PROBABILISTIC SEISMIC STABILITY ANALYSIS OF EARTH SLOPES

by

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Report No. CE-80-2

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Troy, N.Y. 12181

Sponsored by the National Science Foundation

Grant No. ENV 77-16185

December 1980

50272-101

REPORT DOCUMENTATION PAGE	1. REPORT NO. NSF/RA-800385	2.	3. Recipient's Accession No. PB81 158883
4. Title and Subtitle: Probabilistic Seismic Stability Analysis of Earth Slopes			5. Report Date: December 1980
			6.
7. Author(s): D. A-Grivas, J. D. Howland			8. Performing Organization Rept. No. CE-80-2
9. Performing Organization Name and Address: Rensselaer Polytechnic Institute, Department of Civil Engineering, Troy, NY 12181			10. Project/Task/Work Unit No.
			11. Contract(C) or Grant(G) No. (C) (G) ENV7716185
12. Sponsoring Organization Name and Address: Engineering and Applied Science (EAS), National Science Foundation, 1800 G Street, N.W., Washington, D.C. 20550			13. Type of Report & Period Covered
			14.

15. Supplementary Notes

16. Abstract (Limit: 200 words)

A new mathematical model is proposed for determining the reliability of earth slopes during earthquakes. The method provides a probabilistic, pseudo-static, seismic stability analysis. Safety of slopes is measured in terms of probability of failure rather than the customary safety factor, and the numerical values are determined via a Monte Carlo simulation of failure. Significant uncertainties present in conventional methods of analysis have been identified and probabilistic tools introduced for their description and amelioration. This report reviews the objectives and achievements of the project. Models used to account for the variability of soil strength are described and illustrated. Results of a parametric study on the relative strength and seismic parameters are presented in graphs and tables. Finally, an evaluation of the developed model is included with a discussion of its applicability and limitations.

17. Document Analysis a. Descriptors

Earthquakes
Mathematical models
Probability theory
Soil properties
Slopes
Monte Carlo method
Loads

b. Identifiers/Open-Ended Terms

Earthquake Hazards Mitigation

c. COSATI Field/Group

18. Availability Statement	19. Security Class (This Report)	21. No. of Pages
NTIS	20. Security Class (This Page)	22. Price

(See ANSI–Z39.18) See Instructions on Reverse OPTIONAL FORM 272 (4–77) (Formerly NTIS–35) Department of Commerce

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PREFACE

This is the fifth and final report of the project under the general title "Reliability Analysis of Soil Slopes During Earthquakes". This research study is sponsored by the Earthquake Hazard Mitigation Program of the National Science Foundation under Grant No. ENV 77-16185. Dr. Michael Gaus is the program manager of this project of which Professor Dimitrí A-Grivas is the principal investigator.

Four progress reports on the present project have been previously submitted to the National Science Foundation and are referred to in this text as Report Nos. CE-78-5, CE-78-6, CE-78-7 and CE-79-1. Their titles and authorships are as follows:

1. A-Grivas D., J.D. Howland and P. Tolcser, "A Probabilistic Model for Seismic Slope Stability Analysis," Report No. CE-78-5, Dept. of Civil Engineering, R.P.I., p. 82, June 1979.

2. A-Grivas, D., "Program RASSUEL - Reliability Analysis of Soil Slopes Under Earthquake Loading," Report No. CE-78-6. Dept. of Civil Engineering, R.P.I., p. 41, December 1978.

3. A-Grivas, D., R. Dyvik and J.D. Howland, "An Engineering Analysis of the Seismic History of New York State", Report No. CE-78-7, Dept. of Civil Engineering, R.P.I., p. 77, December 1978.

4. A-Grivas, D. and G.F. Nadeau, "Probabilistic Seismic Stability Analysis - A Case Study", Report No. CE-79-1, Dept. of Civil Engineering, R.P.I., p. 34, July 1979.

The authors wish to thank the National Science Foundation for sponsoring this study. Finally, special thanks are extended to Betty Alix for typing this report.

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LIST OF SYMBOLS

English Characters

a	Minimum value of the beta distribution
a_{max}	Maximum acceleration at a site due to an earthquake
a_u	Minimum value of a variable based on the assumption of a uniform distribution
b	Maximum value of the beta distribution
b, c	Parameters of the magnitude-frequency relationship
b_1, b_2, b_3, b_4	Regional attenuation parameters
c	Cohesion intercept of soil strength
c_n	Normalized cohesion intercept
d_x	Shift factor for the variable x
d_y	Shift factor for the variable y
$f(x)$	Probability density function of a variable x
$f(x,y)$	Bivariate density function of variables x and y
$F(x)$	Cumulative density function of a variable x
F_S	Conventional factor of safety
k	Constant
k_x	Scaling factor for c
k_y	Scaling factor for ϕ
m	Richter magnitude
m_o	Lower bound of the earthquake magnitude
m_1	Upper bound of the earthquake magnitude
p_f	Probability of failure
$P[\ ]$	Probability of the event in brackets

r	Radius of the log-spiral failure surface
r_o	Initial radius of the log-spiral failure surface
R	Distance between the source and the site for the point source model
t	Tangent of ϕ
$\tilde{V}$	Normalized variance of a variable x
Var(x)	Variance of a variable x
$\bar{x}$	Mean value of a variable x
$\tilde{x}$	Normalized mean value of a variable x
y'	Transformed variable of the bivariate beta distribution of x and y, independent of x

Greek Characters

α, β	Parameters of the beta (and bivariate beta) distribution
β_1	Coefficient of skewness
β_2	Coefficient of kurtosis
ϵ	Error term in the probabilistic attenuation relation
γ	Parameter of the bivariate beta distribution
γ	Unit weight of soil
λ	Frequency of occurrence of seismic events
$\lambda_{c\phi}$	Dimensionless slope stability parameter
μ_i	i^{th} central moment of a variable
μ_i'	i^{th} moment of a variable about the origin
θ	Orientation of the fault and site for the line source

ϕ	Angle of internal friction of soil strength
ϕ_n	Normalized angle of internal friction of soil strength
ρ	Correlation coefficient
σ	Shear stress along the failure surface
σ_c	Standard deviation of c
σ_ϵ	Standard deviation of ϵ
σ_ϕ	Standard deviation of ϕ
σ_{xy}	Covariance of x and y
τ	Shear strength of soil

ABSTRACT

A new model has been proposed in this study for the determination of the reliability of earth slopes during earthquakes. The developed method provides a probabilistic, pseudo-static, seismic stability analysis of soil slopes. The measure of safety used is the probability of failure (rather than the customary safety factor), the numerical values of which are determined through a Monte Carlo simulation of failure. Significant uncertainties have been identified and probabilistic tools have been introduced for their description and amelioration. Moreover, a comprehensive computer program has been written and applied in a number of case studies.

In the present fifth and final report, a review of the objectives and achievements of this project is first presented. The models employed to account for the variability of soil strength are then described and illustrated in an example. A parametric study is conducted on the relevant strength and seismic parameters and the results are presented in a number of graphs and tables. Finally, an evaluation of the developed model is included together with a discussion of its applicability and limitations.

1. INTRODUCTION

Because of possible excessive life and economic losses incurred by failure, soil slopes (naturally formed or man-built) are among the most important civil engineering structures in earthquake prone areas. This is clearly demonstrated by the considerable amount of literature on the subject, particularly on earthdam slopes (e.g., Newmark, 1965; Seed et al., 1966; Whitman, 1970; Seed et al., 1971, etc.).

Although much experience has already accumulated about the design and performance of soil slopes, geotechnical engineers still face considerable uncertainties when they analyze their stability. These uncertainties reflect the variability of the material parameters, the slope's loading conditions, the location and shape of the failure surface, the particular method used in the analysis, etc. The possibility of an earthquake further complicates the task. Thus, efforts directed towards a more reliable approach for the assessment of the safety of slopes must, of necessity, take into account these uncertainties.

In this study, the safety of slopes is measured in terms of the probability of failure (p_f) rather than the customary factor of safety (F_s). As is the case with other structures, it is assumed that failure occurs when the calculated available strength R of a slope is exceeded by the applied load S; that is,

$$\text{"Failure"} = [R < S] \tag{1-1}$$

The probability of failure is then defined as

$$p_f = P\ [R < S] \tag{1-2}$$

i.e., $P\ [R < S]$ signifies the probability that the applied loading exceeds the available strength.

The resistance R of the soil mass comprising a slope is assumed to be constant during the earthquake loading. This is a reasonable assumption for a wide variety of soils, particularly cohesive ones (Arango and Seed, 1974). The approach taken is not directly applicable to the analysis of the stability of slopes for which R decreases during cyclic loading (e.g. the case of liquefaction of loose saturated sands or sensitive clays).

2. REVIEW OF PREVIOUSLY REPORTED RESEARCH EFFORT

2.1 Objectives of the Present Project

The main objectives of the present research project are as follows:

(a) To describe the additional forces that are imposed upon an earth slope during an earthquake. As these forces are generally random in nature, derivation of the probability density function of the ground acceleration is required. Thus, each acceleration level is exceeded with a certain probability which depends on the earthquake magnitude, the distance between site of slope and earthquake source and a number of empirical regional parameters.

(b) To develop a model for slope failure which accounts for the randomness associated with the location and shape of the failure surface and the variability of soil strength parameters.

(c) To assess the probability of failure (or, its complement, the reliability) of an earth slope with given boundary and material conditions.

(d) To determine the influence of significant seismic and material parameters on the numerical value of the probability of failure of slopes.

(e) To apply the developed analysis to case studies reflecting the seismic characteristics of the State of New York.

(f) To examine the applicability and limitations of the developed method of analysis and to provide guidance for future similar approaches.

2.2 Summary of Completed Tasks

2.2.1 Task 1: Determination of the Probability Density Function of the Maximum Ground Acceleration.

A general expression was developed for the probability density function of the maximum ground acceleration a_{max} at a given site (it appears in Report No. CE-78-5). This was achieved through an application of the method of transformation of variables and use of (a) an appropriate attenuation relationship (i.e., an expression relating a_{max} to the magnitude of the earthquake, the distance between earthquake source and site of slope and a number of empirical regional parameters), and (b) the probability density function of the earthquake magnitude.

Two magnitude-frequency relations (log-linear and log-quadratic) and three types of possible earthquake sources (point, line and area sources) were investigated.

As part of Task 1, an engineering analysis was performed on the seismic history of New York State. Available earthquake data were compiled, evaluated and analyzed. The resulting data list contained 1289 seismic events that took place in the period between 1568 and 1975.

The available data indicated that the seismic activity of New York has been concentrated mainly in three areas: (a) Northern New York (Adirondack and St. Lawrence regions), (b) Western New York (Buffalo-Attica region), and (c) Southern New York and New York

Metropolitan area. Two well identified fault systems exist in the State: the Clerendon-Linden Fault and the Ramapo Fault. These two fault systems are partially associated with the seismic activities in the Western and Southern parts of the State, respectively.

The functional relationships between frequency (λ) and magnitude (m) of earthquakes was investigated by analysing all data available and for a range of magnitude m (in Richter scale) between 2.0 and 7.0 ($2.0 \leqslant m \leqslant 7.0$). This range of magnitude involved a total of 1242 seismic events. On the basis of the results of this task, it was concluded that a log-quadratic frequency-magnitude relationship best represented the available data. This is shown in Fig. 2-1 (from Report No. CE-78-7).

The seismic hazard that corresponds to the log-quadratic frequency-magnitude relationship was determined for a number of time periods and under the assumption that earthquakes occur in accordance with the Poisson model. The case where earthquakes follow a more general Markov process was also investigated. The findings from the two models were compared and the results of this comparison were presented and discussed.

It was found that for smaller return periods, the Markov model gave smaller values for the probability of occurrence of earthquakes of various magnitudes than did the Poisson process. This result is considered to be consistent with the elastic rebound theory.*

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* According to the elastic rebound theory, for an earthquake to occur, elastic strain energy must first be stored in the rock masses along a fault. When an earthquake occurs, the stored energy is released and consequently, the probability of another earthquake taking place before sufficient energy is stored again decreases.

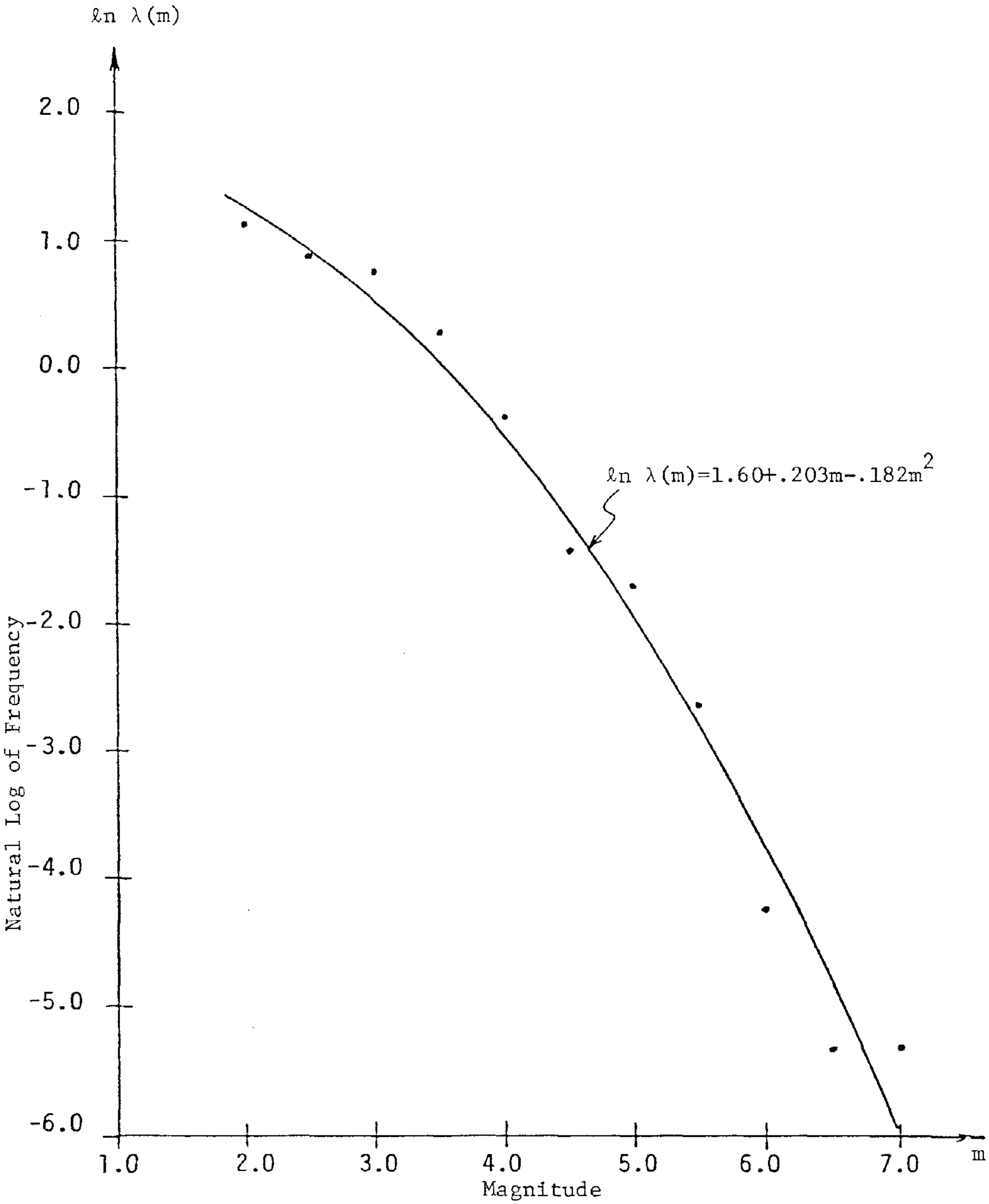


FIGURE 2-1. BEST QUADRATIC RELATIONSHIP BETWEEN $\ell n\ \lambda(m)$ and m

It should be noted that the term "seismic risk" or "seismic hazard", as employed in this study, referred to the probability with which certain values of the earthquake magnitude were exceeded during a given time period. In this sense, they represented an analysis of the data and did not provide seismic loads (usually expressed in terms of peak ground motion parameter) for New York State.

2.2.2 Task 2: Description of Failure Surfaces.

The surfaces along which earth slopes fail have been taken have of an exponential shape (logarithmic spiral), following studies by Rendulic (1935), Frohlich (1953), A-Grivas (1976) and Baker et al. (1977). The analytical expression for a failure surface was given as follows:

$$r = r_o \exp(-\theta t)$$

in which,

r_o = the initial radius of the log spiral (value of r for θ=0),

θ = the angle between r and r_o, and

t = tanϕ, where ϕ = soil's angle of internal friction.

This is shown schematically in Fig. 2-2 (from Report No. CE-78-5).

The location in the interior of the slope mass of a potential failure surface, as given by the above expression, was found to depend on the following three factors:

(a) the position along the slope boundary of the initial radius (Fig. 2-2, Point A),

(b) the location of the center of the log spiral (Point O),

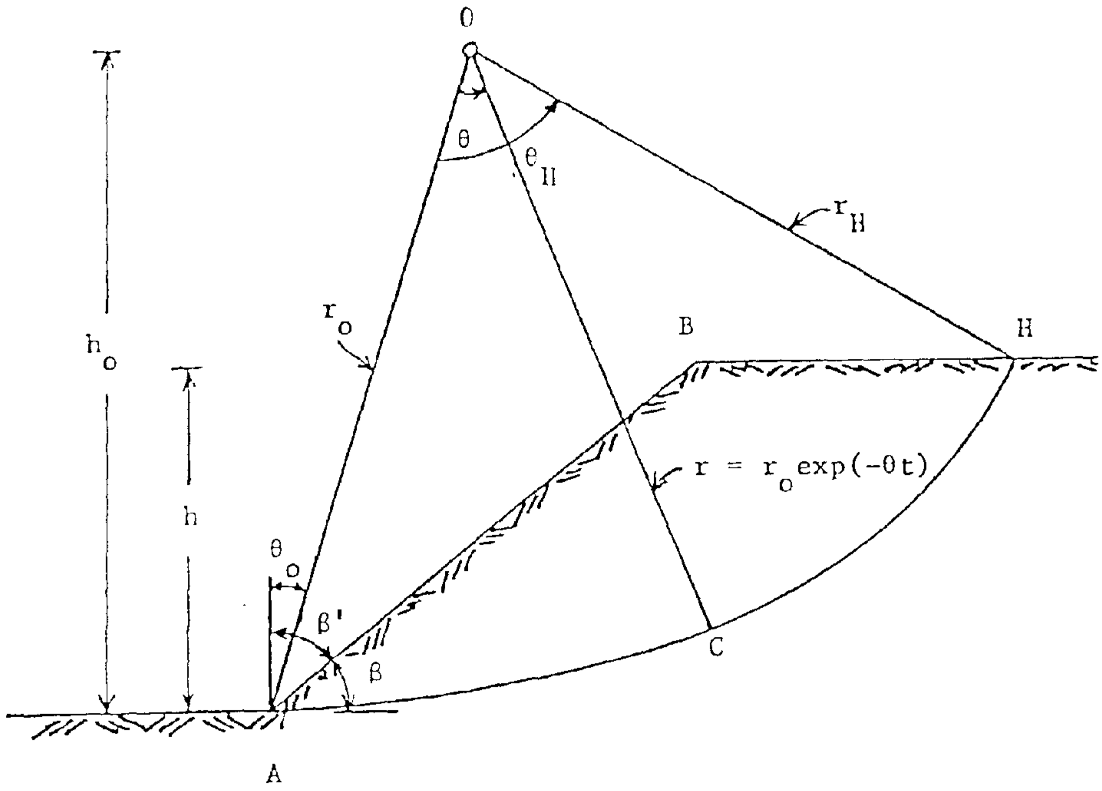


FIGURE 2-2. SHAPE OF FAILURE SURFACE

and

(c) the numerical value of the ϕ-parameter of soil strength.

The expected (mean) location of the failure surface was determined using the Monte Carlo simulation technique. An excellent approximation for the mean failure surface was obtained by employing the point estimates method for probability moments, proposed for the first time by Rosenblueth (1975). In Fig. 2-3 is shown an example of the comparison of the mean failure surface obtained by the two methods.

The details of the manner with which the above failure surfaces were generated and used in the analysis were given in Report No. CE-78-5. It should be noted that the commonly used circular failure surfaces represent a special case of the logarithmic spiral and can be easily employed in the present analysis.

2.2.3 Task 3: Assessment of the Reliability of Slopes.

A probabilistic model was developed for the analysis of the stability of earth slopes under earthquake loading. A detailed description of the model was the content of Report No. CE-78-5. Significant uncertainties associated with conventional pseudo-static methods of seismic stability analysis were recognized and probabilistic tools were introduced for their description and amelioration. In particular, the developed method of analyis accounted for (a) the variability of the material strength parameters, (b) the uncertainty around the exact location of potential failure surfaces, and (c) the uncertainty in the value of the maximum slope acceleration during an earthquake.

The soil material comprising the slope was assumed to be statistically homogeneous with strength parameters (c and $t=\tan\phi$) taken as beta distributed random variables. Using the results of Task 2,

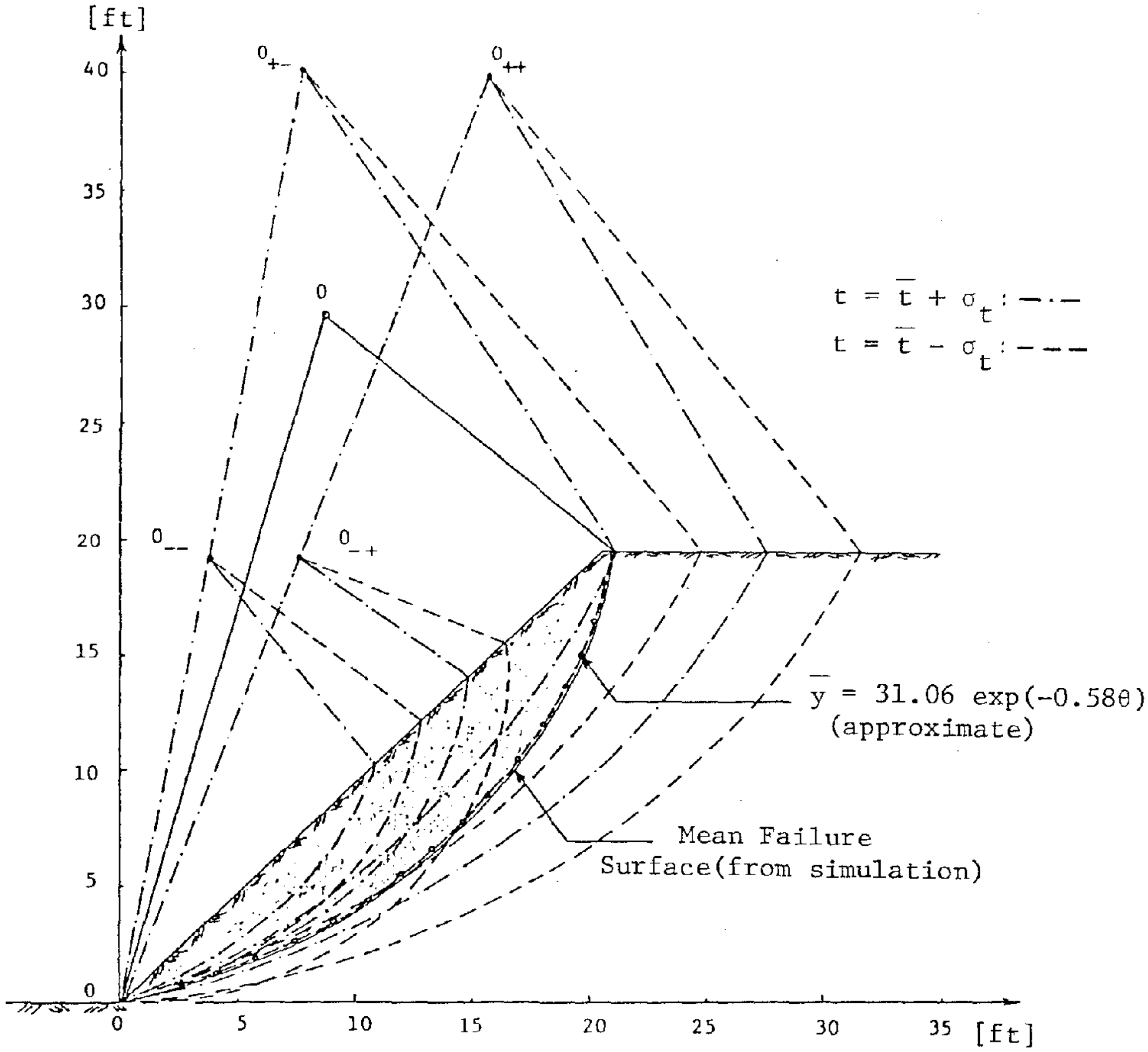


FIGURE 2-3. MEAN FAILURE SURFACE: AN EXAMPLE

potential failure surfaces were considered to have an exponential shape (log-spiral) defined with the aid of three random variables (two geometric factors and the frictional component of soil strength).

The safety of the slope was measured in terms of its probability of failure (p_f) rather than the customary factor of safety. The numerical values of p_f were obtained through a Monte Carlo simulation of failure.

The seismic load was introduced into the analysis using the results of Task 1. The maximum horizontal acceleration (a_{max}) experienced by the slope during an earthquake was assumed to be a random variable the probability distribution of which was found to depend on the earthquake magnitude, the type of earthquake source considered (i.e., point, line, or area source), the distance between the source and the site of the slope and a number of regional parameters. In addition, for the purposes of this study, it was assumed that the slope was rigid and, therefore, the maximum acceleration of the slope mass was taken to be equal to that of the ground.

As a part of Task 3, a computer program was developed to pursue the reliability analysis discussed above. The program was called "RASSUEL" (Reliability Analysis of Soil Slopes Under Earthquake Loading) and the details of its various operations and subroutines were given in Report No. CE-78-6. In Fig. 2-4 is shown the flow chart for program RASSUEL. As a special feature of the program, use was made of the computer graphics facilities available at R.P.I. in order to allow the monitoring of the failure surfaces generated during the Monte Carlo simulation of failure. The graphics option was introduced into the program in a manner that also permitted its use on non-graphics

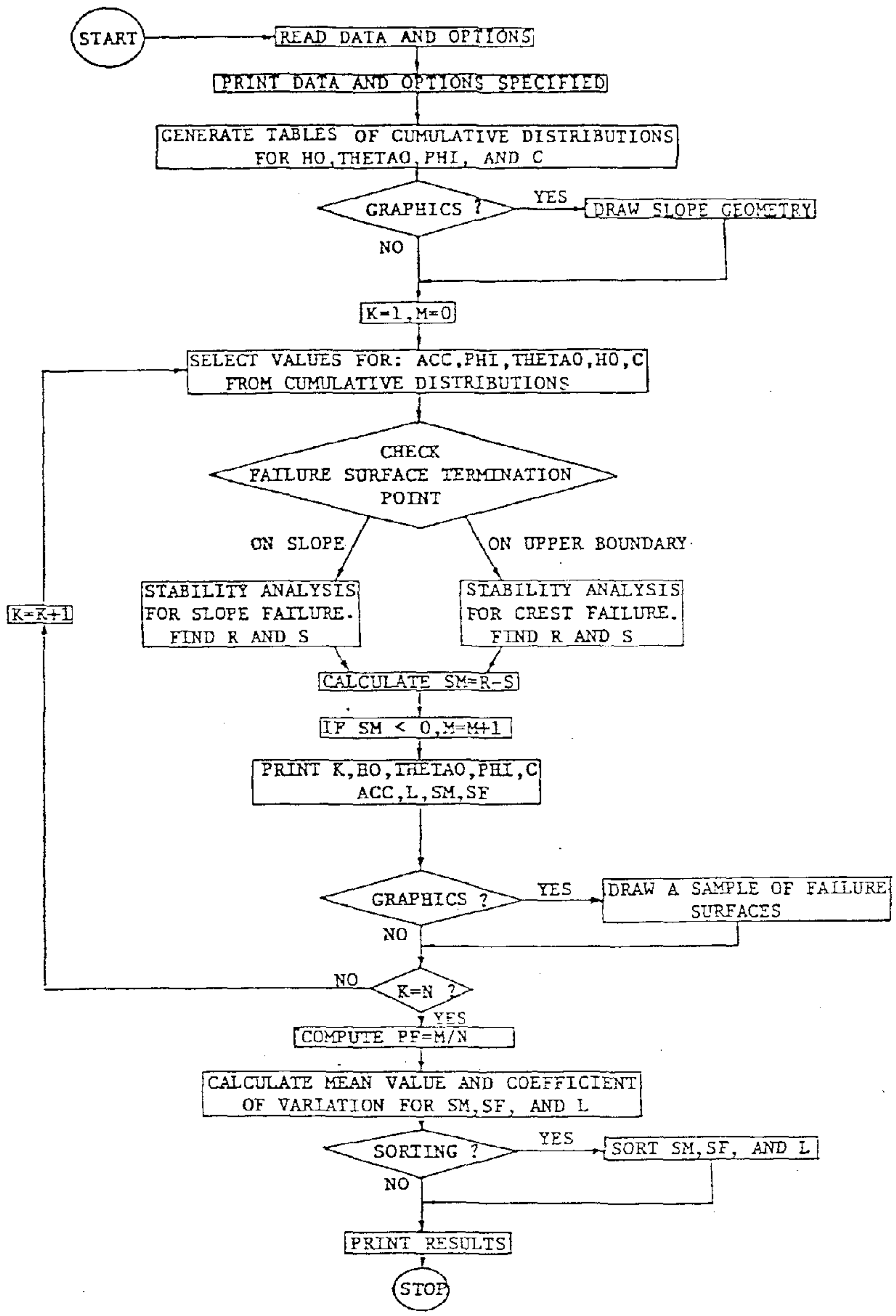


FIGURE 2-4. FLOW CHART FOR PROGRAM 'RASSUEL'

equipped hardware facilities.

2.2.4 Task 4: Case Study.

In a case study, an investigation was made of the reliability of a natural soil slope located near Slingerlands, New York. The details of this case study formed the content of Report No. CE-79-1. Both static and seismic loading conditions were examined. The model developed in the previous tasks was used to determine the probability of failure p_f of the slope for three types of earthquake sources, namely, a point, a line (or, fault), and an area source. The dependence of p_f on siginificant seismic parameters was examined and discussed.

On the basis of the results obtained in the case study, the following conclusions were drawn:

(a) The probability of failure was considered a viable alternative to the factor of safety as a measure of safety of soil slopes.

(b) The present probabilistic model was found useful in assessing the reliability of soil slopes under both static and seismic conditions.

(c) The values of the probability of failure attenuated to the value obtained under static conditions, as the distance between earthquake source and site of the slope increased.

(d) Higher values of the standard deviation σ_ϵ of the "error term" appearing in the attenuation relationship produced larger values for p_f.

(e) The values of the probability of failure of soil slopes were greatly affected by the type of earthquake source used and the values of the associated seismic parameters.

(f) Under the most unfavorable set of circumstances, from among those examined in the present study, the probability of failure of the slope had a value $p_f \simeq 0.35$ (Fig. 10, Report No. CE-78-7) which was more than twice that found under static conditions ($p_f \simeq 0.15$).

3. DESCRIPTION OF SOIL STRENGTH PARAMETERS

3.1 Variability of Soil Strength Parameters

The random variation of the strength parameters c ("cohesion") and ϕ ("angle of the internal friction") has been long recognized in geotechnical engineering (e.g., Schmertmann et al., 1960; Lumb, 1966; etc.) and a number of models have been proposed for their description. These may be classified into two types, namely (a) those which consider strength parameters c and ϕ to be independent of one another, and (b) those which consider c and ϕ to be correlated random variables.

In general, during a conventional limiting equilibrium analysis of slope stability, the strength of soil is represented by the Mohr-Coulomb strength criterion expressed in the form

$$\tau = c + \sigma \tan(\phi) \qquad (3\text{-}1)$$

in which,

τ is the shear strength of soil,

σ is the normal stress along the failure surface,

c is the "cohesion intercept" of soil, and

ϕ is the "angle of internal friction".

For a given soil deposit, the numerical values of c and ϕ may be determined by a variety of tests performed either in the laboratory (e.g., direct shear, triaxial, etc.) or in situ. The variability in the results of these tests depends on many random factors such as sampling disturbance, testing errors and, most importantly, the inherent variability of soil itself. The last factor represents the fundamental source of uncertainty in soil mechanics. It has been found (Lumb, 1966) that its effect on the variability of soil strength is so great that the uncertainties due to the other factors are overwhelmed.

3.2 Independent Strength Parameters

To account for the variabilty in the numerical values of c and ϕ, geotechnical engineers have indentified the two strength parameters as random variables and have proposed probabilistic models for their description. In a pioneering work, Lumb (1966) discovered that random variables c and ϕ followed a normal distribution. This conclusion was drawn from studies on a large amount of test data from soils found in the area of Hong Kong; namely, a soft marine clay, a residual silty sand, an alluvial sandy clay and a residual clayey silt. Additional studies of frequency distributions of soil properties (e.g., Schultze, 1972; Singh and Lee, 1970, etc.) came to support Lumb's conclusion that strength parameters are normal-like variates. In a later work, Lumb (1970) found that the c-parameter of strength followed more closely a beta (or, Pearson's type I) distribution and that only its central portion could be approximated as a normal variate. The use of the beta (rather than the normal) distribution for modelling soil strength parameters was also suggested by Harr (1977). Recognizing the versatility of the beta model, Harr recommended its use to obtain approximations for many data sets whose measures must be positive and of limited range.

When c and ϕ are treated independently, the expression for the beta distribution takes the following form:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \leq x \leq 1 \tag{3-2}$$

in which

α, β are the parameters of the beta distribution (α, $\beta > 0$),

$\Gamma(\ )$ is the gamma function, and

x is the normalized strength parameter receiving values between 0 and 1 ($0 \leqslant x \leqslant 1$); i.e.,

$$x = \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}} \qquad \text{or} \qquad x = \frac{c - c_{min}}{c_{max} - c_{min}}$$

where c_{min}, ϕ_{min} and c_{max}, ϕ_{max} are the minimum and maximum values of c and ϕ, respectively.

The cumulative density function $F_x(x)$ is defined as

$$F_x(x_o) = \int_o^{x_o} f_x(x)\, dx, \quad 0 \leq x_o \leq 1 \tag{3-3}$$

in which x is the strength parameter c or ϕ, and $f_x(x)$ is given in Eqn. (3-2).

Eqn. (3-3) provides the probability with which a strength parameter receives values smaller than, or equal to, a particular value x.

The mean value and second central moment (variance) of a univariate beta distribution may be expressed in terms of the parameters α and β as follows (Harr, 1977):

$$\bar{x} = \frac{\alpha}{\alpha+\beta} \quad (3\text{-}4a)$$

and

$$\mathrm{Var}(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad (3\text{-}4b)$$

In general, when x varies between two limits, say a and b (i.e., $a \leqslant x \leqslant b$), then $\bar{x}$ and Var(x) are equal to

$$\bar{x} = a + \frac{\alpha}{\alpha+\beta}(b-a) \quad (3\text{-}5a)$$

and

$$\mathrm{Var}(x) = (b-a)^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad (3\text{-}5b)$$

When the mean and variance of x are known (e.g., from strength data) , then parameters α and β of the beta distribution can be determined from Eqns. (3-5) as follows:

$$\alpha = \frac{\tilde{x}^2}{\tilde{V}^2}(1-\tilde{x}) - \tilde{x} \quad (3\text{-}6a)$$

$$\beta = \frac{\alpha}{\tilde{x}} - (\alpha + 2) \quad (3\text{-}6b)$$

in which, $\tilde{V} = \dfrac{\mathrm{Var}(x)}{(b-a)^2}$

and $\tilde{x} = \dfrac{\bar{x} - a}{b - a}$

3.3 Illustrative Example

In Table 3-1 are given the results of nine direct shear tests performed on samples of a cohesive soil (Singh and Lee, 1970). Assuming that strength parameters c and ϕ follow a beta distribution, it is asked to determine the values for the parameters α and β and the corresponding expressions for the probability density functions of c and ϕ.

The statistical values (mean, standard deviation and correlation coefficient) of c and ϕ are given in Table 3-2.

As soil strength parameters are by definition non-negative quantities, the minimum value for c and ϕ must always be positive. An estimate of the upper bound for the minimum value a, appearing in Eqns. (3-5) may be obtained by taking the variance of c and ϕ to be equal to that of a uniform distribution. The latter is equal to

$$Var_{uniform} = (b - a)^2/12$$

and, therefore, the upper bound for the minimum value of a is found from the above expression to be

$$a_u = b - \sqrt{12\ Var(x)} \qquad (3\text{-}7)$$

In the case of the data used for the illustrative example, the upper limit of a for the ϕ parameter is found from the Eqn. (3-7) to be 21° while the corresponding values of the parameters for the two beta distributions are

TABLE 3-1

RESULTS OF DIRECT SHEAR TESTS

	Strength Parameter	
No.	c (ksf)	ϕ (degrees)
1	1.042	46.1
2	0.850	37.0
3	0.345	39.4
4	0.000	41.0
5	1.300	25.0
6	0.900	43.0
7	0.720	36.0
8	0.800	27.5
9	0.700	30.0

TABLE 3-2

STATISTICAL VALUES OF c AND ϕ

Statistical Value	Strength Parameter c (ksf)	ϕ (degrees)
Minimum	0.00	25.0
Maximum	1.30	46.1
Mean	0.74	36.1
Standard Deviation	0.38	7.2
Coefficient of Correlations = −0.325		

ϕ-parameter:	α=1.15,	β=0.76
c-parameter:	α=1.06,	β=0.81

Thus, the expressions for the probability density functions of c, $f_c(c)$ and ϕ, $f_\phi(\phi)$, that correspond to the above parameters are

$$f_\phi(\phi) = \frac{\Gamma(1.91)}{\Gamma(1.15)\Gamma(.76)} \left(\frac{\phi-21}{25}\right)^{.15} \left(1-\frac{\phi-21}{25}\right)^{-.24}$$

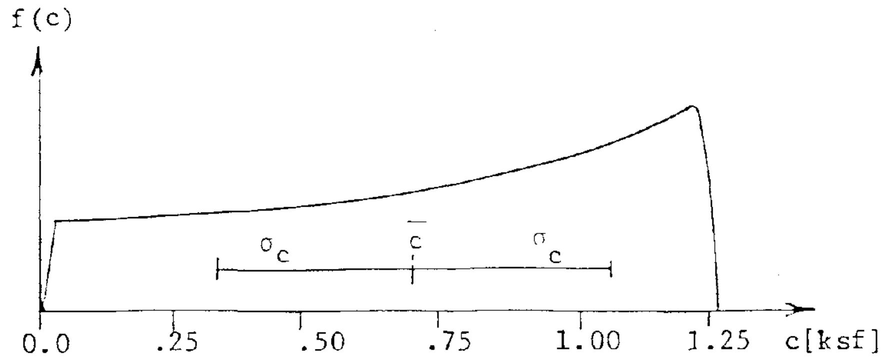
$$f_c(c) = \frac{\Gamma(1.87)}{\Gamma(1.06)\Gamma(.81)} \left(\frac{c}{1.3}\right)^{.06} \left(1-\frac{c}{1.3}\right)^{-.19} \tag{3-8}$$

In Figs. 3-1 are shown the density functions for c and ϕ while the corresponding cumulative density functions appear on Fig. 3-2. The effect on the reliability of a slope of the minimum value of the strength parameters is examined in detail below in Section 4.1.2.

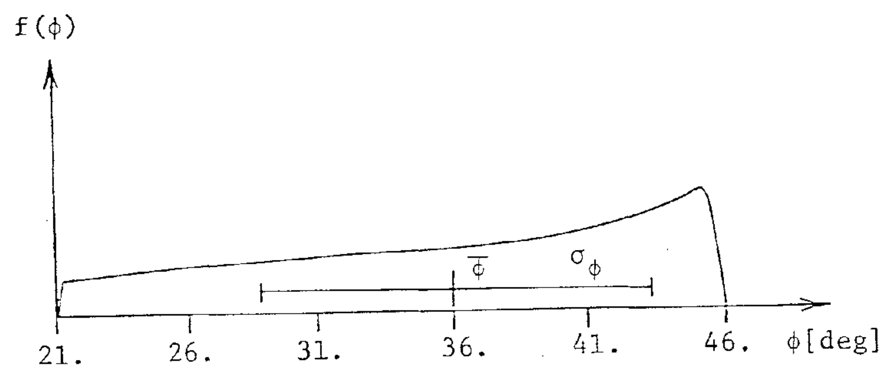
3.4 Correlated Strength Parameters

Statistical examinations of available strength data have revealed the existence of a negative correlation between c and ϕ (e.g., Singh and Lee, 1970; Lumb, 1970; Yuceman and Tang, 1975, etc.). It is therefore necessary to examine the joint variation of c and ϕ and to develop a probabilistic model for its description.

Both the normal and the beta distribution discussed above have multivariate extensions. The bivariate normal density function may be written as (Hogg and Craig, 1970)

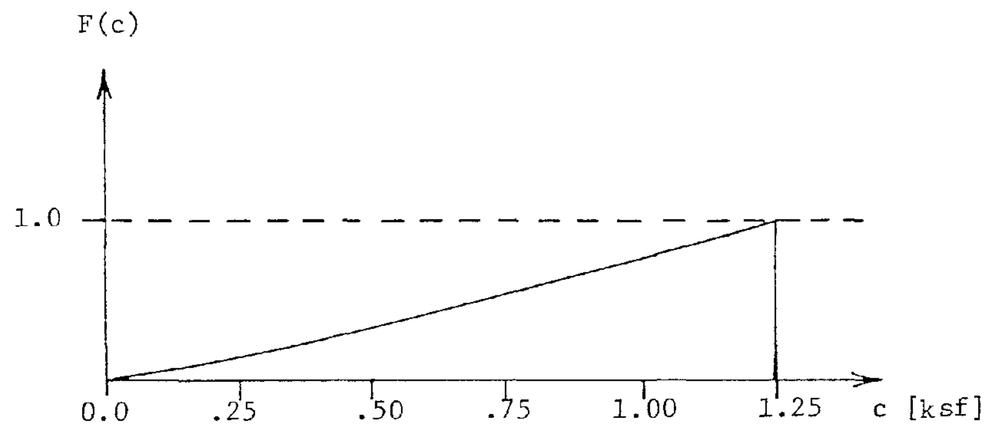


(a) Probability Density Function of c

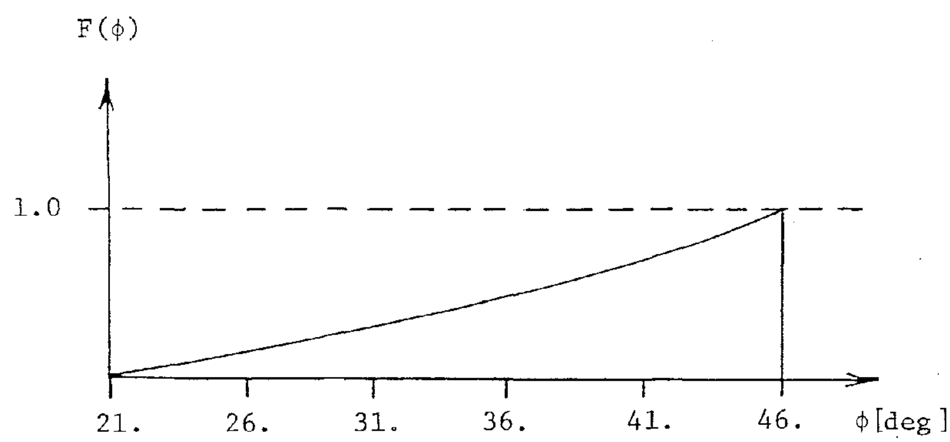


(b) Probability Density Function of ϕ (ϕ_{min} = 21°)

FIGURE 3-1. THE BETA DISTRIBUTION FOR THE STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE



(a) Cumulative Distribution of c



(b) Cumulative Distribution of ϕ

FIGURE 3-2. CUMULATIVE BETA DISTRIBUTIONS FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

$$f(c,\phi) = \frac{1}{2\pi\sigma_c\sigma_\phi\sqrt{1-\rho^2}} \exp(-q/2), \tag{3-9}$$

in which

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{c-\bar{c}}{\sigma_c}\right)^2 - 2\rho\left(\frac{c-\bar{c}}{\sigma_c}\right)\left(\frac{\phi-\bar{\phi}}{\sigma_\phi}\right) + \left(\frac{\phi-\bar{\phi}}{\sigma_\phi}\right)^2\right]$$

$\bar{c}$, σ_c are the mean and standard deviation of c,

$\bar{\phi}$, σ_ϕ are the mean and standard deviation of ϕ, and

ρ is the correlation coefficient of c and ϕ.

Associated with the joint distribution are the marginal and conditional distributions of c and ϕ. When the former follows the bi-variate normal model, the latter are also normal. Thus, for example, the marginal distribution of c is normal with parameters $\bar{c}$ and σ_c, i.e.,

$$f(c) = \frac{1}{\sqrt{2\pi}\,\sigma_c} \exp \frac{1}{2}\left(\frac{c-\bar{c}}{\sigma_c}\right)^2 \tag{3-10}$$

and the conditional distribution $f(\phi|c_o)$ (i.e., ϕ given that $c = c_o$) is also normal with parameters $\bar{\phi} + \rho(c_o-\bar{c})\sigma_\phi/\sigma_c$ and $\sigma_\phi\sqrt{(1-\rho^2)}$, i.e.,

$$f(\phi|c_o) = \frac{1}{\sqrt{2\pi}\,\sigma_\phi\sqrt{1-\rho^2}} \exp \frac{1}{2} \cdot \left[\frac{(\phi-\bar{\phi}_c)^2}{\sigma_\phi^2(1-\rho^2)}\right] \tag{3-11}$$

where

$$\bar{\phi}_c = \bar{\phi} + \rho(c_o-\bar{c})\ \sigma_\phi/\sigma_c$$

In Figs. 3-3, 3-4 and 3-5 are shown graphically the distributions that correspond to Eqns. (3-9), (3-10) and (3-11), respectively, for the case of the data used in the illustrative example of the previous section. Fig. 3-4 provides the normal distribution given in Eqn. (3-9) truncated at the specified limits. In Fig. 3-6 is shown the bivariate normal distribution that corresponds to the same data and for the case where ρ is equal to -0.65. For comparison, the case of uncorrelated c and ϕ (i.e., $\rho = 0$) is shown in Fig. 3-7. From Figs. 3-3, 3-6 and 3-7, it is seen that the concentration of the probability density function along the line connecting c_{max} and ϕ_{max} increases as the value of the correlation coefficient increases. From the same figures, it is also seen that, for the case of the bivariate normal model, a significant portion of the probability density function lies outside the experimentally obtained upper and lower limits of the two variables (in fact, outside the physical limits of the two strength parameters, i.e., $c < 0$ and $\phi < 0$).

The multivariate beta (or, Dirichlet) distribution provides an alternative to the bivariate normal model. This model, used for the first time to describe geotechnical data by A-Grivas and Harrop-Williams (1979), has the following analytical expression:

$$f(x,y) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1} \qquad (3\text{-}12)$$

$$x,y > 0 \text{ and } 0 \le x+y \le 1$$

in which x and y are normalized forms of the two variables and α, β, γ are the three parameters of the bivariate beta distribution (α, β, $\gamma > 0$).

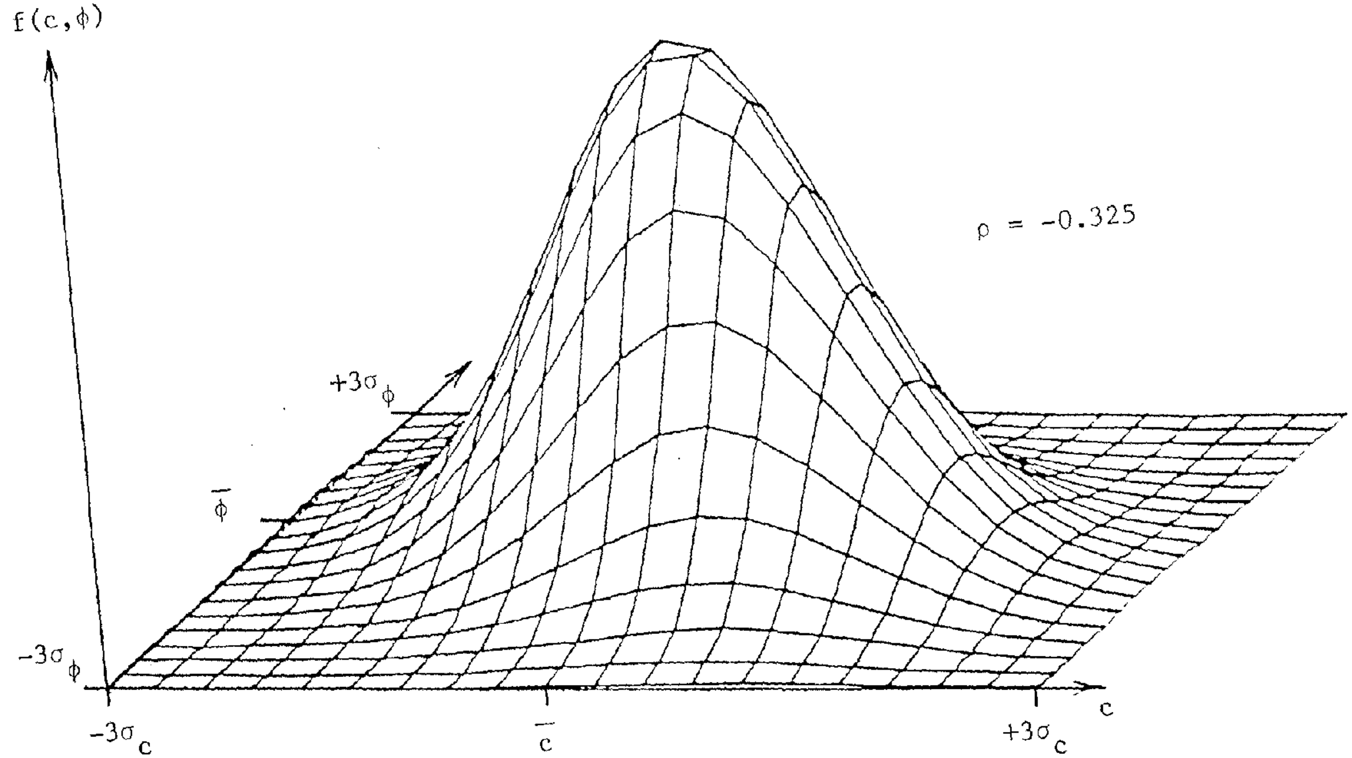


FIGURE 3-3. BIVARIATE NORMAL DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

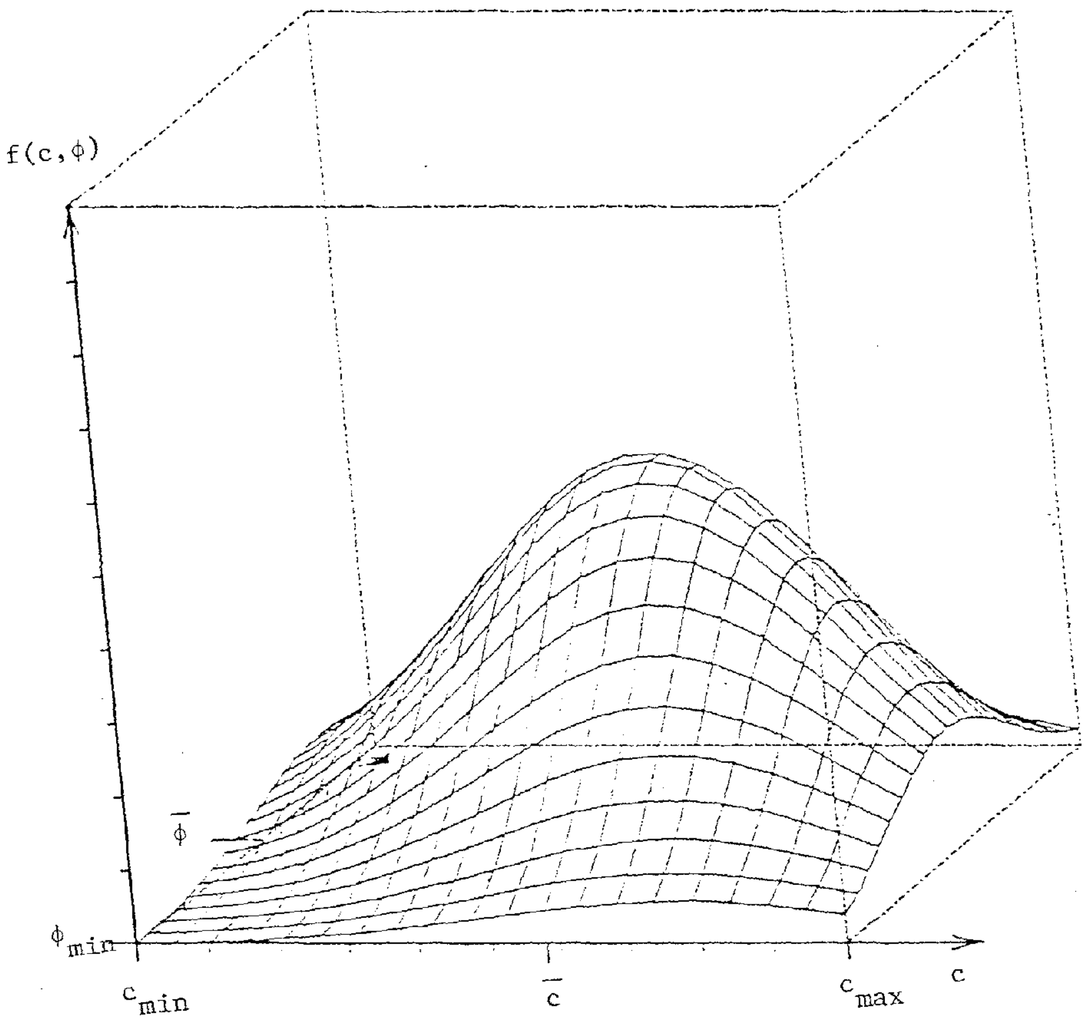
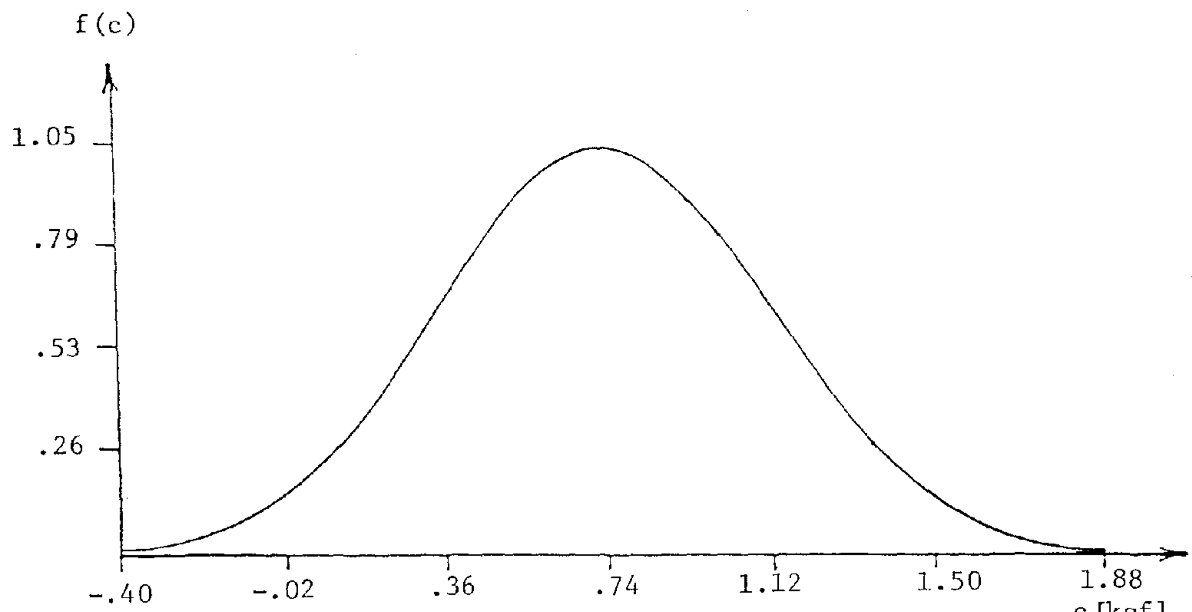
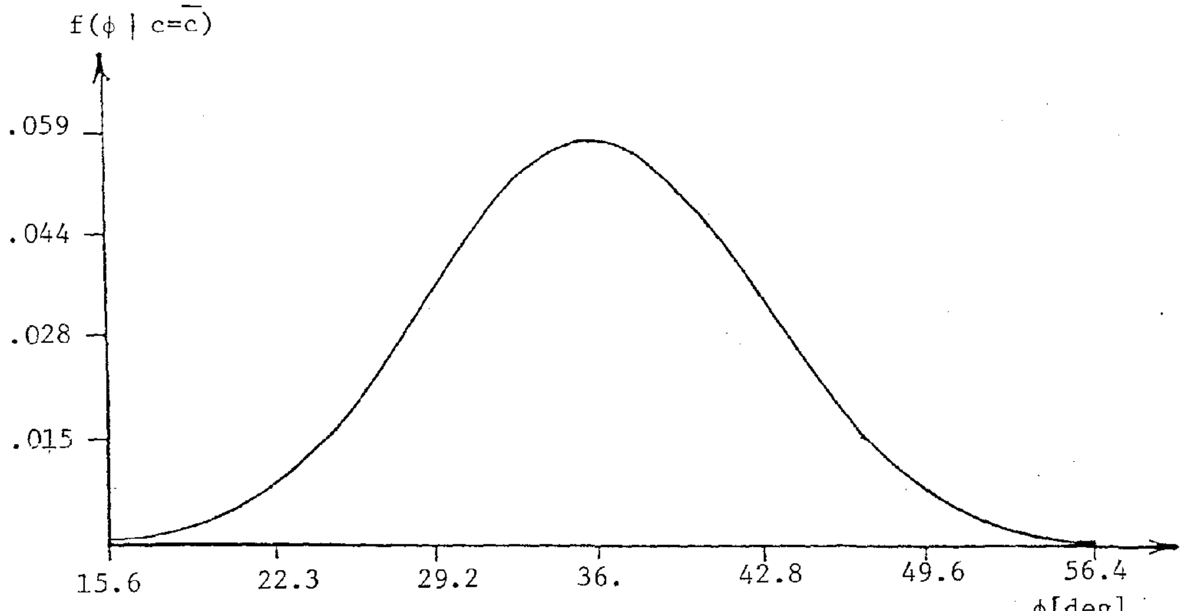


FIGURE 3-4. TRUNCATED BIVARIATE NORMAL DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE ($\rho = -0.325$)



(a) Marginal Distribution of c



(b) Conditional Distribution of ϕ
(for $c_o = \bar{c} = .74$ ksf)

FIGURE 3-5. MARGINAL AND CONDITIONAL DISTRIBUTIONS OF c AND ϕ

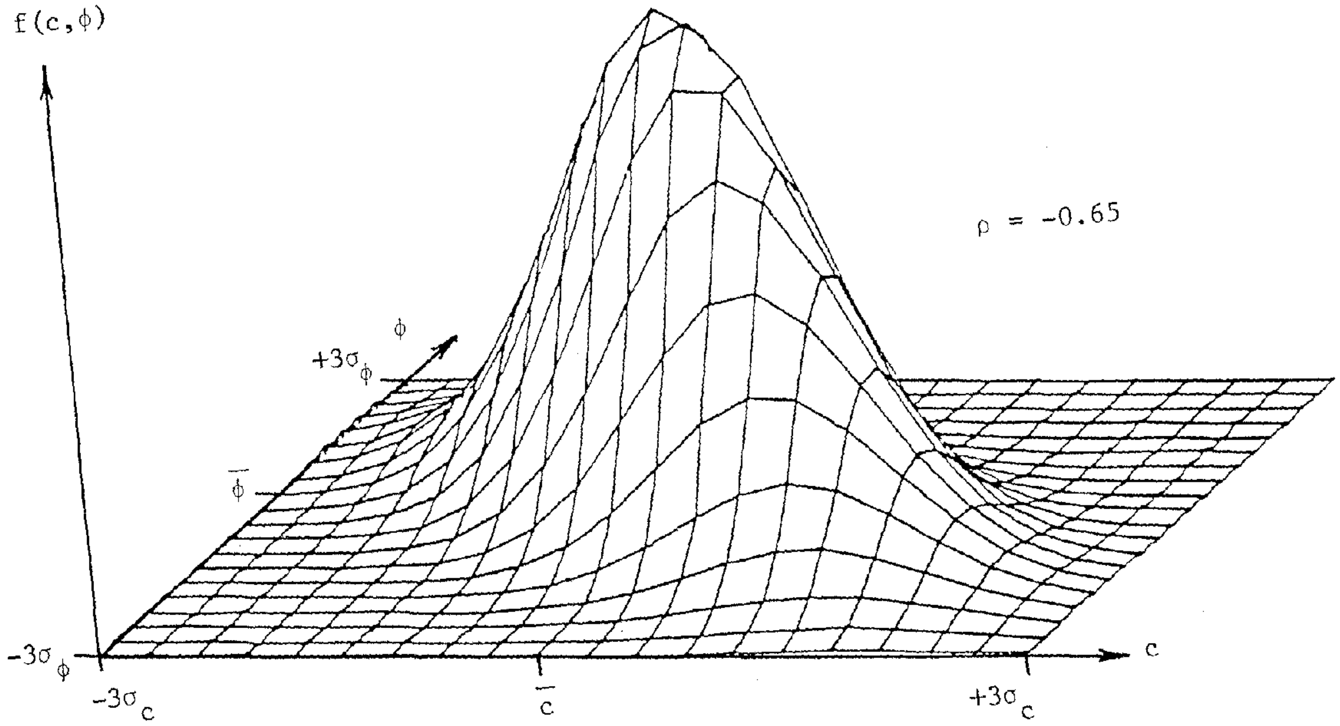


FIGURE 3-6. BIVARIATE NORMAL DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

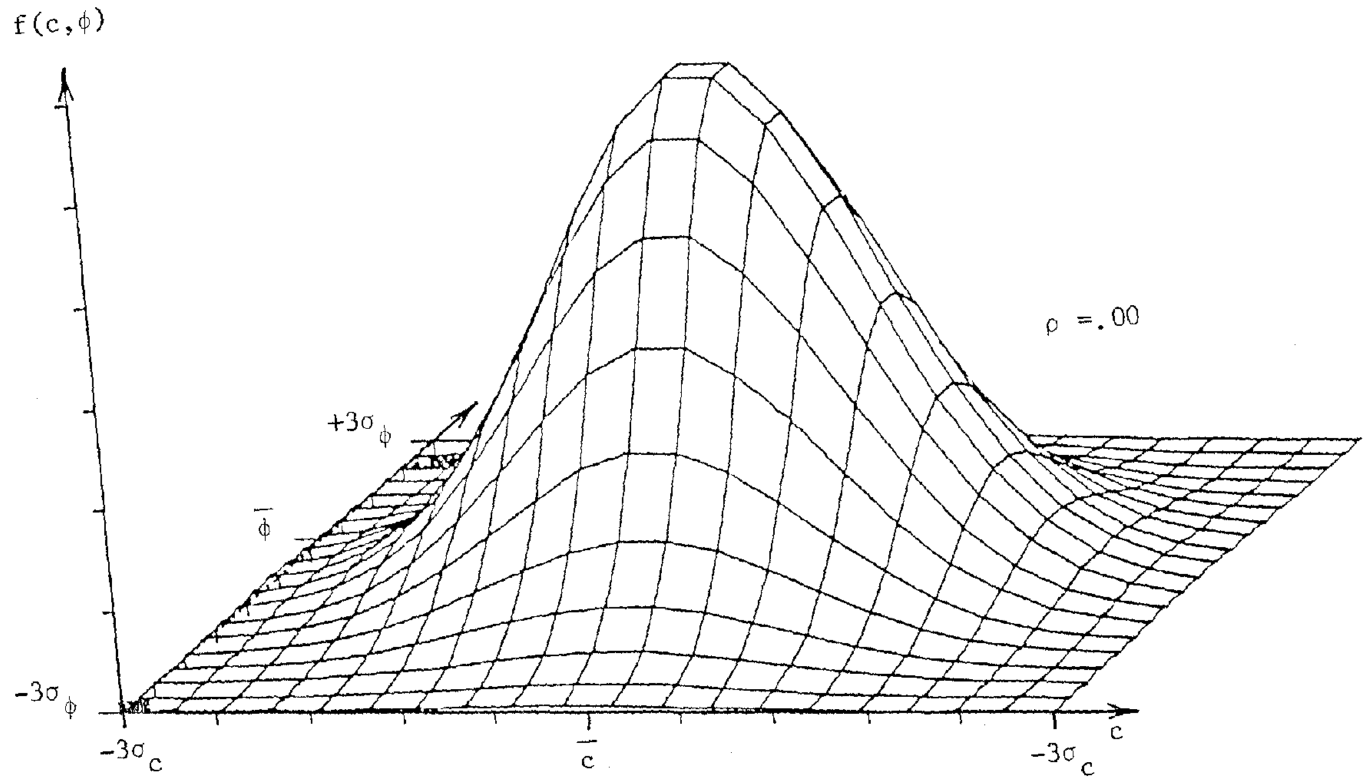


FIGURE 3-7. BIVARIATE NORMAL DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

The mean values ($\bar{x}$, $\bar{y}$), variances (σ_x^2, σ_y^2) and covariance (σ_{xy}) of this distribution can be expressed in terms of the parameters α, β, and γ as follows (Wilks, 1962):

$$\bar{x} = \frac{\alpha}{\Sigma}\ , \quad \sigma_x^2 = \frac{\alpha(\beta+\gamma)}{\Sigma^2(\Sigma+1)}$$

$$\bar{y} = \frac{\beta}{\Sigma}\ , \quad \sigma_y^2 = \frac{\beta(\alpha+\gamma)}{\Sigma^2(\Sigma+1)} \qquad (3\text{-}13)$$

$$\sigma_{xy} = \frac{-\alpha\beta}{\Sigma^2(\Sigma+1)}$$

in which $\Sigma = \alpha + \beta + \gamma$.

The correlation coefficient ρ of x and y is then equal to

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = -\sqrt{\frac{\alpha\beta}{(\alpha+\gamma)(\beta+\gamma)}} \qquad (3\text{-}14)$$

As α, β and γ are all positive by definition, from Eqn. (3-14) one has that the correlation coefficient of the bivariate beta distribution must always be negative.

3.5 Approximation of Strength Data Using the Bivariate Beta Distribution

A procedure is presented here to approximate data on the c and ϕ parameters of strength using the bivariate beta model. In doing so, the task is to determine the three parameters α, β and γ of the model from the five statistical moments (two first order and three second order moments) and four limiting values (two minima and two maxima) of the strength data. This may be achieved in a number of ways, a detailed description of which is presented in Appendix A. The approach followed in this study involves a transformation of c and ϕ using appropriate scaling factors.

Let x and y be the normalized scaled expressions for the two strength parameters c and ϕ, respectively; i.e.,

$$x = \frac{c - c_{min}}{k_x(c_{max} - c_{min})}$$

$$y = \frac{\phi - \phi_{min}}{k_y(\phi_{max} - \phi_{min})} \qquad (3\text{-}15)$$

where k_x and k_y are the scaling factors needed for the transformation of the normalized variables c and ϕ (Appendix A).

Parameters α, β, γ, k_x, and k_y may be calculated directly by transforming the variables (Eqn. (3-15)) and matching the second order moments (i.e., σ_c, σ_ϕ and ρ) of the distribution with the corresponding values obtained from an analysis of the data.

$$\beta = [c_4 \cdot \mathrm{DENX} \cdot \mathrm{DENY}/(\rho \cdot s_x \cdot s_y \cdot (c_3 + c_4 + 1)^2) - 1]/(c_3 + c_4 + 1)$$

$$\gamma = c_3 \cdot \beta$$

$$\alpha = c_4 \cdot \beta \qquad (3\text{-}16)$$

$$k_x = \mathrm{DENX}/(c_{max} - c_{min})$$

$$k_y = \mathrm{DENY}/(\phi_{max} - \phi_{min})$$

where

$$c_1 = (\bar{c}-c_{min})/(\phi-\phi_{min})$$

$$c_2 = \left(\frac{1-c_1\cdot s_y}{\rho\cdot s_x}\right)\left(\frac{\rho\cdot s_y}{c_1-s_x}\right)$$

$$c_3 = -(s_x\cdot c_2)/(s_y\cdot\rho)-1$$

$$c_4 = -c_3/[s_y/(c_2\cdot\rho\cdot s_x)+1]$$

$$DENY=(\bar{\phi}-\phi_{min})\cdot(c_3+c_4+1)$$

$$DENX=DENY/c_2$$

$$s_x = \sigma_c/(c_{max}-c_{min})$$

$$s_y = \sigma_\phi/(\phi_{max}-\phi_{min})$$

Such a formulation allows all five moments, obtained through a statistical analysis of the data, to be preserved along with the lower limits of c and ϕ. The upper bounds of c and ϕ are not maintained as they are found to be functions of the scaling factors k_x and k_y (Appendix A). This shortcoming, however, introduces no significant error in the model.

Both the marginal and the conditional distributions of the bivariate beta are univariate beta types. The marginal distribution for x is beta distributed with parameters (α, $\beta+\gamma$), i.e.,

$$f(x) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\cdot\Gamma(\beta+\gamma)}\, x^{\alpha-1}\cdot(1-x)^{\beta+\gamma+1} \qquad (3\text{-}17)$$

while the conditional distribution of y may be expressed in terms of a transformed variable y' = y/(1-x) which is independent of x. The distribution of y' is as follows:

$$f(y') = f\left(\frac{y}{1-x}\right) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\left(\frac{y}{1-x}\right)^{\beta-1}\left(1-\frac{y}{1-x}\right)^{\gamma-1} \qquad (3\text{-}18)$$

For the data of the illustrative example, presented in Section 3.3, the three parameters α, β and γ and the two scaling factors k_x and k_y are found to be equal to

$$\alpha = 2.71, \ \beta = 2.98, \ \gamma = 5.90$$

$$k_x = 2.43, \ k_y = 2.33$$

In Fig. 3-8 is shown the bivariate beta distribution that corresponds to the data of the illustrative example, while Fig. 3-9 presents the same distribution truncated at the minimum and maximum values of the variables. In Figs. 3-10 and 3-11 are shown the marginal distribution for c and conditional distribution for ϕ, respectively. The bivariate beta distribution for the same data but for a value of the correlation coefficient $\rho = -.65$ appears in Fig. 3-12.

3.6 Comparison of the Various Models

The four distributions presented above (i.e., univariate normal and beta and bivariate normal and beta) provide reasonable models for the two soil strength parameters c and ϕ. The normal and bivariate normal distributions are more convenient to employ, as they require only two (normal) or five (bivariate normal) statistical parameters. The latter are determined from a statistical analysis of available data.

A shortcoming associated with the normal models, which in some cases may have a significant effect, is the extent of the tails of the distribution. The lower and upper limits of the normal (and

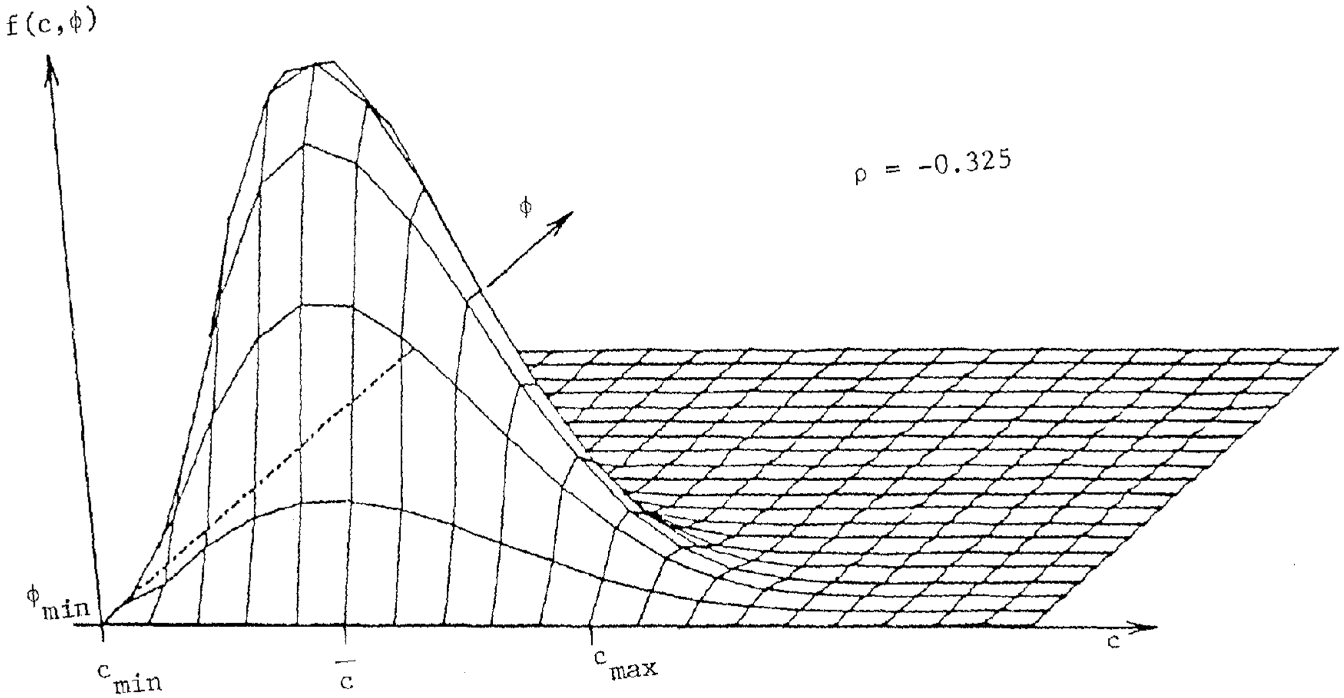


FIGURE 3-8. BIVARIATE BETA DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

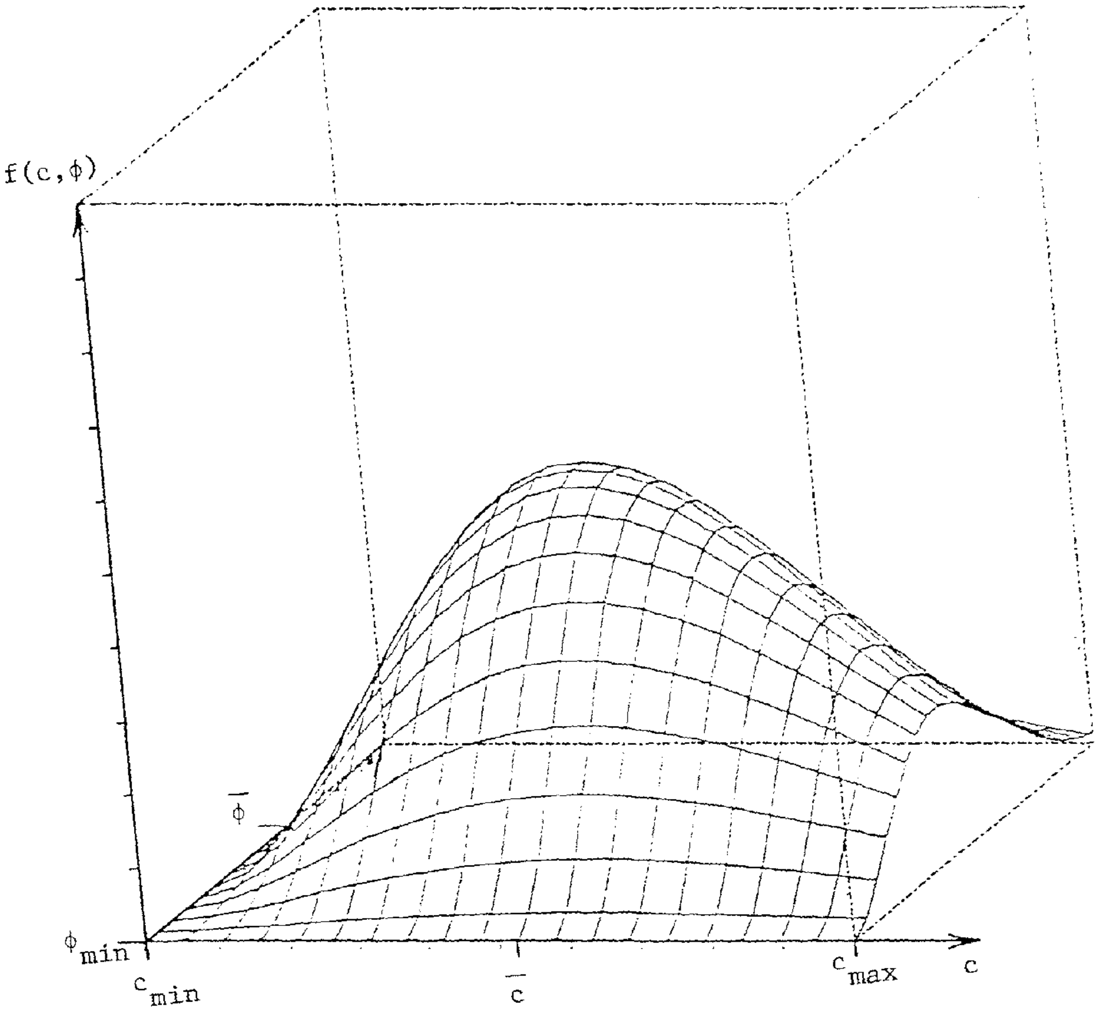
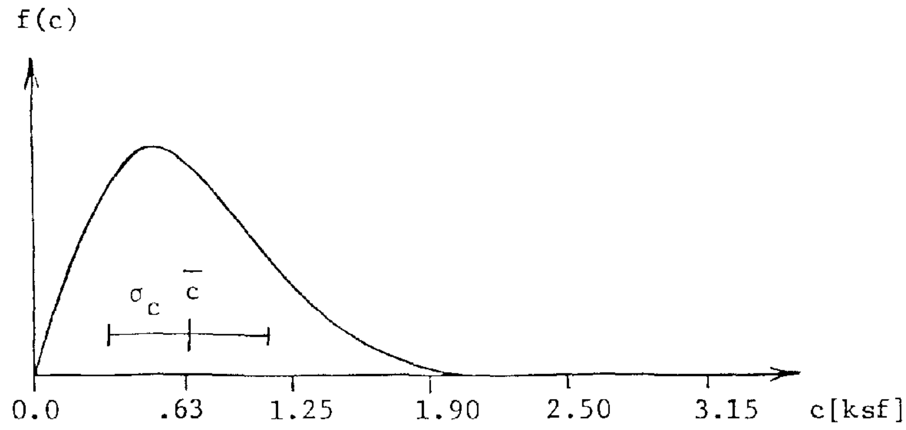
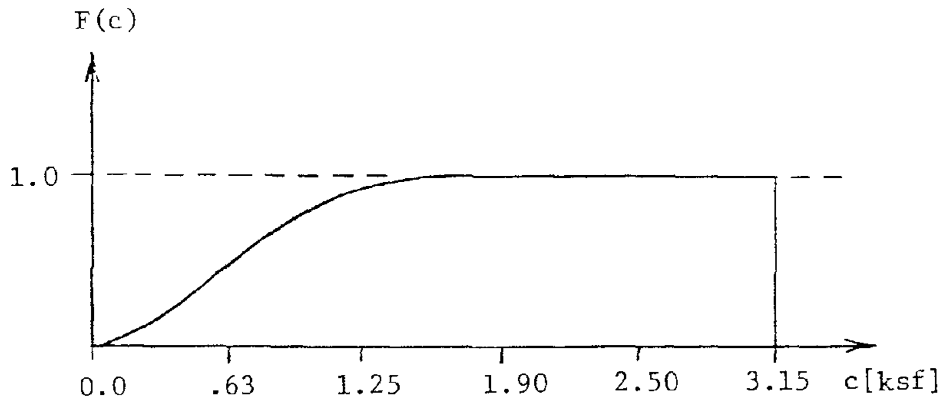


FIGURE 3-9. TRUNCATED BIVARIATE BETA DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE ($\rho = -0.325$)

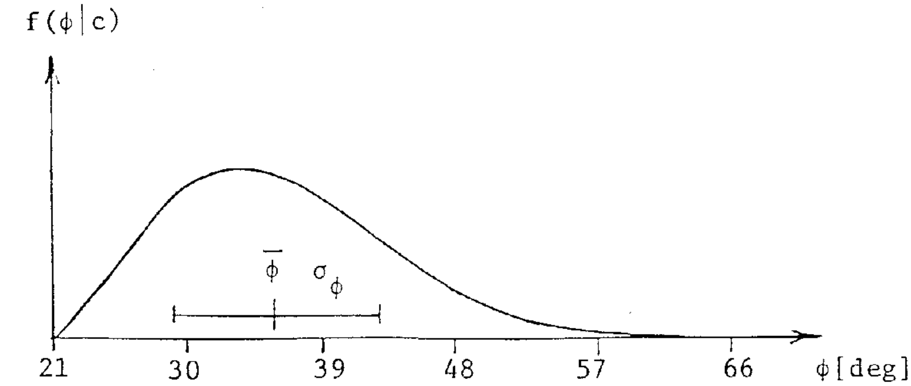


(a) Marginal Density Function of c

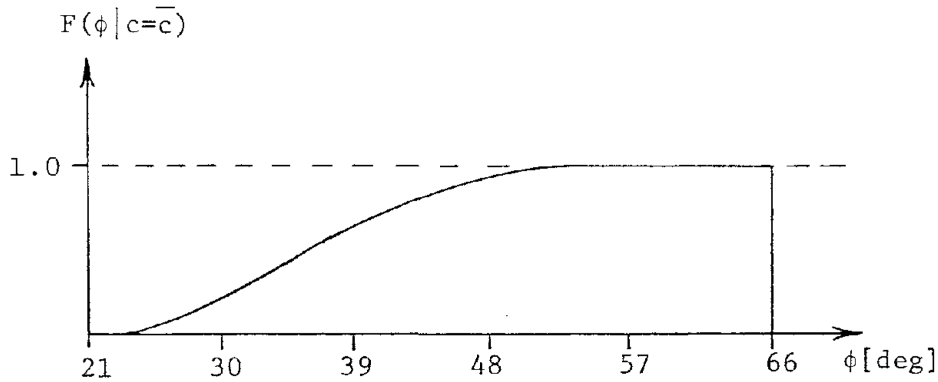


(b) Cumulative Marginal Distribution of c

FIGURE 3-10. MARGINAL BETA DISTRIBUTION FOR c-PARAMETER



(a) Conditional Density Function of ϕ



(b) Cumulative Conditional Distribution of ϕ

FIGURE 3-11. CONDITIONAL BETA DISTRIBUTION FOR ϕ-PARAMETER

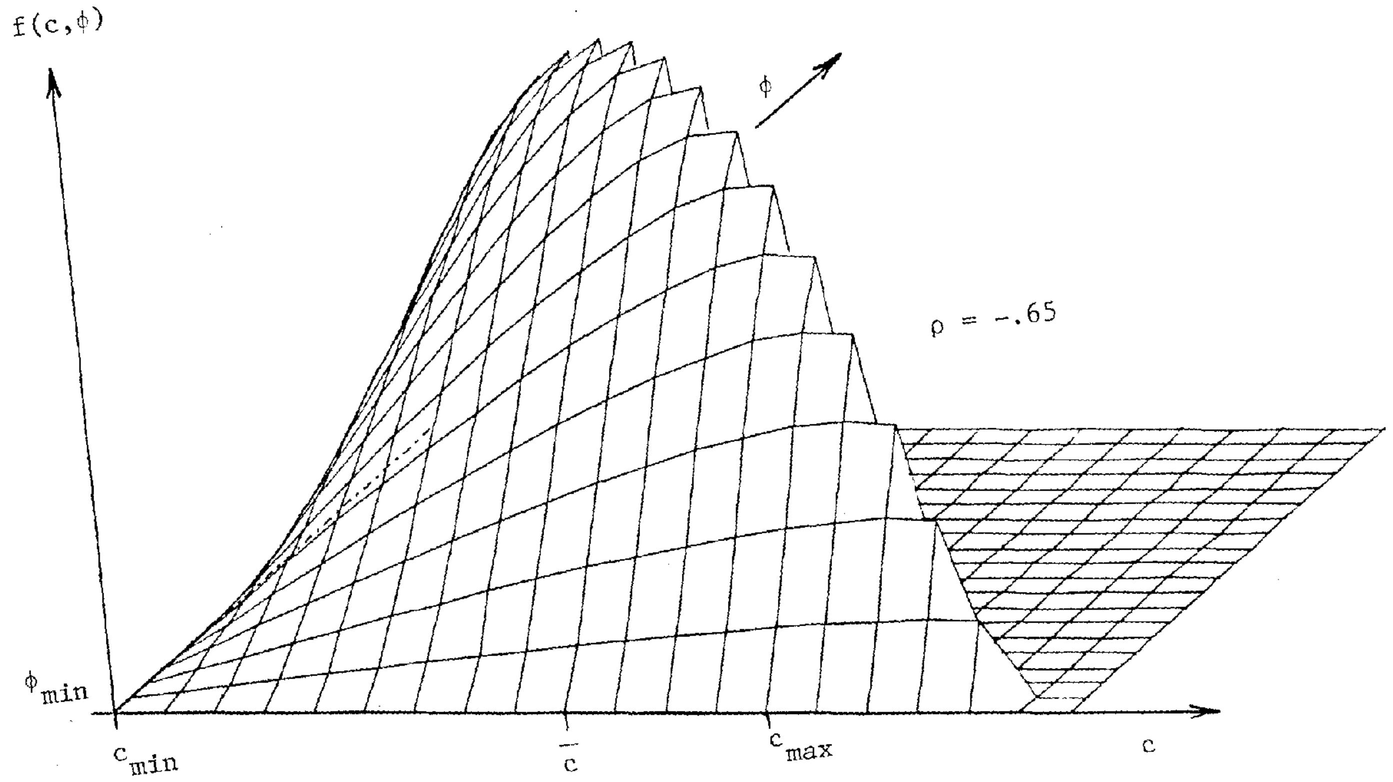


FIGURE 3-12. BIVARIATE BETA DISTRIBUTION FOR STRENGTH PARAMETERS OF ILLUSTRATIVE EXAMPLE

bivariate normal) are $-\infty$ and ∞, respectively, while the actual range of variation of the two strength parameters is significantly smaller and always positive. Moreover, in the case of the illustrative example (Section 3.3), the available data indicate a much flatter distribution than that given by the normal model.

The beta distribution provides a more flexible model as realistic bounds for the random variables can be incorporated. The use of the beta distribution, however, is complicated by the need to select these bounds based on limited data. Using Chebyshev's inequality (for specified sigma bounds) or introducing simply the physical limits of the variables can help overcome this problem. The dependence of the probability of failure of earth slopes on the limits of the strength parameters is investigated below, in Section 4.1.2.

Finally, the use of joint models, such as the bivariate normal and bivariate beta distributions, allows the correlation of the soil strength parameters to be included in the analysis. A comparison of Figs. 3-4, 3-6 and 3-7 indicates that the bivariate normal model depends considerably on the values of the correlation coefficient. The same effect is also true in the case of the bivariate beta model (Figs. 3-8 and 3-12). The sensitivity of the probability of failure to the value of the correlation coefficient is considered in Section 4.2.1.

4. PARAMETRIC STUDY OF SOIL STRENGTH

This section examines the influence of soil strength parameters on the probability of failure of slopes. The latter is determined in accordance with the model developed in this study and presented in Report No. CE-78-5 (Section 2.2.3).

In order to detach the effect of soil strength from that of the failure mode, failure is considered here to occur along the critical (rather than randomly generated) surface.

Two cases are examined, namely: one, in which c and ϕ independent, and another, in which c and ϕ are correlated random variables.

<u>4.1 Independent Soil Strength Parameters</u>

In this part of the parametric strength study, c and ϕ are assumed to be independent random variables following the beta model. Parameters α and β of the beta distribution are calculated using Eqn. (3-6).

More specifically, the effect on the reliability of slopes of the mean value, standard deviation, and third and fourth central moments of the strength parameters are investigated as well as that of their minimum values. The third and fourth central moments are introduced in terms of the coefficient of skewness(β_1) and coefficient of kurtosis(β_2), respectively, defined as

$$\beta_1 = \mu_3 / \sigma^3$$

$$\beta_2 = \mu_4 / \sigma^4 \qquad (4\text{-}1)$$

The coefficient of skewness is a measure of the asymmetry of the distribution, while the coefficient of kurtosis measures its peakedness. A value of the coefficient of kurtosis less than three ($\beta_2 < 3$) indicates a distribution flatter than the normal (as is, for example, the uniform distribution); while a value of β_2 larger than three ($\beta_2 > 3$) corresponds to a distribution more peaked than the normal.

For the beta distribution, the expressions for β_1 and β_2 are given as (Harr, 1977)

$$\beta_1 = \frac{2(\beta-\alpha)(\alpha+\beta+1)}{\alpha\beta(\alpha+\beta+2)}$$

$$\beta_2 = \frac{3(\alpha+\beta+1)[2(\alpha-\beta)^2+\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)} \qquad (4\text{-}2)$$

4.1.1 Effect of the Coefficient of Variation.

In Figs. 4-1 and 4-2 is shown the effect of the coefficient of variation of c and ϕ, respectively, on the probability of failure of the slope. The distributions of both c and ϕ were assumed to be symmetrical around their mean values (i.e., $\alpha_c = \beta_c$ and $\alpha_\phi = \beta_\phi$). Fig. 4-1 corresponds to a mean value and coefficient of variation of ϕ equal to 20° and 10% ,respectively, while Fig. 4-2 to $\bar{c}$ = 130 psf and V_c = 20%. The corresponding values of the dimensionless slope stability parameter $\lambda_{c\phi}$, defined as (Janbu, 1954)

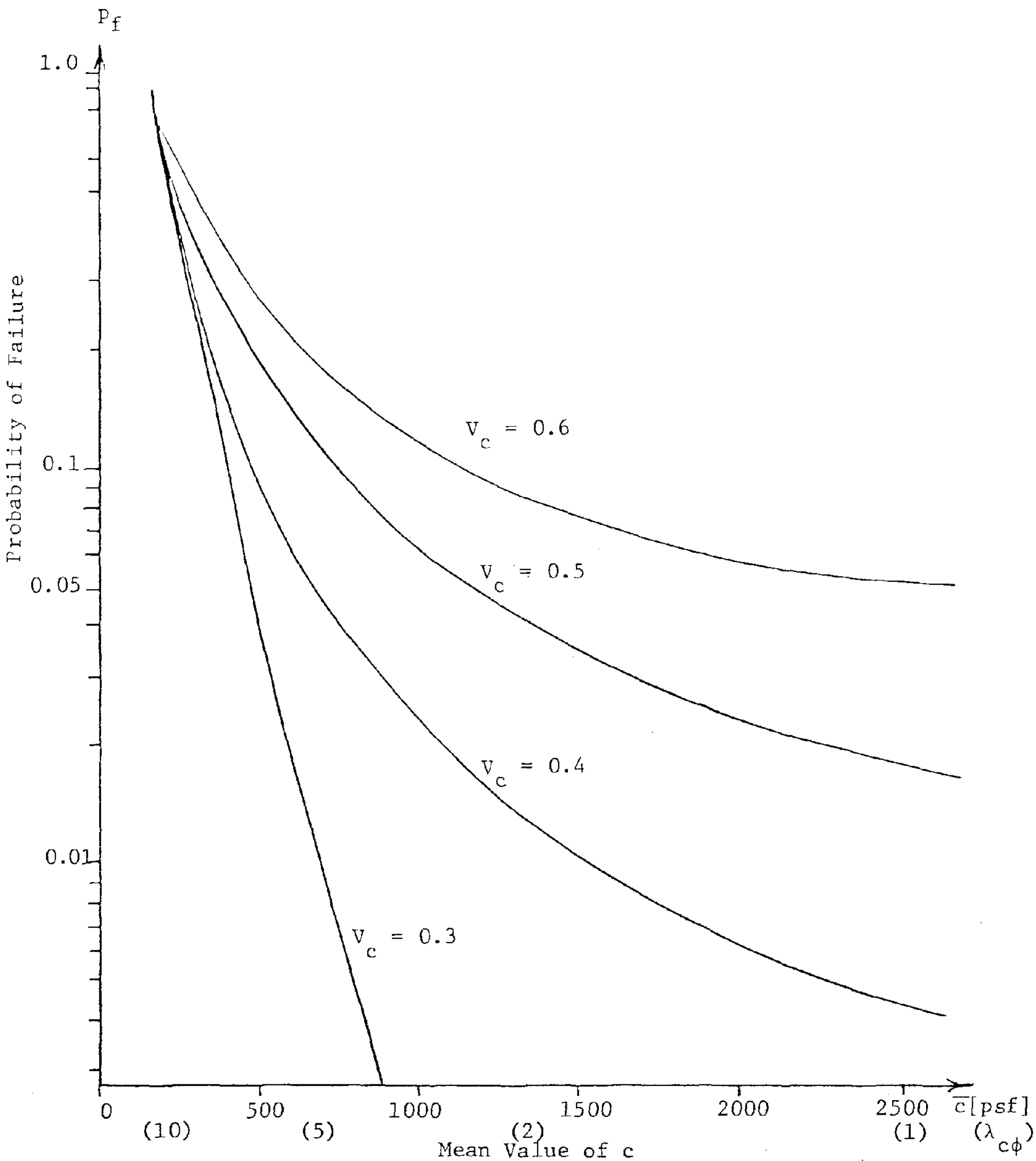


FIGURE 4-1. EFFECT ON THE PROBABILITY OF FAILURE OF THE STATISTICAL VALUES OF c

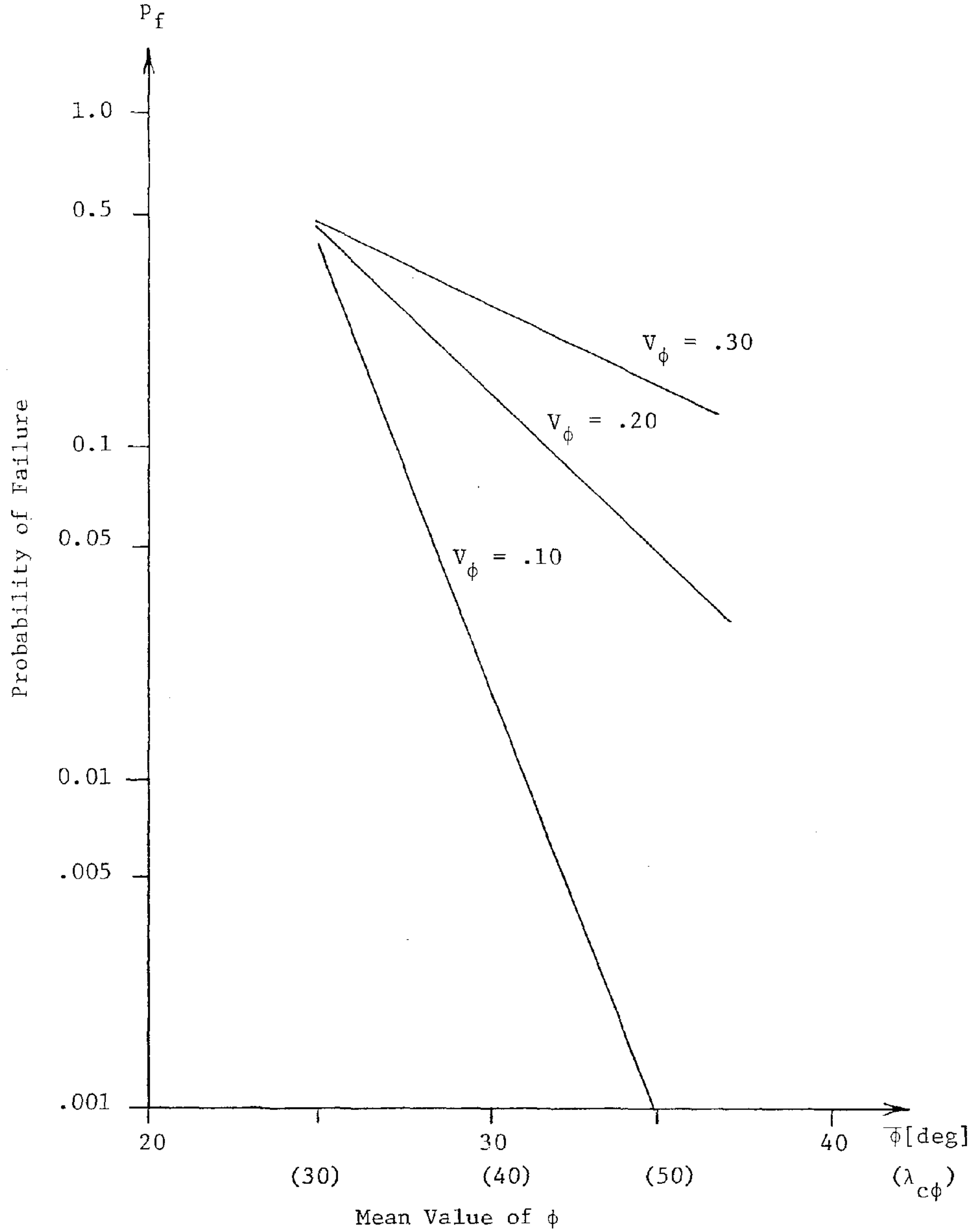


FIGURE 4-2. EFFECT ON THE PROBABILITY OF FAILURE OF THE STATISTICAL VALUES OF ϕ

$$\lambda_{c\phi} = \gamma H \tan(\phi)/c \qquad (4\text{-}3)$$

are also shown in the horizontal axis of the two figures. The value of $\lambda_{c\phi}$ indicates the relative importance of c and ϕ on the stability of the slope. For example, for a slope with $\lambda_{c\phi}$ greater than thirty ($\lambda_{c\phi} > 30$), the ϕ-parameter is the dominant component of strength while, for another, with $\lambda_{c\phi}$ close to zero ($\lambda_{c\phi} \simeq 0$) the c-parameter is dominant.

The dependence of the probability of failure on the mean value ($\bar{c}$) and coefficient of variation (V_c) of c for the special case of a $\phi = 0$ material is shown in Figs. 4-3 and 4-4. Fig. 4-3 corresponds to a slope angle $\beta = 30°$ while Fig. 4-4 to $\beta = 60°$. As only one random variable is present in the $\phi = 0$ analysis, it is convenient to show the results for various values of the mean factor of safety. It is seen that as the mean value of the factor of safety $\bar{F}_s$ approaches one, the probability of failure approaches a value of 50%, for any value of the coefficient of variation V_c.

4.1.2 Effect of Minimum Values.

In Tables 4-1 and 4-2 are given the results of a number of failure simulations examining the effect on the probability of failure of ϕ_{min} and c_{min}, respectively. The mean value and coefficient of variation of ϕ were constant and equal to 36° and 20%, respectively. The minimum values of ϕ used were 21°, 14.4°, and 0°. The values 21° and 14.4° correspond to two and three sigma bounds, respectively, from the mean value, while the third value (0°) is the physical lower bound of the variable. For comparison, the value of the probability of failure that corresponds to a normal distribution of the strength

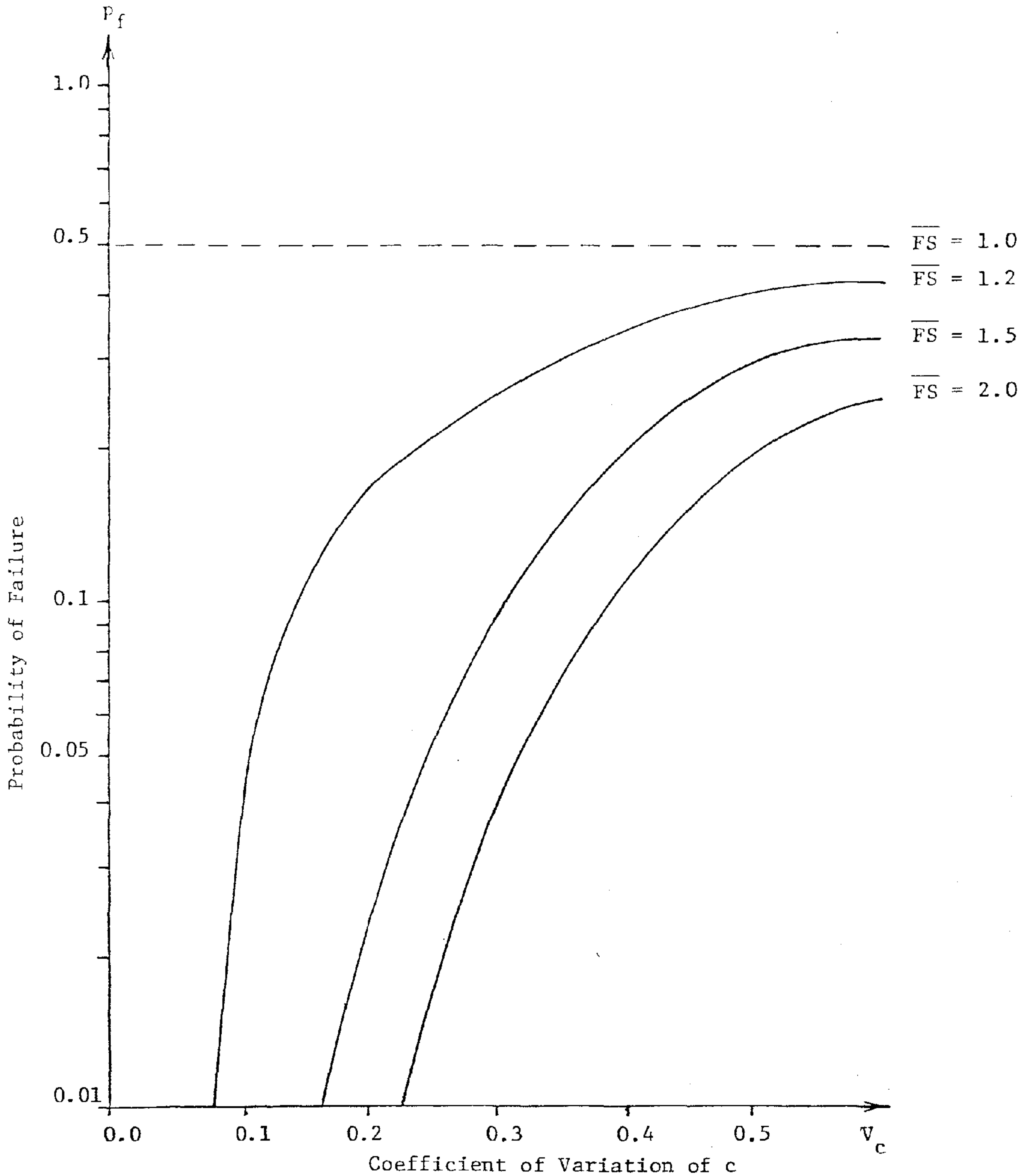


FIGURE 4-3. EFFECT OF V_c ON p_f, UNDRAINED ANALYSIS (β = 30°)

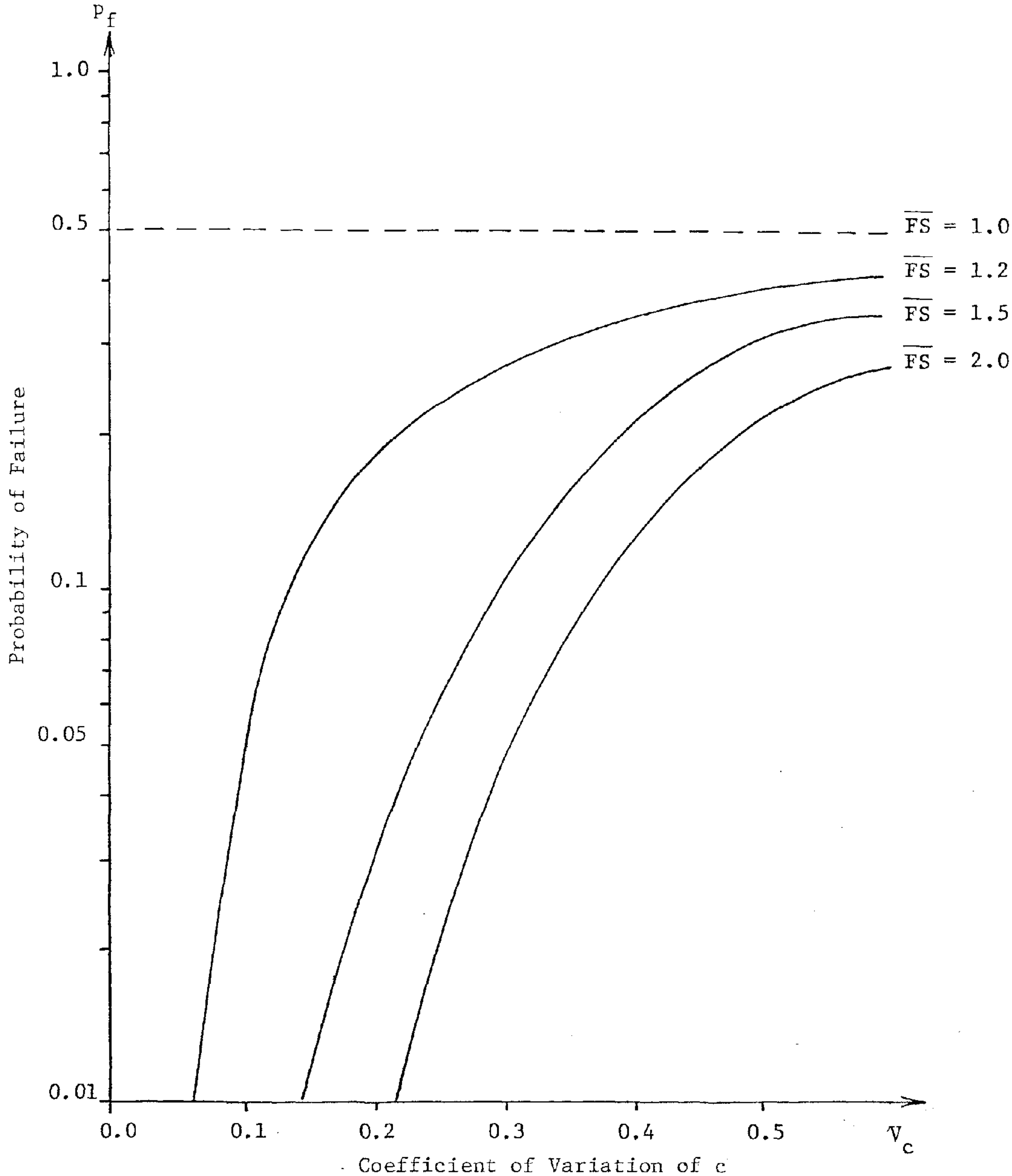


FIGURE 4-4. EFFECT OF V_c ON p_f, UNDRAINED ANALYSIS (β = 60°)

Table 4-1

EFFECT OF THE MINIMUM VALUE OF ϕ

SLOPE DESCRIPTION	PROBABILITY OF FAILURE			
	Beta $\phi_{min}=21°$	Beta $\phi_{min}=14.4°$	Beta $\phi_{min}=0°$	Normal
Run 1 $\lambda_{c\phi}=9.8, \beta=30°$	.055	.052	.052	.044
Run 2 $\lambda_{c\phi}=7.4, \beta=45°$ ($\rho=0$)	.060	.060	.060	.052
Run 2 $\lambda_{c\phi}=7.4, \beta=45°$ ($\rho=-.50$)	.012	.009	.009	.026
Run 2' $\lambda_{c\phi}=7.4, \beta=45°$ (fixed surfaces)	.067	.064	.060	.048
Run 3 $\lambda_{c\phi}=30.3, \beta=35°$	.078	.090	.085	.061
Run 4 $\lambda_{c\phi}=18.2, \beta=30°$ ($p_f=0$ for $\phi_{min}>24°$)	.067	.084	.081	.055

Table 4-2

EFFECT OF THE MINIMUM VALUE OF c

SLOPE DESCRIPTION	PROBABILITY OF FAILURE		
	Beta c_{min}=570psf	Beta c_{min}=380psf	Beta c_{min}=0psf
Run 5 ($\overline{FS}$=1.2) $\lambda_{c\phi}$=0, β=30°	.194	.187	.177
	Beta c_{min}=735psf	Beta c_{min}=490psf	Beta c_{min}=0psf
Run 5' ($\overline{FS}$=1.5) $\lambda_{c\phi}$=0, β=30°	.009	.040	.052

parameters is also included the table.

From Tables 4-1 and 4-2, it is seen that, for given mean values and coefficients of variation of c and ϕ, the probability of failure is rather insensitive to the minimum values of c and ϕ used to determine the parameters of the beta distributions.

4.1.3 Effect of Coefficients of Skewness and Kurtosis.

In Figs. 4-5 and 4-6 is shown the variation of the probability of failure p_f of a slope with the coefficient of skewness β_1 of c and ϕ, respectively. It can be seen that the effect of β_1 on p_f is very small compared to that of the mean values of the two strength parameters.

Finally, in Figs. 4-7 and 4-8 is shown the dependence of the probability of failure p_f on the coefficient of kurtosis β_2 of c and ϕ, respectively. As was the case with the coefficient of skewness, there is no significant effect of β_2 on p_f.

4.2 Correlated Soil Strength Parameters

In this part of the parametric study, c and ϕ are assumed to be correlated random variables. Both the bivariate beta and the bivariate normal models are investigated. The parameters for the bivariate beta distribution are calculated from Eqns. (A-8) of Appendix A.

4.2.1 Effect of the Correlation Coefficient of c and ϕ.

In Table 4-3 are listed the values of the probability of failure p_f of a soil slope that corresponds to various values of the correlation coefficient ρ of the two strength parameters c and ϕ. It can be seen that p_f decreases as ρ decreases. This trend holds for both the normal and beta models of strength parameters and is in agreement

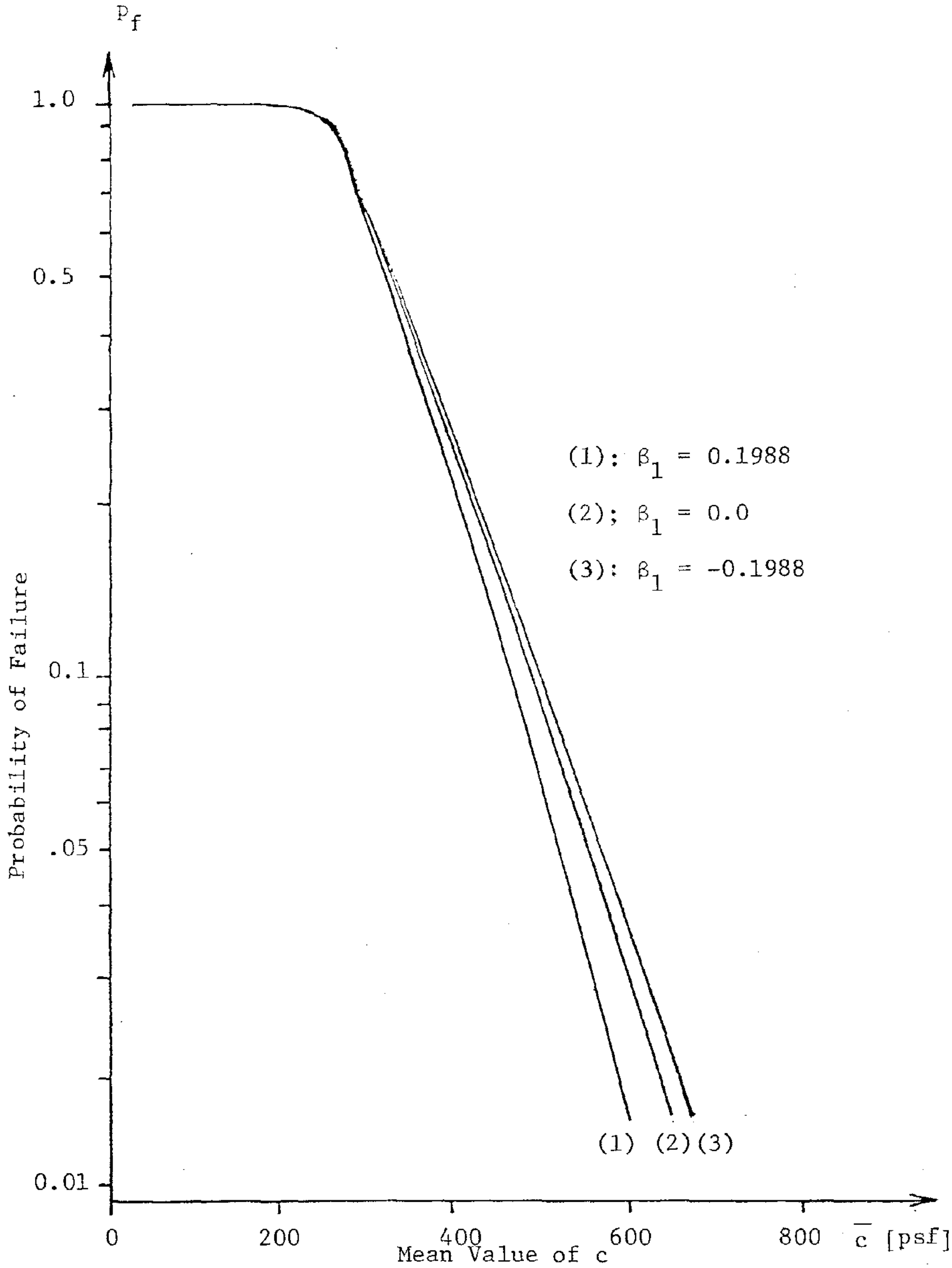


FIGURE 4-5. EFFECT ON THE PROBABILITY OF FAILURE OF COEFFICIENT OF SKEWNESS OF c-PARAMETER

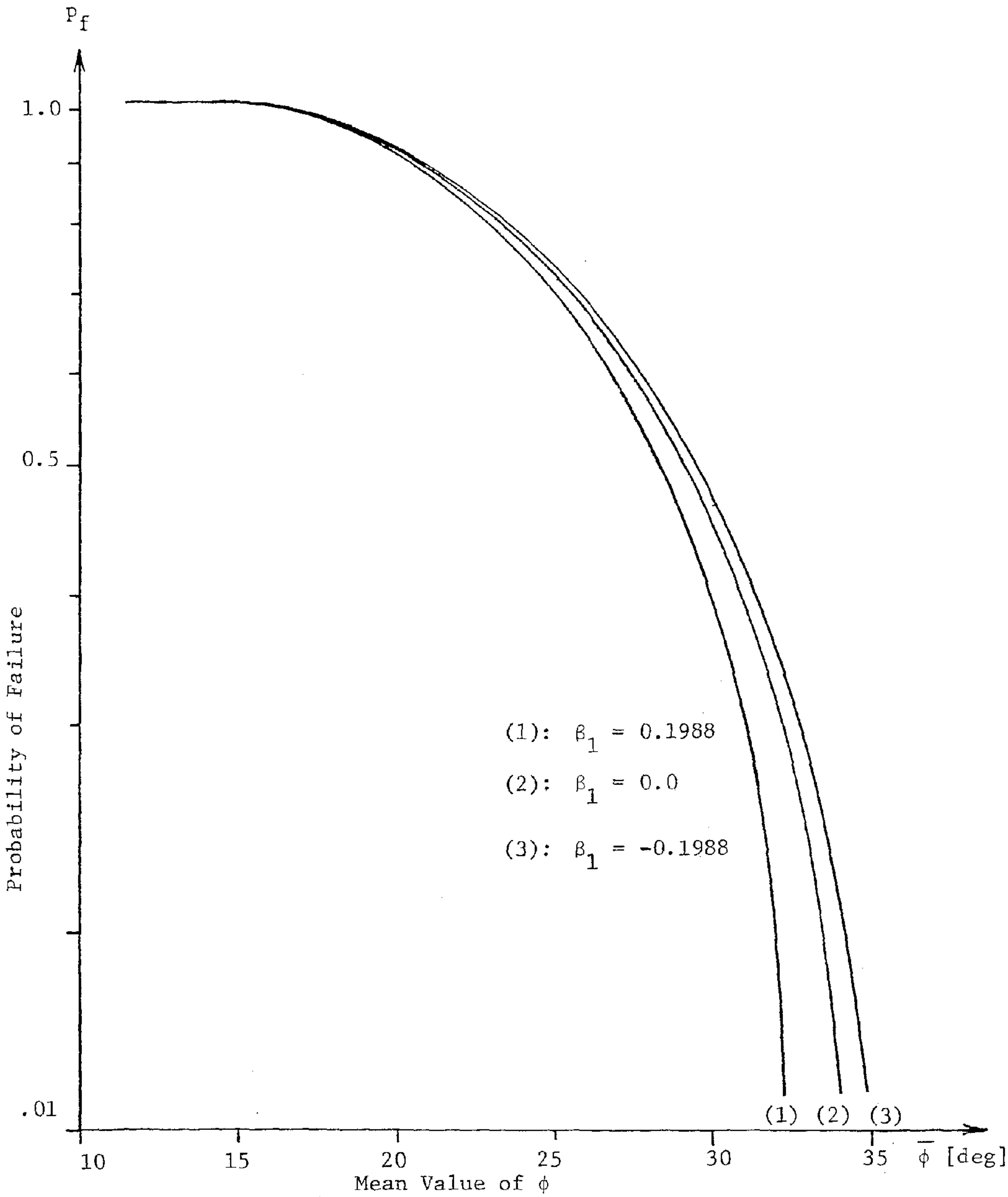


FIGURE 4-6. EFFECT ON PROBABILITY OF FAILURE OF COEFFICIENT OF SKEWNESS OF ϕ-PARAMETER

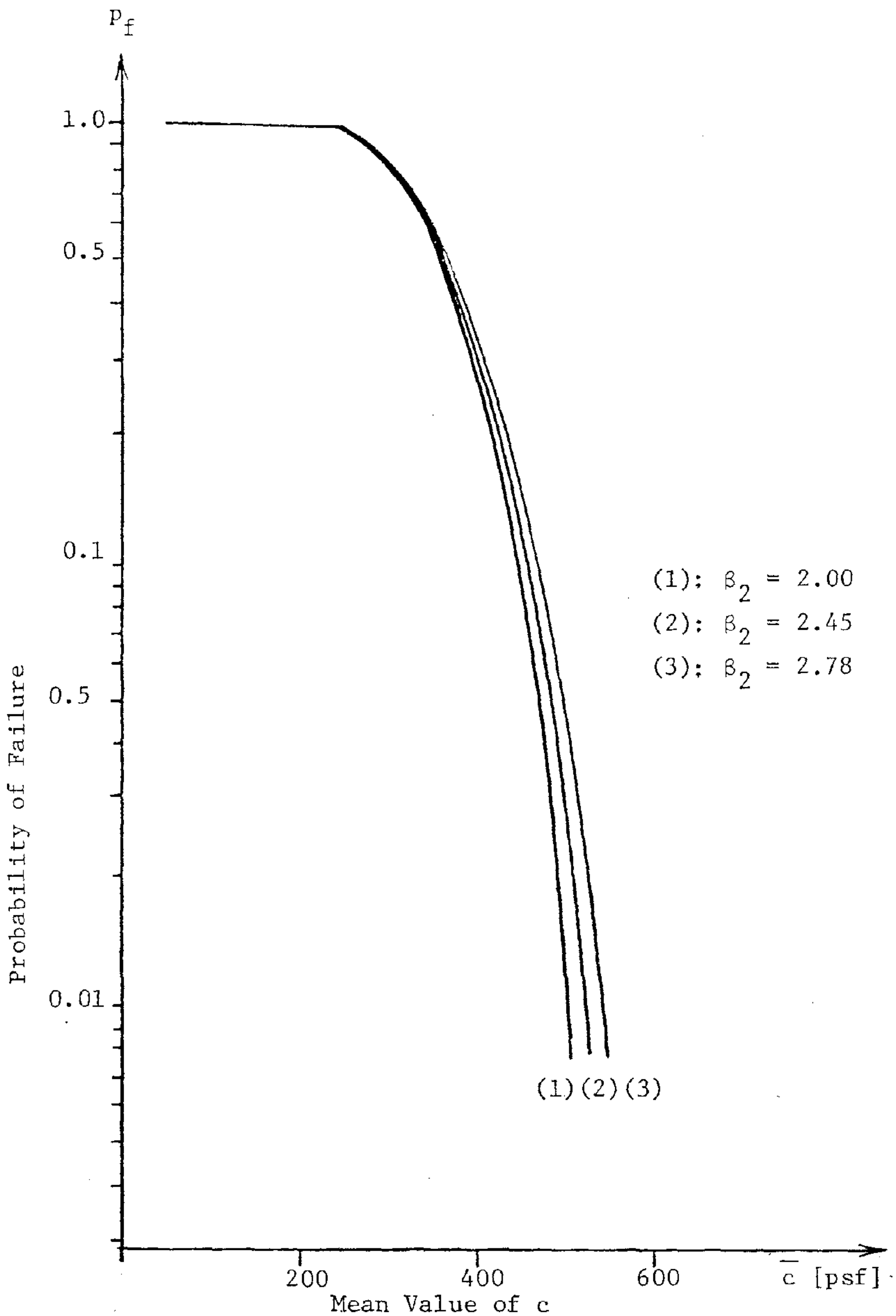


FIGURE 4-7. EFFECT ON PROBABILITY OF FAILURE OF COEFFICIENT OF KURTOSIS OF c-PARAMETER

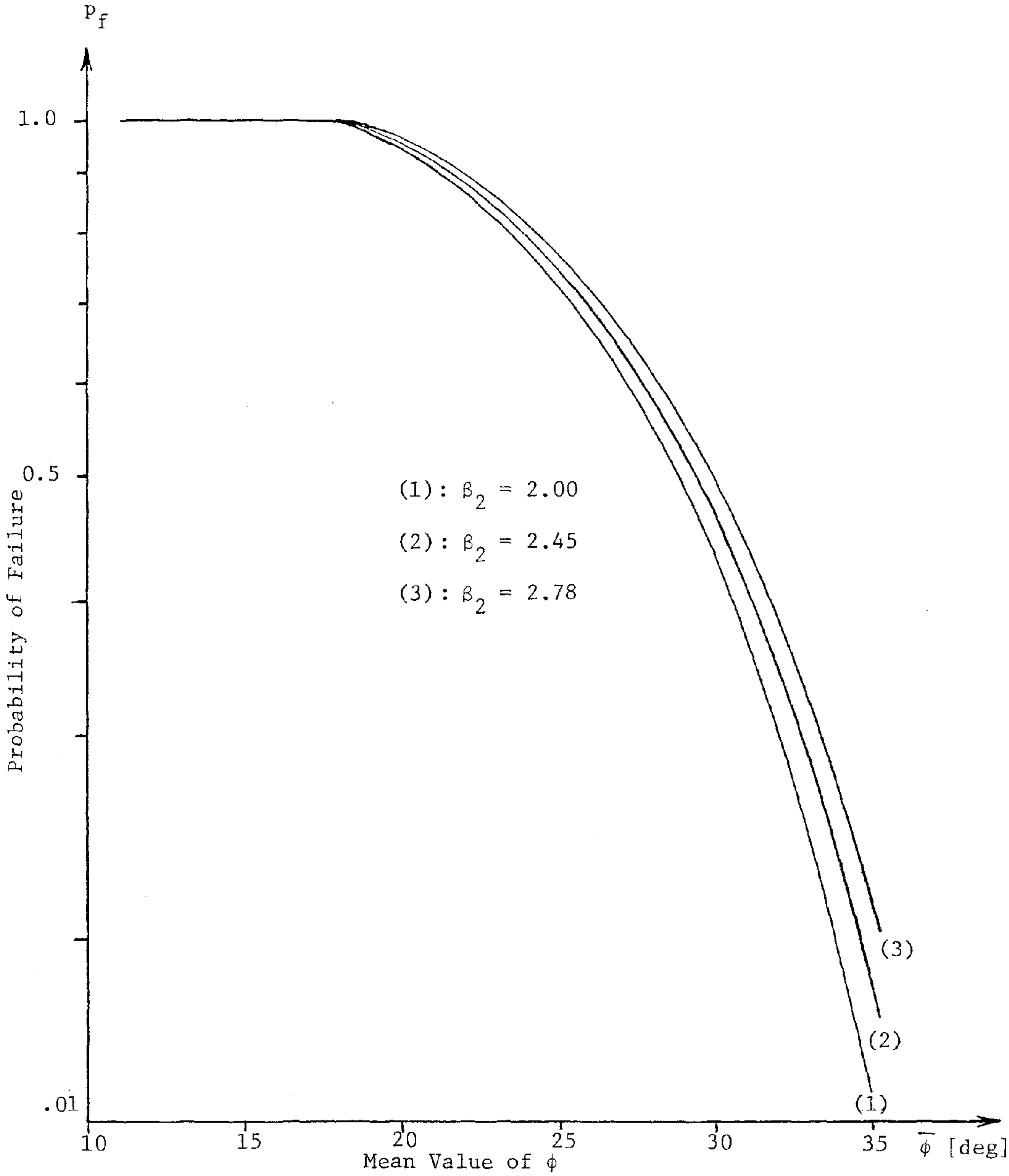


FIGURE 4-8. EFFECT ON PROBABILITY OF FAILURE OF COEFFICIENT OF KURTOSIS OF ϕ-PARAMETER

TABLE 4-3

THE EFFECT OF THE CORRELATION OF c AND ϕ ON THE PROBABILITY OF FAILURE

SLOPE DESCRIPTION	DISTRIBUTION OF C AND PHI	PROBABILITY OF FAILURE					
		CORRELATION COEFFICIENT					
		.00	-.15	-.25	-.325	-.50	-.75
Slope 1	Indep. beta	.055	—	—	—	—	—
H=100' β=30°	Bivariate beta	—	.017	.017			
$\lambda_{c\phi}$=9.8	Bivariate normal	.044	.033	—	.020	.011	.000
Slope 2	Indep. beta	.060	—	—	—	—	—
H=75' β=45°	Bivariate beta	—	.013	.012		.012	
$\lambda_{c\phi}$=7.4	Bivariate normal	.052	.043		.037	.026	
Slope 3	Indep. beta	.521		—		—	—
H=40.5' β=15.5°	Bivariate beta	—		.506		.455	.348
$\lambda_{c\phi}$=6.9, ru=.32	Bivariate normal	.498	.500			.490	.481
Slope 4	Indep. beta	.090		—		—	
H=75' β=35°	Bivariate beta	—		.042		.042	
$\lambda_{c\phi}$=30.3	Bivariate normal	.066		.058		.042	

with previous findings on the subject (Yucemen and Tang, 1975).

For ρ equal to zero, the results indicate that the values of the probability of failure for beta distributed c and ϕ are greater than those corresponding to a normal distribution with the same mean values and coefficients of variation. For values of ρ less than zero, the values of p_f for the normal case decrease linearly with ρ.

4.3 Conclusions from Parametric Study of Soil Strength

On the basis of the results obtained in this parametric study, the following conclusions are drawn:

(a) The mean value and the coefficient of variation of the strength parameters have a much greater effect on the probability of failure of slopes than their minimum values, coefficients of skewness and coefficients of kurtosis.

(b) The correlation coefficient ρ of the two strength parameters has a significant effect on the calculated value of p_f. As ρ becomes more negative, the corresponding value of p_f decreases.

(c) The bivariate normal distribution gives a linear decrease of p_f with increasing ρ. This is not the case for the bivariate beta model.

5. PARAMETRIC STUDY OF SEISMIC LOAD

5.1 Description of Seismic Load

In a pseudo-static stability analysis, the effect of an earthquake on the stability of an earth slope is introduced through the maximum acceleration experienced at the site of the slope. This acceleration depends on many factors such as the amount of energy released during the earthquake (measured in terms of the earthquake magnitude); the law with which this energy attenuates with the distance from the earthquake source; the type of earthquake source involved; the seismic history of the area; local conditions; etc.

In the first report of this series, Report No. CE-78-5, the significant factors mentioned above were analyzed to yield a mathematical expression for the probability density function of the maximum ground acceleration.

The attenuation relationship employed has the form

$$a_{max} = b_1 \, e^{b_2 m} \, (R + b_4)^{-b_3} \qquad (5\text{-}1)$$

in which b_1, b_2, b_3, and b_4 are regional parameters, R is the distance between the source and the site (in km), and m is the earthquake magnitude (in Richter scale).

A log-quadratic frequency magnitude relation was found to best represent the data available for the State of New York (Report No. CE-78-7). The probability density function of the earthquake magnitude was found to have the following expression:

$$f(m) = -k(b-2cm)\exp[b(m-m_o)+c(m^2-m_o^2)], \quad m_o \leq m \leq m_1 \qquad (5\text{-}2)$$

in which b and c are regional constants, m_1 and m_o are the upper and lower limits of the earthquake magnitude, and k is a normalizing constant, so that the cumulative distribution F(m) become equal to one when $m = m_1$. The expression for k is

$$k = \{1-\exp[b(m_1-m_o) + c(m_1^2-m_o^2)]\}^{-1}$$

Parameters b and c were found to be equal to 0.203 and −0.182, respectively, and the lower and upper limits of the magnitude were specified as $m_o = 2$ and $m_1 = 7$. From Eqns. (5-1) and (5-2), using the method of transformation of variables (Hahn and Shapiro, 1967), the density function the maximum acceleration was found to be equal to

$$f(a_{max}) = -\frac{k}{b_2}\,\frac{1}{a_{max}}\,(2cG+b)\exp[b(G-m_o)+c(G^2-m_o^2)] \qquad (5\text{-}3)$$

where

$$G = \frac{1}{b_2}\ln\left[\frac{a_{max}(R+b_4)^{b_3}}{b_1}\right]$$

Two attenuation relationships have been proposed for the Eastern United States (Report No. CE-78-5) and were employed in the present study. Their expressions are

$$a_{max} = 1100\ e^{0.5m}(R+25)^{-1.32} \qquad (5\text{-}4a)$$

$$a_{max} = 1.183\ e^{1.15m}(R)^{-1.0} \qquad (5\text{-}4b)$$

In Figs. 5-1 and 5-2 are shown the cumulative distributions of a_{max} that correspond to Eqns. (5-4a) and (5-4b), respectively.

An error term ϵ has been included in order to account for the difference between predicted and observed values of a_{max}. Introducing ϵ into the expressions of the attenuation relationships, the latter become

$$a_{max} = 1100\ e^{0.5m}(R+25)^{-1.32}\epsilon \qquad (5\text{-}5a)$$

$$a_{max} = 1.183\ e^{1.15m}(R)^{1.0}\epsilon \qquad (5\text{-}5b)$$

where ϵ is taken to be a log-normally distributed random variable with a median value of 1.0 and a standard deviation σ_ϵ varying between 0.5 and 1.0 ($0.5 \leq \sigma_\epsilon \leq 1.0$).

In Figs. 5-3 and 5-4 is shown the resulting cumulative distribution of a_{max} that correspond to Eqn. (5-5a) and for the case where the error term has standard deviations σ_ϵ equal to 0.5 and 1.0, respectively. In Figs. 5-5 and 5-6 is shown the same cumulative distribution for the case of Eqn. (5-5b) with σ_ϵ equal to 0.5 and 1.0, respectively. A summary of the statistical parameters of the acceleration for the above cases (Figs. 5-1 through 5-6) is given in Table 5-1.

5.2 Description of the Earthquake Source

The maximum horizontal ground acceleration as described above corresponds to a point representation of the earthquake source (point-source model). This is shown schematically in Fig. 5-7a. Two additional models have been introduced for the earthquake source,

POINT SOURCE MODEL

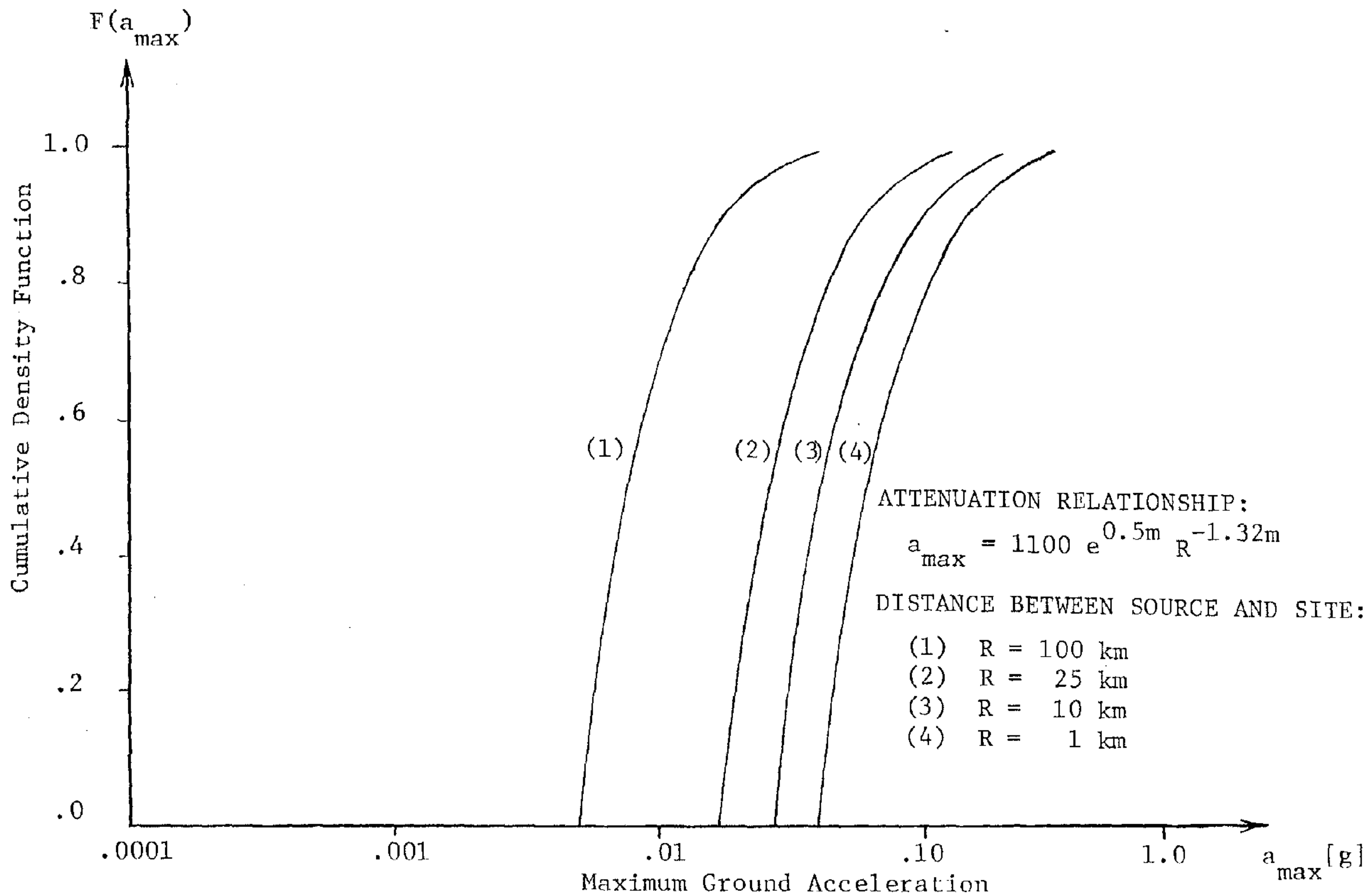


FIGURE 5-1. CUMULATIVE FUNCTION OF GROUND ACCELERATION

POINT SOURCE MODEL

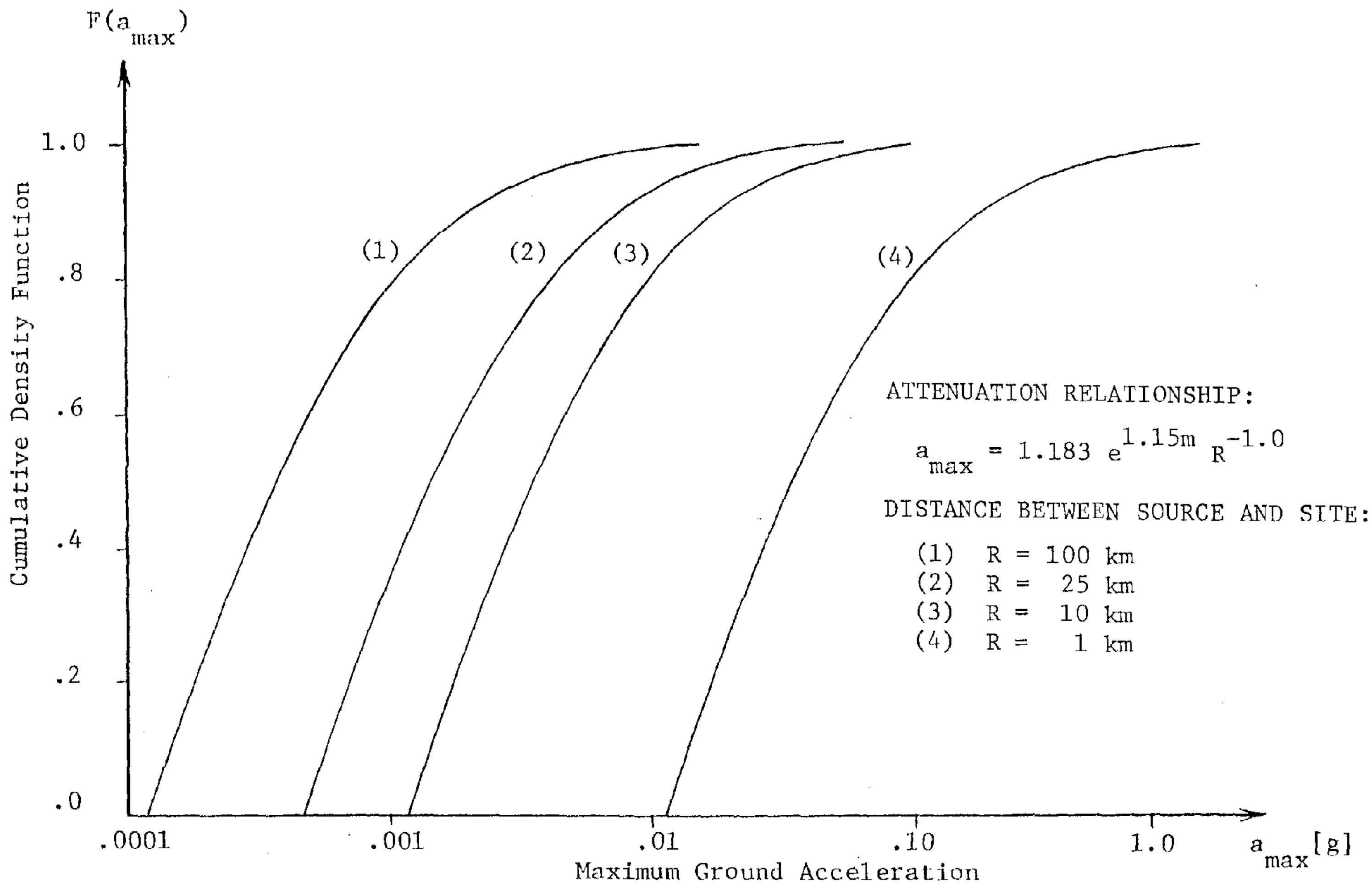


FIGURE 5-2. CUMULATIVE FUNCTION OF GROUND ACCELERATION

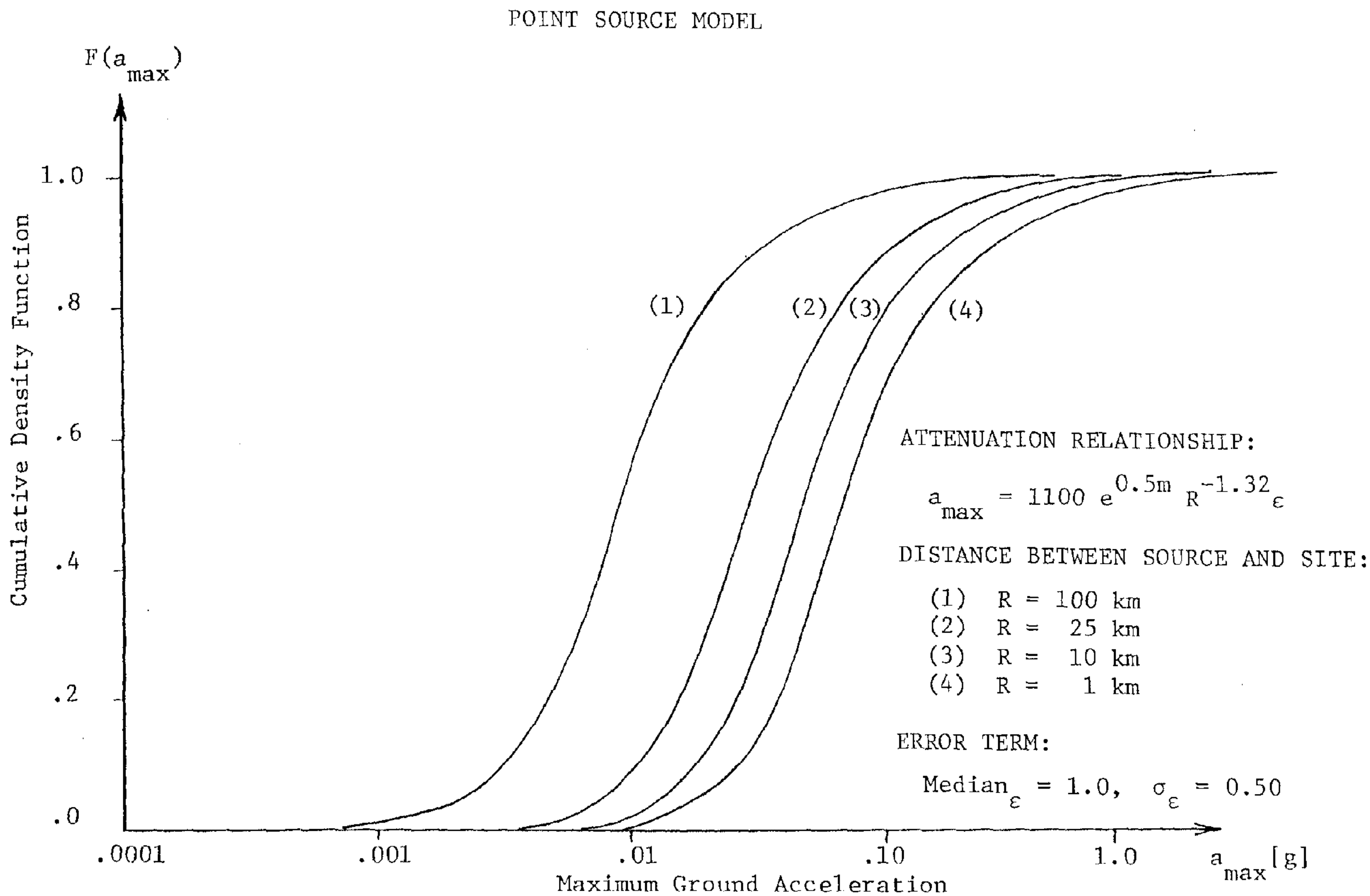


FIGURE 5-3. CUMULATIVE FUNCTION OF GROUND ACCELERATION

POINT SOURCE MODEL

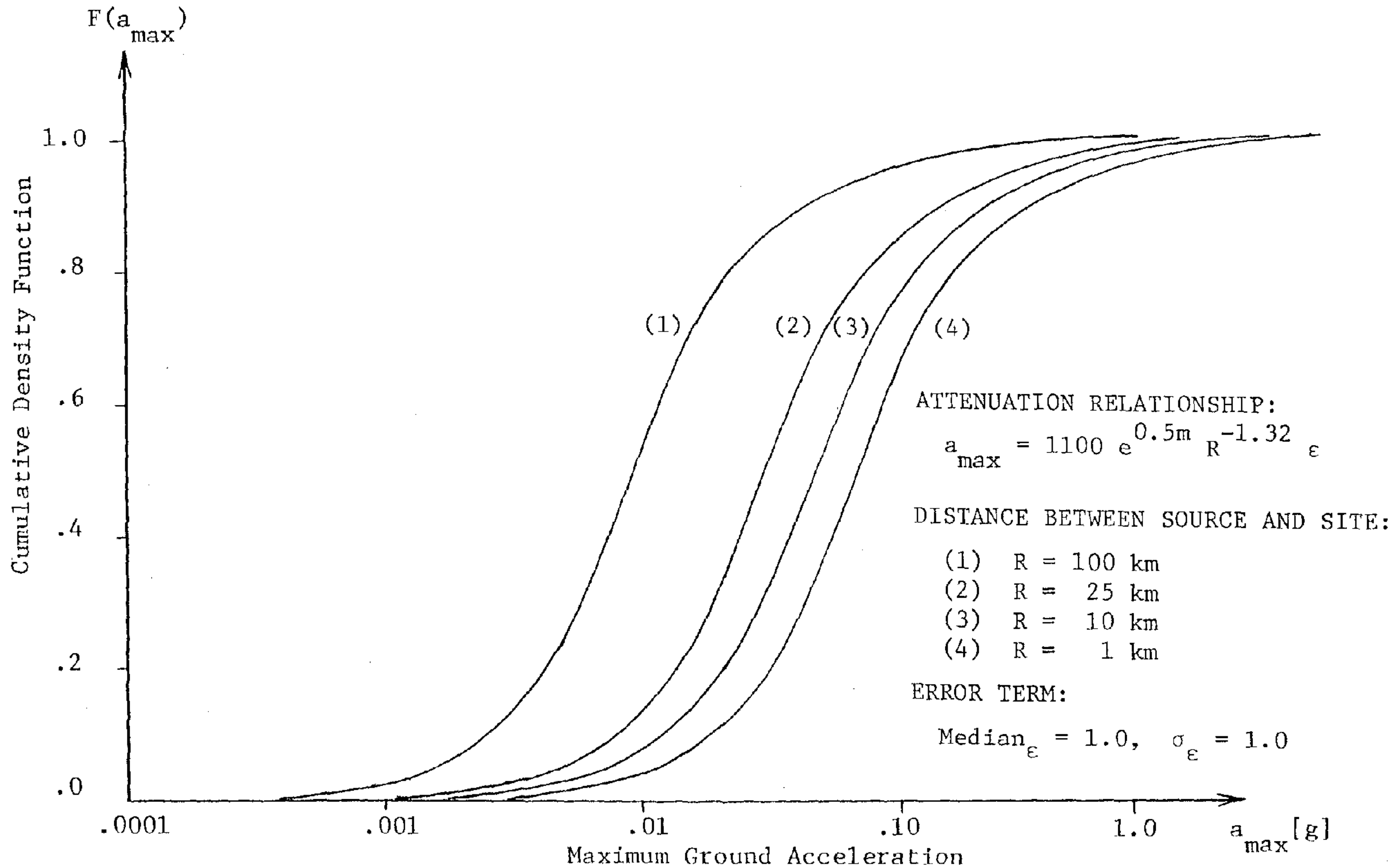


FIGURE 5-4. CUMULATIVE FUNCTION OF GROUND ACCELERATION

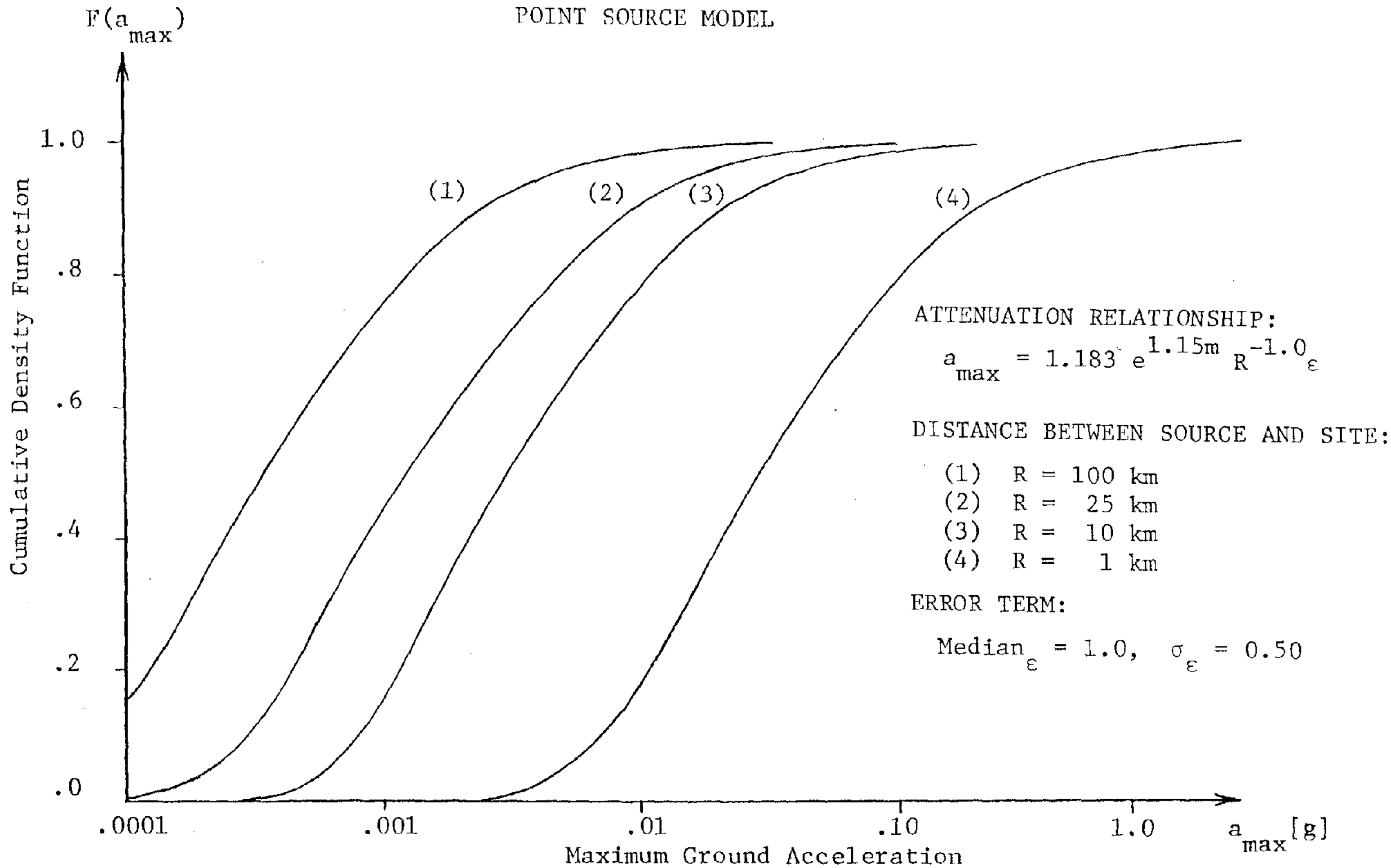


FIGURE 5-5. CUMULATIVE FUNCTION OF GROUND ACCELERATION

POINT SOURCE MODEL

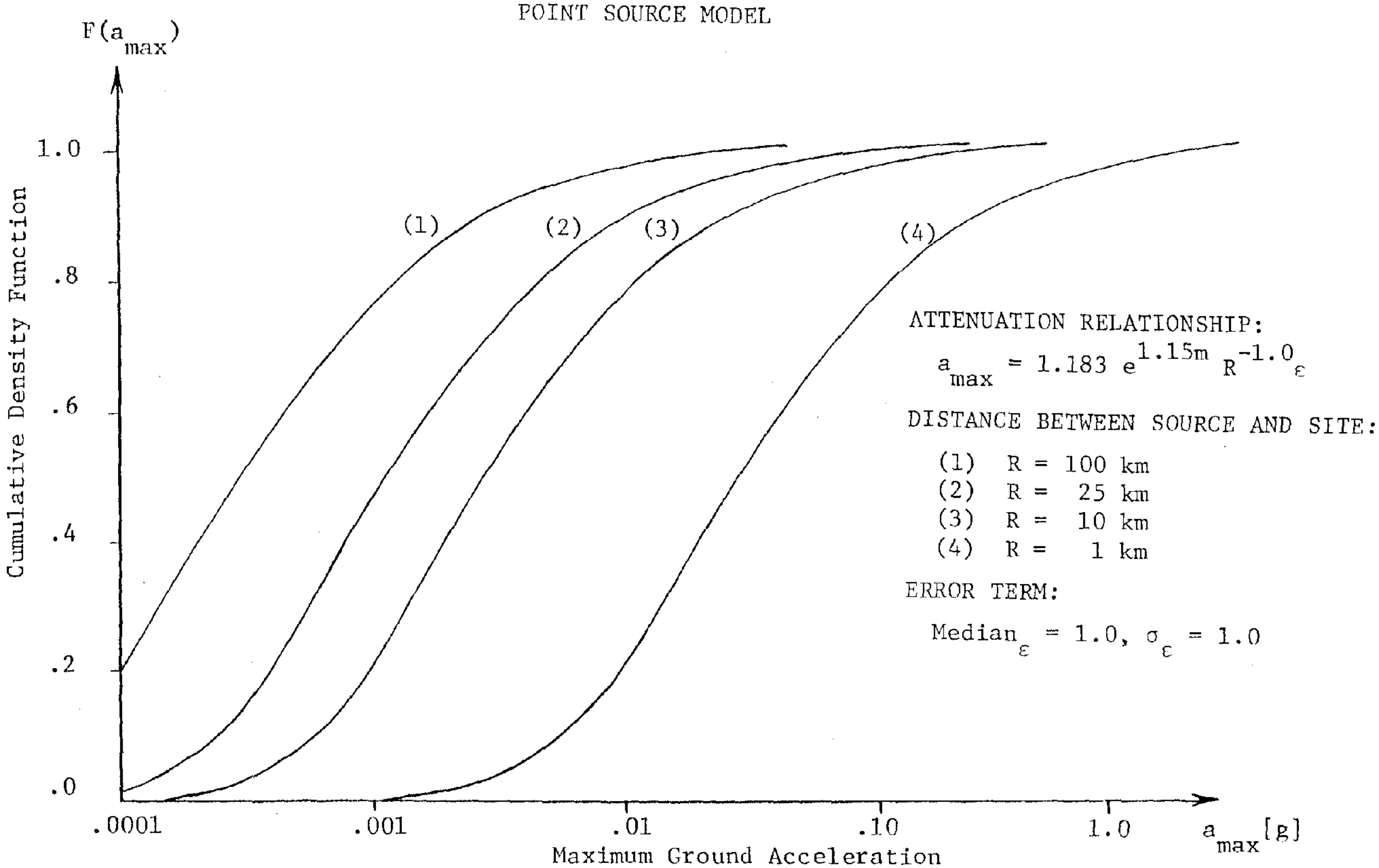
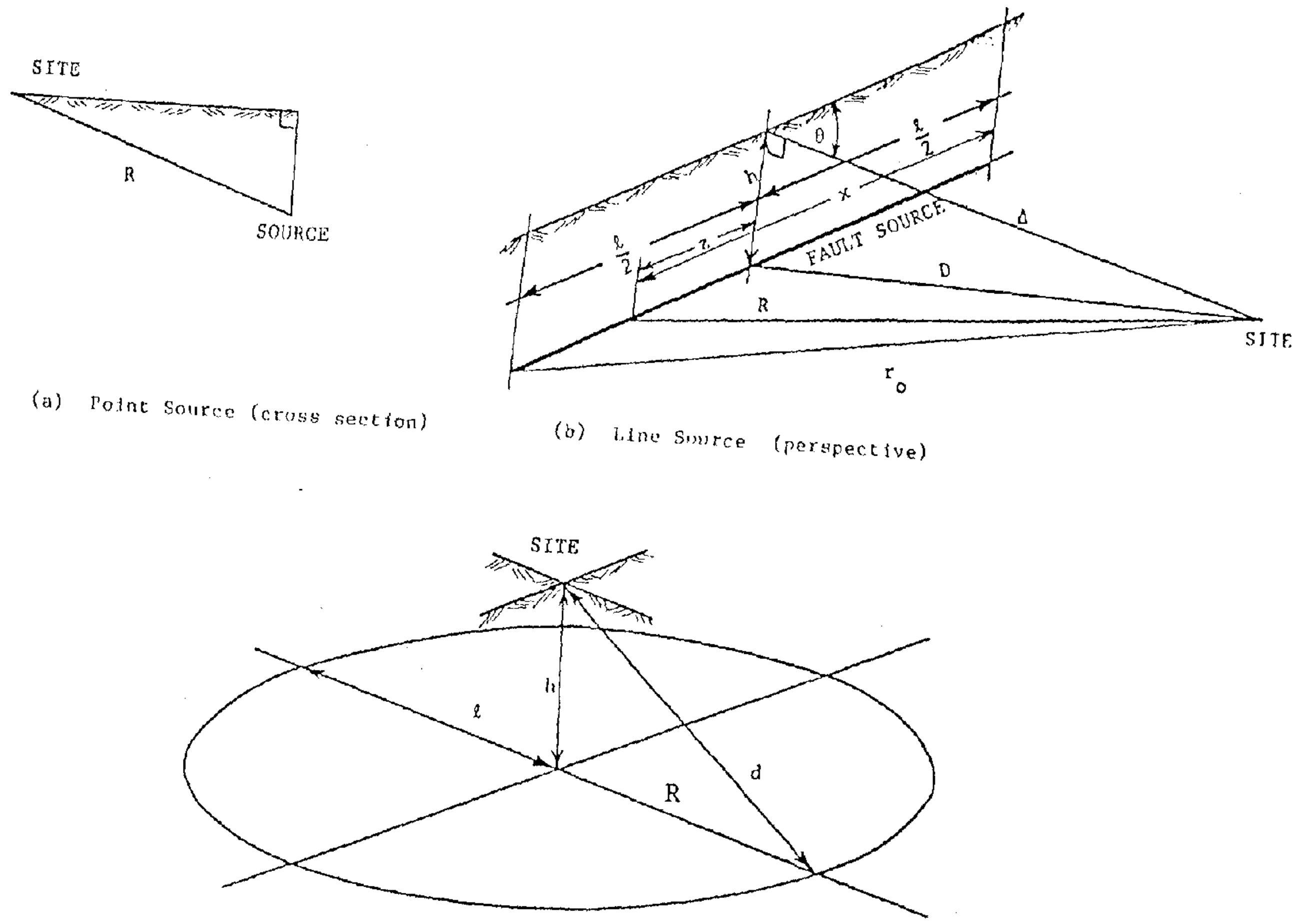


FIGURE 5-6. CUMULATIVE FUNCTION OF GROUND ACCELERATION

TABLE 5-1 STATISTICAL VALUES OF MAXIMUM GROUND ACCELERATION (POINT SOURCE MODEL)

ATTENUATION RELATIONSHIP	ERROR TERM σ_ε	ERROR TERM $\bar{\varepsilon}$	DISTANCE (km)	$\bar{a}_{max}$ (g)	$\sigma_{a_{max}}$ (g)	$V_{a_{max}}$ (g)
(1)	0.00	1.00	1	0.0653	0.0381	58.42
	0.00	1.00	10	0.0438	0.0248	56.59
	0.00	1.00	25	0.0271	0.0144	53.00
	0.00	1.00	100	0.0084	0.0050	59.74
	0.50	1.133	1	0.0703	0.0513	72.91
	0.50	1.133	10	0.0469	0.0336	71.60
	0.50	1.133	25	0.0316	0.0258	81.60
	0.50	1.133	100	0.0090	0.0067	74.52
	1.00	1.649	1	0.0747	0.0626	83.76
	1.00	1.649	10	0.0499	0.0428	85.72
	1.00	1.649	25	0.0343	0.0327	95.42
	1.00	1.649	100	0.0097	0.0085	87.89
(2)	0.00	1.000	1	0.0541	0.1159	214.41
	0.00	1.000	10	0.0052	0.0111	213.40
	0.00	1.000	25	0.0019	0.0037	190.41
	0.00	1.000	100	0.0006	0.0013	220.41
	0.50	1.133	1	0.0574	0.1308	228.04
	0.50	1.133	10	0.0052	0.0097	187.35
	0.50	1.133	25	0.0026	0.0063	238.20
	0.50	1.133	100	0.0006	0.0012	217.22
	1.00	1.649	1	0.0604	0.1426	235.86
	1.00	1.649	10	0.0055	0.0112	204.16
	1.00	1.649	25	0.0029	0.0071	249.01
	1.00	1.649	100	0.0006	0.0014	232.15

ATTENUATION RELATIONSHIP: (1) $1100\ e^{0.5m}(R+25)^{-1.32}\varepsilon$ (2) $1.183\ e^{1.15m}R^{-1.0}\varepsilon$



(a) Point Source (cross section)

(b) Line Source (perspective)

(c) Area Source (perspective)

FIGURE 5-7. SCHEMATIC REPRESENTATION OF THE THREE EARTHQUAKE SOURCES namely the line (or, fault) model and the area model (Cornell, 1968). A line source model (Fig. 5-7b) is used in the case where an active fault has been identified in the area of interest, or when the epicenters of a series of earthquakes lie approximately along a line. An area source (Fig. 5-7c) is used when no particular point or line source can be identified or when earthquakes occur randomly within a certain region.

Occurrences of earthquakes from these sources are assumed to be random events uniformly distributed along a fault or within an area. On the basis of this assumption, the probability density function of the maximum acceleration for the line and area sources were determined. The resulting expressions were presented in Appendix B, Report No. CE-78-5.

The above models for the earthquake source were adopted as part of the Monte Carlo simulation technique employed in the "RASSUEL" model. The most critical of the two attenuation relationship, i.e., that given by Eqn. (5-4a), has been introduced in the program to determine the cumulative density function of the maximum ground acceleration for the cases of line and area sources (Figs. 5-8 to 5-25).

Line Source:

In Figs. 5-8 through 5-13 is shown the cumulative distributions of a_{max} for a fault length equal to 100 km; while in Figs. 5-14 through 5-19 is given the same distribution for a fault length equal to 250 km. Figs. 5-8 through 5-10 and Figs. 5-14 through 5-16 correspond to the case where the orientation of the fault (defined in Fig. 5-7) is at an angle of 45° with respect to the site of the slope, while Figs. 5-11 through 5-13 and Figs. 5-17 through 5-19 correspond to an

LINE SOURCE MODEL

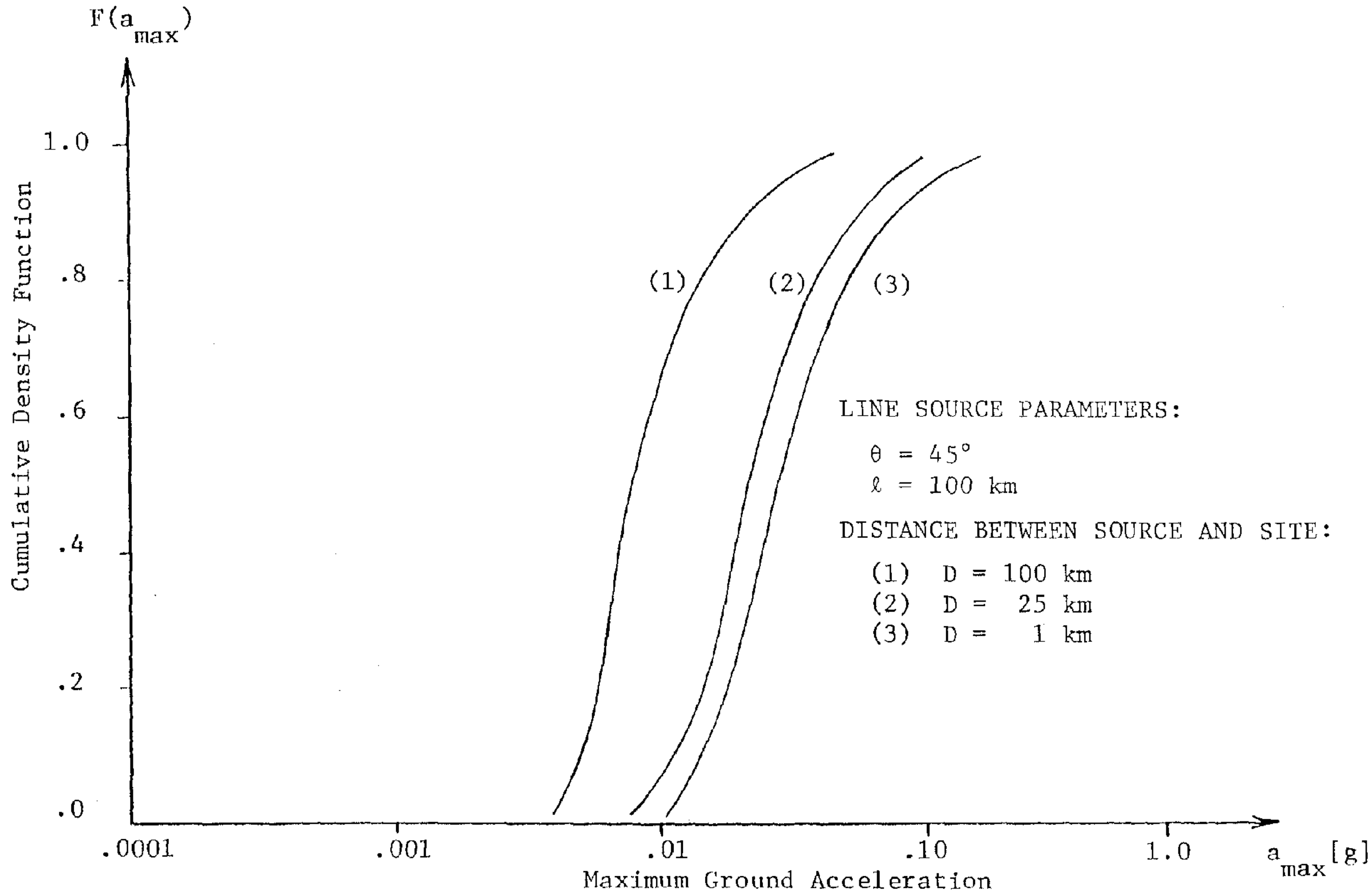


FIGURE 5-8. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

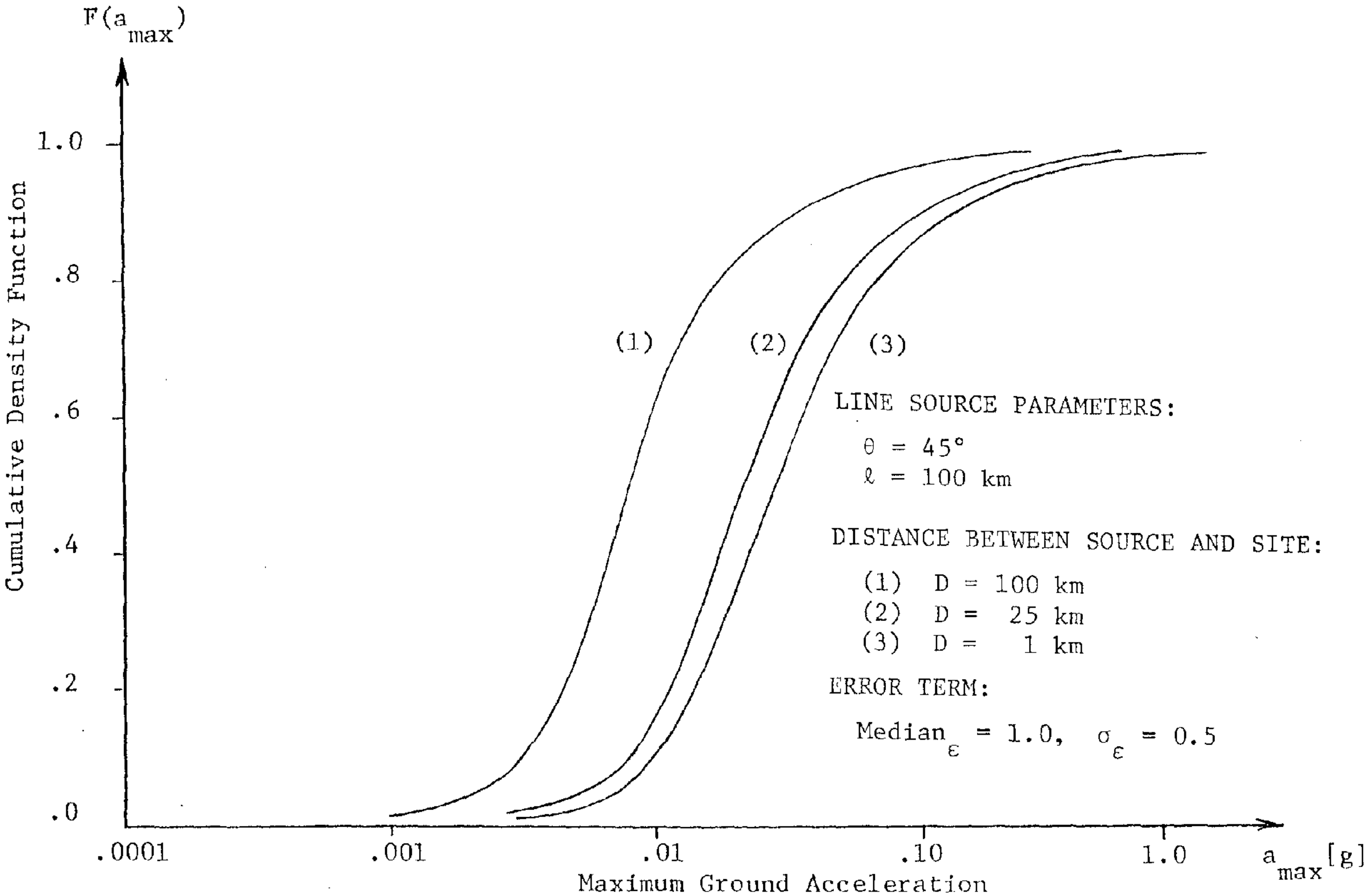


FIGURE 5-9. CUMULATIVE FUNCTION OF GROUND ACCELERATION

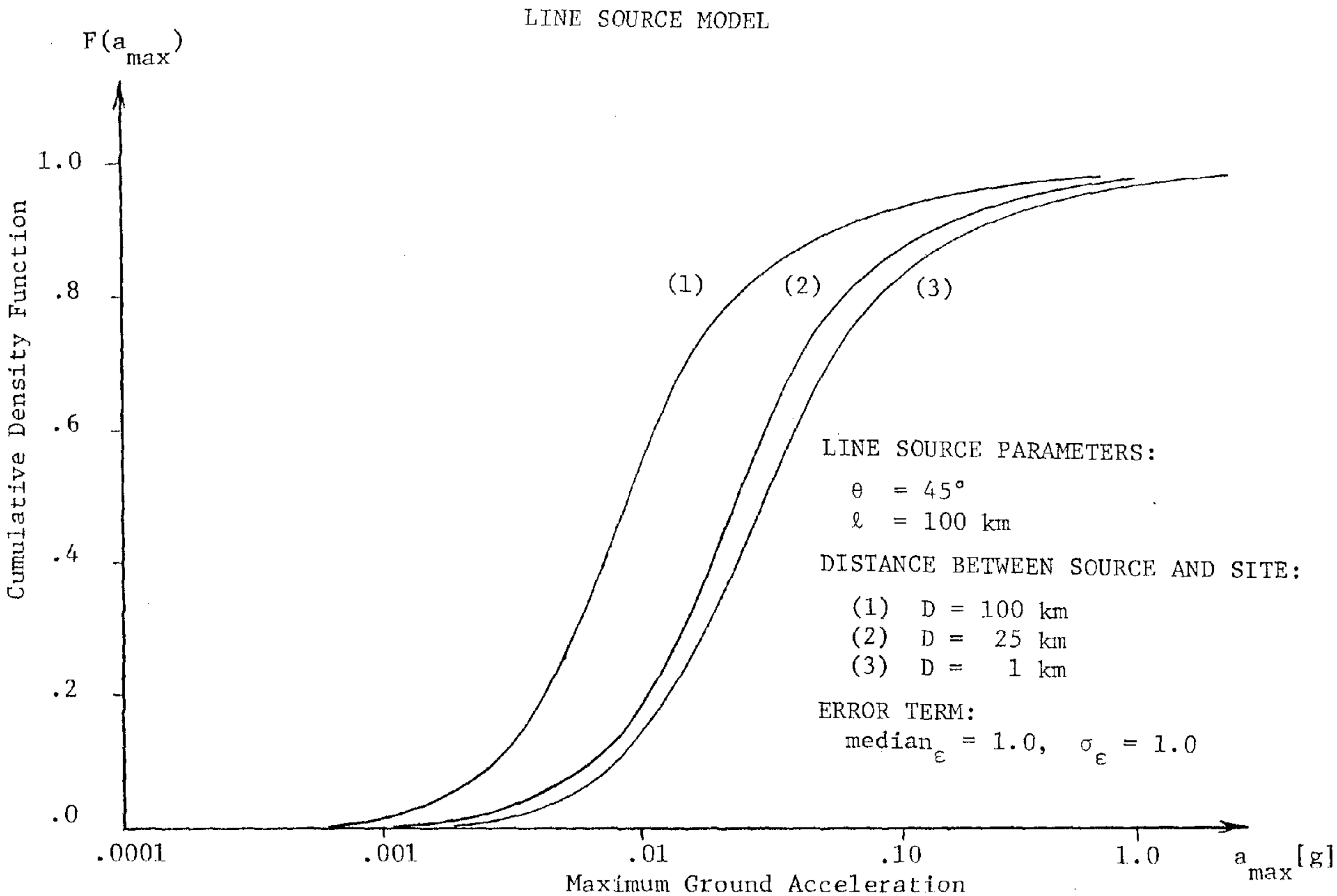


FIGURE 5-10. CUMULATIVE FUNCTION OF GROUND ACCELERATION

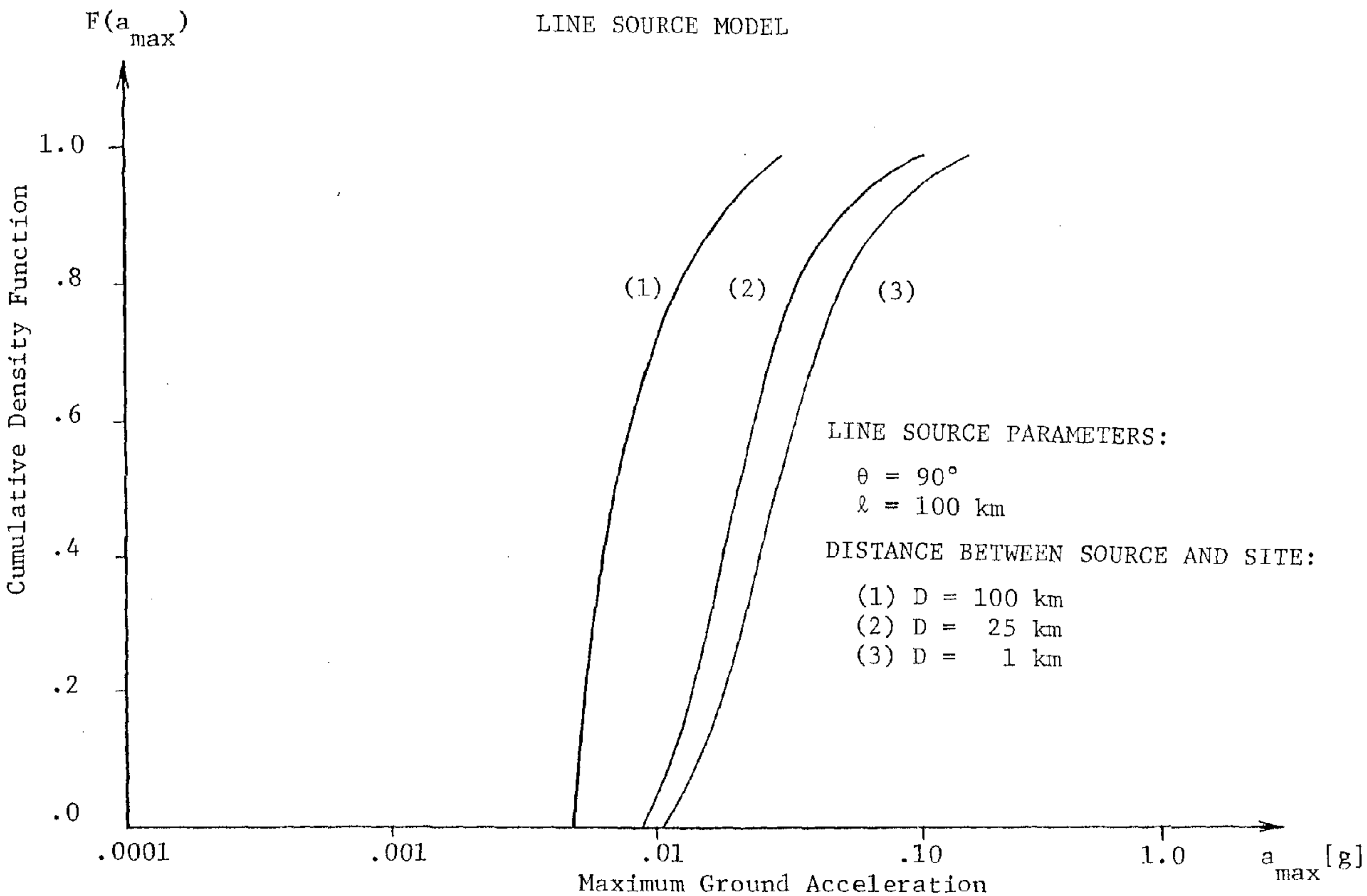


FIGURE 5-11. CUMULATIVE FUNCTION OF GROUND ACCELERATION

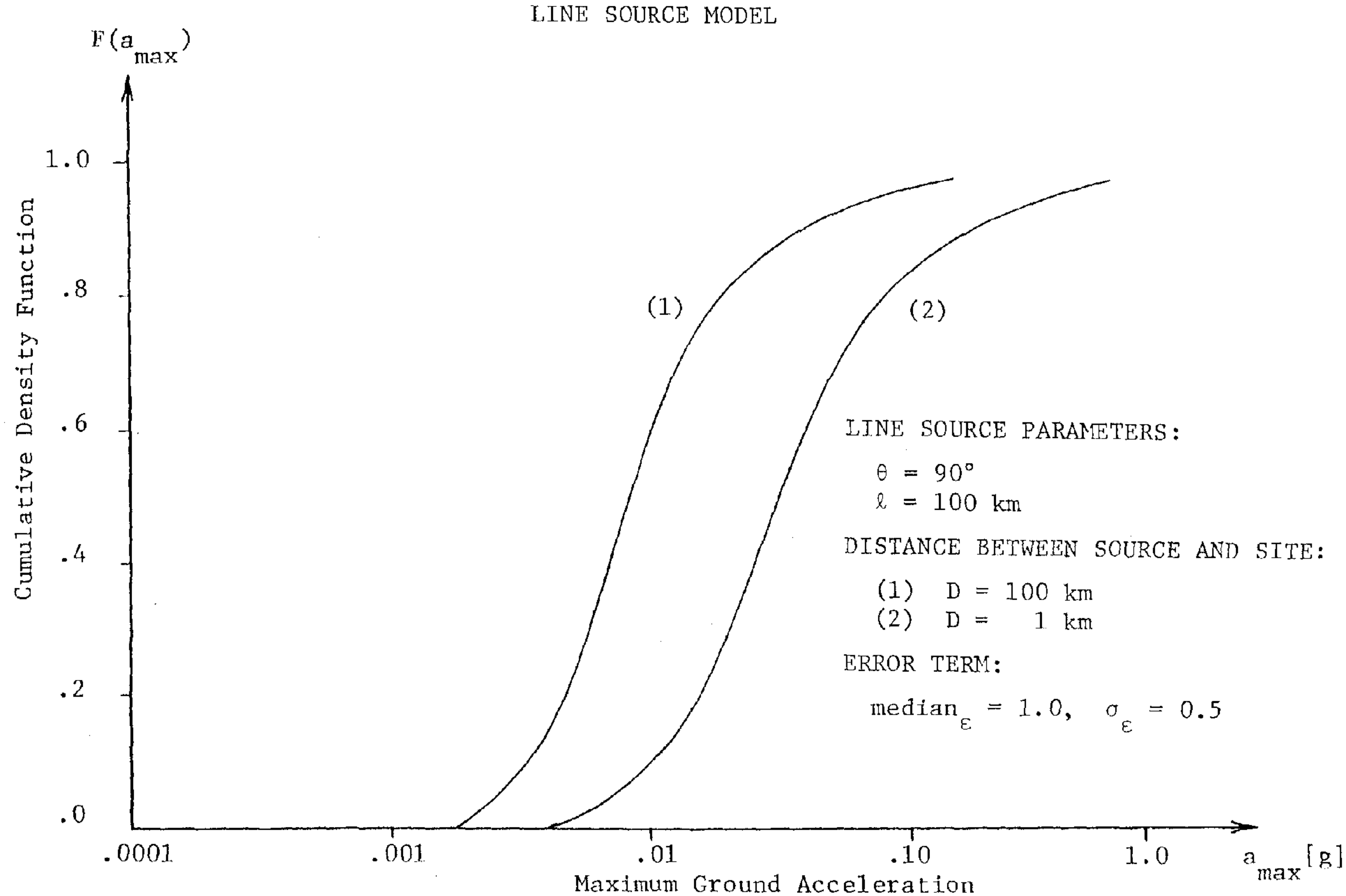


FIGURE 5-12. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

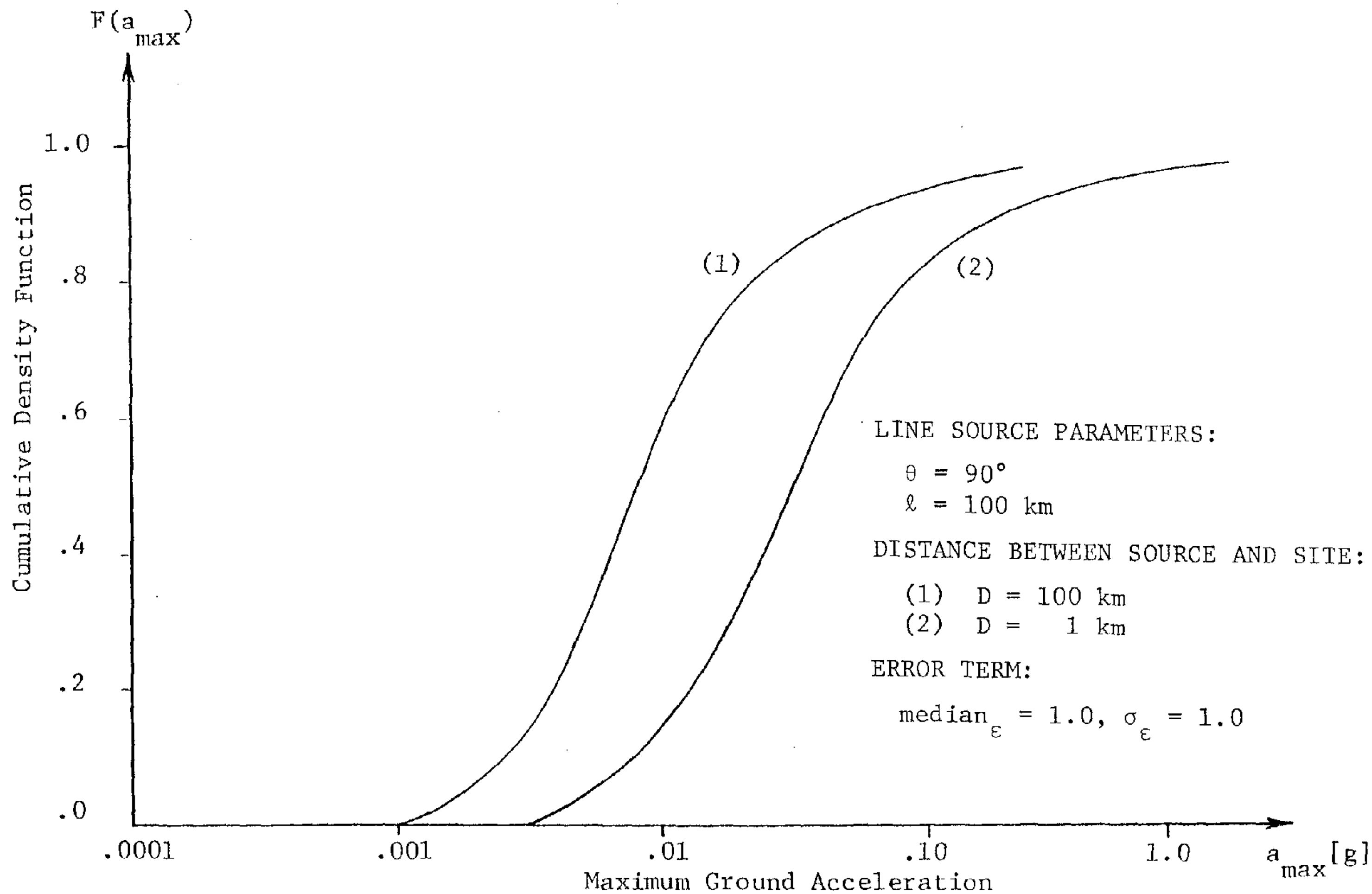


FIGURE 5-13. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

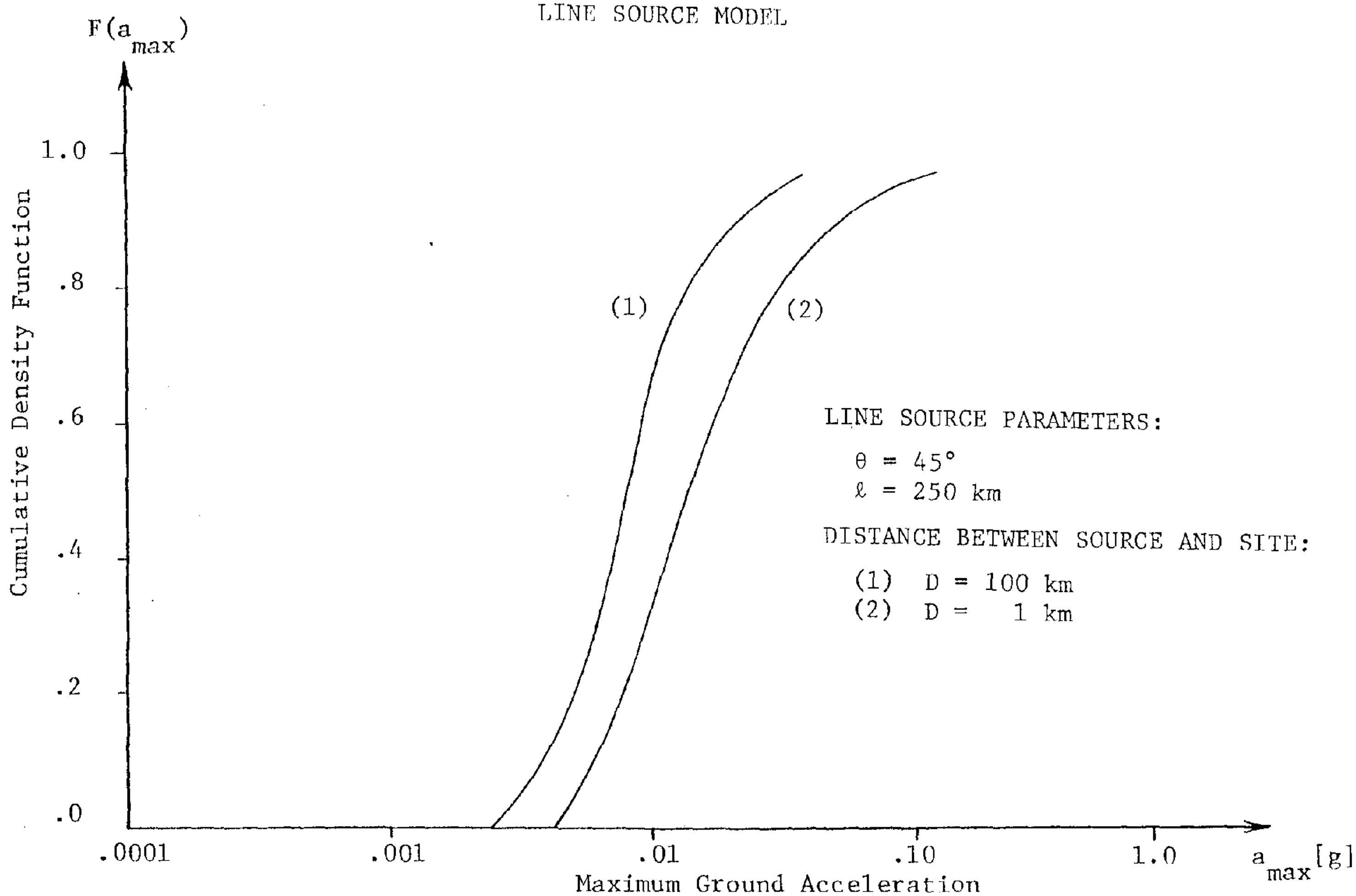


FIGURE 5-14. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

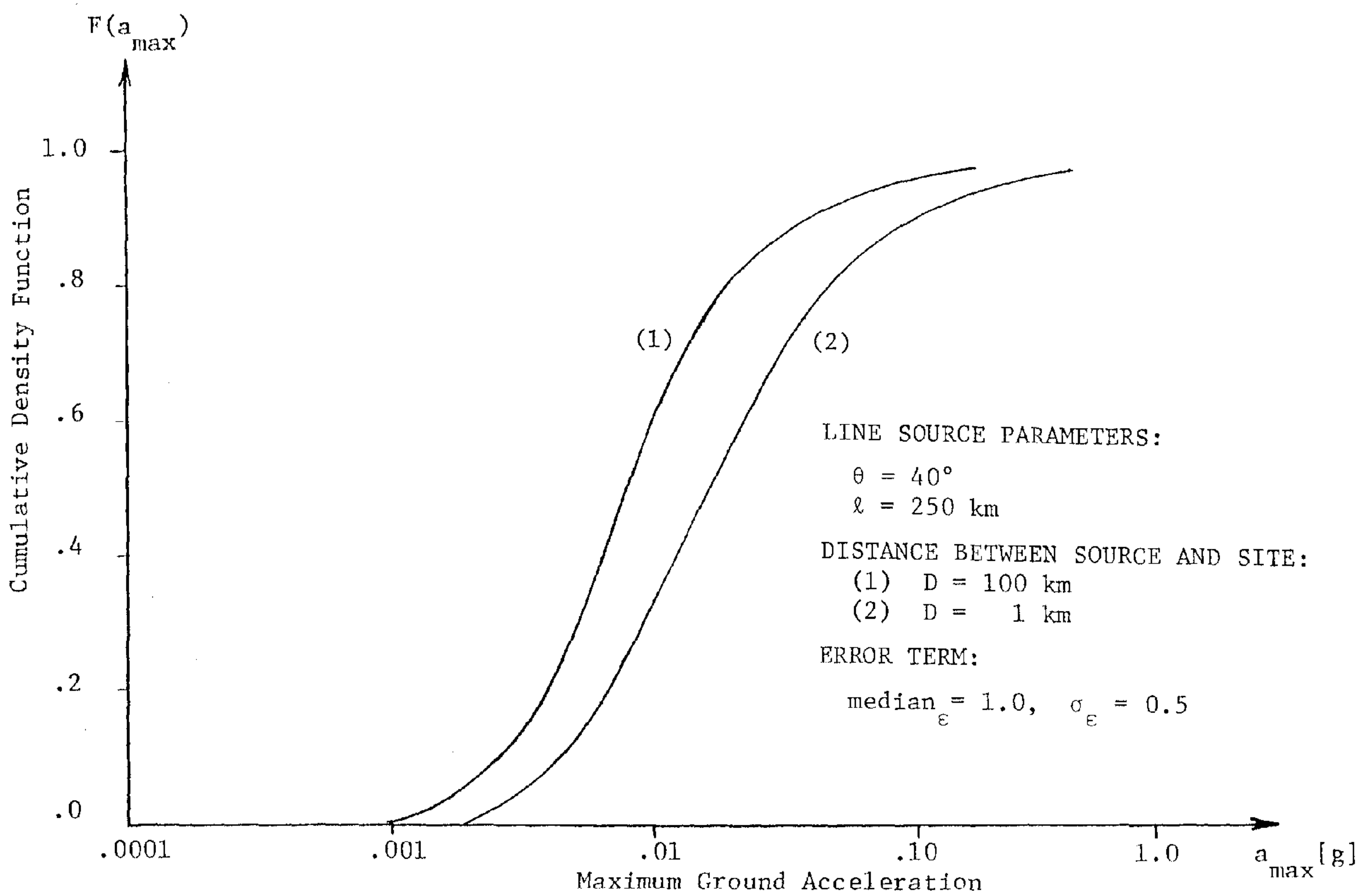


FIGURE 5-15. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

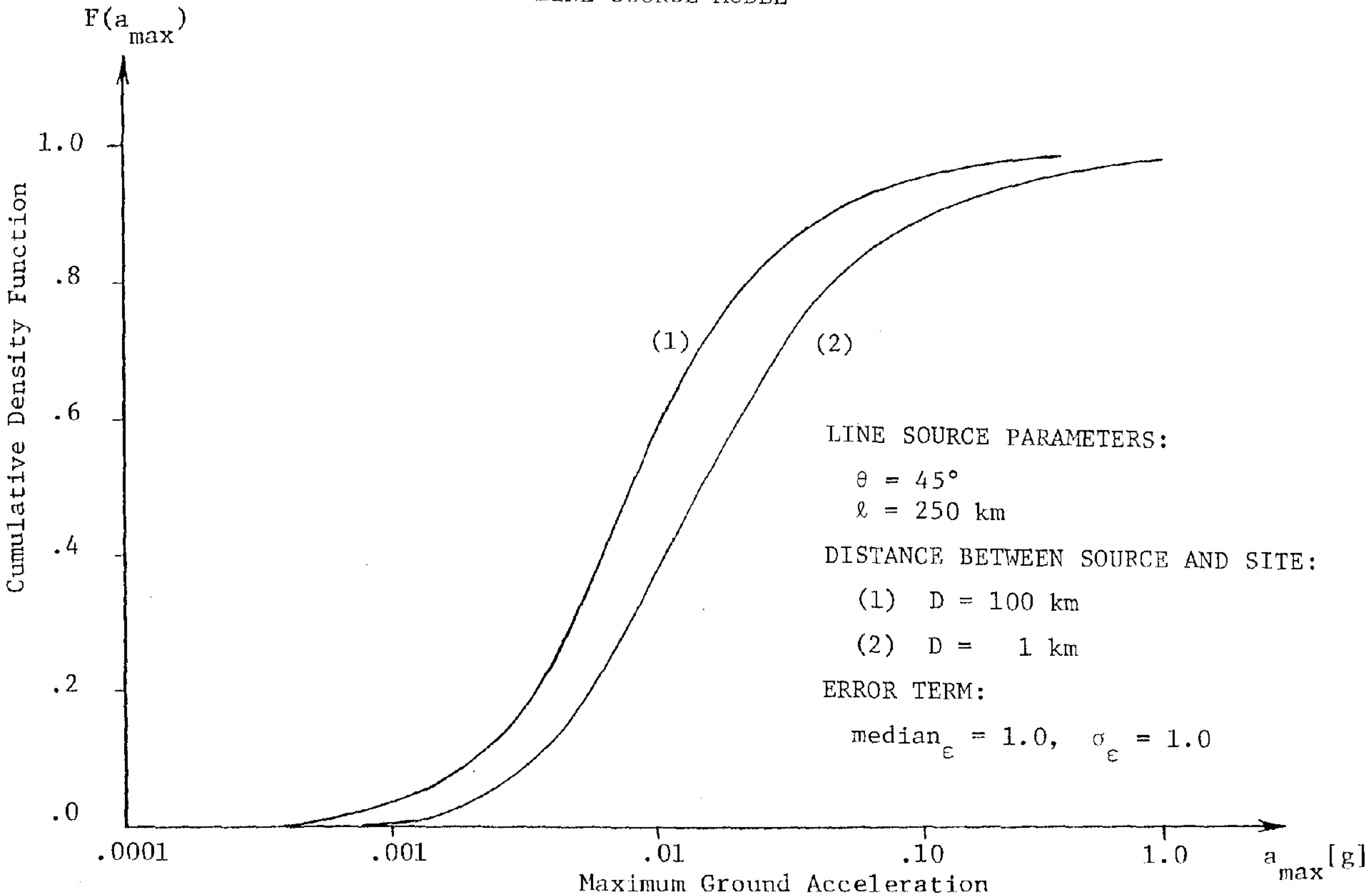


FIGURE 5-16. CUMULATIVE FUNCTION OF GROUND ACCELERATION

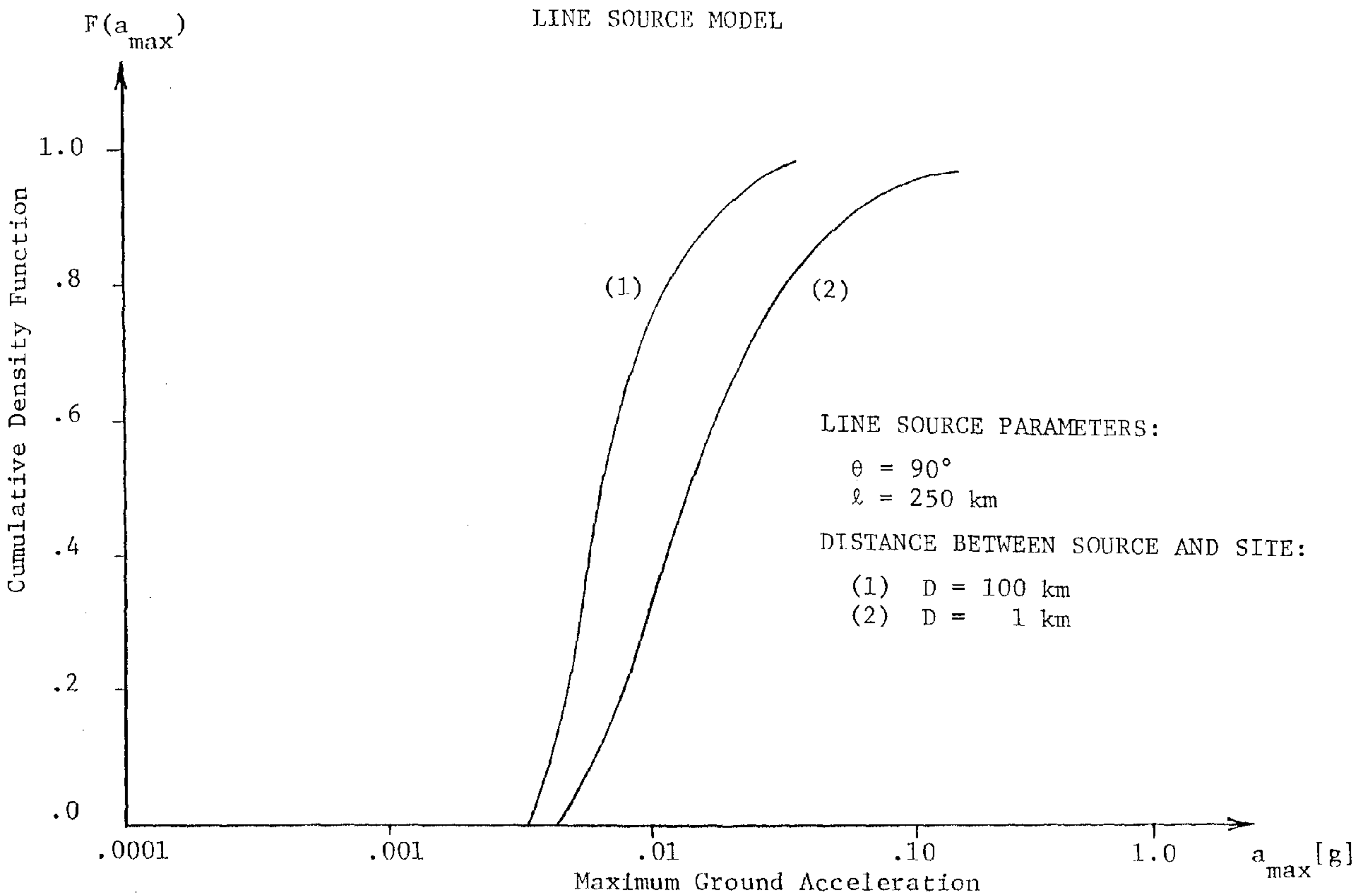


FIGURE 5-17. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MODEL

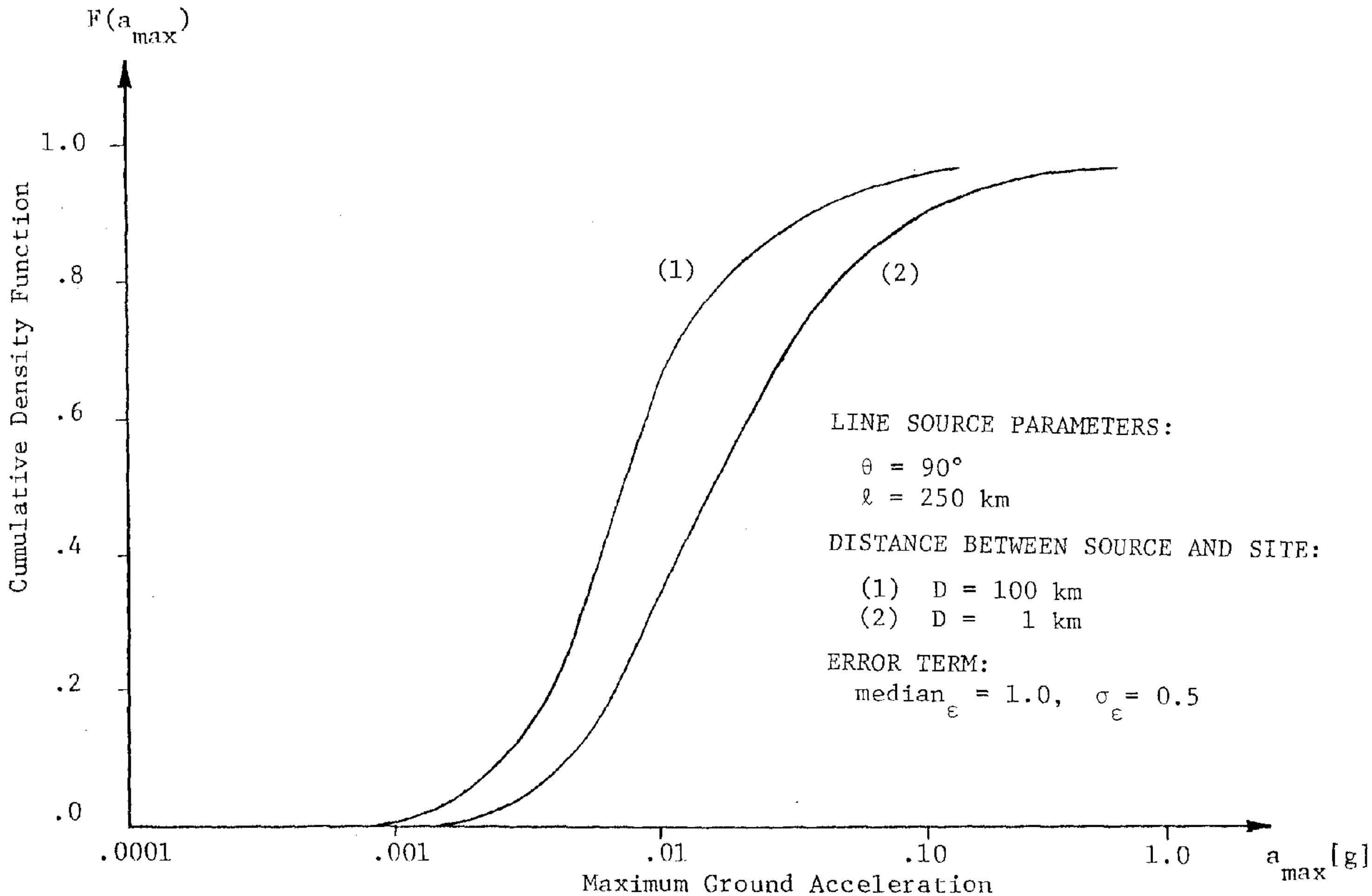


FIGURE 5-18. CUMULATIVE FUNCTION OF GROUND ACCELERATION

LINE SOURCE MCDEL

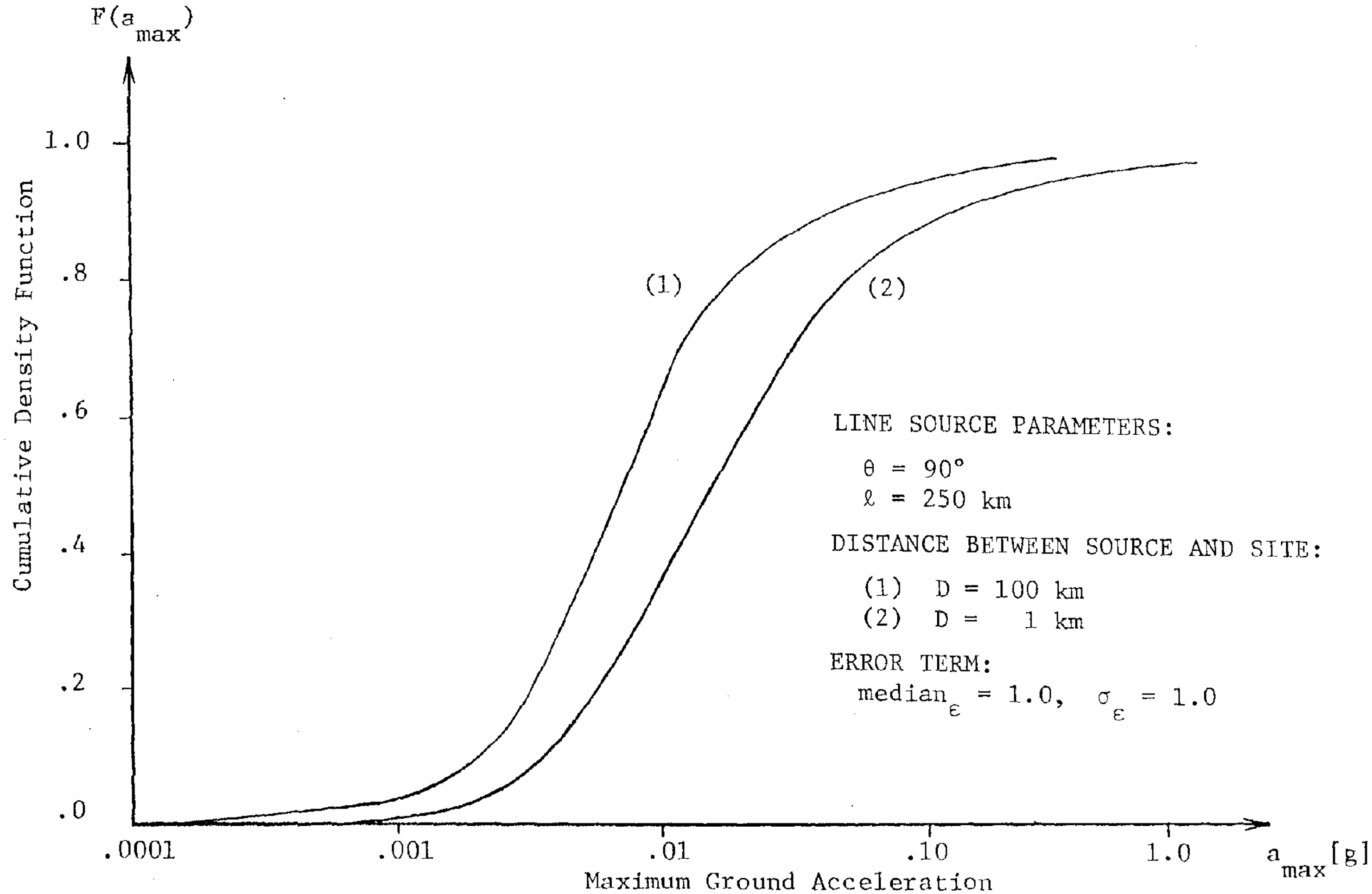


FIGURE 5-19. CUMULATIVE FUNCTION OF GROUND ACCELERATION

orientation angle equal to 90°. Figs. 5-8, 5-11, 5-14 and 5-17 correspond to the case of the deterministic attenuation relationship, given by Eqn. (5-4a); while Figs. 5-9, 5-10, 5-12, 5-13, 5-15, 5-16, 5-18 and 5-19 refer to the attenuation relationship with an error term (having a median equal to 1 and a standard deviation of 0.5 or 1.0). The statistical values of the distributions of a_{max} for the line source are summarized in Tables 5-2 and 5-3.

Area Source:

In Figs. 5-20 and 5-21 are shown the cumulative distribution of a_{max} for depths of the area source equal to 0 km and 20 km, respectively, and for the attenuation relationship given by Eqn. (5-4a).

The case of an attenuation relationship with an error term, Eqn. (5-5a), was also investigated and the results are shown in Figs. 5-22 through 5-25. Finally, in Table 5-4 are listing the statistical values of the maximum accelerations for the case of the area source.

5.3 Conclusions from Parametric Study of Seismic Load

On the basis of the results of this parametric study, the following conclusions are drawn:

(a) From the two attenuation relations examined, Eqn. (4-4a) always results in larger values for the maximum horizontal acceleration.

(b) When an error term is introduced into the expressions for the attenuation relationships, Eqns. (5-5), the corresponding values of a_{max} are larger than those obtained without the error term.

(c) From the results of this study, it appears that the

TABLE 5-2 STATISTICAL VALUES OF MAXIMUM GROUND ACCELERATION (LINE SOURCE MODEL, FAULT LENGTH = 100 km)

ORIENTATION (θ)	ERROR TERM σ_ε	ERROR TERM $\bar{\varepsilon}$	DISTANCE	$\bar{a}_{max}$ (g)	$\sigma_{a_{max}}$ (g)	$V_{a_{max}}$ (g)
45°	0.00	1.00	1	0.0318	0.0231	72.65
	0.00	1.00	10	0.0291	0.0201	69.00
	0.00	1.00	25	0.0231	0.0156	67.63
	0.00	1.00	100	0.0081	0.0046	57.32
	0.50	1.133	1	0.0345	0.0311	90.01
	0.50	1.133	10	0.0323	0.0289	89.37
	0.50	1.133	25	0.0262	0.0230	87.60
	0.50	1.133	100	0.0088	0.0068	77.55
	1.00	1.649	1	0.0370	0.0381	102.88
	1.00	1.649	10	0.0345	0.0354	102.75
	1.00	1.133	25	0.0280	0.0289	103.32
	1.00	1.133	100	0.0094	0.0084	90.37
90°	0.00	1.00	1	0.0315	0.0225	71.41
	0.00	1.00	10	0.0278	0.0193	69.56
	0.00	1.00	25	0.0217	0.0145	67.00
	0.00	1.00	100	0.0079	0.0044	55.37
	0.50	1.133	1	0.0347	0.0321	92.57
	0.50	1.133	10	0.0286	0.0223	77.79
	0.50	1.133	25	0.0238	0.0170	71.19
	0.50	1.133	100	0.0084	0.0058	69.60
	1.00	1.649	1	0.0369	0.0394	106.78
	1.00	1.649	10	0.0305	0.0273	89.48
	1.00	1.649	25	0.0255	0.0210	82.36
	1.00	1.649	100	0.0090	0.0074	82.30

ATTENUATION RELATIONSHIP: $1100\ e^{0.5m}\ (R+25)^{-1.32}\ \varepsilon$

TABLE 5-3 STATISTICAL VALUES OF MAXIMUM GROUND ACCELERATION (LINE SOURCE MODEL, FAULT LENGTH 250 km)

ORIENTATION (θ)	ERROR TERM σ_ε	ERROR TERM $\bar{\varepsilon}$	DISTANCE (km)	$\bar{a}_{max}$ (g)	$\sigma_{a_{max}}$ (g)	$V_{a_{max}}$ (%)
45°	0.00	1.00	1	0.0185	0.0173	93.55
	0.00	1.00	10	0.0184	0.0165	89.81
	0.00	1.00	25	0.0154	0.0124	80.35
	0.00	1.00	100	0.0082	0.0058	71.31
	0.50	1.133	1	0.0204	0.0244	119.80
	0.50	1.133	10	0.0192	0.0238	123.86
	0.50	1.133	25	0.0169	0.0167	98.51
	0.50	1.133	100	0.0088	0.0083	93.90
	1.00	1.649	1	0.0218	0.0285	130.88
	1.00	1.649	10	0.0206	0.0283	136.92
	1.00	1.649	25	0.0181	0.0196	108.29
	1.00	1.649	100	0.0096	0.0108	112.47
90°	0.00	1.00	1	0.0173	0.0160	92.53
	0.00	1.00	10	0.0167	0.0143	85.75
	0.00	1.00	25	0.0146	0.0115	79.03
	0.00	1.00	100	0.0069	0.0041	59.00
	0.50	1.133	1	0.0202	0.0215	106.54
	0.50	1.133	10	0.0195	0.0200	103.02
	0.50	1.133	25	0.0163	0.0163	100.30
	0.50	1.133	100	0.0074	0.0060	81.02
	1.00	1.649	1	0.0215	0.0254	117.67
	1.00	1.649	10	0.0209	0.0245	116.86
	1.00	1.649	25	0.0174	0.0196	112.10
	1.00	1.649	100	0.0078	0.0072	93.05

ATTENUATION RELATIONSHIP : $1100\ e^{0.5m}\ R^{-1.32}\ \varepsilon$

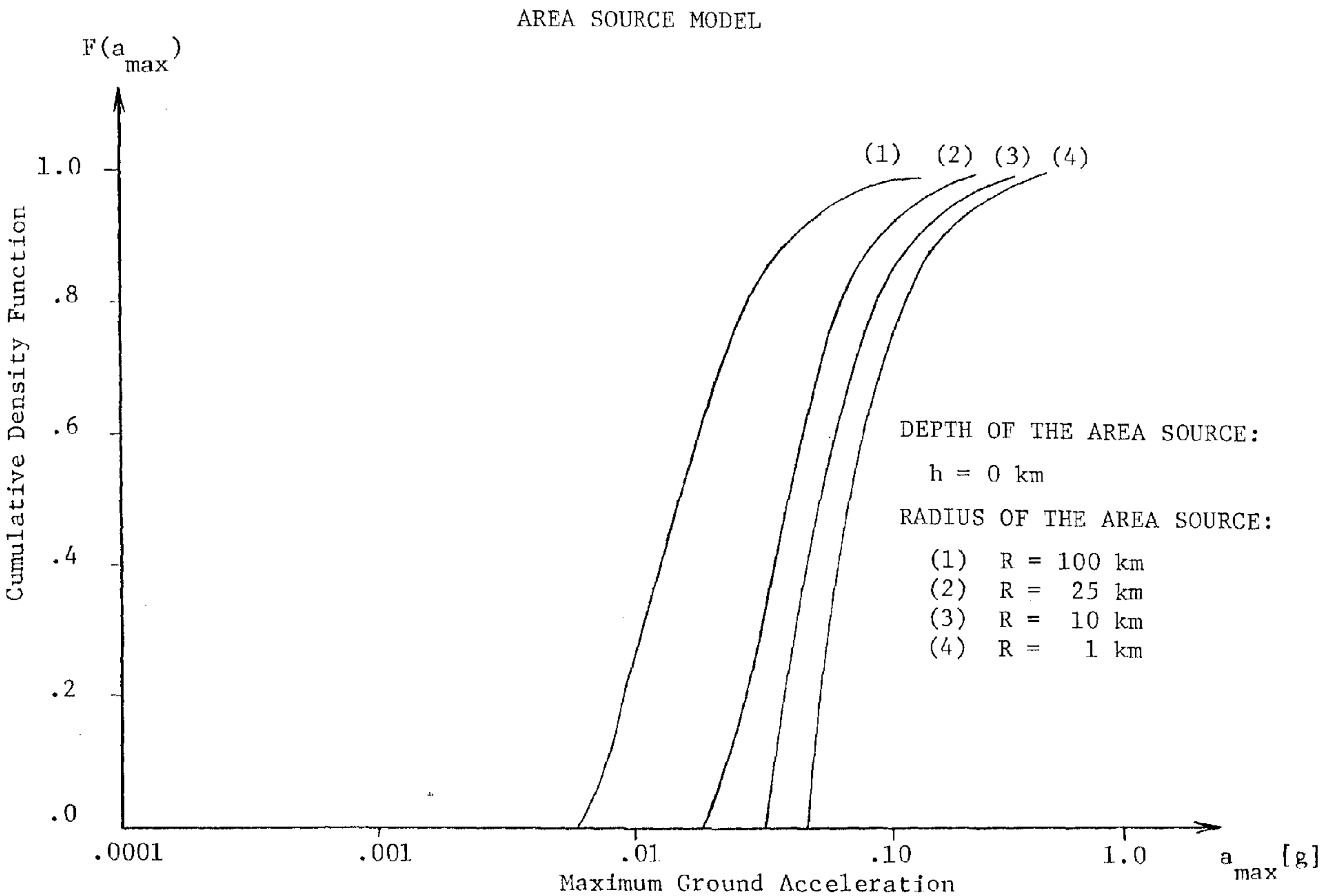


FIGURE 5-20. CUMULATIVE FUNCTION OF GROUND ACCELERATION

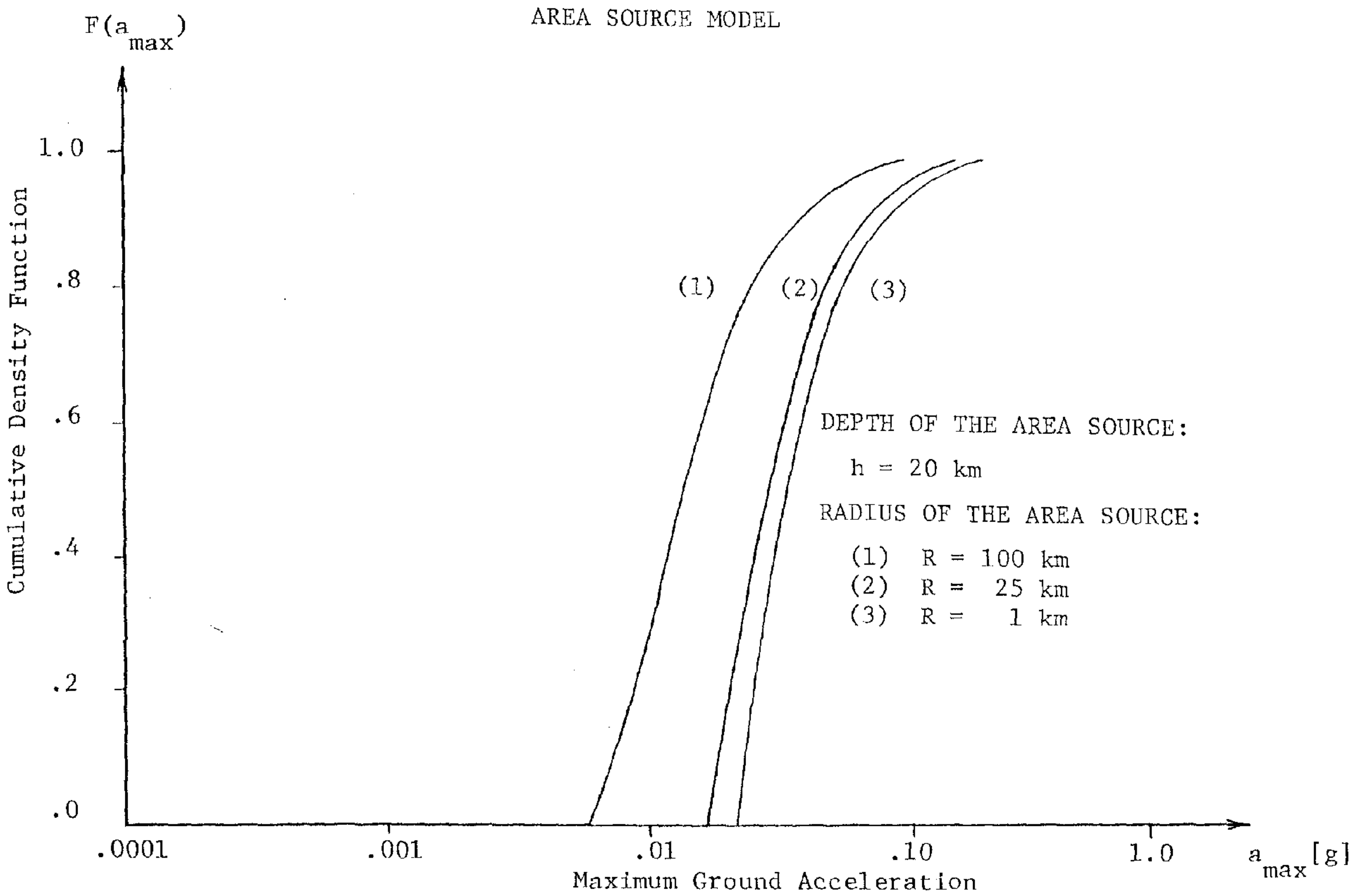


FIGURE 5-21. CUMULATIVE FUNCTION OF GROUND ACCELERATION

AREA SOURCE MODEL

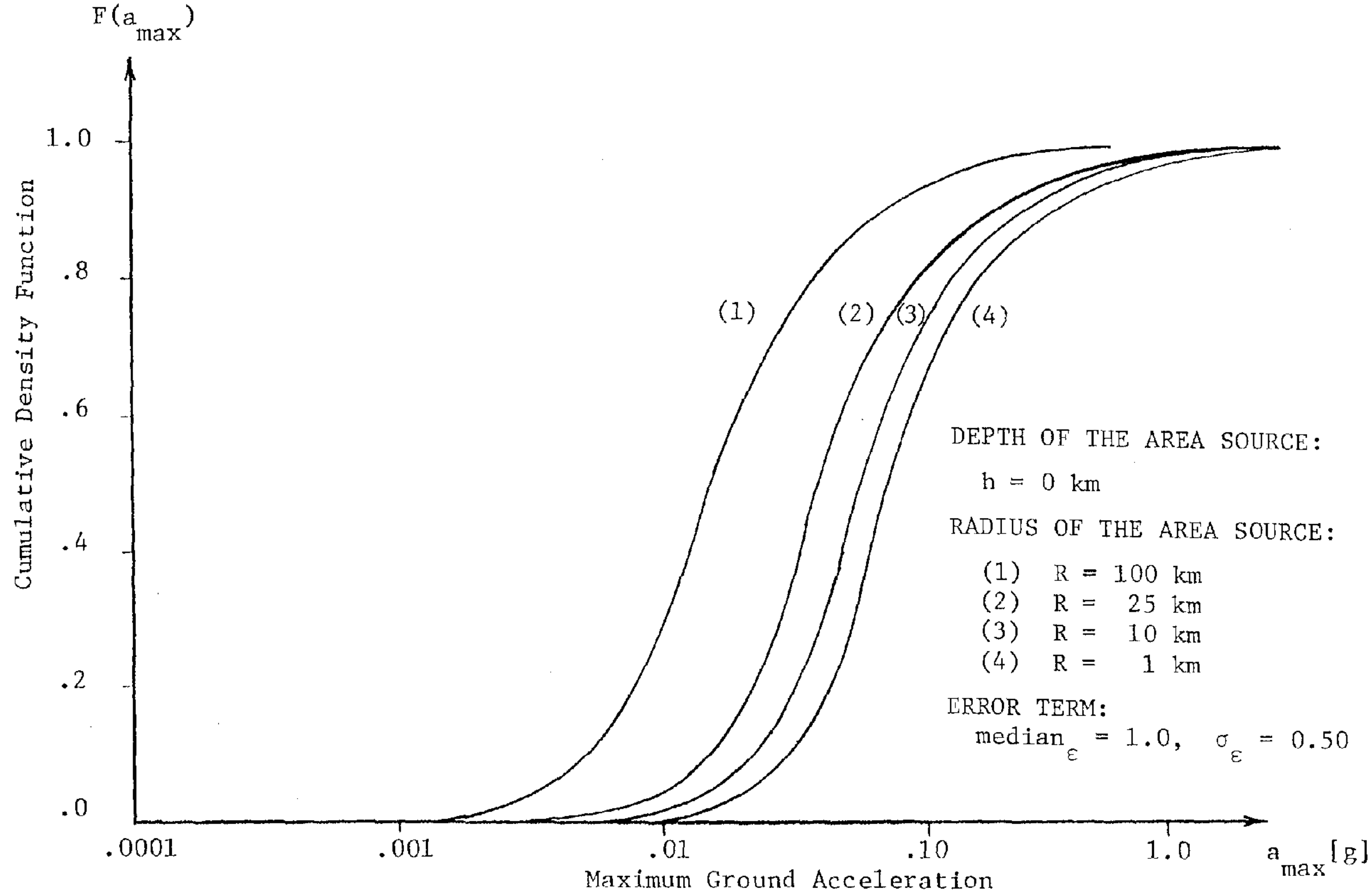


FIGURE 5-22. CUMULATIVE FUNCTION OF GROUND ACCELERATION

AREA SOURCE MODEL

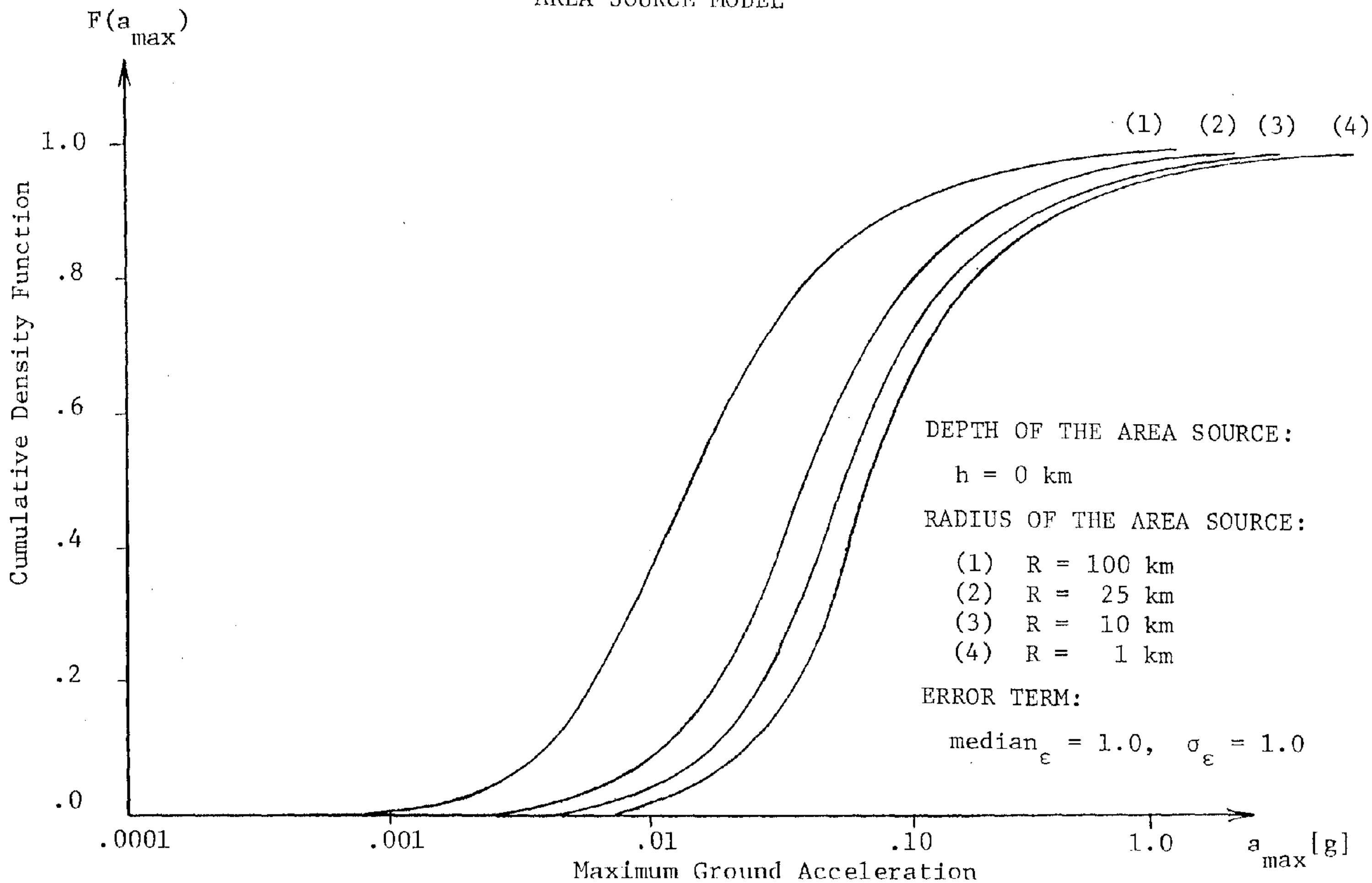


FIGURE 5-23. CUMULATIVE FUNCTION OF GROUND ACCELERATION

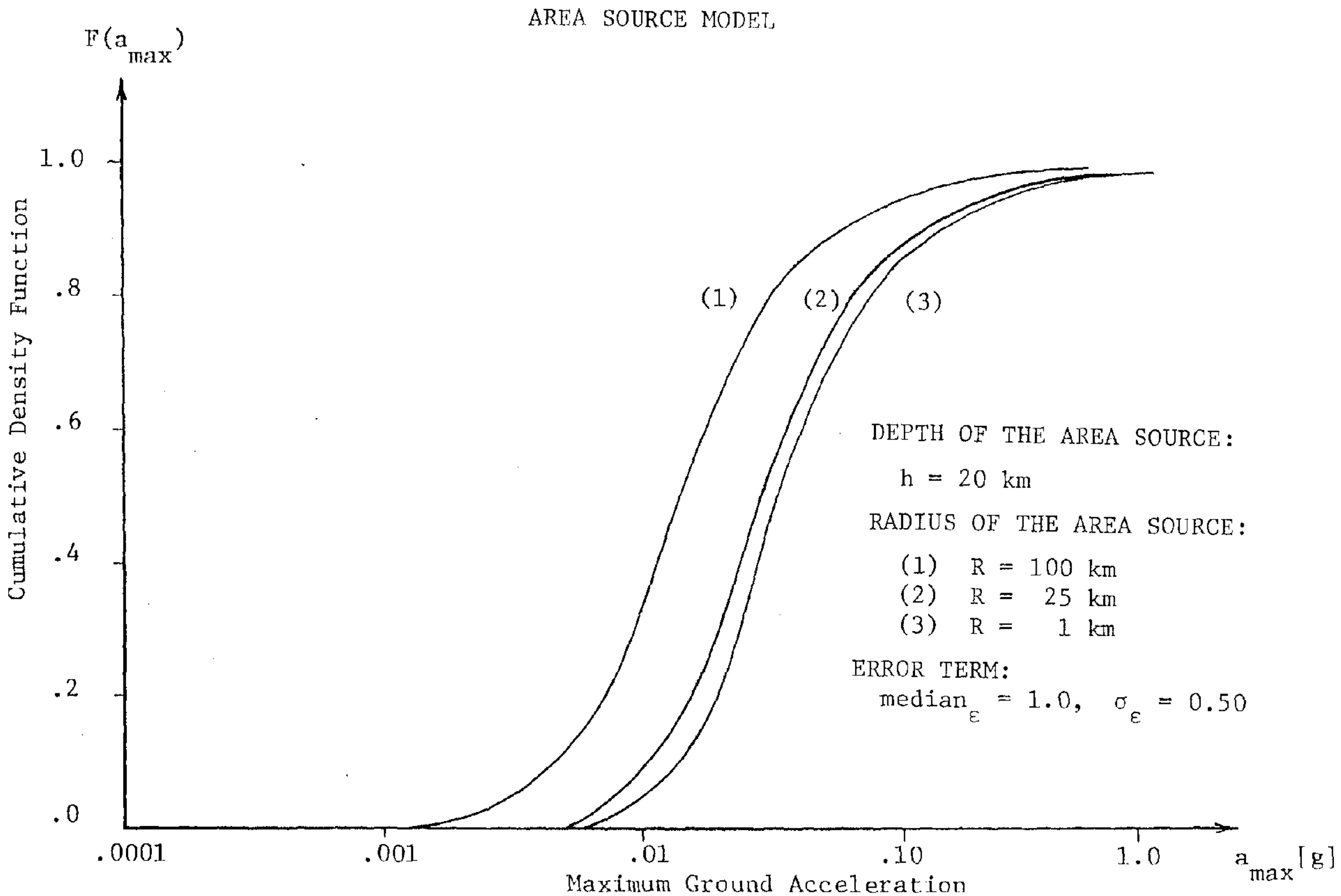


FIGURE 5-24. CUMULATIVE FUNCTION OF GROUND ACCELERATION

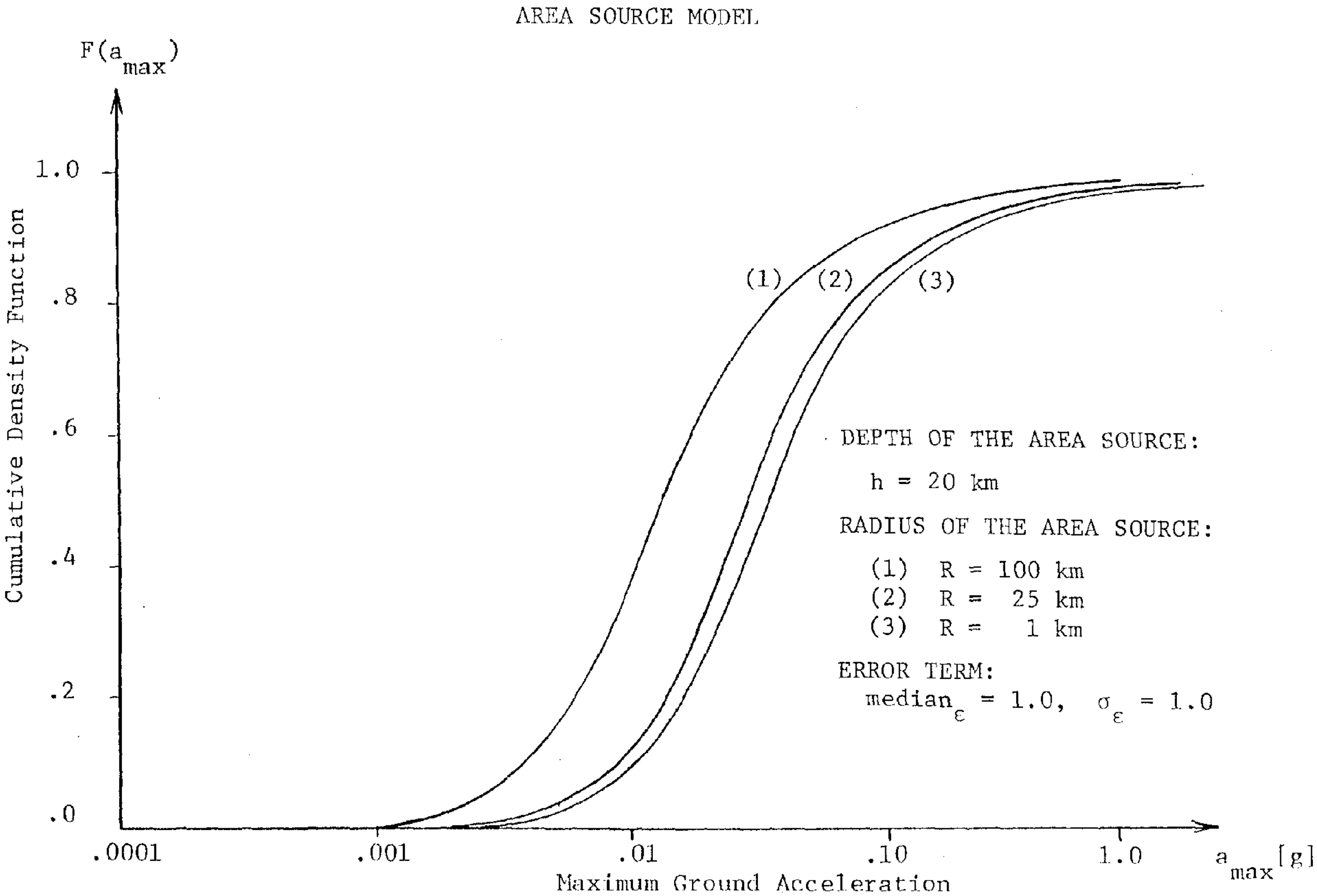


FIGURE 5-25. CUMULATIVE FUNCTION OF GROUND ACCELERATION

TABLE 5-4 STATISTICAL VALUES OF MAXIMUM GROUND ACCELERATION (AREA SOURCE MODEL)

DEPTH	ERROR TERM		RADIUS (km)	$\bar{a}_{max}$ (g)	$\sigma_{a_{max}}$ (g)	$V_{a_{max}}$ (g)
	σ_ε	$\bar{\varepsilon}$				
0.0 km	0.00	1.00	1	0.0663	0.0387	58.33
	0.00	1.00	10	0.0522	0.0314	60.04
	0.00	1.00	25	0.0356	0.0210	58.96
	0.00	1.00	100	0.0142	0.0118	83.27
	0.50	1.133	1	0.0722	0.0560	77.51
	0.50	1.133	10	0.0572	0.0423	73.88
	0.50	1.133	25	0.0396	0.0327	82.61
	0.50	1.133	100	0.0154	0.0160	104.01
	1.00	1.649	1	0.0772	0.0718	92.92
	1.00	1.649	10	0.0614	0.0533	86.64
	1.00	1.649	25	0.0424	0.0400	94.13
	1.00	1.649	100	0.0166	0.0211	126.87
20 km	0.00	1.00	1	0.0027	0.0059	222.26
	0.00	1.00	10	0.0025	0.0044	177.68
	0.00	1.00	25	0.0022	0.0048	215.72
	0.00	1.00	100	0.0008	0.0015	189.33
	0.50	1.133	1	0.0028	0.0054	193.08
	0.50	1.133	10	0.0027	0.0067	254.07
	0.50	1.133	25	0.0023	0.0062	276.31
	0.50	1.133	100	0.0009	0.0024	253.69
	1.00	1.649	1	0.0029	0.0059	201.58
	1.00	1.649	10	0.0028	0.0081	282.44
	1.00	1.649	25	0.0024	0.0080	326.11
	1.00	1.649	100	0.0010	0.0027	270.09

ATTENUATION RELATIONSHIP: $1100\ e^{0.5m}\ R^{-1.32}\ \varepsilon$

orientation (angle θ) of the line source has little effect on the magnitude of the maximum horizontal acceleration.

(d) The length of the fault also has little influence on a_{max}, for a distance between the source and site of the slope approaching 100 km.

(e) For faults closer to the site, the maximum accelerations decrease as the fault length increases.

6. EVALUATION OF THE DEVELOPED MODEL

6.1 Overview of the Model

A new approach has been proposed in this study for the determination of the reliability of earth slopes during earthquakes. The developed method provides a probabilistic, pseudo-static, seismic stability analysis. The safety of slopes is measured in terms of the probability of failure (rather than the customary safety factor), the numerical values of which are determined through a Monte Carlo simulation of failure.

Significant uncertainties that are present in conventional methods of analysis have been identified and probabilistic tools have been introduced for their description and amelioration. More specifically, the developed procedure accounts for (a) the variability in the numerical values of the material strength parameters, c and ϕ; (b) the uncertainty in the exact location of potential failure surfaces within the slope mass; and (c) the uncertainty in the load imposed upon a slope during an earthquake.

The statistical description of the random variables introduced in the model is consistent with recent findings on the subject. Thus, the variability of soil strength parameters is described utilizing and expanding upon results available in the literature, in particular those reported by Lumb (1966, 1970) and Harr (1977). The parameters used to describe the location of the failure surfaces are not directly measurable and have to be estimated empirically. Here, previous experience with the logarithmic failure surfaces is utilized as well as observations on recorded failures (Frohlich, 1953; A-Grivas, 1976). The probability density function of the horizontal ground acceleration is determined on the basis of (a) an analysis of seismic data in the area of interest, yielding the earthquake frequency-magnitude relationship, and (b) an attenuation equation for the peak ground acceleration.

6.2 Comparison with Other Methods

Conventionally, limit equilibrium methods of slope stability analysis involve an iterative (trial-and-error) procedure in which the minimum factor of safety F_S is sought by varying the surface (usually a circular arc) along which failure may occur. Thus, minimization of F_S is achieved through the use of some searching technique and the surface that corresponds to the minimum value of F_S is considered as critical.

More recently, a variational approach was used to minimize F_S and to determine the critical failure surface (e.g., Baker and Garber, 1977). The resulting shape for the latter was that of a logarithmic spiral, similar to the one adopted in the present model.

In general, there is a large number of surfaces along which slope failure is possible. If p_{f_i} represents the probability of failure along a particular surface S_i, i = 1,2,...,N, then the total probability of failure p_f of the slope is equal to

$$p_f = \sum_{i=1}^{N} p_{f_i} \, P[s_i] \tag{6-1}$$

in which $P[S_i]$ is the probability of occurrence of each surface S_i and $\sum_{i=1}^{N} P[s_i] = 1$. Implied in Eqn. (6-1) is that surfaces S_i,

$i=1,2,...,N$, are independent of one another.

The case where a slope may fail along an infinite number of correlated failure surfaces was studied by Catalan and Cornell (1976). The authors pursued a reliability analysis of soil slopes by transforming the problem of a series system with an infinite number of components into a more tractable level-crossing one. Thus, the probability of failure was found by determining the expected number of threshold crossings while assuming that (a) such crossings are rare Poisson events, and (b) the process is Gaussian and stationary in the vicinity of the least reliable failure surface. Failure surfaces were assumed to be circular in shape along which the shear strength of soil was taken to be constant (i.e., conditions corresponding to an undrained analysis).

The method of selecting failure surfaces that was used in the present study was based on the observation that the majority of possible surfaces contribute very little to the summation appearing in Eqn. (6-1); i.e., they have a relatively small probability of occurrence ($P\{S_i\} \simeq 0$). From examinations of recorded slope failures, a critical region was identified within the slope mass and a method was developed for the random generation of potential failure surfaces that lie within this region (Report No. CE-78-5).

An alternative approach to the determination of the probability of failure may be pursued on the basis of one single failure surface, namely that with the highest probability of occurrence. Such an approach does not account for the uncertainty around the exact location of the failure surface and ignores the contribution on the probability of failure of other possible but less probable surfaces. Such a study has been pursued by Alonso (1976), who demonstrated that the critical surface found through a conventional analysis coincides with the one along which the probability of failure receives its maximum value. Also, studies based on a single failure surface (of circular shape) were performed by Wu and Kraft (1970), Gilbert (1977) and Tang, Yuceman and Ang (1976) for undrained conditions; and Yuceman and Tang (1975) and D'Andrea (1979) for long-term (effective stress) conditions. The last two studies were based on a first-order second-moment analysis (Cornell, 1971) of the ordinary method of slices.

The probability of failure for fixed failure surfaces of either circular or exponential shape can be also evaluated using the method developed in this study. This is listed as one of the features of the RASSUEL program (Report No. CE-78-6). In this case, the Monte Carlo simulation is performed by generating random values for only the strength and seismic parameters.

6.3 Limitations of the Model

A main feature of the developed model was the assumption that the soil mass comprising the slope was statistically homogeneous. Thus, soil strength parameters were assumed to be random variables having the same statistical values (i.e., mean value, standard deviation, upper and lower limits and probability density functions) anywhere within the soil medium. This is a reasonable first step beyond the conventional limit equilibrium methods of analysis, in which only a point estimate of the driving forces and resisting forces (or, moments) is considered.

As part of this study, an attempt was made to introduce the spatial variablility of soil strength during the reliability analysis of slopes under both static and seismic conditions. Use was made of the autocorrelation coefficient of the undrained strength of soil slopes ("$\phi = 0$" analysis) and the results were reported in the form of a technical paper (Asaoka and A-Grivas, 1981). An extension of the method for conditions applicable to the long-term (effective stress) stability analysis of slopes is expected to take place during a further development of the model.

A second limitation of the model is that associated with the assumption of constant soil strength during an earthquake. This is believed to be reasonable for a wide variety of soils, particularly cohesive ones (Arango and Seed, 1974). The approximation made above is not directly applicable to the analysis of slopes composed of soils that exhibit shear strength degradation during cyclic loading. This could be the case, for example, of liquefaction of saturated sands or very sensitive clays (Seed, 1968). The applicability and limitations imposed by the above assumption have been recognized by other researchers on the subject:

> "Because of difficulties, it is not at the present time possible to make an accurate determination of the behavior of soils (cohesionless soils and sensitive clays) which are susceptible to liquifaction-like phenomena ... Fortunately, not all soils are susceptible to such phenomena. For many soils, the resistance to shear is largely unaffected by repeated cycles of loading." (Whitman, 1970).

In the present pseudo-static stability analysis, the force system is considered to be applied statically on the slope and its seismic component is measured in terms of the maximum acceleration experienced at the base. That is, the slope mass is assumed to be a rigid body and, thus, the acceleration anywhere within the slope is the same as that along its base.

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Appendix A. USE OF THE BIVARIATE BETA DISTRIBUTION TO MODEL SOIL STRENGTH PARAMETERS

Two important characteristics of the bivariate beta distribution f(x,y) of two normalized random variables x and y are as follows:

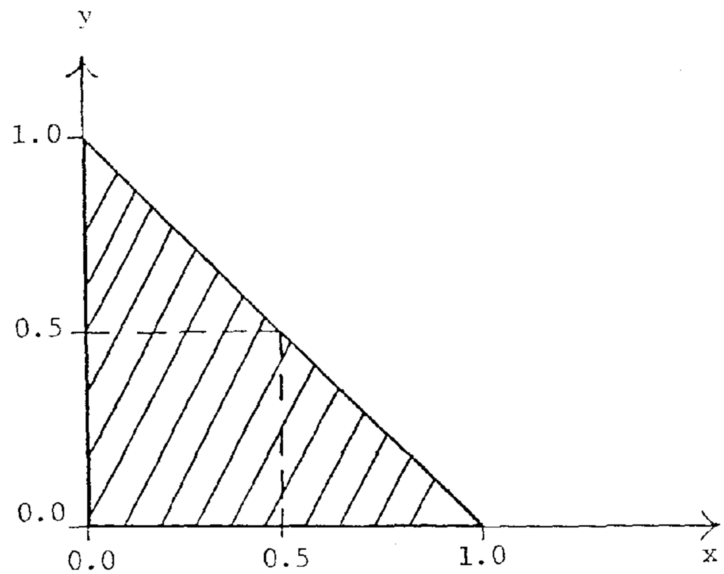
(a) The bivariate beta model has only three parameters, denoted as α, β and γ, which must be determined from the five statistical values of x and y; namely, two mean values ($\bar{x}$, $\bar{y}$), two variances (σ_x^2 , σ_y^2) and a covariance (σ_{xy}).

(b) The bivariate beta distribution, when expressed as (Wilks, 1962)

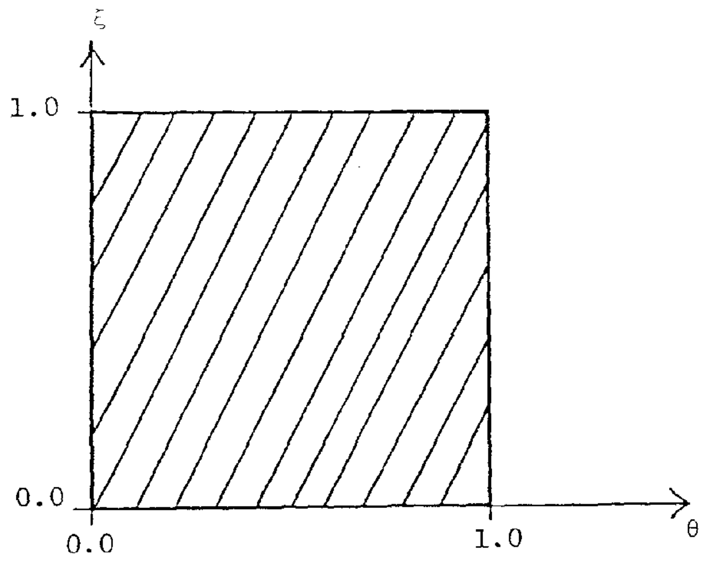
$$f(x,y) = \frac{\Gamma(\alpha,\beta,\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha)} \; x^{\alpha-1} \; y^{\beta-1} (1-x-y)^{\gamma-1} \qquad \text{(A-1)}$$

in which $0 \leqslant x,y \leqslant 1$, and $x + y \leqslant 1$, is defined over a triangular region (shown schematically in Fig. A-1a).

The five statistical values ($\bar{x}$, $\bar{y}$, σ_x^2, σ_y^2 and σ_{xy}) of f(x,y), given in Eqn. (A-1), can be expressed as the following functions of the parameters α, β and γ:



(a) Triangular Region of x and y



(b) Rectangular Region of ξ and θ

FIGURE A-1. REGION OF DEFINITION OF THE TWO VARIABLES OF THE BIVARIATE BETA MODEL

$$\bar{x} = \frac{\alpha}{\Sigma} \qquad \sigma_x^2 = \frac{\alpha(\beta+\gamma)}{\Sigma^2(\Sigma+1)}$$

$$\bar{y} = \frac{\beta}{\Sigma} \qquad \sigma_y^2 = \frac{\beta(\alpha+\gamma)}{\Sigma^2(\Sigma+1)} \tag{A-2}$$

$$\sigma_{xy} = -\frac{\alpha\beta}{\Sigma^2(\Sigma+1)}$$

in which $\Sigma = \alpha + \beta + \gamma$. From the expressions for σ_{xy}, σ_x and σ_y the correlation coefficient ρ of x and y can be obtained as

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{-\alpha\beta}{\sqrt{\alpha\beta(\alpha+\gamma)(\beta+\gamma)}} = -\sqrt{\frac{\alpha\beta}{(\alpha+\gamma)(\beta+\gamma)}} \tag{A-3}$$

The bivariate beta distribution as a model for the correlated soil strength parameters c and ϕ was first introduced by A-Grivas and Harrop-Williams (1979). In order to satisfy the conditions associated with Eqn. (A-1), the authors used normalized expressions of c and ϕ receiving values between 0 and 0.5. Parameters α, β and γ were determined from the expressions corresponding to the two mean values ($\bar{x}$, $\bar{y}$) and the variance (σ_{xy}), given in Eqns. (A-2). Such an approach does not insure that the two individual variances (σ_x^2, σ_y^2) and the correlation coefficient (ρ) are always preserved.

There are several alternative ways to select the three parameters of the bivariate beta model from the five statistical values of x and y, given in Eqn. (A-2). Five such approaches are presented below.

(a) <u>Transformation to Rectangular Variables</u>

The first approach involves a transformation of the triangular region within which x and y are defined (Fig. A-1a) to a rectangular region. If θ and ξ denote the two new variables, the transformation necessary is as follows (Wilks, 1962):

$$x = \theta, \; y = \xi\,(1 - \theta) \qquad \text{(A-4a)}$$

or,

$$\theta = x, \; \xi = y/(1 - x) \qquad \text{(A-4b)}$$

in which $0 \leqslant \theta, \xi \leqslant 1$. When the above expression are introduced into Eqn. (A-1), the joint distribution $f(\theta, \xi)$ of θ and ξ becomes

$$f(\theta,\xi) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\, \theta^{\alpha-1}(1-\theta)^{\beta+\gamma-1}\xi^{\beta-1}(1-\xi)^{\gamma-1} \qquad \text{(A-5)}$$

In Fig. A-1b is shown schematically the region of definition of the transformed variables θ and ξ.

From Eqn. (A-5), one has that $f(\theta, \xi)$ is the product of two separate density functions, one for θ and, another, for ξ, i.e., the new variables θ and ξ are uncorrelated.

(b) Shift Transformation

A second approach is to determine the three parameters α, β and γ of the bivariate model from the three higher moments of x and y, namely the two variances σ_x^2 and σ_y^2 and the covariance σ_{xy}. Thus, from Eqns. (A-2), one has

$$\beta = [-c_2/(\rho s_x s_y (c_1+c_2+1)^2)-1] \;/\; (c_1+c_2+1)$$

$$\alpha = c_2 \cdot \beta \qquad \text{(A-6)}$$

$$\gamma = c_1 \cdot \beta$$

in which the constants c_1, c_2, s_x and s_y appearing in the above expressions are

$$c_1 = -s_x/(\rho s_y) - 1$$

$$c_2 = c_1/[-s_y/(\rho s_x)-1]$$

$$s_x = \sigma_c/(c_{max}-c_{min})$$

$$s_y = \sigma_\phi/(\phi_{max}-\phi_{min})$$

The mean values of x and y obtained from the two marginal distributions f(x) and f(y), are in general different from the given values $\bar{x}$ and $\bar{y}$. To make the two pairs of mean values coincide, the two marginal distributions must be shifted by an appropriate amount. This is equal to the difference between the specified mean values and the mean values calculated from Eqns. (A-2), using parameters α, β and γ from Eqns. (A-6). These two differences called the "shift factors" (one for each of the two variables x and y) are then added to the values of the bivariate beta distribution.

As such a transformation involves only additive constants, the two variances σ_x^2 and σ_y^2 and the covariance σ_{xy} and, therefore the correlation coefficient remain unchanged.

In Fig. A-2 is shown schematically the region of definition of the shifted beta variates x_s and y_s. The two shift factors are denoted as dx and dy. From Fig. A-2, it can be seen that shifting the distribution to match the mean values, results to new limits for the two variables (dx, 1 + dx; dy, 1 + dy). The introduced additional regions of variation of x and y are shown as shaded areas in Fig. A-2. The solid line represents the extent of the bivariate beta distribution of x and y which, of course, covers the range $0 \leqslant x,y \leqslant 1$ and $x + y \leqslant 1$.

Normalizing the strength parameters with the following equations

$$x = c_n = \frac{c - c_{min}}{c_{max} - c_{min}}$$
$$y = \phi_n = \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}} \quad \text{(A-7)}$$

0 and 1 on Fig. A-2 correspond to the normalized minimum and maximum values of c and ϕ specified from the data. It can be seen that in shifting the distribution to match the mean values, the limits of the variables have not been preserved.

(c) Scale Transformation

An alternative to the above method involves the use of two scaling factors. These factors, k_x and k_y are multiplied in the denominators of Eqns. (A-7). Noting that the normalized σ_c, σ_ϕ, $\sigma_{c\phi}$ are now functions of k_x and k_y, α, β, γ, k_x and k_y may be calculated as

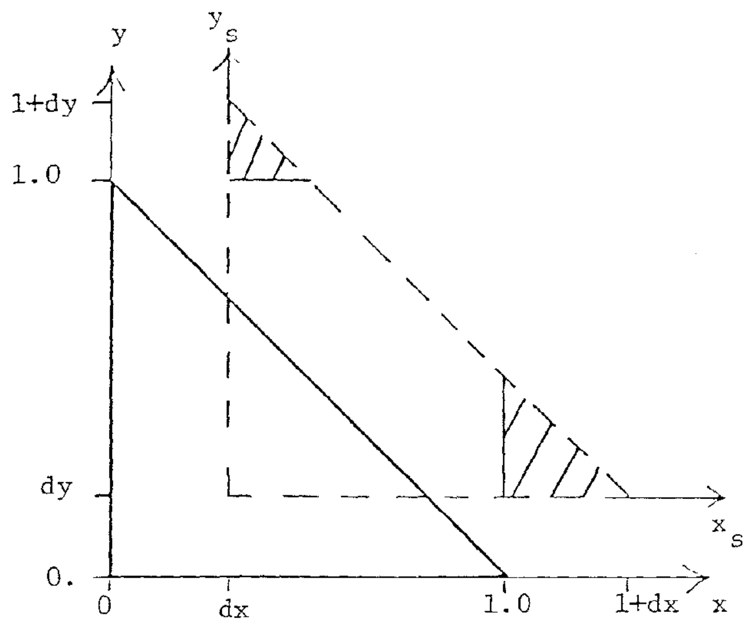


FIGURE A-2. LINEARLY SHIFTED BIVARIATE BETA DISTRIBUTION

$$\beta = [c_4 \cdot DENX \cdot DENY/(\rho \cdot s_x \cdot s_y \cdot (c_3+c_4+1)^2)-1]/(c_3+c_4+1)$$

$$\gamma = c_3 \cdot \beta$$

$$\alpha = c_4 \cdot \beta \qquad \text{(A-8)}$$

$$k_x = DENX/(c_{max}-c_{min})$$

$$k_y = DENY/(\phi_{max}-\phi_{min})$$

where

$$c_1 = (\bar{c}-c_{min})/(\bar{\phi}-\phi_{min})$$

$$c_2 = \left(\frac{1-c_1 \cdot s_y}{\rho \cdot s_x}\right)\left(\frac{\rho \cdot s_y}{c_1-s_x}\right)$$

$$c_3 = -(s_x \cdot c_2)/(s_y \cdot \rho)-1$$

$$c_4 = -c_3/[s_y/(c_2 \cdot \rho \cdot s_x)+1]$$

$$DENY=(\bar{\phi}-\phi_{min}) \cdot (c_3+c_4+1)$$

$$DENX=DENY/c_2$$

and s_x and s_y are defined as in Eqn. (A-6).

In transforming the distribution in this fashion we can see from Fig. A-3 that the specified minimum values of the variables are maintained along with all of the first and second order moments. The maximum values are not fixed and are a function of the scale factors k_x and k_y.

(d) Third Moments

The third moments of a distribution reflect its asymmetry or

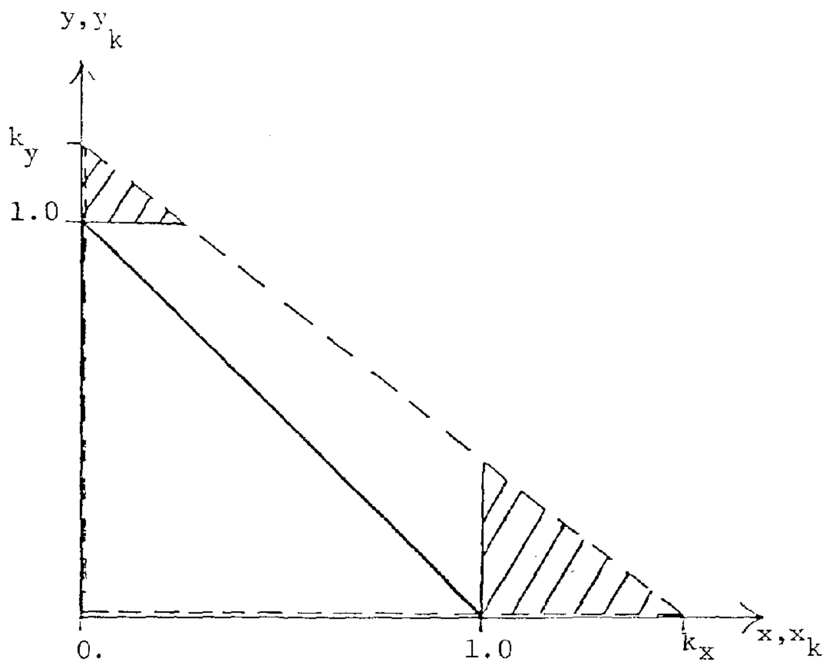


FIGURE A-3. SCALE TRANSFORMATION OF THE BIVARIATE BETA DISTRIBUTION skewness and can be used to manipulate the shape. Third moments are calculated directly from data or inferred from a univariate beta model of x and y.

A general expression for the moments of the bivariate beta distribution is given as (Wilks, 1962)

$$\mu'_{r_x r_y} = \frac{\Gamma(\alpha+r_x)\cdot\Gamma(\beta+r_y)\cdot\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma+r_x+r_y)\cdot\Gamma(\alpha)\cdot\Gamma(\beta)} \qquad \text{(A-9)}$$

In this notation μ_{10} is $\bar{x}$, μ_{02} is σ_y^2, μ_{11} is σ_{xy}, etc., and the third central moments μ_{30}, may then be written as

$$\mu_{30} = E(x-\bar{x})^3 = \mu'_{30} - 3\mu'_{10}\mu'_{20} + 2\mu'^3_{10} \qquad \text{(A-10)}$$

Substituting the appropriate form of Eqn. (A-9) for each term appearing in Eqn. (A-10) and using the recursion relation $\Gamma(a) = (a-1)\Gamma(a-1)$, the third central moment of x is found to be equal to

$$\mu_{30} = [2\alpha(\beta+\gamma)^2-\beta(\alpha+\gamma)]/[s^3+3(s+1)\cdot(s+2)] \qquad \text{(A-11)}$$

where

$$s = \alpha + \beta + \gamma$$

The third central moment of y, μ_{03}, can be obtained by interchanging α and β in Eqn. (A-11).

To satisfy all seven moments, Eqns. (A-2) and (A-11) must be solved simultaneously with respect to α, β, γ, k_x, k_y, d_x and d_y, where k and d are scale and shift factors, respectively. This could not be accomplished algebraicly. Two numerical solutions were attempted: (a) one, to solve directly the system of non-linear equations, and (b) to optimize an approximate solution by minimizing the sum of the squares of the residuals.

When applied, both methods failed to give answers that matched all moments. As the weighting in the optimization method was shifted away from the third moments, the results approximated the solution obtained method by using the scale transformation.

Comparison of the Methods Presented

Four methods associated with the use of the bivariate beta distribution were presented above. Two of these methods fail to preserve the moments specified for the distribution. In the case of the rectangularly transformed distribution, this is true because the variables are no longer correlated. In the case of the third moment approach, there is no exact algebraic solution for the parameters (α, β and γ) and, in general, the numerical solution fails to converge.

The other two methods used do match the five specified moments. The shift transformation, however, is unacceptable for modelling the joint distribution of c and ϕ. This is because the shift tends to distort the limits of the variables for the test data used even though the original moments are maintained.

The scale transformation approach provides the best model for the soil strength parameters. Not only are the moments $\bar{c}$, $\bar{\phi}$, σ_c, σ_ϕ, and ρ reproduced, but the lower limits are fixed at the specified values. This seems particularly important, as the reliability analysis, that these models are being developed for, is primarily focused on the lower tails of the distributions. Here the probability of failure is calculated as the probability of having pairs of c and ϕ the result in a factor of safety less than one.

(e) Truncated Bivariate Beta Distribution

While the scale transformation method does provide an acceptable model, it would still be desirable to have some control over the upper limits of the distributions. One way to accomplish this is to directly truncate the distribution, using a specified upper bound estimated from test data. The truncation will, of course, change the moments originally calculated, if a significant mass of the distribution lies outside of the upper limits specified.

The moments of the truncated distribution are found by direct numerical integration of the moment equations ($\bar{x} = \int_a^b x\, f(x)dx$, etc.). Using calculated moments, the parameters (α, β, γ) needed to produce the correct moments can be determined by iterative refinement. However, in the case of the truncated beta distribution, this is difficult to accomplish and convergence is not guaranteed. Thus, truncation of the bivariate beta distribution proves to be impractical as a means for modelling the soil strength parameters.