure and Fig. 7 shows its acceleration response spectra curves. These artifically generated records are further modified for base line correction to remove some inadvertent long period trends which may have been introduced in the generation process. This correction is however not critical as the time history response results of the soil structure model used in this report obtained for uncorrected and corrected base motion were only imperceptibly different from each other.

A total of 54 earthquake acceleration time histories were generated by this procedure. (The number 54 has no special significance. It was originally meant to be 50.) These acceleration records were integrated to obtain the corresponding velocity time histories to be used in the program CHARSOIL for time history generation of the results.

The acceleration time histories are used for generation of base motion spectra curves. Absolute acceleration, relative velocity and relative acceleration spectra curves are generated for a total of 10 damping ratios ranging from 1% to 50% of the critical. For each damping value, the spectra curves of the time history ensemble were statistically processed to obtain the mean spectra curves. These averaged spectra curves are used as base input in the direct approach. Figs. 8, 9 and 10 show these averaged spectra curves.

Since the proposed approach also uses a peak factor value, the time history response of the oscillators at many spectral frequencies were processed to obtain the mean square response time history. Fig. 11 shows such a time history for an oscillator period of 0.2 secs. at a damping of .01. Such mean square response time histories were then scanned to obtain the maximum mean square response. The peak factor, F, is then defined as

$$F = \frac{\text{response spectrum value}}{\text{maximum root mean squared response}} \tag{44}$$

Fig. 12 shows the variation of the peak factor with frequency and damping. It ranges from a value of about 2.6 at some high frequency to a value of 1.4 at very low frequencies. A value of 2. was selected for use in the direct approach. This is the value at dominant period and damping of the soil structure examined in this report. To examine the sensitivity of the results with respect to the choice of the peak factor, the results at other values are also reported.

### 3.3 Parameters of Soil System

A typical soil strata with 10 layers, 60 ft. deep and with different low strain shear moduli and S-N curve parameters as shown in Table 3, has been analyzed by the two methods. For nonlinear analysis, the stress-strain behavior of each layer is characterized by Ramberg-Osgood curve (20,33) with following expressions:

Backbone Curve:

$$\frac{\gamma}{\gamma_y} = \frac{\tau}{\tau_y} \{1 + \alpha(\frac{\tau}{\tau_y})^{R-1}\} \tag{45}$$

Ascending Branch

$$(\frac{\gamma+\gamma_0}{2\gamma_y}) = \frac{\tau+\tau_0}{2\tau_y} + \alpha(\frac{\tau+\tau_0}{2\tau_y})^R \tag{46}$$

Descending Branch

$$(\frac{\gamma-\gamma_0}{2\gamma_y}) = \frac{\tau-\tau_0}{2\tau_y} + \alpha(\frac{\tau-\tau_0}{2\tau_y})^R \tag{47}$$

in which $\gamma$, $\tau$ = the shear strain and stress, respectively; $\gamma_y$, $\tau_y$, $\alpha$ and R are the parameters of the model. In the analysis $\gamma_y = 2x10^{-4}$, $\alpha = 1$, $R = 3$ have been used. $\tau_y$ is defined in terms of low strain shear modulus $G_0$ as

$$\tau_y = G_0\gamma_y \tag{48}$$

Various branches of the curve are shown in Fig. 3.

In the proposed direct approach, however, equivalent linear properties are defined in terms of strain dependent shear modulus and damping curves (38). For a hysteretic stress strain behavior, the secant modulus is taken as the equivalent shear modulus. Using the equation for the backbone curve, it is seen that the secant modulus, G, at a strain level $\gamma_0$, is obtained as follows:

$$G = \frac{G_0}{1+\alpha(\frac{\tau_0}{\tau_y})^{R-1}} \tag{49}$$

in which $\tau_0$ is the shear stress corresponding to the strain level $\gamma_0$. To obtain $(\tau_0/\tau_y)$ in terms of $(\gamma_0/\gamma_y)$, the following backbone curve equation is solved:

$$\alpha(\frac{\tau_0}{\tau_y})^R + (\frac{\tau_0}{\tau_y}) - \frac{\gamma_0}{\gamma_y} = 0 \quad (50)$$

The area in a hysteresis loop represents the energy loss in a cycle, and thus it has been related to the equivalent viscous damping ratio as follows (28,20)

$$\beta = \frac{1}{4\pi}\frac{\Delta W}{W} \quad (51)$$

in which $\beta$ = the damping ratio, $\Delta W$ = area of the hysteresis loop and $W$ = the maximum strain energy corresponding to the stress and strain levels of $\tau_0$ and $\gamma_0$.

For Eqs. 45-47,

$$\Delta W = 4\,\frac{R-1}{R+1} \cdot \frac{G_0-G}{G} \cdot (\frac{\tau_0}{\tau_y})^2\ \tau_y\tau_y \quad (52)$$

and if the strain energy is defined in terms of $\gamma_0$ and $\tau_0$ as:

$$W = \frac{1}{2}\gamma_0\tau_0$$

$$= \frac{1}{2}\ (\frac{\tau_0}{\tau_y})^2\ \frac{G_0}{G} \quad (53)$$

the equivalent damping ratio, using Eq. 51 is obtained as follows:

$$\beta = \frac{2}{\pi}\frac{R-1}{R+1}\ (1 - \frac{G}{G_0}) \quad (54)$$

Equivalent shear modulus and damping ratio as defined by Eqs. 49 and 54, respectively, are used in the proposed direct approach.

Different values of these parameters are obtained if the strain energy is defined as the area under the backbone curve. That is,

$$W = \frac{1}{2}\ (\frac{\tau_0}{\tau_y})^2\{1 + \frac{2R}{R+1}\frac{G_0-G}{G}\}\ \tau_y\gamma_y \quad (55)$$

Using Eqs. 51, 52 and 55, the damping ratio $\beta$, is defined as follows:

$$\beta = \frac{2}{\pi}\frac{R-1}{R+1}\frac{\frac{G_0-G}{G}}{1+\frac{2R}{R+1}\left(\frac{G_0-G}{G}\right)} \tag{56}$$

Furthermore if the equivalent shear modulus G* is defined as the value which gives the same strain energy as Eq. 55 at strain level $\gamma_0$, then

$$\frac{1}{2} G^*\gamma_0^2 = W \tag{57}$$

or

$$\left(\frac{G^*}{G_0}\right) = \left(\frac{G}{G_0}\right)^2 \left\{1 + \frac{2R}{R+1}\frac{G_0-G}{G}\right\} \tag{58}$$

Eqs. 56 and 58 were also used to define the equivalent properties in proposed direct approach. However, a better correspondence with the nonlinear approach result was obtained when Eqs. 49 and 54 were used.

### 3.4 Construction of System Damping Matrix

Eq. 54 or 58 can be used to obtain the damping ratio corresponding to a level of strain in a finite element. But, how could this damping ratio be used to define the element damping matrix $[C_q]$ and then the global damping matrix [C]? Some methods that have been used or could be used are explored in this section.

#### 3.4.1 Method 1

In absence of any better rational procedure, the Raleigh's damping matrix has often been used to define the damping matrix for a system. That is,

$$[C] = a[M] + b[k] \tag{59}$$

This is a proportional damping matrix. For this matrix the damping ratio in any mode i could be written as

$$2\beta_i = \frac{a}{\omega_i} + b\omega_i \tag{60}$$

However, the damping ratio in a soil structure changes from element to element depending upon the level of strain. To incorporate this feature of soil systems, Idriss et al (17) proposed the use of Eq. 59 at the element level, i.e.,

$$[C_q] = a[m_q] + b[k_q] \tag{61}$$

To define a and b in this equation, they used Eq. 60 with

$$a = \beta\omega_1, \quad b = \beta/\omega_1 \tag{62}$$

in which $\omega_1$ = fundamental frequency of the system. The element damping matrices, $[C_q]$, are then assembled to obtain the global damping matrix [C]. It is seen that if the damping ratios in various elements are the same, system matrix [C] constructed from $[C_q]$s will then give a classical Raleigh damping matrix which can be diagonalized by the undamped normal modes.

This procedure heavily relies on the first mode to determine a and b, which in most cases is a dominant mode. As expected, this generally damps out higher modes by attributing very high damping ratios to them. (For [C] constructed by this method, the modal damping ratios obtained using Eq. 15 were found to be unreasonably high in higher modes. See Table 5.) Thus the response quantities which are affected by higher modes could be in error.

### 3.4.2 Method 2

This is an extension of Method 1 in which constants a and b are defined using two frequencies of the system. If the damping ratio $\beta_i$ remains constant for the two frequencies, using Eq. 60 we obtain

$$a = 2\beta\, \omega_i\omega_j/(\omega_i + \omega_j) \tag{63}$$

$$b = 2\beta/(\omega_i + \omega_j) \tag{64}$$

in which $\omega_i$ and $\omega_j$ are the two chosen frequencies. Again the choice of

$\omega_i$ and $\omega_j$ is arbitrary. The best choice appears to be the first and the last significant frequencies. For the soil system being examined in this investigation, the distribution of damping obtained in various modes with $\omega_i$ and $\omega_j$ equal to the 1st and the last frequencies was more uniform. See Table 5 for modal damping values obtained in a typical case.

An extension of this method to include more than two modal frequencies to define element damping matrix was also investigated. Here, as an extension of the method described by Wilson and Penzien (53) and also discussed by Clough and Penzien (8) for construction of a proportional damping matrix for a system, the use of the following form to define element damping matrix was explored:

$$[C_q] = [m_q] \sum_{i=0}^{n} a_i([m_q]^{-1}[k_q])^i \tag{65}$$

For n = 1, this equation is the same as Eq. 61. To obtain $a_i$, a set of simultaneous equations, as described in the text by Clough and Penzien (8) on page 196, Eq. 13-23, were used. If these equations are used with system [M] and [k] matrices, a system damping matrix with desirable properties can be obtained. However, the use of this method to construct an elemental damping matrix $[C_q]$ gave rather unrealistic modal damping values for the system. Therefore this approach of using more than two frequencies was not investigated any further.

3.4.3 Method 3

Another technique (4) which was investigated is based on the method of least squares. Here we use all frequencies in Eq. 60 to estimate a and b by minimizing the mean square error between the desired and calculated values of $\beta_i$. Different weightage can also be assigned to the errors in various modes. Thus, if the weightage assigned are $W_i$, then the total weighted mean square error is:

$$e_d = \sum_{i=1}^{N} W_1 \left(\frac{a}{\omega_i} + b\omega_i - 2\beta_i\right)^2 \qquad (66)$$

For minimization of $e_d$,

$$\frac{\partial e_d}{\partial a} = \frac{\partial e_d}{\partial b} = 0 \qquad (67)$$

which gives rise to two simultaneous equations as:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} \qquad (68)$$

in which

$$A_{11} = \sum_{i=1}^{N} W_i/\omega_i^2, \quad A_{12} = \sum_{i=1}^{N} W_i,$$

$$A_{21} = A_{12}, \quad A_{22} = \sum_{i=1}^{N} W_i\omega_i^2, \qquad (69)$$

$$C_1 = 2 \sum_{i=1}^{N} W_i\beta_i/\omega_i \text{ and } C_2 = 2 \sum_{i=1}^{N} W_i\beta_i\omega_i.$$

In this investigation, the $\beta_i$ were assumed to be the same for all frequencies. Also, the following four different types of weighting schemes were considered.

$$\begin{aligned} &\text{Scheme 1:} \quad W_i = (\gamma_i\zeta_i(q)R_a(\omega_i)/\omega_i^2)^2 \\ &\text{Scheme 2:} \quad W_i = |\gamma_i\zeta_i(q)R_a(\omega_i)/\omega_i^2| \\ &\text{Scheme 3:} \quad W_i = |\gamma_i\zeta_i(q)| \\ &\text{Scheme 4:} \quad W_i = 1 \text{ (Equal weights)} \end{aligned} \qquad (70)$$

where $\zeta_i(q)$ = strain mode shape for element q; and $R_a(\omega_i)$ = pseudo acceleration response spectra value at frequency $\omega_i$. Thus, the 1st scheme assigns weights in proportion to the contribution of a system mode to the total square-root-of-the-sum-of-the-squares (SRSS) response. Scheme 2 and 3 also assign weights on somewhat similar bases. It was, however, observed that equal weight scheme was the one which resulted into a more uniform distribution of modal damping ratio which were somewhat similar to the ones obtained with the use of Method 2, discussed earlier. As discussed later, these two methods also gave the closest response and safety factor values when compared with the time history analysis results.

For hysteretic damping, the method of complex moduli (27,7) is probably best suited to define the damping method for an element. However, no formulation with such damping is available in which a given set of base response spectra can be directly used. (The frequency domain analysis in which either a time history or a so-called spectrum consistent spectral density function are used as input have, however, been developed (35)). Thus the concept of complex moduli was not used in this investigation.

## 3.5 Cyclic Damage and Factor of Safety Evaluations for a Stress Response Time History

Based on the concept of equivalent uniform stress time history, methods have developed to obtain factor of safety (1,24,39). These methods are equivalent to the theoretical formulation developed in Sec. 2.3 for calculation of factor of safety. However, to be completely consistent and obtain comparable results in the direct and time history approach, a more accurate and methodical procedure was adopted in this investigation to obtain the factor of safety. In this procedure, the cumulative cyclic damage is obtained by using Palmgren-Miner's hypothesis, which is then finally converted to factor of safety using Eq.

41. To obtain cumulative damage, a stress time history is scanned to locate its peaks. The damage corresponding to a peak level of $S_i$ is then defined as = $S_i^b/C$ for a S-N curve with parameters b and C. Thus total damage is equal to

$$D = \frac{1}{C} \sum_{i=1}^{N_p} (S_i^b) \tag{71}$$

in which $N_p$ is the number of peaks on the positive stress side. This is a discrete form of Eq. 26. Since in a stress time history, each positive peak may not have a negative valley, the following alternate expression may also be used

$$D = \frac{1}{2C} ( \sum_{i=1}^{N_p} S_i^b + \sum_{i=1}^{N_v} (|S_i|^b) \tag{72}$$

in which $N_v$ is the number of valleys (or peaks with negative stress). This later procedures requires scanning of a stress time history for negative peaks also. The factor of safety is then obtained using Eq. 41. Both Eqs. 71 and 72 provided almost the same value of the factor of safety in the numerical analysis of a typical stress time history.

# 4. NUMERICAL RESULTS

## 4.1 General

The numerical results have been obtained for a 10-layered strata shown in Fig. 1, with maximum shear modulus, mass density and S-N curve parameters as defined in Table 4, by nonlinear time history analysis approach and by the proposed direct approach. The hysteretic stress-strain law for the soil in various layers is assumed to have been defined by the Ramberg-Osgood relationships with $R = 3$ and $\alpha = 1$, Fig. 3. The cyclic failure rule is assumed to be described by the S-N curves shown in Fig. 2.

The bench-mark numerical results are obtained by nonlinear time history analyses of the strata for an ensemble of 54 artificially generated time histories. The mean of the maximum acceleration of these time histories was 0.1G. Program CHARSOIL, with modification, was used to obtain the nonlinear response results. The modification in CHARSOIL were made to obtain various response quantities such as stress time histories, zero crossing rate, peak response statistics, cyclic damage and factor of safety at desired locations of interest in the strata. More specifically, the maximum stress, peak response statistics, cyclic damage and factor of safety values were obtained at the center of each layer for each input time history of the earthquake motion. Thus in all 54 sets of values were obtained for the 54 time histories of the ensemble. These values were then processed to obtain their mean and coefficient of variation values. The average results are compared with the similar results obtained by the direct approach. To further examine the validity of the proposed direct approach for higher excitation levels, similar set of results are also obtained by time history analyses and direct approach for excitation levels of 0.2g and 0.4g.

For the direct approach, the seismic input consistent with the 54 time histories used in the nonlinear approach, is defined in terms of the psuedo acceleration, relative velocity and relative acceleration response spectra curves. These curves are shown in Figs. 8-10 for average maximum acceleration level of 0.1g. The input curves for acceleration level of 0.2G and 0.4G are obtained by appropriate scaling.

The strain dependent shear modulus and damping ratio curves, used in the direct approach are shown in Figs. 4. These correspond to the Ramberg-Osgood relationships with $R = 3$, and $\alpha = 1$. Eqs. 49 and 54 which define these curves were used in the direct approach.

Not knowing which procedure one should use to obtain the correct damping matrix for a finite element in the direct approach, the results with three methods, as described in Sec. 3.4, are obtained and compared. When Method 1 of Sec. 3.4 is used to construct an elemental damping matrix, the corresponding results are designated as "direct approach-1". Likewise the results for the other two methods are designated as "direct approach-2" and "direct approach-3".

In the following sections, various response results obtained by the direct approaches are evaluated vis-a-vis the bench mark results obtained by the time history analyses.

## 4.2 Maximum Stress Response

Fig. 13 shows the maximum stresses obtained at the centers of the layers by the nonlinear analysis and by the direct approach; the results for all three methods of damping matrix construction used in the direct approach are shown in the figure. The nonlinear analysis curve in the figure represents the average of the maximum stresses obtained for the ensemble of input earthquake motions.

Fig. 13 shows that the results obtained by the direct approach closely follow the stress variation trend obtained in the nonlinear approach. Among the three direct approaches, the best results (when compared with the nonlinear analysis results) are obtained when the elemental damping matrices are formed by the least squares procedures, that is, Method 3, Sec. 3.4. The average difference in the magnitudes of the stresses obtained by direct approach-3 and by the nonlinear approach is about 8%, with the direct approach giving higher values. The percent difference is largest at top. However, it is well recognized that very near the surface the stress and safety predictions by any currently available method are always uncertain and probably are also of little practical significance. The stress difference at other depths, though not large and crucial, can be attributed to: (1) inadequate modeling of energy dissipation characteristics in the analysis; (2) choice of peak factor values used in the analysis; (3) discretization error in the finite element formulation used in the direct approach; and (4) certain soil layer property averaging procedures used in the nonlinear approach (CHARSOIL).

Out of these factors, the energy dissipation modeling procedure is probably the most important. Inadequate modeling of energy dissipation could be due to two reasons: (1) representation of hysteretic behavior by strain dependent damping ratio, (2) the method of formation of an element damping matrix. The effect of the second factor seems much more dominant, as can be seen from Fig. 13; the results are seen to depend upon the method of formation of damping matrix. Although the least square procedure used in Method 3 for this purpose seems to improve the results, the approach itself is arbitrary. Other (probably more rational) methods are available to construct damping matrix (7,27) but they were not implemented here because no direct formulation (the procedure in which input spectra can be used directly) is available which can be used with such damping matrices. It is, therefore, desirable to develop the direct approach further so that these later developments in the definition of damping matrices can also be incorporated.

Figs. 14 and 15 show similar results for higher level of excitations. Comparison of the results obtained by the direct approach and nonlinear approach again seem to establish the validity of the direct approach for higher levels of earthquake excitations. Here again direct approach 3 provides a better estimate of maximum stresses. As the level of excitation increases, the nonlinear soil behavior becomes more predominant. The proposed equivalent linear method, however, seem to handle this nonlinearity rather well. The only effect of this increased nonlinearity seems to be that the direct method-3 now predicts somewhat smaller values of maximum stress near the base. Whether it will do so for all depths of strata is not verified in this investigation, but this small order unconservative prediction of maximum stress near base should be of little practical significance in stability evaluations of such strata.

### 4.3 Factor of Safety Evaluation

The factor of safety values obtained by the direct approaches and nonlinear analysis are shown in Fig. 16 for the mean excitation level of 0.1G. The values in the direct approach are obtained for an equivalent earthquake duration of 2 secs. which is half of the strong motion phase. (This number has been arbitrarily chosen, in absence of a better rational basis. As discussed later, a further inspection of the time history results seem to indicate that the equivalent effective duration is about 1.25 secs. for the earthquake motion considered in this investigation.) It is seen that the factor of safety obtained by the direct approach are consistently lower (a conservative prediction) than those obtained by the time history approach. Also the direct approach-3 in which the damping matrix is formed by the least squares procedure gives a better estimate of factor of safety when compared with the time history results.

The difference in the safety factor values predicted by the direct and time history approach can again be attributed to the factors discussed in Sec. 4.2 which affected the maximum stress. The inadequate modeling of energy dissipation mechanism by damping matrix, besides affecting the stress response, also affects the peak response characteristics, such as band width parameter $\bar{\alpha}$ and thus also the factor of safety values obtained by Eq. 28. Another factor which affects the factor of safety values is the equivalent stationary duration T, as discussed further in Sec. 4.5.

Similar comparisons of the factor of safety values for higher excitation levels are made in Figs. 17 and 18 . Again it is seen that the direct approaches provide a conservative estimate of seismic stability. Also, the direct approach-3 is better than the other two approaches considered in this investigation.

### 4.4 Peak Factor in Direct Approach

A suitable value of peak factor, F, is required in the direct approach to obtain the mean square value of a response quantity from its maximum value which in turn is obtained by the response spectrum analysis. See Eqs. 21-23, 34 and 35. The value of F used in this analysis was obtained as described in Sec. 3.2. A value corresponding to the fundamental frequency of the layered media was used. The effect of higher frequencies on this value was thus ignored, though it could have been included by using several frequency and modal damping dependent peak factors, as in Eqs. 18 and 19.

To see what variations in the calculated maximum stress and factor of safety values will be caused if any other value of peak factor is used, a limited parametric study has been conducted. The results are shown in Figs. 19 and 20 for maximum stress and Figs. 21 and 22 for factor of safety for excitation levels of 0.1G and 0.2G. It is seen that the maximum stress as well as factor of safety predictions are indeed affected by the peak factor value in the direct approach. Thus, use of a correct value is rather important in the prediction of response and safety by the direct appraoch. With time history analysis results as bench-mark the chosen value of 2.0 for the peak factor in this investigation seems to be the most appropriate value.

Various analytical procedures are available to obtain peak factor values (19,51). Whether these methods could provide a consistent value for the analysis performed here, was not investigated further. Since the main purpose of this investigation was to validate the proposed direct approach by comparing its results with the results of a truly nonlinear analysis approach, it was important to choose a value consistent with the seismic input used here to obtain comparable results. This consistency was achieved as discussed in Sec. 3.2. However, since a correct definition of this factor is important in the direct approach, some further research effort to obtain this factor accurately for nonstationary earthquake excitations seems necessary.

### 4.5 Peak Response Statistics and Equivalent Stationary Earthquake Duration

Often a simplified formula proposed by Hou (13) has been used for calculation of the equivalent stationary duration, $\bar{\bar{T}}$ (10,11). According to this formula, the equivalent stationary duration for a nonstationary input motion is approximately defined as the duration of the strong motion phase plus 25% of the remaining earthquake duration. To obtain the equivalent stationary duration for a response time history, a factor which depends upon the damping value has also been suggested as follows:

$$C_r = (23\beta - 180.76\beta^2 + 472.8\beta^3) \quad (73)$$

These guidelines were initially used in this investigation to obtain cyclic damage and factor of safety. It was, however, observed that this provided a very conservative estimate of factor of safety. In fact even a value of 2 secs. for equivalent duration provided a conservative estimate of factor of safety in the direct approach, Figs. 16-18. Since Eq. 28 adopted in the direct approach uses the peak response statistics such as number of peaks/sec., M, and bandwidth parameter, $\bar{\alpha}$, it is of interest to compare these quantities obtained in the two procedures - direct approach and nonlinear time history analysis approach - to resolve the descrepancy in the peak factor results.

The number of peaks, zero crossing, and bandwidth parameter $\bar{\alpha}$ were then obtained for stress time histories of each layer for each of the 54 input earthquake motion time histories. The calculated values were further processed to obtain mean and c.o.v. of these quantities. The mean values are compared with the corresponding values obtained in the direct approach.

Figs. 23 to 28 show the number of peaks/sec. and bandwidth parameter in various layers of the strata obtained by the three direct approaches for the acceleration levels of 0.1, 0.2 and 0.4g. Also shown are the average values obtained in the time history analyses. Again direct approach-3 seems to provide a better prediction, though some differences at higher level excitation, Fig. 25, are noted.

The results obtained in the direct approach are based on stationary formulation, but their comparison with (nonstationary) time history results seems to indicate that the number of peak counts, zero crossings and thus band width parameter are not affected by nonstationarity. This seems plausible. The nonstationarity, however, must strongly affect the probability density function of the peaks; that is, the density function based on stationary assumption, Eq. 27, must have higher density values at higher peak magnitudes than what would be obtained from a realistic nonstationary response time history. Thus the use of Eq. 27 in Eq. 28 in the direct approach will give a higher value of damage and lower value of factor of safety.

Since the expression for the probability density function for peaks for nonstationary earthquake response is not available, it is not clear as to how this conservatism in the direct approach can be removed. However, for the average maximum stress, number of peaks/unit time and the bandwidth parameter obtained for each layer in the time history analyses, the factor of safety values were calculated using Eqs. 28 of the direct approach for different values of $\bar{\bar{T}}$. These values were then compared with the factor of safety values obtained directly in the time history approach. It was found that if $\bar{T}$ in Eq. is taken as 1.25 secs., the two factor of safety values compared very well. Thus it appears that effective equivalent duration of our earthquake motions for the purpose of calculation of factor of safety by Eq. 28 should have been 1.25 secs.

Using this new value of equivalent duration, the factor of safety values are calculated again for the three excitations levels of 0.1G, 0.2G and 0.4G. These are shown in Figs. 29, 30 and 31. In the figures are also shown the values obtained by the time history analysis and the previous values for T = 2 secs. As direct approach-3 provides a better prediction of overall response, the new values only for this approach are shown. This reduction in the equivalent duration further narrows the difference between the factor of safety values obtained by the direct and time history analysis approaches. The remaining difference which is, however, not very large, can be attributed to the factors mentioned in Sec. 4.2.

### 4.6 A Constant Factor Direct Approach

In the equivalent linear time history approach (17,36,37), the strain dependent shear modulus and damping ratio for a finite element or layer are obtained for a level of strain equal to a fraction of the maximum strain. This fraction, herein called as strain ratio, is usually taken as 0.65. In the proposed direct approach also, this procedure of choosing shear modulus and damping ratio in an iteration can be used with significant computational simplifications if the right value of this strain ratio is known. Figs. 32 and 33 show the maximum stress and factor of safety values obtained for a few selected values of the strain ratio ranging from 0.65 to 0.75. The results are seen to depend upon the value of this ratio. In absence of a known value for this ratio, the stochastic linearization technique described in the previous sections should be used.

## 4.7 Variability of Time History Response

No calculations were made to obtain variability of the response by the direct approach. Though it could be done for the maximum stress, there are some analytical difficulties in its evaluation for factor of safety, number of peaks and bandwidth parameter. However, the results of simulation study provided this information. Table 6 shows the coefficient of variation values obtained for the maximum stress, factor of safety, number of peaks and bandwidth parameter obtained for the time history results. Of special significance is the variability of factor of safety which is of the order of 5 percent. This seems to verify the claim made earlier in Sec. 2.3 that for soil system the variability of damage and factor of safety is expected to be small. Thus, the expected value of damage and the factor of safety do provide a good prediction of cyclic damage without much uncertainty.

# 5. Summary & Conclusions

## 5.1 Summary

For seismic stability evaluation of earth structures a direct approach is presented in which seismic design inputs defined in the form of response spectra can be directly used. The proposed approach is similar to the equivalent linear approach in many respects, but it does not require the design input in terms of earthquake motion time histories. As in the equivalent linear approach, it uses equivalent strain dependent damping and shear modulus curves which are assumed to be equivalent to the hysteretic stress strain behavior under cyclic loads. Like the equivalent linear approach, the proposed approach is also iterative in its analysis procedure; in each iteration for a given set of stiffness and damping characteristics a linear analysis is performed. The choice of shear modulus and damping values to be used in each iteration is based on a stochastic linearization procedure, though values corresponding to a strain equal to a constant fraction of the maximum strain can also be used, as is done in the equivalent linear approach.

The analytical model of a system is supposed to be defined in terms of mass, stiffness and damping matrices in this approach. This assumes the decretization of a system into finite elements.

Since it is proposed to use the input response spectra directly in an analysis, it becomes essential to use the normal mode approach. The formulation developed herein adopts the normal mode approach and thus all the response quantities like stress, strain, displacements, etc., are obtained by the (modified) square-root-of-the-sum-of-the-squares (SRSS) procedure.

A SRSS procedure has also been developed in which a nonproportional damping matrix can be used. However, for the example problem considered here, it was unnecessary to use this procedure as the conventional SRSS procedure provided very accurate results.

To find an acceptable procedure for construction of a finite element damping matrix, three somewhat related methods have been investigated for their suitability in this work. Of these three, the one based on the method of least squares is recommended.

For a typical layered soil strata, numerical results are obtained by nonlinear time history analysis for an ensemble of 54 earthquake motions. For comparison and evaluation of the proposed approach, similar results are also obtained by this approach for a consistent set of response spectra curves.

Since the results of a one-dimensional problem only are examined and compared, the following observations, in a strict sense, are applicable to such earth structure problems only. However, the proposed approach is quite general and can be used for two or three dimensional situations as well.

## 5.2 Conclusions & Recommendations

An overall comparison of the results in Chapter 4 indicates that the proposed direct approach provides a rational analytical model for a conservative seismic stability prediction of earth structures. Seismic input in the form of response spectra can be used. This makes the appraoch especially suitable for evaluation of a design.

The response results obtained in the direct approach depend upon the values of peak factor and equivalent earthquake duration used. Some analytical procedures are available to define the inherent values of peak factors built in the response spectra curves. Still some further research effort is warranted to obtain these in a simple usuable form. A correct evaluation of equivalent earthquake duration for its use in the proposed direct approach is also important. A procedure which has been available for sometime now to define equivalent earthquake duration has been found to be rather inadequate. More research effort is required to obtain an equivalent duration which would include the nonstationarity of input, response and peak response characteristics. The numerical results obtained for a typical horizontal strata analyzed here indicate that the equivalent earthquake duration should be much smaller than the strong motion phase of the input motion.

Given the correct values of peak factor and equivalent duration, the direct approach will provide a conservative estimate of response and seismic safety. That is, the calculated stresses will be a little higher and the factors of safety a little lower than the values obtained by a rigourous nonlinear approach.

The definition of a proper damping matrix in the dynamic model of a system has always been elusive. In this investigation also, the need for a continued development to define a better energy dissipation model in a system by damping matrix is identified.

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Table 1: Parameters of Spectral Density Function, $\Phi_g(\omega)$, Eq. 43

| $i$ | $S_i$ ft$^2$-see/rad | $\omega_i$ rad/sec | $\beta_i$ |
|---|---|---|---|
| 1 | .0015 | 13.5 | 0.3925 |
| 2 | .000495 | 23.5 | 0.3600 |
| 3 | .000375 | 39.0 | 0.3350 |

Table 2: Standard Deviation of Relative Displacement Response of the Layered Media by State Vector and Normal Mode Approach for Excitation Spectral Density Function of Eq. 43.

RELATIVE DISPLACEMENT RESPONSE

| * I * | ROOT MEAN SQ.,SQRS STATE VEC. FT. UNIT | ROOT MEANSQ. SQRSN NORMAL MODE FT. UNIT | PERCENT ERROR |
|---|---|---|---|
| 1 | 0.14585726D+00 | 0.14588606D+00 | 0.02 |
| 2 | 0.14585724D+00 | 0.14588605D+00 | 0.02 |
| 3 | 0.13914196D+00 | 0.13916506D+00 | 0.02 |
| 4 | 0.13914195D+00 | 0.13916506D+00 | 0.02 |
| 5 | 0.12771225D+00 | 0.12772740D+00 | 0.01 |
| 6 | 0.12771223D+00 | 0.12772738D+00 | 0.01 |
| 7 | 0.11329987D+00 | 0.11330699D+00 | 0.01 |
| 8 | 0.11330002D+00 | 0.11330713D+00 | 0.01 |
| 9 | 0.97678158D-01 | 0.97679067D-01 | 0.00 |
| 10 | 0.97678121D-01 | 0.97679019D-01 | 0.00 |
| 11 | 0.81790386D-01 | 0.81787351D-01 | -0.00 |
| 12 | 0.81790373D-01 | 0.81787338D-01 | -0.00 |
| 13 | 0.65257763D-01 | 0.65252627D-01 | -0.01 |
| 14 | 0.65257728D-01 | 0.65252571D-01 | -0.01 |
| 15 | 0.48551648D-01 | 0.48546104D-01 | -0.01 |
| 16 | 0.48551663D-01 | 0.48546119D-01 | -0.01 |
| 17 | 0.31742240D-01 | 0.31737759D-01 | -0.01 |
| 18 | 0.31742280D-01 | 0.31737799D-01 | -0.01 |
| 19 | 0.15464687D-01 | 0.15462299D-01 | -0.02 |
| 20 | 0.15464661D-01 | 0.15462273D-01 | -0.02 |

Table 3: Standard Deviation of the Spring Force Response of the Layered Media by State Vector and Normal Mode Approach for Excitation Spectral Density Function of Eq. 43.

SPRING FORCE RESPONCE

| * I * | ROOT MEAN SQ.,SQRS STATE VEC. FT. UNIT | ROOT MEANSQ. SQRSN NORMAL MODE FT. UNIT | PERCENT ERROR |
|---|---|---|---|
| 1 | 0.41119778D+04 | 0.41205777D+04 | 0.21 |
| 2 | 0.41119774D+04 | 0.41205774D+04 | 0.21 |
| 3 | 0.74811827D+04 | 0.74930495D+04 | 0.16 |
| 4 | 0.74811823D+04 | 0.74930492D+04 | 0.16 |
| 5 | 0.66945033D+04 | 0.66991222D+04 | 0.07 |
| 6 | 0.66945026D+04 | 0.66991215D+04 | 0.07 |
| 7 | 0.59957284D+04 | 0.59981738D+04 | 0.04 |
| 8 | 0.59957358D+04 | 0.59981810D+04 | 0.04 |
| 9 | 0.54461769D+04 | 0.54497376D+04 | 0.07 |
| 10 | 0.54461769D+04 | 0.54497356D+04 | 0.07 |
| 11 | 0.49365762D+04 | 0.49411289D+04 | 0.09 |
| 12 | 0.49365447D+04 | 0.49410978D+04 | 0.09 |
| 13 | 0.43786430D+04 | 0.43823762D+04 | 0.09 |
| 14 | 0.43786411D+04 | 0.43823736D+04 | 0.09 |
| 15 | 0.37675512D+04 | 0.37696901D+04 | 0.06 |
| 16 | 0.37675494D+04 | 0.37696900D+04 | 0.06 |
| 17 | 0.31254260D+04 | 0.31232617D+04 | -0.07 |
| 18 | 0.31259210D+04 | 0.31237505D+04 | -0.07 |
| 19 | 0.25803489D+04 | 0.25716643D+04 | -0.34 |
| 20 | 0.25803484D+04 | 0.25716631D+04 | -0.34 |

Table 4: Shear Modulus and S-N Curve Parameters for the Layered Media Model

| Layer No. | Depth ft. | Maximum Shear Modulus ksf | S-N Curve Parameters | | |
|---|---|---|---|---|---|
| | | | b | $C^* = C/S_e^b$ | Curve No. |
| 1 | 6 | 755 | 5.454 | 0.05558 | 1 |
| 2 | 6 | 1400 | 5.454 | 0.05558 | 1 |
| 3 | 6 | 1858 | 5.924 | 0.09700 | 2 |
| 4 | 6 | 2317 | 5.924 | 0.09700 | 2 |
| 5 | 6 | 2786 | 5.924 | 0.09700 | 2 |
| 6 | 6 | 3124 | 5.924 | 0.09700 | 2 |
| 7 | 6 | 3444 | 5.924 | 0.09700 | 2 |
| 8 | 6 | 3700 | 5.764 | 0.23563 | 3 |
| 9 | 6 | 3995 | 5.764 | 0.23563 | 3 |
| 10 | 6 | 4284 | 5.764 | 0.23563 | 3 |

$S_e$ = effective overburden pressure

Table 5: Modal Damping Ratios Obtained for System Damping Matrix Constructed by Various Methods, Sec. 3.4. for Acceleration Level of 0.1G.

| Mode No. | Damping Ratio | | |
|---|---|---|---|
| | Method 1 | Method 2 | Method 3 |
| 1 | .0655 | .0618 | .0789 |
| 2 | .0964 | .0319 | .0406 |
| 3 | .1337 | .0278 | .0350 |
| 4 | .1596 | .0266 | .0333 |
| 5 | .1917 | .0283 | .0354 |
| 6 | .2496 | .0337 | .0422 |
| 7 | .1438 | .0204 | .0248 |
| 8 | .3123 | .0395 | .0495 |
| 9 | .2187 | .0282 | .0346 |
| 10 | .3718 | .0445 | .0559 |
| 11 | .3444 | .0412 | .0513 |
| 12 | .4275 | .0492 | .0618 |
| 13 | .4293 | .0494 | .0616 |
| 14 | .4765 | .0536 | .0672 |
| 15 | .4894 | .0550 | .0688 |
| 16 | .5312 | .0588 | .0736 |
| 17 | .5710 | .0624 | .0780 |
| 18 | .6170 | .0666 | .0833 |
| 19 | .6666 | .0712 | .0892 |
| 20 | .7082 | .0753 | .0941 |

Table 6: Coefficient of Variation of Maximum Stress, Factor of Safety, No. of Peaks per Second and Bandwidth Parameter Obtained in the Nonlinear Approach for Acceleration Level of 0.1G.

| Layer No. | Depth | Coefficient of Variation | | | |
|---|---|---|---|---|---|
| | | Maximum Stress | Factor of Safety | No. of Peaks/ Sec. | Bandwidth Parameter |
| 1 | 3 | .08 | .04 | .09 | .10 |
| 2 | 9 | .08 | .04 | .10 | .08 |
| 3 | 15 | .08 | .05 | .13 | .09 |
| 4 | 21 | .08 | .05 | .13 | .11 |
| 5 | 27 | .08 | .05 | .16 | .12 |
| 6 | 33 | .08 | .05 | .16 | .12 |
| 7 | 39 | .08 | .05 | .12 | .11 |
| 8 | 45 | .08 | .05 | .14 | .12 |
| 9 | 51 | .07 | .05 | .12 | .11 |
| 10 | 57 | .07 | .05 | .13 | .11 |

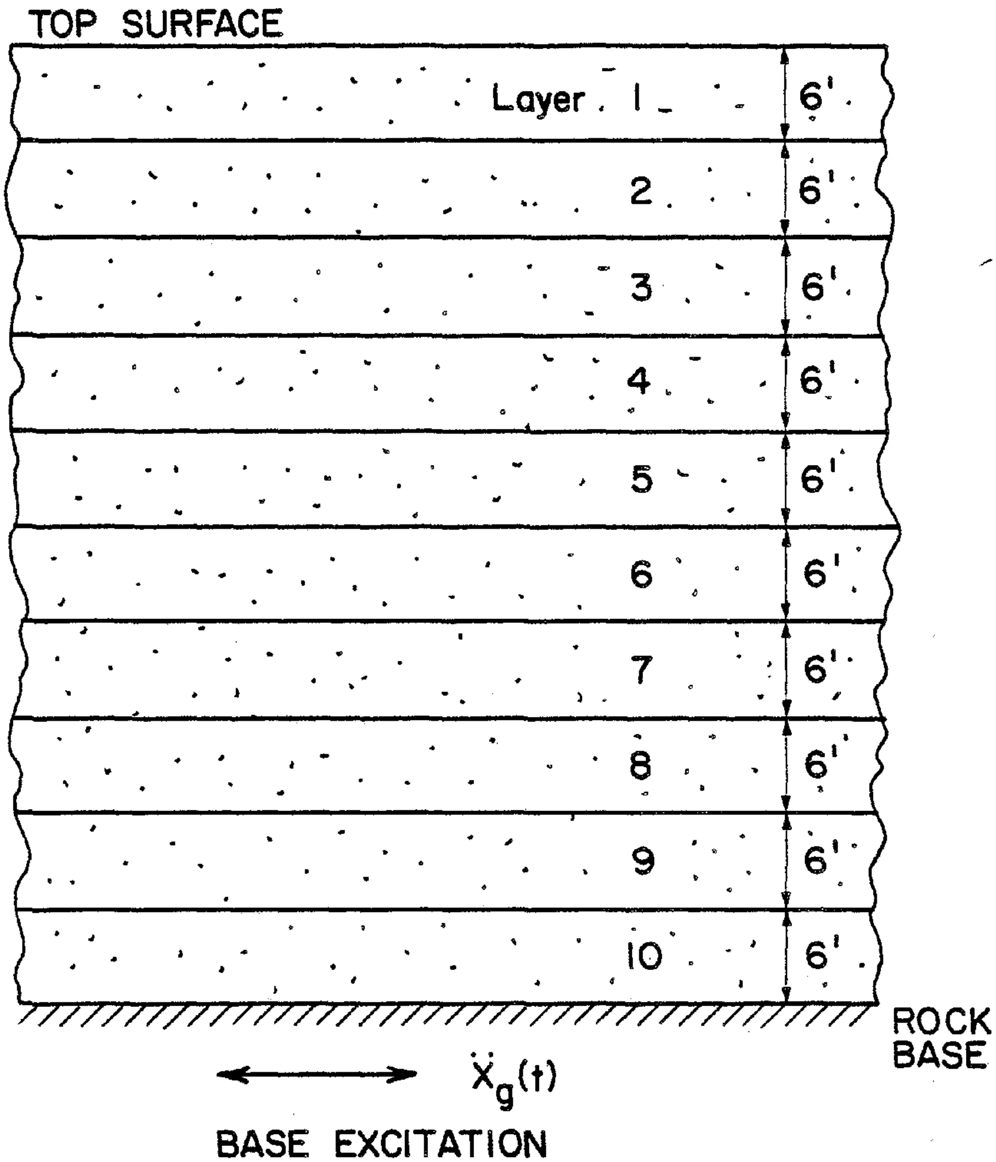


FIG. 1 TEN LAYER SOIL STRATA

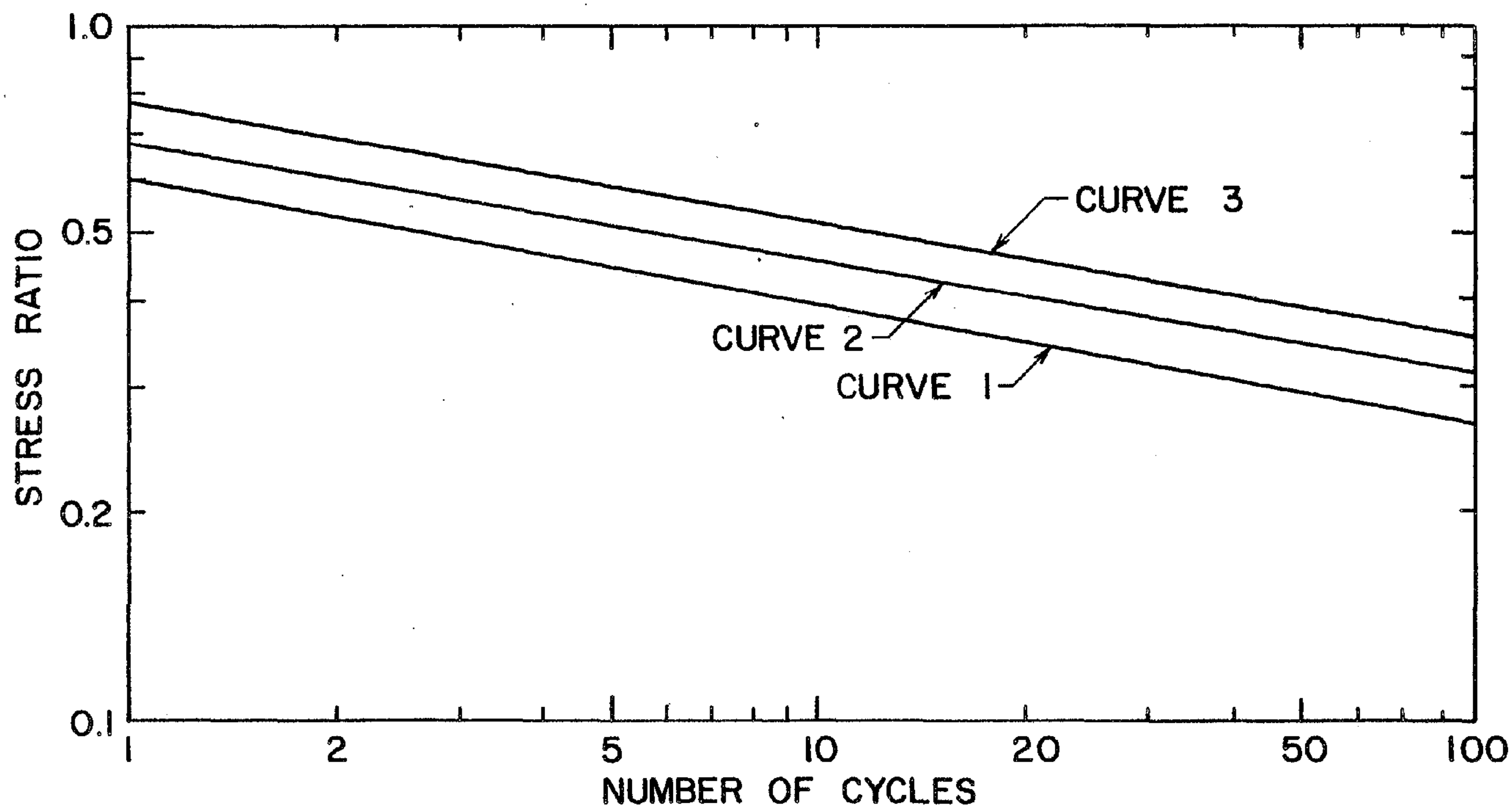


FIG. 2 S-N CURVES FOR VARIOUS LAYERS OF FIG. 1

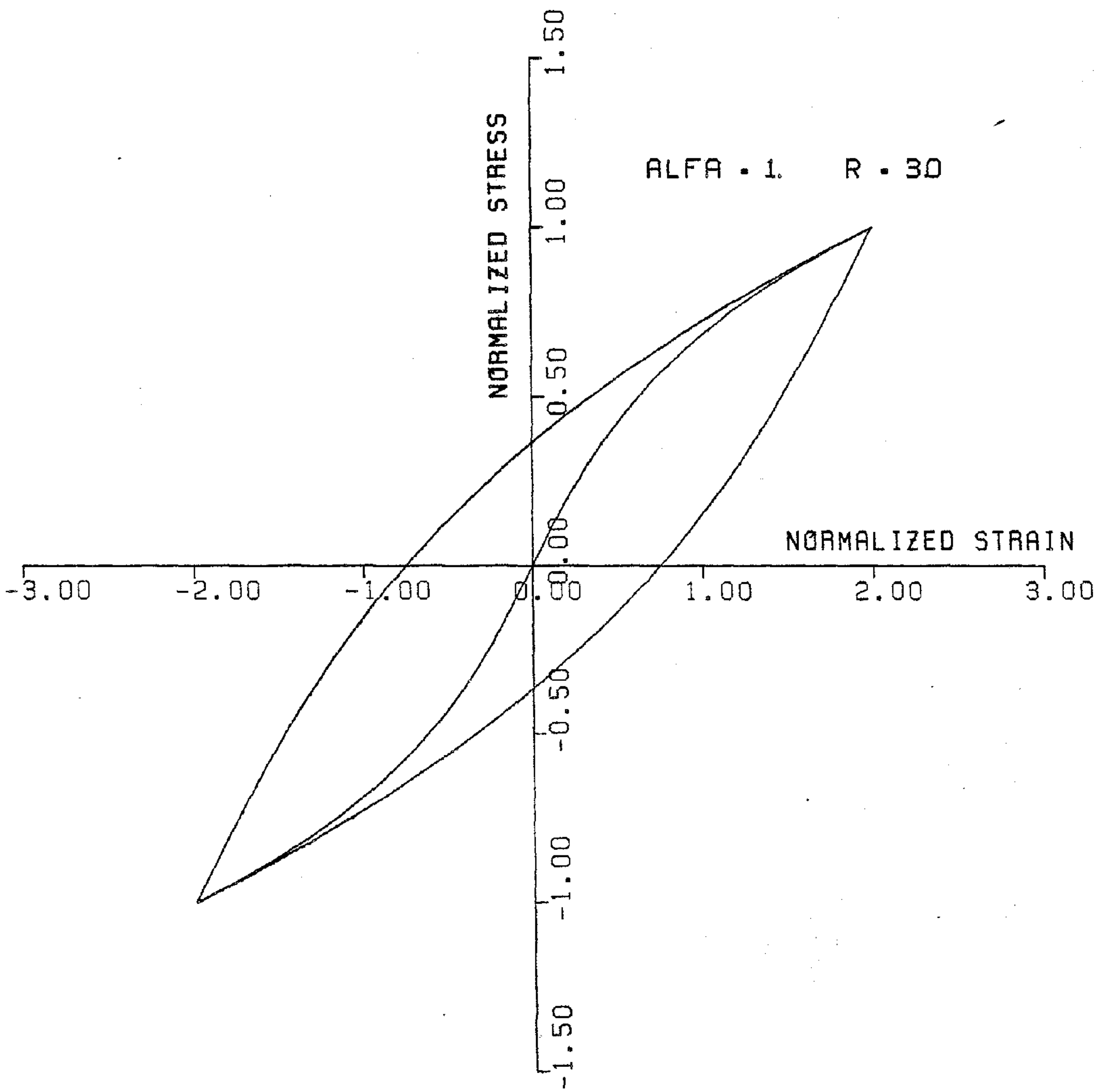


FIG. 3 RAMBER OSGOOD STRESS - STRAIN MODEL

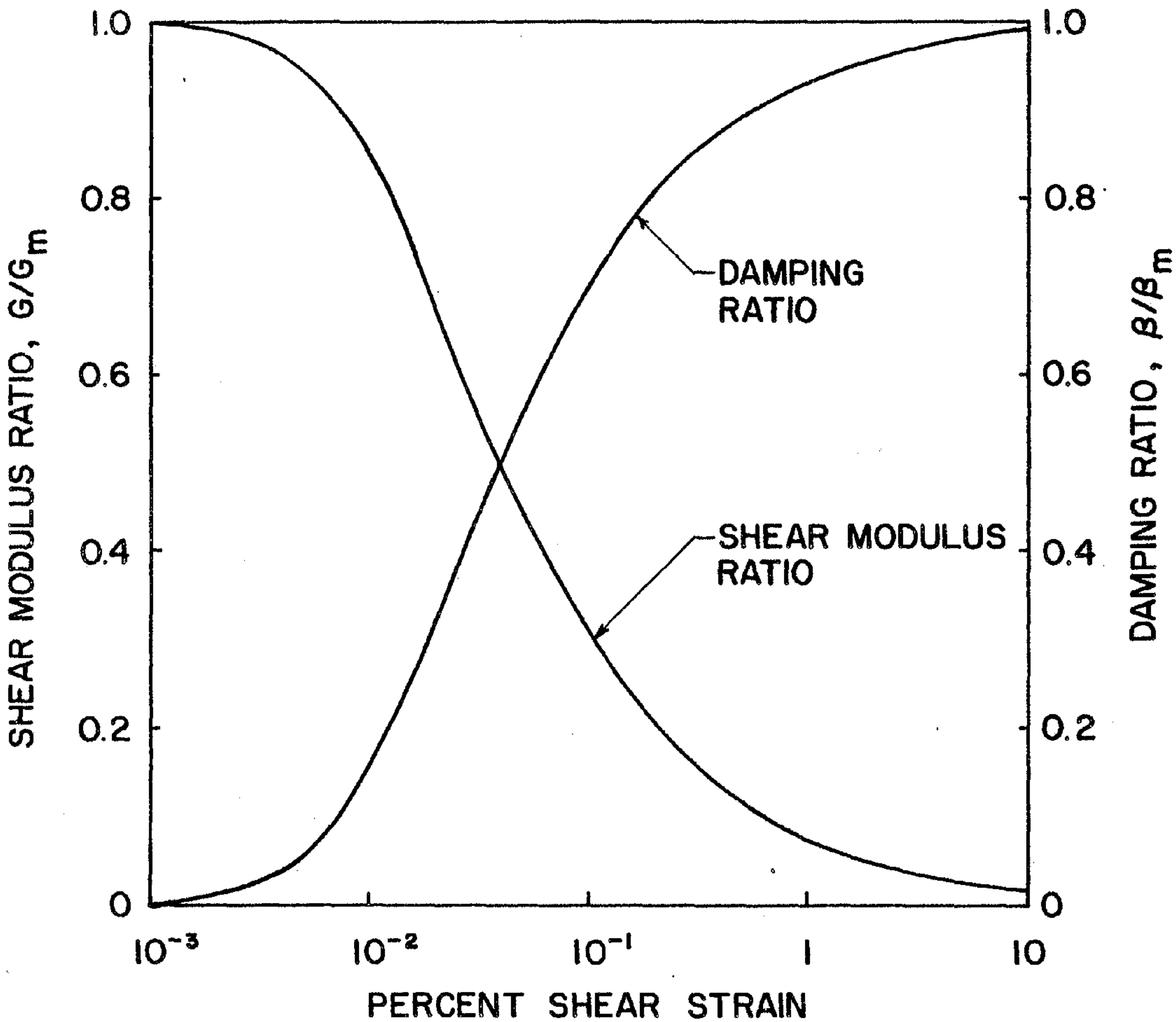


FIG. 4 STRAIN DEPENDENT EQUIVALENT SHEAR MODULUS AND DAMPING RATIOS

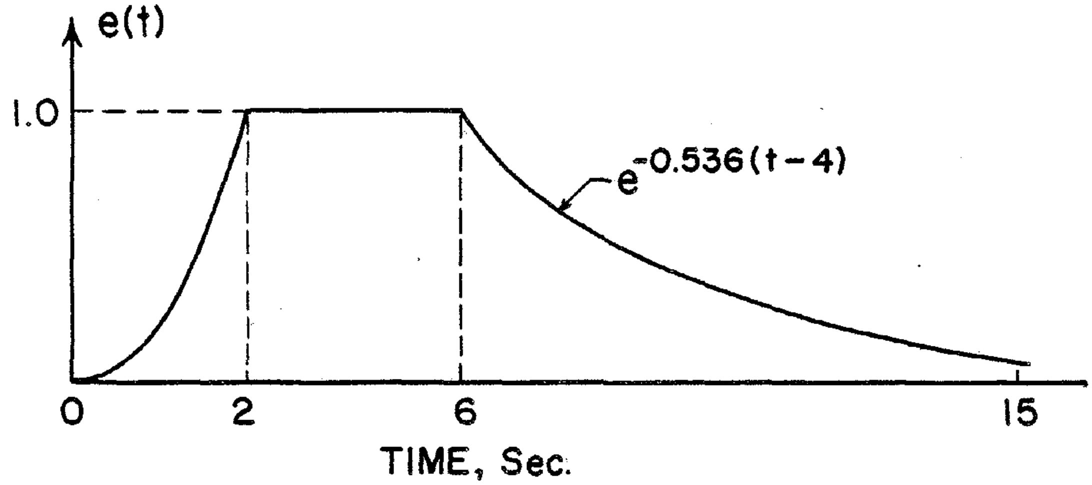


FIG. 5 INTENSITY ENVELOPE FUNCTION

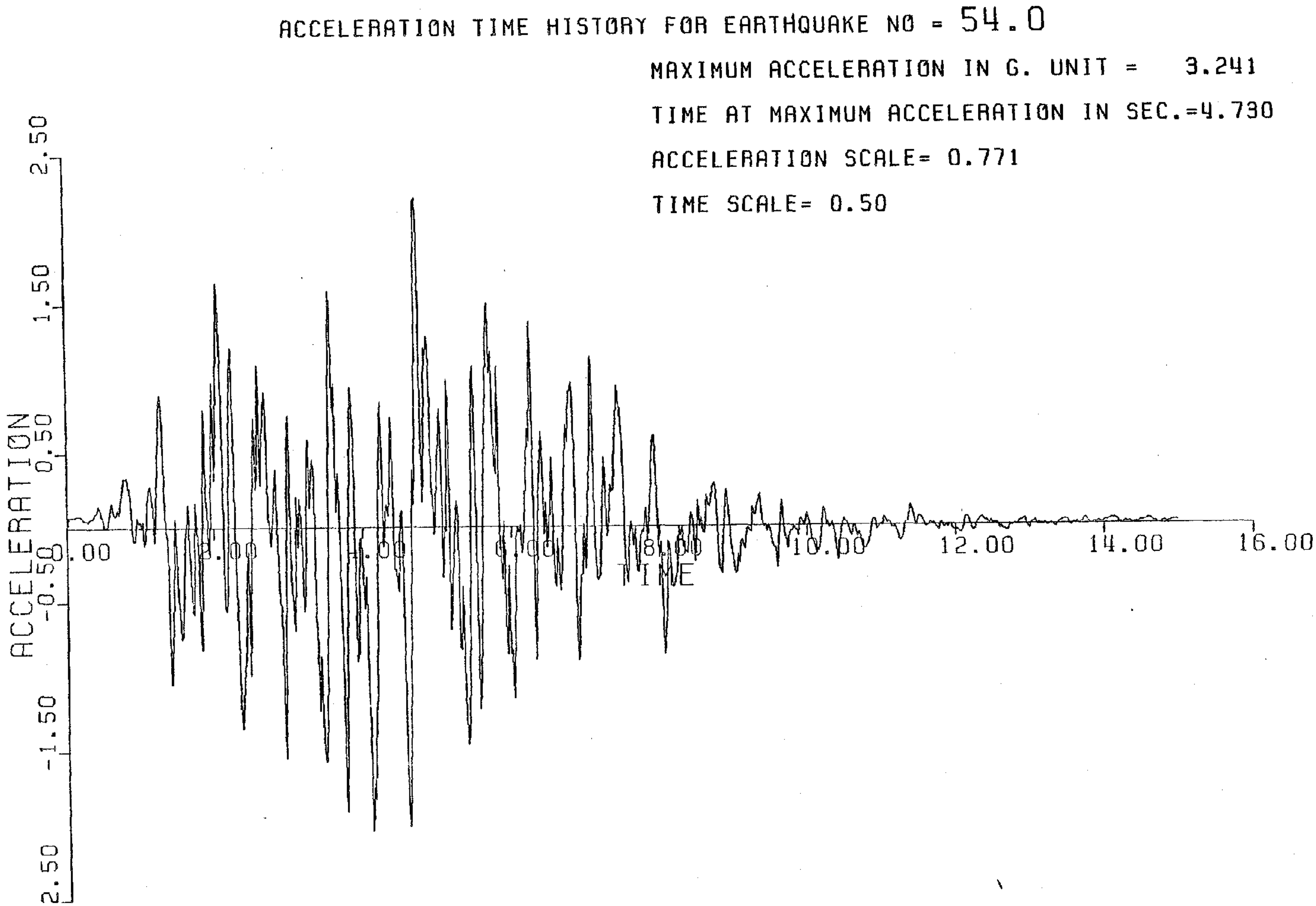


FIG. 6 A TYPICAL SYNTHETIC ACCELERATION TIME HISTORY

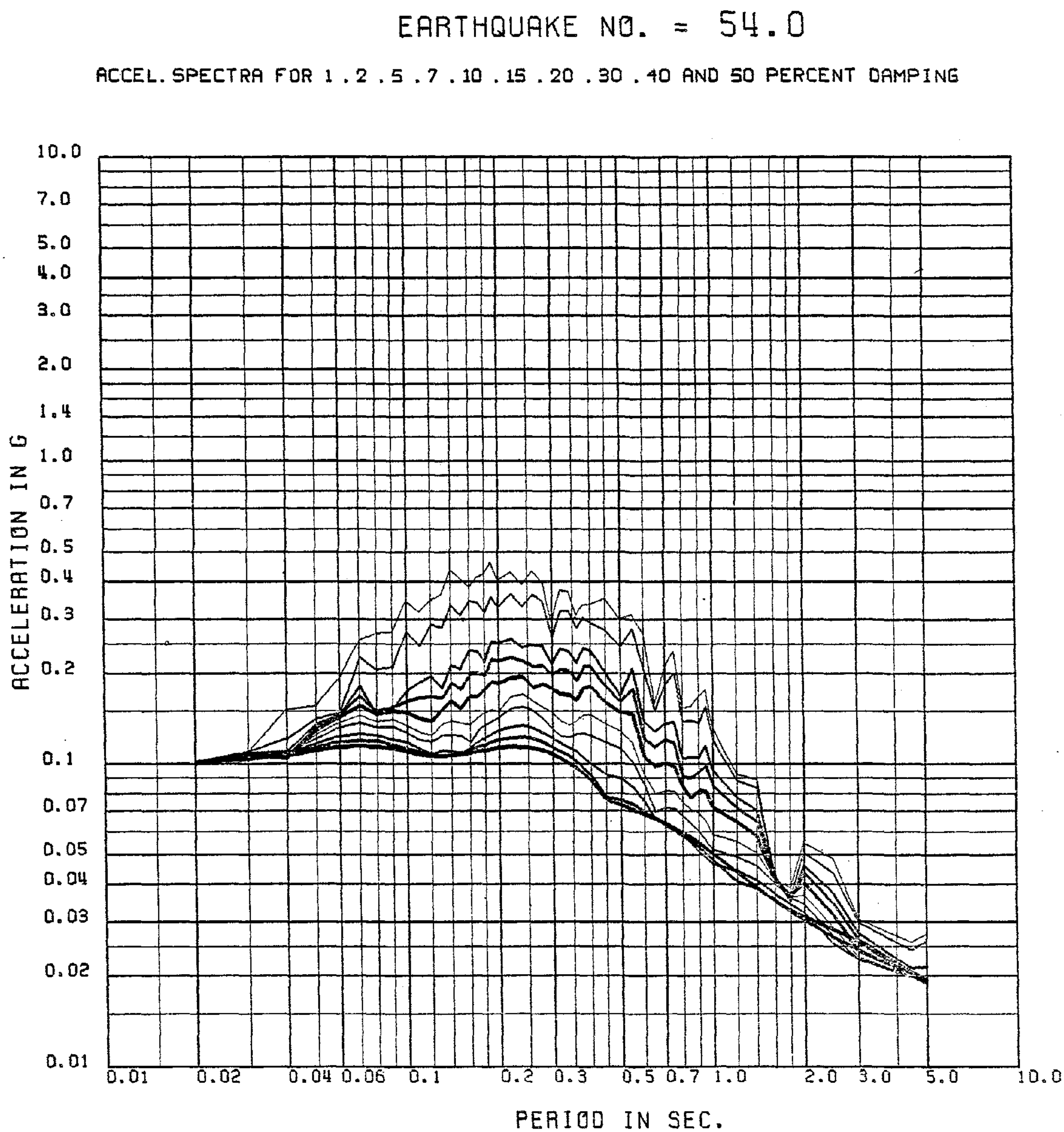


FIG. 7 ACCELERATION RESPONSE SPECTRA FOR TIME HISTORY IN FIG. 6

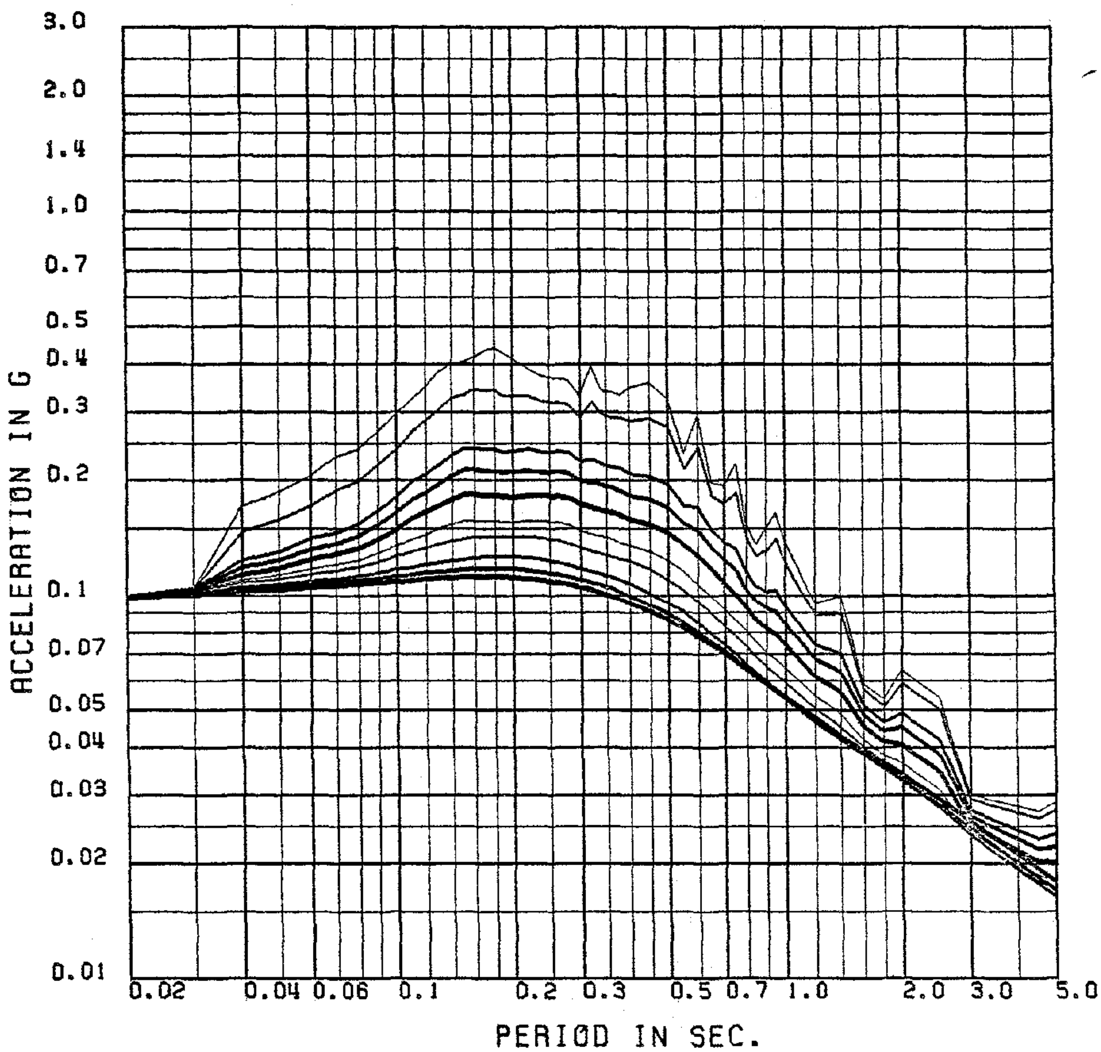


FIG. 8 - MEAN ACCLERATION SPECTRA FOR 1 . 2 . 3 . 4 . 5 . 7 . 10 . 15 . 20 . 30 . 40 AND 50 PERCENT DAMPING FOR 54 EARTHQUAKES .

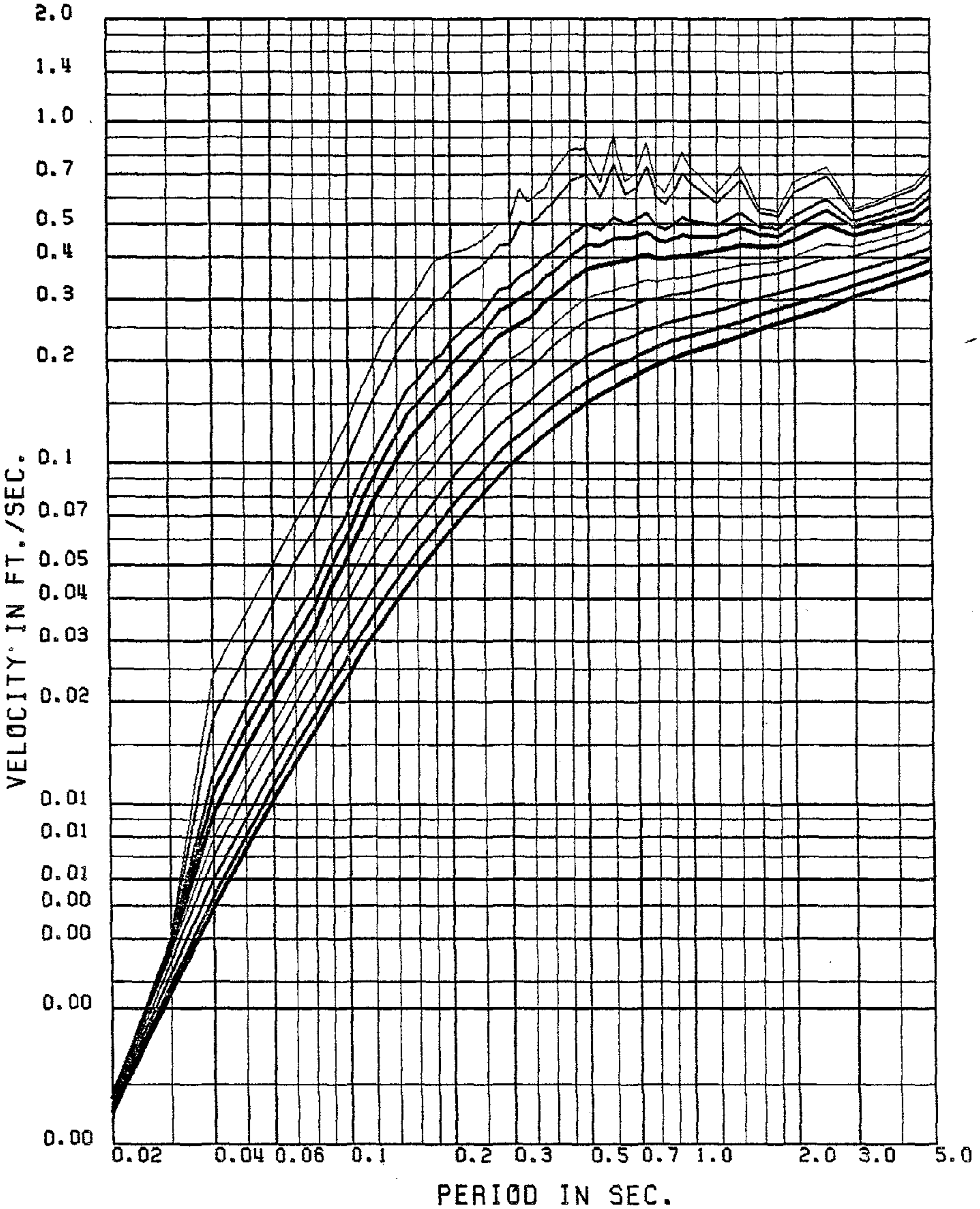


FIG. 9 - MEAN REL. VELOC. SPECTRA FOR 1 . 2 . 3 . 4 . 5 . 7 . 10 . 15 . 15 . 20 . 30 . 40 AND 50 PERCENT DAMPING FOR 54 EARTHQUAKES .

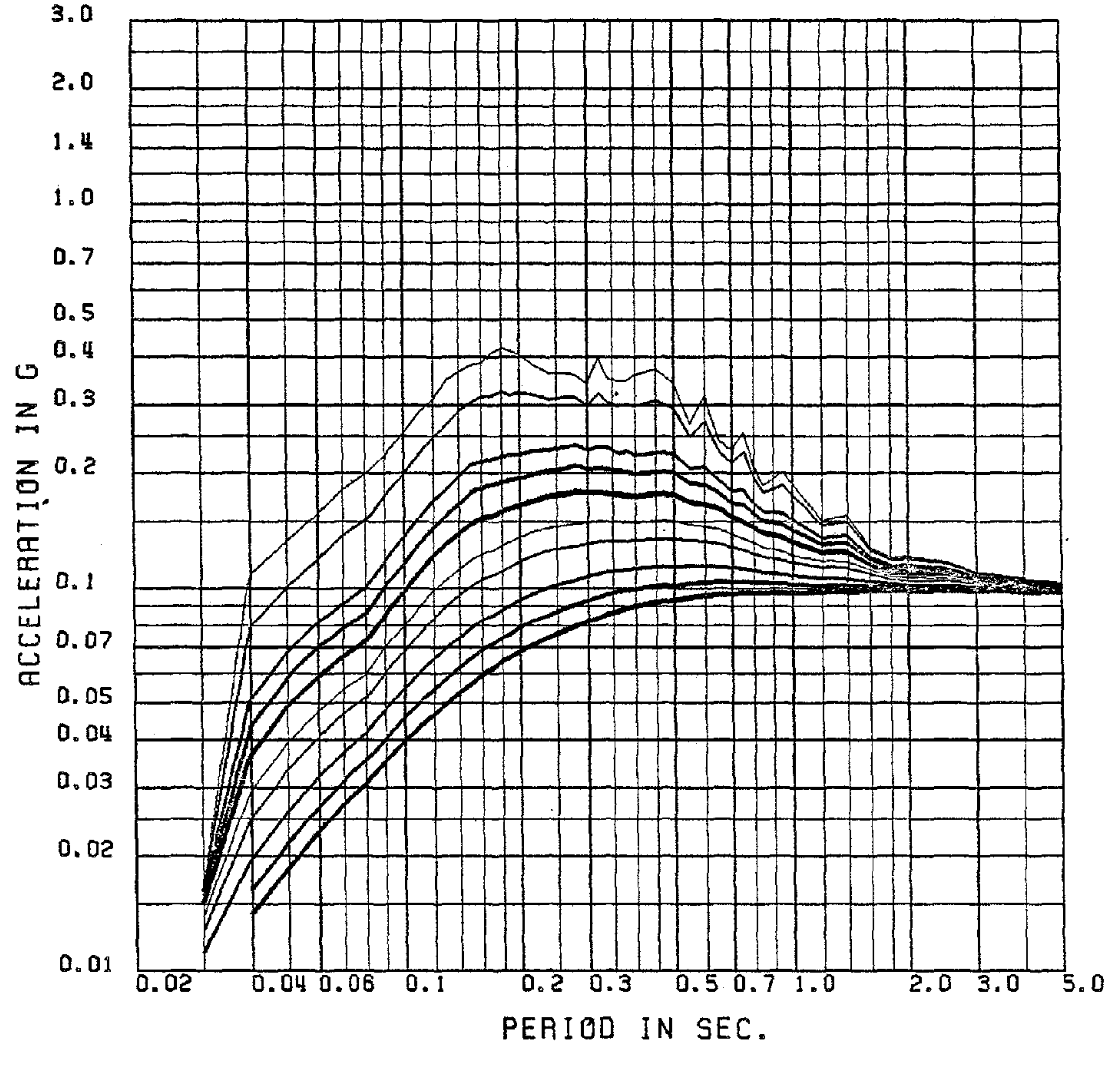


FIG. 10 - MEAN REL. ACCL. SPECTRA FOR 1 . 2 . 3 . 4 . 5 . 7 . 10 . 15 . 20 . 30 . 40 AND 50 PERCENT DAMPING FOR 54 EARTHQUAKES .

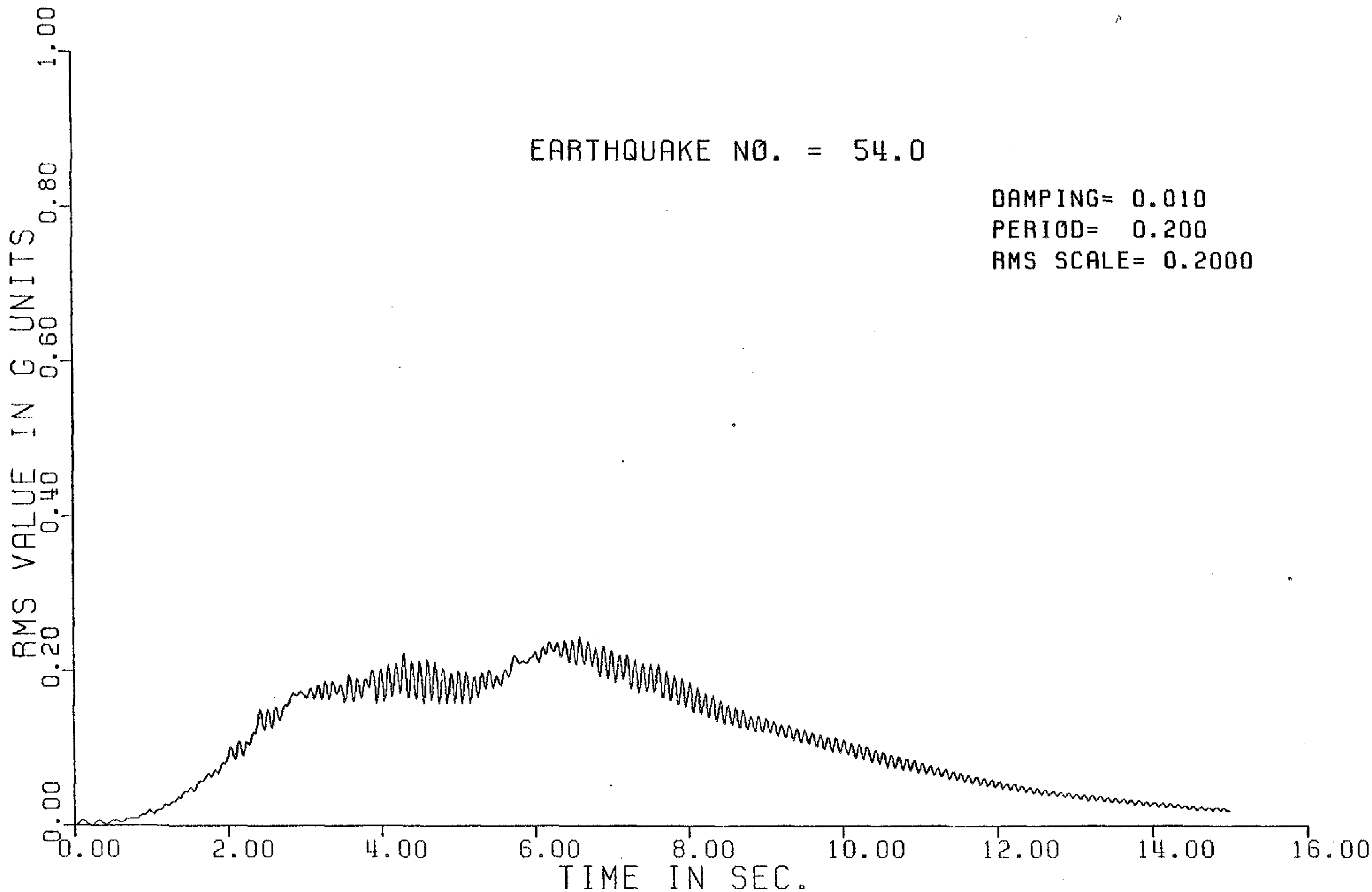


FIG. 11 ROOT MEAN SQUARE ACCELERATION RESPONSE TIME HISTORY FOR OSCILLATOR PERIOD OF 0.2 SECS. AND DAMPING RATIO = .01 (AVERAGE OF 54 SIMULATIONS)

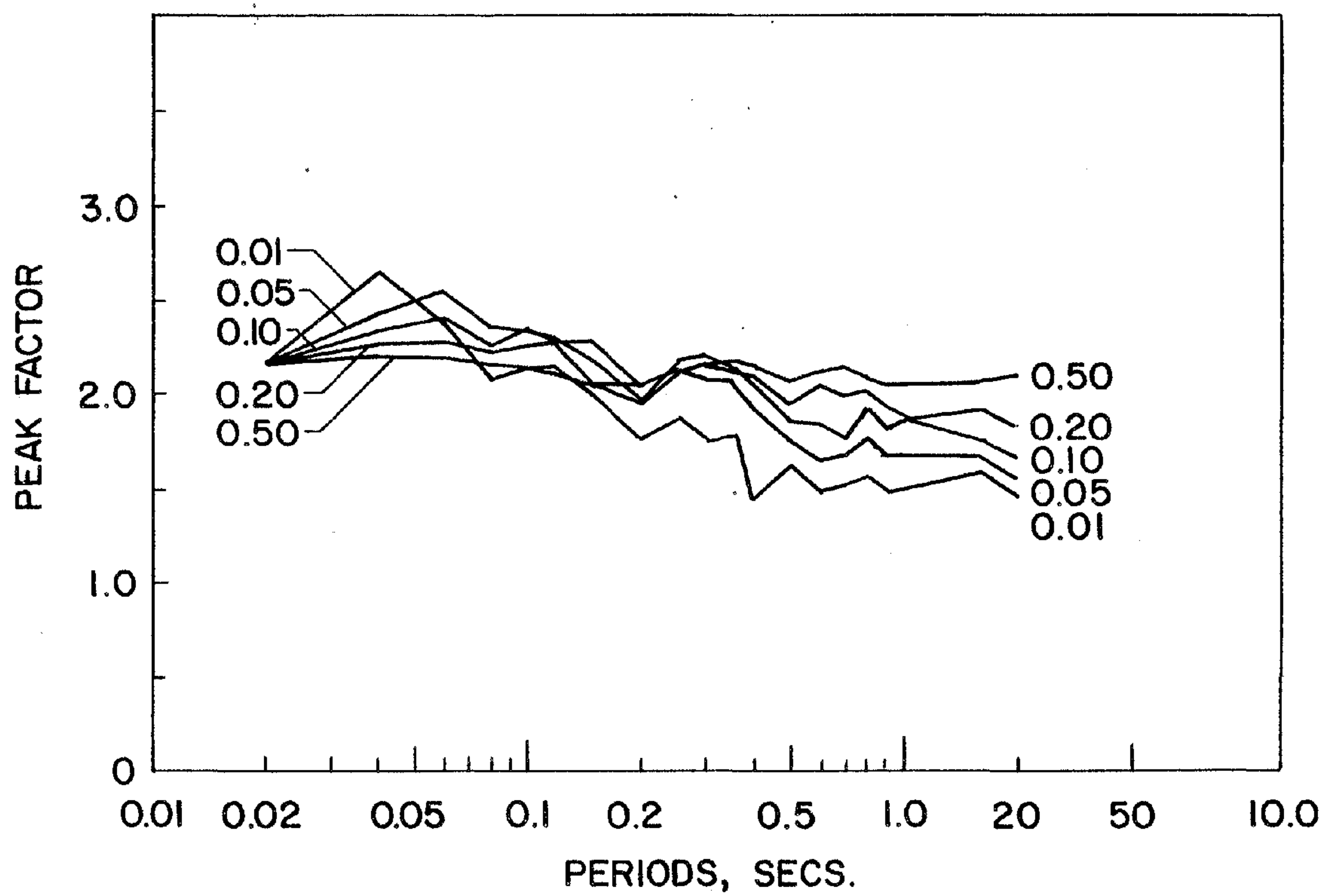


FIG. 12 VARIATION OF PEAK FACTOR WITH OSCILLATOR DAMPING AND PERIOD (BASED ON 54 ARTIFICIAL TIME HISTORIES)

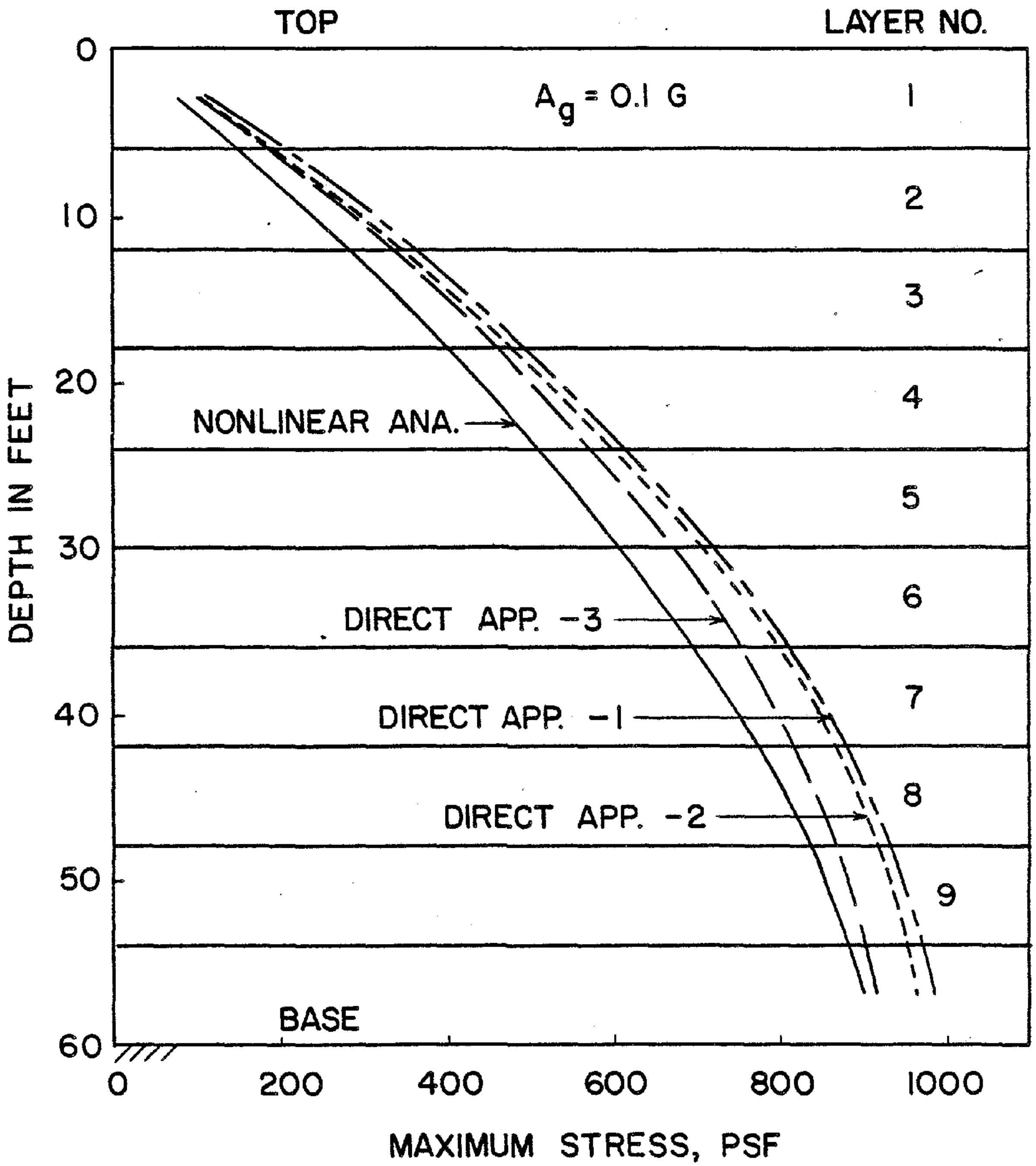


FIG. 13 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.1G

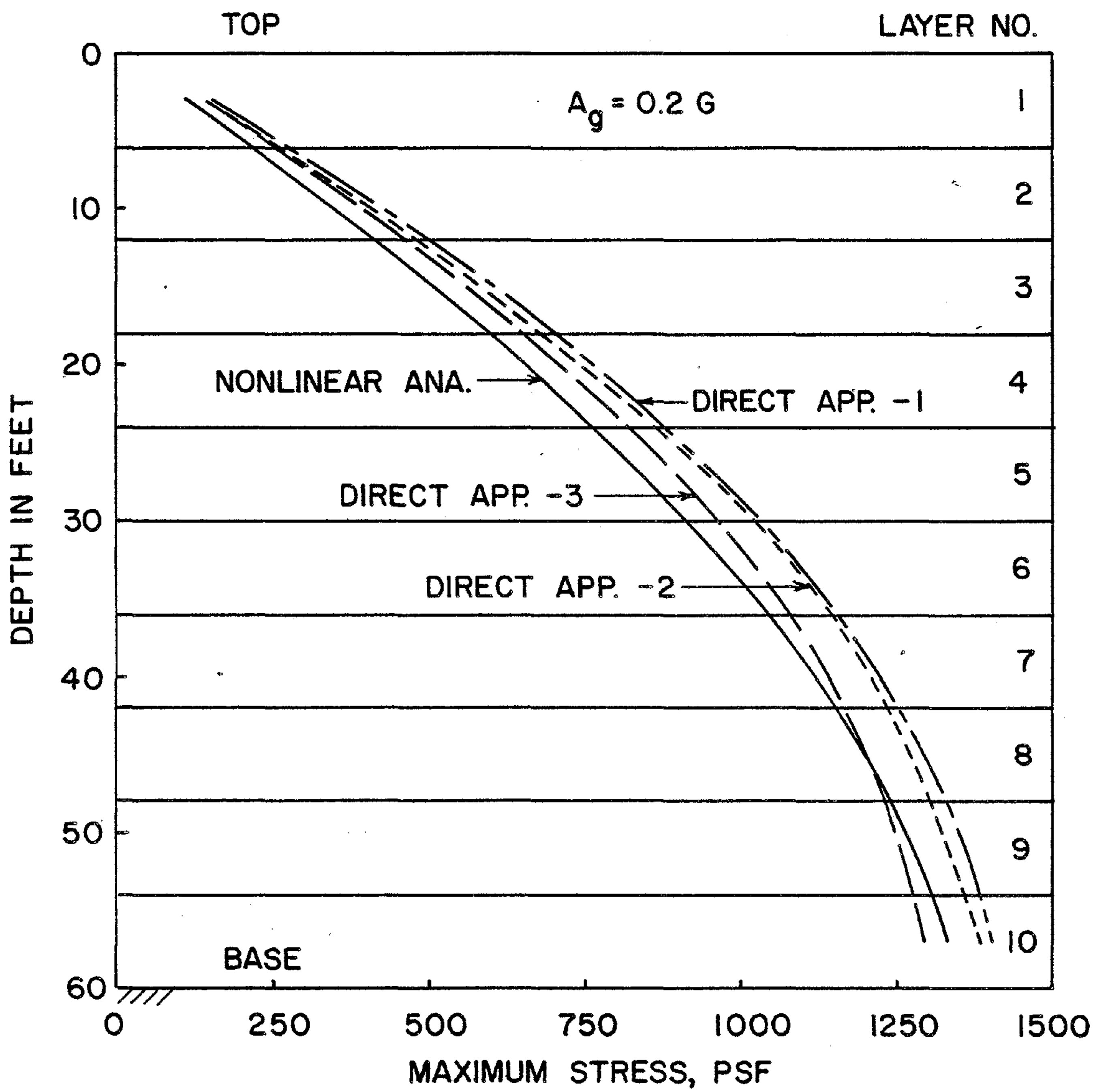


FIG. 14 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.2G

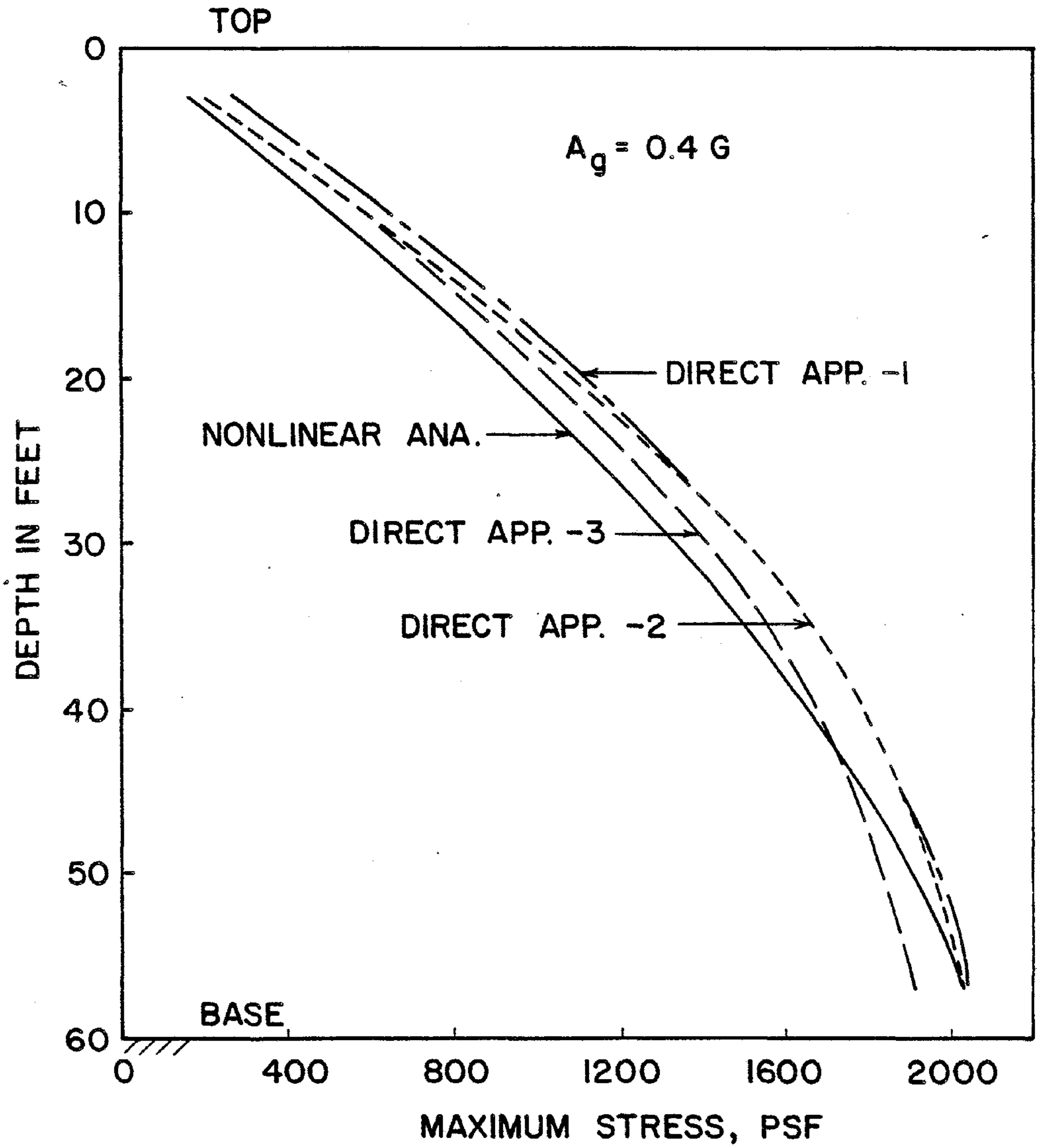


FIG. 15 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.4G

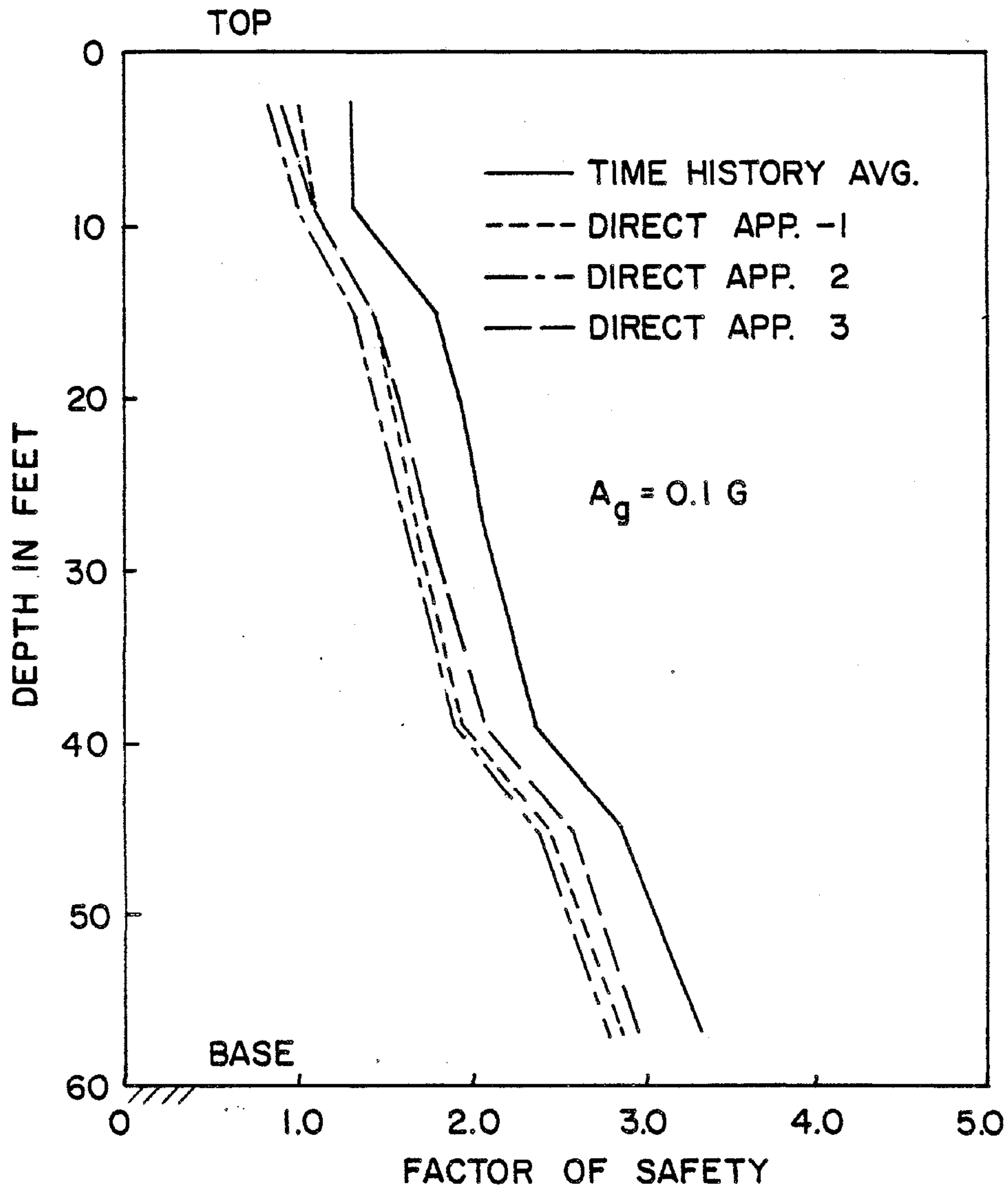


FIG. 16 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSIS (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.1G

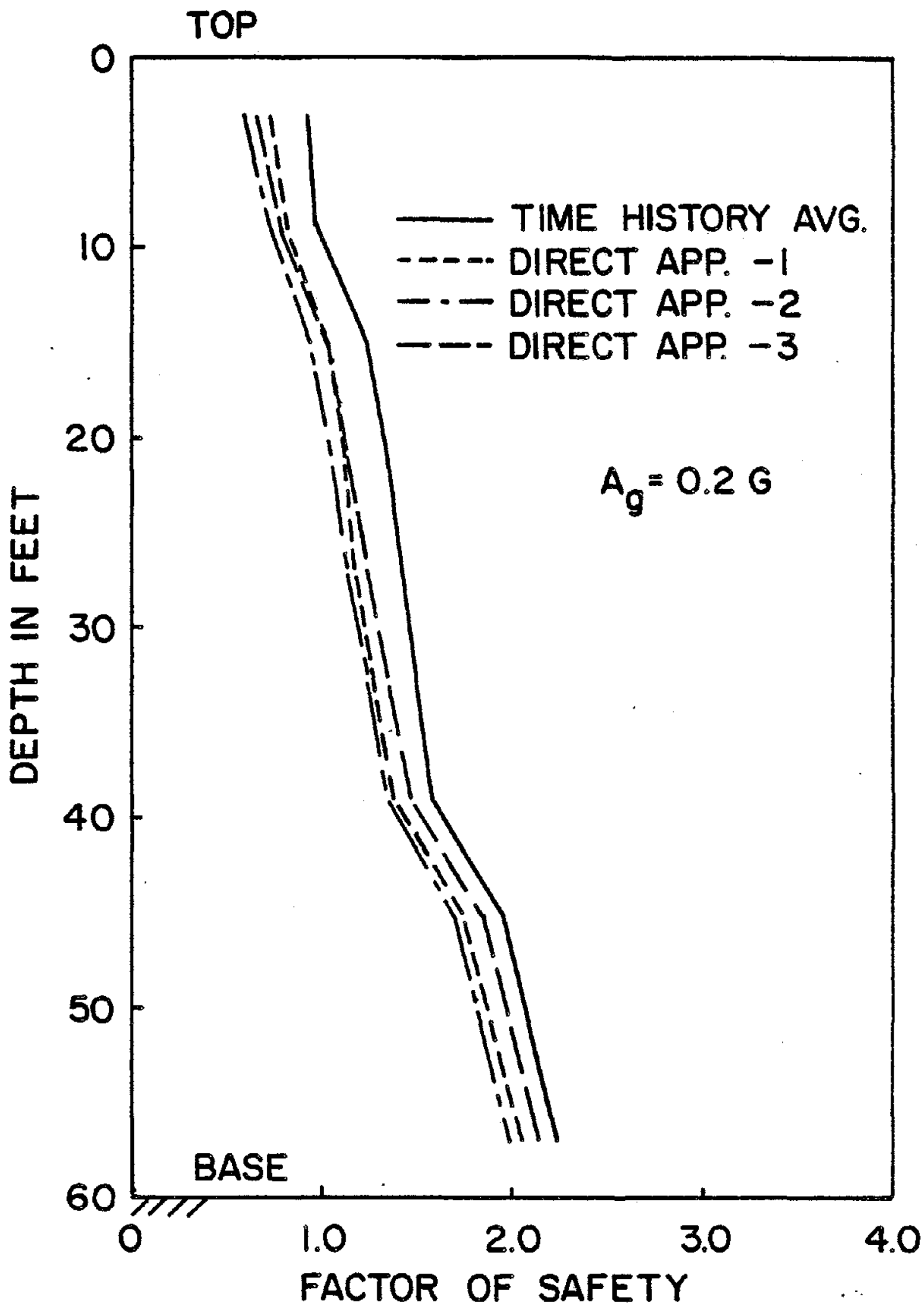


FIG. 17 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.2G

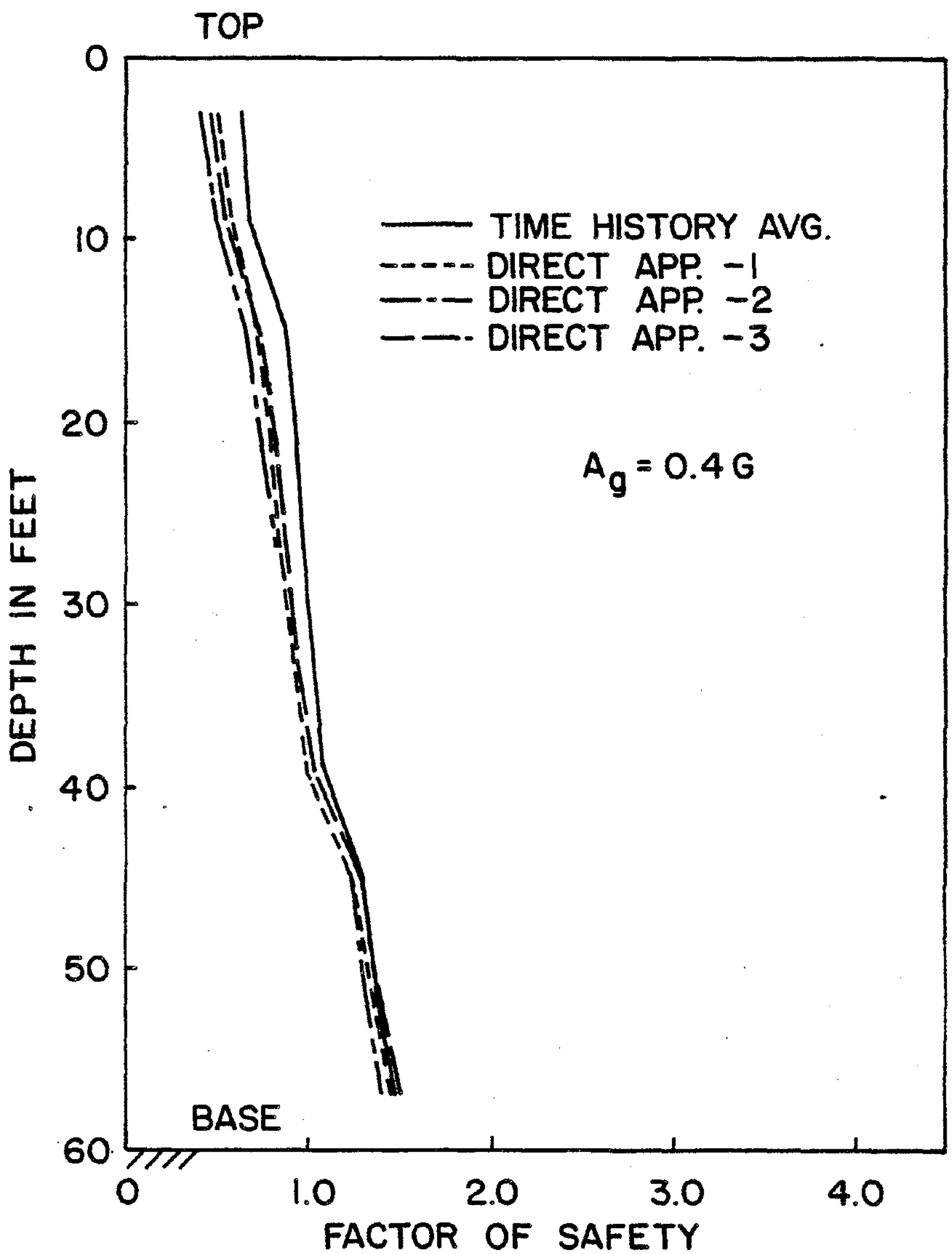


FIG. 18 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH AND TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.4G

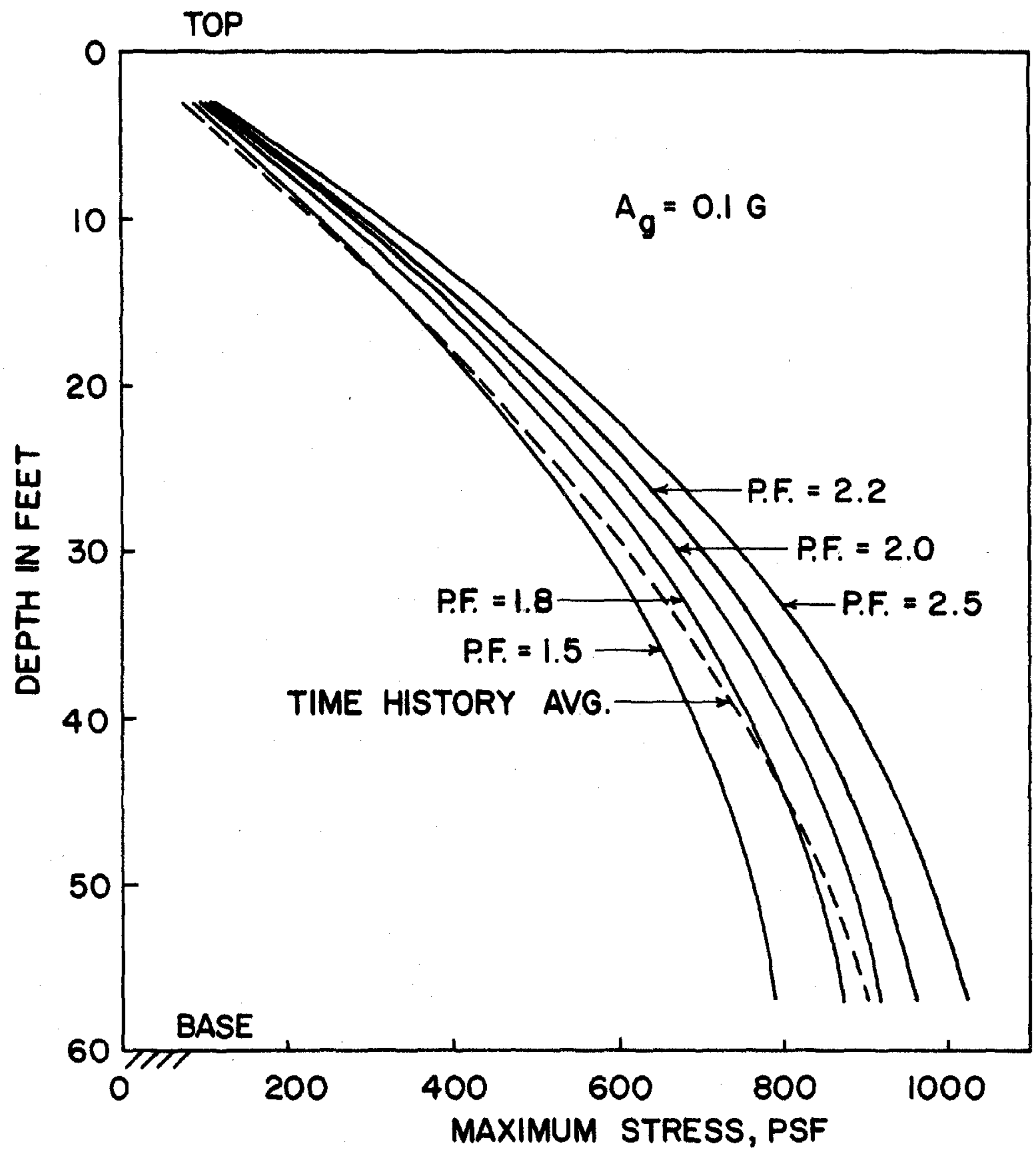


FIG. 19 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH-3 FOR DIFFERENT VALUES OF PEAK FACTORS FOR MEAN ACCELERATION LEVEL OF 0.1G

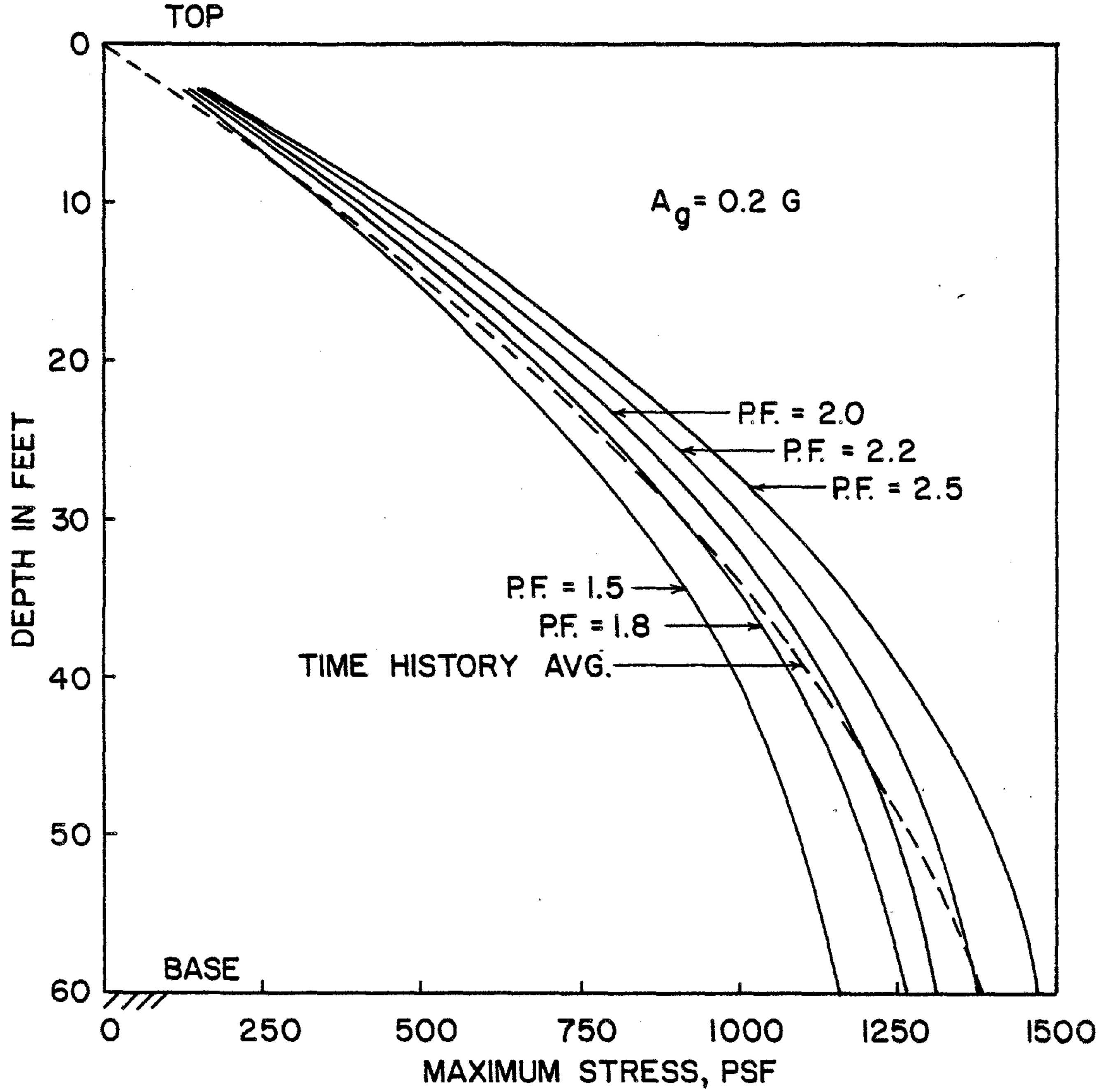


FIG. 20 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH-3 FOR DIFFERENT VALUES OF PEAK FACTORS FOR MEAN ACCELERATION LEVEL OF 0.2G

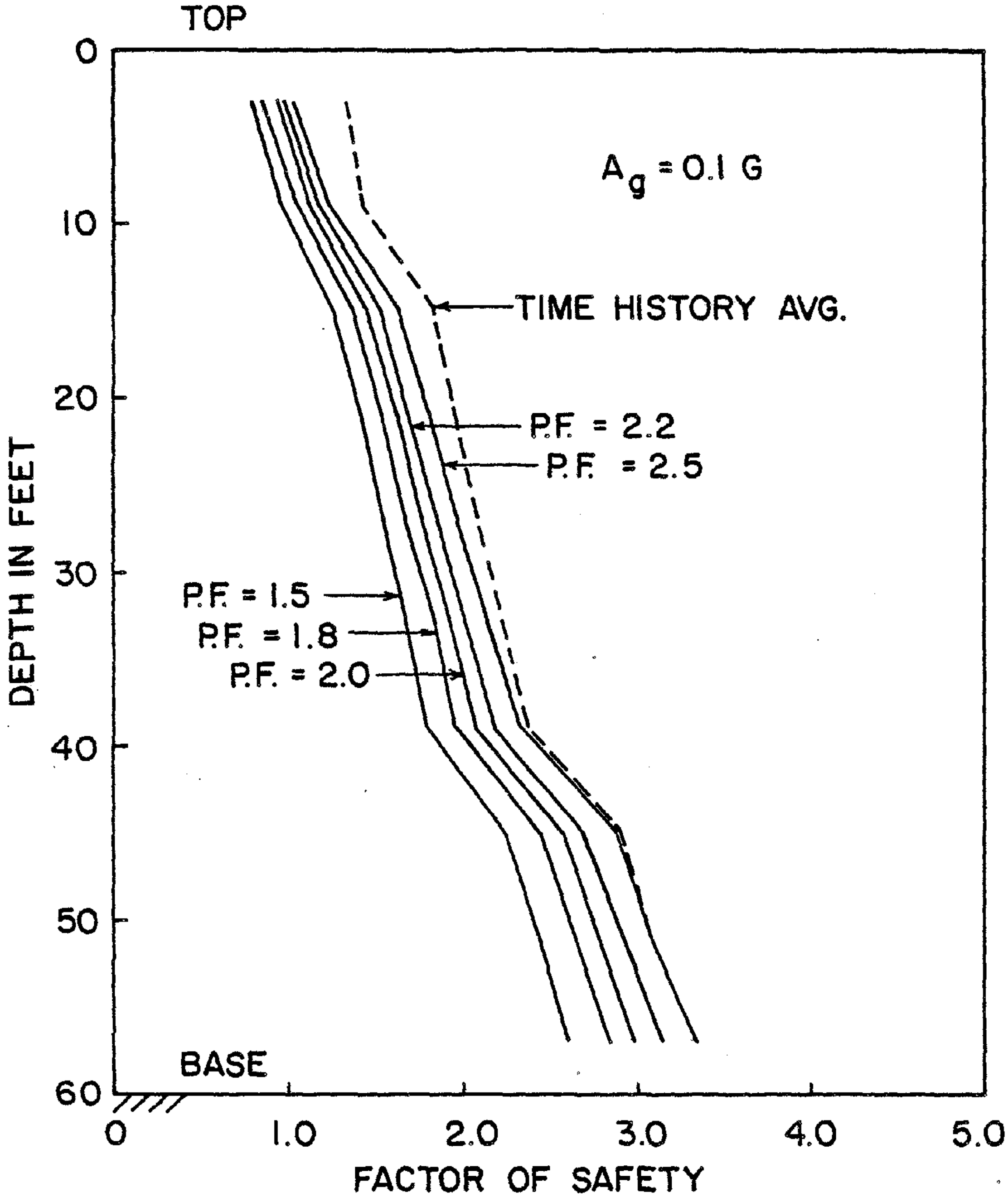


FIG. 21 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH-3 FOR DIFFERENT VALUES OF PEAK FACTORS FOR MEAN ACCELERATION LEVEL OF 0.1G

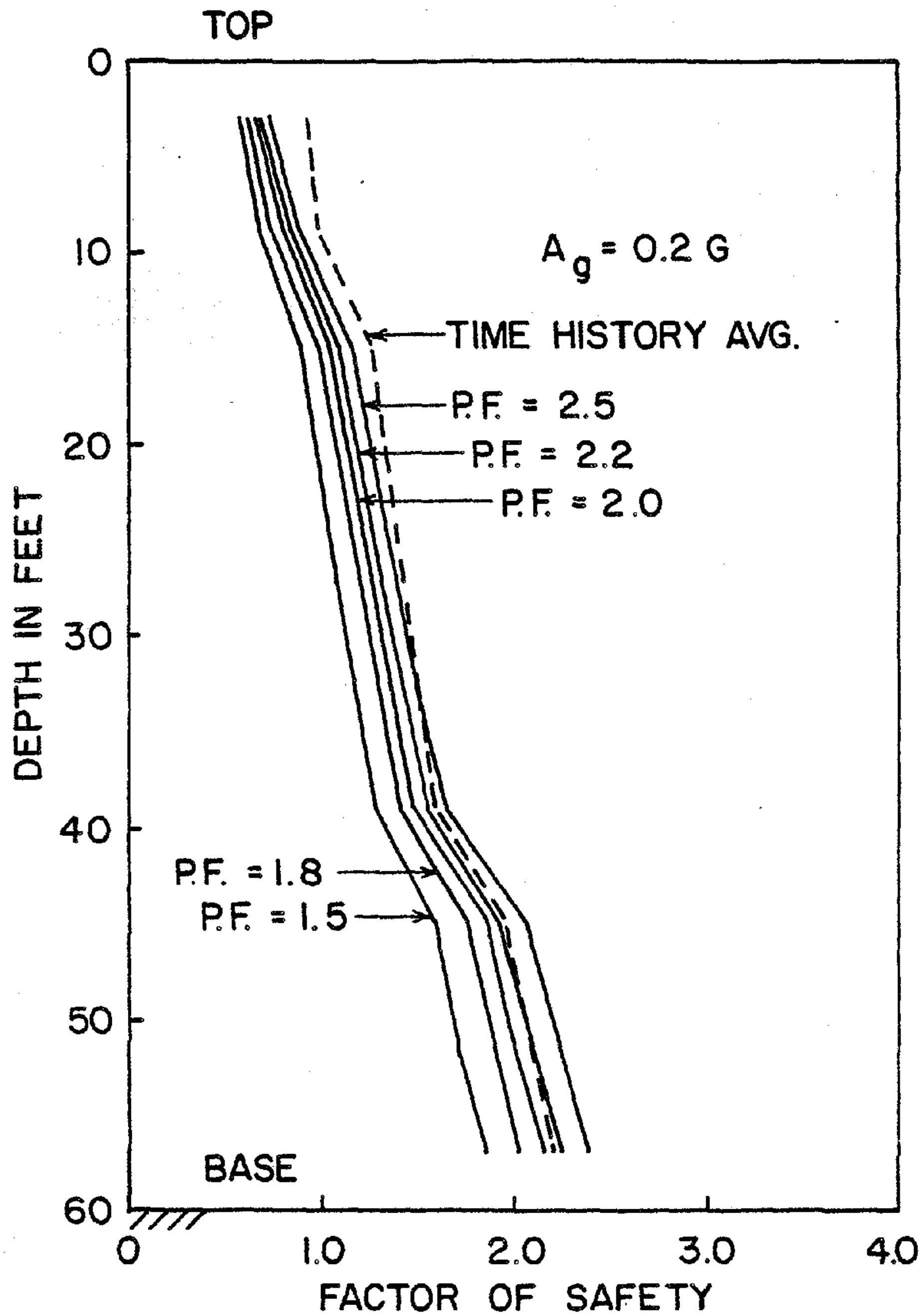


FIG. 22 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH-3 FOR DIFFERENT VALUES OF PEAK FACTOR FOR MEAN ACCELERATION LEVEL OF 0.2G

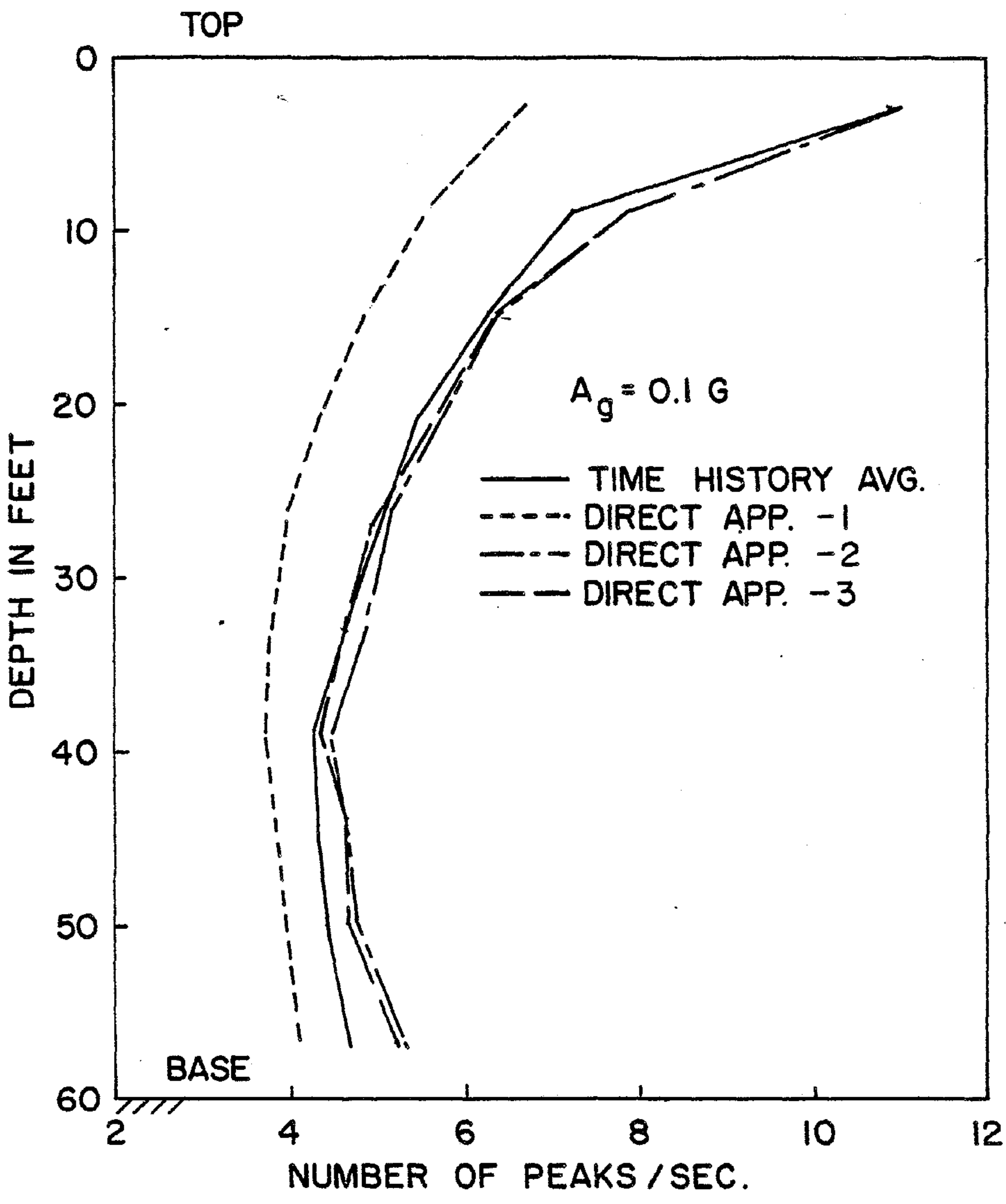


FIG. 23 AVERAGE NUMBER OF PEAKS/SECOND OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.1G

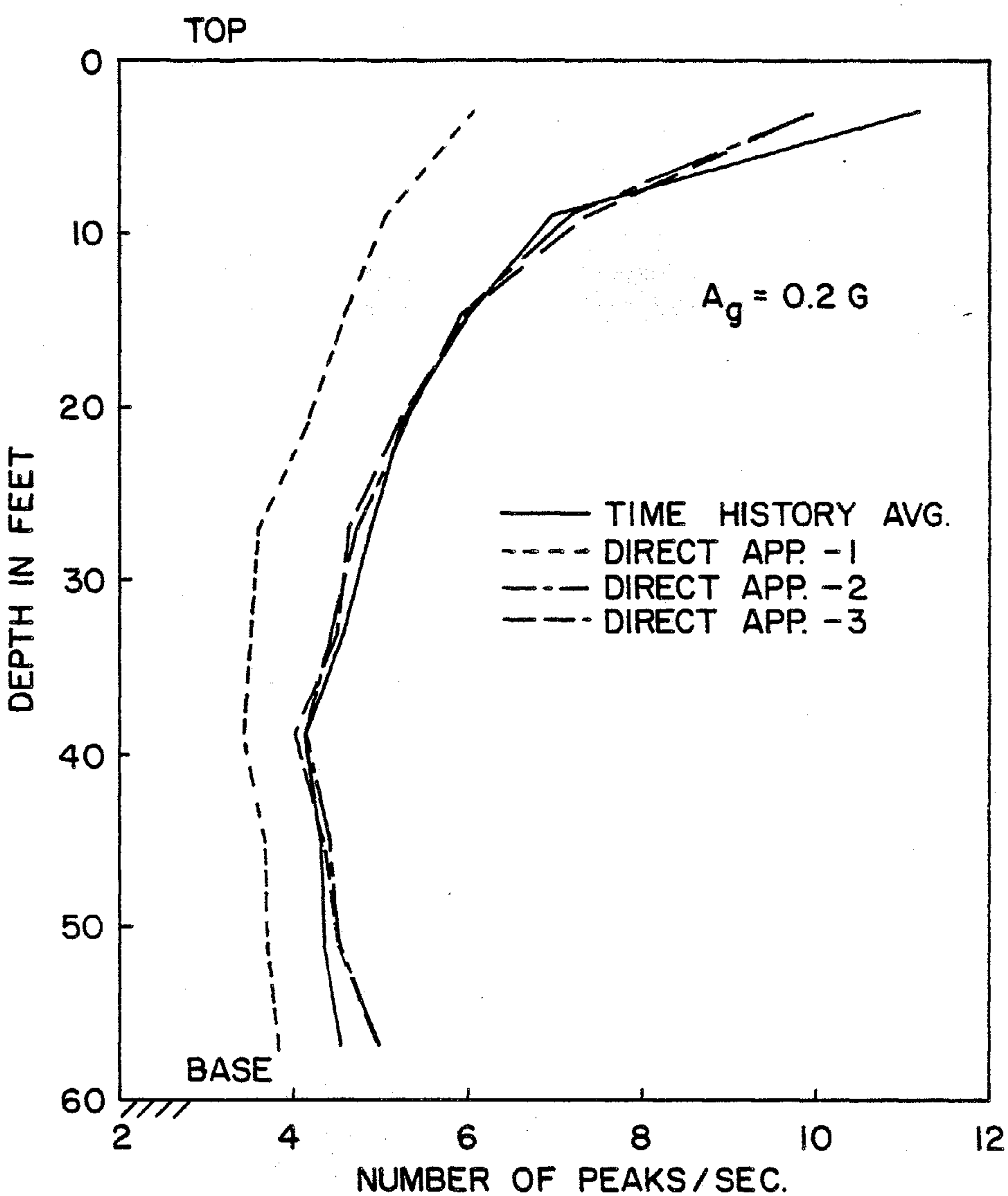


FIG. 24 AVERAGE NUMBER OF PEAKS PER SECONDS OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.2G

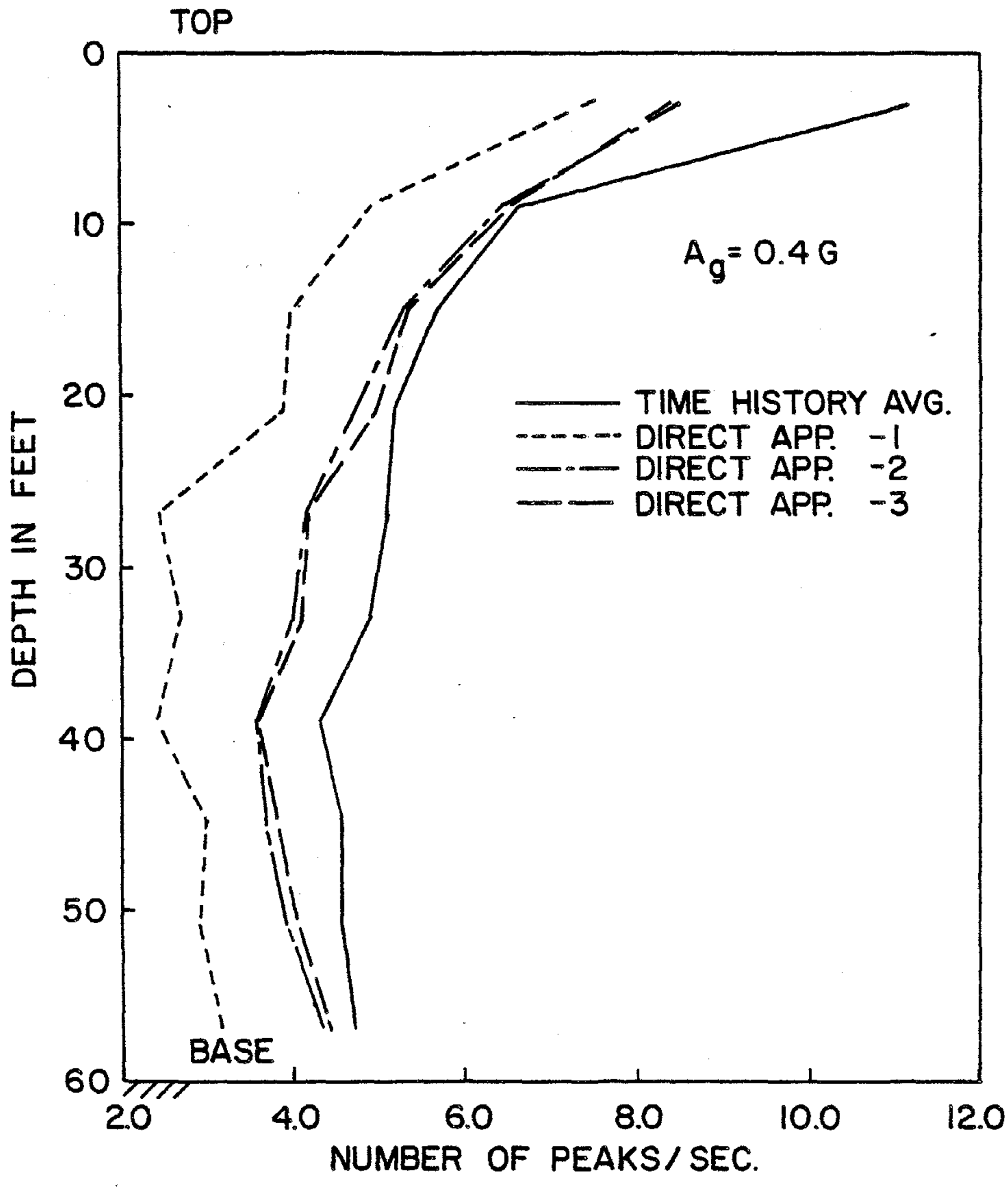


FIG. 25 AVERAGE NUMBER OF PEAKS PER SECONDS OBTAINED BY DIRECT APPROACH AND NONLINEAR TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.4G

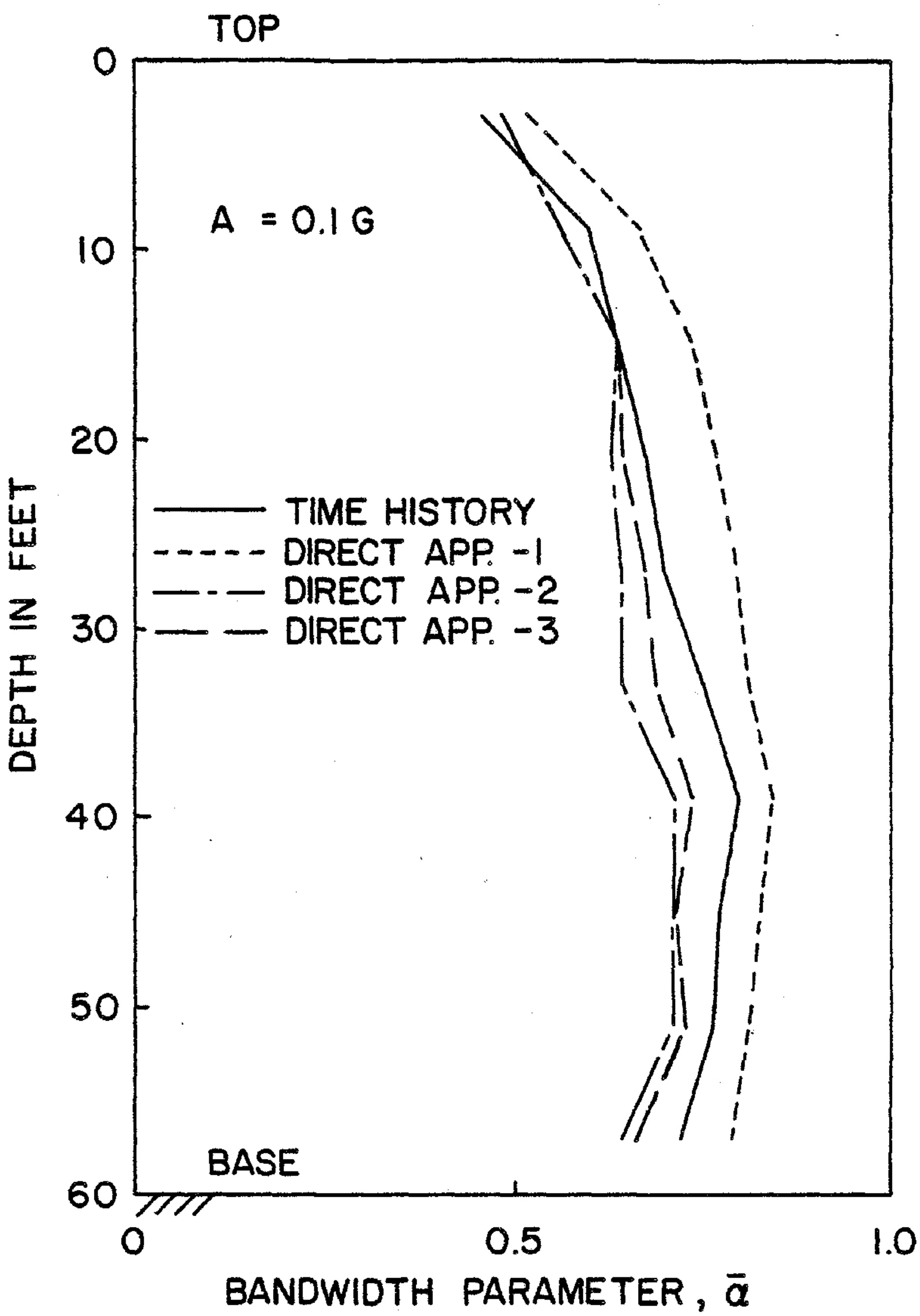


FIG. 26 BANDWIDTH PARAMETER OBTAINED BY DIRECT APPROACH AND BY TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.1G

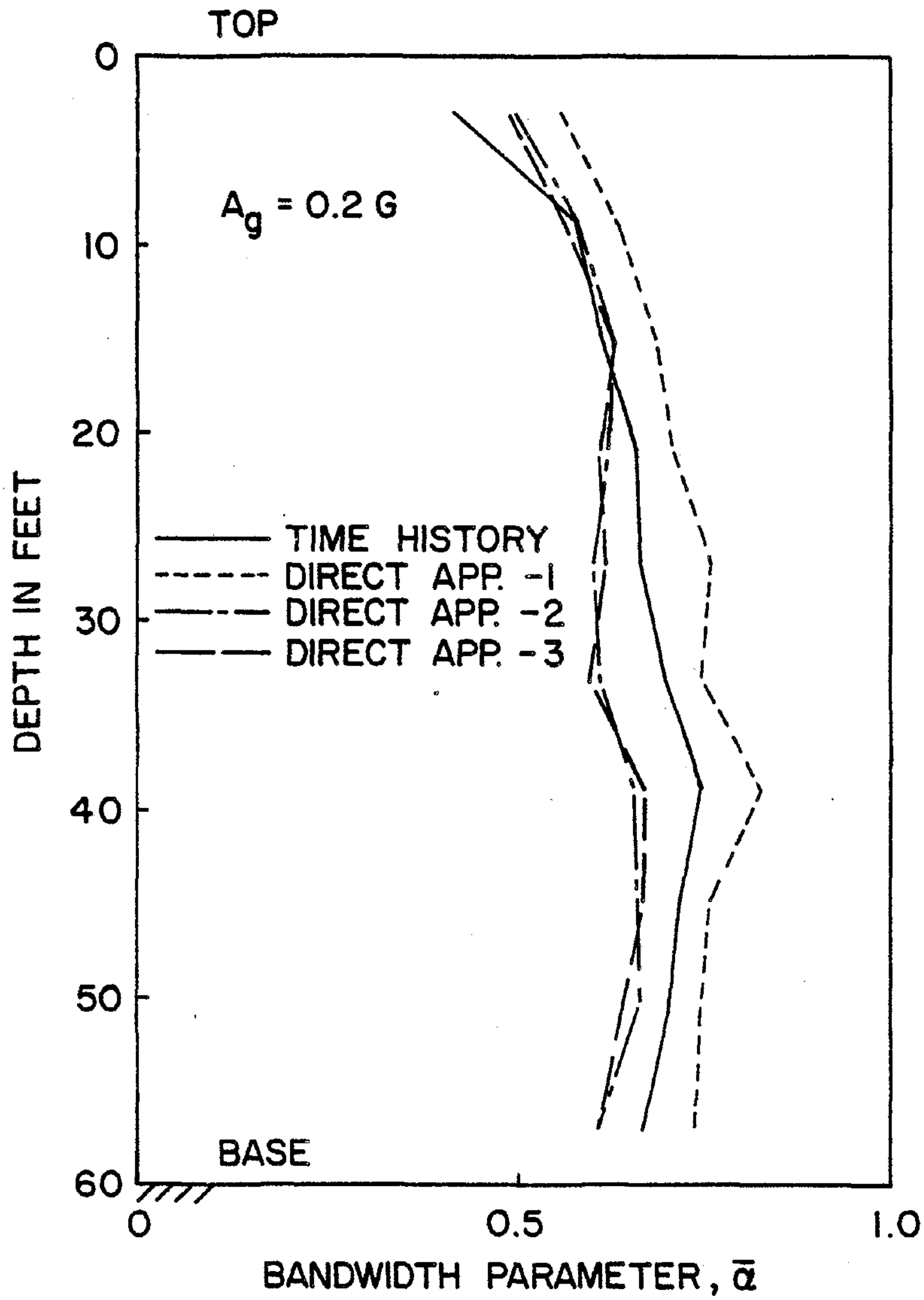


FIG. 27 BAND WIDTH PARAMETER OBTAINED BY DIRECT APPROACH AND TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.2G

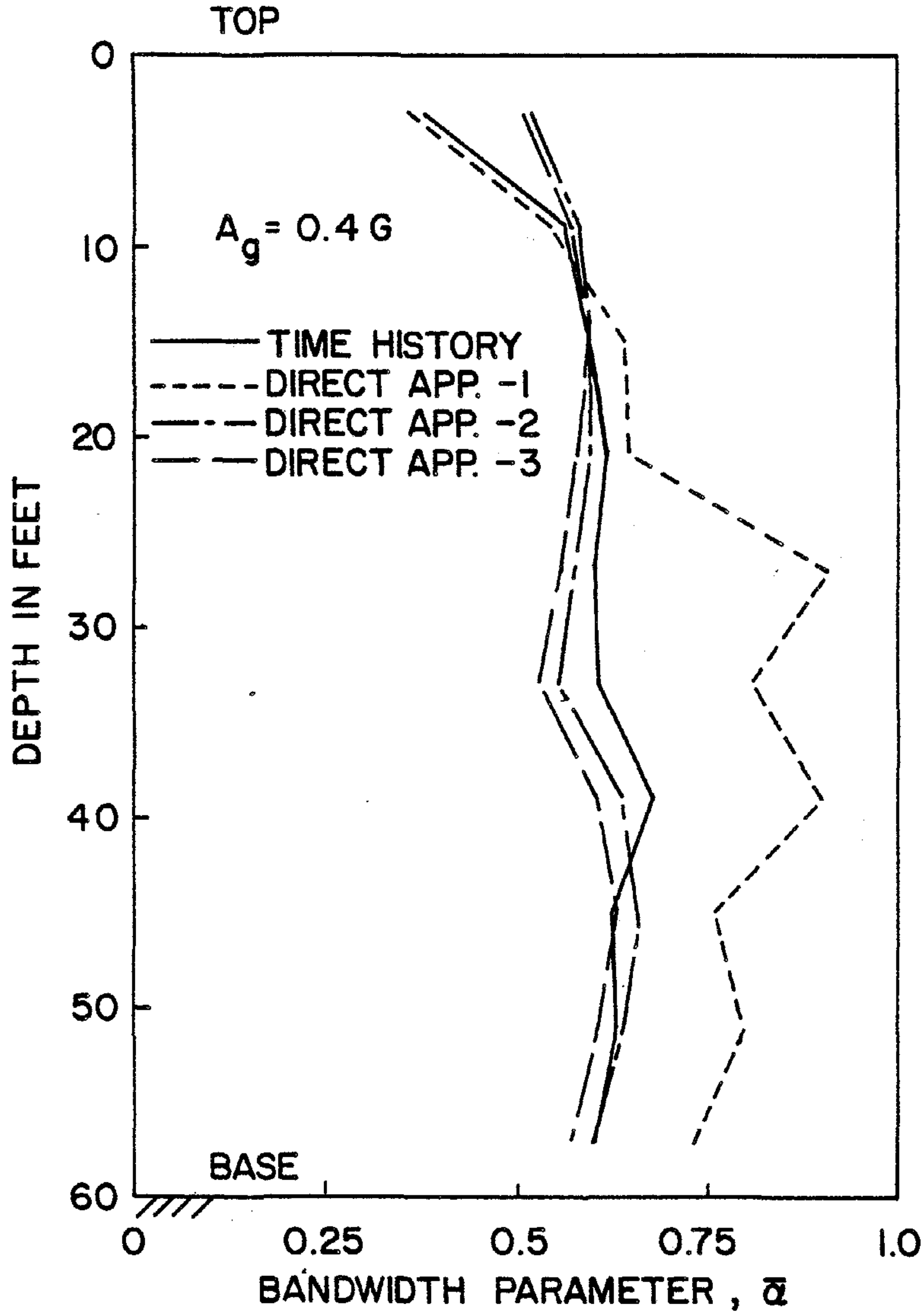


FIG. 28 BAND WIDTH PARAMETER OBTAINED BY DIRECT APPROACH AND TIME HISTORY ANALYSES (54 SIMULATIONS) FOR MEAN ACCELERATION LEVEL OF 0.4G

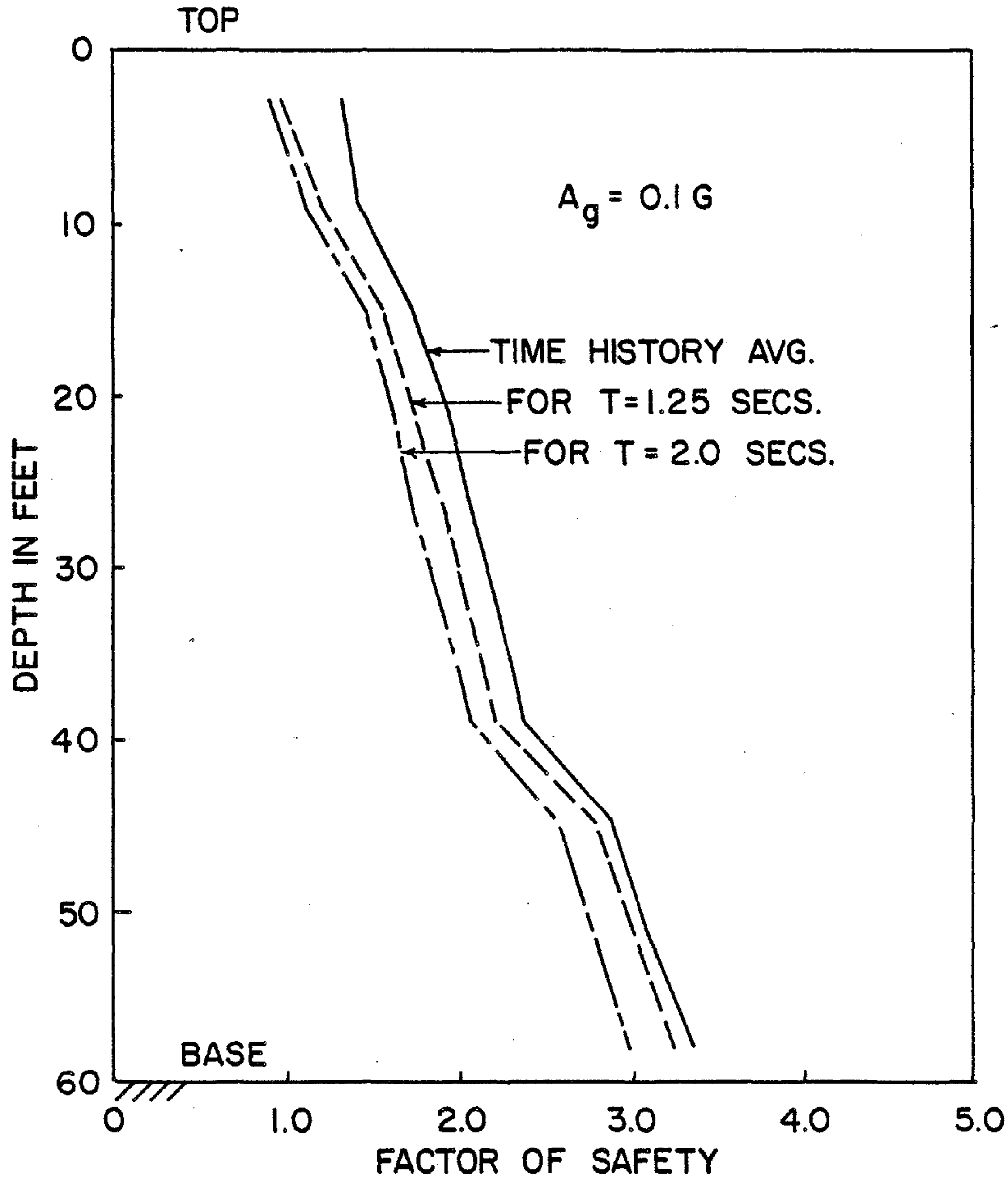


FIG. 29 FACTOR OF SAFETY OBTAINED BY TIME HISTORY ANALYSES AND DIRECT APPROACH-3 FOR EQUIVALENT EARTHQUAKE DURATION OF 2.0 AND 1.25 SECS. FOR ACCELERATION LEVEL OF 0.1G

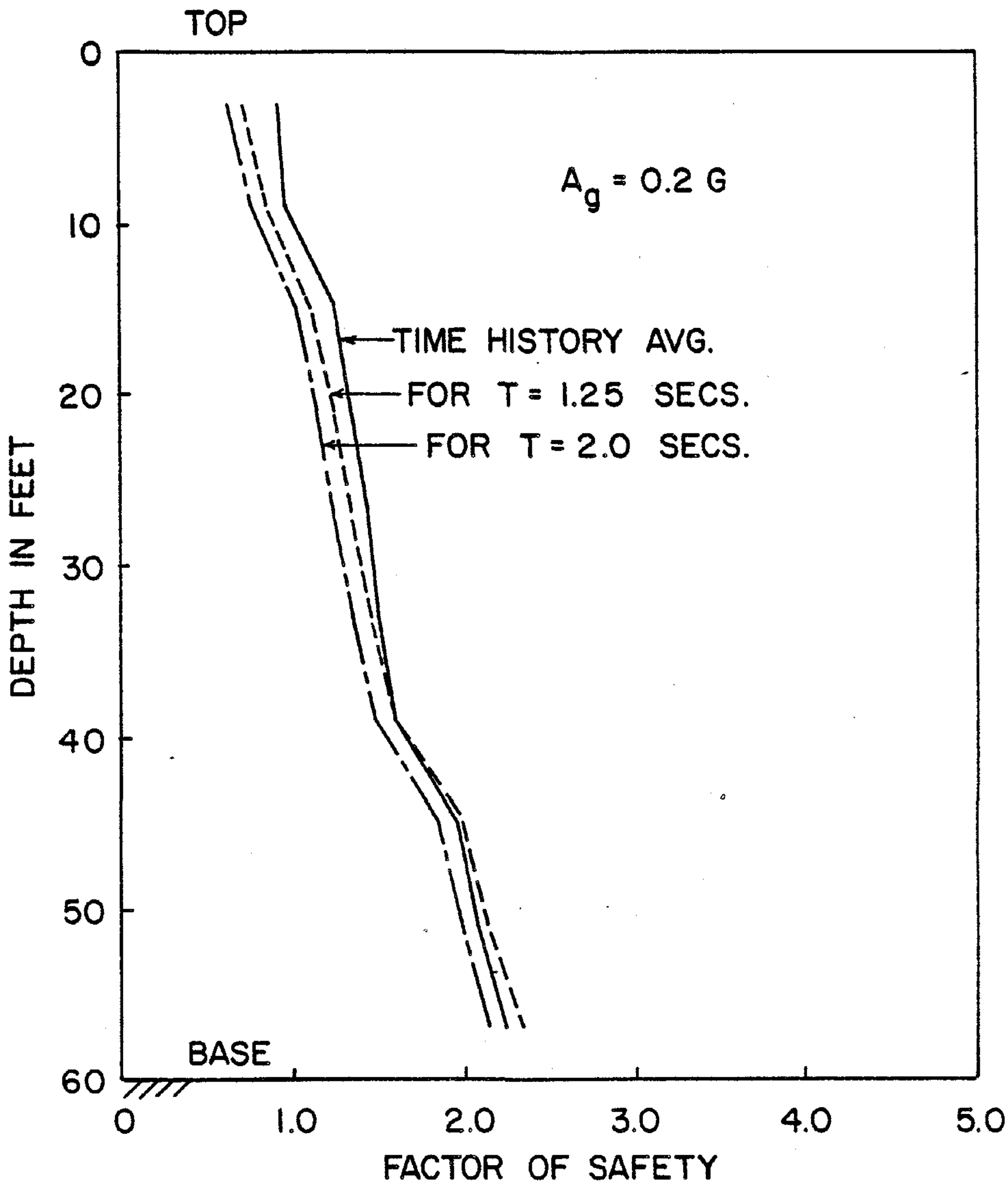


FIG. 30 FACTOR OF SAFETY OBTAINED BY TIME HISTORY ANALYSES AND DIRECT APPROACH-3 FOR EQUIVALENT DURATIONS OF 2.0 AND 1.25 SECS. FOR ACCELERATION LEVEL OF 0.2G

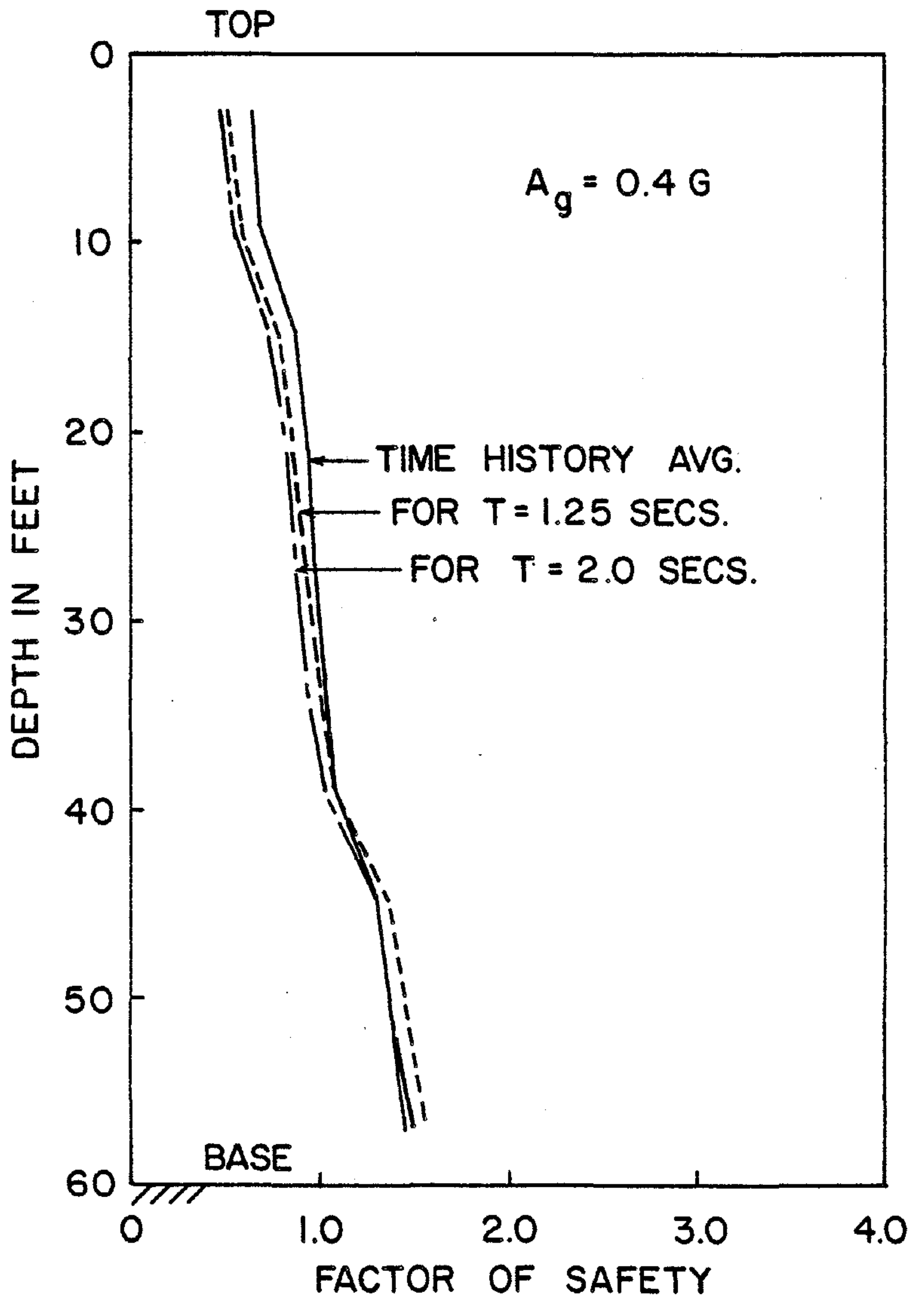


FIG. 31 FACTOR OF SAFETY OBTAINED BY TIME HISTORY ANALYSES AND DIRECT APPROACH-3 FOR EQUIVALENT EARTHQUAKE DURATIONS OF 2.0 AND 1.25 SECS. FOR ACCELERATION LEVEL OF 0.4G

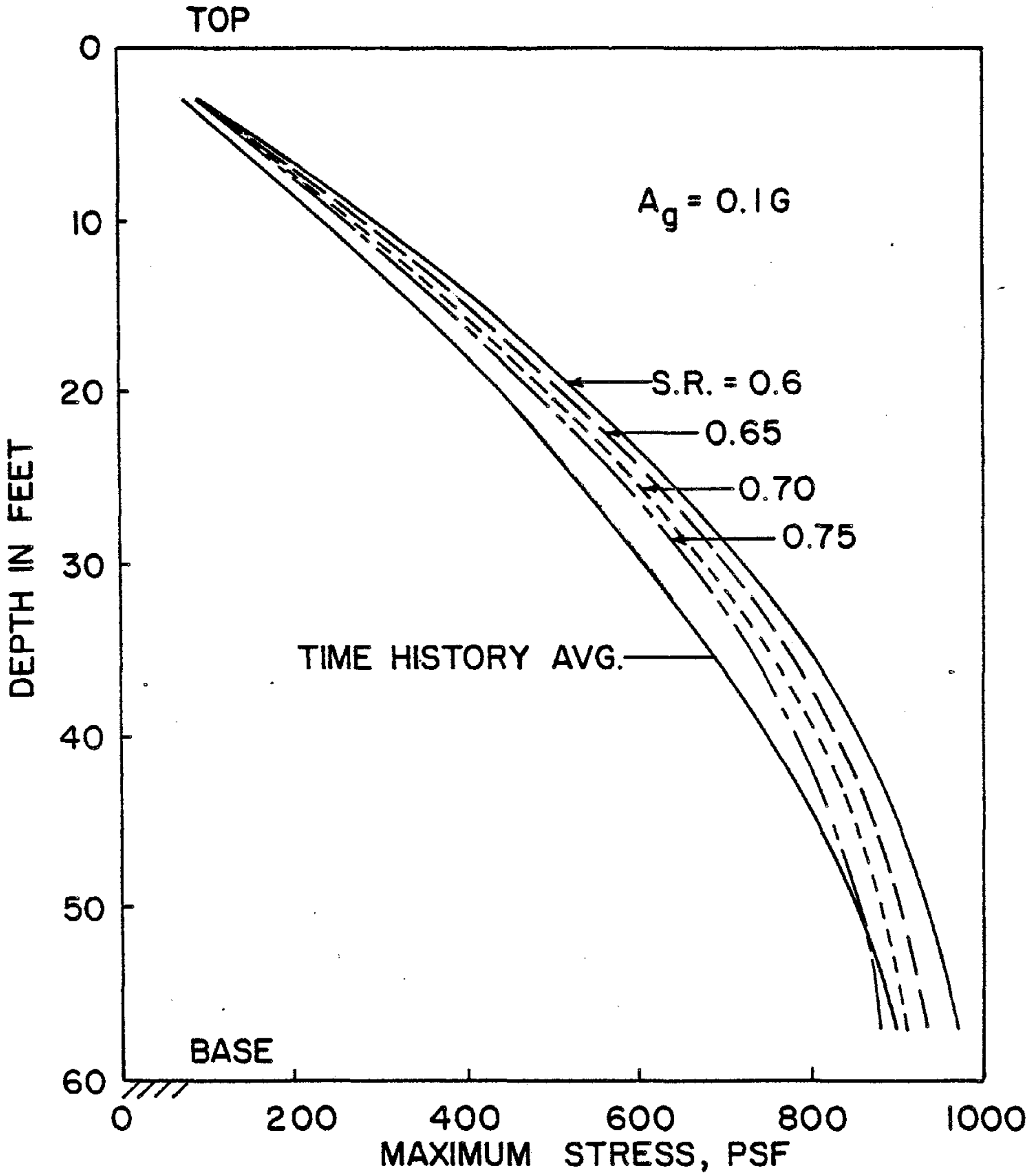


FIG. 32 MAXIMUM SHEAR STRESS OBTAINED BY DIRECT APPROACH-3 WITH SHEAR MODULUS AND DAMPING OBTAINED FOR DIFFERENT VALUES OF STRAIN RATIO (S.R.) - MEAN ACCLERATION LEVEL = 0.1G

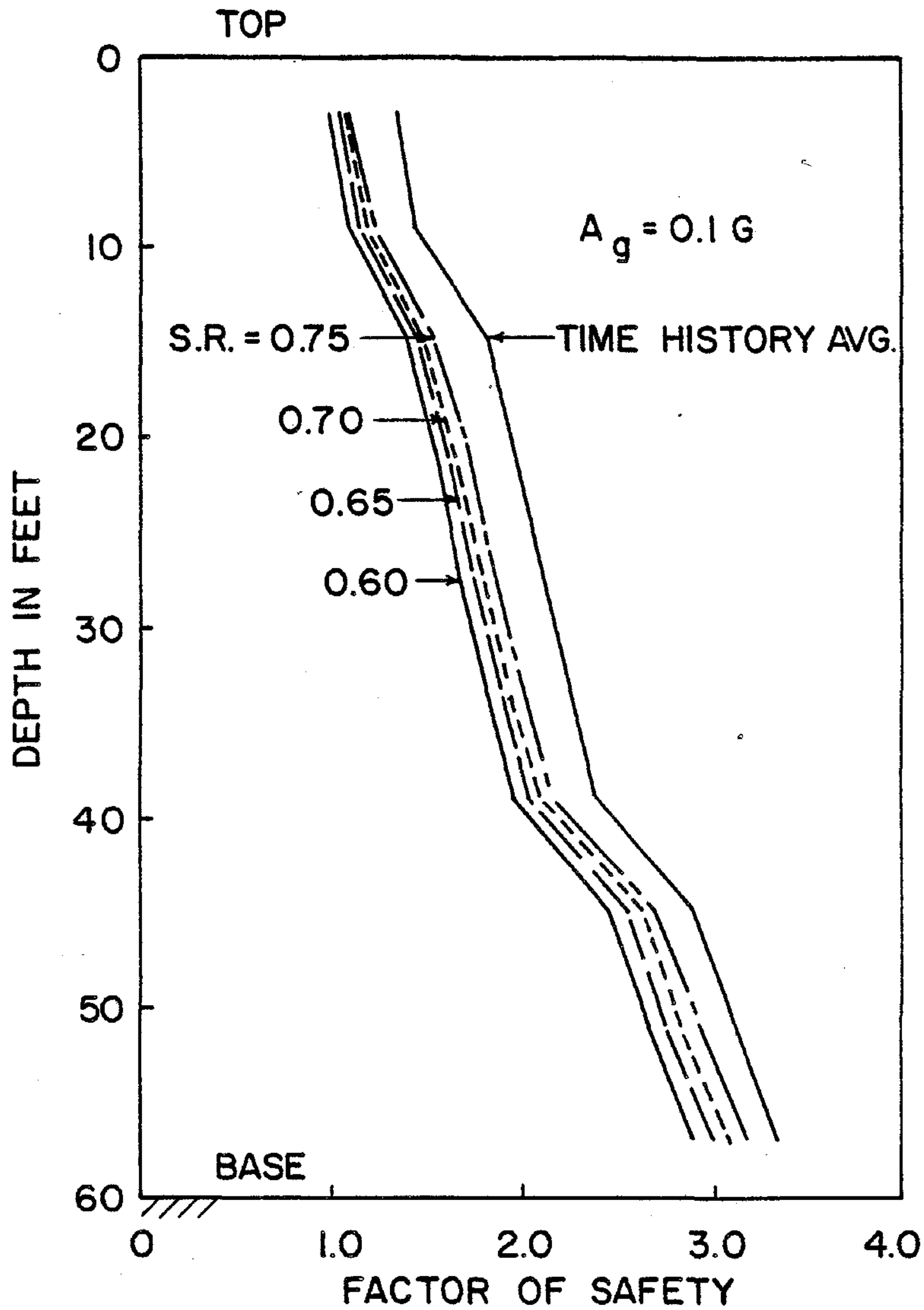


FIG. 33 FACTOR OF SAFETY OBTAINED BY DIRECT APPROACH-3 WITH SHEAR MOD. AND DAMPING OBTAINED FOR DIFFERENT VALUES OF STRAIN RATIO (S.R.) - MEAN ACCELERATION LEVEL = 0.1G

## APPENDIX - I

### Correlation Between $x_r(t)$ and $\dot{x}_s(t)$:

To evaluate the effect of correlation between displacement at one node and velocity at another on Eqs. 6a,b, the following equations were used to obtain the magnitude of correlation coefficient.

Correlation Coefficient:

$$\rho_{x_r \dot{x}_s} = \frac{E[x_r(t)\dot{X}_s(t)]}{\sigma_{x_r}\,\sigma_{\dot{x}_s}} \tag{I.1}$$

$$E[x_r(t)\dot{x}_s(t)] = -2 \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \phi_j(r)\phi_k(s)$$

$$\int_{-\infty}^{\infty} \Phi_g(\omega)[\omega_j\omega_k(\beta_k\omega_j - \beta_j\omega_k)\omega^2 - (\omega_k\beta_k - \omega_j\beta_j)\omega^4]$$

$$|H_j(\omega)|^2 |H_k(\omega)|^2 d\omega \tag{I.2}$$

The derivation of Eq. I.2 assumes a stationary response. For a given spectral density function, this equation can be integrated by a suitable integration procedures. To express it in terms of base response spectra values, the following equation can be used:

$$E[x_r(t)\dot{x}_s(t)] = -\frac{2}{F^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_j \gamma_k \phi_j(r)\phi_k(s)$$

$$\omega_k[\{A'' R_a^2(\omega_j) + B''\omega_j^2 r^2 R_v^2(\omega_j)\}/\omega_j^4$$

$$+ \{C'' R_a^2(\omega_k) + D''\omega_k^2 R_v^2(\omega_k)\}/\omega_k^4] \tag{I.3}$$

in which

$$A'' = -r^2[(1 - r^4)A_2 + 2\{1-2\beta_k^2 - r^2(1-2\beta_j^2)\}A_1]/\Delta$$

$$B'' = -[r^2\{2(1-2\beta_k^2)r^2 - 2(1-2\beta_j^2)\}A_2 - (1-r^4)A_1]/\Delta$$

$$C'' = - A/r^4$$

$$D'' = - B$$

$$\Delta = (1-r^4)^2 + 2r^2\{(1-2\beta_k^2)r^2 - (1-2\beta_j^2)\}\{1-2\beta_k^2 - r^2(1-2\beta_j^2)\}$$

$$A_1 = \beta_k r^2 - \beta_j r; \quad A_2 = r\beta_j - \beta_k \qquad (I.4)$$

## APPENDIX - II

Response Evaluation by SRSS for Nonproportionally Damped Systems

The system damping matrix constructed from element damping matrices will usually be nonclassical or nonproportional. That is matrix $[\Phi]^T[C]$ $[\phi]$ will have off diagonal terms. Analysis is, however, considerably simplified if these off-diagonal terms are neglected. To examine what effect this ommission of off-diagonal terms will have on the system response, a method has been developed in this investigation to obtain exact response for the case where off-diagonal terms are included. This exact method is described in the paper given in this appendix. The paper will appear in the December 1980 issue of the Journal of Engineering Mechanics Division, ASCE (46).

The method is cast in the form of the conventional square-root-of-the-sum-of-the-squares (SRSS) procedure, and it can use a given set of response spectra curves directly as seismic design input.

Some response results for the layered soil system being considered in this investigation are obtained by this newly developed approach and compared with the results obtained by the approximate approach in which the off-diagonal terms are ignored. Tables 2 and 3 show these results. The error is of the order of 0.35 percent. This small magnitude of error in the response justified the use of the approximate approach, at least for the soil systems being examined in this investigation.

# SEISMIC RESPONSE BY SRSS FOR NONPROPORTIONAL DAMPING

by

Mahendra P. Singh[1], M. ASCE

## Introduction

Seismic analysis of structures with mass and (or) stiffness proportional damping matrix can be conveniently performed by normal mode approach. Especially, for obtaining the design response for a design input prescribed in the form of ground response spectra, the method of, so-called, square-root-of-the-sum-of-the-squares (SRSS) or modified SRSS can be used for such a special damping case. Theoretical basis of using SRSS to combine the maximum modal responses to obtain the design response is fairly well established now. See references 1, 17 and 19.

The difficulty arises when the damping matrix of the system is not proportional to either mass and (or) stiffness matrix or is not of a special form as discussed by Caughey (3). Such a damping is usually called nonproportional or nonclassical. The normal modes (undamped modes) of the system then cannot be used to decouple the equations of motion. Cases where the damping matrix of the system are nonproportional are encountered rather quite often (24). Whenever the structural model to be analyzed consists of elements made of different material with significantly different energy dissipation characteristics, the total damping matrix assembled from individual elemental damping matrices are usually non-proportional. Such cases occur whenever combined soil and structure analytical models are used to incorporate soil-

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structure interaction effects in the dynamic analysis (4, 5, 18). Also in the equivalent linear analysis of earth structures with strain-dependent soil properties, the total damping matrix of the system will usually be nonproportional.

Since the damping matrix can not be decoupled in such cases, very often time-history analysis using numerical integration techniques are used if the time history of the earthquake input is known. In some cases undamped normal modes can still be used with advantage for such time history analyses (4). However, to obtain design response for a prescribed seismic design input like ground response spectra, the time history approach becomes rather impractical and expensive since an ensemble of time histories representing all possible ground motion must be considered to obtain a reliable estimate of the design response.

Other approximate approaches have been proposed where normal modes can still be used. In the most commonly used approach, the off-diagonal terms representing modal coupling through damping matrix are completely ignored (2, 18) and diagonal elements of the (transformed) matrix are used to obtain the modal damping ratios. In other approaches, the modal damping ratios are obtained based on some weighting schemes (13, 18, 25) or other techniques (6, 21, 22). These modal damping ratios are then used in the decoupled modal equations for time history analysis or with the method of SRSS to obtain the design response. The rationale behind the development of some of these simplified procedures appear to be reasonable. Yet, however, these approaches are subjective and approximate for a given general damping case. The results substantiating (and/or contradicting) various proposed approaches have appeared in the literature. See References 2 and 18 on one side and References 4 and 7 on the other. The reported contradictions are probably due to the choice of different example problems used for illustrations. Thus, in general, the applicability of various proposed approaches to all possible cases of systems and responses cannot possibly be verified and claimed.

For the purpose of obtaining design response for inputs prescribed in the forms of ground spectra, a method is developed in this paper in which nonproportional damping matrix can be used in essentially an SRSS type of approach. For this, first a formulation for the analysis of a nonproportional damping system is developed for seismic inputs characterized by power spectral density functions or other random forms. This formulation is then extended for its use with inputs prescribed by ground response spectra. A similar extension of the random vibration formulation is made in the proportional damping case to justify the use of SRSS for modal response combinations (1, 17, 19). The proposed approach is, therefore, a form of modified conventional SRSS approach where nonproportional damping effects can be included properly.

## Analytical Formulation

The equations of motion of a multi-degree-of-freedom structure subjected to a base excitation due to seismic ground disturbances can be written in the following matrix form:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = -[M]\{r\}\ddot{X}_g \qquad (1)$$

in which [M] = mass matrix, [C] = damping matrix not necessarily proportional, [K] = stiffness matrix, {x} = vector of relative displacement with respect to the base, {r} = influence coefficient vector (5), and $\ddot{X}_g(t)$ = base acceleration time history. The dots over a vector represent its time derivative.

If [C] is nonproportional or nonclassical (3), Eqs. 1 can not be decoupled using the normal modes. For a correct modal analysis of such system, a 2n-dimensional state vector approach is commonly used. See, for example, the paper by Foss (8) and the texts by Frazer, Duncan and Collar (9), Hurty and Rubinstein (10) and Meirovitch (15). This state vector approach has been used by Itoh (11) for time history analysis of a nuclear power plant system with nonproportional damping.

In the state vector approach, Eq. 1 is cast into a 2n-dimensional equations (with the help of an identity equation, Ref. 9) of the following form:

$$[A]\{\dot{y}\} + [B]\{y\} = -[D]\left\{\frac{0}{r}\right\}\ddot{X}_g \tag{2}$$

in which

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}, \quad B = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

$$[D] = \begin{bmatrix} [0] & [0] \\ [0] & [M] \end{bmatrix} \tag{3}$$

are 2n x 2n dimension matrices (n = number of generalized degrees-of-freedom of the system), and {y} is a 2n x 1 dimension vector as follows

$$\{y\} = \begin{Bmatrix} \{\dot{x}\} \\ \{x\} \end{Bmatrix} \tag{4}$$

Solution of Eq. 2 can be obtained in terms of eigenvalues and eigenvectors of its homogeneous equations. This requires solution of the following 2n x 2n dimensional eigenvalue problem:

$$p[A][\Phi] + [B][\Phi] = \{0\} \tag{5}$$

where p = eigenvalue and $[\Phi]$ = 2n x 2n matrix of eigenvectors $\{\phi_j\}$.

Solution of this eigenvalue problem will give 2n-complex eigenvalues and a corresponding number of eigenvectors. Complex values are accompanied by their conjugates. Some good algorithms are available for solution of such eigenvalue problems. Required subroutine programs can be found on most computing systems, such as in IMSL Library (12).

These eigenvectors are orthogonal with respect to matrices [A] and [B]. Using the expansion theorem with the following transformation:

$$\{y\} = [\Phi]\{z\} \tag{6}$$

and orthogonal properties of the eigenvectors with respect to [A] and [B] matrices, Eqs. 1 can be transformed into a decoupled set of equations as follows:

$$[A^*]\{\dot{z}\} + [B^*]\{z\} = -\,[\Phi]'[D]\{\tfrac{0}{r}\}\ddot{X}_g(t) \tag{7}$$

where $[A^*] = [\Phi]'[A][\Phi]$ and $[B] = [\Phi]'[B][\Phi]$ are diagonal matrices, with their diagonal elements related as follows:

$$A^*_j = -\,p_j\, B^*_j \tag{8}$$

where $p_j$ is the $j^{\underline{th}}$ eigenvalue of Eq. 5. Herein, a prime (') over a matrix or vector represents its transpose.

A decoupled $j^{\underline{th}}$ equation of Eqs. 7 is, therefore,

$$\dot{z}_j - p_j z_j = F_j \ddot{X}_g(t) \tag{9}$$

where $F_j$ is an element of complex vector {F} defined as follows:

$$\{F\} = -\,[A^*]^{-1}[\Phi]'[D]\{\tfrac{0}{r}\} \tag{10}$$

The solution of Eq. 9 can be written as

$$z_j(t) = F_j \int_0^t \ddot{X}_g(\tau) e^{p_j(t-\tau)}\, d\tau \tag{11}$$

wherefrom, the solution for {y} is obtained from Eq. 6. The lower half of this vector represents the relative displacement {x}. Thus using only the lower half of modal matrix [Φ], the vector {x} can be written as

$$\{x\} = \sum_{j=1}^{2n} \{\phi_j\} z_j \tag{12a}$$

$$\{x\} = \sum_{j=1}^{2n} \{\phi_j\} F_j \int_0^t \ddot{X}_g(\tau) e^{p_j(t-\tau)} d\tau \tag{12b}$$

$$\{x\} = \sum_{j=1}^{2n} \{q_j\} \int_0^t \ddot{X}_g(\tau) e^{p_j(t-\tau)} d\tau \tag{12c}$$

in which $\{\phi_j\}$ the jth complex mode shape (lower part only) and $\{q_j\} = F_j\{\phi_j\}$.

The $p_j$s, which are complex eigenvalues, will have a negative real part for a well supported stable structural system. If a complex eigenvalue analysis of a system with proportional damping matrix is performed, the real and imaginary parts, $\xi_j$ and $\theta_j$, respectively, of an eigenvalue will be related to the natural frequency and damping ratio as follows

$$p_j = -\xi_j + i\theta_j$$

$$= -\beta_j\omega_j + \omega_j\sqrt{1-\beta_j^2}\, i \tag{13}$$

where $\omega_j$ = jth natural frequency, $\beta_j$ = damping ratio (ratio of actual damping to the critical damping) in the jth mode and $i = \sqrt{-1}$. Thus the imaginary part defines the damped natural frequency of the system and the real part describes the decay characteristics of the response as governed by the damping ratio $\beta_j$.

The same interpretation can also be used to characterize the real and imaginary parts of a particular eigenvalue $p_j$ in a nonproportional damping case, as was done by Minami and Sakurai (16) in their investigation of soil-structure interaction effect. Thus, $\omega_j$ obtained from the solution of real and imaginary parts as

$$\omega_j = \sqrt{\xi_j^2 + \theta_j^2} \tag{14}$$

are the undamped natural frequencies of the system. Also $\beta_j$ obtained as

$$\beta_j = \xi_j / \sqrt{\xi_j^2 + \theta_j^2} \tag{15}$$

could be interpreted to represent the modal damping ratio. This realization helps in transforming the response obtained from Eq. 12 in terms of the modal response.

For the ground motion, $\ddot{X}_g(t)$, being characterized by a random process the autocorrelation characteristics of any response $x_u(t)$ can be obtained as follows

$$E\{x_u(t_1)x_u(t_2)\} = \sum_{j=1}^{2n} \sum_{k=1}^{2n} q_j(u)q_k(u) \int_0^{t_1} \int_0^{t_2} E\{\ddot{X}_g(\tau_1)\ddot{X}_g(\tau_2)\}$$

$$e^{\{p_j(t_1-\tau_1)+p_k(t_2-\tau_2)\}} d\tau_1 d\tau_2 \tag{16}$$

For a zero mean process, Eq. 16 also gives the covariance function of the response process $x_u(t)$. For a given description of the autocorrelation function $E[\ddot{X}_g(\tau_1)\ddot{X}_g(\tau_2)]$, Eq. 16 could possibly be evaluated to obtain the covariance of the response. If the ground motion can be assumed to be represented by a stationary random process with a spectral density function, $\Phi_g(\omega)$, Eq. 16 can be written as

$$E\{x_u(t_1)x_u(t_2)\} = \sum_{j=1}^{2n} \sum_{k=1}^{2n} q_j q_k \int_{-\infty}^{\infty} \Phi_g(\omega) \int_0^{t_1} \int_0^{t_2} \exp[i\omega(\tau_1-\tau_2)$$

$$+ p_j(t_1 - \tau_1) + p_k(t_2 - \tau_2)] d\tau_1 d\tau_2 d\omega \tag{17}$$

in which $q_j(u)$ has been replaced by $q_j$. Henceforth the same notation will be adopted in the paper.

To put this equation in the form so that the modal response can be easily identified, the summation over 2n terms is represented as a summation over n terms in which complex conjugate terms are considered in pairs as follows:

$$E[x_u(t_1)x_u(t_2)] = \sum_{j=1}^{n}\sum_{k=1}^{n}\int_{-\infty}^{\infty}\Phi_g(\omega)\int_0^{t_1}\int_0^{t_2} e^{i\omega(\tau_1-\tau_2)}[q_j e^{p_j(t_1-\tau_1)} + q_j^* e^{p_j^*(t_1-\tau_1)}]\,[q_k e^{p_k(t_2-\tau_2)} + q_k^* e^{p_k^*(t_2-\tau_2)}]d\tau_1 d\tau_2 d\omega \quad (18)$$

where asterisk (*) as a superscript represents the complex conjugate of a quantity. To further simplify this equation and put it in a more usuable form, the terms with j=k and j≠k (called cross terms here) will now be evaluated separately.

A term with j=k, represented here as $R_j$, consists of four terms as follows:

$$R_j = \int_{-\infty}^{\infty}\Phi_g(\omega)\int_0^{t_1}\int_0^{t_2}\Big(q_j^2\exp\{i\omega(\tau_1-\tau_2)+p_j(t_1-\tau_1)+p_j(t_2-\tau_2)\}$$
$$+ q_j^{*2}\exp\{i\omega(\tau_1-\tau_2)+p_j^*(t_1-\tau_1)+p_j^*(t_2-\tau_2)\}$$
$$+ q_jq_j^*\Big[\exp\{i\omega(\tau_1-\tau_2)+p_j(t_1-\tau_1)+p_j^*(t_2-\tau_2)\}$$
$$+ \exp\{i\omega(\tau_1-\tau_2)+p_j^*(t_1-\tau_1)+p_k(t_2-\tau_2)\}\Big]\Big)d\tau_1 d\tau_2 d\omega \quad (19)$$

The integrals over $\tau_1$ and $\tau_2$ can be carried out separately with $\omega$ as a parameter. For the situation of stationary response, i.e. when $t_1 \to \infty$, $t_2 \to \infty$, but $(t_1 - t_2) \to \tau$, the integrals in Eq. 19 are obtained as follows:

$$R_j(\tau) = \int_{-\infty}^{\infty}\Phi_g(\omega)e^{i\omega\tau}\left[\frac{q_j^2}{(p_j^2+\omega^2)} + \frac{q_j^{*2}}{(p_j^{*2}+\omega^2)} + q_jq_j^*\left\{\frac{1}{(-p_j+i\omega)(-p_j^*-i\omega)}\right.\right.$$
$$\left.\left. + \frac{1}{(-p_j^*+i\omega)(-p_j-i\omega)}\right\}\right]d\omega \quad (20)$$

With $p_j$ expressed in terms of its real and imaginary parts as in Eq. 13, and $q_j$ as

$$q_j = a_j + ib_j \tag{21}$$

and combining separately the 1st and 2nd parts and the last two parts in the parenthesis in Eq. 20 and after some simplification, this equation can be put into a recognizable form of integrations on $\omega$ as follows:

$$R_j(\tau) = \int_{-\infty}^{\infty} \Phi_g(\omega)(4a_j^2\omega^2 + 4A_j\omega_j^2)|H_j(\omega)|^2 e^{i\omega\tau}d\omega \tag{22}$$

in which

$$A_j = b_j^2 + (a_j^2 - b_j^2)\beta_j^2 - 2a_jb_j\beta_j\sqrt{1-\beta_j^2} \tag{23}$$

and $H_j(\omega)$ is the well known frequency transfer function of a single-degree-of-freedom oscillator defined as

$$H_j(\omega) = \frac{1}{(\omega_j^2-\omega^2)+2i\beta_j\omega_j\omega} \tag{24}$$

Similarity of Eq. 22 with the one encountered in the case of a proportional damping matrix (19) may be noted.

Algebraic manipulations of the integrals in the cross terms with $j \neq k$ are more involved but similar. Here again, a cross term will consist of four terms as follows:

$$\begin{aligned} R_{jk}(t_1,t_2) = \int_{-\infty}^{\infty} \Phi_g(\omega) \int_0^{t_1}\int_0^{t_2} e^{i\omega(\tau_1-\tau_2)} \Big[ & q_jq_k \exp\{p_j(t_1-\tau_1) \\ & + p_k(t_2-\tau_2)\} + q_j^*q_k^* \exp\{p_j^*(t_1-\tau_1) + p_k^*(t_2-\tau_2)\} \\ & + q_jq_k^* \exp\{p_j(t_1-\tau_1) + p_k^*(t_2-\tau_2)\} \\ & + q_j^*q_k \exp\{p_j^*(t_1-\tau_1) + p_k(t_2-\tau_2)\}\Big] d\tau_1 d\tau_2 d\omega \end{aligned} \tag{25}$$

These integrals over $\tau_1$ and $\tau_2$ can be evaluated separately for each term. For the stationary response case, i.e., when $t_1 \to \infty$, $t_2 \to \infty$ and $(t_1-t_2) = \tau$, the above expression can be written as

$$R_{jk}(\tau) = \int_{-\infty}^{\infty} \Phi_g(\omega)e^{i\omega\tau} \left[\frac{q_jq_k}{(-p_j+i\omega)(-p_k-i\omega)} + \frac{q_j^*q_k^*}{(-p_j^*+i\omega)(-p_k^*-i\omega)}\right.$$

$$+ \frac{q_j q_k^*}{(-p_j+i\omega)(-p_k^*-i\omega)} + \frac{q_j^* q_k}{(-p_j^*+i\omega)(-p_k-i\omega)} \Bigg] d\omega \tag{26}$$

Substituting $q_j$, $q_k$, $p_j$ and $p_k$ in terms of their real and imaginary parts, and after some algebraic manipulations, Eq. 26 can be put into its following form:

$$R_{jk}(\tau) = \int_{-\infty}^{\infty} \Phi_g(\omega) e^{i\omega\tau} [X'_{jk}(\omega) + iY'_{jk}(\omega)] H_j(\omega) H_k^*(\omega) d\omega \tag{27}$$

in which

$$X'_{jk}(\omega) = 4a_j a_k \omega^2 + 4\omega_j \omega_k [a_j a_k \beta_j \beta_k + b_j b_k \sqrt{1-\beta_j^2}\sqrt{1-\beta_k^2} - a_j \beta_j b_k \sqrt{1-\beta_k^2} - a_k \beta_k b_j \sqrt{1-\beta_j^2}] \tag{28}$$

$$Y'_{jk}(\omega) = 4\omega\{a_j a_k (\omega_k \beta_k - \omega_j \beta_j) - (a_j b_k \omega_k \sqrt{1-\beta_k^2} - b_j a_k \omega_j \sqrt{1-\beta_j^2})\}$$

Eq. 27 can be further transformed into the real forms of $H_j(\omega)$ and $H_k^*(\omega)$ as

$$R_{jk}(\tau) = \int_{-\infty}^{\infty} \Phi_g(\omega) e^{i\omega\tau} [X_{jk}(\omega) + iY_{jk}(\omega)] |H_j(\omega)|^2 |H_k(\omega)|^2 d\omega \tag{29}$$

in which

$$X_{jk}(\omega) = X'_{jk} U_{jk} - Y'_{jk} V_{jk},$$
$$Y_{jk}(\omega) = X'_{jk} V_{jk} + Y'_{jk} U_{jk} \tag{30}$$

in which, in turn, $U_{jk}$ and $V_{jk}$ are defined as

$$U_{jk} = (\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2) + 4\beta_j \beta_k \omega_j \omega_k \omega^2$$
$$V_{jk} = 2\omega\{\omega_k \beta_k (\omega_j^2 - \omega^2) - \omega_j \beta_j (\omega_k^2 - \omega^2)\} \tag{31}$$

Noting that $X_{jk} = X_{kj}$ and $Y_{jk} = -Y_{kj}$, the two cross terms $R_{jk}(\tau)$ and $R_{kj}(\tau)$ when combined together will eliminate $Y_{jk}$ from the final expression. Using Eqs. 22 and 29, the autocovariance function of the stationary response, $E[x_u(t+\tau)X_u(t)]$, can be obtained. For $\tau = 0$, it gives the mean square response as follows:

$$E[x_u^2] = \sum_{j=1}^{n} 4\int_{-\infty}^{\infty} (a_j^2\omega^2 + A_j\omega_j^2|H_j(\omega)|^2\Phi_g(\omega)d\omega$$

$$+ 2 \sum_{j=1}^{n} \sum_{k=j+1}^{n} \int_{-\infty}^{\infty} X_{jk}(\omega)|H_j(\omega)|^2|H_k(\omega)|^2\Phi_g(\omega)d\omega \qquad (32)$$

where $A_j$ and $X_{jk}(\omega)$ are given by Eqs. 23 and 30, respectively. Again similarity of this expression, with an equivalent response expression for a proportional damping case (19) may be noted. This expression can be used to obtain the response for an input prescribed in the form of spectral density function. For commonly used Kanai-Tajimi type of spectral density functions, the integrations in the above equation can be routinely obtained by the method of residues. For the degitized form of spectral density functions also, where variation between any continguous discrete points can be assumed to be linear, the integrations in Eq. 32 are obtained without much difficulty.

Design Response by SRSS

Eq. 32 represents a random vibration form of the (modified) square-root-of-the-sum-of-the-squares approach for obtaining the root mean square response. The terms under single summation represent the contributions from individual modes whereas the terms under double summation represent the interaction effect between the modes. To obtain the design response, the mean square response obtained from Eq. 32 should be amplified by an appropriate value of the peak factor. The peak factor could depend upon an acceptable level of probability of exceedance.

For a seismic input defined in the form of ground response spectra curves, which define maximum (or design) response in each mode, the modal peak factors are implicitly defined. Assuming that the peak factor is a constant value for each mode, and following the development given in Ref. 19, Eq. 32 can be used to obtain the design response in terms of modal response spectra values, as follows (also see Appendix I)

$$R_x^2(u) = \sum_{j=1}^{n} 4[a_j^2 F(\omega_j) + A_j] I_1(\omega_j)/\omega_j^2$$

$$+ 2 \sum_{j=1}^{n} \sum_{k=j+1}^{n} \{[Q'/r^2 + F(\omega_j)P'] I_1(\omega_j)/\omega_j^2$$

$$+ [S' + F(\omega_k)R'] I_1(\omega_k)/\omega_k^2\} \quad (33)$$

in which $R_x(u)$ is the design response, $r = \omega_j/\omega_k$ and P', Q', R' and S' are obtained from the solution of the following simultaneous equations

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ u & 1 & s & 1 \\ v & u & t & s \\ o & v & o & t \end{bmatrix} \begin{Bmatrix} P' \\ Q' \\ R' \\ S' \end{Bmatrix} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} \quad (34)$$

where the terms u, v, s, t and $W_1$, $W_2$, $W_3$ and $W_4$ are defined in Appendix I. The values of $I_1(\omega_j)$ and $F(\omega_j)$, as discussed in Ref. 20, are defined in terms of frequency integrals as follows.

$$I_1 = C_f^2 \int_{-\infty}^{\infty} \omega_j^4 \Phi_g(\omega) |H_j(\omega)|^2 d\omega \quad (35a)$$

$$F(\omega_j) = I_2(\omega_j)/I_1(\omega_j) \quad (35b)$$

in which $C_f$ is the peak factor and $I_2(\omega_j)$ is related to the relative velocity response of the oscillator of frequency $\omega_j$ and is defined as

$$I_2(\omega_j) = C_f^2 \int_{-\infty}^{\infty} \omega_j^2 \omega^2 \Phi_g(\omega) |H_j(\omega)|^2 d\omega \quad (36)$$

The characteristics of $I_1(\omega_j)$, $I_2(\omega_j)$ and $F(\omega_j)$ are discussed in Ref. 20. For a Newmark type of smooth ground response spectra, Ref. 23, these terms may be defined by the following equations.

$$I_1(\omega_j) \cong \frac{R^2(\omega_j)}{1+4\beta_j^2} \tag{37}$$

$$I_2(\omega_j) \cong I_1(\omega_j) - \frac{\ln\omega_j - \ln\omega_\ell}{\ln\omega_u - \ln\omega_\ell} A_g^2 \tag{38}$$

in which $R(\omega_j)$ = ground response spectrum value at frequency $\omega_j$ and damping ratio $\beta_j$, $\omega_\ell$ = the control frequency at which the ground spectra starts to drop (such as 9 cps in R.G.-1.60 NRC spectra, Ref. 23), $\omega_u$ = the frequency beyond which there is no spectral amplification of the ground motion (such as 33 cps for the NRC spectra), and $A_g$ is the maximum ground acceleration.

Eq. 33 defines the form of SRSS which should be used to combine modal responses to obtain the design response in a nonproportional damping case. This equation is similar to the modified SRSS equation (19) which is used in a proportional damping case. In the latter case, the contribution of the double summation terms (also called cross terms) can be neglected if the frequencies are far apart, i.e., when ratio $r = \omega_j/\omega_k$ is significantly different from 1.0. In such a case, the correlation between modes is insignificant. Such a simplified statement, however, cannot be made in the case of nonproportional damping. As can be seen from the cross terms in Eqs. 32 or 33, this correlation is affected not only by the closeness of frequencies but also by the phase-lag relationship between the modes. This effect is brought in through the real and imaginary components, $a_j$ and $b_j$, of the complex modes. In general, therefore, for a nonproportional damping case the cross terms should be considered in the analysis. The form of Eq. 33, however, should not present any special difficulty in doing so in an analysis procedure.

The design response of any other quantity of interest, such as member forces, stress, story shear, etc., can also be obtained, if appropriate complex modal response values are used in Eqs. 12 in place of $\{\phi_j\}$'s. For a linear structure, such response quantities are linearly related to displacement mode shapes through the stiffness matrix [K]. However, the elastic and the inertia forces now are not related as simply as they are in the case of proportional damping, Ref. 5. Thus the use of easy-to-use diagonal mass matrix [M], therefore, cannot be made to obtain the elastic force response.

In some engineering structures, the higher modes do not contribute much to the total response, and, therefore, in the proportional damping case the summation to obtain the response is usually performed only over a first few modes. Extension of this procedure to a nonproportional damping case appears to be acceptable but may have to be verified by calculations involving increasing number of modes.

Numerical Results

To verify Eq. 32 (and thus Eq. 33 also), the response results for a system with proportional damping were obtained by the method proposed in this paper, Eq. 32, and by the conventional normal mode approach (which also gives mathematically exact response in this case). A Kanai-Tajimi type, power spectral density function was used as a seismic input. The response results obtained by the two approaches were the same, which numerically verified the validity of Eq. 32.

As an example of a nonproportional damping case, an artificial problem of a three story building with a spring and dashpot to represent the soil half space effect was analyzed. The elements of the mass, stiffness and damping matrices used are given in Appendix II. Here the assumed damping matrix could not be diagonalized by undamped normal mode shapes of the structure. The exact root mean square response results obtained for displacements, elastic forces and the story shears from Eq. 32 are shown in Table 1. The seismic input in the form of Kanai-Tajimi type of power spectral density function, as defined in Ref. 19, has been used. Corresponding results obtained by the approximate normal mode approach in which off-diagonal terms are neglected are also shown in the table for comparison. The magnitude of the percent error in the values obtained by approximate undamped normal mode approach may be noted. This difference will, of course, depend upon the parameters of the system and the type of response obtained and thus should not be considered typical. Larger differences can possibly be obtained for systems where the modal frequencies are close and modes of the response quantity interact with each other significantly. No further attempt was made to construct systems where more severe differences in the approximate and exact solutions could be shown. However, such cases can possibly be encountered in many practical situations, and the use of the method proposed in this paper can be made to obtain accurate response results. Also, no attempt was made to evaluate various approximate procedures described in the literature on the subject vis-a-vis the exact procedure presented here; such an evaluation may not be conclusive as it will be strongly affected by the structural model and response quantities chosen in the evaluation process.

Summary of Proposed Procedure

For the benefit of an analyst who may not wish to go through the details of the formulation given in the paper, the steps required for the implementation of the proposed SRSS procedure to obtain the design response for a prescribed ground spectra are outlined below.

Step 1: Obtain the complex eigenvalues and eigenvectors of the 2n x 2n dimensional eigenvalue problem defined by Eq. 5. As these values occur in complex conjugate pairs, consider only one of them. Only a first few sets may have to be considered.

Step 2: Obtain modal frequencies, $\omega_j$, and damping ratios, $\beta_j$, using Eqs. 14 and 15 for each eigenvalue selected in Step 1.

Step 3: Obtain the complex values of $F_j$ for each selected mode using Eq. 10.

Step 4: Obtain the eigenvectors of the response quantity of interest by appropriate linear transformation relating the response quantity with the relative displacement modes $\{\phi_j\}$, e.g.

$$\{g_j\} = [T]\{\phi_j\} \tag{39}$$

in which $\{g_j\}$ = the jth modal vector of the response quantity and [T] is the appropriate transformation matrix. Only lower half of the eigenvectors which represent the relative displacement quantities should be considered.

Step 5: Obtain $\{q_j\} = F_j\{g_j\}$ and obtain the real and imaginary parts of $\{q_j\} = \{a_j + ib_j\}$.

Step 6: Obtain $I_1(\omega_j)$ and $F(\omega_j)$ using Eqs. 35, 37 and 38 corresponding to each modal frequency $\omega_j$ and damping ratio $\beta_j$.

Step 7: For each pair of cross terms obtain P', Q', R' and S' from the solution of Eqs. 34 with the definitions of various terms therein given in Eqs. 43 of Appendix I.

Step 8: Sum up the modal responses according to Eq. 33 to obtain the total design response.

Summary and Remarks

A modified SRSS approach is presented to obtain the seismic design response of systems with nonproportional damping. Nonproportional damping effects are included exactly in a mathematical sense in the proposed approach. Seismic input defined in the form of ground response spectra or spectral density function can be conveniently used. To cast the procedure in the form of SRSS, the assumption of stationarity of earthquake inputs is made. Since the earthquake motions are nonstationary in character, this assumption is a point for criticism. However, the analytical basis of the conventional method of SRSS or its other modified forms, which provide acceptable and reliable estimates of design response of proportional systems, is also the same, and the validity of such a basis has been verified by numerical simulation studies. In this regard, therefore, the use of the SRSS proposed here for nonproportional systems should be as accurate as the conventional SRSS for proportional systems. Furthermore, the use of response spectrum values in Eq. 33 does introduce the nonstationary effect as these values are obtained from prescribed ground spectra which are generated from ensemble of recorded accelerrograms.

The proposed approach requires the solution of a complex eigenvalue problem. Also, the size of the eigenvalue problem is twice as large as the size for a proportional damping case. This obviously means a significant increase in computational expense, especially if the system is large. However, efficient and accurate computational algorithms of such eigenvalue problems are now available on most computing systems. Therefore, accessibility of such algorithms and a desire to obtain a more analytically accurate design response should justify additional computational cost. More accurate response can also be obtained by time history analysis of a smaller size problem. However, a large number of such calculations with a large ensemble of time histories as input may be necessary to establish a credible value of the design response. This may more than offset the computational cost in the favor of the proposed SRSS scheme.

Acknowledgements

The assistance provided by M. Ghafory-Ashtiany, research assistant at VPI & SU, in numerical verification of the proposed approach is acknowledged. This work is supported by the National Science Foundation through Grant No. PFR-7823095, with William W. Hakala as its Program Manager. This support is gratefully acknowledged. Any opinions, findings and conclusions or recommendations expressed in this publication are those of the writer and do not necessarily reflect the views of the National Science Foundation.

## APPENDIX I - DESIGN RESPONSE

The design response corresponding to the mean square response defined by Eq. 32 for a peak factor value of $C_f$ can be written as follows:

$$R_x^2(u) = C_f^2 \sum_{j=1}^{n} \int_{-\infty}^{\infty} 4(a_j^2\omega^2 + A_j\omega_j^2)|H_j(\omega)|^2\Phi_g(\omega)d\omega$$

$$+ 2C_f^2 \sum_{j=1}^{n} \sum_{k=j+1}^{n} \int_{-\infty}^{\infty} X_{jk}(\omega)|H_j(\omega)|^2|H_k(\omega)|^2\Phi_g(\omega)d\omega \quad (40)$$

Assuming that the peak factor relating an oscillator root mean square response to the ground response spectrum value is also $C_f$, a term under single summation of Eq. 32, in view of Eqs. 35 and 36, can be written as

$$4C_f^2 \int_{-\infty}^{\infty} (a_j^2\omega^2 + A_j\omega_j^2)|H_j(\omega)|^2\Phi_g(\omega)d\omega$$

$$= 4(a_j^2F(\omega_j) + A_j)I_1(\omega_j)/\omega_j^2 \quad (41)$$

To express a cross term into the form given in Eq. 33, the integrand is resolved into partial fractions as,

$$X_{jk}(\omega)|H_j(\omega)|^2|H_k(\omega)|^2 = \frac{P'\omega^2 + Q'\omega_k^2}{(\omega_j^2-\omega^2)^2+4\beta_j^2\omega_j^2\omega^2} + \frac{R'\omega^2 + S'\omega_k^2}{(\omega_k^2-\omega^2)^2+4\beta_k^2\omega_k^2\omega^2} \quad (42)$$

in which P', Q', R' and S' are obtained from solution of four simultaneous Eqs. 34, with various terms therein defined as follows:

$$\left.\begin{aligned} & s = -2r^2(1 - 2\beta_j^2), \quad t = r^4 \\ & u = -2(1 - 2\beta_k^2), \quad v = 1 \\ & W_1 = D_1, \quad W_2 = C_1D_1 + D_2 + E_2 \\ & W_3 = C_1D_2 + C_2D_1 + E_3, \quad W_4 = C_2D_2 \end{aligned}\right\} \quad (43)$$

$$C_1 = -(1 + r^2 - 4\beta_j\beta_k r) \ , \quad C_2 = r^2$$

$$D_1 = 4a_j a_k$$

$$D_2 = 4r(a_j a_k \beta_j \beta_k + b_j b_k \sqrt{1-\beta_j^2}\ \sqrt{1-\beta_k^2} - a_j b_k \beta_k \sqrt{1-\beta_j^2} - b_j a_k \beta_j \sqrt{1-\beta_k^2})$$

$$E_2 = -8(\beta_j r - \beta_k)[a_j a_k(\beta_k - \beta_j r) - (a_j b_k \sqrt{1-\beta_k^2} - b_j a_k r\sqrt{1-\beta_j^2})]$$

$$E_3 = -8r(\beta_k r - \beta_j)[a_j a_k(\beta_k - \beta_j r) - (a_j b_k \sqrt{1-\beta_k^2} - b_j a_k r\sqrt{1-\beta_j^2})]$$

The integrals of these partial fractions can be defined in terms of response spectrum values with the help of Eq. 35 and 36 to give the double summation term of Eq. 33.

Refinements in which the peak factor $C_f$ is considered to depend upon the frequency $\omega_j$ can also be incorporated in this formulation; however, they are considered unnecessary unless the design response quantity is strongly affected by the higher frequency modes. See Ref. 12.

APPENDIX II - MASS, STIFFNESS AND DAMPING MATRICES OF EXAMPLE PROBLEM

The elements of the mass, stiffness and damping matrices used in the example problem are given as follows:

$$[M] = \begin{bmatrix} 300 & 0 & 0 & 0 \\ 0 & 24.0 & 0 & 0 \\ 0 & 0 & 18.0 & 0 \\ 0 & 0 & 0 & 12.0 \end{bmatrix} \text{kips-sec}^2\text{/ft}$$

$$[K] = \begin{bmatrix} 38160 & -2160 & 0 & 0 \\ -2160 & 3600 & -1440 & 0 \\ 0 & -1440 & 2160 & -720 \\ 0 & 0 & -720 & 720 \end{bmatrix} \text{kips/ft.}$$

The damping matrix was constructed from the fixed base structural damping matrix which had 5% damping in all the modes and a soil damper with damping constant C = 1050 kips-sec/ft. This resulted into a combined damping matrix as shown below

$$[C] = \begin{bmatrix} 1081.14 & -20.73 & -7.01 & -3.39 \\ -20.73 & 28.32 & -6.73 & -0.85 \\ -7.01 & -6.73 & 18.07 & -4.33 \\ -3.39 & -.85 & -4.33 & 8.58 \end{bmatrix} \text{kips-sec/ft.}$$

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## APPENDIX IV - NOTATIONS

| | |
|---|---|
| $a_j, b_j$ | - real and imaginary parts of $q_j$ defined in Eq. 21 |
| $A_g$ | - maximum ground acceleration due to gravity |
| $A_j$ | - a factor as defined in Eq. 23 |
| [A],[B] | - 2n x 2n matrix as defined in Eq. 3 |
| C | - damping constant |
| $C_1, C_2$ | - factors defined in Eq. 43 |
| $C_f$ | - peak factor |
| [C] | - damping matrix |
| $D_1, D_2$ | - factors defined in Eq. 43 |
| $E_2, E_3$ | - factors defined in Eq. 43 |
| $F(\omega_j)$ | - factor defined in Eq. 35 |
| $\{g_j\}$ | - vector for a response quantity defined in Eq. 39 |
| $H_j(\omega)$ | - relative displacement frequency transfer function Eq. 24 |
| i | - complex number $\sqrt{-1}$ |
| $I_1(\omega_j), I_2(\omega_j)$ | - factors defined by Eqs. 34 and 36 or 37 and 38 |
| [K] | - stiffness matrix |
| [M] | - mass matrix |
| $p_j$ | - complex eigenvalue of Eq. 5 |
| P',Q',R',S' | - factors defined by Eq. 34 |
| $q_j(u)$ | - uth element of vector $\{q_j\}$ |
| $\{q_j\}$ | - jth element of a response quantity, defined by Eq. 39 |
| r | - frequency ratio = $\omega_j/\omega_k$ |
| s,t,u and v | - defined by Eqs. 43 in Appendix I |
| $t_1, t_2$ | - time variables |
| $U_{jk}$ and $V_{jk}$ | - defined by Eqs. 31 |

| | |
|---|---|
| $x_u(t)$ | - relative displacement response of a point u |
| $\ddot{X}_g(t)$ | - ground acceleration |
| $X'_{jk}(\omega)$, $Y'_{jk}(\omega)$ | - factors defined by Eqs. 28 |
| $X_{jk}(\omega)$, $Y_{jk}(\omega)$ | - factors defined by Eq. 30 |
| $\{y\}$ | - state vector, Eq. 4 |
| $z_j$ | - jth complex valued principal coordinate, Eq. 9 |
| $\beta_j$ | - damping ratio in the jth mode, Eq. 15 |
| $\xi_j$ | - real part of $p_j$ |
| $\{\phi_j\}$ | - jth relative displacement mode shape |
| $[\Phi]$ | - modal matrix |
| $\theta_j$ | - complex part of $p_j$ |
| $\tau_1, \tau_2$ | - dummy time variable |
| $\omega$ | - frequency variable |
| $\omega_j$ | - jth mode frequency |

Table 1 - Root-Mean Square Response of the Structural System with Mass, Stiffness and Damping Matrices as shown in Appendix II

| | Displacement in ft-units | | | Elastic Forces kips | | | Story Shears kips | | |
|---|---|---|---|---|---|---|---|---|---|
| Node or Spring No. | Exact Approach Eq. 32 | Approx. Approach | Percent Diff. | Exact Approach $x10^{-3}$ | Approx. Approach $10^{-3}$ | Percent Diff. | Exact Approach $x10^{-3}$ | Approx. Approach $x10^{-3}$ | Percent Diff. |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| 1 | 0.1700 | .1802 | 5.9 | 5.983 | 6.370 | 6.5 | 6.121 | 6.488 | 6.0 |
| 2 | .6367 | .6623 | 4.0 | .784 | .915 | 16.7 | 1.250 | 1.305 | 4.4 |
| 3 | 1.1448 | 1.1489 | 0.3 | .714 | .735 | 2.8 | .880 | .887 | 0.8 |
| 4 | 1.7411 | 1.7657 | 1.4 | .670 | .724 | 8.1 | .670 | .724 | 8.1 |

## Abstract

A method is presented to obtain seismic design response of linearly behaving structures with nonproportional damping characteristics. For such systems, time history analyses are usually performed to obtain accurate seismic response. To obtain design response, such procedures can become expensive and cumbersome as an ensemble of time histories may have to be considered as seismic input. Other procedures have been developed to ascertain appropriate values of modal dampings so that modal analysis approach and thus commonly adopted square-root-of-the-sum-of-the-squares (SRSS) procedures can be used. These approaches are, however, approximate and numerical results in favor as well as against the use of these approximate procedures have appeared in the literature. The approach presented here considers nonproportional damping effects exactly in analytical sense. It uses the state-vector formulation and is based on random vibration principles. In its final form the approach is similar to conventional SRSS approach, and thus ground response spectrum can be directly used to obtain a design respose. Details of the proposed procedure are outlined for its direct implementation by a potential user.

Civil Engineering Abstract

A modal analysis method which can include nonproportional damping effects accurately is presented to obtain seismic design response of linearly behaving structures. The format of the proposed method is the same as that of the conventional response spectrum SRSS approach. Thus prescribed design response spectra can be used directly to obtain a design response.

Key Words - Earthquakes, Seismic Design Response, Response Spectra, Dynamic Analysis, Damping, Nonproportional Damping, Modal Analysis, Structural Analysis, Dynamics, Vibrations

## APPENDIX III

<u>Expected Values $E\{G_q(\varepsilon)x_r x_s\}$ and $E[G_q(\varepsilon)x_r^2]$:</u>

It is assumed that $\varepsilon$, $x_r$ and $x_s$ are jointly and individually normal random variables. If the excitation is a Gaussian random process, the response of the equivalent linear system represented by Eq. 2 in a give iteration will also be normal. In such a case the assumption of normality for $\varepsilon$, $x_r$ and $x_s$ will be valid. The expected value $E\{G_q(\varepsilon)x_r x_s\}$ can be written in terms of the joint probability density function of the variables as follows

$$E\{G_q(\varepsilon)x_r x_s\} = \int_\varepsilon \int_{x_r} \int_{x_s} G_q(\varepsilon)x_r x_s\ f\ (x_r, x_s, \varepsilon)dx_r dx_s d\varepsilon$$

$$= \int_\varepsilon G_q(\varepsilon)\{\int_{x_r} \int_{x_s} x_r x_s\ f\ (x_r, x_s|\varepsilon)dx_r dx_s\}f(\varepsilon)d\varepsilon \qquad \text{(III.1)}$$

$$= \int_\varepsilon G_q(\varepsilon)\ E(x_r x_s|\varepsilon)\ f\ (\varepsilon)d\varepsilon \qquad \text{(III.2)}$$

For jointly normal variables with zero mean, the conditional mean values and covariance can be written as follows (32):

$$\mu^c_{x_r} = E(x_r|\varepsilon) = \frac{E(x_r\varepsilon)}{\sigma^2_\varepsilon}\ \varepsilon;\ \mu^c_{x_a} = E(x_s|\varepsilon) = \frac{E(x_s\varepsilon)}{\sigma^2_\varepsilon}\ \varepsilon$$

$$C^c_{x_r x_s} = E[(x_r-\mu^c_{x_r})(x_s-\mu^c_{x_s})|\varepsilon] \qquad \text{(III.3)}$$

$$= E(x_r x_s) - \frac{E(x_r\varepsilon)E(x_s\varepsilon)}{\sigma^2_\varepsilon}$$

In terms of these conditional mean values and the covariance, the conditional expected value $E[x_r x_s|\varepsilon]$ can be written as follows:

$$E[x_r x_s|\varepsilon] = \text{cov}(x_r, x_s|\varepsilon) + E(x_r|\varepsilon)E(x_s|\varepsilon)$$

$$= E(x_r x_s) - \frac{E(x_r\varepsilon)E(x_s\varepsilon)}{\sigma^2_\varepsilon} + \frac{E(x_r\varepsilon)E(x_s\varepsilon)}{\sigma^4_\varepsilon}\ \varepsilon^2 \qquad \text{(III.4)}$$

Substituting for this conditional expected value in Eq. III.2 we obtain

$$E\{G_q(\varepsilon)x_r x_s\} = A \int_\varepsilon G_q(\varepsilon)f(\varepsilon)d\varepsilon + B \int_\varepsilon \varepsilon^2 G_q(\varepsilon)f(\varepsilon)d\varepsilon \qquad \text{(III.5)}$$

in which

$$A = E(x_r x_s) - E(x_r\varepsilon)E(x_s\varepsilon)/\sigma_\varepsilon^2$$

$$B = E(x_r\varepsilon)E(x_s\varepsilon)/\sigma_\varepsilon^4 \qquad \text{(III.6)}$$

The values of $E(x_r x_s)$, $E(x_r\varepsilon)$, $\sigma_\varepsilon^2$, etc. can be obtained using Eqs. 21-23. For the strain dependent shear modulus curves, such as shown in Fig. 4, the integrals in Eq. III.5 are obtained by numerical integrations of the density function $f(\varepsilon)$ over $G_q(\varepsilon)$ and $\varepsilon^2 G_q(\varepsilon)$. For a Gaussian input, the density function of strain $\varepsilon$ is a zero mean Gaussian random variable with density function defined as

$$f(\varepsilon) = \exp(-\varepsilon^2/2\sigma_\varepsilon^2)/(\sigma_\varepsilon\sqrt{2\pi}) \qquad \text{(III.7)}$$

The expected value of $G_q(\varepsilon)X_s^2$ are obtained similarly:

$$E[G_q(\varepsilon)X_s^2] = \int_\varepsilon G_q(\varepsilon)E(X_s^2|\varepsilon)f(\varepsilon)d\varepsilon \qquad \text{(III.8)}$$

For jointly normal $X_s$ and $\varepsilon$

$$E(X_s^2|\varepsilon) = \sigma_s^2(1-\rho^2) + \frac{\rho^2\sigma_s^2}{\sigma_\varepsilon^2}\varepsilon^2 \qquad \text{(III.9)}$$

where $\rho$ = correlation coefficient

$$= E(X_s\varepsilon)/\sigma_s\sigma_\varepsilon \qquad \text{(III.10)}$$

Thus:

$$E[G_q(\varepsilon)X_s^2] = \sigma_s^2(1-\rho^2)E[G_q(\varepsilon)] + \frac{\rho^2\sigma_s^2}{\sigma_\varepsilon^2}E[\varepsilon^2 G_q(\varepsilon)] \qquad \text{(III.11)}$$