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DYNAMIC SOIL-STRUCTURE INTERACTION

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16. Abstract (Limit: 200 words)

The report provides a method for a simple extension of response-spectrum procedures in seismic building analysis to include dynamic soil-structure interaction. The analysis utilizes soil impedance functions for a structural foundation. Most of the computational results refer to a rigid mat foundation. The damping that arises in a soil-structure system will generally be nonclassical. A rapid approximate scheme is developed for computing nonclassical modes of a soil-structure system. The computation, which is extremely effective, uses a classical mode of the structure with its base fixed. The differences of the initial and improved modes give direct measures of the significance of the interaction effects. The response-spectrum approach, which is the most reasonable and convenient for the case of classical damping, is found to be applicable to a soil-structure system, including the nonclassical damping that is present. Several examples are presented to demonstrate the capability of the solution method.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Statement of the Soil-Structure Interaction Problem

A soil-structure interaction problem arises when one seeks to relax the rigid ground assumption in a conventional seismic analysis. Major effects of the interaction are flexibility arising from soil compliance and an energy feedback in the form of wave propagation into the soil during vibrations of the structure under investigation. It is especially important to consider interaction effects for massive, stiff, and lightly damped structures. When inelastic deformations of structure occur, the soil-structure interaction effects are expected to be of less significance.

The objective of this investigation is to develop a method for a simple extension of the response-spectrum procedures in seismic building analysis to include dynamic soil-structure interaction. This solution method stems from the so-called impedance approach, which will be explained later. It is intended to permit the design engineer to include the interaction effects in the kind of seismic analysis with which he is familiar.

The problem of accounting for soil-structure interaction was formulated by Seed, Whitman, and Lysmer (1977)* as follows: Given the earthquake ground motions that would occur on the surface of the ground

* Names followed by dates of publication in parentheses refer to the entries in the List of References at end of the text proper.

if the structure were not present (the so-called control, design, or free-field motions), find the dynamic response of the structure. The soil-structure interaction effects, however, should not be confused with the so-called site effects or the effects of the development of unstable soil conditions such as soil liquefaction or excessive settlements. The site effects refer to the fact that the characteristics of the free-field ground motions depend on the properties of a selected site; whereas the interaction effects refer to the fact that the dynamic response of a structure built on that site depends on both the characteristics of the free-field ground motions and the interrelationship of the structural characteristics and the properties of the underlying soil deposits (Veletsos, 1977).

Since the characteristics of the actual free-field motions of the "next" earthquake are never known in a deterministic sense, implicit in the problem is, then, a statistical specification of ground motions. As mentioned, another factor affecting the soil-structure interaction is the (generally nonlinear) characteristics of the soil in the earthquake environment. The determination of these soil properties is not an easy task. In addition, a sensitivity analysis covering ranges of soil properties is always necessary for engineering purposes, regardless of the method of analysis used. Thus, the soil-structure interaction problem is a nonlinear problem of a three-dimensional, infinite degree-of-freedom system subjected to nondeterministic transient disturbances.

It is not surprising that such a complex problem as soil-structure interaction has been one of the most discussed and controversial problems in seismic analysis (e.g. Ad Hoc Group, 1979; Hadjian, 1976; Hall and

Kissenpfenning, 1976; Whitman, 1975; Hadjian, Luco, and Tsai, 1974). One reason for this controversy is that even the most refined analyses possible today fail to provide conclusive solutions. Two simplified methods of approach, namely the so-called finite element approach and the impedance approach, are prominent in the literature. Depending on the problem at hand both approaches are valuable and necessary. For instance, no one would expect the impedance approach to give information about liquefaction of the soil under a building. However, in many cases, the simple impedance approach permits a good engineering approximation of the soil-structure interaction effects.

1.2 Finite Element Approach

As noted by Desai and Abel (1972), the term "finite element method" was first used by Clough (1960). Among many other applications, finite elements have been applied to model the soil in a soil-structure interaction problem (e.g. Lysmer, 1979; Gomez-Masso, Lysmer, Chen, and Seed, 1979; Lysmer, Udaka, Tsai, and Seed,1975). Usually, a large mass of the soil near the structure is discretized by two-dimensional plain-strain elements or axisymmetric solid elements. The design motions defined at one point on the ground are first assumed to be identical over the ground surface and then deconvoluted vertically downward by the theory of one-dimensional wave propagation in order to generate corresponding motions at the horizontal base of the soil model which is assumed to be rigid. Finally, the corresponding base motions are used as input motions to idealized soil-structure model. The assumption of one-dimensional wave propagation over a long distance is, of course, questionable.

Frequently, the number of degrees of freedom for the soil model far exceeds that for the structure which is the real subject of the investigation. Thus the overall efficiency of this type of approach may be very poor (Clough and Penzien, 1975). Indeed, it is very costly, if not impossible, to carry out a nonlinear time-history analysis, a statistical analysis considering design motions, or a sensitivity analysis covering ranges of soil properties.

1.3 Impedance Approach

In the impedance approach, the foundation of the structure is assumed to be rigid. The rigid foundation has, of course, only a few degrees of freedom. The supporting soil is regarded as a half space of a linear solid which may be both viscoelastic and nonhomogeneous (Luco, 1976). Soil impedance functions (force-displacement relationships) for the foundation are found to be frequency dependent. Numerical values of these functions for a variety of cases have been given in the literature in recent years (e.g. Veletsos and Wei, 1971; Luco and Westmann, 1971; Veletsos and Verbic, 1973; Wong, 1975; Luco, 1976; Kausel and Ushijima, 1979). Approximations taking into account the effects of nonlinearity of soil properties have also been given in the literature (Veletsos, 1977; ATC-3 Code, 1978; Rosset, Whitmann, and Dobry, 1973).

Since soil compliance contributes no more than six degrees of freedom to a soil-structure model, a frequency domain analysis can be used more efficiently than the finite element approach does. The frequency domain analysis, however, fail to provide physical insight to the interaction effects, as compared to a time-domain modal analysis conventionally used for the case of rigid soil. In computation, the frequency domain analysis requires solving a set of simultaneous linear algebraic equation for each selected value of frequency. Also, it is usually necessary to select an important range of frequency in order to carry out computations.

Some simple mechanical system with frequency independent properties have been used to approximate impedance functions over a limited and important range of frequency (e.g., Newmark and Rosenblueth, 1971; Richert, Hall, and Woods, 1970; Whitman and Richart, 1967). This type of approximation will be discussed in more detail in Section 4.4. Using this approximation, a linear elastic soil-structure model with constant parameters can be constructed. An important consequence of this is that a time domain analysis can be performed. Various time-domain solution methods suggested in the literature will be reviewed in Section 1.5. The solution method proposed here is also a time-domain analysis.

Some aspects of the impedance approach should be noted here. Even if the structure investigated were massless, the foundation of the structure would not experience a motion identical to the free field ground motion. In other words, the size and rigidity of the foundation modifies the high-frequency part of excitations due to actual spatial variation of the free field ground motion. (Hall, Morgan, and Newmark, 1978). This phenomenon is called "kinematic soil-structure interaction"*

* This name is not an especially fortunate choice. The stiffness of the foundation is important here. Only for a rigid foundation is the term "kinematic" especially helpful.

(Kausel, Whitman, Morray, and Elsabee, 1978), and is distinct from "dynamic soil-structure interaction" as defined in the previous paragraphs.

Usually, the effects of kinematic soil-structure interaction are neglected in practice. This is equivalent to saying that the design ground motion is some sort of average free-field motion in the immediate vicinity of the foundation site under consideration (Whitman, 1975). Special engineering judgment must be used when dealing with large structure and with possible ground motions exhibiting pronounced high-frequency excitations, e.g. nuclear reactor containments subjected to close-in earthquakes (Newmark, 1976).

1.4 Objective of the Present Study

As mentioned in Sections 1.1 and 1.3, a solution method will be proposed to improve the current time-domain, elastic analysis using the impedance approach to the problem of dynamic soil-structure interaction. The typical way of modeling for this type of analysis is summarized as follows. The structure investigated may have a general finite element idealization. The base of the structure is limited to be a mat foundation, which is assumed to be a rigid body resting on soil modeled by some mechanical system. This mechanical "soil" system may consist of a spring, a dashpot, and a mass for each possible degree of freedom of the rigid foundation of the structure (Newmark and Rosenblueth, 1971). Since soil deformations are not of primary concern, this type of foundation impedance representation of the half space can be very useful for practical purposes.

The basic difficulty encountered in analyzing this type of soil-structure model relates to the fact that the damping is nonclassical due to relatively large dashpots in the soil model representing both material damping of soil and geometric damping (energy feedback into the half space). In other words, the problem is a nonclassical damping problem.*

In the case of classical damping, the response-spectrum approach has been regarded as the most reasonable and convenient for elastic aseismic design (Biggs, Hansen, and Holley, 1977). However, it has been commented that the response-spectrum approach apparently could not be applied to a (soil-structure) system with damping coupling (Clough and Penzien, 1975).

Although the nonclassical damping problem can be solved using the so-called nonclassical modal analysis (Foss, 1958), the determination of the nonclassical modes (see Section 2.2.1) of a general, viscously damped system requires a great deal of computational effects (Clough and Mojtahedi, 1976; Clough and Penzien, 1975).

One main aspect of this investigation is to develop a method for a simple extension of the response-spectrum procedure in elastic seismic analysis to include dynamic soil-structure interaction. Another phase of the study seeks to develop an effective and efficient numerical scheme for computing the nonclassical modes of a soil-structure system. The nonclassical mode shapes and frequencies are required in order to apply nonclassical modal analysis or a response-spectrum analysis in the case of nonclassical damping.

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* This has been termed "the damping coupling problem" in the literature (e.g. Clough and Penzien, 1975). Since the presence or absence of coupling depends on the coordinate system used, this could be confusing. What is meant, of course, is that in the undamped modal coordinates, damping coupling is present.

Some attention will be given to alternatives for approximating frequency variation of foundation impedances by a simple mechanical model. Further, the possibility of relaxing the assumption of a rigid mat will be discussed briefly.

1.5 Previous Solution Methods

The basic theory for solving the problem of nonclassical damping was first developed by Routh (1905) in his method of multipliers for initial value problems. A well-known and elegant solution method for transient problems is associated with the name of Foss (1958). The Foss method is a nonclassical modal analysis. As shown in Chapter 2, a small further step combining the Routh method and Gantmacher's transformation (Gantmacher, 1960) yields a useful form of solution for transient problems. The problem, then, becomes a simple extension of the normal mode method valid for the case of classical damping.

Since it may require a great deal of efforts to compute the nonclassical modes needed in a nonclassical modal analysis, Clough and Mojtahedi (1976) suggested solving coupled equations of motion formulated by a use of the lowest classical modes of an undamped soil-structure system by direct integration over time history. Applying Foss's method to the problem of dynamic soil-structure interaction, Jennings and Bielak (1973) defined a kind of modified excitation.

These two methods as well as the nonclassical modal analysis require a time history. However, there is no real advantages, in general, in using a time history analysis as compared with a response-spectrum approach for multi-degree-of-freedom systems, unless one is faced with an actual deterministic input (Newmark and Hall, 1977).

Since the response-spectrum method had not been extended to a problem with damping coupling (or with nonclassical damping), some simplified analyses neglected coupling terms in the generalized damping matrix. Rosset, Whitman, and Dobry (1973) used an undamped mode of a soil-structure system and calculated for it a weighted damping ratio. Similar analyses have been done by Novak (1974) and Rainer (1975). However, there is no indication of when soil-structure interaction need not be taken into account. In order to examine the effects of interaction or to carry out a sensitivity analysis covering ranges of soil properties, the solution of a large eigenvalue problem over and over again may be required.

Another simplified method uses the shape of the fundamental mode of a structure on a fixed base and assigns to it an effective modal damping and frequency to reflect the effects of interaction (Veletsos, 1977; Bielak, 1976; Jennings and Bielak, 1973). The results of this approach have been adopted in the Tentative Provisions for the Development of Seismic Regulations for Buildings prepared by the Applied Technology Council (ATC-3 Code, 1978). The studies were carried out for the case of a shear-beam building with a rigid mat. Caution might be needed when applying this method to other types of buildings and foundations.

1.6 Organization of the Study

In the first portion of Chapter 2, the general theory of a viscously damped, dynamic system with constant parameters is presented. This provides the theoretical basis for the solution method of this study. An initial value problem is solved using Routh's orthogonality relation

(Routh, 1905) and Gantamacher's transformation (Gantamacher, 1960). The problem is then generalized to a transient problem. This theory for non-classical modal analysis turns out to be a logical extension of the classical modal analysis.

The useful form of solution suggests an extension of the response-spectrum procedures to the case of nonclassical damping. A description of possible approximations for this extension is then presented in the remaining part of Chapter 2.

In Chapter 3, a general numerical scheme is developed for computing nonclassical modes by an iterative method.

The general numerical scheme is applied specifically to the problem of dynamic soil-structure interaction in the first portion of Chapter 4. A method is proposed for an effective and efficient computation of the nonclassiscal modes of a soil-structure system starting from the classical modes of the structure with its base fixed.

The remaining part of Chapter 4 shows how the modeling technique of the soil and a founcation can be further improved. Some attention is given to alternatives for approximating the frequency variation of the foundation impedances by a simple mechanical model. Moreover, the possibility of relaxing the assumption of a rigid mat is discussed briefly.

Some numerical results of sample problems will be shown in Chapter 5. The sample problems include computations of nonclassical modes for a simple, idealized system and for two soil-structure systems with rigid-mat or spread-footing foundation. In addition, an example is given to illustrate the computation required in using the extended response-spectrum procedure.

In order to show the generality of the modal improvement technique, an example is given in Appendix A for studying the effects of lateral-torsional coupling. In Appendix B, a useful improvement of the Holzer method for an undamped system is presented although the computation of classical modes is not of primary concern of this study.

1.7 Nomenclature

The symbols used in this study are defined in the text when they first appear. For convenient reference, the more important symbols are summarized here in alphabetical order. Some symbols are assigned more than one meaning; however, in the context of their use there are no ambiguities.

A, a	constants
a_o	dimensionless frequency parameter, $= \frac{R\,\Omega}{V_s}$
{B}	complex eigenvector
[C], $[C_o]$	damping matrix, damping matrix for superstructure
C_Δ, C_θ	soil dashpots
{F} f(t)	applied forces
{h}	vector of story altitudes
I	mass moment of inertia
I_θ	virtual mass moment of inertia of soil
J	constant, $= \{1\}^T [M_o] \{h\}$
[K], $[K_o]$	stiffness matrix, stiffness matrix of superstructure
K_Δ, K_θ	soil springs
K_o, K_{rs}*	static impedance, frequency-dependent complex impledance
[M], $[M_o]$	mass matrix, mass matrix of superstructure

m_Δ	virtual mass of soil
m_t	total mass of building
$\{P\}$, $\{p\}$, p	(real) constant vectors and constant
p	constant, = $\{1\}^T [M_o] \{\phi\}$ in Section 4.2 and 4.3
PSV	pseudovelocity, = $\Omega \cdot$ SD
$\{Q\}$, $\{q\}$, q	(real) constant vector and constant
	constant, = $\{h\}^T [M_o] \{\phi\}$ in Section 4.2 and 4.3
$\{R\}$	residual vector
R	radius of disk
SD, SV	maximum relative displacement and velocity
S_f, T_f	base shear and rocking moment
t, τ	time
$\{U_t\}$, $\{U\}$	total and relative displacement vector
$\ddot{u}_g$	ground acceleration
V_s	shear wave velocity
v_r, W_r	constants, defined in Eqs. (2.26) and (2.28)
χ, $\{\chi\}$	displacement and displacement vector
$\{\chi_o\}$, $\{\dot{\chi}_o\}$	initial displacement and velocity vector
$y_s(t)$, $y_c(t)$	integrals, defined by Eqs. (2.26) and (2.28)
β	damping ratio
δ	variation operator
Δ, θ	foundation translation and rocking due to interaction
λ	eigenvalue, = ω^2
μ	complex eigenvalue, = $- \beta\Omega \pm i\Omega \sqrt{1 - \beta^2}$
ν	Poission's ratio
ξ	damping ratio of superstructure

ρ	mass/unit volume
$\{\phi\},[\Phi]$	(real) modal vector and matrix
$\{\psi\}$	complex eigenvector
$\{\tilde{\psi}\}$	approximate quantity of $\{\psi\}$
$\{\overline{\psi}\}$	conjugate quantity of $\{\psi\}$
Ω, ω	frequency

CHAPTER 2

THEORY OF VISCOUS DAMPING

2.1 General

It is well known that an undamped, linear, dynamic system possesses exactly the same number of normal modes as degrees of freedom. Each normal mode has associated with it a natural frequency and a characteristic shape. When properly released into a state of free vibration, the system can vibrate in any one of its normal mode. A knowledge of normal mode shapes and frequencies is basic to an understanding of the dynamic response of a system under any kind of excitation.

Two major advantages of the normal mode method result from the convenient properties of the normal modes. First, the complicated problem of a multi-degree-of-freedom system can readily be transformed into a set of simple problems of single-degree-of-freedom systems using the orthogonality relationships among mode shapes. Secondly, a good approximation for displacements can often be achieved by including only a few modes, especially in seismic analysis of buildings.

The phenomenon of resonance is closely related to natural frequencies. A state of resonance occurs under harmonic excitations when the excitation frequency coincides with one of the natural frequenices of an undamped, linear, dynamic system. In actual systems, the presence of damping limits the amplitudes of system responses at resonance to finite values.

The presence of damping is also very important in determining responses of a system subjected to transient disturbances. The presentation in the first portion of this chapter summarizes the general theory of a linear, dynamic system having viscous damping. The theory provides the solution method and theoretical basis of the proposed method of this work. Since the theory is a logical extension of the theory of the normal mode method, it is useful for one to note how the derived results reduce in the undamped case to the familiar forms of the normal mode method. The exact theory of viscous damping is followed by a description of possible approximations for a use of the response-spectrum approach in seismic analysis.

2.2 Free Vibration

We shall consider small motions about a stable equilibrium of a discrete, linear, dynamic system with viscous damping. The equations of motion for free vibration of an N degree-of-freedom system with constant parameters can be written in matrix notation as

$$[M]\,\{\ddot{\chi}\} + [C]\,\{\dot{\chi}\} + [K]\,\{\chi\} = \{0\} \tag{2.1}$$

where the vectors $\{\ddot{\chi}\}$, $\{\dot{\chi}\}$, $\{\chi\}$, and $\{0\}$ are the acceleration, velocity, displacement, and null vectors; and the N X N matrices [M], [C], and [K] are the mass, damping, and stiffness matrices. The origin of each coordinate of displacements is taken to correspond with the configuration of equilibrium. The mass and stiffness matrices must be real, symmetric, and positive-definite by virtue of the physical meaning of kinetic energy and the stability of the equilibrium. The damping matrix must also be real, symmetric, and postive semi-definite for a viscously damped system (Lord Rayleigh, 1894). It degenerates to a null matrix for an undamped system.

2.2.1 Nonclassical Modes

We seek solutions of Eq. (2.1) in the form

$$\{\chi\} = Ae^{\mu t}\{\psi\} \tag{2.2}$$

where the constants A, μ and the constant vector $\{\psi\}$ are complex and time invariant, and the independent variable t is time. A characteristic vector $\{\psi\}$ is called a nonclassical mode. It is worth noting that the product of matrices $[C]\,[M]^{-1}[K]$ is symmetric if and only if the non-classical modes can be taken as real and identical to the corresponding normal modes for zero damping (Caughey and O'Kelly, 1965). In these cases, both the modes and the damping are termed classical. When this occurs, the same transformation that diagonalizes the mass and stiffness matrices also diagonalizes the damping matrix.

2.2.2 Roots of Secular Equation

Substituting Eq. (2.2) into Eq. (2.1) and dividing by $Ae^{\mu t}$, we arrive the following equation which defines the complex eigenproblem involved.

$$\mu^2\,[M]\,\{\psi\} + \mu[C]\,\{\psi\} + [K]\,\{\psi\} = \{0\} \tag{2.3}$$

Nontrivial solutions for $\{\psi\}$ exist if, and only if,

$$\det\,|\,\mu^2\,[M] + \mu[C] + [K]\,| = 0 \tag{2.4}$$

Now, the secular equation above is an algebraic equation in μ of degree 2N with real coefficients. There must be N pairs of conjugate roots of μ.

For simplicity, we assume the roots are distinct. If the value of a root μ is substituted into Eq. (2.3), the resulting homogeneous system of linear equation in N unknowns has a coefficient matrix with rank N-1. Therefore, an arbitrary nonzero constant can be assigned to the value of one unknown in the corresponding vector $\{\psi\}$. The values of other unknowns are then uniquely determined (and can be found by means of Cramer's rule). In other words, nonclassical modes exist and are not determined in magnitude. If μ and $\{\psi\}$ satisfy Eq. (2.3), so do their conjugates $\bar{\mu}$ and $\{\bar{\psi}\}$. Thus, there also exist N conjugate pairs of nonclassical modes.

Premultiplying Eq. (2.3) by $\{\bar{\psi}\}^T$, the transpose of the conjugate vector of $\{\psi\}$, yields

$$(\{\bar{\psi}\}^T [M] \{\psi\}) \mu^2 + (\{\bar{\psi}\}^T [C] \{\psi\}) \mu + (\{\bar{\psi}\}^T [K] \{\psi\}) = 0 \qquad (2.5)$$

The first and third coefficients of the above second-degree equation in μ are positive because of the positive-definiteness and symmetry of the mass and stiffness matrices. This can be easily proved by separating the real and imaginary parts of conjugate nonclassical modes into two real vectors. Also, the second coefficient is non-negative. Thus, roots of Eq. (2.5) can be written as

$$\mu = -\beta\Omega \pm i\,\Omega\sqrt{1-\beta^2} \qquad (2.6)$$

where

$$\Omega = \left(\frac{\{\overline{\psi}\}^T [K] \{\psi\}}{\{\overline{\psi}\}^T [M] \{\psi\}} \right)^{1/2} > 0 \,, \tag{2.7}$$

$$\beta = \frac{1}{2\Omega} \cdot \frac{\{\overline{\psi}\}^T [C] \{\psi\}}{\{\overline{\psi}\}^T [M] \{\psi\}} \geq 0 \,, \tag{2.8}$$

and $i^2 = 1$. The imaginary part of μ represents the damped angular frequency of oscillation about the equilibrium configuration (see also Eq. (2.2)). If the damping matrix is positive-definite, the negative real part of μ characterizes the decay rate of the oscillations. For convenience, β and Ω will be called modal damping and frequency respectively. Notice that we did not assume vibration of any sorts. For usual cases, in which the value of modal damping is smaller than unit, the solution will represent a vibratory motion.

2.2.3 General Solution

The general solution of Eq. (2.1) is then of the form

$$\{\chi\} = \sum_{r=1}^{2N} A_r \, e^{\mu_r t} \, \{\psi_r\} \tag{2.9a}$$

Differentiating the displacement vector above with respect to time gives the velocity vector as

$$\{\dot{\chi}\} = \sum_{r=1}^{2N} A_r \, \mu_r \, e^{\mu_r t} \, \{\psi_r\} \tag{2.10a}$$

The 2N complex unknowns in modal amplitudes A_r are determined by the 2N complex initial values of displacements and velocities.

In usual applications, the initial values are all real. It will be shown later by Eq. (2.19) or Eq. (2.20) that for these cases, there exist N conjugate pairs of modal amplitudes such that

$$\{\chi\} = \sum_{r=1}^{N} (A_r e^{\mu_r t} \{\psi_r\} + \bar{A}_r e^{\bar{\mu}_r t} \{\bar{\psi}\}) \tag{2.9}$$

and

$$\{\chi\} = \sum_{r=1}^{N} (A_r \mu_r e^{\mu_r t} \{\psi_1\} + \bar{A}_r \bar{\mu}_r e^{\bar{\mu}_r t} \{\bar{\psi}_r\}) \tag{2.10}$$

Now, the 2N real unknowns in modal amplitudes A_r are determined by the 2N real initial values of displacements and velocities.

The general solution, Eq. (2.9), can be transformed into the following form of solution (Gantmacher, 1960) by separating real and imaginary parts of complex constants and vectors, and by proper arrangements.

$$\{\chi\} = \sum_{r=1}^{N} e^{-\beta_r \Omega_t t} \cdot (\{P_r\} \operatorname{Sin} \Omega_r \sqrt{1 - \beta_r^2}\, t + \{Q_r\} \operatorname{Cos} \Omega_r \sqrt{1 - \beta_r^2}\, t) \tag{2.11}$$

where $\{P_r\}$ and $\{Q_r\}$ are real vectors defined by

$$\{P_r\} = -2\left\{(\mathrm{Im}A_r)(\mathrm{Re}\{\Psi_r\}) + (\mathrm{Re}A_r)(\mathrm{Im}\{\Psi_r\})\right\} \tag{2.11}$$

$$\{Q_r\} = 2\left\{(\mathrm{Re}A_r)(\mathrm{Re}\{\Psi_r\}) - (\mathrm{Im}A_r)(\mathrm{Im}\{\Psi_r\})\right\} \tag{2.13}$$

In Eqs. (2.12) and (2.13) above, the notations "Re" and "Im" are read "the real part of" and "the imaginary part of". Thus, in a pair of conjugate nonclassicla modes, there are two real displacement shapes, {P} and {Q}, "chasing" one another with a 90 degree phase difference. There exist no stationary nodal points, in general, as those of a classical mode.

2.3 Orthogonality

Premultiplying Eq. (2.3) by $\{\Psi_s\}^T$, the transpose of a nonclassical mode, we get

$$\mu_r^2\{\Psi_s\}^T[M]\{\Psi_r\} + \mu_r\{\Psi_s\}^T[C]\{\Psi_r\} + \{\Psi_s\}^T[K]\{\Psi_r\} = 0 \tag{2.14a}$$

or, equivalently,

$$\mu_s^2\{\Psi_r\}^T[M]\{\Psi_s\} + \mu_s\{\Psi_r\}^T[C]\{\Psi_s\} + \{\Psi_r\}^T[K]\{\Psi_s\} = 0 \tag{2.14b}$$

Since the matrices are symmetric, the difference of Eq. (2.14 a) and Eq. (2.14b) gives

$$(\mu_r + \mu_s)\{\Psi_r\}^T[M]\{\Psi_s\} + \{\Psi_r\}^T[C]\{\Psi_s\} = 0, \text{ if } \mu_r \neq \mu_s \tag{2.15}$$

Similarly, elimination of the second terms on the left side of Eq. (2.14a) and Eq. (2.14b) by proper multiplication and subtraction gives

$$\mu_r\mu_s\{\Psi_r\}^T[M]\{\Psi_S\} - \{\Psi_r\}^T[K]\{\Psi_S\} = 0, \text{ if } \mu_r \neq \mu_s \tag{2.16}$$

Equations (2.15) and (2.16) are the orthogonality conditions for nonclassical modes, first obtained by Routh using his method of multipliers* (Routh, 1905). These conditions are exactly the same as those of Foss's method (Foss, 1958). In the undamped case, Eq. (2.15) and Eq. (2.16) degenerate to the so-called M-Orthogonality and K-Orthogonality of the normal mode method.

2.4 Initial-Value and Transient Problems

2.4.1 Initial-Value Problems

Given prescribed values of initial displacements and initial velocities at time zero, $\{x_o\}$ and $\{\dot{x}_o\}$, Eq. (2.9a) and Eq. (2.10a) reduce to

$$\{\dot{x}_o\} = \{\dot{x}\}_{t=o} = \sum_{r=1}^{2N} A_r\{\Psi_r\} \tag{2.17}$$

and

$$\{\dot{x}\} = \{\dot{x}\}_{t=o} = \sum_{r=1}^{2N} A_r\mu_r\{\Psi_r\} \tag{2.18}$$

By the orthogonality relation Eq. (2.15), it can be shown after substitutions and expansions that

* It seems that this contribution of Routh has never been noted in the literature.

$$A_r = \frac{\mu_r\{\Psi_r\}^T[M]\{\chi_o\} + \{\Psi_r\}^T[M]\{\dot{\chi}_o\} + \{\Psi_r\}^T[C]\{\chi_o\}}{2\mu_r\{\Psi_r\}^T[M]\{\Psi_r\} + \{\Psi_r\}^T[C]\{\Psi_r\}} \tag{2.19}$$

Similarly, by the orthogonality relation Eq. (2.16),

$$A_r = \frac{\mu_r\{\Psi_r\}^T[M]\{\dot{\chi}_o\} - \{\Psi_r\}^T[K]\{\chi_o\}}{\mu_r^2\{\Psi_r\}^T[M]\{\Psi_r\} - \{\Psi_r\}T[K]\{\Psi_r\}} \tag{2.20}$$

In the most general cases where $\{\chi_o\}$ and $\{\dot{\chi}_o\}$ contain 2N complex constants, 2N complex unknowns of A_r in Eq. (2.7) can be found by the use of either Eq. (2.19) or Eq. (2.20). In usual applications of initial value problems, however, the initial values, $\{\chi_o\}$ and $\{\dot{\chi}_o\}$, are all real. In these cases, the 2N complex unknown will be in N conjugate pairs since the replacement of μ and $\{\Psi\}$ in the right side of Eq. (2.19) or Eq. (2.20) by their conjugates $\bar{\mu}$ and $\{\bar{\Psi}\}$ yields $\bar{A}_r$, the conjugate of A_r in the left side of equation.

In summary, we have N conjugate pairs of parameters A_r, μ_r, and $\{\Psi_r\}$ for usual applications of initial value problems. The general solution of Eq. (2.1) can then be expressed by Eq. (2.9) as the following

$$\{\chi\} = \sum_{r=1}^{N} (A_r e^{\mu_r t}\{\Psi_r\} + \bar{A}_r e^{\bar{\mu}_r t}\{\bar{\Psi}_r\}) \tag{2.9}$$

Given 2N real initial values $\{\chi_0\}$ and $\{\dot{\chi}_o\}$, the 2N real unknowns (N conjugate pairs) A_r can be found by either Eq. (2.19) or Eq. (2.20) provided that the complex eigenproblem Eq. (2.3) has been solved in advance.

2.4.2 Transient Problems

Given applied forces $\{F\}f(t)$ with real vector $\{F\}$ independent of time, the instantaneous momentum input to a dynamic system at time τ and within interval $d\tau$ is

$$[M]\{\dot{\chi}\}_{t=\tau} = \{F\}f(\tau)d\tau \tag{2.21}$$

Determination of the consequent responses due to the particular pulse $\{F\}f(\tau)d\tau$ is then an initial value problem. The equivalent initial velocities are determined by the above equation and are real. The initial displacements are null. By Eq. (2.11) and Eq. (2.19), resulting displacements due to the particular pulse at time τ are

$$\{\chi\}_{t>\tau} = \sum_{r=1}^{N} \{p_r\} \frac{1}{\Omega_r\sqrt{1-\beta_r^2}} e^{-\beta_r\Omega_r(t-\tau)} \mathrm{Sin}\,\Omega_r\sqrt{1-\beta_r^2}(t-\tau)f(\tau)d\tau$$

$$+\sum_{r=1}^{N} \{q_r\} \frac{1}{\Omega_r\sqrt{1-\beta_r^2}} e^{-\beta_r\Omega_r(t-\tau)} \mathrm{Cos}\,\Omega_r\sqrt{1-\beta_r^2}(t-\tau)f(\tau)d\tau \tag{2.22}$$

where

$$\{p_r\} = -2\Omega_r\sqrt{1-\beta_r^2}\,\left[(\mathrm{Im}\alpha_r)(\mathrm{Re}\{\Psi_r\}) + (\mathrm{Re}\alpha_r)(\mathrm{Im}\{\Psi_r\})\right], \tag{2.23a}$$

$$\{q_r\} = 2\Omega_r\sqrt{1-\beta_r^2}\,\left[(\mathrm{Re}\alpha_r)(\mathrm{Re}\{\Psi_r\}) - (\mathrm{Im}\alpha_r)(\mathrm{Im}\{\Psi_r\})\right], \tag{2.23b}$$

and

$$\alpha_r = \frac{\{\Psi_r\}^T\{F\}}{2\mu_r\{\Psi_r\}^T M\{\Psi_r\} + \{\Psi_r\}^T C\{\Psi_r\}} \tag{2.23c}$$

By the linearity of the above equations, transient displacements of a system initially at rest can be obtained by integrating Eq. (2.22) over the time history of the applied forces.

$$\{x\} = \sum_{r=1}^{N} \{p_r\} \int_{0-}^{t} \frac{1}{\Omega_r \sqrt{1-\beta_r^2}} e^{-\beta_r \Omega_r (t-\tau)} \operatorname{Sin}\Omega_r \sqrt{1-\beta_r^2}\,(t-\tau) f(\tau) d\tau$$

$$+ \sum_{r=1}^{N} \{q_r\} \int_{0}^{t} \frac{1}{\Omega_r \sqrt{1-\beta_r^2}} e^{-\beta_r \Omega_r (t-\tau)} \operatorname{Cos}\Omega_r \sqrt{1-\beta_r^2}\,(t-\tau) f(\tau) d\tau \qquad (2.24)$$

The solution, Eq. (2.24), is exact for a viscously damped, linear, dynamic system initially at rest. If the system is not initially at rest, a suitable solution of an initial value problem should be added to Eq. (2.24).

The theory of viscous damping has now been completed as a logical extension of the theory of the normal mode method. By a different approach, Foss (1958) derived a solution equivalent to Eq. (2.24). However, the present form of solution is more convenient in engineering applications such as a use of the response-spectrum approach discussed in the next section. For usual cases of small dissipation, one may also expect the (Euclidean) vector norm of a vector $\{p_r\}$ is larger than that of the corresponding vector $\{q_r\}$, which vanishes in cases of classical damping.

2.5 Response-Spectrum Approach

The recorded ground accelerations of past earthquakes or earthquake models provide a basis of the rational design of structures to resist earthquakes. Responses of a viscously damped, linear, dynamic system to earthquake ground motions can be formulated by the close-form expression Eq. (2.24). For elastic aseismic design, however, we are interested in determination of the maximum* value of certain responses, rather than detailed description of the response over the whole time history. For cases of classical damping, the response-spectrum approach is the most convenient for this purpose. Of course, this approach gives only approximation to the peak response. The purpose of this section is to show how the response-spectrum approach can also be applied to cases of nonclassical damping, without introducing further approximations than those made in usual cases of classical damping.

2.5.1 Basic Definitions

In a deformation spectrum, the maximum relative displacement SD (relative to the ground) is plotted for single-degree-of-freedom system, as a function of system frequency Ω, system damping ratio β, and a selected time history of the ground acceleration $\ddot{U}_g(t)$.

$$SD = \max_t \left| y_s(t) \right| \tag{2.25}$$

where

$$y_s(t) = \int^t \frac{-1}{\Omega\sqrt{1-\beta^2}} e^{-\beta\Omega(t-\tau)} \operatorname{Sin}\Omega\sqrt{1-\beta^2}(t-\tau)\, \ddot{U}_g(\tau)d\tau \tag{2.26}$$

Differentiating the above equation with respect to time yields the relative velocity as the following

---

* In this section, the word "maximum" is understood as meaning "maximum among absolute values of the quantity under consideration".

$$\dot{y}_s(t) = -\beta\Omega y_s(t) + \sqrt{1-\beta^2}\, y_c(t) \tag{2.27}$$

in which

$$y_c(t) = \int_0^t \frac{-1}{\Omega\sqrt{1-\beta^2}}\, e^{-\beta\Omega(t-\tau)} \cos\Omega\sqrt{1-\beta^2}(t-\tau)\, \ddot{U}_g(\tau)d\tau \tag{2.28}$$

Notice that $y_c(t)$ and $y_s(t)$ differ from one another only by a cosine or a sine in the integrand. Both of them are essential to the general solution of transient displacements of a nonclassically damped system, Eq. (2.24).

Theoretically speaking, the maximum of $y_c(t)$ can also be plotted in the same manner as we have done for the maximum relative displacement SD. However, such information is not currently available in general. For immediate uses, we shall seek approximate values via a relation between the following velocity terms. The maximum relative velocity SV and the pseudovelocity PSV are defined by

$$SV = \max\left|\dot{y}_s(t)\right| \tag{2.29}$$

and

$$PSV = \Omega . SD \tag{2.30}$$

2.5.2 Conventional Approximations

For usual cases of small dissipation in earthquake engineering, the pseudovelocity PSV is found nearly equal to the maximum relative valocity SV for systems with moderate or high frequencies (Newmark, 1970), i.e.

$$SV \simeq PSV \tag{2.31}$$

or, equivalently,

$$\max \left| -\beta\Omega y_s(t) + \Omega\sqrt{1-\beta^2}\, y_c(t) \right| \simeq SD$$

For the maximum of $y_c(t)$, this gives possible values ranging approximately from

$$\frac{1-\beta}{\sqrt{1-\beta^2}} \cdot SD \text{ to } \frac{1+\beta}{\sqrt{1-\beta^2}} \cdot SD$$

Thus, implicit in Eq. (2.31) is

$$\max \left| y_c(t) \right| \simeq SD \tag{2.32}$$

In fact, Eqs. (2.32) and (2.31) are identical in the undamped case. The approximation Eq. (2.32) is recommended for aseismic design purposes, at least, for the time being. In general, this will not introduce a significant error except at the relatively low frequencies, say lower than 1 Hz (Hudson, 1962).

The approximation, Eq. (2.31), has been used in earthquake engineering for single-degree-of-freedom systems. For multi-degree-of-freedom system with classical damping, approximations conventionally used for combining modal maxima to predict a peak response of system are the absolute-sum method and the SRSS method. In the absolute-sum method, maximum responses of the classical modes are added up to yield an upper bound to the solution. The approximation of the peak response by the square root of the sum of the square (SRSS) of maximum responses of the classical modes is commonly used, with special consideration given to closely spaced modes, which may be arithmetically additive (Rosenblueth and Elorduy, 1969; Nuclear Regulatory Commission, 1976).

2.5.3 Modal Maximum

Determination of maximum responses of a pair of nonclassical modes is not so obvious as that of a classical mode. This can be seen from the following equation representing the transient displacements occurring at one point and extracted from a pair of nonclassical modes (see Eq. (2.24)).

$$\chi = p \cdot y_s(t) + q \cdot y_c(t) \tag{2.33}$$

In the above equation, the p and q are the interested values taken from two real vector {p} and {q} in Eq. (2.24); and $y_s(t)$ and $y_c(t)$ are defined by Eqs. (2.26) and (2.28).

We shall discuss three possible approaches of approximation, for the prediction of the maximum displacement of the above equation. For the prediction of maximum values of other response quantities, the p and q in Eq. (2.33) shall have other suitable interpretations.

The first approach follows the reasoning of the absolute-sum method. Adopting the approximation Eq. (2.32), we have

$$\max |\chi(t)| \simeq SD.(|p| + |q|) \tag{2.34}$$

This may give an upper bound to the solution. Although $y_s(t)$ and $y_c(t)$ have exactly the same value of frequency, the above equation needs not be the only choice of approximation. As shown in Eq. (2.27), maxima of $y_x(t)$ and $y_c(t)$ do not occur at the same time. When $y_s(t)$ reaches its maximum value SD, $y_c(t)$ takes the value

$$\frac{\beta}{\sqrt{1-\beta^2}} \cdot SD,$$

which is much less than SD, the value assumed for the maximum of $y_c(t)$.

The SRSS rule of approximation provides a less conservative prediction to the solution. Adopting the approximation Eq. (2.32), we get

$$\max\left|x(t)\right| \simeq SD.\sqrt{p^2 + q^2} \qquad (2.35)$$

However, we cannot assert whether this prediction will be conservative or not.

A third approach is aimed at giving a close bound to the solution, for some important special cases. First, we seek a good lower bound of the solution. When $y_s(t)$ reaches its stationary values, occuring at time t_D, setting Eq. (2.27) equal to zero yields

$$y_c(t_D) = \frac{\beta}{\sqrt{1-\beta^2}} y_s(t_D) \qquad (2.36)$$

Substituting the above equation into Eq. (2.33) gives

$$x(t_D) = (p + q \frac{\beta}{\sqrt{1-\beta^2}}) \cdot y_s(t_D) \qquad (2.37)$$

The maximum of $\chi(t_D)$ is, of course, not greater than the maximum of $\chi(t)$. Using Eq. (2.25), we find

$$\max \left|\chi(t)\right| \geq \left| p + q \frac{\beta}{\sqrt{1-\beta^2}} \right| \cdot \mathrm{SD} \tag{2.38}$$

For an upper bound of the solution, we consider stationary values of $\chi(t)$, occurring at time t_m. Setting the first derivative of $\chi(t)$, Eq. (2.33), with respect to time equal to zero yields

$$\begin{aligned} 0 &= p\,(-\beta\Omega) y_s(t_m) + p\,(\Omega\sqrt{1-\beta^2}) y_c(t_m) \\ &+ q\,(-\beta\Omega) y_c(t_m) - q\,(\Omega\sqrt{1-\beta^2})\, y_s(t_m) \\ &- q\,\frac{\ddot{U}_g(t_m)}{\Omega\sqrt{1-\beta^2}} \end{aligned} \tag{2.39}$$

or, equivalently,

$$y_c(t_m) = \frac{P\cdot\frac{\beta}{\sqrt{1-\beta^2}} + q}{P-q\cdot\frac{\beta}{\sqrt{1-\beta^2}}} \cdot y_s(t_m) + \frac{q}{P-q\cdot\frac{\beta}{\sqrt{1-\beta^2}}} \cdot \frac{\ddot{U}_g(t_m)}{\Omega^2(1-\beta^2)} \tag{2.40}$$

if the denominators do not vanish. Substituting the above equation back into Eq. (2.33) gives

$$\chi(t_m) + \left(P + q \cdot \frac{P\cdot\frac{\beta}{\sqrt{1-\beta^2}} + q}{P-q\cdot\frac{\beta}{\sqrt{1-\beta^2}}}\right) y_s(t_m) + \frac{q}{P-q\cdot\frac{\beta}{\sqrt{1-\beta^2}}} \cdot \frac{\ddot{U}_g(t_m)}{\Omega^2(1-\beta^2)} \tag{2.41}$$

The maximum of the above equation is the exact solution. However, we can only assert that the maxima of $y_s(t_m)$ and $\ddot{U}_g(t_m)$ are not greater than SD and the maximum ground acceleration $\ddot{U}_{g\ max}$ respectively. Nevertheless, we have the following upper bound of the solution.

$$\max\left|x(t)\right| \leq \left| P + q \cdot \frac{P \cdot \frac{\beta}{\sqrt{1-\beta^2}} + q \pm q \cdot \frac{\ddot{U}_g max}{\Omega^2(1-\beta^2) \cdot SD}}{P - q \cdot \frac{\beta}{\sqrt{1-\beta^2}}} \right| \cdot SD \qquad (2.42)$$

in which a proper sign should be taken for the "$\pm$" above in order to give the greater value for the right side of equation. Combining Eq. (2.42) and Eq. (2.38), we find

$$\left|P\left(1 + \frac{q}{P}\frac{\beta}{\sqrt{1-\beta^2}}\right)\right| \leq \frac{\max\left|x(t)\right|}{SD} \leq$$

$$\leq \left| P\left(1 + \frac{q}{P} \cdot \frac{\frac{\beta}{\sqrt{1-\beta^2}} + \frac{q}{P} \pm \frac{q}{P} \cdot \frac{U_g max}{SD \cdot \Omega^2(1-\beta^2)}}{1 - \frac{q}{P}\frac{\beta}{\sqrt{1-\beta^2}}}\right|\right. \qquad (2.43)$$

We may now summarize the three approaches discussed above. The exact expression Eq. (2.43) is specially useful to usual cases of small dissipation if

$$\left|\frac{q}{P}\right| << 1 \qquad (2.44)$$

and

$$\left|\frac{\ddot{U}_g\max}{SD\Omega^2}\right| < 1 \qquad (2.45)$$

The inequality Eq. (2.44) may be expected to hold in the majority of cases since the parameter q vanishes in cases of classical damping. Also, from experience in using response-spectrum (Figures 2.1 to 2.5), we do expect a general relation as Eq. (2.45) if the frequency is higher than, say, 2Hz. For the frequency region between 2 and 8 Hz in a design spectrum, for example, the left side of Eq. (2.45) may take a value ranging from 1/3 to 2/3 for a damping ratio ranging from 0.01 to 0.10 (Newmark and Hall, 1977). Either the absolute-sum or the SRSS methods can be used for any possible values of p and q. Because the approximation Eq. (2.32) is used for the time being, Eqs. (2.34) and (2.35) should not be applied to systems with relatively low frequencies. Nevertheless, all the three approaches are applicable to a wide range of practical design of aseismic structures (such as nuclear power plants). Some numerical examples will be given in Section 5.4.

2.5.4. Peak Response

Consideration has been given to the prediction of the maximum response of a pair of nonclassical modes. One reason of doing this is that information of maximum values of the function $y_c(t)$ is not currently available. Also we observed that although the $y_c(t)$ and $y_s(t)$ in a pair of nonclassical modes have exactly the same value of frequency, their maximum values do not occur at the same time for usual applications of small dissipation. Thus, the absolute-sum method need not be the only choice of computation, and may be too conservative.

The remaining problem is how to combine maximum responses of different pairs of nonclassical modes to predict the peak response of a nonclassically damped system. It is expected that the same rules of conventional approximations in combining (classical) modal maxima (see Section 2.5.2) are also applicable for this purpose. There appears to be no more objection to doing this here than for classical damping. If the frequencies are very different, the maxima may not occur at the same time, and the SRSS method can be used. If the frequencies are close or even the same, the maxima may well occur at the same time. In these cases, either the absolute-sum method or the extension of the SRSS method given for closely spaced modes (Rosenblueth and Elorduy, 1969; Nuclear Regulatory Commission, 1976) can be used.

CHAPTER 3

EIGENVALUE PROBLEMS

3.1 General

Before the theory of viscous damping in Chapter 2 can be applied, it is first necessary to solve the complex eigenvalue problem defined by Eq. (2.3). Although methods for the numerical solution for the normal modes of an undamped system have been developed extensively in the literature (Wilkison, 1965), the computation of the nonclassical modes of a viscously damped system has attracted the attention of relatively few research workers (Foss, 1958; Hurty and Rubinstein, 1967). The purpose of this chapter is to develop a computational scheme for solving complex eigenvalue problems. It will be shown that one conventional method for finding classical modes can be extended to the nonclassical case.

The method developed is presented in Section 3.3 as a modified version of the Robinson-Harris method. The original method (Robinson and Harris, 1971), summarized in Section 3.2, improves an approximate classical mode and frequency extremely effectively by an application of the Newton-Raphson technique. The approximate method, in the classical case, can remove some small coupling that is present if the modes are only approximate. The scheme will be altered to treat nonclassical damping in which coupling may include damping coupling.

3.2 Robinson-Harris Method

The general eigenvalue problem for an undamped, linear, dynamic system is in the form of

$$-\lambda\ [M]\ \{\phi\} + [K]\ \{\phi\} = \{0\} \tag{3.1}$$

where the matrices [M]and [K] are the mass and stiffness matrices; the eigenvalue λ is square of a natural frequency ω; and the eigenvector $\{\phi\}$

is a classical mode.

The method developed by Robinson and Harris (1971) is an application of the Newton-Raphson technique of improving an eigenvalue λ and the corresponding eigenvector $\{\phi\}$ from approximations of the eigenvalue and eigenvector, $\tilde{\lambda}$ and $\{\tilde{\phi}\}$. If an approximate eigenvalue and eigenvector are substituted into Eq. (3.1), we get a residual vector $\{R\}$ instead of a null vector.

$$-\tilde{\lambda}\ [M]\ \{\phi\} + [K]\ \{\tilde{\phi}\} = \{R\} \tag{3.2}$$

The object is to remove the residual by changing both $\{\tilde{\phi}\}$ and $\tilde{\lambda}$.

Let

$$\lambda = \tilde{\lambda} + \delta\lambda \tag{3.3}$$

and

$$\{\phi\} = \{\tilde{\phi}\} + \delta\{\phi\} \tag{3.4}$$

in which a variation, δf, represents an unknown quantity. By substituting Eq. (3.3) and Eq. (3.4) into Eq. (3.1) and neglecting second- and higher-order terms in δ quantities, we find the linear system

$$\left[[K] - \tilde{\lambda}[M]\right]\delta\ \{\phi\} + \delta\lambda[M]\{\tilde{\phi}\} = -\{R\} \tag{3.5}$$

where the residual $\{R\}$ is determined from Eq. (3.2). In Eqs. (3.5) above, the number of unknowns exceeds the number of equations by one, therefore, an additional equation is generally needed to get a solution. The additional equation is taken as

$$\{\tilde{\phi}\}^T[M]\delta\{\phi\} = 0 \tag{3.6}$$

This guarantees that the allowable change in the eigenvector is orthogonal to the approximate eigenvector with respect to the mass matrix. The side condition Eq. (3.6) prevents unlimited change in the eigenvector in its own direction. This would occur in the absence of a side condition such as

Eq. (3.6) since the eigenvector is not determined in magnitude.

The resulting set of simultaneous linear algebraic equation formed from Eqs. (3.5) and Eq. (3.6) can be written in partitioned form as

$$\begin{bmatrix} [K] - \tilde{\lambda}\,[M] & -[M]\,\{\tilde{\phi}\} \\ -\{\tilde{\phi}\}^T\,[M] & 0 \end{bmatrix} \begin{Bmatrix} \delta\,\{\phi\} \\ \delta\lambda \end{Bmatrix} = \begin{Bmatrix} \{-R\} \\ 0 \end{Bmatrix} \tag{3.7}$$

The set of simultaneous linear algebraic equation may be solved by Gaussian elimination, or by any other suitable technique, to yield correction quantities $\delta\{\phi\}$ and $\delta\lambda$. A better approximation of the eigenvalue and eigenvector can be obtained by adding the corrections to the corresponding values of originally approximated quantities, Eq. (3.3) and Eq. (3.4). The whole process may be repeated (with the new quantities on the left side of Eq. (3.3) and Eq. (3.4) as the new approximations). The residual vector will be reduced and the procedure is terminated when the residual vector is within a prescribed allowable tolerance.

It has been proved formally (Robinson and Harris, 1971) that the convergence of the eigenvalue and eigenvector is more rapid than a second order process. The generality of the method is illustrated by many sample problems in the original paper cited above.

3.3 Complex Eigenvalue Problems

We shall proceed to develop a modified version of the Robinson-Harris method for a linear, dynamic system with viscous damping. The complex eigenvalue problem under consideration is defined by Eq. (2.3).

$$\mu^2\,[M]\,\{\psi\} + \mu[c]\,\{\psi\} + [k]\,\{\psi\} = \{0\} \tag{2.3}$$

in which the eigenvector $\{\psi\}$ is a nonclassical mode, and the eigenvalue μ determines the corresponding modal damping β and frequency Ω.

$$\mu = -\beta\Omega \pm i\Omega\sqrt{1 - \beta^2} \tag{2.6}$$

It is obvious that Eq. (2.3) would reduce to Eq. (3.1) if the damping matrix [c] were null.

If an approximate eigenvalue $\tilde{\mu}$ and approximate eigenvector $\{\tilde{\psi}\}$ are substituted into Eq. (2.3) above, we have

$$\tilde{\mu}^2 [M]\{\psi\} + \tilde{\mu}\,[c]\{\tilde{\psi}\} + [K]\,\{\tilde{\psi}\} = \{R\} \tag{3.8}$$

Again, the objective is to remove the residual vector $\{R\}$ in the above equation by changing both $\{\tilde{\Psi}\}$ and $\tilde{\mu}$.

Let

$$\mu = \tilde{\mu} + \delta\mu \tag{3.9}$$

and

$$\{\Psi\} = \{\tilde{\Psi}\} + \delta\{\Psi\} \tag{3.10}$$

in which a variation, δf, represents an unknown correction quantity. After expanding Eq. (2.3) above by Eq. (3.9) and (3.10), and then neglecting second-and higher-order terms in δ quantities, we have

$$(\tilde{\mu}^2[M] + \tilde{\mu}[c] + [K])\delta\{\Psi\} + \delta\mu(2\tilde{\mu}[M]\{\tilde{\Psi}\} + [c]\{\tilde{\Psi}\}) = -\{R\} \tag{3.11}$$

the residual $\{R\}$ is being found from Eq. (3.8). In Eqs. (3.11) above, the number of unknowns exceeds the number of equations by one, therefore, an additional equation is generally needed to get a solution.

We now seek a side condition to prevent unlimited drift of the eigenvector, which is not determined in magnitude. By the orthogonality relation Eq. (2.15), one expression, Eq. (2.19), for solving the initial value problems is found as

$$A = \frac{\mu \{\Psi\}^T[M]\{\chi o\} + \{\Psi\}^T[M]\{\dot{\chi} o\} + \{\Psi\}^T[c]\{\chi o\}}{2\mu\{\Psi\}^T[M]\{\Psi\} + \{\Psi\}^T[c]\{\Psi\}} \tag{2.19}$$

By substituting $\{\tilde{\Psi}\}$ and $\tilde{\mu}\{\tilde{\Psi}\}$ for initial values in the above equation and setting the modal amplitude A equal to unity, we have

$$2\mu\{\Psi\}^T[M]\{\Psi\} + \{\Psi\}^T[c]\{\Psi\} = (\mu + \tilde{\mu})\{\Psi\}^T[M]\{\tilde{\Psi}\} + \{\Psi\}^T[c]\{\tilde{\Psi}\} \tag{3.12}$$

This is equivalent to saying that $\delta\{\Psi\}$ has no component parallel to $\{\Psi\}$. After expanding the above equation by Eqs. (3.9) and (3.10) and then neglecting second-and higher-order terms in δ quantities, we find

$$(2\tilde{\mu}\{\tilde{\Psi}\}^T[M] + \{\tilde{\Psi}\}^T[c])\delta\{\Psi\} + (\{\tilde{\Psi}\}^T[M]\{\tilde{\Psi}\})\delta\mu = 0 \tag{3.13}$$

The combination of Eqs. (3.11) and Eqs. (3.13) results in a set of simultaneous linear algebraic equation, which may be expressed in partitioned form as

$$\left[\begin{array}{c|c} \tilde{\mu}^2[M] + \tilde{\mu}[c] + [K] & 2\tilde{\mu}[M]\{\tilde{\Psi}\} + [c]\{\tilde{\Psi}\} \\ \hline (2\tilde{\mu}[M]\{\tilde{\Psi}\} + [c]\{\tilde{\Psi}\})^T & \{\tilde{\Psi}\}^T[M]\{\tilde{\Psi}\} \end{array}\right] \left\{\begin{array}{c} \delta\{\Psi\} \\ \hline \delta\mu \end{array}\right\} \left\{\begin{array}{c} -\{R\} \\ \hline 0 \end{array}\right\} \tag{3.14}$$

An alternative form of Eq. (3.14) above is

$$\left[\begin{array}{c|c} \tilde{\mu}^2[M] + \tilde{\mu}[c] + [K] & 2\tilde{\mu}[M]\{\tilde{\Psi}\} + [c]\{\tilde{\Psi}\} \\ \hline (2\tilde{\mu}[M]\{\tilde{\Psi}\} + [c]\{\tilde{\Psi}\})^T & \{\tilde{\Psi}\}^T[M]\{\tilde{\Psi}\} \end{array}\right] \left\{\begin{array}{c} \{\tilde{\Psi}\} + \delta\{\Psi\} \\ \hline \delta\mu \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \alpha \end{array}\right\} \tag{3.15}$$

where

$$\alpha = 2\tilde{\mu}\{\tilde{\Psi}\}^T[M]\{\tilde{\Psi}\} + \{\tilde{\Psi}\}^T[c]\{\tilde{\Psi}\} \tag{3.16}$$

An approximate eigenvalue $\tilde{\mu}$ and approximate eigenvector $\{\tilde{\Psi}\}$ can be improved effectively by iterative solution of Eqs. (3.14) (or Eqs. (3.15)).

For most practical applications involving small dissipation, the first approximations of nonclassical modes of a damped system can, of course, be taken as classical modes $\{\phi\}$ of the corresponding undamped system, and the first approximations of the eigenvalue $\tilde{\mu}$ as

$$\tilde{\mu} = i\omega \tag{3.17}$$

where ω represents the corresponding natural frequency. A sample example presented in Section 5.2 shows that one iteration of the proposed method gives a satisfactory solution for design purposes, while a procedure of successive iterations converges rapidly to the exact solution sought.

For rare cases of large dissipation, it may be necessary to imagine the system formed by successively increasing damping and iterate only for fairly small changes of damping. In all cases, the method developed for computing nonclassical modes can be regarded as an extension of usual solution methods of classical modes.

CHAPTER 4

DYNAMIC SOIL-STRUCTURE INTERACTION

4.1 General

Before treating the problem of dynamic soil-structure interaction, we shall review briefly the preparatory work done in the previous chapters. As mentioned in Chapter 1, for all but very large structures, a simple and realistic way to model soil compliance in seismic building analysis is to assume the foundation of a building to be a rigid body resting on soil modeled by some mechanical system. This mechanical system may consist of a spring, a dashpot, and a mass for each possible degree of freedom of the rigid foundation of a structure (Newmark and Rosenblueth, 1971). Once a linear model of the soil-structure system is set up in this manner for elastic design purposes, the solution methods using the response-spectrum approach (or the nonclassical-modal analysis) presented in Chapter 2 are applicable. As a prerequisite for applying these solution methods, information on nonclassical modes can be computed effectively by the use of the numerical technique developed in Chapter 3.

The main purpose of this chapter is to show how this general numerical scheme can be applied specifically to the dynamic soil-structure interaction problem. In Section 4.2, the general form of the governing equations of motion for the problem is given. In Section 4.3, the proposed method takes advantage of the specific form of the equations to incorporate the Ritz method into general numerical scheme of the modified Robinson-Harris method. Nonclassical modes of a soil-structure system can then be computed effectively by using information obtained from the ordinary calculations of classical modes of the corresponding structure with a fixed base.

Another purpose of this chapter is to show how the modeling technique of the soil and foundation can be improved somewhat. Alternative approximations of soil impedance for a rigid foundation are discussed briefly in Section 4.4. The possibility of relaxing the assumption of a rigid foundation is discussed in Section 4.5.

4.2 Equations of Motion

We shall set up the governing equations of motion for a linear soil-structure system with the soil modeled by a mass-spring-dashpot system for each possible degree of freedom of the rigid foundation of the structure. For reasons of simplicity only, we limit ourselves to the two-dimensional problem of a shear-beam building with its mat foundation resting on the soil. Each discrete point of the structural model has only one horizontal degree of freedom, except for one extra rocking degree of freedom at the rigid foundation (Fig. 4.1). The free-field ground motion is assumed in the horizontal direction. The reasoning used here in deriving the proposed method applies, however, to a three dimensional problem of a general structure with a rigid foundation resting on soil modeled by simple mechanical devices as described above.

It is well known that if the soil were rigid, the equations of motion of a damped, linear, dynamic structure subjected to earthquake ground excitations could be written in a general form as

$$[M_o]\{\ddot{u}_t\} + [C_o]\{\dot{u}\} + [K_o]\{u\} = \{0\} \tag{4.1}$$

In the above equation, the $n \times n$ matrices $[M_o]$, $[C_o]$, and $[K_o]$ are the mass, damping, and stiffness matrices of a structure with its base fixed. The total displacement vector $\{u_t\}$ differs from the relative displacement vector $\{u\}$ by a vector describing rigid-body displacements of the structure caused by the ground displacement $u_g(t)$, i.e.

$$\{u_t\} = u_g\{1\} + \{u\} \tag{4.2}$$

in which $\{1\}$ is a column vector of ones for the shear-beam building problem.

Since the soil is compliant, the rigid foundation may experience both a rigid body translation $\Delta(t)$ and a rigid body rotation $\theta(t)$ relative to the free-field ground translational motion. Consequently, the whole structure may undergo corresponding rigid body motions. Thus, a more general form of the above equation is (see Fig. 4.1)

$$\{u_t\} = u_g\{1\} + \Delta\{1\} + \theta\{h\} + \{u\} \tag{4.3}$$

in which $\{h\}$ is a vector consisting of the heights of the discrete points of the structure above the foundation-soil interface. Because the newly introduced rigid-body displacements of the structure do not cause any structural deformation, the equations of motion of the structure remain in the same form as Eq. (4.1). Substituting Eq. (4.3) into Eq. (4.1) yields

$$[M_o]\{\ddot{u}\} + [M_o]\{1\}\ddot{\Delta} + [M_o]\{h\}\ddot{\theta} + [C_o]\{\dot{u}\} + [K_o]\{u\} = -\ddot{u}_g[M_o]\{1\} \tag{4.4}$$

These rigid body motions do, however, introduce two extra unknowns $\Delta(t)$ and $\theta(t)$. Therefore, two additional equations of motions are needed to obtain a solution.

The equations of motion of the rigid foundation provide the two additional equations needed. Setting the sum of horizontal forces equal to zero yields

$$\{1\}^T[M_o]\{\ddot{u}_t\} + m_f(\ddot{u}_g + \ddot{\Delta}) + S_f = 0 \tag{4.5}$$

Where m_f is the mass of the rigid foundation. The quantity $S_f(t)$ is the total base shear exerted on the rigid foundation by the supporting soil. Similarly, from the overturning moments at the interface, (with no vertical motion of the foundation)

$$\{h\}^T[M_o]\{\ddot{u}_t\} + I_f\ddot{\theta} + T_f = 0 \tag{4.6}$$

In the above equation, I_f is the sum of the mass moments of inertia of each story-mass about a rotational axis at its own level and $T_f(t)$ is the total rocking moment resisted by the soil. The determination of numerical values of soil impedance functions, which relate the S_f and T_f to the unknowns Δ and θ, have been the subject of many studies in recent years (see Section 1.3).

If the soil is modeled by a spring, dashpot, and mass for each degree-of-freedom of the rigid foundation, S_f and T_f may be approximated by

$$\begin{Bmatrix} S_f \\ T_f \end{Bmatrix} = \begin{bmatrix} m_\Delta & \\ & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{\Delta} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} C_\Delta & \\ & C_\theta \end{bmatrix} \begin{Bmatrix} \dot{\Delta} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} K_\Delta & \\ & K_\theta \end{bmatrix} \begin{Bmatrix} \Delta \\ \theta \end{Bmatrix} \tag{4.7}$$

where the elements of the coefficient matrices are real constants given in the literature (e.g., Newmark and Rosenblueth, 1971). Here m_Δ and I_θ are often termed the "virtual mass" and the "virtual mass moment of inertia" of the soil. Alternative approximations of the S_f and T_f are discussed briefly in Section 4.5. Substitution of Eqs. (4.7) and (4.3) into both Eq. (4.5) and Eq. (4.6) yields

$$\{1\}^T[M_o]\{\ddot{u}\} + (m_t + m_\Delta)\ddot{\Delta} + J\ddot{\theta} + C_\Delta\dot{\Delta} + K_\Delta\Delta = - m_t\ddot{u}_g \tag{4.8}$$

and

$$\{h\}^T[M_o]\{\ddot{u}\} + J\ddot{\Delta} + (I_t + I_\theta)\ddot{\theta} + C_\theta\dot{\theta} + K_\theta\theta = -J\ddot{u}_g \tag{4.9}$$

in which

$$m_t = \{1\}^T[M_o]\{1\} + m_f \, , \tag{4.10}$$

$$I_t = \{h\}^T[M_o]\{h\} + I_f \, , \tag{4.11}$$

and

$$J = \{1\}^T[M_o]\{h\} \tag{4.12}$$

The physical meaning of m_t, I_t, and J are the total mass of the structure, the total mass moment of inertia for the structure undergoing rigid-body rocking motion, and the corresponding first moment of mass. Note that the virtual mass and virtual mass moment of inertia do not appear in the right sides of Eqs. (4.8) and (4.9).

The combination of Eqs. (4.4), (4.8), and (4.9) results in a set of simultaneous equations of motion, which may be expressed in matrix form as

$$[M]\{\ddot{\chi}\} + [C]\{\dot{\chi}\} + [K]\{\chi\} = -\ddot{u}_g\{F\} \tag{4.13}$$

where

$$[M] = \left[\begin{array}{c|c|c} [M_o] & [M_o]\{1\} & [M_o]\{h\} \\ \hline \{1\}^T[M_o] & m_t+m_\Delta & J \\ \hline \{h\}^T[M_o] & J & I_t+I_\theta \end{array}\right] , \tag{4.14}$$

$$[C] = \left[\begin{array}{c|c|c} [C_o] & & \\ \hline & C_\Delta & \\ \hline & & C_\theta \end{array}\right] , \tag{4.15}$$

$$[K] = \left[\begin{array}{c|c|c} [K_o] & & \\ \hline & K_\Delta & \\ \hline & & K_\theta \end{array}\right] , \qquad (4.16)$$

$$\{\chi\} = \left\{\begin{array}{c} \{u\} \\ \hline \Delta \\ \hline \theta \end{array}\right\} , \qquad (4.17)$$

and

$$\{F\} = \left\{\begin{array}{c} [M_o]\{1\} \\ \hline m_t \\ \hline J \end{array}\right\} \qquad (4.18)$$

Note that the original $n \times n$ matrices $[M_o]$, $[C_o]$, and $[K_o]$ of the superstructure are bordered by two rows and two columns to form a new $(n+2) \times (n+2)$ matrices $[M]$, $[C]$, and $[K]$ of the soil-structure system. Also, the relative displacement $\{u\}$ is bordered below by two entries to form $\{\chi\}$. This is a typical pattern of the resulting set of governing equations of motion, Eq. (4.13), of a soil-structure system.

4.3 Proposed Method

In a state of free vibration, the set of governing equations of motion of Eq. (4.13) for a soil-structure system reduces to

$$[M]\{\ddot{\chi}\} + [C]\{\dot{\chi}\} + [K]\{\chi\} = \{0\} \qquad (4.19)$$

This is in the same form as equation (2.1). The corresponding complex eigenproblem is then in the same form as the general equation (2.3), i.e.

$$\mu^2[M]\{\psi\} + \mu[C]\{\psi\} + [K]\{\psi\} = \{0\} \qquad (4.20)$$

The eigenvector $\{\psi\}$ represents a nonclassical mode shape. The eigenvalue μ gives both the frequency Ω and the damping ratio β of the nonclassical mode, i.e.,

$$\mu = -\beta\Omega \pm i\Omega\sqrt{1-\beta^2} \tag{4.21}$$

The complex eigenproblem can, of course, be solved by direct use of the modified Robinson-Harris method developed in Section 3.3, i.e.

$$\left[\begin{array}{c|c} \tilde{\mu}^2[M] + \tilde{\mu}[C] + [K] & 2\tilde{\mu}[C]\{\tilde{\psi}\} + [K]\{\tilde{\psi}\} \\ \hline (2\tilde{\mu}[M]\{\tilde{\psi}\}+[C]\{\tilde{\psi}\})^T & \{\tilde{\psi}\}^T[M]\{\tilde{\psi}\} \end{array}\right] \left\{\begin{array}{c} \{\psi\} + \delta\{\psi\} \\ \hline \delta\mu \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \alpha \end{array}\right\} \tag{4.22}$$

where

$$\alpha = 2\tilde{\mu}\{\tilde{\psi}\}^T[M]\{\tilde{\psi}\} + \{\tilde{\psi}\}^T[C]\{\tilde{\psi}\} + \{\tilde{\psi}\}^T[C]\{\tilde{\psi}\} \tag{4.23}$$

Initial approximations of eigenvalue and eigenvector, $\tilde{\mu}$ and $\{\tilde{\psi}\}$, will be improved effectively after solving Eq. (4.22). The improved eigenvalue $\tilde{\mu} + \delta\mu$ and eigenvector $\{\tilde{\psi}\} + \delta\{\psi\}$ can be used as the initial approximations for the next iteration. A procedure of successive iterations converges rapidly to the exact solution. To begin the process, the initial approximations can be taken from a classical mode (either of a soil-structure system or of the structure with a fixed base).*

We shall develop an alternative procedure specifically for the dynamic soil-structure interaction problem. The motivation for doing this is that for the present problem, the general procedure requires solving a set of simultaneous linear algebraic equation of order n+3,

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*in fact, the procedure can be applied to general problems of a linear dynamic system with viscous damping. In other words, it is not limited to the dynamic soil-structure interaction problem.

Eq. (4.22). There are n degrees of freedom assumed for the superstructure, and two at the rigid foundation. Also, one side condition is, in general, needed in the solution. It is sometimes necessary to take a large n in order to model a structure accurately. In such cases it is useful to develop a more economical computational scheme. The $(n+3) \times (n+3)$ combined matrix in the left side of Eq. (4.22) will be transformed into a bordered matrix by a transformation of coordinates as shown in the following paragraphs.

The complex eigenproblem Eq. (4.20) is characterized by the matrices [M], [C], and [K]. These matrices are defined specifically by Eqs. (4.14), (4.15), and (4.16) for the dynamic soil-structure interaction problem, where it is seen that the $n \times n$ matrices $[M_o]$, $[C_o]$, and $[K_o]$ of the superstructure are submatrices in the upper left corners. We recall how the submatrices can be transformed into diagonal matrices. In engineering practice, the matrices $[M_o]$ and $[K_o]$ are first set up. For the undamped system, n classical modes exist. Each classical mode has associated with it a natural frequency ω and a characteristic shape $\{\phi\}$. Let the $n \times n$ matrix $[\Phi]$ consist of n columns of the n classical mode shapes. By definition,

$$[\Phi]^T[M_o][\Phi] = [\diagdown m_r \diagdown] \tag{4.24}$$

and

$$[\Phi]^T[K_o][\Phi] = [\diagdown m_r\omega_r^2 \diagdown] \tag{4.25}$$

in which the generalized masses m_r are defined by

$$m_r = \{\phi_r\}^T[M_o]\{\phi_r\} \tag{4.26}$$

The determination of the damping matrix $[C_0]$ is not as obvious as for the mass and stiffness matrices, $[M_0]$ and $[K_0]$. In practice in earthquake engineering, the damping of the superstructure is usually assumed to be classical, and defined implicitly by n prescribed values of modal damping ratio ξ_r such that

$$[\Phi]^T[C_0][\Phi] = \left\lceil 2\xi_r\omega_r m_r \right\rfloor \tag{4.27}$$

We may now proceed to modify the complex eigenproblem Eq. (4.20) by a transformation of coordinates. Let the complex vector $\{B\}$ be defined by

$$\{\psi\} = \left[\begin{array}{c|c|c} [\Phi] & & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right] \{B\} \tag{4.28}$$

The vector $\{B\}$ exists since the combined matrix in the above equation is not singular. Premultiplying Eq. (4.20) by

$$\left[\begin{array}{c|c|c} [\psi] & & \\ \hline & 1 & \\ \hline & & 1 \end{array}\right]^T$$

and using Eqs. (4.14), (4.15), (4.16), (4.24), (4.25), and (4.27), we find

$$\mu^2[M]\{B\} + [C]\{B\} + [K]\{B\} = \{0\} \tag{4.29}$$

in which

$$[M] = \left[\begin{array}{cccc|c|c} m_1 & & & & P_1 & q_1 \\ & m_2 & & & P_2 & q_2 \\ & & \ddots & & \vdots & \vdots \\ & & & m_n & P_n & q_n \\ \hline P_1 & P_2 & \cdots & P_n & m_t + m_\Delta & J \\ \hline q_1 & q_2 & \cdots & q_n & J & I_t + I_\theta \end{array}\right] , \tag{4.30}$$

$$[C] = \left[\begin{array}{cccc|c|c} 2\xi_1\omega_1 m_1 & & & & & \\ & 2\xi_2\omega_2 m_2 & & & & \\ & & \ddots & & & \\ & & & 2\xi_n\omega_n m_n & & \\ \hline & & & & C_\Delta & \\ \hline & & & & & C_\theta \end{array}\right] , \tag{4.31}$$

$$[K] = \left[\begin{array}{cccc|c|c} m_1\omega_1^2 & & & & & \\ & m_2\omega_2^2 & & & & \\ & & \ddots & & & \\ & & & m_n\omega_n^2 & & \\ \hline & & & & K_\Delta & \\ \hline & & & & & K_\theta \end{array}\right] , \tag{4.32}$$

$$P_r = \{\phi_r\}^T[M_o]\{1\} \tag{4.33}$$

and

$$q_r = \{\phi_r\}^T[M_o]\{h\} \tag{4.34}$$

The resulting complex eigenproblem Eq. (4.30) is now characterized by the $(n+2) \times (n+2)$ matrices $[M]$, $[C]$, and $[K]$ defined by Eqs. (4.30), (4.31), and (4.32). The eigenvalue μ is, of course, unchanged by the transformation of coordinates.

Application of the modified Robinson-Harris method to the transformed eigenproblem Eq. (4.29) then yields

$$\left[\begin{array}{c|c} \tilde{\mu}^2[M] + \tilde{\mu}[C] + [K] & 2\tilde{\mu}\,[M]\{\tilde{B}\} + [C]\{\tilde{B}\} \\ \hline (2\tilde{\mu}[M]\{\tilde{B}\} + [C]\{\tilde{B}\})^T & \{B\}^T[M]\{\tilde{B}\} \end{array}\right] \left\{\begin{array}{c} \{\tilde{B}\} + \delta\{B\} \\ \hline \delta\mu \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline \alpha \end{array}\right\}$$

where

$$\alpha = 2\tilde{\mu}\{\tilde{B}\}^T[M]\{\tilde{B}\} + \{\tilde{B}\}^T[C]\{\tilde{B}\} \tag{4.36}$$

For the dynamic soil-structure interaction problem, Eq. (4.35) and (4.28)are proposed to replace Eq. (4.22) for computing nonclassical modes if the number n is large. By Eqs. (4.30), (4.31), and (4.32),

$$\tilde{\mu}^2[M] + \tilde{\mu}[C] + [K] =$$

$$\left[\begin{array}{cccc|c|c}
\eta_1 & & & & \tilde{\mu}^2 p_1 & \tilde{\mu}^2 q_1 \\
 & \eta_2 & & & \tilde{\mu}^2 p_2 & \tilde{\mu}^2 q_2 \\
 & & \ddots & & \vdots & \vdots \\
 & & & \eta_n & \tilde{\mu}^2 p_n & \tilde{\mu}^2 q_n \\
\hline
\tilde{\mu}^2 p_1 & \tilde{\mu}^2 p_2 & \cdots & \tilde{\mu}^2 p_n & \mu^2(m_t+m_\Delta)+\tilde{\mu}C_\Delta+K_\Delta & \tilde{\mu}^2 J \\
\hline
\tilde{\mu}^2 q_1 & \tilde{\mu}^2 q_2 & \cdots & \tilde{\mu}^2 q_n & \tilde{\mu}^2 J & \tilde{\mu}^2(I_t+I_\theta)+\mu C_\theta+K_\theta
\end{array}\right] \tag{4.37}$$

where $\eta_i = m_i(\tilde{\mu}^2 + 2\,\xi_i\,\omega_i + \omega_i^2)$.

Thus, the newly-formed $(n+3) \times (n+3)$ matrix on the right side of Eq. (4.35) is a bordered matrix, which contains non-zero off-diagonal terms only in the last three rows and columns. The set of simultaneous equations of Eq. (4.35) can be solved effectively after a procedure of condensation. If none of the first n diagonal terms in Eq. (4.36) is zero, the problem of solving Eq. (4.35) can be reduced to solving a set of simultaneous linear algebraic equation of order 3, instead of order n+3. However, if one of the diagonal terms is zero, it is necessary to solve a set of simultaneous equation of order 4. This occurs when the approximate eigenvalue is initially taken as

$$\tilde{\mu} = -\xi_r\omega_r + i\omega_r\sqrt{1-\xi_r^2}\ , \qquad r = 1,2,\ldots,n \tag{4.38}$$

In all cases, the computational effort in solving simultaneous equations is reduced significantly. Some aspects of using the proposed method will be discussed in the next paragraph.

Let us express the eigenvector {B} more explicitly by

$$\{B\} = \{b_1 \; b_2 \; . \; . \; . \; b_n \; b_\Delta \; b_\theta\}^T \tag{4.39}$$

The equation (4.28) can then be rewritten as

$$\{\psi\} = (\sum_{r=1}^{n} \begin{Bmatrix} \{\phi_r\} \\ 0 \\ 0 \end{Bmatrix}) \quad b_\Delta \begin{Bmatrix} \{0\} \\ 1 \\ 0 \end{Bmatrix} \quad b_\theta \begin{Bmatrix} \{0\} \\ 0 \\ 1 \end{Bmatrix} \tag{4.40}$$

Modal values corresponding to one classical mode (say the rth) of the superstructure can be taken as the initial approximation for Eq. (4.35). The $\tilde{\mu}$ can be taken from Eq. (4.38), and the $\{\tilde{B}\}$ from a vector having zero entries except for a unity at the rth term of vector (see Eqs. (4.39) and (4.40)). By using Eq. (4.35), n out of a total of n+2 (pairs of) nonclassical modes can be obtained. Because the soil is compliant and not rigid, each computed frequency will be smaller than the corresponding one approximated initially. The two (pairs of nonclassical) modes omitted are those of the highest frequencies (Bisplinghoff, Ashley, and Halfman, 1965). This is not a real disadvantage of the proposed method, for in earthquake engineering applications, the lower modes are usually of more interest. By the Ritz method, some of the b_r (for large r) in Eqs. (4.39) and (4.35) can be dropped as an approximation. However, an accurate approximation of the rth nonclassical mode may require information on as many as 2r ($2r \leq n$) classical modes of the superstructure. This is the result of experience in applying subspace iteration (Bathe and Wilson, 1976), which is also an application of the Ritz coordinate-reduction.

In summary, the proposed method provides a fast numerical scheme for solving the equation (4.22). This is made possible by taking advantage of the specific pattern of the governing equations of motion for the dynamic soil-structure interaction problem. After a procedure of transformation of coordinates, two major schemes of coordinate-reduction can be used. One of these is a procedure of condensation. The other is an approximation by imposing suitable constraints, i.e. the Ritz coordinate-reduction. Approximations of both eigenvalue and eigenvector will then be improved after solving a set of only 4 (or 3) simultaneous equations. The improvement is extremely effective because the modified Robinson-Harris method is a variant of the Newton-Raphson technique. Numerical examples will be given in Chapter 5.

4.4 Frequency Dependence of Impedance

The determination of the soil impedance for a rigid foundation has been the subject of many studies in recent years (see Section 1.3). The soil impedance functions are found to be frequency dependent. The frequency dependence of impedance introduces some difficulties in a time-domain analysis. The purpose of this section is to discuss

1. The general form of the impedance functions,
2. Problems occurring in a time-domain analysis,
3. Cause of the frequency dependence, and
4. Some possible approximation for a time-domain analysis.

Consider the case of steady-state vibrations of a rigid massless foundation rigidly attached to the supporting soil, which is modeled as a linear semi-infinite solid. The rigid foundation has six degrees of freedom in a three-dimensional problem. Under a sinusoidal load, a steady-state displacement of the foundation will also be a sinusoidal function of time. These two quantities, the load and displacement, have the same frequency Ω. Also, there will be a phase difference, in general. If only two degrees of freedom of the foundation are considered for reasons of simplicity, we have the following general expression

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} e^{i\Omega t} = \left[\begin{array}{c|c} K_{11}^* & K_{12}^* \\ \hline K_{21}^* & K_{22}^* \end{array}\right] \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} e^{i\Omega t} \tag{4.41}$$

where f_r and d_r are real quantities of amplitudes of forces and displacements. The coefficient matrix above must be symmetric due to the dynamic reciprocal theorem (Love, 1927). Elements of the coefficient matrix depend, in general, on the selected value of the frequency. The elements are complex numbers since there are phase differences. Let

$$K_{rs}^* = K_{rs} + i \ C_{rs} \quad , \qquad r,s = 1,2 \tag{4.42}$$

where K_{rs} and C_{rs} are real numbers. Equation (4.41) can now be expanded as

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} e^{i\Omega t} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} e^{i\Omega t} + \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} e^{i\Omega t} i\Omega \tag{4.43}$$

If elements of the coefficient matrices above are frequency indpendent, the expression can be generalized to

$$\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \begin{Bmatrix} d_1(t) \\ d_2(t) \end{Bmatrix} + \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right] \begin{Bmatrix} \dot{d}_1(t) \\ \dot{d}_2(t) \end{Bmatrix} \tag{4.44}$$

This would be very convenient for a time-domain analysis.

Difficulties arise, however, in a time-domain analysis due to the frequency dependence of the stiffness and damping matrices of Eq. (4.44). The problem encountered in a direct-integration analysis is that system parameters representing the soil are functions of the time history of response of the rigid foundation. In a nonclassical-modal analysis, nonclassical modes can still be determined by the iterative use of Eq. (4.22) or (4.35), even though the impedance functions are frequency dependent. However, the modes computed are really nonclassical modes of different dynamic systems with constant parameters. It is important to note that the orthogonality relations among these modes, Eqs. (2.15) and (2.16), are not applicable. Consequently, neither the solution method developed in Chapter 2 nor the Foss method (Foss, 1958) can be applied to a system with frequency-dependent parameters. The recommendation of the Foss method in a recent ASCE state-of-the-art report on soil-structure interaction (Ad Hoc Group, 1979) should, therefore, be used cautiously.

The difficulty encountered above does not necessarily mean that an engineering solution cannot be obtained by a time-domain analysis with some suitable approximations. The problem of frequency dependence of impedance functions arises whenever a dynamic system of many (or infinitely many) degrees of freedom is idealized by a model with fewer degrees of freedom. In other words, it is a common problem in dynamic substructure analysis. For example, the dynamic impedance of a real column with distributed mass depends on its own dynamic responses, and thus on the frequency content of the dynamic load. Nevertheless, frequency-independent column models have been constructed successfully in earthquake engineering by the so-called lumped-mass method. For the problem of estimating soil impedance for a rigid foundation, we shall discuss briefly what approximations may be made in order to obtain a frequency-independent model for engineering purposes.

In aseismic design of buildings, the lower modes are often of most interest. The range of lower frequencies is then of greatest importance. Thus, one way of approximation is to take the coefficient matrices of Eq. (4.44) from those of Eq. (4.43) evaluated at one selected frequency. The frequency is usually taken as the fundamental frequency of a soil-structure system with the soil modeled in this manner. This can be most easily achieved by the iterative use of the proposed method with the soil impedance functions updated by the computed frequency. Numerical examples are shown in Chapter 5. For some special cases, the selected frequency may also be taken as that of another mode of a soil-structure system with constant parameters.

For a foundation with no embedment, off-diagonal terms of the coefficient matrices of Eq. (4.43) are found to be negligible (Veletsos and Wei, 1971). Another way of approximation has been to introduce an extra parameter that permits a better adjustment of impedance over a limited range of frequencies (Newmark and Rosenblueth, 1971). This has been shown in Eq. (4.8) by use of a virtual mass and a virtual mass moment of inertia. This is equivalent to allow a parabolic variation of soil stiffness with frequency, since

$$K^* = (K_\Delta - \Omega^2 m_\Delta) + i\Omega C_\Delta \tag{4.45}$$

A virtual mass can also be added to approximate off-diagonal terms of the coefficient matrices if appropriate.

The technique of introducing a virtual mass can be extended by including some extra mechanical components in a soil model. A good approximation, for example, has been achieved for the case of a rigid massless disk "welded" to the soil modeled by an elastic half space (Veletsos and Verbic, 1973; Meek and Veletsos, 1973). The approximation is summarized below. Let the diagonal terms of the coefficent matrix of Eq. (4.41) be expressed by

$$K^* = K_o(k + ia_o\ C) \tag{4.46}$$

where

K_o = static impedance evaluated at zero frequency

k, C = dimensionless real parameters representing frequency variations of impedance

a_o = dimensionless frequency parameter

$$= \frac{R\Omega}{V_s} \tag{4.47}$$

R = radius of disk

V_s = shear-wave velocity of the elastic half space.

Semi-empirical approximations of impedance are as follows, for the horizontally excited disk:

$$k = 1 \tag{4.48}$$

$$c = \alpha_1; \tag{4.49}$$

for the disk in rocking motion,

$$k = 1 - \beta_1 \frac{(\beta_2 a_o)^2}{1+(\beta_2 a_o)^2} - \beta_3 a_o{}^2 \tag{4.50}$$

$$c = \beta_1\beta_2 \frac{(\beta_2 a_o)^2}{1+(\beta_2 a_o)^2} \quad ; \tag{4.51}$$

and for the vertically excited disk,

$$k = 1 - \gamma_1 \frac{(\gamma_2 a_o)^2}{1+(\gamma_2\ a_o)^2} - \gamma_3 a_o{}^2 \tag{4.52}$$

$$c = \gamma_4 + \gamma_1\gamma_2 \frac{(\gamma_2 a_o)^2}{1+(\gamma_2 a_o)^2} \tag{4.53}$$

where α_1, β_r, and γ_r are numerical coefficients depend on Poisson's ratio ν as shown in Table 4.1. Under harmonic excitation, the vertical impedance can be reproduced by using a combination of some simple mechanical components, as shown by Fig. 4.2. The values of c_1, m_2, m_3, and c_4 in the figure are to be taken as

$$c_1 = K_o\gamma_1\ (\gamma_2 \frac{R}{V_s}) \quad , \tag{4.54}$$

$$m_2 = K_o\gamma_1\ (\gamma_2 \frac{R}{V_s})^2 \quad , \tag{4.55}$$

$$m_3 = K_o\gamma_3\ (\frac{R}{V_s})^2 \quad , \tag{4.56}$$

and

$$c_4 = K_o\gamma_4\ (\frac{R}{V_s}\ ) \quad . \tag{4.57}$$

An approximation for the rocking impedance can also be obtained after replacing γ_r in Eqs. (4.54) to (4.57) by β_r. By this type of approximations, an engineer can adjust soil impedance functions over an important range of frequency, and yet use modal analyses for a frequency-independent model of a soil-structure system.

4.5 Flexibility of the Foundation

In the impedance approach to a dynamic soil-structure interaction problem, the conventional assumption of rigid soil is relaxed. However, the foundation of the structure is usually assumed to be rigid. By the assumption of a rigid foundation, interaction between the soil and foundation is limited to a few degrees of freedom. This may be sufficient for usual applications in aseismic design of multi-story buildings or nuclear power plants. For the case of a wide structure with a thin mat or spread footings, a more refined analysis considering flexibility of the foundation may be needed. In this section, we shall examine the possibility of introducing foundation flexibility. The discussion will be restricted to an undamped structure. With this restriction, we separate the problem from that of defining the damping matrix for a superstructure-foundation system.

Consider the case of a general two-dimensional structure with spread footings. Each footing has a rotational, horizontally translational, and vertically translational displacement relative to the free-field ground translational motions. The interaction displacements of all footings are expressed by the $s \times 1$ vector $\{\Delta\}$ (where s is three times of the number of footings). Let the $n \times 1$ vector $\{\eta_r\}$ represent the displacements of the superstructure caused by imposing a unit displacement of the Δ_r type. The equations of motion of the superstructure in free vibration are

$$[M_o]\,\{\ddot{u}_t\} + [K_o]\,\{u\} = \{0\} \tag{4.58}$$

where

$$\{u_t\} = \{u\} + [\eta]\,\{\Delta\} \tag{4.59}$$

The n × s matrix $[\eta]$ consists of s columns of $\{\eta_r\}$. If the soil were rigid, the vector $\{\Delta\}$ would be a null vector. In general, there are s extra unknowns of Δ_r in the equations of motion above.

Application of the principle of virtual work provides the other equations of motion needed. Let the virtual displacements be taken as a unity value of Δ_r and the corresponding vector $\{\eta_r\}$. Then, we have s equations of motion in free vibration as

$$\{\eta_r\}^T[M_o]\{\ddot{u}_t\} + 1 \cdot m_{f_r}\ddot{\Delta}_r + 1 \cdot S_r = 0, \quad r = 1,2,\ldots,s \tag{4.60}$$

where the quantities m_{f_r} are masses and mass moments of inertia of the footings, and $S_r(t)$ are reactions of the soil exerted on the footings.

Combining Eqs. (4.58) and (4.60) in matrix form yields

$$\left[\begin{array}{c|c} [M_o] & [M_o][\eta] \\ \hline [\eta]^T[M_o] & [\eta]^T[M_o][\eta] + \lceil m_{f_r} \rfloor \end{array}\right] \left\{\begin{array}{c} \{\ddot{u}\} \\ \hline \{\ddot{\Delta}\} \end{array}\right\} +$$

$$\left[\begin{array}{c|c} [K_o] & \\ \hline & \lceil S_r(t) \rfloor \end{array}\right] \left\{\begin{array}{c} \{u\} \\ - \\ \{1\} \end{array}\right\} = 0 \tag{4.61}$$

The reactions of the soil S_r at one footing can be related to the unknowns Δ_r at that footing by the same approximation as used for a rigid mat in the last section. Note that the matrices $[M_o]$ and $[K_o]$ for the superstructure are now bordered by s rows and s columns to form the new (n+s) × (n+s) matrices in Eq. (4.61) for the soil-structure system. Thus, the proposed method can be applied for computing nonclassical modes if frequency-independent models of the soil are used.

If the number s is large, we may also want to apply the Ritz method to reduce the number of degrees of freedom at the footings. One of the generalized coordinates can be selected by taking the displacements of the footings from those of a rigid-body displacement shape of the whole structure. Numerical examples of this method will be given in Section 5.6. For a more refined analysis, other generalized coordinates can also be chosen.

For a mat foundation, there are an infinite number of degrees of freedom, in general. The Ritz method must be applied somehow in order to carry out numerical computations. In the impedance approach, the displacement of the mat can be approximated by suitably selected generalized coordinates. The first few generalized coordinates are, of course, to be taken from shapes of rigid-body displacements of the mat. This is precisely what we have done previously by assuming a rigid mat. For a flexible mat, other curvilinear shapes can also be included. We shall not dwell further on this because soil impedance functions for these cases are not currently available in the literature. These impedance functions, however, can be calculated by the same numerical techniques as those proposed for the case of a rigid mat (e.g. Wong and Luco, 1976).

CHAPTER 5

NUMERICAL RESULTS

5.1 Objectives and Scope

In this chapter, the results of solving some sample problems are presented. These problems are solved using the technique introduced in Chapters 2, 3, and 4. The main objective of this chapter is to demonstrate the capability of the technique. However, some conclusions on the effects of dynamic soil-structure interaction can be drawn.

In Section 5.2, the complex eigenproblem of a simple mechanical system with nonclassical damping is solved using the general numerical scheme of Eq. (3.15).

In Section 5.3, the proposed method, Eq. (4.35) is applied to solve the complex eigenproblem of a sample soil-structure model. The assumption of a rigid-mat foundation is used. The convergence of the solution is examined. In Section 5.4, the (absolute) maximum of a repsonse within a conjugate pair of nonclassical modes is predicted using the methods developed in Section 2.5.3 for earthquake engineering applications. This computation is essential to the response-spectrum approach for a dynamic system with nonclassical damping.

In Section 5.5, some of the effects of dynamic soil-structure interaction are discussed. In Section 5.6, a comparison of the effects of a spread-footing foundation to those of a rigid-mat foundation is made.

5.2 Nonclassical Damping

Figure 5.1 shows a simple mechanical model with many degrees of freedom (say n). This model will also be used later on for the undamped superstructure of a uniform shear-beam building. All the springs in the model have the same spring constant k. The mass lumped at the free end of the model is one half of the typical mass, m, lumped at the other points. The exact solution of the n classical modes of the model is readily available (von Kârmân and Biot, 1936) as shown below. A natural frequency (say rth) and the associated mode shape are

$$\omega_r = \sqrt{2-2\cos\alpha_r} \cdot \sqrt{k/m} \tag{5.1}$$

and

$$\{\phi_r\} = \{\sin j\alpha_r\} \quad , \quad j = 1,2,\ldots,n \tag{5.2}$$

where

$$\alpha_r = \frac{2r-1}{2n}\,\pi \tag{5.3}$$

Consider the case of a four degree-of-freedom model, i.e. n=4. The values of m and k are taken as unity and ten. The four classical modes of the undamped system can easily be obtianed from Eqs. (5.1) to (5.3). If a dashpot of unit value is added in parallel to the spring attached to the fixed end of the model, the system becomes nonclassically damped.

The nonclassical modes of the damped system are computed iteratively using the modified Robinson-Harris method, Eq. (3.15). To begin the process of computing a nonclassical mode, a single classical mode of the original undamped system is taken as the initial approximation. The convergence of the frequencies and damping ratios of the four (pairs of) nonclassical modes is shown in Table 5.1.

The calculated modal damping ratios range from 0.5 to 6 percent of critical. Thus, this is an example of small dissipation.

As can be seen from the table, one iteration of the numerical scheme gives a very good approximation to the exact solution of a modal damping ratio and frequency. Although the results are not shown here, the same is true for the associated mode shape.

The data in Table 5.1 also show that a procedure of successive iterations converges rapidly to the exact solution sought. This is expected because Eq. (3.15) is a modification of the Newton-Raphson technique. The procedure turns out to be more rapid than a second order process. This rate of convergence is consistent with that of the Robinson-Harris method, which has been proved formally to be of order 2.41 (Robinson and Harris, 1971).

Lastly, one point of academic interest is that the frequencies of the first two modes are increased due to the introduction of the nonclassical damping. The phenomenon has been explained analytically (Caughey and O'Kelly, 1961).

5.3 Rigid-Mat Foundation

Figure 5.2 shows the sample soil-structure model investigated in this work for the problem of dynamic soil-structure interaction. A four- and a fifteen-story uniform shear-beam building are examined, representing relatively low- and high-rise buildings. Each story has only one horizontal degree of freedom. The rigid foundation of a structure has two degrees of freedom, one horizontal translation and one rocking rotation.

In this section, a rigid-mat foundation is used. The superstructure is assumed to be undamped. The energy dissipates in the forms of both material damping of the soil and radiational wave propagation into the half space of the soil. The soil is soft, having a shear-wave velocity of 500 feet per second. Detailed descriptions of the numerical data used for the structure and soil impedance functions for the rigid foundation are given in the next two paragraphs. The effects of both varying degrees of soil compliance and different sources of energy dissipation will be discussed later on in Section 5.5.

The size and the average mass density of the superstructure are shown in Fig. 5.3. The mass lumped at either the roof or the base is one half of the typical story-mass. The classical modes of the superstructure with a fixed base are calculated using Eqs. (5.1) to (5.3). In the computation, the stiffness of a story is defined implicitly by prescribing a value for the fundamental period as shwon in the figure. In rocking motion of the foundation, the quantity I_f in Eq. (4.6), the sume of the mass moments of inertia of each story-mass about a rotational axis at its own level, is assumed to be zero.

The static impedance of an elastic half space for a rigid disk is shown in Table 5.2. For the noncircular mat, an equivalent radius is taken such that the equivalent disk has the same area as the rigid mat. The mass density and Poisson's ratio of the elastic half space are shown in Fig. 5.2. The half-space impedance functions for the rigid mat are assumed to be constant, and are evaluated from Eqs. (4.46) to (4.51) at one selected frequency. This frequency is the fundamental frequency of a soil-structure system. It is obtained by the iterative use of the proposed method, Eq. (4.35). To include the material damping of the soil, Eq. (4.46) is multiplied by a complex constant 1+i(2x0.05) to simulate a hysteretic damping of 0.05 (Kausel, Whitman, Morray, and Elsabee, 1978).

The results of the application of the proposed method to the soil-structure model with the four-story building are presented below. The convergence of the modal frequency and damping ratio of the fundamental mode is shown in Table 5.3, while that of the second mode is shown in Table 5.4. As can be seen, for either mode, one or two iterations of Eq. (4.35) give a very good approximation to the exact solution even for the present case of a soft soil. The convergence of the second mode is more rapid than that of the fundamental mode. One reason for this is that the values of soil "springs and dashpots" are updated during each iteration of the computation for the fundamental mode, while they are known constants throughout the computation for the second mode.

The results for the case of the fifteen-story building are shown in Table 5.5 and 5.6. For this relatively high builidng, the rate of convergence of the solution is about the same as that of the previous cases. One thing new in the present case is that only some of the classical modes of the superstructure are used. In other words, the Ritz method has been applied. Using only 3 out of a total of 15 classical modes of the superstructure, the converged results are not much different from those of the exact solution shown also in the tables. This is true even for the second mode of the soil-structure system.

5.4 Modal Maximum

As shown by Eq. (2.24), transient displacements of a conjugate pair of nonclassical modes of a dynamic system initially at rest are

$$\{\chi\} = \{p\}\, y_s(t) + \{q\}\, y_c(t) \qquad (5.4)$$

As defined by Eqs. (2.26) and (2.28), $y_s(t)$ and $y_c(t)$ are functions of a modal frequency, a modal damping, and a time history of earthquake accelerations $\ddot{u}_g(t)$. The vectors {p} and {q} are dimensionless and real. The vector {q} vanishes in the case of classical damping.

For the present problem of dynamic soil-structure interaction, {p} and {q} are expanded as:

$$\{p\} = \left(\sum_{r=1}^{n} v_r \begin{Bmatrix} \{\phi_r\} \\ 0 \\ 0 \end{Bmatrix} \right) + v_\Delta \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + v_\theta \begin{Bmatrix} \{h\} \\ 0 \\ 1 \end{Bmatrix} \frac{1}{H} \tag{5.5}$$

and

$$\{q\} = \left(\sum_{r=1}^{n} w_r \begin{Bmatrix} \{\phi_r\} \\ 0 \\ 0 \end{Bmatrix} \right) + w_\Delta \begin{Bmatrix} \{1\} \\ 1 \\ 0 \end{Bmatrix} + w_\theta \begin{Bmatrix} \{h\} \\ 0 \\ 1 \end{Bmatrix} \frac{1}{H} \tag{5.6}$$

where v's and w's are real values shown in Tables 5.3 to 5.6; H is the height of a structure. The unit value of a v or w represents the contribution of a unit horizontal displacement at the roof of the structure due to the corresponding type of displacement (see Eq. (5.2)).

We shall take an example of computing the (absolute) maximum of some quantity related to the structural deformation. The quantities v_Δ, v_θ, w_Δ, and w_θ are neglected since they represent rigid-body motions of the structure. In Tables 5.3, v_r and w_r are all small except v_1 and w_1, which are 1.0314 and 0.15545. Thus, a typical case of computing structural deformation is

$$\chi = 1.0314\, y_s(t) + 0.15545\, y_c(t) \tag{5.7}$$

Three possible methods have been presented in Section 2.5 for predicting the (absolute) maximum of the above equation. This computation of a peak response of a conjugate pair of nonclassical modes is essential to the response-spectrum approach for the case of nonclassical damping.

By the absolute-sum method, Eq. (2.34) gives

$$\max_t |\chi|/SD \simeq 1.1869 \tag{5.8}$$

where SD is the (absolute) maximum of $y_s(t)$, the corresponding spectral displacement. By the SRSS method, Eq. (2.35) gives

$$\max |\chi|/SD \simeq 1.0430 \tag{5.9}$$

Assume

$$\left| \frac{\ddot{u}_{g\ max}}{SD\ \Omega^2(1-\beta^2)} \right| = 1 \tag{5.10}$$

for the present case of a frequency of 3.5 Hz and a damping ratio of 14.5 percent of critical. Equation (2.43) then gives

$$1.0543 \le \frac{\max|\chi|}{SD} \le 1.1027 \tag{5.11}$$

If the mean value, 1.0785, of the extremes in the above equation is taken, its error must be less than 2.3 percent. In this case, the SRSS method gives an error ranging from 1.0 to 5.7 percent on the unsafe side. The error of the absolute-sum method ranges from 7.6 to 12.6 percent on the safe side. The conventional approach assuming a rigid soil gives the quantity v_1 a value of 1.2568 (see Table 5.3 for the initial approximation). The error then ranges from 14.0 to 19.2 percent on the safe side if the effects of the modal damping ratio and frequency are neglected.

5.5 Some Interaction Effects

In addition to the soft soil described above with a shear-wave velocity of 500 feet per second, a firm soil (2000 ft/sec) and an intermediate soil (1000 ft /sec) are used in this section in order to study the effects of various degrees of soil compliance. Also, three different models for energy dissipation are considered. First, only radiational damping of an elastic half space is considered. The second type of damping is the same as that assumed in Section 5.3 including both material and radiational damping of the soil. In the third case, the superstructure damping is also included. The damping of the superstructure is assumed to be classical and is taken as 2 percent of critical for each classical modes of the superstructure. The results of the fundamental mode of a soil-structure model are shown in Table 5.7 for the four-story building (see Section 5.3), and in Table 5.8 for the fifteen-story building.

The data show that for an undamped low-rise building, the modal damping is primarily due to radiational damping of the half space. This contribution of damping ranges from less than 1 percent of critical for a firm soil site to more than 12 percent of critical for a soft soil site. For an undamped high-rise building, however, the modal damping is primarily due to material damping of the soil. This contribution of damping is usually less than a few percent of critical. The damping ratio of the superstructure is not directly additive in order to compute the system damping ratio. However, a good approximation can be obtained if the superstructure damping ratio is first multiplied by a reduction factor. This reduction factor varies with cube of the ratio of decrease in frequency (Veletsos, 1977). The above general trend of the system damping is consistent with the parameter study of Veletsos cited, which has been adopted to the tentative provisions of the ATC-3 code (1978).

In addition to a system damping, the ratio of decrease in v_1 sheds light on the interaction effect to structural deformation. As has been predicted (Veletsos, 1977) for shear-beam buildings, this ratio can be approximated for most cases by the square of the ratio of decrease in frequency. However, the error may reach 11 percent on the unsafe side for the case of a four-story building founded on a soft soil. Furthermore, the contribution of structural deformation due to the w_1-type motion is not negligible in this case (see Section 5.4). Lastly, it is obvious that the system frequency decreases as the shear-wave velocity of the soil decreases.

5.6 Spread-Footing Foundation

In this section, the rigid-mat foundation used in the previous section is replaced by a foundation consisting of isolated spread footings (see Fig. 5.4). In a two-dimensional problem, each footing may have three degrees of freedom, i.e. two translations and one rocking rotation. Thus, the number of degrees of freedom for a spread-footing foundation is three times the number of footings. Nevertheless, the Ritz method can be applied, in order to reduce the degrees of freedom, by assuming some suitable displacement shapes for the foundation. In the rocking motion of the structure, for example, the whole foundation may take a shape as shown by Fig. 5.5A, which is different from Fig. 5.5B, a possible shape for the case of a flexible-mat foundation.

Each instantaneous displacement shape of a spread-footing foundation is, of course, affected by the stiffness of the superstructure above. For a shear-beam building, in which girders are quite stiff, it is common to assume each story has only one horizontal degree of freedom in a two-dimensional problem with no vertical ground motions. In this special case, it may be acceptable to assume the displacements of the footings are in the shape which would exist if the footings were connected by a rigid mat. In other words, it may be acceptable to assume that the foundation has only two degrees of freedom, one horizontal translation and one rocking rotation. For reasons of simplicity, this assumption will be used here in order to compare the effects of a spread-footing foundation and a rigid-mat foundation for shear-beam buildings.

The horizontal impedance function for the foundation is the sume of those for the spread footings. The rocking impedance function for the foundation, however, is primarily due to the vertical impedance functions for the footings, instead of the rocking ones. One reason for this is that the static impedance for a rigid disk in rocking is a function of the cubic of the radius of the disk (see Table 5.2). The vertical impedance function is evaluated from Eqs. (4.52) and (4.53). The results of the fundamental mode of a soil-structure system are shown in Table 5.9 for the four-story building (see Section 5.3), and in Table 5.8 for the fifteen-story building.

Compared to a rigid-mat construction, a footing foundation has relatively flexible soil "springs" and thus gives a lower system frequency. Consequently, the quantity v_1 is also smaller, which is related to the square of the ratio of decrease in frequency due to interaction. Also, the effectiveness of the apparent damping of a superstructure is reduced since it is related to the cube of the ratio of decrease in frequency.

For both low- and high-rise buildings with a footing foundation, material damping of the soil contributes more to a system damping than radiational damping of the half space does. As shown in Table 5.9 and 5.10 for various cases, the calculated system damping ratios are all less than a few percent of critical.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

6.1 Conclusions

Three general conclusions can be drawn from the results of Chapters 2, 3, 4, and 5. The first conclusion is related to the application of the response-spectrum method to a soil-structure system. The second deals with the computation of the nonclassical modes of soil-structure system. The last one concerns the generality of some of the solution methods presented.

It has been generally thought that the response-spectrum approach was not applicable to a soil-structure system with its nonclassical damping. However, as explained in Chapter 2, such computations can be carried out. The key to this application of the response-spectrum approach is the prediction of a maximum response corresponding to a conjugate pair of nonclassical modes. This turns out to be quite possible (see Section 2.5). The numerical results in Section 5.4 show that the error involved is less than 2.3 percent even when the effects of interaction are large. Since this key problem can be solved accurately and simply, there seems to be no more objection to applying the response-spectrum approach to a soil-structure system with nonclassical damping than a system with classical damping that is imagined to be on a rigid base.

No matter whether the damping is classical or nonclassical, knowledge of the modes is needed in order to apply a response-spectrum procedure (or a modal analysis). For a large system, it is the computation of modes that dominates the computational effort required. An effective and efficient numerical scheme has been proposed in Section 4.3 for the specific form of system arising in a study of dynamic soil-structure interaction using the impedance approach. In order to compute a nonclassical mode of a soil-structure system, a classical mode of the structure with its base fixed is used as the initial approximation. The differences between the initial and improved modes give direct measures of the significance of interaction effects.

As indicated by the numerical results in Chapter 5, only one or two iterations in the computation are needed to give a very good approximation to a nonclassical mode even if the effects of interaction are large. In addition, the convergence is more rapid than a second order process. This rapid rate of convergence is also to be expected from a theoretical point of view (see Chapter 3). It is concluded that the range of applicability and the degree of accuracy of the proposed method are more than satisfactory for solving the problem of dynamic soil-structure interaction. Moreover, a sensitivity analysis covering ranges of soil properties (and foundation impedances) can be carried out very conveniently by the method proposed in Section 4.3.

Although this work is aimed at solving the problem of dynamic soil-structure interaction, the theory of viscous damping in Chapter 2 and the numerical scheme in Section 3.3 are applicable to a general, viscously damped system with constant parameters. In other words, their applications are not limited to the problem of dynamic soil-structure interaction.

6.2 Recommendations for Further Study

When using the impedance approach to study the effects of dynamic soil-structure interaction, a superstructure may have a great number of degrees of freedom and even can be described by a general finite element idealization. However, the foundation of the structure is usually limited to be a rigid mat, that is, a subsystem with very few degrees of freedom. As explained in Section 4.5, this limitation can be removed. The application of the principle of virtual work and the Ritz method gives the governing equations of motion for a soil-structure system with almost any type of foundation. The examples in Chapter 5 show only the most simplified cases in which a foundation has two degrees of freedom. It would be very desirable to study the effects of the flexibility of the foundation by including more degrees of freedom for the foundation. The significance of the effect of these added degrees of freedom will depend on the size and rigidity of the structure investigated. For a spread-footing foundation, the displacements may be very different from the shapes assumed in Section 5.6. For a flexible-mat foundation, soil impedance functions for a curvilinear-shape foundation remain to be calculated.

The general form of impedance functions of a linear solid (or system) is discussed in Section 4.4. These impedance functions are, in general, frequency dependent. The problem of using some simple system to approximate the impedance functions over some range of frequency deserves further study. This is an important subject because the frequency dependence of an impedance is a common problem in linear, dynamic, substructure analysis.

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TABLE 4.1 VALUES OF α_1, β_r, AND γ_r IN EQUATIONS (4.50) - (4.53) (after Veletsos and Verbic, 1973)

Quantity	$\nu = 0$	$\nu = 1/3$	$\nu = 0.45$	$\nu = 0.5$
α_1	0.775	0.65	0.60	0.60
β_1	0.8	0.8	0.8	0.8
β_2	0.525	0.5	0.45	0.4
β_3	0.	0.	0.023	0.027
γ_1	0.25	0.35	--	0.
γ_2	1.0	0.8	--	0.
γ_3	0.	0.	--	0.17
γ_4	0.85	0.75	--	0.85

TABLE 5.1 INCREMENTS OF FREQUENCIES AND DAMPING RATIOS

	Iterations				
	0	1	2	3	4
β_1	.0	.29591 E-01	.68725 E-05	.11624 E-12	
Ω_1	.12339 E 01	.26051 E-02	.91376 E-05	-.37659 E-12	
β_2	.0	.60667 E-01	.78418 E-03	-.12657 E-06	-.44409 E-15
Ω_2	.35137 E 01	.17988 E-01	.40832 E-03	-.17846 E-06	.14211 E-13
β_3	.0	.39768 E-01	.11096 E-02	-.37472 E-06	.17319 E-13
Ω_3	.52587 E 01	-.26098 E-01	-.23697 E-02	.44972 E-05	.28422 E-13
β_4	.0	.52227 E-02	-.31041 E-93	.78295 E-09	.27756 E-16
Ω_4	.62030 E 01	-.11572 E-01	.35276 E-04	.20400 E-08	.0

TABLE 5.2 STATIC STIFFNESS FOR A RIGID DISK

Horizontal	$\frac{8\ GR}{2-\nu}$
Rocking	$\frac{8\ GR^3}{3\ (1-\nu)}$
Vertical	$\frac{4\ GR}{1-\nu}$

where $G = \rho V_s^2$

TABLE 5.3. CONVERGENCE OF FUNDAMENTAL MODE
(4-STORY BUILDING)

	Iterations					
	0	1	2	3	4	5
Ω	.26180 E 02	.22650 E 02	.22064 E 02	.22072 E 02	.22072 E 02	.22072 E 02
β	.0	.14305 E 00	.14870 E 00	.14570 E 00	.14570 E 00	.14570 E 00
v_1*	.12568 E 01	.94351 E 00	.10340 E 01	.10314 E 01	.10314 E 01	.10314 E 01
v_2	.0	.12972 E-02	.23713 E-02	.23553 E-02	.23553 E-02	.23553 E-02
v_3	.0	.15236 E-02	.18251 E-02	.17972 E-02	.17975 E-02	.17975 E-02
v_4	.0	-.14058 E-03	-.11541 E-03	-.11207 E-03	-.11213 E-03	-.11213 E-03
v_Δ	.0	.90601 E-01	.11860 E 00	.11835 E 00	.11836 E 00	.11836 E 00
v_θ	.0	.24371 E 00	.33263 E 00	.33165 E 00	.33167 E 00	.33167 E 00
w_1*	.0	.13976 E 00	.15896 E 00	.15543 E 00	.15545 E 00	.155045E 00
w_2	.0	-.40822 E-02	-.38300 E-02	-.38516 E-02	-.38509 E-02	-.38509 E-02
w_3	.0	-.17680 E-02	-.10186 E-02	-.10069 E-02	-.10070 E-02	-.10070 E-02
w_4	.0	-.81272 E-05	-.11256 E-03	-.11554 E-03	-.11547 E-03	-.11547 E-03
w_Δ	.0	-.12962 E 00	-.17272 E 00	-.17180 E 00	-.17182 E 00	-.17182 E 00
w_θ	.0	-.19232 E 00	-.19068 E 00	-.18078 E 00	-.18093 E 00	-.18093 E 00

* The quantities v's and w's are defined in Eqs. (5.5) and (5.6).

TABLE 5.4. CONVERGENCE OF SECOND MODE
(4-STORY BUILDING)

	Iterations			
	0	1	2	3
Ω	.74554 E 02	.74029 E 02	.74107 E 02	.74107 E 02
β	.0	.36207 E-01	.37141 E-11	.37141 E-01
v_1*	.0	.10165 E-01	.13192 E-01	.13191 E-01
v_2	.37415 E 00	.38015 E 00	.38818 E 00	.38815 E 00
v_3	.0	-.92063 E-04	.29619 E-03	.29683 E-03
v_4	.0	.34742 E-03	.44995 E-03	.44999 E-03
v_Δ	.0	.38479 E-02	.21689 E-03	.21756 E-03
v_θ	.0	-.16744 E-01	-.13371 E-01	-.13369 E-01
w_1*	.0	.31315 E-01	.32238 E-01	.32233 E-01
w_2	.0	-.17273 E-01	-.31132 E-01	-.31133 E-01
w_3	.0	-.59144 E-02	-.60322 E-02	-.60311 E-02
w_4	.0	-.22824 E-02	-.23148 E-02	-.23145 E-02
w_Δ	.0	-.59492 E-01	-.62273 E-01	-.62265 E-01
w_θ	.0	.57630 E-01	.60798 E-01	.60791 E-01

* The quantities v's and w's are defined in Eqs. (5.5) and (5.6).

TABLE 5.5 CONVERGENCE OF FUNDAMENTAL MODE (15-STORY BUILDING)

	Iterations					Exact Solution
	0	1	2	3	4	
Ω	.69813 E 01	.51506 E 01	.51370 E 01	.51373 E 01	.51373 E 01	.51373 E 01
β	.0	.33767 E-01	.31700 E-01	.31750 E-01	.31749 E-01	.31749 E-01
v_1*	.12721 E 01	.64133 E 00	.69036 E 00	.69037 E 00	.69037 E 00	.69037 E 00
v_2	.0	-.48674 E-02	-.24334 E-02	-.24349 E-02	-.24349 E-02	-.24349 E-02
v_3	.0	.13162 E-02	.76727 E-03	.76731 E-03	.76731 E-03	.76731 E-03
v_Δ	.0	.41641 E-01	.48960 E-01	.48946 E-01	.48946 E-01	.48946 E-01
v_θ	.0	.61828 E 00	.64657 E 00	.64645 E 00	.64645 E 00	.64645 E 00
w_1*	.0	.29839 E-01	.40186 E-01	.40278 E-01	.40277 E-01	.40274 E-01
w_2	.0	-.48359 E-03	-.42231 E-03	-.42428 E-03	-.42428 E-03	-.42428 E-03
w_3	.0	.26539 E-03	-.67052 E-04	-.67137 E-04	-.67137 E-04	-.67142 E-04
w_Δ	.0	-.18945 E-01	-.16455 E-01	-.16455 E-01	-.16455 E-01	-.16455 E-01
w_θ	.0	-.45556 E-01	-.32026 E-01	-.32084 E-01	-.32082 E-01	-.32085 E-01

* The quantities v's and w's are defined in Eqs. (5.5) and (5.6).

TABLE 5.6 CONVERGENCE OF SECOND MODE (15-STORY BUILDING)

	Iterations				Exact Solution
	0	1	2	3	
Ω	.20868 E 02	.20105 E 02	.20086 E 02	.20086 E 02	.20087 E 02
β	.0	.33305 E-01	.34556 E-01	.34558 E-01	.34697 E-01
v_1*	.0	.16282 E 00	.17122 E 00	.17123 E 00	.17189 E 00
v_2	.42091 E 00	.44656 E 00	.46473 E 00	.46475 E 00	.46600 E 00
v_3	.0	-.14385 E-04	.69413 E-03	.69323 E-03	.67572 E-03
v_Δ	.0	.29923 E-01	.31790 E-01	.31790 E-01	.31665 E-01
v_θ	.0	-.22515 E 00	-.23552 E 00	-.23553 E 00	-.23605 E 00
w_1*	.0	-.56023 E-02	-.59951 E-02	-.60037 E-02	-.63598 E-02
w_2	.0	-.57339 E-02	-.11878 E-01	-.11899 E-01	-.12869 E-01
w_3	.0	-.63023 E-02	-.61494 E-02	-.61487 E-02	-.61930 E-02
w_Δ	.0	-.56674 E-01	-.62357 E-01	-.62365 E-01	-.62847 E-01
w_θ	.0	.94993 E-01	.10252 E 00	.10253 E 00	.10366 E 00

* The quantities v's and w's are defined in Eqs. (5.5) and (5.6).

TABLE 5.7 DATA FOR THE FUNDAMENTAL MODE

(4 -STORY BUILDING WITH RIGID MAT)

		SOIL TYPE		
		FIRM	INTERMEDIATE	SOFT
	I*	.35678 E-02	.22403 E-01	.12418 E 00
β	II*	.51453 E-02	.27977 E-01	.14575 E 00
	III*	.24266 E-01	.45184 E-01	.15713 E 00
	I	.98377	.94366	.83319
$\frac{\Omega}{\omega}$	II	.98426	.94603	.84309
	III	.98392	.94450	.83801
	I	.97462	.91900	.78315
$\frac{v_1}{1.2568}$	II	.97677	.92775	.82066
	III	.97581	.92409	.80753

* I = Radiational Damping Only

II = Radiational + Soil Material Damping

III = Radiational + Soil Material + Structure Apparent Damping

TABLE 5.8 DATA FOR THE FUNDAMENTAL MODE

(15-STORY BUILDING WITH RIGID MAT)

		SOIL TYPE		
		FIRM	INTERMEDIATE	SOFT
β	I*	.27743 E-03	.18693 E-02	.86019 E-02
	II*	.27213 E-02	.10475 E-01	.31749 E-01
	III*	.21278 E-01	.25567 E-01	.39718 E-01
$\frac{\Omega}{\omega}$	I	.97502	.90940	.73488
	II	.97528	.91016	.73587
	III	.97512	.90967	.73522
$\frac{v_1}{1.2721}$	I	.95063	.82690	.53991
	II	.95158	.82973	.54270
	III	.95103	.82832	.54163

* I = Radiational Damping Only

II = Radiational + Soil Material Damping

III = Radiational + Soil Material + Structure Apparent Damping

TABLE 5.9 DATA FOR THE FUNDAMENTAL MODE

(4-STORY BUILDING WITH SPREAD FOOTINGS)

		SOIL TYPE		
		FIRM	INTERMEDIATE	SOFT
β	I*	.44128 E-03	.24016 E-02	.70809 E-02
	II*	.55059 E-02	.18281 E-01	.40243 E-01
	III*	.22557 E-01	.29607 E-01	.44224 E-01
$\frac{\Omega}{\omega}$	I	.94786	.82804	.58976
	II	.94832	.82903	.59030
	III	.94805	.82845	.58995
$\frac{v_1}{1.2568}$	I	.90985	.70951	.36726
	II	.91160	.71278	.36802
	III	.91065	.71127	.36784

* I = Radiational Daming Only

II = Radiational + Soil Material Damping

III = Radiational + Soil Material + Structure Apparent Damping

TABLE 5.10 DATA FOR THE FUNDAMENTAL MODE

(15-STORY BUILDING WITH SPREAD FOOTINGS)

		SOIL TYPE		
		FIRM	INTERMEDIATE	SOFT
β	I*	.20466 E-03	.87490 E-03	.19142 E-02
	II*	.88351 E-02	.23731 E-01	.40564 E-01
	III*	.23896 E-01	.31742 E-01	.42731 E-01
$\frac{\Omega}{\omega}$	I	.90910	.73675	.47785
	II	.90975	.73767	.47822
	III	.90937	.73718	.47806
$\frac{v_1}{1.2721}$	I	.82431	.53859	.22497
	II	.82666	.54078	.22500
	III	.82548	.53993	.22511

* I = Radiational Damping Only

II = Radiational + Soil Material Damping

III = Radiational + Soil Material + Structure Apparent Damping

IMPERIAL VALLEY EARTHQUAKE MAY 18, 1940 - 2037 PST

IIIA001 40.001.0 EL CENTRO SITE IMPERIAL VALLEY IRRIGATION DISTRICT COMP SOOE

DAMPING VALUES ARE 0, 2, 5, 10 AND 20 PERCENT OF CRITICAL

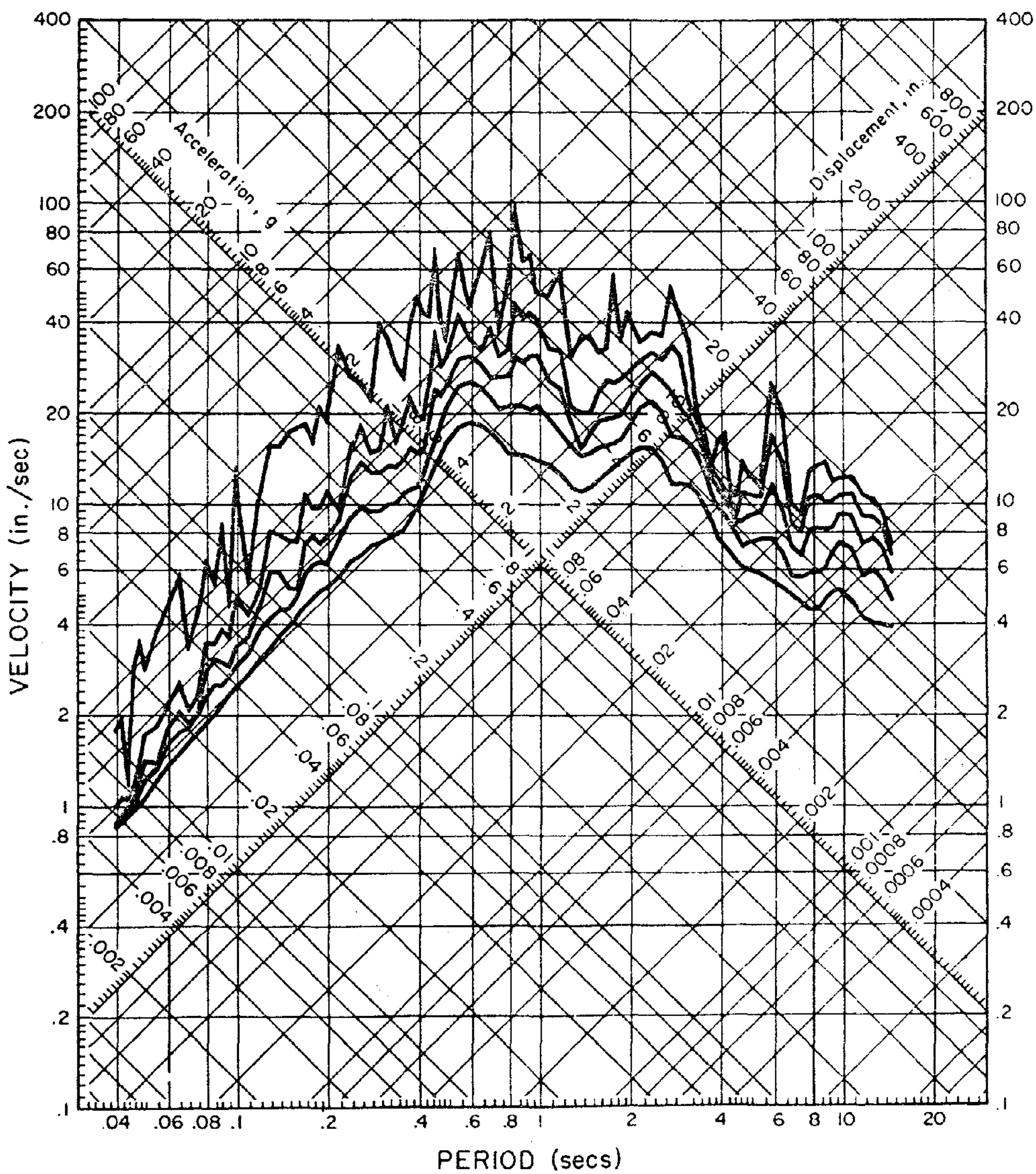


FIG. 2.1 A RESPONSE SPECTRUM
(after Earthquake Engineering Research Laboratory, 1972)

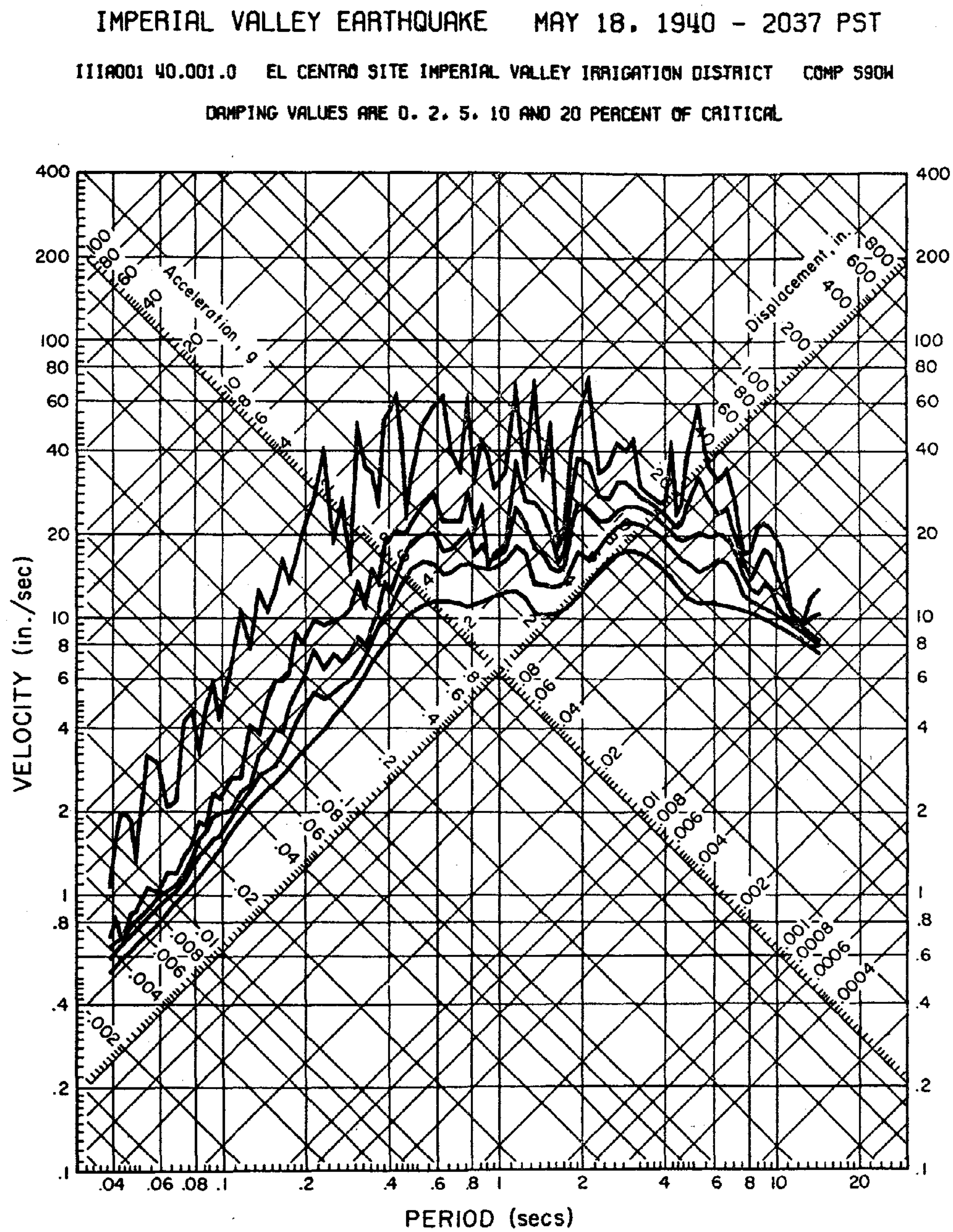


FIG. 2.2 A RESPONSE SPECTRUM
(after Earthquake Engineering Research Laboratory, 1972)

SPECTRUM AMPLIFICATION FACTORS FOR HORIZONTAL ELASTIC RESPONSE

Damping, % Critical	One Sigma (84.1%)			Median (50%)		
	A	V	D	A	V	D
0.5	5.10	3.84	3.04	3.68	2.59	2.01
1	4.38	3.38	2.73	3.21	2.31	1.82
2	3.66	2.92	2.42	2.74	2.03	1.63
3	3.24	2.64	2.24	2.46	1.86	1.52
5	2.71	2.30	2.01	2.12	1.65	1.39
7	2.36	2.08	1.85	1.89	1.51	1.29
10	1.99	1.84	1.69	1.64	1.37	1.20
20	1.26	1.37	1.38	1.17	1.08	1.01

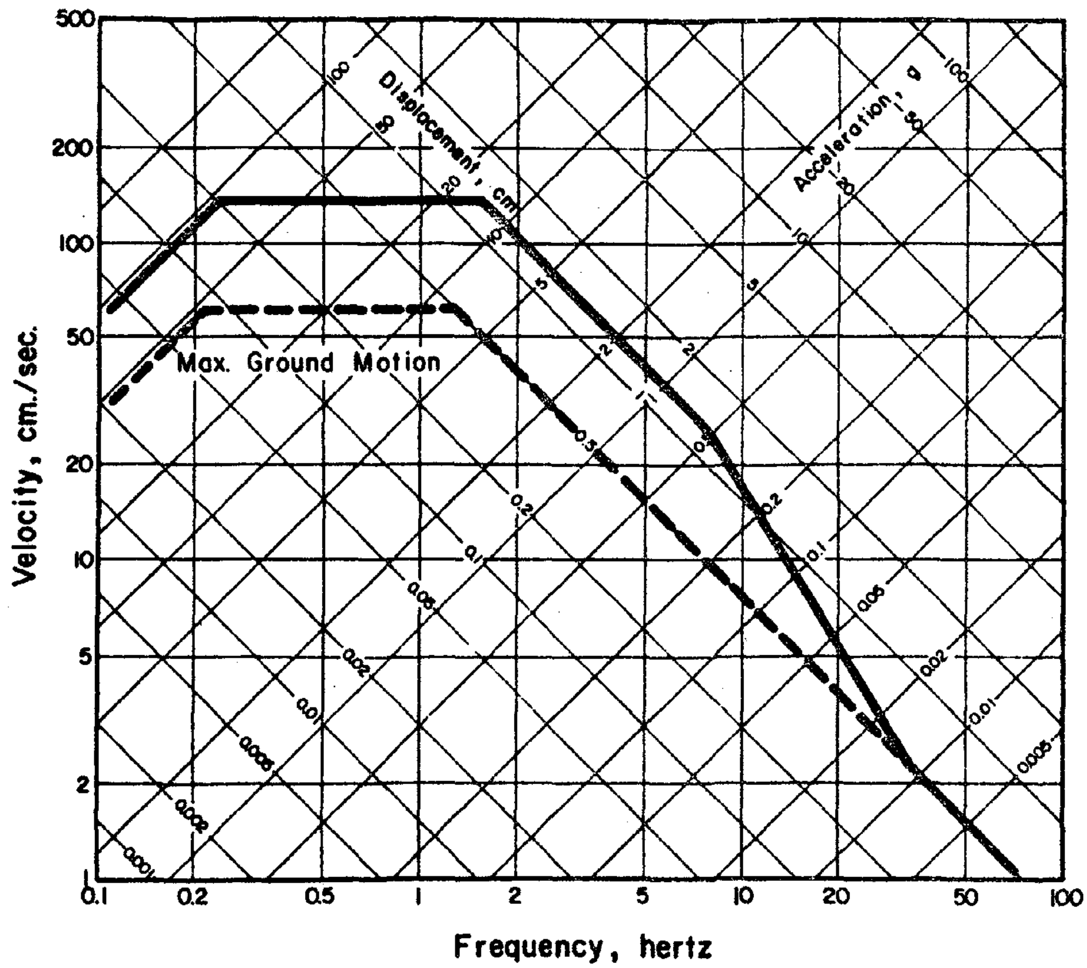


FIG. 2.3 ELASTIC DESIGN SPECTRUM (0.5g MAX. ACCEL., 5% DAMPING, 84.1% CUMULATIVE PROPABILITY)
(after Newmark and Hall, 1977)

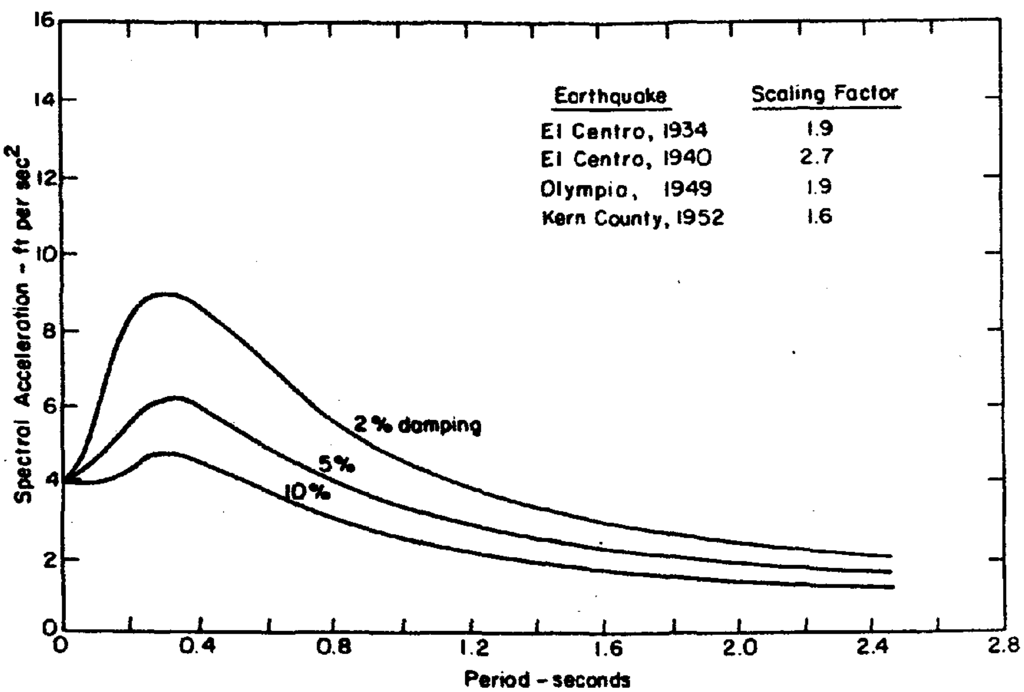


FIG. 2.4 AVERAGE ACCELERATION SPECTRA
(after Housner, 1959)

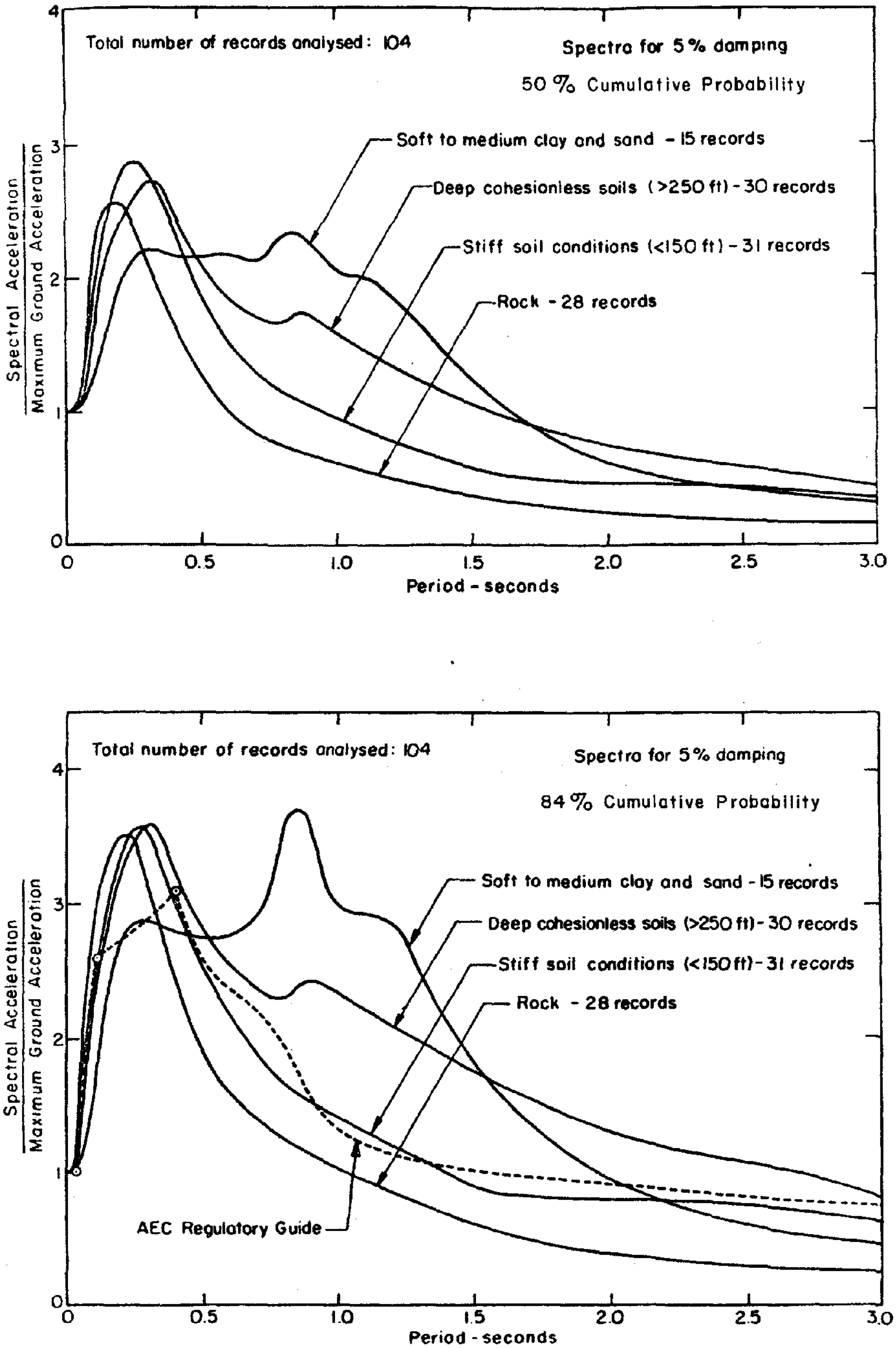


Fig. 2.5 AVERAGE ACCELERATION SPECTRA FOR DIFFERENT SITE CONDITIONS (after Seed, Ugas, and Lysmer, 1974)

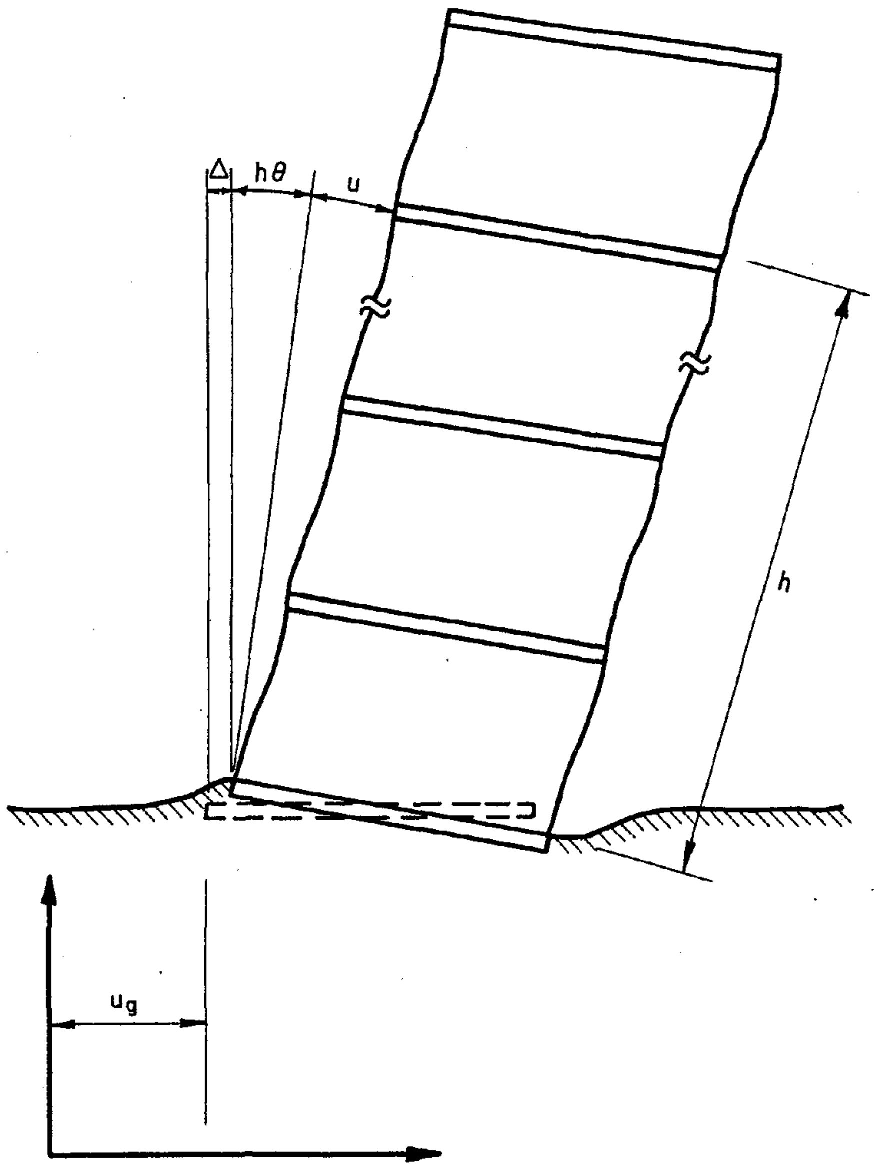


FIG. 4.1 DISPLACEMENTS OF A SHEAR-BEAM BUILDING WITH A RIGID-MAT FOUNDATION

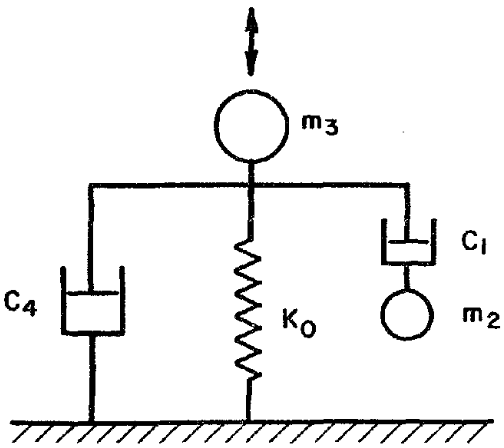


FIG. 4.2 A SOIL MODEL OF MEEK AND VELETSOS (1973)

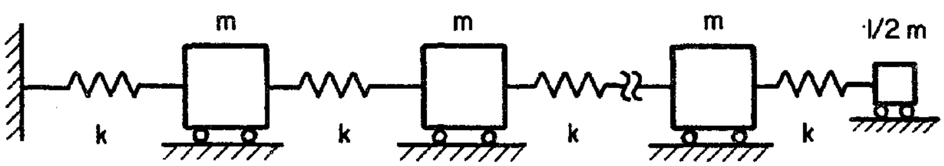


FIG. 5.1 AN UNDAMPED MECHANICAL MODEL

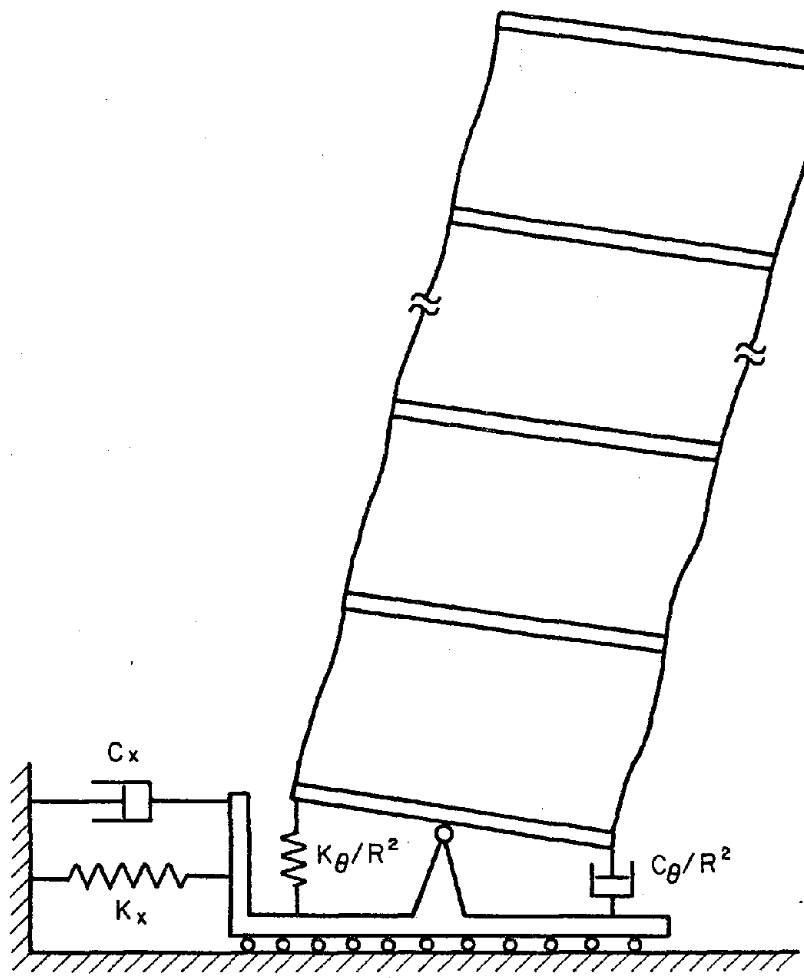


FIG. 5.2 SOIL-STRUCTURE INTERACTION MODEL FOR A SHEAR-BEAM BUILDING WITH A RIGID-MAT FOUNDATION

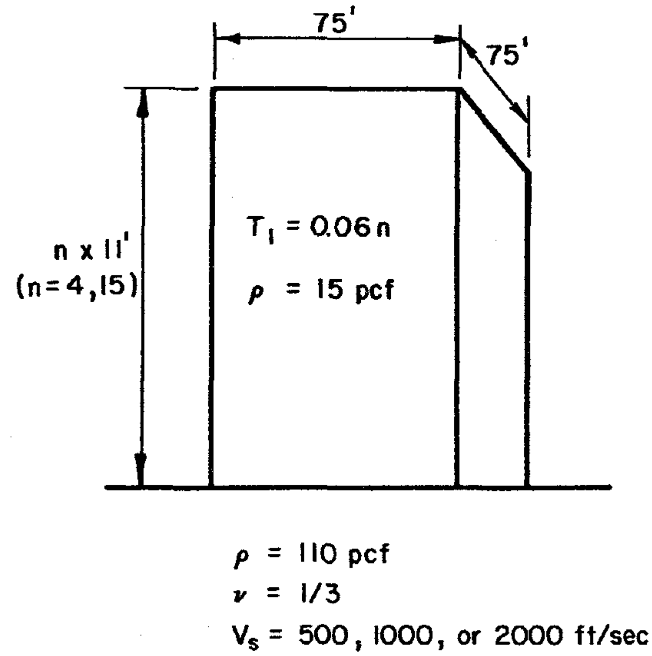


FIG. 5.3 PROPERTIES OF A SAMPLE SOIL-STRUCTURE SYSTEM

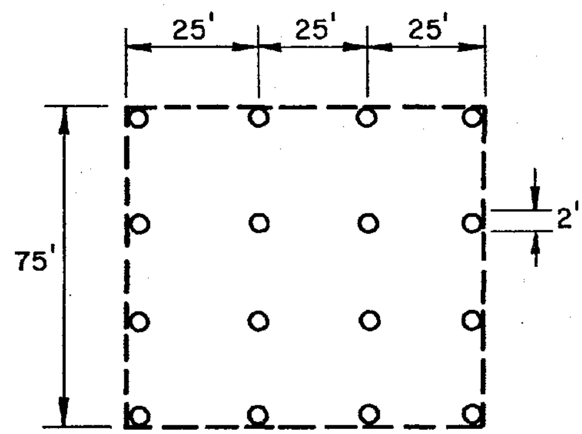
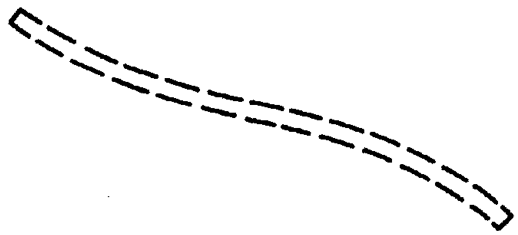


FIG. 5.4 PLAN VIEW OF ISOLATED SPREAD FOOTINGS



(A) Spread-Footing Foundation

(B) Flexible-Mat Foundation

FIG. 5.5 DISPLACEMENT SHAPES OF SPREAD-FOOTING AND FLEXIBLE-MAT FOUNDATIONS

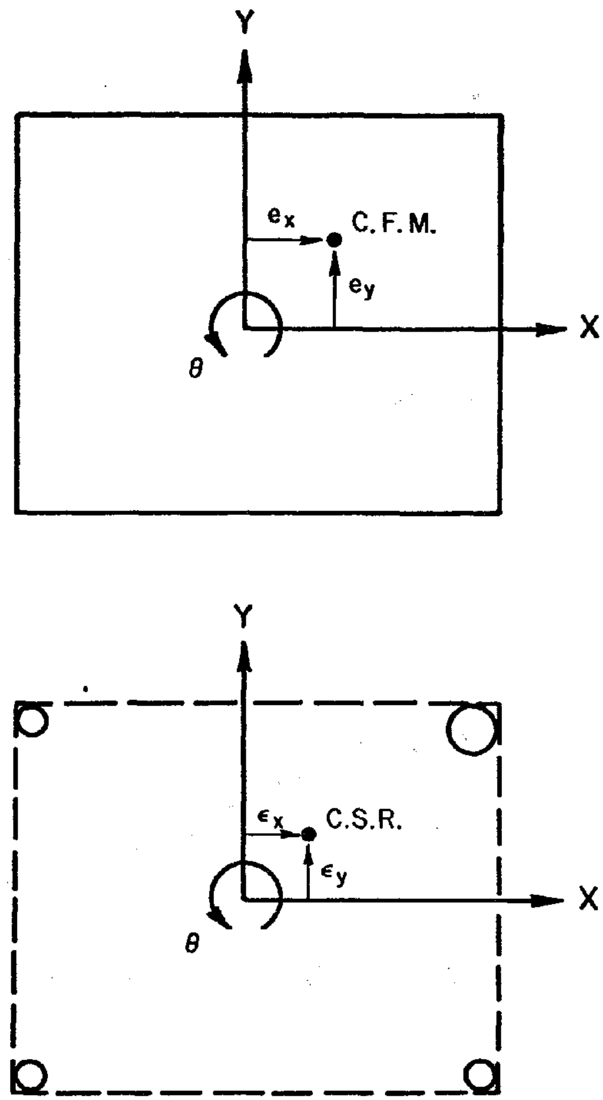


FIG. A.1 ECCENTRICITIES OF CENTERS OF FLOOR MASS AND STORY RIGIDITY

APPENDIX A

LATERAL-TORSIONAL MODAL COUPLING

A computational scheme is presented in the section for finding the coupled lateral-torsional modes of a shear-beam building using modes obtained from the usual, uncoupled lateral and torsional analysis. The difference between an uncoupled mode and improved mode will give direct measures of the effects of lateral-torsional modal coupling. This modal improvement is very useful in earthquake engineering for assessing the effects of lateral-torsional coupling. Since the basic idea of this approach has been discussed in Section 4.3, we shall not detail this presentation here. Attention will be given to introducing the general form of the eigenvalue problem involved.

We shall examine the case of an undamped, multi-story, shear-beam building with its base fixed. Each floor of the building is assumed to have only three degrees of freedom, one torsional rotation and two horizontal translations in perpendicular directions (see Fig. A. 1). For a building with n stories, the building system has a total of 3n degrees of freedom.

Consider first the special case of a two-story building (n=2). The equations of motion for this six degrees of freedom system in free vibration can be written in matrix notation as

$$\left\{ 0 \right\} = \left[\begin{array}{cc|cc|cc} m_2 & & & & -m_2e_{y2} & \\ & m_1 & & & & -m_1e_{y1} \\ \hline & & m_2 & & m_2e_{x2} & \\ & & & m_1 & & m_1e_{x1} \\ \hline \text{sym.} & & & & I_2 & \\ & & & & & I_1 \end{array}\right] \left\{\begin{array}{c} \ddot{u}_{x2} \\ \ddot{u}_{x1} \\ \ddot{u}_{y2} \\ \ddot{u}_{y1} \\ \ddot{u}_{\theta 2} \\ \ddot{u}_{\theta 1} \end{array}\right\} +$$

$$
\left[\begin{array}{cc|cc|c|c}
k_{x2} & -k_{x2} & & & -k_{x2}\varepsilon_{y2} & k_{x2}\varepsilon_{y2} \\
 & k_{x2}+k_{x1} & & & k_{x2}\varepsilon_{y2} & -(k_{x2}\varepsilon_{y2}+k_{x1}\varepsilon_{y1}) \\
\hline
 & & k_{y2} & -k_{y2} & k_{y2}\varepsilon_{x2} & -k_{y2}\varepsilon_{x2} \\
 & & & k_{y2}+k_{y1} & -k_{y2}\varepsilon_{x2} & (k_{y2}\varepsilon_{x2}+k_{y1}\varepsilon_{x1}) \\
\hline
\text{sym.} & & & & k_{\theta 2} & -k_{\theta 2} \\
 & & & & & k_{\theta 2}+k_{\theta 1}
\end{array}\right]
\begin{Bmatrix} u_{x2} \\ u_{x1} \\ u_{y2} \\ u_{y1} \\ u_{\theta 2} \\ u_{\theta 1} \end{Bmatrix}
\qquad \text{(A.1)}
$$

In the above equation, the displacements, the u's, and the eccentricity quantities, the e's and ε's, are defined in Fig. A.1. In the case of zero eccentricities, Eqs. (A.1.) reduce to the three usual, uncoupled sets of equations of motion. The notation for the m's and k's in these three sets of equation is self-explanatory.

For a general n-story building, the eigenvalue problem is expressed in matrix notation as

$$
-\omega^2
\begin{bmatrix}
[M]_x & [0] & [M]_{x\theta} \\
 & [M]_y & [M]_{y\theta} \\
\text{sym.} & & [M]_\theta
\end{bmatrix}
\begin{Bmatrix} \{u\}_x \\ \{u\}_y \\ \{u\}_\theta \end{Bmatrix}
+
\begin{bmatrix}
[K]_x & [0] & [K]_{x\theta} \\
 & [K]_y & [K]_{y\theta} \\
\text{sym.} & & [K]_\theta
\end{bmatrix}
\begin{Bmatrix} \{u\}_x \\ \{u\}_y \\ \{u\}_\theta \end{Bmatrix}
= 0 \qquad \text{(A.2)}
$$

The diagonal n x n sub-matrices above do not involve the eccentricities e's and ε's. If the conventional assumption of zero eccentricities is used, the off-diagonal n x n sub-matrices are null. And the 3n uncoupled lateral and torsional modes can be computed by the Holzer table (or some other suitable method), so that

$$[M]_\alpha [\Phi]_\alpha [\ ^\backprime w_r^2\ _\backprime]_\alpha = [K]_\alpha [\Phi]_\alpha, \qquad \begin{array}{l} \alpha = \ , y, \theta \\ r = 1, 2, \ldots, n \end{array} \tag{A.3}$$

where an nxn matric $[\Phi]$ consists of n columns of the uncoupled lateral or torsional modes. When the eccentricities are nonzero, lateral-torsional couplings occur among these uncoupled modes.

The eigenvalue problem Eqs. (A.2) can be modified by a transformation of coordinates. Let

$$\begin{Bmatrix} \{u\}_x \\ \{u\}_y \\ \{u\}_\theta \end{Bmatrix} = \begin{bmatrix} [\Phi]_x & & \\ & [\Phi]_y & \\ & & [\Phi]_\theta \end{bmatrix} \begin{Bmatrix} \{v\}_x \\ \{v\}_y \\ \{v\}_\theta \end{Bmatrix} \tag{A.4}$$

After transforming the coordinates of Eqs. (A.2) to those of the type, we have

$$-\omega^2 \begin{bmatrix} [M]_x^1 & [0] & [M]_{x\theta}^1 \\ & [M]^1 & [M]_{y\theta}^1 \\ \text{sym} & & [M]_\theta^1 \end{bmatrix} \begin{Bmatrix} \{v\}_x \\ \{v\}_y \\ \{v\}_\theta \end{Bmatrix} + \begin{bmatrix} [K]_x^1 & [0] & [K]_{x\theta}^1 \\ & [K]_y^1 & [K]_{y\theta}^1 \\ \text{sym.} & & [K_\theta]^1 \end{bmatrix} \begin{Bmatrix} \{v\}_x \\ \{v\}_y \\ \{v\}_\theta \end{Bmatrix} = \{0\} \tag{A.5}$$

The diagonal nxn sub-matrices above are now diagonal matrices since

$$[M]_\alpha^1 = [\phi]_\alpha^T [M]_\alpha [\phi]_\alpha, \tag{A.6}$$

$$[K]_\alpha^1 = [\phi]_\alpha^T [K]_\alpha [\phi]_\alpha, \quad \alpha = x, y, \theta \tag{A.7}$$

Thus, the two 3n x 3n matrices in Eqs. (A.5) are bordered matrices. Each one contains a 2n x 2n diagonal matrix bordered by n columns and n rows. Although the order of the simultaneous equation for using the Robinson-

Harris method is $3n + 1$, it can be reduced to $n+1$ after a procedure of condensation. Furthermore, if the Ritz method is applied by using only s uncoupled torsional modes ($s < n$), it is then reduced to $s+1$. These procedures increase efficiency further in the effective numerical scheme used.

APPENDIX B

RAYLEIGH-HOLZER METHOD

The Holzer's table (Holzer, 1921) has been used extensively for computing the natural modes of a multi-story shear-beam building (say n-story) with its base fixed (Newmark and Rosenblueth, 1971; Clough and Penzien, 1975). The building is assumed to have only one degree of freedom at each floor. The modes may be usual, uncoupled lateral or torsional modes. For an assumed frequency ω, this method gives an approximate mode shape $\{\phi\}$.

As noted by Crandall and Strang (1957), Lord Rayleigh (1894) suggested that the Rayleigh quotient obtained from a calculated shpae of the Holzer's table* be used for next trial in the iteration, i.e.

$$\omega^{1^2} = \frac{\{\phi\}^T[K]\ \{\phi\}}{\{\phi\}^T[M]\ \{\phi\}} \tag{B.1}$$

where ω^1 is the frequency suggested for next trial; the matrices [M] and [K] are mass and stiffness matrices. With the interpretation of energy, this formula in Newmark and Rosenblueth's notation becomes

$$\omega^{1^2} = \omega^2 \frac{\sum_{r=1}^{n} Q_r\ \Delta Z_r}{\sum_{r=1}^{n} F_r\ Z_r} \tag{B.2}$$

where F_r = inertia force,

Q_r = story forces,

Z_r = story displacements, and

ΔZ_r = relative story displacements.

---

*Rayleigh's suggestion preceded the Holzer method by more than 27 years.

These two equations are often used in hand calculations. It would be beneficial if these equations could be simplified computationally.

Since ω and $\{\phi\}$ are only approximate, in general, there are residual forces $\{R\}$ in the following expression

$$- \omega^2 \; [M] \; \{\phi\} + [K] \; \{\phi\} = \{R\} \tag{B.3}$$

In this case of a shear-beam building, $\{R\}$ has only one nonzero term R_n, the residual force (usually) at the top of building. A consequence of Eq. (B.3) is

$$\{\phi\}^T \; [K] \; \{\phi\} = \omega^2 \; \{\phi\}^T \; [M] \; \{\phi\} + \{\phi\}^T \; \{R\} \tag{B.4}$$

Substituting Eq. (B.4) into Eq. (B.1) yields

$$\omega^{1^2} = \omega^2 + \frac{\{\phi\}^T \; \{R\}}{\{\phi\}^T \; [M] \; \{\phi\}} \tag{B.5}$$

With the interpretation of energy, this equation can be written as

$$\omega^{1^2} = \omega^2 \left(1 - \frac{R_n \; Z_n}{\sum_{r=1}^{n} F_r \; Z_r} \right) \tag{B.6}$$