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A LINEARIZATION TECHNIQUE FOR THE DYNAMIC RESPONSE OF NONLINEAR CONTINUA

By

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ABSTRACT

The efforts of this dissertation are directed toward the development of a technique for understanding the dynamic response of structural elements governed by nonlinear partial differential equations. This technique is based on the concepts of the eqiuvalent linearization method which relies on obtaining an optimal linear set of equations to model the original nonlinear set.

In this method, the linearization is performed at the continuum level. At this level, the equivalent linear stiffness and damping parameters are physically realizable and are defined in such a way that the method can be easily be incorporated into finite element computer codes.

Three different approaches to the method are taken with each approach based on the minimization of a distinct difference between the nonlinear system and its linear replacement. Existence and uniqueness properties of the minimization solutions are established.

The method is specialized for the treatment of steady-state solutions to harmonic excitation and of stationary response to random excitation. Procedures for solving the equivalent linearization are also discussed.

The method is applied to three specific examples: one dimensional, hysteretic shear beams, thin plates governed by nonlinear equations of motion and the same nonlinear thin plates but with cutouts. Solutions via the equivalent linearization method using the stress difference minimization compare well with Galerkin's method and numerical integration. The last example is easily handled by the continuum equivalent linearization technique, whereas other methods prove to be inadequate.

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I. INTRODUCTION

Linear models of dynamic systems have enjoyed widespread use throughout the history of engineering analysis. Due to the relative simplicity of such models and the amount of attention given to this type of analysis, a vast amount of both qualitative and quantitative information is available on the response of such systems. In fact, the simplicity of linear analysis is often the sole impetus for using a linear model in the first place.

However, the need for nonlinear analysis becomes particularly apparent when the dynamic systems experience large amplitudes of vibration. In particular, it is recognized that large scale civil engineering structures behave nonlinearly when excited by damaging earthquakes, wind loadings and wave loadings. Nonlinear, dynamic models are also needed for the analysis of aircraft and aerospace structural components when excited by strong acoustical loadings. The necessity for nonlinear structural models may arise from geometric considerations, nonlinear elastic and inelastic material behavior and/or from the inadequacy of linear models in describing the energy dissipation in the structure.

The difficulties of nonlinear analysis lie not so much in the lack of good mathematical models, but more in the inability to solve the equations of motion. Only in rare cases can exact solutions to the system equations be obtained, and generally, existing analytical techniques produce only qualitative information on the response. Therefore, recourse to numerical techniques or approximate analysis is generally made. Although purely numerical techniques provide detailed information on the response of specific systems, they tend to be expensive and are not well suited to general studies. Hence, emphasis in this dissertation will be placed on approximate analytical techniques. The inclusion of

purely numerical analysis will be limited to the provision of checks on the analytical solutions.

Considerable attention has been given to the understanding of the response of nonlinear discrete systems. The discrete models originate either from systems where discrete dynamic components are physically apparent or from spatial discretization of a continuous systems. The linearization procedures are then applied to the resulting ordinary differential equations of motion.

A discussion of some commonly used techniques of nonlinear analysis directed toward use in earthquake engineering is presented in reference [20]. For the analysis of systems subjected to deterministic excitation, most of these approximate techniques rely on the assumption of a particular solution form. The solutions are written in terms of undetermined coefficients for which the approximate method produces solutions. The classical methods of harmonic balance, energy balance and equivalent linearization can be used to obtain first-order approximate solutions for steady-state response of the system when excited by a harmonic input. Higher order approximations to such solutions can be obtained through the use of perturbation techniques, asymptotic methods [4] and Galerkin's method [27] by the inclusion of several terms in the approximation. The specification of the solution form in the above methods precludes transient solutions, nor can stability information on steady-state solutions can be obtained. The method of slowly varying parameters [20] does, however, permit such analysis.

A number of these approximate techniques used on systems described by the discrete, deterministic theory have been adapted for use on the nonlinear, stochastic vibration problem. A complete review of these methods can be found in references [20] and [7]. Three of the most widely used methods are the previ-

ously mentioned perturbation and equivalent linearization techniques along with the Fokker-Planck equation approach. The latter approach is one in which the joint probability density functions of the displacements and velocities are obtained, either exactly or through an approximate technique, from the corresponding Fokker-Planck equation.

When a nonlinear continuum model is used to describe the dynamic system, complexities are introduced in that the model now contains a spatial domain as well as the temporal domain present in a discrete model. Existing studies deal with the nonlinear analysis of structural elements such as beams, strings, plates, membranes and shells. Three rather comprehensive surveys of existing work with these nonlinear structural elements can be found in references [41], [42] and [11]. In most studies of these types of nonlinear continua, the partial differential equations of motion are first spatially discretized by an approximate technique such as modal decomposition, Galerkin's method or the finite element method. Then the previously described approximate discrete techniques are directly applicable to the resulting set of nonlinear, ordinary differential equations.

The approach taken in this dissertation for the analysis of nonlinear continua is to perform the linearization *prior to* the spatial discretization of the partial differential equations. In particular, the nonlinear system is replaced by a linear auxiliary system containing undetermined, spatially distributed damping and stiffness parameters. These parameters are then determined in such a way as to model the amplitude dependent energy dissipation and stiffness properties of the nonlinearity in the original system. The solution to the auxiliary system can then be established in terms of the equivalent linear parameters. This is a continuum analog to the discrete equivalent linearization method that has been previously mentioned.

It is recognized that the finite element method is widely used in structural analysis, especially for structural members with complicated boundary conditions and spatially distributed material properties. The motivation for defining the equivalent linear system at the continuum level is for the ease of implementation into finite element computer codes. Analysis presented in this dissertation has been directed toward such an implementation.

In Chapter II, a set of general equations of motion are defined for a linearizable system for which zero-mean solutions exist. A brief survey of existing techniques for solving such a system is presented. For later use, the Galerkin method and a Ritz technique are described in detail. The formulation for equivalent linearization method for discrete systems is presented to provide a basis for the new continuum equivalent linearization approach.

The general formulation for the continuum equivalent linearization method is developed in Chapter III. Three different formulations of the method are presented and discussed. The existence and uniqueness of solutions for the equivalent linear parameters are investigated for the three problem formulations. Considerations of the finite element discretization of the linear, auxiliary system are also presented.

The general equivalent linearization relations from Chapter III are specialized in Chapter IV for two types of analysis: steady-state solutions for harmonic excitation and stationary response to random excitation. Mechanization of the method is discussed for both types of analysis. The free vibration, amplitude-frequency relations can be obtained as a special case of the steady-state, harmonic analysis, and a technique is presented for solving the equivalent linear equations. A method for obtaining resonant response of the harmonically excited system is also introduced.

In Chapter V three example studies of steady-state, harmonic response is presented. In each study, the fundamental nonlinear response mode is investigated using the continuum equivalent linearization method, and results are compared with those from other existing techniques.

The response of a one-dimensional shear beam composed a hysteretic yielding material is investigated in section 5.1. The convergence of the equivalent linear solution with refinement of the mesh for the spatial domain is discussed. Results from the equivalent linearization analysis are compared with those from Galerkin's method.

Section 5.2 contains the nonlinear analysis of Kirchhoff plates described by the Berger formulation of the Von Karman nonlinear equations of motion. The linear auxiliary equation is discretized using the Bogner-Fox-Schmidt shape functions in the finite element formulation. The results from the three equivalent linearization problem formulations are compared with those from Galerkin's method and numerical integration of the equations of motion.

In section 5.3, a further example of the nonlinear plate of section 5.2 is treated: the response of a rectangular plate with *cutouts*. The matching of the additional boundary conditions at the hole makes analysis by standard techniques difficult, while usage of the finite element-equivalent linearization techniques can be as routine as for the plate without a hole. The effect of the hole size on the resonant frequencies and forced response has been determined by the equivalent linearization technique, and results are compared with those found using a Ritz formulation of the nonlinear problem.

II. GOVERNING EQUATIONS AND REVIEW OF SOME EXISTING SOLUTION TECHNIQUES

2.1 Introduction

The first section of this chapter deals with the definition of the class of nonlinear continuous systems that will be investigated in this and subsequent chapters. The wide variety of vibration problems encountered in engineering practice makes the definition of an all-encompassing system of equations difficult, if not impossible. In spite of this obstacle, an attempt will be made to cover as many nonlinear problems as possible, while retaining a sufficient amount of conciseness to permit some general statements concerning their solutions.

Limitations placed on the system operators, boundary conditions, etc., as presented in section 2.3, should be viewed in the spirit of conciseness and not so much as being necessary conditions to obtain solutions to a specific problem. For example, the familiar property of self-adjointness of the differential operator in a eigenvalue problem is sufficient for the useful property of eigenfunction orthogonality, but some systems which are not self-adjoint can surely be solved.

In section 2.4, a number of existing techniques for solving nonlinear vibration response of continua are discussed. The list of methods is not intended to be all-inclusive; only those techniques that will be explored in later chapters are reviewed. These techniques, typical of all those encountered by the author, are formulated in such a way that the system is first spatially discretized, and the resulting time dependent relations are then linearized. This contrasts with the new technique to be presented in Chapter III in which the linearization is performed in the continuous domain, followed by the spatial discretization.

2.2 Governing Equations

A number of dynamic systems can be described by partial differential equations (PDE's) in which the independent variables are space and time. A subclass of the PDE's are those in which the PDE is approximately linear for small motion. For larger amplitude of motion, the nonlinear effect becomes more pronounced with the degree of nonlinearity depending on the size of subsequent motion. Herein, consideration will be given to a further subclass of equations in which the nonlinearities are independent of inertial stresses and will depend only on dissipative and restoring stresses of the system. Let $\mathcal{S}(\mathbf{x})$ and $\partial\mathcal{S}(\mathbf{x})$ represent the spatial domain and boundary of the spatial domain, respectively, described by spatial coordinates $\mathbf{x}$, and the time parameter be represented by t. Consider a continuous system in $\mathcal{S}(\mathbf{x})$ described by the set of partial differential equations:

$$\nabla\cdot\tau^L(\mathbf{a},\mathbf{w}) + \nabla\cdot\tau^N(\mathbf{c},\mathbf{w},\dot{\mathbf{w}}) + \dot{\mathbf{Q}}^L(\mathbf{b},\mathbf{w}) + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{p}(\mathbf{x},t) \tag{2.1}$$

where a *dot* above a variable (or operator) denotes differentiation with respect to t, $\nabla\cdot(\,)$ is the divergence operator with respect to the spatial coordinates $\mathbf{x}$, and

- $\mathbf{w} = \mathbf{w}(\mathbf{x},t)$ is the dependent variable representing "displacements"
- $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are sets of "material" and "geometry" properties of the system and are functions only of $\mathbf{x}$
- $m(\mathbf{x})$ is the "mass" distribution of the system
- $\mathbf{p}(\mathbf{x},t)$ is the applied "load"
- τ^L is a **linear** "stress" operator working only on the displacements $\mathbf{w}$ and "stiffness" parameters $\mathbf{a}$
- τ^N is an operator (or functional) which is **nonlinear** in displacements and/or velocities, $\dot{\mathbf{w}}$ and involves the nonlinear "stiffness"

parameters **b**

- $\dot{\mathbf{Q}}^L$ is a **linear** "damping" operator working on the displacements **w** and damping parameters **c**

On every point on the boundary, $\partial\, \mathbb{S}(\mathbf{x})$, let either of the following hold:

$$\mathbf{B}^g(\mathbf{w}) = \mathbf{0} \tag{2.2}$$

$$\mathbf{B}^n(\mathbf{w}) = \mathbf{0} \tag{2.3}$$

where $\mathbf{B}^g(\mathbf{w})$ and $\mathbf{B}^n(\mathbf{w})$ are sets of geometric and natural boundary conditions, respectively, which are linear homogeneous operators containing derivatives normal to and along $\partial\, \mathbb{S}(\mathbf{x})$.

Before continuing the discussion of the nonlinear system of equations, a few remarks need to be made on notation and terminology:

1. The **bold** terms in equations (2.1) thru (2.3) represent sets of indexed elements. The number and range of indices depend on the given system under investigation. For example, the linear "stress" operation τ^L may be a scalar τ^L, a vector τ_i^L or a two tensor τ_{ij}^L , depending on the number of indices needed to describe the stress field. The boldface notation was chosen here for the sake of generality.

2. The terms enclosed in quotes in the text following equation (2.1), such as "stress", should be thought of as generic descriptions. For different systems, τ^L, for example, may have different physical meanings, but in each case τ^L would have the character of stress. In further discussion, the generic name will be used without being restricted to any physical interpretation.

3. In order for the mathematical statement of the nonlinear problem, equations (2.1) - (2.3), to be well-posed, appropriate initial conditions

need to be specified for $\mathbf{w}$ and $\dot{\mathbf{w}}$. However, attention will be focused mainly on obtaining steady state solutions when $\mathbf{p}$ is harmonic in time or on stationary solutions for a temporally random $\mathbf{p}$. For these types of analysis, the periodicity or stationarity conditions will alleviate the need for specifying initial conditions.

2.3 System Properties

The following restrictions will be placed on the nonlinear continuous system described by equations (2.1) - (2.3). These properties will be used to simplify analysis in order that specific statements can be made about the response of the system.

2.3.1 Linearizability of the system

Most existing techniques, along with the technique to be presented in the next chapter, for solving nonlinear systems of the type introduced in the preceding section assume that the system response is nearly linear for small amplitudes of motion. In fact, the response of this linear system is often used for a first a approximation to the nonlinear response. The linearizability has already been partially satisfied by the statement of the problem in equations (2.1); that is, the restoring stress has been written in a linear part, τ^L, and a nonlinear part, τ^N. It only remains that the qualitative "effect" of τ^L be predominant over τ^N as the amplitude of vibration becomes small. The idea of linearizability is dependent on the approximate method used, and therefore detailed discussion of this point will be deferred to later discussion of the specific techniques, whenever possible. In all further discussion, the *linearized system* will be taken to be equations (2.1) with τ^N omitted, or

$$\nabla \cdot \tau^L + \dot{\mathbf{Q}}^L + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{p}(\mathbf{x}, t) \tag{2.4}$$

2.3.2 Positive definiteness and self-adjointness of linearized system

Definition - Inner product

For general variables, or operators, **a**, **b** and **c**, the scalar inner product $(\,)\cdot(\,)$ is defined to have the following properties:

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot\mathbf{a} > 0$ if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{a} = \mathbf{0}$ iff $\mathbf{a} = \mathbf{0}$
3. $(\mathbf{a} + \mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$

Definition - Positive Definite

For a differential operator $L(\alpha, \mathbf{u})$ and for α and $\mathbf{u}$ defined on the spatial domain $\mathcal{D}(\mathbf{x})$, L is said to be *positive definite* if

$$\int_{\mathcal{D}} \mathbf{u}\cdot L(\alpha, \mathbf{u})\, d\,\mathcal{D}(\mathbf{x}) > 0$$

and for $\alpha \neq 0$, the integral is zero for $\mathbf{u} = 0$. The integral can vanish for $\mathbf{u} \neq 0$ iff $\alpha = 0$.

Definition - Self Adjoint

For L and **u** described above and for **v** also defined on $\mathcal{D}(\mathbf{x})$, L is said to be *self- adjoint* if

$$\int_{\mathcal{D}} \mathbf{u}\cdot L(\mathbf{v})\, d\,\mathcal{D}(\mathbf{x}) = \int_{\mathcal{D}} \mathbf{v}\cdot L(\mathbf{u})\, d\,\mathcal{D}(\mathbf{x})$$

It should be noted that whether or not an operator is positive definite and self-adjoint can be established by integration by parts.

According to the definitions presented above, the divergence of the linear stress operator, $\nabla\cdot\tau^{L}$, will be required to be both positive definite and self

adjoint in all further discussion of the nonlinear system (2.1). In addition, the mass distribution $m(\mathbf{x})$ will be required to satisfy

$$m(\mathbf{x}) > 0 \tag{2.5}$$

Consider the eigenvalue problem of the undamped, linearized system in equation (2.4):

$$\nabla \cdot \tau^L(\mathbf{u}_r) = \lambda_r m(\mathbf{x})\mathbf{u}_r \tag{2.6}$$

where λ_r and $\mathbf{u}_r$ are the r^{th} eigenvalue and r^{th} set of eigenfunctions. As can be found in a number of vibrations texts, for example reference [34], the above properties for $\nabla \cdot \tau^L$ and $m(\mathbf{x})$ permit the following relationships for the eigenvalues of (2.6):

1. The eigenvalues are positive.

$$\lambda_r > 0 \quad ; r = 1, 2,... \tag{2.7}$$

2. For distinct eigenvalues λ_r and λ_s, $\mathbf{u}_r$ and $\mathbf{u}_s$ are orthogonal and can be normalized in such a way that

$$\int_{\mathcal{B}} m(\mathbf{x})\, \mathbf{u}_r \cdot \mathbf{u}_s \, d\mathcal{B}(\mathbf{x}) = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{2.8}$$

for r,s = 1,2,...

3. A Galerkin discretization of the eigenvalue problem in equation (2.6) will produce symmetric, positive definite "mass" and "stiffness" matrices.

2.3.3 Form of damping operator

The damping operator $\dot{\mathbf{Q}}^L(\mathbf{w})$ will be assumed to be from the class of Rayleigh damping operations, where $\mathbf{Q}^L$ can be written as a linear combination of $\nabla \cdot \tau^L$ and $m(\mathbf{x})\mathbf{w}$, or

$$\mathbf{Q}^L = \gamma_1 \nabla \cdot \tau^L + \gamma_2 m(\mathbf{x})\mathbf{w} \tag{2.9}$$

with γ_1 and γ_2 being constant coefficients.

Again referring to the eigenfunctions $\mathbf{u}_r$ and eigenvalues λ_r of the linearized equation (2.6) and using properties 1 and 2 from section 2.3.2, the following orthogonality relations can be established for the Rayleigh damping operator of equation (2.9):

$$\int \mathbf{u}_r \cdot \mathbf{Q}^L(\mathbf{u}_s)\, d \quad (\mathbf{x}) = \begin{cases} \lambda_r \gamma_1 + \gamma_2 & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{2.10}$$

provided $\lambda_r \neq \lambda_s$ for $r \neq s$; $r, s = 1, 2, \ldots$ Furthermore, since $\mathbf{Q}^L$ is a linear combination of $\nabla \cdot \tau^L$ and $m(\mathbf{x})\mathbf{w}$, a Galerkin discretization of the linearized equation will produce a symmetric, positive definite damping matrix.

2.3.4 Symmetry of nonlinear stress operator - steady-state harmonic response

Definition - Symmetric Operator

If $\mathbf{u}(\mathbf{x}, t)$ is a periodic function of period T in time, then the operation $\tau^N(\mathbf{u})$ is said to be *symmetric* if

$$\int_0^T \tau^N(\mathbf{u})\, dt = \mathbf{0} \tag{2.11}$$

Consider the case of the nonlinear system of equations (2.1) - (2.3) where the right hand side of equation (2.1) is harmonic in time. Then,

$$\nabla \cdot \tau^L + \nabla \cdot \tau^N + \dot{\mathbf{Q}}^L + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{r}(\mathbf{x})\cos\omega t \tag{2.12}$$

$$B^g(\mathbf{w}) = \mathbf{0} \tag{2.12a}$$

$$B^n(\mathbf{w}) = \mathbf{0} \tag{2.12b}$$

If a steady-state solution of (2.12) exists, let the solution, $\mathbf{w}$, be sought which is periodic of period T, where $T = \frac{2\pi}{\omega}$.

Since $\mathbf{w}(\mathbf{x}, t)$ is periodic in time, it can be expanded in a temporal Fourier series [52]

$$\mathbf{w}(\mathbf{x}, t) = \tfrac{1}{2}\mathbf{b}_0 + \sum_{n=1}^{\infty} [\mathbf{a}_n(\mathbf{x})\sin\frac{2n\pi}{T}t + \mathbf{b}_n(\mathbf{x})\cos\frac{2n\pi}{T}t] \tag{2.13}$$

where $\mathbf{a}_n(\mathbf{x})$ and $\mathbf{b}_n(\mathbf{x})$; $n=1, 2,...,$ are the Fourier coefficients of $\mathbf{w}$. Then, if τ^N is symmetric and $\nabla \cdot \tau^L$ is positive definite, it can be shown that $\mathbf{b}_0(\mathbf{x}) = \mathbf{0}$.

In all further discussion of the steady-state solution of equation (2.12), it will be assumed that the nonlinearity is symmetric. The significance of this, as shown above, is that it will not be necessary to include a constant (in time) *offset* function in the approximate solution for the steady-state, periodic solutions of the nonlinear system. The inclusion of the offset function will generally involve more complexities in the analysis, as can be seen in the treatment of non-symmetric nonlinearities of discrete systems in reference [45].

2.4 Review of Existing Solution Techniques

Exact solutions to the class of nonlinear systems described in equations (2.1) - (2.3) are generally not available. Therefore, recourse to approximate techniques is usually made. Existing techniques fall into three general classifications: those which are purely numerical, those which are purely analytical and those which are both analytical and numerical in character. Methods which are purely numerical are generally quite expensive, and generalities to solutions of the same system using different parameters are lost. Therefore,

further discussion will be focused only on purely analytical and numerical-analytical methods, except when purely numerical techniques are desired to serve as a check on other solution procedures.

A different technique is generally used to solve the temporal portion of the problem than is used to solve the spatial part. The method chosen for each part of the problem will depend on the type of analysis to be performed (stability, steady-state response, etc.), on the boundary conditions to be satisfied or on simply a personal preference of the analyst. Table 2.1 lists a number of publications that deal with the nonlinear vibration of continua by purely analytical or numerical-analytical methods. An attempt has been made in the table to include methods representative of current and past work, but it is by no means inclusive of all work.

In the following section, three methods will be discussed in detail: Galerkin's method applied to both spatial and temporal problems, a Ritz approximation in space and Hamilton's principle in time, and spatial discretization followed by the method of equivalent linearization on the temporal problem.

2.4.1 Galerkin's method (periodic solutions)

Consider the nonlinear system given by equation (2.12) which is forced by a load that is harmonic in time with period $T = \frac{2\pi}{\omega}$. Let a set of functions $\Phi_i(\mathbf{x})$; $i=1, 2,...,N$, be *comparison* functions, where a comparison function has the following properties:

1. $\Phi_i(\mathbf{x})$; $i=1, 2,...,N$, satisfy *all* the boundary conditions of (2.12); i.e. $B^g(\Phi_i) = 0$ and $B^n(\Phi_i) = 0$

2. $\Phi_i(\mathbf{x})$; $i=1, 2,...,N$, are linearly independent and represent the first N of a group of functions $\Phi_i(\mathbf{x})$ $i=1, 2,...,N,...$ which is complete in the

Type of System	Type of Analysis	Spatial Method	Temporal Method	Ref.
planar string	forced random	modal analysis	equivalent linearlization	[10]
planar string non autonomous	forced periodic	modal analysis	perturbation	[33]
1-D rod hysteretic	traveling waves	perturbation	harmonic balance	[9]
non-planar bending beam	stability of forced, periodic	Galerkin	perturbation	[17]
bending beam	sub-harmonics forced, periodic	modal analysis	Galerkin	[50]
bending beam	free vibration	modal analysis	exact	[53]
bending beam	stability of forced, periodic	finite elements	perturbation	[6]
plate	free vibration	Ritz-Kantrorovitch	finite difference	[18]
plate	forced periodic	Lagrange's equations	equivalent linearization	[31]
plate	forced periodic	exact	multiple scales	[46]
plate on elastic foundation	free vibration	Galerkin	exact	[14]
plate with initial stress	forced periodic	Galerkin	Galerkin	[12]
plate	free vibration	perturbation	Hamilton's principle	[40]
cylindrical shell	forced periodic	Lagrange's equations	harmonic balance	[15]
2-D plane strain hysteretic	forced random	finite elements	equivalent linearization	[43]

Table 2.1 Summary of Existing Work on the Nonlinear Vibration of Continua

given region.

Furthermore, since solutions are sought which are periodic with period $\frac{2\pi}{\omega}$, let $f_i(t)$ be a set of functions which are periodic with a period $\frac{2\pi}{\omega}$. Since the set of functions [$\sin\omega t$, $\cos\omega t$] form a basis for such periodic functions, let

$$f_i(t) = u_i \cos\omega t + v_i \sin\omega t \tag{2.14}$$

Using these properties of Φ_i and f_i, the solution $\mathbf{w}(\mathbf{x},t)$ will be approximated by $\hat{\mathbf{w}}(\mathbf{x},t)$, where

$$\hat{\mathbf{w}}(\mathbf{x},t) = \sum_{i=1}^{N} \Phi_i(\mathbf{x}) f_i(t) \tag{2.15}$$

Since $\hat{\mathbf{w}}$ is not an exact solution, substitution of $\hat{\mathbf{w}}$ into the equations of motion leaves a non-vanishing residual, $\boldsymbol{\varepsilon}$, where:

$$\boldsymbol{\varepsilon} = \nabla \cdot \boldsymbol{\tau}^L(\hat{\mathbf{w}}) + \nabla \cdot \boldsymbol{\tau}^N(\hat{\mathbf{w}}, \dot{\hat{\mathbf{w}}}) + \dot{\mathbf{Q}}^L + m(\mathbf{x})\ddot{\hat{\mathbf{w}}} - \mathbf{r}(\mathbf{x})\cos\omega t \tag{2.16}$$

With $\hat{\mathbf{w}}$ defined as in equation (2.19), Galerkin's method says that the best solution for u_i and v_i ; $i=1, 2,\ldots,N$, is given by the solution to the following set of equations (see reference [27]):

$$\int_0^T \int_{\mathcal{B}} \boldsymbol{\varepsilon} \cdot \frac{\partial \hat{\mathbf{w}}}{\partial u_k} \, d\mathbf{x} \, dt = 0 \tag{2.17}$$

$$\int_0^T \int_{\mathcal{B}} \boldsymbol{\varepsilon} \cdot \frac{\partial \hat{\mathbf{w}}}{\partial v_k} \, d\mathbf{x} \, dt = 0$$

for $k=1, 2,\ldots,N$.

Inserting equations (2.15) and (2.16) into equations (2.17) and performing the integrations gives:

$$\sum_{j=1}^{N} \left[-\omega^2 M_{ij}^G + K_{ij}^G\right] u_j + \omega \sum_{j=1}^{n} B_{ij}^G v_j + \mathcal{F}_{Gi}^G = f_i^G \tag{2.18}$$

$$-\omega \sum_{j=1}^{N} B_{ij}^{G} u_j + \sum_{j=1}^{N} [-\omega^2 M_{ij}^{G} + K_{ij}^{G}]\, v_j + \mathfrak{F}_{Si}^{G} = 0$$

where

$$M_{ij}^{G} = \int_{\mathcal{D}} m(\mathbf{x}) \Phi_i \cdot \Phi_j \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18a}$$

$$B_{ij}^{G} = \int_{\mathcal{D}} \dot{\mathbf{Q}}^{L}(\Phi_j) \cdot \Phi_i \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18b}$$

$$K_{ij}^{G} = \int_{\mathcal{D}} [\nabla \cdot \tau^{L}(\Phi_j)] \cdot \Phi_i \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18c}$$

$$\mathfrak{F}_{Ci}^{G} = \int [\nabla \cdot \mathbf{C}(\mathbf{u}, \mathbf{v})] \cdot \Phi_i \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18d}$$

$$\mathfrak{F}_{Si}^{G} = \int_{\mathcal{D}} [\nabla \cdot \mathbf{S}(\mathbf{u}, \mathbf{v})] \cdot \Phi_i \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18e}$$

$$\mathbf{C}(\mathbf{u}, \mathbf{v}) = \frac{1}{\pi} \int_{0}^{2\pi} \tau^{N}(\mathbf{u}, \mathbf{v}, \vartheta) \cos\vartheta \, d\vartheta \tag{2.18f}$$

$$\mathbf{S}(\mathbf{u}, \mathbf{v}) = \frac{1}{\pi} \int_{0}^{2\pi} \tau^{N}(\mathbf{u}, \mathbf{v}, \vartheta) \sin\vartheta \, d\vartheta \tag{2.18g}$$

$$f_i^{G} = \int_{\mathcal{D}} \mathbf{r}(\mathbf{x}) \cdot \Phi_i \, d\, \mathcal{D}(\mathbf{x}) \tag{2.18h}$$

$$\vartheta = \omega t \tag{2.18i}$$

Recall the properties of the nonlinear system given in section (2.3), particularly those of positive definiteness, self-adjointness and Rayleigh damping. Now suppose that Φ_k ; $k=1, 2,\ldots,N$, are the first N sets of eigenfunctions to the linearized problem, or

$$\nabla \cdot \tau^{L}(\Phi_k) = \lambda_k^2 \, m(\mathbf{x}) \Phi_k \tag{2.19}$$

where λ_k^2 represents the corresponding k^{th} eigenvalue. Using (2.19), the orthogonality property of equation (2.8), and the Rayleigh damping definiton , the matrices of equations (2.18a) - (2.18c) become

$$M_{ii}^{G} = 1$$

$$B_{ii}^{G} = 2\zeta_i\lambda_i \tag{2.20}$$

$$K_{ii}^{G} = \lambda_i^2$$

$$M_{ij}^{G} = B_{ij}^{G} = K_{ij}^{G} = 0 \quad i \neq j$$

where ζ_i is the damping coefficient of the i^{th} linear mode and $\zeta_i \equiv \frac{1}{2}(\frac{1}{\lambda_i}\gamma_1 + \lambda_i\gamma_2)$.

All future references to the Galerkin solution for the periodic nonlinear problem will be solutions to the nonlinear algebraic equations of (2.18), (2.18d) - (2.18h) and (2.20). Note that by using the linear eigenfunctions for comparison functions, the linear portion is *uncoupled*, thus leaving only the nonlinear vectors $\mathfrak{F}_C^G$ and $\mathfrak{F}_S^G$ to couple the equations (2.18).

2.4.2 Ritz method (periodic solutions)

The Ritz method will proceed from the *variational form* of the nonlinear problem , whereas the preceding Galerkin method started with the system being described by a differential equation. Let the generalized Lagrangian, L, of the system (2.12) be written as:

$$L = T - U + W \tag{2.21}$$

where T is the kinetic energy, U the potential energy and W the virtual work done on the system by non-conservative forces. The generalized Hamilton's principle [16] says that from time t_1 to time t_2, a point in the system will follow a path which extremizes the integral of L over time from t_1 to t_2, or

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{2.22}$$

In the Ritz method, the solution will be approximated by a function

$$\tilde{w}(x,t) = \sum_{i=1}^{N} \Phi_i(x)[u_i \cos\omega t + v_i \sin\omega t] \tag{2.23}$$

where here, Φ_i ; $i=1, 2,...,N$, are *admissable* functions. Admissable functions are similar to comparison functions except that they need only satisfy the *geometric* boundary conditions; that is, for the system of equations (2.12), only the following need be satisfied:

$$B^g(\Phi_i) = 0 \quad ; i=1, 2,...,N$$

Surely trial functions which satisfy all of the boundary conditons could be used, but the use of admissable functions enlarges the class of functions from which to choose. In fact, the satisfaction of geometric boundary conditions is not necessarily required to effect a solution when dealing with the variational form of the problem. Discussion of this can be found in references [48] and [34].

The variation in equation (2.22) is accomplished by setting

$$\left.\begin{aligned} \frac{\partial}{\partial u_k} \int_0^{2\frac{\pi}{\omega}} (T - U + W)\, dt = 0 \\ \frac{\partial}{\partial v_k} \int_0^{2\frac{\pi}{\omega}} (T - U + W)\, dt = 0 \end{aligned}\right\} \qquad (2.24)$$

The kinetic energy for the system in equations (2.12) is given by

$$T(\tilde{\mathbf{w}}) = \tfrac{1}{2}\int_{\mathcal{D}} m(\mathbf{x})\dot{\tilde{\mathbf{w}}}\cdot\dot{\tilde{\mathbf{w}}}\, d\ \mathcal{D}(\mathbf{x}) \qquad (2.25)$$

Suppose that the potential energy U can be written as:

$$U = U^L + U^{NL}$$

$$= \tfrac{1}{2}\int_{\mathcal{D}} \tilde{\tau}^L\cdot\tilde{\varepsilon}^L\, d\ \mathcal{D}(\mathbf{x}) + U^{NL} \qquad (2.26)$$

where U^L is the potential energy in the linearized problem, $\tilde{\tau}^L$ and $\tilde{\varepsilon}^L$ are linear operations of $\mathbf{w}$ and U^{NL} is the contribution to the potential energy by τ^N.

Using the linearity of $\tilde{\tau}^L$ and $\tilde{\varepsilon}^L$, along with equations (2.23), (2.25) and (2.26), the Ritz equations can be written as:

$$\sum_{j=1}^{N} [-\omega^2\, m_{ij}^R + k_{ij}^R] u_j = \mathcal{F}_{Ci}^R \qquad (2.27)$$

$$\sum_{j=1}^{N}[-\omega^2 \mathcal{M}_{ij}^R + \mathcal{K}_{ij}^R]v_j = \mathcal{F}_{Si}^R$$

where

$$\mathcal{M}_{ij}^R = \int_{\mathcal{D}} m(\mathbf{x})\Phi_i \cdot \Phi_j \, d\mathcal{D}(\mathbf{x}) \tag{2.27a}$$

$$\mathcal{K}_{ij}^R = \tfrac{1}{2}\int_{\mathcal{D}} [\tilde{\tau}^L(\Phi_i)\cdot\tilde{\varepsilon}^L(\Phi_j) + \tilde{\tau}^L(\Phi_j)\cdot\tilde{\varepsilon}^L(\Phi_i)] \, d\mathcal{D}(\mathbf{x}) \tag{2.27b}$$

$$\mathcal{F}_{Ci}^R = \frac{1}{\pi}\frac{\partial}{\partial v_i}\int_0^{2\pi}[W(\tilde{w}) - U^{NL}(\tilde{w})]\, d\vartheta \tag{2.27c}$$

$$\mathcal{F}_{Si}^R = \frac{1}{\pi}\frac{\partial}{\partial v_i}\int_0^{2\pi}[W(\tilde{w}) - U^{NL}(\tilde{w})]\, d\vartheta \tag{2.27d}$$

$$\vartheta = \omega t \tag{2.27e}$$

If the system investigated is conservative and linear, that is if W and U^{NL} are omitted from (2.27c) and (2.27d), the system of equations reduce to the eigenvalue problem:

$$\sum_{j=1}^{N}[-\omega^2 \mathcal{M}_{ij}^R + \mathcal{K}_{ij}^R]\, u_j = 0 \tag{2.28}$$

It can be shown that the matrices $\mathcal{M}^R$ and $\mathcal{K}^R$ in (2.27a) and (2.27b) are identical to those one would obtain by directly applying the well-known Rayleigh-Ritz method to the linear, conservative system. Therefore, this method of performing a Ritz approximation to Hamilton's principle can be shown equivalent to a well established method for linear problems.

2.4.3 Spatial discretization and equivalent linearization

Let attention be returned to the general nonlinear system of equations given by (2.1) - (2.3). The first step of this method is to discretize the spatial portion by any acceptable technique, such as via the finite element method, Galerkin's method, or modal expansion in terms of the linearized eigenfunctions. This will

generally lead to a set of ordinary differential equations in time of the form:

$$M\ddot{\mathbf{d}} + C(\mathbf{b})\dot{\mathbf{d}} + K(\mathbf{a})\mathbf{d} + \mathfrak{F}(\mathbf{c}, \mathbf{d}, \dot{\mathbf{d}}) = \mathbf{f}(t) \tag{2.29}$$

where $\mathbf{d} = \mathbf{d}(t)$ are discretized N-vector displacements. M, C and K are the mass, damping and stiffness matrices of size NxN, resulting from the linear operators in (2.1), while $\mathbf{f}(t)$ represents the discretized forcing term. $\mathfrak{F}(\mathbf{b}, \mathbf{d}, \dot{\mathbf{d}})$ is an N-vector which is nonlinear in displacements $\mathbf{d}$ and velocities $\dot{\mathbf{d}}$ and originates from the discretization of the nonlinear operator $\nabla \cdot \tau^N$. Recall that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are stiffness, damping and nonlinearity parameters, respectively, in the original nonlinear system. The explicit form of the matrices and vectors in (2.29) will depend on the method of discretization.

By way of obtaining appproximate solutions to equations (2.29), let an auxiliary set of *linear* equations be given by:

$$M\ddot{\mathbf{d}} + (C+C^e)\dot{\mathbf{d}} + (K+K^e)\mathbf{d} = \mathbf{f}(t) \tag{2.30}$$

where C^e and K^e are matrices independent of time. As will be seen later in this section, these matrices will be adjusted in a prescribed manner such that the difference between the nonlinear equations of (2.29) and the auxiliary, or *equivalent linear*, equations in (2.30) will be minimized.

Let $\mathbf{d}^*$ be a vector function of time and a member of a class of functions containing the solutions to the linear, auxiliary equations (2.30). Let Δ represent the difference between the original nonlinear system and the linear auxiliary system both operating on the same function $\mathbf{d}^*(t)$, or

$$\Delta = \mathfrak{F}(\mathbf{d}^*, \dot{\mathbf{d}}^*) - C^e\dot{\mathbf{d}}^* - K^e\mathbf{d}^* \tag{2.31}$$

The central idea of equivalent linearization, as applied to equations (2.29), is to minimize the difference term Δ over all functions $\mathbf{d}^*$ belonging to $\mathbb{C}$. The usual procedure is to first average over time the Euclidean norm of Δ. Then the

minimization of Δ is stated as

$$\mathcal{G}(\Delta^T\Delta) = \textit{minimum} \qquad \text{for all } \mathbf{d}^* \in \mathcal{C} \tag{2.32}$$

where Δ^T denotes the transpose of Δ and $\mathcal{G}$ is a linear, time invariant operation.

The minimization is then accomplished by extremizing $\mathcal{G}(\Delta^T\Delta)$ with respect to all elements of C^e and K^e. If C^e_{ij} and K^e_{ij} ; $i,j=1, 2,...,N$, represent the elements of C^e and K^e, the extremization is expressed as:

$$\frac{\partial}{\partial C^e_{ij}} \mathcal{G}(\Delta^T\Delta) = 0 \tag{2.33}$$

$$\frac{\partial}{\partial K^e_{ij}} \mathcal{G}(\Delta^T\Delta) = 0 \tag{2.34}$$

Since $\mathbf{d}^*(t)$ belongs to the class of functions which satisfy the linear auxiliary equations, there exists an implicit relationship between the solution $\mathbf{d}$ of auxiliary equations (2.30) and the equivalent linear matrices C^e and K^e in equations (2.33) and (2.34). These equations are then solved in an iterative fashion for $\mathbf{d}$, C^e and K^e.

For the sake of brevity, the above description of the generalized equivalent linearization method for ordinary differential equations is by no means complete. A more thorough treatment of the method can be found in references [45] and [21]. One result, from the investigation reported in [45] on the invertibility of relations (2.33) and (2.34), deserves comment at this point. It was shown that when the dimension of the solution space described by equaled or was greater than 2N, the relations in (2.33) and (2.34) became singular. Therefore, if solutions for C^e and K^e exist, they are not unique. However, it was demonstrated that the non-uniqueness does not affect the quality of the solutions in that all values of C^e and K^e which satisfy the singular relationships do

an equally good job of minimizing $G(\Delta^T\Delta)$. It should be noted, however, that in the mechanization of the general method, one encounters the task of solving a singular set of equations. This singularity of equations (2.33) and (2.34) will create numerical difficulties. The ideas of existence and uniqueness of the minimization relationships resulting from the extension of equivalent linearization to partial differential equations will be investigated in Chapter III.

2.5 Remarks

The class of nonlinear partial differential equations that will be investigated has been presented in equations (2.1) - (2.3). Simplifying assumptions of linearizability, positive definiteness, self-adjointness, form of damping to be considered, and symmetry of the nonlinearity have been stated and discussed.

A brief survey of existing techniques for solving nonlinear continua problems that are either purely analytical or numerical-analytical has been presented. Galerkin's method, the Ritz approximation to Hamilton's principle and the method of equivalent linearization applied to the spatially discretized problem have been discussed in further detail. These methods reduce the original nonlinear partial differential equations to a set of of nonlinear *algebraic* equations for which general solution techniques exist. The algebraic equations developed via Galerkin's method (2.18) and the Ritz method (2.27) will be used to obtain solutions to particular problems for comparison with solutions generated by the new equivalent linearization technique to be developed in Chapter III. It should be noted that the striking difference between the Galerkin and Ritz method is that the Ritz method is applied to the *variational* form of the system, while the Galerkin method starts with the problem stated in the *differential* form. The variational statement of the problem permits the usage of admissable functions instead of the comparison functions that are needed for Galerkin's method.

This difference gives the edge to the Ritz method on problems in which satisfying the natural boundary conditions becomes difficult, if not impossible.

III. EQUIVALENT LINEARIZATION FOR CONTINUOUS SYSTEMS

3.1 Introduction

The concept of equivalent linearization was introduced in Chapter II in the application of the method to a set of nonlinear ordinary differential equations. The principle of the method was first presented in reference [4] for the case of a single ordinary differential equation with harmonic excitation. Although this original work derived the equivalent linear relationships through the methods of power balance and harmonic balance, the relationships are now typically developed by the residual minimization procedure discussed in Chapter II. The method was then extended to stochastic equations in references [5] and [8], and a generalization to multi-degree-of-freedom systems has been presented in reference [21]. For the most part, equivalent linearization has been used exclusively for determining periodic solutions and for stationary random solutions. One exception to this is recent work in reference [45] in which the method was applied to transient response of nonlinear single-degree-of-freedom systems.

In this chapter, the idea of equivalent linearization is be extended to the class of nonlinear systems by equations (2.1) - (2.3). This method differs significantly from the discretization-equivalent linearization technique described in section 2.4.3, in that here, the linearization is performed before the spatial discretization. More specifically, the auxiliary system for this method is a set of partial differential equations containing dissipative and stiffness parameters to be determined in such a way that a difference between the original nonlinear system and the linear auxiliary is minimized. The difference term here is not only a function of time but also a function of space. Therefore, its minimization involves complexities not encountered in its discrete analog.

This chapter deals with the mechanics of constructing such an equivalent linear solution and the properties of their solutions. The first sections are concerned with the definitions of the linear auxiliary system, system difference terms and the averaging operators to be used on the difference terms. Following this in section 3.5, the minimization of the equation difference terms is discussed. Discretization of the resulting equivalent linear system and properties of the equivalent linear solutions are discussed in sections 3.6 and 3.7.

3.2 Class of Auxiliary Systems

Let the linear auxiliary system for the equivalent linearizaton method be defined as:

$$\nabla \cdot \tau^L(\mathbf{a}, \mathbf{w}) + \nabla \cdot \tau^L(\alpha, \mathbf{w}) + \dot{\mathbf{Q}}^L(\mathbf{b}, \mathbf{w}) + \dot{\mathbf{Q}}^L(\beta, \mathbf{w}) + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{p}(\mathbf{x}, t) \tag{3.1}$$

$$\mathbf{B}^g(\mathbf{w}) = \mathbf{0} \tag{3.2}$$

$$\mathbf{B}^n(\mathbf{w}) = \mathbf{0}$$

where $\nabla \cdot \tau^L(\mathbf{a}, \mathbf{w})$, $\dot{\mathbf{Q}}^L(\mathbf{b}, \mathbf{w})$, $m(\mathbf{x})$, $\mathbf{p}$, $\mathbf{B}^g$ and $\mathbf{B}^n$ are as defined in section 2.2. $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are the equivalent linear stiffness and damping parameters, respectively, for the auxiliary system.

The form of the equivalent linear damping $\dot{\mathbf{Q}}^L(\beta, \mathbf{w})$ is more restrictive than for $\dot{\mathbf{Q}}^L(\mathbf{b}, \mathbf{w})$. $\dot{\mathbf{Q}}^L(\beta, \mathbf{w})$ is limited to strain related damping; that is, in the Rayleigh damping definition of equation (2.9), $\gamma_1 = 0$ or

$$\mathbf{Q}^L(\beta, \mathbf{w}) = \nabla \cdot \tau^L(\beta, \mathbf{w}) \tag{3.3}$$

It is clear that the form of the linear auxiliary system arises from the replacement of the nonlinear operation $\nabla \cdot \tau^N$ in the nonlinear equation (2.1) by the linear operation $\nabla \cdot \tau^L(\alpha, \mathbf{w}) + \dot{\mathbf{Q}}^L(\beta, \mathbf{w})$. Then if equation (3.1) is to model the response of equation (2.1), $\nabla \cdot \tau^L$ and $\dot{\mathbf{Q}}^L$ must model the response dependent

stiffness and dissipation properties of $\nabla \cdot \tau^N$ in some prescribed way.

3.3 Definition of Equation Differences

Let $\mathbf{w}^* = \mathbf{w}^*(\mathbf{x}, t)$ be a set of functions belonging to a class of functions $\mathbb{C}$ to which solutions of (3.1) belong. The difference terms defined in this section are differences between given nonlinear operations of the system (2.1) and given linear operations of the auxiliary (3.1), both operating on the function $\mathbf{w}^*$. Three difference terms, constituting what are referred to as problem formulations I, II and III, are defined as the following.

Definition - Differential Stress Difference (Formulation I)

Since $\mathbf{w}^*(\mathbf{x}, t)$ is not necessarily a solution of either the nonlinear system or of the auxiliary system, substitution of $\mathbf{w}^*$ into equations (2.1) and (3.1) will leave non-vanishing residual terms. The difference of these residual terms, Δ^I, is written as

$$\Delta^I \equiv \nabla \cdot \tau^N(\mathbf{c}, \mathbf{w}^*, \dot{\mathbf{w}}^*) - \nabla \cdot \tau^L(\alpha, \mathbf{w}^*) - \nabla \cdot \dot{\tau}^L(\beta, \mathbf{w}^*) \tag{3.4}$$

Definition - Energy Difference (Formulation II)

Let $\mathcal{S}(\mathbf{w}^*)$ be a linear spatial operation on $\mathbf{w}^*$ such that for $\mathbf{w}^* \neq \mathbf{0}$

$$\int \tau^L(\alpha, \mathbf{w}^*) \cdot \mathcal{S}(\mathbf{w}^*)\, d(\mathbf{x}) = 0 \qquad \text{iff } \alpha = \mathbf{0}$$

Then the difference term for formulation II, Δ^{II}, is defined as:

$$\Delta^{II} \equiv [\tau^N(\mathbf{c}, \mathbf{w}^*, \dot{\mathbf{w}}^*) - \tau^L(\alpha, \mathbf{w}^*) - \dot{\tau}^L(\beta, \mathbf{w}^*)] \cdot \mathcal{S}(\mathbf{w}^*) \tag{3.5}$$

where the $(\ ) \cdot (\ )$ scalar inner product is a defined in section 2.2.

Definition - Stress Difference (Formulation III)

For formulation III, let the system difference term, Δ^{III}, be defined as :

$$\Delta^{III} \equiv \tau^N(c, \mathbf{w}^*, \dot{\mathbf{w}}^*) - \tau^L(\alpha, \mathbf{w}^*) - \dot{\tau}^L(\beta, \dot{\mathbf{w}}^*) \tag{3.6}$$

For the sake of clarity, a few observations should be made on the preceding defintions.

1. Δ^I for formulation I is directly analogous to the equation difference term for ordinary differential equations (2.31).

2. If $\mathcal{S}(\mathbf{w}^*)$ is interpreted as the strain operator and if the system is elastic, then Δ^{II} would represent the difference in the strain energy density of τ^N and τ^L operating on the same function $\mathbf{w}^*$. Although inelastic systems are not to be excluded from the analysis, Δ^{II} will still be referred to as an "energy" difference.

3. Note that Δ^I and Δ^{II} are functionally related to Δ^{III}. Δ^I is simply the divergence of the stress difference term Δ^{III}, while Δ^{II} is equivalent to $\Delta^{III} \cdot \mathcal{S}$.

3.4 Averaging Operators

In establishing an equivalent linear solution, one is left with the task of minimizing a given difference, Δ, between the nonlinear system and the equivalent linear system. Minimizing on a point by point basis will generally engender too many constraints on the solution to make a solution possible. One alternative is to minimize some measure of the difference: for example, its averaged value over time and space. That precisely is the approach taken here.

For a given difference term, Δ, let its scalar inner product, as defined in section 2.3.2, be given by $\Delta \cdot \Delta$. Then the averaged value of $\Delta \cdot \Delta$ will be given by $G_{tx}(\Delta \cdot \Delta)$, with $G_{tx}(\ )$ defined as below

Definition - Averaging Operator

For a given scalar function, $u = u(\mathbf{x},t)$, let $\mathcal{G}_{tx}(u)$ be defined as follows:

$$\mathcal{G}_{tx}(u) = \mathcal{G}_t(\mathcal{G}_x(u)) \qquad (3.7)$$

$$= \mathcal{G}_x(\mathcal{G}_t(u))$$

where

$$\mathcal{G}_x(u) = \int_{\mathcal{D}} u \, d\mathcal{D}(\mathbf{x}) \qquad (3.8)$$

and $\mathcal{G}_t(u)$ has the following properties

1. Time invariance: $\frac{d}{dt}\ \mathcal{G}_t(u)$
2. Linearity: $\mathcal{G}_t(u+v) = \mathcal{G}_t(u) + \mathcal{G}_t(v)$
3. Positive definiteness: $\mathcal{G}_t(u^2) > 0$ for $u \neq 0$ and $\mathcal{G}_t(0) = 0$

3.5 System Difference Minimization

In order to determine the equivalent linear stiffness and damping parameters, $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$, it is necessary to minimize the averaged system difference between systems (2.1) and (3.1). Let Δ represent one of the system differences defined in section 3.3 and $\mathcal{G}_{tx}(\Delta \cdot \Delta)$ be the averaging operation defined in section 3.4. Formally, this minimization can be stated as the extremization of $\mathcal{G}_{tx}(\Delta \cdot \Delta)$ over all functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$, or

$$\underset{w.r.t.\ \alpha,\beta}{\delta} \ \mathcal{G}_{tx}(\Delta(\alpha,\beta,\mathbf{w}^*) \cdot \Delta(\alpha,\beta,\mathbf{w}^*)) = 0 \quad \text{for all } \mathbf{w}^* \in \mathcal{C} \qquad (3.9)$$

One approach in effecting the extremization in equation (3.9) would be the use of direct analytical variation via calculus of variation. For example, consider the system difference defined for formulation III. Using the property of the inner product operator $(\ ) \cdot (\ )$ given in section 2.3.2 and the linearity of

G_{tx} and τ^L, the variation with respect to α can be written as:

$$\underset{w.r.t\,\alpha}{\delta} \; G_{tx}(\Delta^{III} \cdot \Delta^{III}) = 2 \; G_{tx}(\Delta^{III} \cdot \delta\Delta^{III})$$

$$= -2 \; G_{tx}([\tau^N(\mathbf{w}^*) - \tau^L(\alpha, \mathbf{w}^*) - \dot{\tau}^L(\beta, \mathbf{w}^*)] \cdot \; \tau^L(\delta\alpha, \mathbf{w}^*) \;)$$

The variation with respect to β can be written in a similar way. Therefore the extremizaton in (3.9) can be written as

$$G_{tx}([\tau^N(\mathbf{w}^*) - \tau^L(\alpha, \mathbf{w}^*) - \dot{\tau}^L(\beta, \mathbf{w}^*)] \cdot \tau^L(\delta\alpha, \mathbf{w}^*)) = 0 \tag{3.10}$$

$$G_{tx}([\tau^N(\mathbf{w}^*) - \tau^L(\alpha, \mathbf{w}^*) - \dot{\tau}^L(\beta, \mathbf{w}^*)] \cdot \dot{\tau}^L(\delta\beta, \mathbf{w}^*)) = 0 \tag{3.11}$$

Integration by parts over $\mathcal{D}(\mathbf{x})$ of (3.10) and (3.11) will produce a set of Euler equations in α and β with $\mathbf{x}$ being the independent variable. The Euler equations for α and β will generally be partial differential equations for which general solutions are not readily available. Therefore the exact minimization has not significantly simplified the solution procedure for the original nonlinear partial differential equations. In what follows, an approximate method for performing the extremization of (3.9) will be explored which will produce algebraic relationships for the minimizing stiffness and damping parameters.

Suppose the equivalent linear parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are approximated by the following functions

$$\alpha(\mathbf{x}) \approx \sum_{i=1}^{M} \varphi_i(\mathbf{x})\hat{\alpha}_i \tag{3.12}$$

$$\beta(\mathbf{x}) \approx \sum_{i=1}^{M} \varphi_i(\mathbf{x})\hat{\beta}_i \tag{3.13}$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are unknown constants and $\varphi_i(\mathbf{x})$; $i=1, 2,\ldots,M$, are linearly independent functions of $\mathbf{x}$.

Sufficient conditions for the minimization of $G_{tx}(\Delta \cdot \Delta)$ with respect to the approximations to α and β given in (3.12) and (3.13) are

$$\frac{\partial}{\partial \hat{\alpha}_k} G_{tx}(\Delta \cdot \Delta) = 0 \tag{3.14}$$

$$\frac{\partial}{\partial \hat{\beta}_k} G_{tx}(\Delta \cdot \Delta) = 0 \tag{3.15}$$

The question of whether the $\hat{\alpha}$ and $\hat{\beta}$ that satisfy (3.14) and (3.15) truly minimize $G_{tx}(\Delta \cdot \Delta)$, and not *maximizes* $G_{tx}(\Delta \cdot \Delta)$, will be examined in a later section.

Let attention be focused on developing the minimizing relations (3.14) and (3.15) for the specific problem formulations. Up to this point, the equivalent linear parameters α and β were assumed to be *sets* of parameters for the sake of generality. To expedite the developing of the minimizing relationships, let it be assumed that α and β are sets of one parameter each, or simply α and β. Similar relationships to those in the following sections can be established for larger sets of α and β.

3.5.1 Formulation I minimization

For problem formulation I, the differential stress difference Δ^I was defined as

$$\Delta^I = \nabla \cdot \tau^N(\mathbf{w}^*, \dot{\mathbf{w}}^*) - \nabla \cdot \tau^L(\alpha, \mathbf{w}^*) - \nabla \cdot \dot{\tau}^L(\beta, \mathbf{w}^*) \tag{3.16}$$

Since τ^L is a linear operation,

$$\tau^L(\hat{\alpha}, \mathbf{w}^*) = \sum_{j=1}^{M} \tau_j^L \hat{\alpha}_j \tag{3.17}$$

where τ_j^L is a shorthand notation for $\tau^L(\varphi_j, \mathbf{w}^*)$. Similarly, $\tau^N(\mathbf{w}^*, \dot{\mathbf{w}}^*)$ will be written as τ^N, with the operation on $\mathbf{w}^*$ and $\dot{\mathbf{w}}^*$ implied.

Using (3.16) and (3.17) and the linear property of G_{tx} in the first minimizing

relation (3.14) gives:

$$\frac{\partial}{\partial \hat{\alpha}_k} \mathbb{G}_{tx}(\Delta^I \cdot \Delta^I) = 2\, \mathbb{G}_{tx}(\Delta^I \cdot \frac{\partial \Delta^I}{\partial \hat{\alpha}_k})$$

$$0 = \sum_{i=1}^{M} \mathbb{G}_{tx}([\nabla \cdot \tau^N - \nabla \cdot \tau_i^I \hat{\alpha}_i - \nabla \cdot \dot{\tau}_i^I \hat{\beta}_i] \cdot [\nabla \cdot \tau_k^I]) \tag{3.18}$$

Equation (3.18) can be written in matrix form as:

$$A^I \hat{\alpha} + C^I \hat{\beta} = \mathbf{G} \tag{3.19}$$

where

$$\left.\begin{aligned} A_{ij}^I &= \mathbb{G}_{tx}([\nabla \cdot \tau_i^I] \cdot [\nabla \cdot \tau_j^I]) \\ C_{ij}^I &= \mathbb{G}_{tx}([\nabla \cdot \dot{\tau}_j^I] \cdot [\nabla \cdot \tau_i^I]) \\ G_i^I &= \mathbb{G}_{tx}([\nabla \cdot \tau^N] \cdot [\nabla \cdot \tau_i^I]) \end{aligned}\right\} \tag{3.20}$$

$$\hat{\alpha},\ \hat{\beta} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_M)^T \quad \text{and} \quad (\hat{\beta}_1, \ldots, \hat{\beta}_M)^T \tag{3.21}$$

Similiarly, the minimization relation (3.15) becomes

$$D^I \hat{\alpha} + B^I \hat{\beta} = \mathbf{H}^I \tag{3.22}$$

where

$$\left.\begin{aligned} D_{ij}^I &= \mathbb{G}_{tx}([\nabla \cdot \tau_j^I] \cdot [\nabla \cdot \dot{\tau}_i^I]) \\ B_{ij}^I &= \mathbb{G}_{tx}([\nabla \cdot \dot{\tau}_i^I] \cdot [\nabla \cdot \dot{\tau}_j^I]) \\ H_i^I &= \mathbb{G}_{tx}([\nabla \cdot \tau^N] \cdot [\nabla \cdot \dot{\tau}_i^I]) \end{aligned}\right\} \tag{3.23}$$

Equations (3.19) and (3.22) can simplified further by observing the form of matrices C^I and D^I along with properties of the operator $\mathbb{G}_{tx}$. First note that

$$C^I = (D^I)^T \tag{3.24}$$

where $(D^I)^T$ is the transpose of the matrix D^I. For any two functions $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$ on $\mathcal{S}(\mathbf{x})$, and since $G_{tx}(\,)$ is linear and time invariant,

$$\frac{\partial}{\partial t} G_{tx}(\mathbf{u}\cdot\mathbf{v}) = G_{tx}(\dot{\mathbf{u}}\cdot\mathbf{v}) + G_{tx}(\mathbf{u},\dot{\mathbf{v}})$$

$$= 0$$

Therefore

$$G_{tx}(\dot{\mathbf{u}}\cdot\mathbf{v}) = - \quad _{tx}(\mathbf{u}\cdot\dot{\mathbf{v}}) \tag{3.25}$$

Looking at equations (3.20) and (3.23), it is clear that (3.25) implies

$$C^I = -(D^I)^T \tag{3.26}$$

Equations (3.24) and (3.26) together say that

$$C_{ij} = D_{ij} = 0 \quad ; \; i,j=1,2,\ldots,M \tag{3.27}$$

Therefore the equations for the formulation I parameters $\hat{\alpha}$ and $\hat{\beta}$ are uncoupled from each other.

3.5.1 Formulation II minimization

The system difference for formulation II, the scalar Δ^{II}, was defined to be

$$\Delta^{II} = [\tau^N - \tau^L(\alpha) - \dot{\tau}^L(\beta)]\cdot\mathcal{S}$$

Using (3.14) and the shorthand notation of $\mathcal{S}$ for $\mathcal{S}(\mathbf{w}^*)$, the minimization relations (3.14) and (3.15) can be written as :

$$A^{II}\hat{\alpha} + C^{II}\hat{\beta} = G^{II} \tag{3.28}$$

$$D^{II}\hat{\alpha} + B^{II}\hat{\beta} = H^{II} \tag{3.29}$$

where

$$
\left.
\begin{aligned}
A_{ij}^{II} &= G_{tx}([\tau_i^L \cdot \mathcal{S}\ ][\tau_j^L \cdot \mathcal{S}\ ]) \\
B_{ij}^{II} &= G_{tx}([\dot{\tau}_i^L \cdot \mathcal{S}\ ][\dot{\tau}_j^L \cdot \mathcal{S}\ ]) \\
C_{ij}^{II} &= G_{tx}([\dot{\tau}_j^L \cdot \mathcal{S}\ ][\tau_i^L \cdot \mathcal{S}\ ]) \\
D_{ij}^{II} &= G_{tx}([\tau_j^L \cdot \mathcal{S}\ ][\dot{\tau}_i^L \cdot \mathcal{S}\ ]) \\
G_i^{II} &= G_{tx}([\tau^N \cdot \mathcal{S}\ ][\tau_i^L \cdot \mathcal{S}\ ]) \\
H_i^{II} &= G_{tx}([\tau^N \cdot \mathcal{S}\ ][\dot{\tau}_i^L \cdot \mathcal{S}\ ])
\end{aligned}
\right\} \qquad (3.30)
$$

As was true for the formulation I minimization relations, the matrix D^{II} is the transpose of C^{II}, or

$$D_{ij}^{II} = C_{ji}^{II} \qquad (3.31)$$

However C^{II} and D^{II} are not null matrices, as was the case for formulation I. Consequently, the equations for $\hat{\alpha}$ and $\hat{\beta}$ remain coupled for a general system. As will be seen in section 3.8, the uncoupling of the equations is sufficient condition to guarantee the existence of unique solutions for $\hat{\alpha}$ and $\hat{\beta}$.

There does exist, however, at least one special definiton of the strain operation, $\mathcal{S}$, and of the basis functions, $\varphi_i(\mathbf{x})$, which permit an uncoupling of the minimization equations (3.28) and (3.29). It can be shown that, if $\varphi_i(\mathbf{x})$; $i=1, 2,\ldots,M$ are such that C^{II} and D^{II} are diagonal matrices and the operation $\mathcal{S}$ is proportional to τ^L, C^{II} and D^{II} vanish identically, thereby uncoupling the minimization relations.

3.5.3 Formulation III minimization

The formulation III system difference was given by

$$\Delta = \tau^N - \tau^L(\alpha) - \dot{\tau}^L(\beta) \tag{3.32}$$

In the same manner as for formulations I and II, the minimization relations can be shown to give the following equations for $\hat{\alpha}$ and $\hat{\beta}$:

$$A^{III}\hat{\alpha} + C^{III}\hat{\beta} = \mathbf{G}^{III} \tag{3.33}$$

$$D^{III}\hat{\alpha} + B^{III}\hat{\beta} = \mathbf{H}^{III} \tag{3.34}$$

where

$$\left.\begin{aligned} A_{ij}^{III} &= G_{tx}(\tau_i^L \cdot \tau_j^L) \\ B_{ij}^{III} &= G_{tx}(\dot{\tau}_i^L \cdot \dot{\tau}_j^L) \\ C_{ij}^{III} &= G_{tx}(\dot{\tau}_j^L \cdot \tau_i^L) \\ D_{ij}^{III} &= G_{tx}(\tau_j^L \cdot \dot{\tau}_i^L) \\ G_i^{III} &= G_{tx}(\tau^N \cdot \tau_i^L) \\ H_i^{III} &= G_{tx}(\tau^N \cdot \dot{\tau}_i^L) \end{aligned}\right\} \tag{3.35}$$

for i,j=1,2,...,M

Here again (3.35) shows that

$$C^{III} = (D^{III})^T \ . \tag{3.36}$$

Equation (3.36) and the time invariance of $G_{tx}(\ )$ implies that

$$C_{ij}^{III} = D_{ij}^{III} = 0 \quad ; \ i,j=1,2,...,M$$

which uncouples the equations for $\hat{\alpha}$ and $\hat{\beta}$, as was the case with formulation I.

3.6 Examination of the Minimum

In determining the values of the equivalent linear parameters $\hat{\alpha}$ and $\hat{\beta}$ which made the averaged system difference an extreme value, there is no guarantee that the extreme value is truly a minimum and not actually a maximum. It will be shown in this section that the values of $\hat{\alpha}$ and $\hat{\beta}$ established from equations (3.14) and (3.15) do in fact make the averaged system difference term $\mathcal{G}_{tx}(\Delta \cdot \Delta)$ a minimum.

Again let Δ represent the general system difference term from either formulation I, II or III, and $\mathcal{G}_{tx}$ have the properties specified in section 3.4. Recall the definitions of Δ for the three problem formulations given in section 3.3. All three definitions can be written in the following general form, when the approximations to α and β of (3.12) and (3.13) are considered:

$$\Delta = \mathcal{F}^N(\mathbf{w}, \dot{\mathbf{w}}) - \sum_{k=1}^{M} \mathcal{F}_k^a(\mathbf{w})\hat{\alpha}_k - \sum_{k=1}^{M} \mathcal{F}_k^b(\dot{\mathbf{w}})\hat{\beta}_k \tag{3.37}$$

where $\mathcal{F}^N$, is a nonlinear operation on $\mathbf{w}$ and $\dot{\mathbf{w}}$. $\mathcal{F}_k^a$ and $\mathcal{F}_k^b$; $k=1, 2,\ldots,M$ are linear operations on $\mathbf{w}$ and $\dot{\mathbf{w}}$.

Then $\Delta \cdot \Delta$ is a scalar, *quadratic* polynomial in the parameters $\hat{\alpha}$ and $\hat{\beta}$. Therefore its mixed partial derivatives with respect to $\hat{\alpha}_k$ and $\hat{\beta}_k$; $k=1, 2,\ldots,M$, of order higher than 2 vanish. With $\hat{\alpha}$ and $\hat{\beta}$ satisfying the extremization relationships of (3.14) and (3.15), let another set of parameters $\hat{\alpha}'$ and $\hat{\beta}'$ be defined as:

$$\hat{\alpha}'_k = \hat{\alpha}_k + \delta\hat{\alpha}_k \tag{3.38}$$

$$\hat{\beta}'_k = \hat{\beta}_k + \delta\hat{\beta}_k$$

Using the shorthand notation of $\Delta' \equiv \Delta(\hat{\alpha}', \hat{\beta}')$ and $\Delta \equiv \Delta(\hat{\alpha}, \hat{\beta})$, the averaged inner product $\mathcal{G}_{tx}(\Delta' \cdot \Delta')$ can be expanded in a Taylor series expansion about $\hat{\alpha}$

and $\hat{\beta}$ as

$$G_{tx}(\Delta' \cdot \Delta') = G_{tx}(\Delta \cdot \Delta)$$

$$+ \sum_{p=1}^{M} \frac{\partial}{\partial \hat{\alpha}_p} G_{tx}(\Delta \cdot \Delta)\delta\hat{\alpha}_p + \frac{\partial}{\partial \hat{\beta}_p} G_{tx}(\Delta \cdot \Delta)\delta\hat{\beta}_p$$

$$+ \tfrac{1}{2} \sum_{m,p=1}^{M} \frac{\partial\, G_{tx}(\Delta \cdot \Delta)}{\partial \hat{\alpha}_m\, \partial \hat{\alpha}_p}\delta\hat{\alpha}_m\, \delta\hat{\alpha}_p + \tfrac{1}{2} \sum_{m,p=1}^{M} \frac{\partial\, G_{tx}(\Delta \cdot \Delta)}{\partial \hat{\beta}_m\, \partial \hat{\beta}_p}\delta\hat{\beta}_m\, \delta\hat{\beta}_p$$

$$+ \sum_{m,p=1}^{M} \frac{\partial\, G_{tx}(\Delta \cdot \Delta)}{\partial \hat{\alpha}_m\, \partial \hat{\beta}_p}\delta\hat{\alpha}_m\, \delta\hat{\beta}_p \qquad (3.39)$$

But, from (3.14) and (3.15),

$$\left.\begin{aligned} \frac{\partial}{\partial \hat{\alpha}_p} G_{tx}(\Delta \cdot \Delta) = 0 \\ \frac{\partial}{\partial \hat{\beta}_p} G_{tx}(\Delta \cdot \Delta) = 0 \end{aligned}\right\} \qquad (3.40)$$

The higher order derivatives can be written , with the use of (3.40) as:

$$\frac{\partial^2}{\partial \hat{\alpha}_m \partial \hat{\alpha}_p} G_{tx}(\Delta \cdot \Delta) = 2\, G_{tx} \frac{\partial \Delta}{\partial \hat{\alpha}_m} \cdot \frac{\partial \Delta}{\partial \hat{\alpha}_p}$$

$$= 2\, G_{tx}(\, \mathfrak{F}_m^a \cdot\, \mathfrak{F}_p^a) \qquad (3.41)$$

$$\frac{\partial^2}{\partial \hat{\beta}_m \partial \hat{\beta}_p} G_{tx}(\Delta \cdot \Delta) = 2\, G_{tx} \frac{\partial \Delta}{\partial \hat{\beta}_m} \cdot \frac{\partial \Delta}{\partial \hat{\beta}_p}$$

$$= 2\, G_{tx}(\, \mathfrak{F}_m^b \cdot\, \mathfrak{F}_p^b) \qquad (3.42)$$

$$\frac{\partial^2}{\partial \hat{\alpha}_m \partial \hat{\beta}_p} G_{tx}(\Delta \cdot \Delta) = 2\, G_{tx} \frac{\partial \Delta}{\partial \hat{\alpha}_m} \cdot \frac{\partial \Delta}{\partial \hat{\beta}_p}$$

$$= 2\, G_{tx}(\, \mathfrak{F}_m^a \cdot\, \mathfrak{F}_p^b) \qquad (3.43)$$

Using equations (3.40) - (3.43) in (3.39) and simplifying through the linearity of G_{tx} gives

$$G_{tx}(\Delta' \cdot \Delta') = G_{tx}(\Delta \cdot \Delta) + G_{tx}(\mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) \cdot \mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta})) \tag{3.44}$$

where

$$\mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) = \sum_{m=1}^{M} (\mathfrak{I}_m^a \delta\hat{\alpha}_m + \mathfrak{I}_m^b \delta\hat{\beta}_m)$$

The positive definite properties of G_{tx} and $(\ )\cdot(\ )$ prescribes that

$$\left.\begin{aligned} &G_{tx}(\mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) \cdot \mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta})) > 0 \\ &\qquad \text{for } \mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) \neq 0 \end{aligned}\right\} \tag{3.45}$$

and

$$\left.\begin{aligned} &G_{tx}(\mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) \cdot \mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta})) = 0 \\ &\qquad \text{iff} \quad \mathbf{f}(\delta\hat{\alpha}, \delta\hat{\beta}) = 0 \end{aligned}\right\} \tag{3.46}$$

Hence

$$G_{tx}(\Delta(\hat{\alpha}', \hat{\beta}') \cdot \Delta(\hat{\alpha}', \hat{\beta}')) \geq G_{tx}(\Delta(\hat{\alpha}, \hat{\beta}) \cdot \Delta(\hat{\alpha}, \hat{\beta})) \tag{3.47}$$

In words, equation (3.47) says that for parameters $\hat{\alpha}$ and $\hat{\beta}$ that make $G_{tx}(\Delta \cdot \Delta)$ an extremum, the corresponding extreme value is no larger than any other parameters $\hat{\alpha}'$ and $\hat{\beta}'$. Therefore $G_{tx}(\Delta(\hat{\alpha}, \hat{\beta}) \cdot \Delta(\hat{\alpha}, \hat{\beta}))$ is a global minimum. Note, however, that this does not preclude the possibility of more than one set of $\hat{\alpha}$ and $\hat{\beta}$ which minimizes $G_{tx}(\Delta \cdot \Delta)$. If non-unique solutions do exist, then relation (3.46) implies a linear dependence of the operations $\mathfrak{I}_m^a$ and $\mathfrak{I}_m^b$; $m=1, 2, \ldots, M$, with the form of $\mathfrak{I}_m^a$ and $\mathfrak{I}_m^b$ of course depending on the problem formulation. The existence and uniqueness of the solutions for $\hat{\alpha}$ and

$\hat{\beta}$ will be investigated in a later section for the three problem formulations.

3.7 Spatial Discretization of the Auxiliary Equations

In solving the type of equations given by the linear auxiliary system (3.1), approximate techniques are usually employed. This is especially true when the problem contains inhomogeneous material properties and/or has complex boundary conditions that need to be satisfied. The former will always be true for the equivalent linearization method due to the manner in which the linear parameters are discretized.

The finite element method enjoys widespread usage in solving the types of problems described by (3.1). The underlying principle of the method is the subdivision of the total spatial domain , $\mathcal{D}(\mathbf{x})$, into subdomains, or elements $\mathcal{D}^e(\mathbf{x})$. Within each element the displacements are discretized in such a way that they can be written in terms of linearly independent basis, or "shape", functions of the spatial coordinates, or

$$\mathbf{w}(\mathbf{x},t) = N(\Psi)\mathbf{d}(t) \tag{3.48}$$

where N is a matrix of a set of shape functions $\Psi(\mathbf{x})$ and $\mathbf{d}$ is a set of time dependent nodal coefficients. A Galerkin procedure, with $\mathbf{w}$ defined as in (3.48), can be then be used to approximate the solution to (3.1). This procedure leads to the following discretized equation

$$\mathcal{M}\ddot{\mathbf{d}}(t) + \mathcal{C}\dot{\mathbf{d}}(t) + \mathcal{K}\mathbf{d}(t) = \mathbf{f}(t) \tag{3.49}$$

where the vector $\mathbf{f}$ and the matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are integral functionals of the shape functions Ψ and their derivatives.

The striking difference between the equations (3.49) from the finite element method, and an analogous set derived from the general Galerkin formulation, lies in the shape functions Ψ. The basis functions used for the finite element

method are simple polynomials which are defined over only one element. With such a compact support, boundary conditions are easier to impose locally, along the edge of an element, rather than globally along a more complicated boundary. The accuracy of the method is increased by a refinement of the subdivisions, and not by the classical Galerkin method of including more complex basis functions. The advantage of this is in the ease of implementation into a computer code; that is, the computer instructions remain nearly the same as the mesh is refined.

In order to determine suitable shape functions for a particular problem, the convergence properties of the finite element method must be considered. Convergence of the method, as the mesh of elements is refined, depends on the continuity of the shape functions used. In particular, the stiffness matrix K in equation (3.49) will generally be the strain-energy integral for the system, which will involve a number, say m, of derivatives of the shape functions. Then, if the m-1 derivatives are continuous across the elements, the integrand is finite and the strain-energy integral exists. This is a simplified explanation of one criterion of convergence, which is sufficient *continuity* of the shape functions [54]. The use of shape functions which do not satisfy this continuity requirement does appear in the literature (for example [49]), but attention here will only be given to shape functions which satisfy continuity.

Another requirement for convergence of the finite element method is mathematical *completeness*. From a physical standpoint, completeness implies that the finite element mesh must be capable of modeling a constant strain condition throughout the domain. Completeness can be established for isoparametric elements (see [54]): that is, elements which use the same shape functions on the spatial coordinates as on the displacement field. For the case of non-isoparametric elements, the constant strain condition can be established

through an element "patch test" (reference [48]).

Continuity of the shape functions is also a consideration in the minimization procedure for determining the equivalent linear parameters $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$. For example, in the formulation III minimization relations of equations (3.33) - (3.35), a typical integration that must be performed is:

$$A_{ij}^{III} = \mathcal{C}_t \left(\int \tau^L(\varphi_i, \mathbf{w}) \cdot \tau^L(\varphi_j, \mathbf{w}) \, d\mathcal{V}(\mathbf{x}) \right) \tag{3.50}$$

From (3.50), it is seen that τ^L must be continuous in the mean square sense for derivatives of $\mathbf{w}$ up to n-1, where n is the number of spatial derivatives of $\mathbf{w}$ in τ^L. Generally n will be greater than m, where m has previously been defined as the number of derivatives in the strain energy integral. Therefore, continuity requirements of the displacement field imposed by equivalent linearization tend to be more restrictive than those of the finite element method.

3.8 Existence and Uniqueness of Equivalent Linear Parameters

In section 3.5, a set of relationships were established between the equivalent linear parameters and the solution of the linear auxiliary system such that the difference between given properties of the nonlinear system and the auxiliary system are minimized. The preceding section dealt with the spatial discretization of the auxiliary system, which produces a second set of relations between the equivalent linear solution and the equivalent linear stiffness and damping parameters. Before these ideas are applied to specific problems, some consideration must be given to whether this technique is capable of producing reasonable solutions. Therefore this section deals with the question of whether the minimization technique yields equations for which solutions exist and, if solutions exist, whether they are unique.

The equations for the equivalent linear parameters $\hat{\alpha}$ and $\hat{\beta}$ which minimized the system differences for problem formulations I, II and III can be written in the general form (see equations (3.19), (3.22), (3.28), (3.29), (3.33) and (3.34)):

$$\begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \begin{Bmatrix} \hat{\alpha} \\ \hat{\beta} \end{Bmatrix} = \begin{Bmatrix} \mathbf{G} \\ \mathbf{H} \end{Bmatrix} \tag{3.51}$$

where the matrices A, B and C and the vectors G and H depend on the problem formulation and are functions of the auxiliary equation solutions.

As shown in section 3.5.1 - 3.5.3, the matrix C always vanishes for formulations I and III. Therefore, for the case of uncoupled equations, the existence of unique solutions for $\hat{\alpha}$ and $\hat{\beta}$ depend on the invertibility of matrices A and B. The question of whether a matrix is non-singular can be answered by showing that it is sign definite; that is, if all the eigenvalues of a matrix tend away from zero, then its determinant will never vanish.

The following claims will show that the matrices A and B for all three problem formulations are positive definite according to the definition:

Definition - Matrix Positive Definiteness

A matrix T of size M xM will said to be *positive definite* if for an $M-$ vector $\mathbf{r}$

$$\mathbf{r}^T T \mathbf{r} > 0 \quad ; \text{ if } \mathbf{r} \neq \mathbf{0} \tag{3.52}$$

$$\mathbf{r}^T T \mathbf{r} = 0 \quad ; \text{ iff } \mathbf{r} = \mathbf{0}$$

where $\mathbf{r}^T$ is the transpose of $\mathbf{r}$.

Claim - Positive Definiteness of $\mathbf{A^I}$ and $\mathbf{B^I}$

For matrices A^I and B^I defined in equations (3.20) and (3.23) with $\mathbf{w} \neq \mathbf{0}$, $\dot{\mathbf{w}} \neq \mathbf{0}$ for all $\mathbf{x}$ and t, A^I and B^I are positive definite.

Proof

Recall that for formulation I, A^I and B^I were derived to be

$$A^I_{ij} = \mathcal{G}_{tx}([\nabla \cdot \tau^L(\varphi_i, \mathbf{w})] \cdot [\nabla \cdot \tau^L(\varphi_j, \mathbf{w})])$$

$$B^I_{ij} = \mathcal{G}_{tx}([\nabla \cdot \dot{\tau}^L(\varphi_i, \mathbf{w})] \cdot [\nabla \cdot \dot{\tau}^L(\varphi_j, \mathbf{w})])$$

Then

$$\mathbf{r}^T A^I \mathbf{r} = \sum_{i,j=1}^{M} r_i \; \mathcal{G}_{tx}([\nabla \cdot \tau^L(\varphi_i, \mathbf{w})] \cdot [\nabla \cdot \tau^L(\varphi_j, \mathbf{w})]) r_j$$

$$= \mathcal{G}_{tx}([\nabla \cdot \tau^L(\hat{r}, \mathbf{w})] \cdot [\nabla \cdot \tau^L(\hat{r}, \mathbf{w})]) \tag{3.53}$$

where $\hat{r} = \sum_{i=1}^{M} \varphi_i(\mathbf{x}) r_i$. Note that the linearity of $\mathcal{G}_{tx}$ and τ^L were used in obtaining (3.53). Taking into account the positive definitenes of $\mathcal{G}_{tx}$ and the inner product operation, equation (3.53) says that:

$$\left.\begin{aligned} &\mathbf{r}^T A^I \mathbf{r} \geq 0 \\ &\mathbf{r}^T A^I \mathbf{r} = 0 \quad \text{iff} \quad \nabla \cdot \tau^L(\hat{r}, \mathbf{w}) = 0 \end{aligned}\right\} \tag{3.54}$$

Forming the inner product of $\mathbf{w}$ and $\nabla \cdot \tau^L$, integrating over $\mathcal{D}(\mathbf{x})$, and using the last equation in (3.54) gives

$$\int \mathbf{w} \cdot [\nabla \cdot \tau^L(\hat{r}, \mathbf{w})] d\mathcal{D}(\mathbf{x}) = 0 \tag{3.55}$$

Since $\nabla \cdot \tau^L(\hat{r}, \mathbf{w})$ is positive definite according to section 2.3.2, equation (3.55) can be satisfied for $\mathbf{w} \neq \mathbf{0}$ iff

$$\hat{r} = \sum_{i=1}^{M} \varphi_i(\mathbf{x}) r_i = 0 \tag{3.56}$$

The basis functions $\varphi_i(\mathbf{x})$; $i=1, 2, \ldots, M$, are linearly independent. Hence

$\mathbf{r} = [\tau_1, \tau_2, \cdots, \tau_M]^T = \mathbf{0}$, and therefore A^I is positive definite.

Now looking at B^I,

$$\mathbf{r}^I B^I \mathbf{r} = \mathrm{G}_{tx}([\nabla \cdot \dot{\tau}^L(\hat{\tau}, \mathbf{w})] \cdot [\nabla \cdot \dot{\tau}^L(\hat{\tau}, \mathbf{w})])$$

and by the same arguments as with A^I,

$$\left.\begin{aligned} &\mathbf{r}^T B^I \mathbf{r} \geq 0 \\ &\mathbf{r}^T B^I \mathbf{r} = 0 \quad \text{iff} \quad \nabla \cdot \dot{\tau}^L(\hat{\tau}, \mathbf{w}) = \mathbf{0} \end{aligned}\right\} \tag{3.57}$$

τ^L is a linear operation. Therefore $\dot{\tau}^L(\hat{\tau}, \mathbf{w}) = \tau^L(\hat{\tau}, \dot{\mathbf{w}})$, and with (3.57)

$$\int \dot{\mathbf{w}} \cdot [\nabla \cdot \tau^L(\hat{\tau}, \dot{\mathbf{w}})]\, d\mathcal{B}(\mathbf{x}) = 0 \tag{3.58}$$

Again for positive definite $\nabla \cdot \tau^L$, $\dot{\mathbf{w}} \neq \mathbf{0}$ and linearly independent φ_i, $\mathbf{r} \equiv \mathbf{0}$ if (3.57) holds, and B^I is positive definite. **Q.E.D.**

Claim

The matrices A^{II} and B^{II} developed for formulation II are positive definite.

Claim

The matrices A^{III} and B^{III} developed for formulation III are positive definite.

The proofs for the preceding two claims are quite similar to the proof for formulation I and therefore are not included here.

In conclusion, solutions for the formulation I and III equivalent parameters always exist for non-zero displacements and are unique. For the special case described in section 3.5.2 in which the equations for $\hat{\alpha}$ and $\hat{\beta}$ uncouple, unique solutions for formulation II parameters exist. Unfortunately, attempts to show existence and uniqueness of the general formulation II minimization solutions have not been successful.

IV. SPECIAL CASES OF THE GENERAL APPROACH

Chapter III dealt with the development of the general equivalent linearization approach for nonlinear continuous systems. The relationships between the equivalent linear nodal parameters, $\hat{\alpha}$ and $\hat{\beta}$ and the displacements $\mathbf{w}$ which belong to the class of functions $\mathcal{C}$ were established. Describing $\mathcal{C}$ are the basis functions which are solutions to the linear auxiliary system given by equations (3.1) and (3.2). In this chapter, the equivalent linear parameter relationships are specialized for two types of excitation, $g(t)$, for the nonlinear system of equations (2.1) written in the form

$$\nabla \cdot \tau^L(\mathrm{a}) + \nabla \cdot \tau^N(\mathrm{c}) + \dot{\mathbf{Q}}^L(\mathrm{b}) + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{p}(\mathbf{x})g(t) \tag{4.1}$$

The linear auxiliary equations are written as

$$\nabla \cdot \tau^L(\mathrm{a}) + \nabla \cdot \tau^L(\alpha) + \dot{\mathbf{Q}}^L(\mathrm{b}) + \nabla \cdot \dot{\tau}^L(\beta) + m(\mathbf{x})\ddot{\mathbf{w}} = \mathbf{p}(\mathbf{x})g(t) \tag{4.2}$$

where, as before, the functional dependence of τ^L, τ^N and $\mathbf{Q}^L$ on $\mathbf{w}$ and $\dot{\mathbf{w}}$ is understood. For the excitation $g(t)$ in (4.1) and (4.2), attention is focused on the cases of monofrequency harmonic and stationary Gaussian excitation.

Interest in harmonic response often arises when rotating machinery serves as the excitation of the structural element, or when adjoining structural elements filter the excitation to a single frequency input. In addition to this, the forced vibration tests of structural elements to determine stiffness and energy dissipation properties frequently use harmonic excitation. Therefore, steady-state response of a nonlinear system has significant relevance to engineering problems.

The random nature of excitations due to earthquakes and acoustical noise is also of engineering importance. If the analyst considers a large number of independent records of the random process where no one record's contribution

is more significant than the others, the central limit theorem says that process can be considered *normal*, or Gaussian. If the average and second moments of these records can be assumed to be good substitutes for the actual records of the process, then the use of a Gaussian excitation would appear to be a good approach to these engineering problems.

For simplicity of notation, the displacements $\mathbf{w}(\mathbf{x},t)$ will be assumed to be simply a *scalar* function $w(\mathbf{x},t)$. The examples to be considered in the next chapter will fit into this category. Therefore use of relations developed can be applied to the examples in a straight forward manner. Extension of the results of this chapter can be made to problems requiring a vector representation of displacements but only at the expense of notation.

4.1 Steady-State Harmonic Response

Consider the case where $g(t) = \cos\omega t$ in the nonlinear equation and auxiliary linear equation in (4.1) and (4.2). The nonlinearity, τ^N, of equation (4.1) has been assumed to be symmetric in accordance to the definition of section 2.3.4. Therefore, if the linear auxiliary system of equations (4.2) is to model the nonlinear system, the steady-state, linear solution can be expanded in the first terms of its Fourier series as

$$w(\mathbf{x},t) = U(\mathbf{x})\cos\omega t + V(\mathbf{x})\sin\omega t \tag{4.3}$$

Hence, the set of basis functions for steady-state harmonic response is $[\cos\omega t, \sin\omega t]$.

In discretizing the auxiliary system, the functions $U(\mathbf{x})$ and $V(\mathbf{x})$ from equation (4.3) will be approximated by

$$U(\mathbf{x}) \approx \sum_{i=1}^{N} \psi_i(\mathbf{x}) u_i \tag{4.4}$$

$$V(\mathbf{x}) \approx \sum_{i=1}^{N} \psi_i(\mathbf{x}) v_i \tag{4.5}$$

where ψ_i is a set of N spatial basis functions and u_i and v_i are undetermined coefficients for i=1,2,...,N. Then the discretization can be carried out through the finite element method, Galerkin's method, modal analysis, etc. Equating the coefficients of the sine and cosine functions will give a set of matrix algebraic equations for **u** and **v** in terms of the equivalent linear parameters $\hat{\alpha}$ and $\hat{\beta}$ as:

$$[-\omega^2 \mathcal{M} + \mathcal{K}(\hat{\alpha})]\mathbf{u} + \omega\, \mathcal{C}(\hat{\beta})\mathbf{v} = \mathbf{f} \tag{4.6}$$

$$-\omega\, \mathcal{C}(\hat{\beta})\mathbf{u} + [-\omega^2 \mathcal{M} + \mathcal{K}(\hat{\alpha})]\mathbf{v} = \mathbf{0} \tag{4.7}$$

where $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are the mass, damping and stiffness matrices, respectively. **f** is the discretized excitation term. The form of the matrices and vector will depend on the method used for discretization. Equations (4.6) and (4.7) can then be solved for the "in-phase" and "quadrature" (w.r.t. the excitation) solutions **u** and **v**, respectively, once the stiffness and damping parameters $\hat{\alpha}$ and $\hat{\beta}$ are known.

The system difference minimization procedure produces a set of relations for $\hat{\alpha}$ and $\hat{\beta}$ in terms of the auxiliary equation solution w, as were developed in section 3.5. These equations were written in terms of the general temporal averaging operator $\mathcal{G}_t$. Since the steady-state solution for the auxiliary system (4.2) with $g(t)$ harmonic is periodic with period

$$T = \frac{2\pi}{\omega}, \tag{4.8}$$

a logical choice of $\mathcal{G}_t$ which satisfies the properties of $\mathcal{G}_t$ in section 3.4 is

$$\mathcal{G}_t(\cdot) \equiv \int_0^T (\cdot)\, dt \tag{4.9}$$

Using this definition of $\mathcal{G}_t$, the minimization relations

$$\left.\begin{aligned} \frac{\partial}{\partial \hat{\alpha}_k} \mathcal{G}_{tx}(\Delta\cdot\Delta) = 0 \\ \frac{\partial}{\partial \hat{\beta}_k} \mathcal{G}_{tx}(\Delta\cdot\Delta) = 0 \end{aligned}\right\} \tag{4.10}$$

for k=1,2,....,M, will be specialized for steady-state harmonic analysis for the three problem formulations.

Recall first that the equations (4.10) could be written in the general matrix equation form of:

$$\begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \begin{Bmatrix} \hat{\alpha} \\ \hat{\beta} \end{Bmatrix} = \begin{Bmatrix} \mathbf{G} \\ \mathbf{H} \end{Bmatrix} \tag{4.11}$$

where the form of matrices A,B and C and vectors **G** and **H** depend on the problem formulation. In fact, for formulations I and III, the matrix C vanishes. These matrices will be discussed in the following sections.

4.1.1 Formulation I - steady-state harmonic response

Equations (3.20) and (3.23) give the form of A^I ,B^I, $\mathbf{G}^I$ and $\mathbf{H}^I$ for formulation I. Since τ^L is a linear operation and with w defined in (4.3), the linear stress operator becomes:

$$\tau_i^L(w) = \tau_i^L(U)\cos\omega t + \tau_i^L(V)\sin\omega t \tag{4.12}$$

where $\tau_i^L(w) = \tau^L(\varphi_i, w)$

Using (4.9), (4.12), the definition of $\mathcal{G}_x$ and the orthogonality of $\sin\omega t$ and $\cos\omega t$ over one period of oscillation gives

$$A_{ij}^I = \mathcal{G}_x(\mathcal{G}_t([\nabla\cdot\tau_i^L(w)]\cdot[\nabla\cdot\tau_j^L(w)]))$$

$$= \frac{\pi}{\omega}\int_{\Omega}\left\{[\nabla\cdot\tau_i^L(U)]\cdot[\nabla\cdot\tau_j^L(U)] + [\nabla\cdot\tau_i^L(V)]\cdot[\nabla\cdot\tau_j^L(V)]\right\} d\Omega(\mathbf{x})$$

$$\equiv \frac{\pi}{\omega}\hat{A}\,I_{ij} \tag{4.13}$$

$$B_{ij}^{I} = \mathcal{G}_x(\,\mathcal{G}_t([\nabla\cdot\tau_i^I(\dot{w})]\cdot[\nabla\cdot\tau_j^I(\dot{w})]))$$

$$= \pi\omega\int_{\mathcal{S}}\left\{[\nabla\cdot\tau_i^I(U)]\cdot[\nabla\cdot\tau_j^I(U)] + [\nabla\cdot\tau_i^I(V)]\cdot[\nabla\cdot\tau_j^I(V)]\right\}d\mathcal{S}(\mathbf{x})$$

$$= \pi\omega\hat{A}\,I_{ij} \tag{4.14}$$

$$C_i^I = \mathcal{G}_x(\,\mathcal{G}_t([\nabla\cdot\tau^N(w,\dot{w})]\cdot[\nabla\cdot\tau_i^I(w)]))$$

$$= \frac{\pi}{\omega}\int_{\mathcal{S}}\left\{[\nabla\cdot\mathbf{C}^{(1)}(U,V)]\cdot[\nabla\cdot\tau_i^I(U)] + [\nabla\cdot\mathbf{S}^{(1)}(U,V)]\cdot[\nabla\cdot\tau_i^I(V)]\right\}d\mathcal{S}(\mathbf{x})$$

$$\equiv \frac{\pi}{\omega}\hat{C}\,I_i \tag{4.15}$$

$$H_i^I = \mathcal{G}_x(\,\mathcal{G}_t([\nabla\cdot\tau^N(w,\dot{w})]\cdot[\nabla\cdot\tau_i^I(\dot{w})]))$$

$$= \pi\int_{\mathcal{S}}\left\{[\nabla\cdot\mathbf{C}^{(1)}(U,V)]\cdot[\nabla\cdot\tau_i^I(v)] - [\nabla\cdot\mathbf{S}^{(1)}(U,V)]\cdot[\nabla\cdot\tau_i^I(U)]\right\}d\mathcal{S}(\mathbf{x})$$

$$\equiv \pi\hat{H}\,I_i \tag{4.16}$$

where

$$\mathbf{C}^{(1)}(U,V) = \frac{1}{\pi}\int_0^{2\pi}\tau^N(U\cos\vartheta, V\sin\vartheta)\,\cos\vartheta\,d\vartheta \tag{4.17}$$

$$\mathbf{S}^{(1)}(U,V) = \frac{1}{\pi}\int_0^{2\pi}\tau^N(U\cos\vartheta, V\sin\vartheta)\,\sin\vartheta\,d\vartheta \tag{4.18}$$

$$\vartheta = \omega t$$

The formulation I equations for $\hat{\alpha}$ and $\hat{\beta}$ can then be written as

$$\hat{A}^I\hat{\alpha} = \hat{G}^I \tag{4.19}$$

$$\hat{A}^I\hat{\beta} = \frac{1}{\omega}\hat{H}^I \tag{4.20}$$

A few observations are pertinent to the above expressions for $\hat{\alpha}$ and $\hat{\beta}$:

1. The solutions $\hat{\alpha}$ and $\hat{\beta}$ depend on the first Fourier coefficients of the nonlinearity τ^N as seen by equations (4.17) and (4.18). A number of other approximate methods including Galerkin's method (see equations (2.18f) and (2.18g)) also require the evaluation of such coefficients.

2. In solving for $\hat{\alpha}$ and $\hat{\beta}$, only one matrix $\hat{A}^I$ needs to be computed and inverted. Thus a saving of computation effort has appeared for steady-state harmonic analysis.

3. With $\hat{A}^I$ and $\hat{H}^I$ not being explicit functions of ω, equation (4.20) shows that the effect of the equivalent linear damping, $\hat{\beta}$, will diminish with higher frequencies.

4.1.2 Formulation II - steady-state harmonic response

For the steady-state solutions of w given by (4.3), the stress and strain operations can be written as

$$\tau_i^L(w) = \tau_i^L(U)\cos\omega t + \tau_i^L(V)\sin\omega t \tag{4.21}$$

$$\mathbf{g}(w) = \mathbf{g}(U)\cos\omega t + \mathbf{g}(V)\sin\omega t \tag{4.22}$$

where $\tau_i^L(w) = \tau^L(\varphi_i, w)$. With the above relations, the general matrices and vectors for the formulaton II minimization take on the following special form for steady-state solutions:

$$A_{ij}^{II} = \int \{ [\tau_i^L(U)\cdot \mathbf{S}(V) + \tau_i^L(V)\cdot \mathbf{S}(U)][\tau_j^L(U)\cdot \mathbf{S}(V) + \tau_j^L(V)\cdot \mathbf{S}(U)]$$

$$+ [\tau_i^L(U)\cdot \mathbf{S}(U)][3\tau_j^L(U)\cdot \mathbf{S}(U) + \tau_j^L(V)\cdot \mathbf{S}(V)]$$

$$+ [\tau_i^L(V)\cdot \mathbf{S}(V)][\tau_j^L(U)\cdot \mathbf{S}(U) + 3\tau_j^L(V)\cdot \mathbf{S}(V)] \} \, d\,\mathcal{S}(\mathbf{x}) \tag{4.23a}$$

$$C_{ij}^{II} = \omega \int \{ -[\tau_i^L(U)\cdot \mathbf{S}(U) - \tau_i^L(V)\cdot \mathbf{S}(V)][\tau_j^L(U)\cdot \mathbf{S}(V) + \tau_j^L(V)\cdot \mathbf{S}(U)]$$

$$- [\tau_i^L(U)\cdot \mathbf{S}(V)][\tau_j^L(U)\cdot \mathbf{S}(U) + 3\tau_j^L(V)\cdot \mathbf{S}(V)]$$

$$+ [\tau_i^L(V)\cdot \mathbf{S}(U)][3\tau_j^L(U)\cdot \mathbf{S}(U) + \tau_j^L(V)\cdot \mathbf{S}(V)] \} \, d\,\mathcal{S}(\mathbf{x}) \tag{4.23b}$$

$$D_{ij}^{II} = C_{ji}^{II} \tag{4.23c}$$

$$B_{ij}^{II} = \omega^2 \int \{ [\tau_i^L(U)\cdot \mathbf{S}(U) - \tau_i^L(V)\cdot \mathbf{S}(V)][\tau_j^L(U)\cdot \mathbf{S}(U) - \tau_j^L(V)\cdot \mathbf{S}(V)]$$

$$+ [\tau_i^L(U)\cdot \mathbf{S}(V)][3\tau_j^L(U)\cdot \mathbf{S}(V) - \tau_j^L(V)\cdot \mathbf{S}(U)]$$

$$- [\tau_i^L(V)\cdot \mathbf{S}(U)][\tau_j^L(U)\cdot \mathbf{S}(V) - \tau_j^L(V)\cdot \mathbf{S}(U)] \} \, d\,\mathcal{S}(\mathbf{x}) \tag{4.23d}$$

$$C_i^{II} = \int [\mathbf{C}^{(1)} \cdot \mathbb{S}(U)][3\tau_i^L(U) \cdot \mathbb{S}(U) + \tau_i^L(V) \cdot \mathbb{S}(V)]$$

$$+ [(\mathbf{C}^{(1)} + \mathbf{C}^{(3)}) \cdot \mathbb{S}(V) + (\mathbf{S}^{(1)} + \mathbf{S}^{(3)}) \cdot \mathbb{S}(U)][\tau_i^L(U) \cdot \mathbb{S}(V) + \tau_i^L(V) \cdot \mathbb{S}(U)]$$

$$+ [\mathbf{C}^{(3)} \cdot \mathbb{S}(U) + \mathbf{S}^{(3)} \cdot \mathbb{S}(V)][\tau_i^L(U) \cdot \mathbb{S}(U) - \tau_i^L(V) \cdot \mathbb{S}(V)]$$

$$+ [\mathbf{S}^{(1)} \cdot \mathbb{S}(V)][\tau_i^L(U) \cdot \mathbb{S}(U) + 3\tau_i^L(V) \cdot \mathbb{S}(V)] \; d\,\mathbb{S}(\mathbf{x}) \tag{4.23e}$$

$$H_i^{II} = \omega \int -[\mathbf{C}^{(1)} \cdot \mathbb{S}(U)][\tau_i^L(U) \cdot \mathbb{S}(V) - 3\tau_i^L(V) \cdot \mathbb{S}(U)]$$

$$+ [(-\mathbf{C}^{(1)} + \mathbf{C}^{(3)}) \cdot \mathbb{S}(V) - (\mathbf{S}^{(1)} + \mathbf{S}^{(3)}) \cdot \mathbb{S}(U)][\tau_i^L(U) \cdot \mathbb{S}(U) - \tau_i^L(V) \cdot \mathbb{S}(V)]$$

$$+ [\mathbf{C}^{(3)} \cdot \mathbb{S}(U) + \mathbf{S}^{(3)} \cdot \mathbb{S}(V)][\tau_i^L(U) \cdot \mathbb{S}(V) + \tau_i^L(V) \cdot \mathbb{S}(U)]$$

$$- [\mathbf{S}^{(1)} \cdot \mathbb{S}(V)][3\tau_i^L(U) \cdot \mathbb{S}(V) - \tau_i^L(V) \cdot \mathbb{S}(U)] \; d\,\mathbb{S}(\mathbf{x}) \tag{4.23f}$$

where

$$\mathbf{C}^{(3)} = \frac{1}{\pi} \int_0^{2\pi} \tau^N(U, V, \vartheta) \cos 3\vartheta \, d\vartheta \tag{4.23g}$$

$$\mathbf{S}^{(3)} = \frac{1}{\pi} \int_0^{2\pi} \tau^N(U, V, \vartheta) \sin 3\vartheta \, d\vartheta \tag{4.23h}$$

and $\mathbf{C}^{(1)}$ and $\mathbf{S}^{(1)}$ are as defined in (4.17) and (4.18).

It should be noted that the formulation II minimization requires not only the

first Fourier coefficients $\mathbf{C}^{(1)}$ and $\mathbf{S}^{(1)}$, as did formulation I, but also the third coefficients $\mathbf{C}^{(3)}$ and $\mathbf{S}^{(3)}$. It appears then that this formulation extracts more information from the nonlinearity than formulation I.

4.1.3 Formulation III - steady-state harmonic response

Looking back at the general minimization relations for formulation I, (3.20) and (3.23), and for formulation III, (3.30), one can see that the matrices and vectors A^{III}, B^{III}, $\mathbf{G}^{III}$ and $\mathbf{H}^{III}$ are the same as those for formulation I if the $\nabla \cdot \tau^L$ operation is replaced by τ^L. Using this equivalence, the minimization relations for steady-state harmonic response with formulation III can be written directly using those of formulation I, equations (4.13) - (4.20). That is

$$\hat{A}^{III}\hat{\alpha} = \hat{\mathbf{G}}^{III} \tag{4.24}$$

$$\hat{A}^{III}\hat{\beta} = \frac{1}{\omega}\hat{\mathbf{H}}^{III} \tag{4.25}$$

where

$$\hat{A}_{ij}^{III} = \int [\tau_i^L(U)\cdot \tau_j^L(U) + \tau_i^L(V)\cdot \tau_j^L(V)]\, d \quad (\mathbf{x})$$

$$\hat{G}_i^{III} = \int [\mathbf{C}^{(1)}\cdot \tau_i^L(U) + \mathbf{S}^{(1)}\cdot \tau_i^L(V)]\, d \quad (\mathbf{x}) \tag{4.26}$$

$$\hat{H}_i^{III} = \int [\mathbf{C}^{(1)}\cdot \tau_i^L(V) - \mathbf{S}^{(1)}\cdot \tau_i^L(U)]\, d \quad (\mathbf{x})$$

with $\mathbf{C}^{(1)}$ and $\mathbf{S}^{(1)}$ as defined in (4.17) and (4.18).

4.2 Stationary Random Response

For the case of $g(t)$ in equation (4.1) being a stationary, Gaussian excitation, stationary solutions to (4.1) with zero mean value will be sought. Since the replacement auxiliary system is linear and the input is Gaussian, it is well known that the response w will also be Gaussian.

Because $w(\mathbf{x},t)$ is a random process, the temporal averaging operator for this type of analysis must be a stochastic average. If the process is also *ergodic* the stochastic operator may be written as an ensemble average, or

$$\mathcal{G}_t(\ \mathcal{G}_x(\Delta\cdot\Delta)) = E[\ \mathcal{G}_x(\Delta\cdot\Delta)] \tag{4.27}$$

where $E[\cdot]$ is the mathematical expectation operator.

Before the minimizaton relations are developed for stationary random response, the solution will be expanded in terms of deterministic spatial basis functions ψ_k and stochastic coefficients $d_k(t)$; $k=1, 2,\ldots,N$, as

$$w(\mathbf{x},t) = \sum_{k=1}^{N} \psi_k(\mathbf{x}) d_k(t) \tag{4.28}$$

If $\mathbf{L}$ and $\mathbf{M}$ represent linear operators on w, then, with (4.28), the following can be written:

$$\left.\begin{aligned} E[\mathbf{L}(w)] &= \sum_{k=1}^{N} \mathbf{L}(\psi_k) E[d_k] \\ E[\mathbf{L}(w)\cdot\mathbf{M}(w)] &= \sum_{k,l=1}^{N} \mathbf{L}(\psi_k)\cdot\mathbf{M}(\psi_l) E[d_k d_l] \end{aligned}\right\} \tag{4.29}$$

Recall that w has a zero mean in time. Therefore

$$E[w] = \sum_{K=1}^{N} \psi_k E[d_k] = 0$$

and since the ψ_k are linearly independent

$$E[d_k] = 0 \quad ; \; k=1, 2, \ldots, N \tag{4.30}$$

The zero mean property of $\mathbf{d}$ along with the properties of stationarity and normality permit further statements to be made about certain averaged values of $\mathbf{d}$. It can be shown that for a stationary random vector $\mathbf{d}(t)$ which is differentiable

$$\left.\begin{aligned} E[\mathbf{d}\dot{\mathbf{d}}^T] &= -E[\dot{\mathbf{d}}\mathbf{d}^T] \\ E[\mathbf{d}\ddot{\mathbf{d}}^T] &= E[\ddot{\mathbf{d}}\mathbf{d}^T] = -E[\dot{\mathbf{d}}\dot{\mathbf{d}}^T] \end{aligned}\right\} \tag{4.31}$$

The normality of $\mathbf{d}$ with (4.30) can be used to show that for a function $q = q(y)$ (reference [1]):

$$E[\mathbf{y}\, q(\mathbf{y})] = E[\mathbf{y}\mathbf{y}^T] E[\nabla_{\mathbf{y}} q(\mathbf{y})] \tag{4.32}$$

where $\nabla_{\mathbf{y}} = [\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_N}]^T$ These properties can also be used to show that the higher order moments of $\mathbf{d}$ can be expressed in terms of the second order moments [30]. In particular,

$$\begin{aligned} E[d_1 d_2 d_3 d_4] = {} & E[d_1 d_2] E[d_3 d_4] + E[d_1 d_3] E[d_2 d_4] \\ & + E[d_1 d_4] E[d_2 d_3] \end{aligned} \tag{4.33}$$

Using the discretized form of $w(\mathbf{x}, t)$ given by (4.28), the linear auxiliary equation (4.2) can be spatially discretized via a number of methods to produce a set of stochastic differential equations for $\mathbf{d}$ in the form of

$$\mathcal{M}\ddot{\mathbf{d}} + \mathcal{C}(\beta)\dot{\mathbf{d}} + \mathcal{K}(\alpha)\mathbf{d} = \mathbf{f} g(t) \tag{4.34}$$

where as before the matrices $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ and the vector $\mathbf{f}$ depend on the discretization method.

The relationships (4.31) for a stationary random vector permits the discretized auxiliary equation to be written in terms of its correlation matrices as (see

reference [51]):

$$AX + XA^T = B \tag{4.35}$$

where

$$A = \begin{bmatrix} 0 & -I \\ \mathcal{M}^{-1}\,\mathcal{K}(\alpha) & \mathcal{M}^{-1}\,\mathcal{C}(\beta) \end{bmatrix}$$

$$X = \begin{bmatrix} E[\mathbf{d}\mathbf{d}^T] & E[\mathbf{d}\dot{\mathbf{d}}^T] \\ E[\dot{\mathbf{d}}\mathbf{d}^T] & E[\dot{\mathbf{d}}\dot{\mathbf{d}}^T] \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & E[\mathbf{d}\mathbf{f}^T]\,\mathcal{M}^{-T} \\ \mathcal{M}^{-1}E[\mathbf{f}\mathbf{d}^T] & \mathcal{M}^{-1}E[\mathbf{f}\dot{\mathbf{d}}^T] + E[\dot{\mathbf{d}}\mathbf{f}^T]\,\mathcal{M}^{-T} \end{bmatrix}$$

where 0 and I are the null and identity matrices, respectively.

If $g(t)$ is white noise, i.e. having a constant spectral density, the matrices $E[\mathbf{f}\mathbf{d}^T]$ and $E[\mathbf{f}\dot{\mathbf{d}}^T]$ on the right hand side of (4.35) can be written as:

$$\left.\begin{aligned} E[\mathbf{f}\mathbf{d}^T] &= 0 \\ E[\mathbf{f}\dot{\mathbf{d}}^T] &= \pi\,\mathcal{M}^{-1}W \end{aligned}\right\} \tag{4.36}$$

where W is the spectral density of $g(t)$.

Therefore the stationary random solutions to the linear auxiliary equations with Gaussian excitation can be written in terms of the correlation matrices $E[\mathbf{d}\mathbf{d}^T]$, $E[\mathbf{d}\dot{\mathbf{d}}^T]$ and $E[\dot{\mathbf{d}}\dot{\mathbf{d}}^T]$ where these matrices are related to the equivalent linear parameters α and β through (4.35). The minimization relationships will now be expressed in terms of these correlation matrices.

4.2.1 Formulation I - stationary random response

The coefficient matrix, A^I , for the equivalent linear stiffness parameters $\hat{\alpha}$ were given in equation (3.21) as

$$A^I_{ij} = \mathcal{C}_{tx}([\nabla \cdot \tau^L_i(w)] \cdot [\nabla \cdot \tau^L_j(w)])$$

where, again $\tau^L_i(w) = \tau^L(\varphi_i, w)$ and $\varphi_i(\mathbf{x})$ are the basis functions for α and β. With $w(\mathbf{x}, t)$ spatially discretized as

$$w(\mathbf{x}, t) = \sum_{k=1}^{N} \psi_k(\mathbf{x}) d_k(t)$$

the divergence of the linear stress operator τ^L can be written as

$$\nabla \cdot \tau^L_i(w) = \nabla \cdot \sum_{k=1}^{N} \tau^L_i(\psi_k) d_k$$

Then for stationary random response $\mathcal{C}_t(\,) = E[\,]$ and

$$A^I_{ij} = \sum_{k,l=1}^{N} E_{kl} \int [\nabla \cdot \tau^L_i(\psi_k)] \cdot [\nabla \cdot \tau^L_j(\psi_l)] \, d\, \mathcal{D}(\mathbf{x}) \tag{4.37}$$

where $E_{kl} \equiv E[d_k d_l]$

Similarly the coefficent matrix for the damping parameters $\hat{\beta}$, B^I, along with the right hand side vectors G^I and H^I for both $\hat{\alpha}$ and $\hat{\beta}$ are:

$$B^I_{ij} = \sum_{k,l=1}^{N} E_{kl} \int [\nabla \cdot \tau^L_i(\psi_k)] \cdot [\nabla \cdot \tau^L_j(\psi_l)] \tag{4.38}$$

$$G^I_i = \sum_{k=1}^{N} \int E[(\nabla \cdot \tau^N(w, \dot{w})) \cdot (\nabla \cdot \tau^L_i(\psi_k)) d_k] \, d\, \mathcal{D}(\mathbf{x}) \tag{4.39}$$

$$H^I_i = \sum_{k=1}^{N} \int E[(\nabla \cdot \tau^N(w, \dot{w})) \cdot (\nabla \cdot \tau^L_i(\psi_k)) \dot{d}_k] \, d\, \mathcal{D}(\mathbf{x}) \tag{4.40}$$

where, for example, $E_{\dot{k}l} \equiv E[\dot{d}_k d_l]$.

From the property given in equation (4.32) for a Gaussian vector $\mathbf{d}$ with a zero mean

$$E[(\nabla \cdot \tau^N(w,\dot{w})) \cdot (\nabla \cdot \tau_i^I(\psi_k))d_k] = \sum_{l=1}^{N} E[\frac{\partial}{\partial d_l}(\nabla \cdot \tau^N(\psi, \mathbf{d}, \dot{\mathbf{d}}))] \cdot (\nabla \cdot \tau_i^I(\psi_k)) \; E_{kl}$$

Therefore (4.39) can be written as

$$C_i^I = \sum_{k,l=1}^{N} E_{kl} \int E[\frac{\partial}{\partial d_l}(\nabla \cdot \tau^N(\psi, \mathbf{d}, \dot{\mathbf{d}}))] \cdot (\nabla \cdot \tau_i^I(\psi_k)) \; d \; \mathcal{S}(\mathbf{x}) \tag{4.41}$$

and in a similar way (4.40) becomes

$$H_i^I = \sum_{k,l=1}^{N} E_{kl} \int E[\frac{\partial}{\partial \dot{d}_l}(\nabla \cdot \tau^N(\psi, \mathbf{d}, \dot{\mathbf{d}}))] \cdot (\nabla \cdot \tau_i^I(\psi_k)) \; d \; \mathcal{S}(\mathbf{x}) \tag{4.42}$$

Generally the nonlinearity $\nabla \cdot \tau^N$ can be written in a power series of $\mathbf{d}$ and $\dot{\mathbf{d}}$ as

$$\nabla \cdot \tau^N = \sum_{\substack{m_1 m_2 \cdots \\ n_1 n_2 \cdots}}^{M} \mathcal{F}_{m_1 m_2 \cdots, n_1 n_2 \cdots}(\psi) d_{m_1} d_{m_2} \cdots \dot{d}_{n_1} \dot{d}_{n_2} \cdots$$

which says that $E[\frac{\partial}{\partial d_l}(\nabla \cdot \tau^N)]$ and $E[\frac{\partial}{\partial \dot{d}_l}(\nabla \cdot \tau^N)]$ can be expressed in terms of the higher order moments of the jointly distributed Gaussian vectors $\mathbf{d}$ and $\dot{\mathbf{d}}$. From reference [30], it can be shown that such higher order moments of Gaussian processes can always be expressed in terms of second order moments, for example equation (4.33) of this chapter. Hence the formulation I minimization relationships for stationary, Gaussian response can be written in terms of the correlation matrices for the response of the linear auxiliary solutions.

4.2.2 Formulation II - stationary random response

Using a similar specialization as performed in the last section, the formulation II minimization for stationary random response can be written as:

$$A^{II}\hat{\alpha} + C^{II}\hat{\beta} = C^{II}$$

$$D^{II}\hat{\alpha} + B^{II}\hat{\beta} = H^{II}$$

where

$$A_{ij}^{II} = \sum_{k,l,m,n=1}^{N} \Lambda_{klmn}^{ij}[E_{kl}E_{mn} + E_{km}E_{ln} + E_{kn}E_{lm}]$$

$$C_{ij}^{II} = \sum_{k,l,m,n=1}^{N} \Lambda_{klmn}^{ij}[E_{kl}E_{mn} + E_{km}]E_{ln} + E_{kn}E_{lm}]$$

$$D_{ij}^{II} = C_{ji}^{II}$$

$$B_{ij}^{II} = \sum_{k,l,m,n=1}^{N} \Lambda_{klmn}^{ij}[E_{kl}E_{mn} + E_{km}E_{ln} + E_{kn}E_{lm}]$$

$$G_i^{II} = \sum_{k,l,m,n=1}^{N} [E_{kl}E_{mn} + E_{km}E_{ln} + E_{kn}E_{lm}]$$

$$\int [E[\frac{\partial}{\partial d_l}\tau^N] \cdot \mathcal{S}(\psi_k)][\tau_i^L(\psi_m) \cdot \mathcal{S}(\psi_n)]d\mathcal{A}(\mathbf{x})$$

$$H_i^{II} = \sum_{k,l,m,n=1}^{N} [E_{kl}E_{mn} + E_{km}E_{ln} + E_{kn}E_{lm}]$$

$$\int [E[\frac{\partial}{\partial \dot{d}_l}\tau^N] \cdot \mathcal{S}(\psi_k)][\tau_i^L(\psi_m) \cdot \mathcal{S}(\psi_n)]d\mathcal{A}(\mathbf{x})$$

$$\Lambda_{klmn}^{ij} = \int [\tau_i^L(\psi_k) \cdot \mathcal{S}(\psi_l)][\tau_j^L(\psi_m) \cdot \mathcal{S}(\psi_n)]d\mathcal{A}(\mathbf{x})$$

4.2.3 Formulation III - stationary random response

In section 4.1.3, the formulation III minimization equations for steady- state harmonic response were obtained from the corresponding formulation I relations by replacing $\nabla \cdot \tau^N$ and $\nabla \cdot \tau^L$ by τ^N and τ^L, respectively. Again this similarity

between the two formulations is used to obtain the formulation III relations.

Replacing $\nabla \cdot \tau^N$ and $\nabla \cdot \tau^L$ in equations (4.37), (4.38), (4.41) and (4.42) by τ^N and τ^L gives

$$A^{III}\hat{\alpha} = G^{III} \tag{4.43}$$

$$B^{III}\hat{\beta} = H^{III} \tag{4.44}$$

where

$$\left.\begin{aligned}
A^{III}_{ij} &= \sum_{k,l=1}^{N} E_{kl} \int [\tau^L_i(\psi_k) \cdot \tau^L_j(\psi_l)]\, d\ \mathcal{S}(\mathbf{x}) \\
B^{III}_{ij} &= \sum_{k,l=1}^{N} E_{kl} \int [\tau^L_i(\psi_k) \cdot \tau^L_j(\psi_l)]\, d\ \mathcal{S}(\mathbf{x}) \\
G^{III}_{i} &= \sum_{k,l=1}^{N} E_{kl} \cdot \int E\,[\frac{\partial}{\partial d_l} \tau^N(\psi, \mathbf{d}, \dot{\mathbf{d}})] \cdot \tau^L_i(\psi_k)\, d\ \mathcal{S}(\mathbf{x}) \\
H^{III}_{i} &= \sum_{k,l=1}^{N} E_{kl} \int E\,[\frac{\partial}{\partial \dot{d}_l} \tau^N(\psi, \mathbf{d}, \dot{\mathbf{d}})] \cdot \tau^L_i(\psi_k)\, d\ \mathcal{S}(\mathbf{x})
\end{aligned}\right\} \tag{4.45}$$

Remarks made in section 4.2.1 are herein applicable.

4.3 *Discussion of Solution Procedures*

This chapter has dealt with the development of two sets of equations describing the response of a nonlinear dynamic system via the equivalent linearization technique for the two special cases of steady-state periodic solutions and stationary random response. The first set of these equations arise from the minimization of a given system difference with respect to the equivalent linear stiffness and damping parameters α and β, respectively. The second set results from the spatial discretization of the auxiliary linear system. These two sets of equations are interrelated through α and β and the linear displacements w. This suggests that such equations will have to be solved in an iterative fashion. Techniques for solving different types of vibration problems described by the equations developed in this chapter will now be discussed.

4.3.1 Free vibration response

Consider the case of a nonlinear, conservative system excited only by prescribed initial displacements. Since the system is conservative, the response $W(\mathbf{x},t)$ will continue to oscillate at a frequency, say ω. Then the linear auxiliary response wll be governed by the singular matrix equation

$$[-\omega^2 \mathcal{M} + \mathcal{K}(\alpha, \hat{\alpha})]\mathbf{u} = \mathbf{0} \tag{4.46}$$

which is constrained by the minimization equations

$$A(\mathbf{u})\hat{\alpha} = \mathbf{G}(\mathbf{u}) \tag{4.47}$$

where

- $w(\mathbf{x},t) = (\sum_{i=1}^{N} \psi_k(\mathbf{x}) u_k) \cos\omega t$
- $\mathcal{M}$, $\mathcal{K}$ are the mass and stiffness matrices, respectively, for the discretized auxiliary equation

- a is the stiffness parameter of the linearized equation
- $\alpha(\mathbf{x}) = \sum_{i=1}^{N} \varphi_i(\mathbf{x})\hat{\alpha}_k$ is the equivalent linear stiffness parameter
- $A(\mathbf{u})$ and $\mathbf{G}(\mathbf{u})$ depend on the problem formulation used

If the constraint between $\hat{\alpha}$ and $\mathbf{u}$ in equation (4.47) is not imposed, equation (4.46) is simply a linear eigenvalue problem in ω and $\mathbf{u}$, and therefore the frequency of vibration is *independent* of the amplitude of the response. However the inclusion of this constraint introduces the well known dependence of frequency on the amplitude of vibration. To determine this dependence, an iterative scheme will be introduced.

Let ω_0 be the lowest value of ω which satisfies (4.46) with $\hat{\alpha} = \mathbf{0}$ (i.e. the linearized problem) and $\mathbf{e}_0$ the corresponding eigenvector. Let c be a measure of the amplitude of response, and define the first approximation of $\mathbf{u}$ be equal to

$$\mathbf{u}_1 = c\,\mathbf{e}_0 \ . \tag{4.48}$$

Using (4.48), the first approximation to $\hat{\alpha}$ can be obtained from (4.47)

$$\hat{\alpha}_1 = A^{-1}(\mathbf{u}_1)\mathbf{G}(\mathbf{u}_1) \ . \tag{4.49}$$

Substituting (4.49) into (4.46) gives the next approximation to ω:

$$[-\omega_1^2\, \mathcal{M} + \mathcal{K}\,(a, \hat{\alpha}_1)]\mathbf{e}_1 = \mathbf{0}$$

This iteration continues with

$$\mathbf{u}_k = c\,\mathbf{e}_{k-1} \tag{4.50}$$

until the nonlinear frequency on the k^{th} step, ω_k , converges to prescribed accuracy or the mode shape $\mathbf{e}_k$ satisfies some convergence criterion, where

$$[-\omega_k^2\, \mathcal{M} + \mathcal{K}\,(a, \hat{\alpha}_k)]\mathbf{e}_k = \mathbf{0} \tag{4.51}$$

$$\hat{\alpha}_k = A^{-1}(\mathbf{u}_k)\mathbf{G}(\mathbf{u}_k) \tag{4.52}$$

This method can be extended to higher modes of nonlinear vibration. As in the case of the first mode, the complete eigenvalue problem of (4.51) need not be solved. A number of approximate eigenvalue/eigenvector techniques allow for the extraction of the first few modes without having to solve for the higher ones.

4.3.2 Forced response to harmonic excitation

As described in section (4.1), the equivalent linear response $w(\mathbf{x},t)$ is given by the solutions to

$$\begin{bmatrix} [-\omega^2 \mathcal{M} + \mathcal{K}(a,\hat{\alpha})] & [\omega\, \mathcal{C}(b,\hat{\beta})] \\ [-\omega\, \mathcal{C}(b,\hat{\beta})] & [-\omega^2 \mathcal{M} + \mathcal{K}(a,\hat{\alpha})] \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix} \tag{4.53}$$

and constrained by

$$\begin{bmatrix} A(\mathbf{u},\mathbf{v}) & C(\mathbf{u},\mathbf{v}) \\ C^T(\mathbf{u},\mathbf{v}) & B(\mathbf{u},\mathbf{v}) \end{bmatrix} \begin{Bmatrix} \hat{\alpha} \\ \hat{\beta} \end{Bmatrix} = \begin{Bmatrix} \mathbf{G} \\ \mathbf{H} \end{Bmatrix} \tag{4.54}$$

where $w(\mathbf{x},t) = \boldsymbol{\psi}^T\mathbf{u}\cos\omega t + \boldsymbol{\psi}^T\mathbf{v}\sin\omega t$. The rest of the terms in (4.53) and (4.54) are as defined before.

A "secant" method" method might appear to be a logical way to solve equations (4.53) and (4.54). That is, for a given value of ω and an initial guess for $\mathbf{u}$ and $\mathbf{v}$, $\hat{\alpha}$ and $\hat{\beta}$ are determined from (4.54). This approximation is then used to obtain a next approximation for $\mathbf{u}$ and $\mathbf{v}$ in (4.53) and so on until $\mathbf{u}$ and $\mathbf{v}$ converge according to a prescribed criterion. It has been found, however that such a scheme is unstable, especially for the case of small damping. In this case, the coefficient matrix for $\mathbf{u}$ and $\mathbf{v}$ in (4.53) can become nearly singular for ω near a natural frequency of the linearized response.

Equations (4.53) and (4.54) together are nonlinear in $\mathbf{u}$, $\mathbf{v}$, $\hat{\alpha}$ and $\hat{\beta}$. Hence

the well established Newton-Raphson method for nonlinear algebraic equations can be used to obtain a nonlinear frequency response curve. In the case where good initial guesses for $\mathbf{u}$, $\mathbf{v}$, $\hat{\alpha}$ and $\hat{\beta}$ are difficult to obtain, a hybrid technique such as the Levenberg/Marquardt method (see references [38] and [39]) might be used. This method, which combines features of both Newton's method and the method of steepest descent, has convergence properties less dependent on the initial guess.

If $\mathbf{u}$ and $\mathbf{v}$ are N - vectors and $\hat{\alpha}$ and $\hat{\beta}$ are M - vectors, solution via the above two methods will require the solution of a $2(N+M) \times 2(N+M)$ system of equations a number of times for only one point on the response curve. For large problems, such a number of inversions may prove to be prohibitively expensive in computing a nonlinear frequency spectra. It is often the case that only the resonant response of say the first nonlinear mode of response is desired for a given applied load. A new method will now be presented for establishing a relationship between applied load in the forced problem and the response of the system under free vibration. This method will be restricted to systems with conservative nonlinearities.

In equation (4.53), let the excitation vector $\mathbf{f}$ on the right hand side be written as

$$\mathbf{f} = p_e \mathfrak{F} \tag{4.55}$$

where p_e can be thought of as a spatially constant applied pressure. Since (4.53) is a linear system, resonance occurs when the response is nearly 90° out of phase with the excitation (i.e. $\mathbf{u} \approx \mathbf{0}$) provided the frequencies are well-separated and the damping is small. With $\hat{\beta} = \mathbf{0}$, for a conservative nonlinearity in the original system, equations (4.53) and (4.54) reduce to

$$\omega\, \mathcal{C}(b)\mathbf{v} = p_e\, \mathcal{F} \tag{4.56}$$

$$[-\omega^2 \mathcal{M} + \mathcal{K}(a, \hat{\alpha})]\mathbf{v} = \mathbf{0} \tag{4.57}$$

$$A(\mathbf{v})\hat{\alpha} = \mathbf{G}(\mathbf{v}) \tag{4.58}$$

But equations (4.57) and (4.58) are simply the equivalent linear free vibration problem of (4.46) and (4.47). Premultiplying (4.56) by $\mathbf{v}^T$ and solving for p_e gives

$$p_e = \omega \frac{\mathbf{v}^T \mathcal{C}\, \mathbf{v}}{\mathbf{v}^T \mathcal{F}} \tag{4.59}$$

Equation (4.59) can be thought of as a system power balance relationship at resonance. Here the input power $p_e(\omega \mathbf{v}^T)\, \mathcal{F}$ is balanced by the dissipated power $(\omega \mathbf{v}^T)\, \mathcal{C}\, (\omega \mathbf{v})$.

Consequently, the damped resonant response can be obtained once the free vibration response is known. The steps for such a method are as follows: First the free vibration is solved for $\mathbf{v}$ using (4.57) and (4.58) through the method in section 4.3.1. Then the applied load required for the system to resonate at ω with displacement $\mathbf{v}$ and damping given by $\mathcal{C}$ can be found by using equation (4.59). This will be considerably more cost efficient than solving equations (4.53) and (4.54) directly.

4.3.3 Forced response to stationary random excitation

The equivalent linear equations for stationary random response to Gaussian excitation were written in the form of equation (4.35) and the minimization relations developed in sections 4.2.1 - 4.2.3. These equations can be solved iteratively by a secant method such as the one described in section 4.3.2. The auxiliary system equation of (4.35) is the familiar Liapunov matrix equation which often arises in stability theory of linear systems. References [25] and [35] discuss some computationally efficient algorithms for solving Liapunov's equation.

V. EXAMPLE STUDIES

5.1 Response of a One-Dimensional Yielding Continuum

Many tall structures can be modeled by a continuous shear beam where the base of the beam is excited by a prescribed function of time. Such a system is shown in figure 5.1.1. For large amplitude of response, the need may arise for a mathematical model which includes nonlinearities due to geometric considerations and/or the behavior of the material.

Consider the case of nonlinear material behavior in the beam. Let the spatial domain $\mathcal{D}(x)$ of the beam be $x \in [0, 1]$ and the boundary $\partial\ \mathcal{D}(x)$ be at $x=0$ and $x=1$. If the bending effects are negligible compared to the effects of shearing strains, the equation of motion and boundary conditions for the harmonically excited beam of figure 5.1.1 may be written as

$$-\ddot{w} + \frac{\partial}{\partial x}[a(x)\frac{\partial w}{\partial x}] + \frac{\partial}{\partial x}[\tau^N(w,\dot{w})] = r\cos\omega t \tag{5.1.1}$$

$$w(0,t) = 0 \tag{5.1.2}$$

$$\frac{\partial w}{\partial x}(1,t) + \tau^N(w(1,t), \dot{w}(1,t)) = 0 \tag{5.1.3}$$

in which $w = w(x,t)$ is the deflection of the beam from its unstrained equilibrium position at a distance x from the base. τ^N is a nonlinear stress function/functional of the strain and possibly strain rate, $a(x)$ is the linear stiffness and r is the amplitude of the excitation. The boundary conditons in equations (5.1.2) and (5.1.3) are those of a beam fixed at the base and stress free at the top.

The linear auxiliary equation to be used in solving (5.1.1) - (5.1.3) is written in the following form:

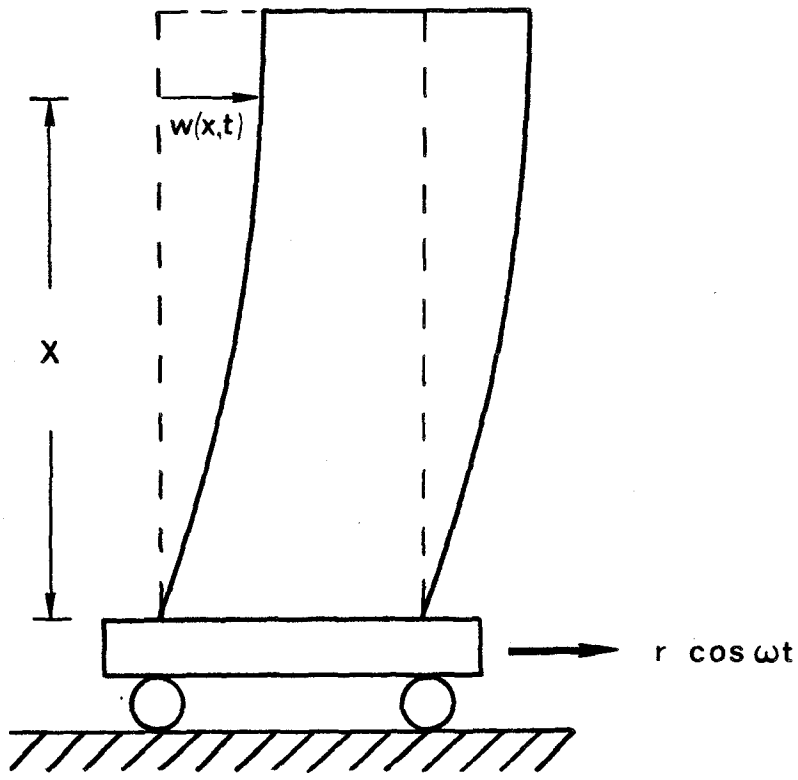


Figure 5.1.1 Shear Structure with Harmonic Base Excitation

$$-\ddot{w} + \frac{\partial}{\partial x}[a(x)\frac{\partial w}{\partial x}] + \frac{\partial}{\partial x}[\alpha(x)\frac{\partial w}{\partial x}] + \frac{\partial}{\partial x}[\beta(x)\frac{\partial \dot{w}}{\partial x}] = r\cos\omega t \qquad (5.1.4)$$

where $\alpha(x)$ and $\beta(x)$ are the equivalent linear stiffness and damping parameters, respectively. The other terms in (5.1.4) are as defined for (5.1.1) and the boundary conditions of (5.1.2) and (5.1.3) also apply to (5.1.4).

It should be noted that equations (5.1.1) and (5.1.4) are simply special cases of the nonlinear and linear auxiliary systems of equation (2.1) and (3.1). The positive definiteness and self-adjointness of $\frac{\partial \tau^L}{\partial x} = \frac{\partial}{\partial x}[a(x)\frac{\partial w}{\partial x}]$ can be shown using the boundary conditions stated in (5.2) and (5.3). The stress operations, τ^L and τ^N, and the displacements are scalar quantities, which implies that the dot product defined in Chapter II is simply a scalar multiplication.

The equivalent linearization approach developed in Chapters III and IV can be applied directly to this problem. The use of problem formulation III for this nonlinear system will be detailed in the following sections.

5.1.1 Minimization relations and discretization of the auxiliary equations

The relationships for $\alpha(x)$ and $\beta(x)$ which minimize the system difference term for steady state harmonic solutions in formulation III were developed in section 4.1.3 and are given by equations (4.24) - (4.26). Recall that for the one-dimensional continuum model of equation (5.1.1), the linear stress operation $\tau^L(\varphi_i, w)$ is given by

$$\tau^L(\varphi_i, w) = \varphi_i(x)\frac{\partial w}{\partial x} \qquad (5.1.5)$$

where φ_i ; $i=1, 2,\ldots,M$, are the linear independent basis functions for the equivalent linear parameters $\alpha(x)$ and $\beta(x)$, as specified in equations (3.12) and (3.13).

Let the spatial domain of the beam be subdivided into "elements"

$\mathcal{D}_e(x)$: $x \in (\frac{e-1}{M}, \frac{e}{M})$ for $e=1, 2,...,M$. Furthermore let the basis functions be defined as

$$\varphi_e(x) = \begin{cases} 1 & \text{if } x \in \mathcal{D}_e \\ 0 & \text{if } x \notin \mathcal{D}_e \end{cases} \tag{5.1.6}$$

Therefore the basis functions are defined in such a way that $\alpha(x)$ and $\beta(x)$ are *constant* within each element but discontinuous across the element boundaries.

Substituting (5.1.5) and (5.1.6) into (4.24) - (4.26) gives the following explicit relations for the elemental equivalent linear parameters of formulation III in terms of the equivalent linear solution $w(x,t)$:

$$\hat{\alpha}_i = \frac{\int_{\mathcal{D}_i} [C(U,V)U_{,x} + S(U,V)V_{,x}]\,dx}{\int_{\mathcal{D}_i} [U_{,x}^2 + V_{,x}^2]\,dx} \tag{5.1.7}$$

$$\hat{\beta}_i = \frac{\int_{\mathcal{D}_i} [C(U,V)V_{,x} - S(U,V)U_{,x}]\,dx}{\omega \int_{\mathcal{D}_i} [U_{,x}^2 + V_{,x}^2]\,dx} \tag{5.1.8}$$

where

$$w(x,t) = U(x)\cos\omega t + V(x)\sin\omega t$$

$$U_{,x},\ V_{,x} = \frac{\partial U}{\partial x},\ \frac{\partial V}{\partial x}$$

$$C(U,V) = \frac{1}{\pi}\int_0^{2\pi} \tau^N(U,V,\vartheta)\cos\vartheta\, d\vartheta$$

$$S(U,V) = \frac{1}{\pi}\int_0^{2\pi} \tau^N(U,V,\vartheta)\sin\vartheta\, d\vartheta$$

$$\vartheta = \omega t$$

Let $\psi_i(x)$ and λ_i^2 ; $i=1, 2, \ldots$ be the eigenfunctions and eigenvalues of the linearized form of equation (5.1.1). The auxiliary linear system will now be discretized in terms of the N eigenfunctions; that is

$$U(x) = \sum_{i=1}^{N} \psi_i(x) u_i = \psi^T \mathbf{u} \tag{5.1.9}$$

$$V(x) = \sum_{i=1}^{N} \psi_i(x) v_i = \psi^T \mathbf{v} \tag{5.1.10}$$

Substituting (5.1.9) and (5.1.10) into (5.1.4), premultiplying by $\psi(x)$, integrating over $\mathcal{S}(x)$ using orthogonality properties of $\psi(x)$, and equating coefficients of $\cos\omega t$ and $\sin\omega t$ gives the modally discretized form of the linear auxiliary equation as

$$\Lambda \mathbf{u} + \mathrm{A}\dot{\mathbf{u}} + \mathrm{B}\mathbf{v} = \tau \mathbf{f} \tag{5.1.11}$$

$$\Lambda \mathbf{v} + \mathrm{A}\dot{\mathbf{v}} - \mathrm{B}\mathbf{u} = \mathbf{0} \tag{5.1.12}$$

where

$$\Lambda = [\,\Lambda_i\,] \tag{5.1.13}$$

$$\Lambda_i = \lambda_i^2 - \omega^2 \tag{5.1.14}$$

$$\mathrm{A} = \sum_{k=1}^{M} \hat{\alpha}_k \int_{\mathcal{S}_k} \psi\, \psi_{,xx}^T \, dx \tag{5.1.15}$$

$$\mathrm{B} = \omega \sum_{k=1}^{M} \hat{\beta}_k \int_{\mathcal{S}_k} \psi\, \psi_{,xx}^T \, dx \tag{5.1.16}$$

$$\mathbf{f} = \int \psi \, d\,\mathcal{S}(\mathbf{x}) \tag{5.1.17}$$

$$\psi_{,xx} = [\frac{\partial^2 \psi_1}{\partial x^2}, \cdots, \frac{\partial^2 \psi_N}{\partial x^2}]^T \tag{5.1.18}$$

It will be assumed that the linearized form of equation (5.1.1) has spatially constant stiffness properties, i.e. $a \neq a(x)$. Then its eigenvalues and normalized eigenfunctions take on the form:

$$\left.\begin{aligned} \lambda_i &= \frac{\pi}{2}(2i-1)\sqrt{a} \\ \psi_i(x) &= \sqrt{2}\sin\frac{\pi}{2}(2i-1)x \end{aligned}\right\} \qquad (5.1.19)$$

for i=1,2,... . The spatial basis functions, ψ , used in all further analysis will be written as in(5.1.19) unless otherwise stated.

The solution of the equivalent linearized problem is the solution to the set of equations given in (5.1.7), (5.1.8), (5.1.11) and (5.1.12). These can be solved once the nonlinear stress functional τ^N is specified. In the next section, a form of τ^N will be introduced to model a yielding material behavior in the one-dimensional beam.

5.1.2 Specification of the yielding model

In reference [24], a model for yielding behavior in one dimension has been presented which consists of an infinite collection of ideal elastic-plastic elements with continuous distributed yield levels. The same author, in reference [23], extended this yielding model to cyclic loading for a yielding level distribution which is constant for strains up to a level, say $W_{,x}^u$ and zero for strains levels above $W_{,x}^u$ where

$$W_{,x}^u = \frac{1}{\sqrt{3\mu}} \qquad (5.1.20)$$

and μ is a constant specifying the nonlinearity of the material. The resulting cyclic stress functional, as given in [23], is an integral relationship in terms of the strain level $W_{,x}$. This stress-strain relationship is shown in figure 5.1.2 for $\mu = 0.2$.

A solution of the nonlinear vibration problem using the above yielding model has been obtained in reference [23] by the application of Galerkin's method. In developing a solution, the same Fourier coefficients $C(U,V)$ and $S(U,V)$ as in the

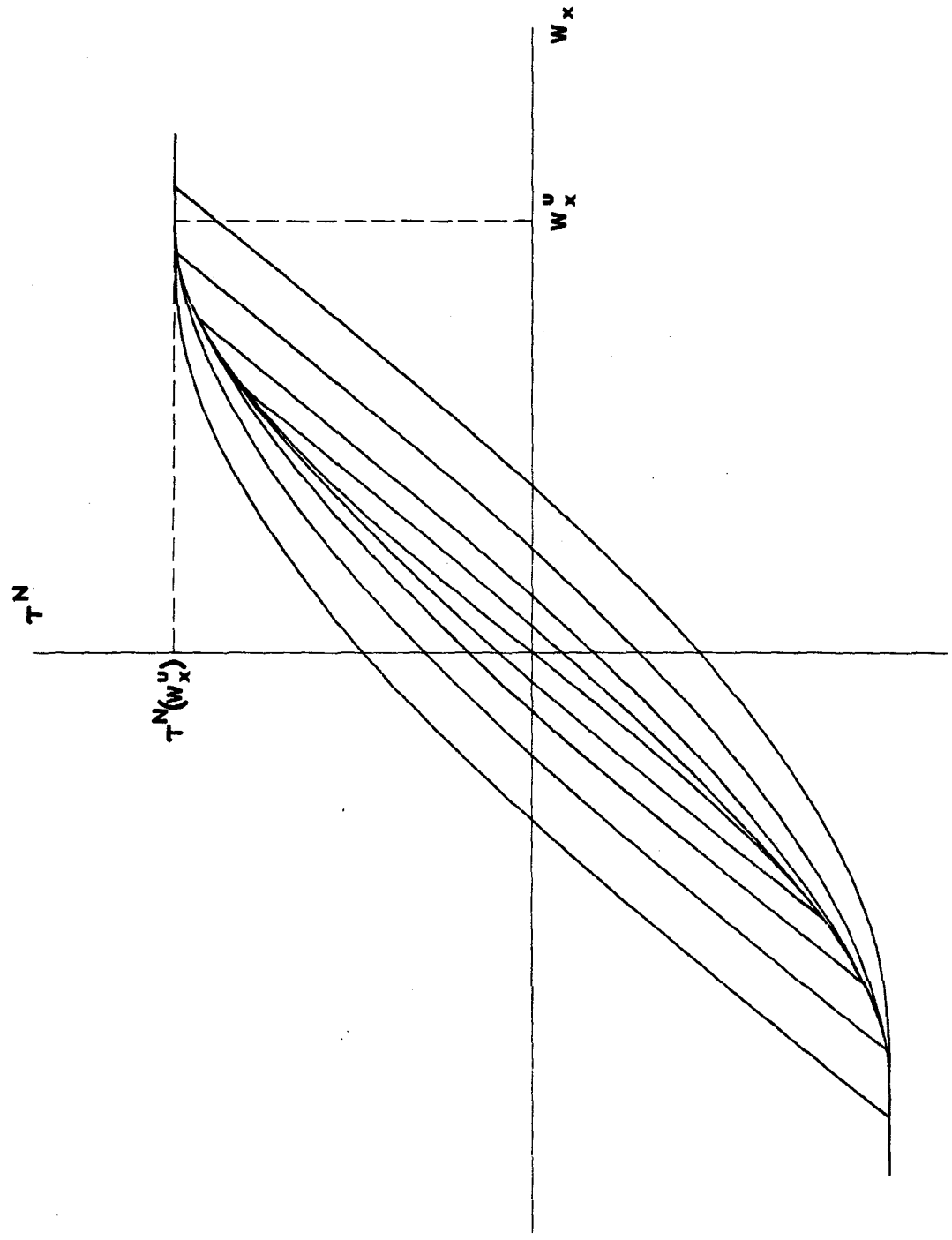


Figure 5.1.2 Hysteresis Loops for Yield Model

equivalent linear parameter relations in (5.1.7) and (5.1.8) of this chapter were evaluated as

$$C(U_{,x}, V_{,x}) = -\frac{15}{16}\mu\,[U_{,x}^2 + V_{,x}^2]^{\frac{3}{2}} \tag{5.1.21}$$

$$S(U_{,x}, V_{,x}) = -\frac{2}{\pi}\mu\,[U_{,x}^2 + V_{,x}^2]^{\frac{3}{2}}$$

for $W_{,x} \leq W_{,x}^u$ and where $W_{,x} = [U_{,x}^2 + V_{,x}^2]^{\frac{1}{2}}$. The relationships for C and S become more complicated when $W_{,x} > W_{,x}^u$. Therfore, as in reference [23], only strain levels satisfying $W_{,x} \leq W_{,x}^u$ will be considered herein.

Substituting (5.1.21) into (5.1.7) and (5.1.8) with the use of the expansions of $U(x)$ and $V(x)$ in (5.1.9) and (5.1.10) give the following relations for the discretized equivalent linear parameters:

$$\hat{\alpha}_i = \frac{-\mu \sum_{m=1}^{N} (\frac{15}{16}u_m + \frac{2}{\pi}v_m)\ \mathcal{K}_m^{(i)}}{\mathcal{L}^{(i)}} \tag{5.1.22}$$

$$\hat{\beta}_i = \frac{-\frac{\mu}{\omega} \sum_{m=1}^{N} (\frac{15}{16}v_m - \frac{2}{\pi}u_m)\ \mathcal{K}_m^{(i)}}{\mathcal{L}^{(i)}} \tag{5.1.23}$$

where

$$\mathcal{K}_m^{(i)} = \sum_{k,l=1}^{N} (u_k u_l + v_k v_l)^{\frac{3}{2}} \int_{\mathcal{D}_i} (\psi_{k,x}\psi_{l,x})^{\frac{3}{2}} \psi_{m,x}\, dx$$

$$\mathcal{L}^{(i)} = \sum_{k,l=1}^{N} (u_k u_l + v_k v_l) \int_{\mathcal{D}_i} \psi_{k,x}\psi_{l,x}\, dx$$

$$\psi_{k,x} = \frac{\partial \psi_k}{\partial x}$$

5.1.3 Convergence of equivalent linear parameters

Recall from Chapter III that the equations for the discretized equivalent parameters $\hat{\alpha}$ and $\hat{\beta}$ were established by an approximation to the exact minimization in equation (3.9). The question may arise as to how well the approximations

$$\alpha(x) \approx \sum_{i=1}^{M} \varphi_i(x)\hat{\alpha}_i \tag{5.1.24}$$

$$\beta(x) \approx \sum_{i=1}^{M} \varphi_i(x)\hat{\beta}_i \tag{5.1.25}$$

compare with the values of α and β that satisfy (3.9). In this section, it will be shown for the basis functions φ_i ; $i=1, 2,\ldots,M$, defined in (5.1.6) how the approximate values for α and β approach their exact counterparts as $M \to \infty$. For the special yielding nonlinearity specified in the last section, the convergence rate will also be given.

The formulation III stiffness parameters $\hat{\alpha}_i$ for the one-dimensional problem of this chapter were shown to be (equation (5.7)):

$$\hat{\alpha}_i = \frac{\int_{\mathcal{U}_i} f(x)\,dx}{\int_{\mathcal{U}_i} g(x)\,dx} \tag{5.1.26}$$

where

$$f(x) = C(U,V)U_{,x} + S(U,V)V_{,x}$$

$$g(x) = U_{,x}^2 + V_{,x}^2$$

If the minimization of (3.9) is carried out *exactly* for the formulation III equation difference, it can be shown that $\alpha(x)$ must satisfy the relationship:

$$\alpha(x) = \frac{f(x)}{g(x)} \tag{5.1.27}$$

Recall that the displacement functions $U(x)$ and $V(x)$ were expanded in terms of the basis functions $\psi_i(x)$; $i=1, 2,...,N$, as shown in equations (5.1.9) and (5.1.10). Suppose that the functions $\psi_i(x)$ are such that $f(x)$ and $g(x)$ are continuous except at say the boundaries of $\mathcal{D}_i(x)$, where they may be discontinuous but finite. These types of discontinuities may arise if, for example, the functions $\psi_i(x)$ are finite element shape functions and $f(x)$ and $g(x)$ are differential functions of $\psi_i(x)$. Let x_i ; $i=1, 2,...,M$, be the equally spaced coordinates of the possible discontinuities. Then α at the discontinuities will be defined to be

$$\alpha(x_i) \equiv \tfrac{1}{2} [\alpha(x_i^-) + \alpha(x_i^+)] \tag{5.1.28}$$

where $x_i^- = \lim_{\varepsilon \to 0}(x_i - \varepsilon)$ and $x_i^+ = \lim_{\varepsilon \to 0}(x_i + \varepsilon)$.

Since $f(x)$ and $g(x)$ are continous on $x \in (x_i, x_{i+1})$, the mean value theorem can be applied to $\hat{\alpha}_i$ in (5.1.26) to show that

$$\lim_{h \to 0} \hat{\alpha}_i = \frac{f(x_i^+)}{g(x_i^+)} \tag{5.1.29}$$

and

$$\lim_{h \to 0} \hat{\alpha}_{i-1} = \frac{f(x_i^-)}{g(x_i^-)} \tag{5.1.30}$$

where $h = x_{i+1} - x_i = x_i - x_{i-1} = \frac{1}{M}$. Therefore, from (5.1.28) - (5.1.30)

$$\lim_{h \to 0} \tfrac{1}{2} [\hat{\alpha}_i + \hat{\alpha}_{i-1}] = \tfrac{1}{2} [\frac{f(x_i^+)}{g(x_i^+)} + \frac{f(x_i^-)}{g(x_i^-)}]$$

$$\equiv \alpha(x_i) \tag{5.1.31}$$

Equation (5.1.31) says that the approximate equivalent linear stiffness

parameters $\hat{\alpha}$ converge as $h \to 0$ to their exact counterpart α in an "average" sense across a discontinuity of the functions $f(x)$ and $g(x)$. If no discontinuities exist, it is clear that $f(x_i^+) = f(x_i^-)$ and $g(x_i^+) = g(x_i^-)$, implying that $\hat{\alpha}_i$ converges exactly to $\alpha(x_i)$ or

$$\lim_{h \to 0} \hat{\alpha}_i = \alpha(x_i)$$

Similar statements can be made about the convergence of $\hat{\beta}_i$ to $\beta(x_i)$

Consider now the equivalent linear parameter relations (5.1.22) and (5.1.23) for the yielding nonlinearity, with $U(x)$ and $V(x)$ expanded in terms of only one mode of the linearized problem. That is, from (5.1.9), (5.1.10) and (5.1.19), U and V are written as

$$U(x) = [\sqrt{2} \sin \frac{\pi}{2} x \,] u_1$$

$$V(x) = [\sqrt{2} \sin \frac{\pi}{2} x \,] v_1$$

The equivalent linear parameters then reduce to

$$\hat{\alpha}_i = c_a \frac{\int_{\mathcal{D}_i} \cos^4 \frac{\pi}{2} x \, dx}{\int_{\mathcal{D}_i} \cos^2 \frac{\pi}{2} x \, dx} \tag{5.1.32}$$

$$\hat{\beta}_i = c_b \frac{\int_{\mathcal{D}_i} \cos^4 \frac{\pi}{2} x \, dx}{\int_{\mathcal{D}_i} \cos^2 \frac{\pi}{2} x \, dx} \tag{5.1.33}$$

where

$$c_a = -2\mu (\frac{\pi}{2})^2 (u_1^2 + v_1^2)^{\frac{1}{2}} [\frac{15}{16} u_1 + \frac{2}{\pi} v_1]$$

$$c_b = -2\frac{\mu}{\omega}\left(\frac{\pi}{2}\right)^2 (u_1^2 + v_1^2)^{\frac{1}{2}} \left[\frac{15}{16}v_1 - \frac{2}{\pi}u_1\right]$$

The exact minimization of the formulation III equation difference with respect to α and β gives:

$$\alpha(x) = c_a \cos^2\frac{\pi}{2}x \tag{5.1.34}$$

$$\beta(x) = c_b \cos^2\frac{\pi}{2}x \tag{5.1.35}$$

Let e be defined as the mean square value of the error between $\alpha(x)$ and its approximation $\alpha(x) \approx \sum_{i=1}^{M} \varphi_i(x)\hat{\alpha}_i$, or

$$e = \int_0^1 [\alpha(x) - \sum_{i=1}^{M} \varphi_i(x)\hat{\alpha}_i]^2 \, dx$$

$$= c_a^2 E_M$$

where $E_M \equiv \sum_{i=1}^{M} \int_0^1 [\cos^2\frac{\pi}{2}x - \frac{\int_{\mathcal{D}_i} \cos^4\frac{\pi}{2}x\,dx}{\int_{\mathcal{D}_i} \cos^2\frac{\pi}{2}x\,dx}]^2 \, dx$

The rate of convergence of $\sum_{i=1}^{M} \varphi_i(x)\hat{\alpha}_i \to \alpha$ will be determined by investigating how $E_M \to 0$ as $M \to \infty$. Figure 5.1.3 shows the relationship of $\log(\frac{1}{M})$ vs. $\log E_M$. For large values of M , the slope of this curve is about 2; therefore, the error decreases at a nearly quadratic rate as $\frac{1}{M} \to 0$.

From (5.1.33) and (5.1.35) it is clear that the convergence of $\sum_{i=1}^{M} \varphi_i(x)\hat{\beta}_i \to \beta(x)$ as $M \to \infty$ is governed by the same error function E_M. Therefore the approximate equivalent linear damping parameters also converge at a quadratic rate.

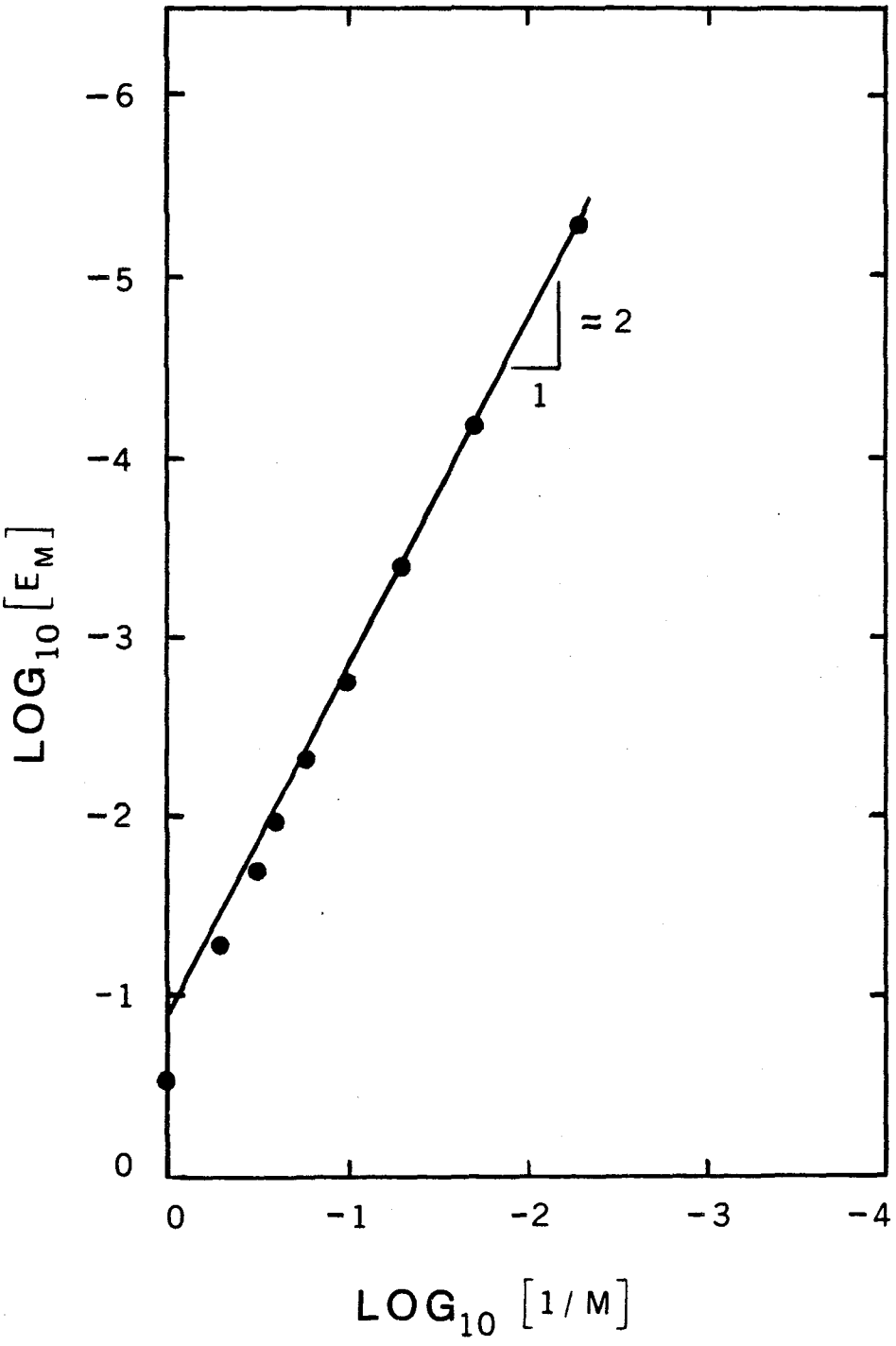


Figure 5.1.3 Convergence Rate for Equivalent Linear Parameters

5.1.4 Numerical results

In this section, a one term expansion on $w(x,t)$ will be used to calculate the response of the yielding shear beam by means of the equivalent linearization technique and Galerkin's method. The equivalent linearization equations to be solved are equations (5.1.11), (5.1.12), (5.1.22) and (5.1.23) with N = 1 while the Galerkin equations can be found in reference [22].

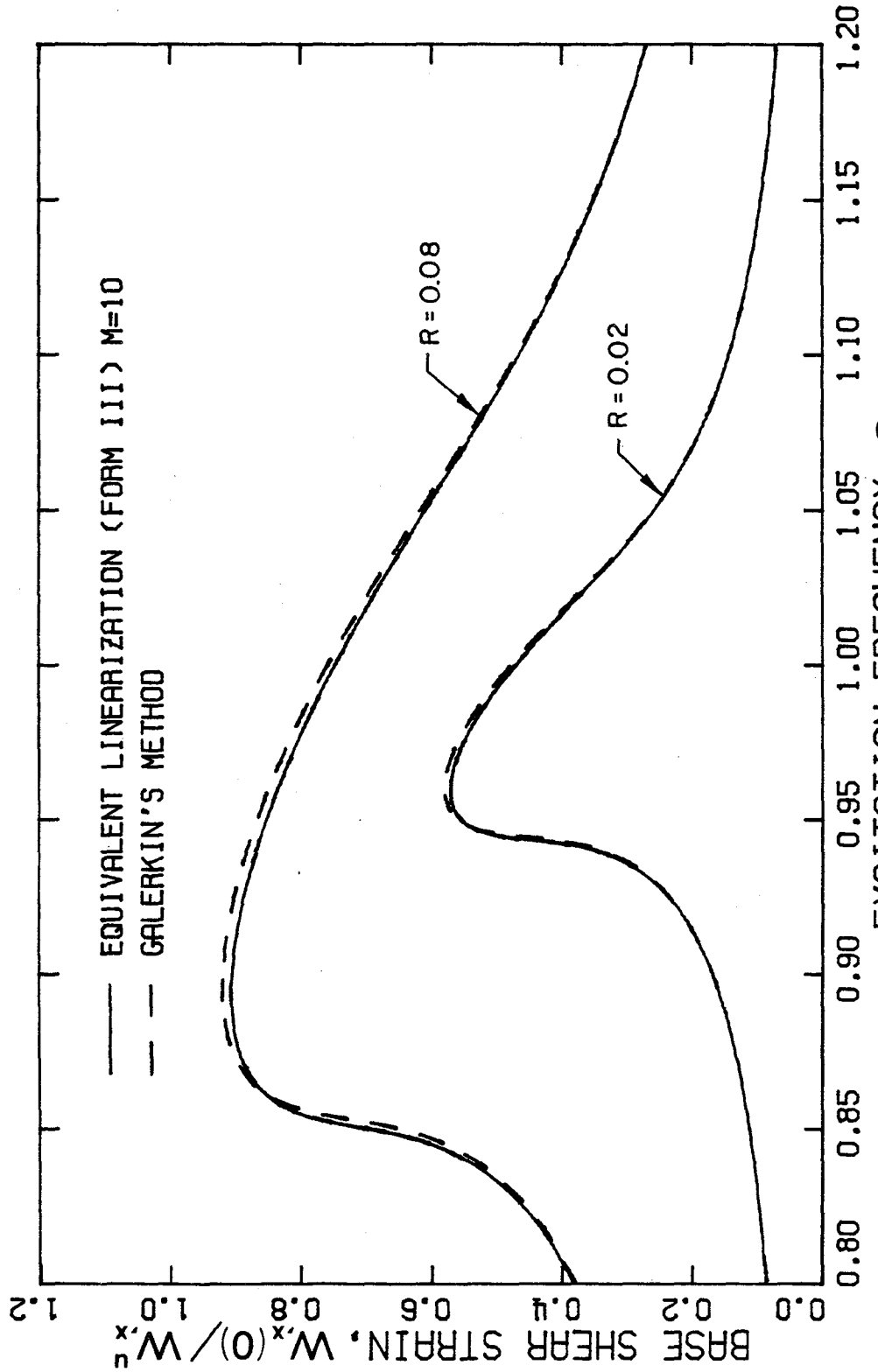
Figure 5.1.4 shows the base shear strain ratio $\frac{W_{,x}}{W^u_{,x}}$ obtained from the equivalent linearization method with M = 10 and Galerkin's method in terms of the dimensionless excitation frequency Ω, where

$$\Omega \equiv \frac{\omega}{\lambda_1} \tag{5.1.36}$$

These response curves are shown on a somewhat expanded scale in order that the characteristics of the two types of solutions can be seen in detail. As can be seen, with M = 10 the equivalent linearization technique predicts a slightly lower level of response and lower resonant frequency than Galerkin's method. This difference is more pronounced for larger levels of excitation R where

$$R \equiv \frac{r}{\lambda_1^2} \tag{5.1.37}$$

Figure 5.1.5 shows the effect of mesh subdivision for the equivalent linear parameters on the peak response of the beam. In the last section, it was shown that as $M \to \infty$, the equivalent linear parameters obtained by the approximate minimization technique converged to their counterparts obtained by the exact minimization technique. An interesting result is that when the exact linear parameters are used in the one term expansion of the auxiliary linear equations, the equivalent linear solutions are exactly the same as the one term Galerkin solution. This is displayed in figure 5.1.5 by the coincidence of the


EQUIVALENT LINEARIZATION (FORM III) M=10
GALERKIN'S METHOD
R=0.08
R=0.02
BASE SHEAR STRAIN, $W_{,x}(0)/W^{c}_{,x}$
EXCITATION FREQUENCY, Ω
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.80
0.85
0.90
0.95
1.00
1.05
1.10
1.15
1.20

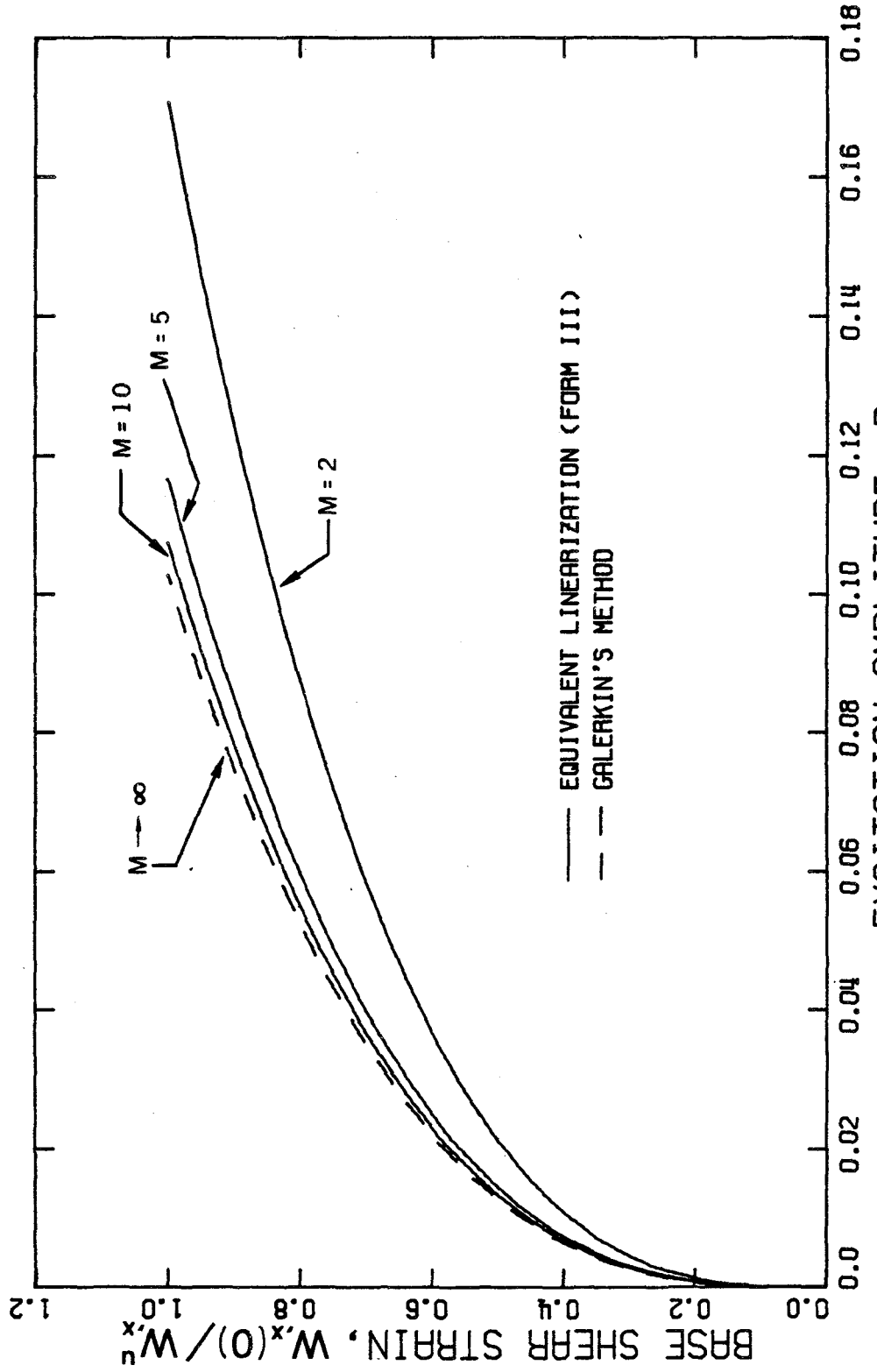


Figure 5.1.5 Peak Response Amplitude as a Function of Excitation Amplitude for $\mu = 0.2$

$M \to \infty$ equivalent linearization curve with the Galerkin response. Therefore, for this special case, there appears to be an equivalence between the two methods.

5.2 Steady-State Harmonic Response of a Nonlinear Plate

A typical assumption used in the vibration of thin plates governed by the Poisson-Kirchoff theory is that the amplitude of response is small in comparison to the thickness of the plate. This assumption permits the use of a linear equation of motion for describing the response of the plate. Whenever the deflection becomes of the same order of magnitude as the thickness, geometric nonlinearities arise from the coupling of the membrane and bending stresses. In this case a linear model of the response of the plate would not be appropriate.

The nonlinear theory used in this example is based on the so-called Berger approximation to the coupled nonlinear Von Karman equations for static analysis. Berger's analysis, which originally appeared in reference [2], is essentially based upon neglecting the strain energy due to the second invariant of the middle surface. This assumption permits the nonlinear equations for the in-plane and transverse displacements to become uncoupled. The work presented in reference [36] extended the Berger approximation to the dynamic equations of motion.

5.2.1 Equations of motion

The nonlinear equations of motion can be derived through the application of the generalized Hamilton's principle (see section 2.4.2) which states that

$$\delta \int_{t_1}^{t_2} (T + W - U)\, dt = 0 \tag{5.2.1}$$

where T, U, and W are the kinetic energy, potential energy and non-conservative virtual work of the system, respectively. Let the spatial domain of the plate be defined for rectangular coordinates $\mathbf{x} = (x_1, x_2)$ on $\mathcal{S}(\mathbf{x})$ and the boundary $\partial\, \mathcal{S}(\mathbf{x})$. Then if the in-plane kinetic energy is negligible compared to that due to transverse motion, the kinetic energy can be written as:

$$T = \tfrac{1}{2} \int_{\mathcal{L}} \rho \dot{w}^2 \, d\mathcal{L}(\mathbf{x}) \tag{5.2.2}$$

and the work done by an externally applied, harmonic load is

$$W = -[\int_{\mathcal{L}} p(\mathbf{x}) w \, d\mathcal{L}(\mathbf{x})] \cos\omega t \tag{5.2.3}$$

where $w = w(\mathbf{x}, t)$ is the transverse displacement of the plate for which $\dot{w} = \frac{\partial w}{\partial t}$, ρ is the mass per unit area and $p(\mathbf{x})$ is the distributed applied load. From reference [2], the potential energy of the plate which neglects the second invariant of strain energy can be written for a non-homogeneous elastic material as:

$$U = \tfrac{1}{2} \int_{\mathcal{L}} [M_{\gamma\delta}(D, w) w_{,\gamma\delta} + \frac{12D}{h^2} e^2] \, d\mathcal{L}(\mathbf{x}) \tag{5.2.4}$$

where the summation convention is implied on $\gamma, \delta = 1, 2$ and

$$M_{11} = D \, (w_{,11} + \nu w_{,22})$$

$$M_{12} = D \, (1-\nu) w_{,12} = M_{21}$$

$$M_{22} = D \, (\nu w_{,11} + w_{,22})$$

$$e = u_{,1} + v_{,2} + \tfrac{1}{2} w_{,\alpha} w_{,\alpha}$$

= the first invariant of the strains at the midplane surface

u, v = in-plane displacements of the plate

in the x_1 and x_2 directions, respectively

ν = Poisson's ratio for the material

h = thickness of the plate

$$D = \frac{Eh^3}{12\,(1-\nu^2)} = \text{flexural rigidity of the plate}$$

E = modulus of elasticity for the material

$$w_{,\gamma} = \frac{\partial w}{\partial x_\gamma}$$

Using the energy relations (5.2.2) - (5.2.4) in the variational equation (5.2.1) and integrating (5.2.1) by parts produces the following set of equations of motion for the plate:

$$(\frac{12D}{h^2}e)_{,\gamma} = 0 \quad \text{for } \gamma = 1, 2 \tag{5.2.5}$$

$$M_{\gamma\delta,\gamma\delta}(D,w) - (\frac{12D}{h^2}ew_{,\gamma})_{,\gamma} + \rho\ddot{w} = p(\mathbf{x})\cos\omega t \tag{5.2.6}$$

Equation (5.2.5) states that the quantity $\frac{12D}{h^2}e$ is not a function of the spatial coordinates. Therefore, the integration of $\frac{12D}{h^2}e$ over the domain $\mathcal{D}(\mathbf{x})$ gives:

$$\int_{\mathcal{D}} \frac{12D}{h^2}e \; d\;\mathcal{D}(\mathbf{x}) = \frac{12D}{h^2}e \int_{\mathcal{D}} d\;\mathcal{D}(\mathbf{x})$$

$$= \frac{12D}{h^2}e\,(Area)$$

or

$$\frac{12D}{h^2}e = \frac{N(w)}{(Area)} \tag{5.2.8}$$

where $Area$ is the surface area of the plate and

$$N(w) \equiv \int_{\mathcal{D}} \frac{12D}{h^2}e \; d\;\mathcal{D}(\mathbf{x}) \tag{5.2.9}$$

It should be noted that if $D \neq D(\mathbf{x})$, $h \neq h(\mathbf{x})$, and if the inplane displacements vanish at the boundaries of the plate, $N(w)$ becomes a functional involving only the transverse displacements, or

$$N(w) = \frac{12D}{h^2}\int_{\mathcal{S}} e \; d\;\; \mathcal{S}(\mathbf{x})$$

$$= \frac{6D}{h^2}\int_{\mathcal{S}} w_{,\gamma} w_{,\gamma} \; d\;\; \mathcal{S}(\mathbf{x}) \tag{5.2.9a}$$

Substitution of (5.2.8) with (5.2.9a) into equation of motion (5.2.6) gives

$$M_{\gamma\delta,\gamma\delta}(D,w) - \mu[\int_{\mathcal{S}} w_{,\gamma} w_{,\gamma} \; d\;\; \mathcal{S}(\mathbf{x})]w_{,\gamma\gamma} + \rho\ddot{w} = p(\mathbf{x})\cos\omega t \tag{5.2.10}$$

where $\mu \equiv \dfrac{6D}{(Area)h^2}$.

Equation (5.2.10) is the dynamic analog to the uncoupled form of Berger's equations discussed in reference [2]. The damped response of the plate will be given by the solution to (5.2.10) with the addition of an absolute , or "mass proportional", damping term $b\dot{w}(\mathbf{x},t)$ where $b = b(\mathbf{x})$.

5.2.2 Equivalent linear system

The linear auxiliary equation of motion which serves as a replacement for the damped form of the nonlinear equation of motion (5.2.10) is defined to be:

$$M_{\gamma\delta,\gamma\delta}(D,w) + M_{\gamma\delta,\gamma\delta}(\alpha,w) + b\dot{w} + \rho\ddot{w} = p\cos\omega t \tag{5.2.11}$$

where $\alpha = \alpha(\mathbf{x})$ is the equivalent linear flexural rigidity of the plate and all other terms are as defined before.

Referring to the notation used in chapters II and III, it can be seen by inspection that the stress operators for the nonlinear and linear auxiliary plate equations are are vector operations. Specifically τ^N and τ^L can be written as

$$\left.\begin{aligned} \tau^N &= (\tau_1^N,\ \tau_2^N)^T \\ \tau^L &= (\tau_1^L,\ \tau_2^L)^T \end{aligned}\right\} \qquad (5.2.12)$$

where

$$\left.\begin{aligned} \tau_\gamma^N &= -\mu w_{,\gamma} \int_{\mathcal{D}} w_{,\delta} w_{,\delta}\, d\mathcal{D}(\mathbf{x}) \\ \tau_\gamma^L &= M_{\gamma\delta,\delta} \end{aligned}\right\} \qquad (5.2.13)$$

for $\gamma = 1, 2$.

Recall that problem formulation II defined in Chapter III required the use of a strain-type function $\mathbf{S}(w)$. For the analysis of the plate response using formulation II, $\mathbf{S}(w)$ will be a vector operation defined as

$$\mathbf{S} = (\mathcal{S}_1,\ \mathcal{S}_2)^T \qquad (5.2.14)$$

where

$$\mathcal{S}_\gamma(w) = w_{,\gamma} \qquad (5.2.15)$$

The nonlinearity in the equation of motion (5.2.10) influences the stiffness of the plate and not the energy dissipation. It should therefore be expected that the inclusion of the equivalent linear damping parameter β will not be necessary. By not including β in the minimization relations, the approximation for the equivalent linear flexural rigidity α

$$\alpha(\mathbf{x}) \approx \sum_{i=1}^{M} \varphi_i(\mathbf{x}) \hat{\alpha}_i$$

must satisfy the following minimization relationship:

$$A\hat{\alpha} = \mathbf{G} \qquad (5.2.16)$$

where $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_M)^T$. The form of the matrix A and vector $\mathbf{G}$ will depend on the problem formulation used and are given in section 4.1 for the

steady-state response of the general system. It should be observed that in section 3.8 the matrix A was shown to be invertible for all three problem formulations, and henceforth unique solution for $\hat{\alpha}$ and $\hat{\beta}$ always exist.

Let attention now be focused on developing the specific form of A and $\mathbf{G}$ in (5.2.16) for problem formulations I, II and III. The linear operations τ^L , $\nabla \cdot \tau^L$ and from (5.2.13) and (5.2.15) are expressed as:

$$\tau^L_\gamma(\varphi_i, w) = M^{(i)}_{\gamma\alpha,\delta}(U)\cos\omega t + M^{(i)}_{\gamma\alpha,\delta}(V)\sin\omega t \tag{5.2.17}$$

$$\nabla \cdot \tau^L(\varphi_i, w) = M^{(i)}_{\gamma\alpha,\gamma\delta}(U)\cos\omega t + M^{(i)}_{\gamma\alpha,\gamma\delta}(V)\sin\omega t \tag{5.2.18}$$

$$\mathbb{S}_\gamma(w) = U_{,\gamma}\cos\omega t + V_{,\gamma}\sin\omega t \tag{5.2.19}$$

where

$$w(\mathbf{x}, t) = U(\mathbf{x})\cos\omega t + V(\mathbf{x})\sin\omega t \tag{5.2.20}$$

$$M^{(i)}_{\gamma\delta}(U) \equiv M_{\gamma\delta}(\varphi_i, U) \tag{5.2.21}$$

The time integrals $C^{(1)}_\gamma$, $C^{(3)}_\gamma$, $S^{(1)}_\gamma$ and $S^{(3)}_\gamma$; $\gamma = 1, 2$, defined in equations (4.17), (4.18), (4.23g) and (4.23h), respectively, can be written for the plate problem as :

$$C^{(1)}_\gamma(U,V) = -\frac{\mu}{4}\,[3\,\eta(U,U) + \eta(V,V)]U_{,\gamma} + 2\,\eta(U,V)V_{,\gamma} \tag{5.2.22}$$

$$S^{(1)}_\gamma(U,V) = -\frac{\mu}{4}\,2\,\eta(U,V)U_{,\gamma} + [\eta(U,U) + 3\,\eta(V,V)]V_{,\gamma} \tag{5.2.23}$$

$$C^{(3)}_\gamma(U,V) = -\frac{\mu}{4}\,[\eta(U,U) - \eta(V,V)]U_{,\gamma} - 2\,\eta(U,V)V_{,\gamma} \tag{5.2.24}$$

$$S^{(3)}_\gamma(U,V) = -\frac{\mu}{4}\,2\,\eta(U,V)U_{,\gamma} + [\eta(U,U) - \eta(V,V)]V_{,\gamma} \tag{5.2.25}$$

where

$$\eta(U,V) = \int_{\mathcal{D}} U_{,\delta} V_{,\delta}\, d\mathcal{D}(\mathbf{x}) \tag{5.2.26}$$

The preceding relations, (5.2.17) - (5.2.26) are then used to derive the matrix A and vector G for the three problem formulations, as shown below. As before, repeated Greek subscripts will imply summation on indices from 1 to 2.

Formulation I

$$A^I_{ij} = \int_{\mathcal{D}} [M^{(i)}_{\gamma\alpha,\gamma\delta}(U) M^{(j)}_{\xi\eta,\xi\eta}(U) + M^{(i)}_{\gamma\alpha,\gamma\delta}(V) M^{(j)}_{\xi\eta,\xi\eta}(V)]\, d\mathcal{D}(\mathbf{x}) \tag{5.2.27}$$

$$C^I_i = -\frac{\mu}{4}\Big\{[3\ \eta(U,U) + \eta(U,V)] \int_{\mathcal{D}} M^{(i)}_{\gamma\alpha,\gamma\delta}(U) U_{,\xi\xi}\, d\mathcal{D}(\mathbf{x}) \tag{5.2.28}$$

$$+ 2\ \eta(U,V) \int_{\mathcal{D}} M^{(i)}_{\gamma\alpha,\gamma\delta}(U) V_{,\xi\xi} + M^{(i)}_{\gamma\alpha,\gamma\delta}(V) U_{,\xi\xi}\, d\mathcal{D}(\mathbf{x})$$

$$+ [\ \eta(U,U) + 3\ \eta(V,V)] \int_{\mathcal{D}} M^{(i)}_{\gamma\alpha,\gamma\delta}(V) V_{,\xi\xi}\, d\mathcal{D}(\mathbf{x})\Big\}$$

Formulation II

$$A^{II}_{ij} = \int_{\mathcal{D}} \Big\{[M^{(i)}_{\gamma\alpha,\delta}(U) V_{,\gamma} + M^{(i)}_{\gamma\alpha,\delta}(V) U_{,\gamma}]\, [M^{(j)}_{\xi\eta,\eta}(U) U_{,\xi} + M^{(j)}_{\xi\eta,\eta}(V) V_{,\xi}] \tag{5.2.29}$$

$$+ [M^{(i)}_{\gamma\alpha,\delta}(U) U_{,\gamma}]\, [3M^{(j)}_{\xi\eta,\eta}(U) U_{,\xi} + M^{(j)}_{\xi\eta,\eta}(V) V_{,\xi}]$$

$$+ [M^{(i)}_{\gamma\alpha,\delta}(V) V_{,\gamma}]\, [M^{(j)}_{\xi\eta,\eta}(U) U_{,\gamma} + 3M^{(j)}_{\xi\eta,\eta}(V) V_{,\gamma}]\Big\} d\mathcal{D}(\mathbf{x})$$

$$C^{II}_i = -\frac{\mu}{4} \int_{\mathcal{D}} \Big\{[(3\ \eta(U,U) + \eta(V,V)) U_{,\xi} U_{,\xi} + 2\ \eta(U,V) V_{,\xi} U_{,\xi}] \tag{5.2.30}$$

$$[3M^{(i)}_{\gamma\alpha,\delta}(U) U_{,\gamma} + M^{(i)}_{\gamma\alpha,\delta}(V) V_{,\gamma}]$$

$$+[2(\eta(U,U)-\eta(V,V))U_{,\xi}V_{,\xi}-4\,\eta(U,V)U_{,\xi}U_{,\xi}][M^{(i)}_{\gamma\alpha,\delta}(U)V_{,\gamma}-M^{(i)}_{\gamma\alpha,\delta}(V)U_{,\gamma}]$$

$$+[\,\eta(U,U)-\eta(V,V)][U_{,\xi}U_{,\xi}+V_{,\xi}V_{,\xi}][M^{(i)}_{\gamma\alpha,\delta}(U)U_{,\gamma}-M^{(i)}_{\gamma\alpha,\delta}(V)V_{,\gamma}]$$

$$+[2\,\eta(U,V)U_{,\xi}V_{,\xi}+(\,\eta(U,U)+3\,\eta\,(V,V))V_{,\xi}V_{,\xi}]$$

$$[M^{(i)}_{\gamma\alpha,\delta}(U)U_{,\gamma}+3M^{(i)}_{\gamma\alpha,\delta}(V)V_{,\gamma}]\Big\}d\ \mathcal{A}(\mathbf{x})$$

Formulation III

$$A^{III}_{ij}=\int_{\mathcal{A}}[M^{(i)}_{\gamma\alpha,\delta}(U)M^{(j)}_{\gamma\eta,\eta}(U)+M^{(i)}_{\gamma\alpha,\delta}(V)M^{(j)}_{\gamma\eta,\eta}(V)]\,d\ \mathcal{A}(\mathbf{x}) \tag{5.2.31}$$

$$G^{III}_{i}=-\frac{\mu}{4}\Big\{[3\,\eta\,(U,U)+\eta\,(V,V)]\int_{\mathcal{A}}M^{(i)}_{\gamma\alpha,\delta}(U)U_{,\gamma}\,d\ \mathcal{A}(\mathbf{x}) \tag{5.2.32}$$

$$+2\,\eta\,(U,V)\int_{\mathcal{A}}[M^{(i)}_{\gamma\alpha,\delta}(U)V_{,\gamma}+M^{(i)}_{\gamma\alpha,\delta}(V)U_{,\gamma}]\,d\ \mathcal{A}(\mathbf{x})$$

$$+[\,\eta(U,U)+3\,\eta\,(V,V)]\int_{\mathcal{A}}M^{(i)}_{\gamma\alpha,\delta}(V)V_{,\gamma}\,d\ \mathcal{A}(\mathbf{x})\Big\}$$

5.2.3 Finite element analysis of the auxiliary equations

The finite element method was briefly discussed in Chapter III. A more complete discussion of the mechanics of constructing the finite element equations can be found in a number of finite element textbooks. Only the results of the discretization will be presented here.

Recall from equation (5.2.11) that the auxiliary equation for the thin plate problem was chosen to be:

$$M_{\gamma\delta,\gamma\delta}(D,w) + M_{\gamma\delta,\gamma\delta}(\alpha,w) + b\dot{w} + \rho\ddot{w} = p\cos\omega t \tag{5.2.33}$$

where repeated Greek subscripts imply summation of the indices from 1 to 2 and with all other terms and notation are as before. If the displacements w are written as

$$w(\mathbf{x},t) = \sum_{j=1}^{N} \psi_j(\mathbf{x}) d_j(t) \quad , \tag{5.2.34}$$

then the finite element equations can be written as:

$$\mathcal{M}\ddot{\mathbf{d}}(t) + \mathcal{C}\dot{\mathbf{d}}(t) + \mathcal{K}\mathbf{d}(t) = \mathcal{F}\cos\omega t \tag{5.2.35}$$

where

$$\mathcal{M}_{ij} = \int \rho\psi_i\psi_j \, d\mathcal{A}(\mathbf{x}) \tag{5.2.36}$$

$$\mathcal{C}_{ij} = \int b\,\psi_i\psi_j \, d\mathcal{A}(\mathbf{x}) \tag{5.2.37}$$

$$\mathcal{K}_{ij} = \int \mathcal{B}_i R \, \mathcal{B}_j \, d\mathcal{A}(\mathbf{x}) \tag{5.2.38}$$

$$\mathcal{F}_i = \int p(\mathbf{x})\psi_i \, d\mathcal{A}(\mathbf{x}) \tag{5.2.39}$$

$$\mathcal{B}_i = (\psi_{i,11}, \psi_{i,22}, \psi_{i,12})^T \tag{5.2.40}$$

$$R = \begin{bmatrix} (D+\alpha) & \nu & 0 \\ \nu & (D+\alpha) & 0 \\ 0 & 0 & 2(1-\nu)(D+\alpha) \end{bmatrix} \tag{5.2.41}$$

and $\psi_i(\mathbf{x})$ and $d_i(t)$; $i=1, 2, \ldots, N$, being the shape functions and nodal "displacements", respectively.

The domain of the plate is to be divided into N_{el} , four-node quadrilateral elements with each element having nodal points as $\mathbf{x} = \mathbf{x}^a$; a=1, 2, 3, 4 . Consider a transformation from a set of local coordinates $\xi = (\xi_1, \xi_2)$ to the global set $\mathbf{x}$ given by

$$\mathbf{x}(\xi) = \sum_{a=1}^{4} N_a(\xi)\mathbf{x}^a \tag{5.2.42}$$

where

$$N_a(\xi) = \frac{1}{4}(1 + \xi_1^a\xi_1)(1+\xi_2^a\xi_2)$$

and the local coordinates of the nodes ξ^a ; a=1, 2, 3, 4 , are as shown in figure 5.2.1. The shape functions for the displacement w will now be defined in terms of the local coordinates, with the understanding that the transformation of the shape functions to global coordinates is accomplished through (5.2.42).

Recall from section 3.7 that one sufficient condition for convergence of the finite element method is the continuity of the m-1 derivatives of the shape functions, where m is the highest order derivative in the strain energy integral (5.2.38). From (5.2.40) it is clear that m=2 for Kirchoff plates, and consequently the shape functions must have continuous *first* derivatives across the element boundaries. A set of shape functions developed in reference [3] constructed from Hermite cubic functions satisfies this requirement. Using these shape functions, the unknown parameters at each node are w, $\frac{\partial w}{\partial x_1}$, $\frac{\partial w}{\partial x_2}$, $\frac{\partial^2 w}{\partial x_1 \partial x_2}$ which leads to a total of 16 degrees-of-freedom/element. Let $\hat{\mathbf{d}}_a$; a=1, 2, 3, 4 be defined as

$$\hat{\mathbf{d}}_a = (w, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial^2 w}{\partial x_1 \partial x_2})^T \quad \text{at } \xi = \xi^a$$

Then the set of nodal parameters $\mathbf{d}_e$; $e=1, 2, \ldots, N_{el}$ will be ordered as

$$\mathbf{d}_e^T = (\hat{\mathbf{d}}_1, \hat{\mathbf{d}}_2, \hat{\mathbf{d}}_3, \hat{\mathbf{d}}_4)^T$$

With this, the shape functions in local coordinates from reference [3] are as shown in table 5.2.1.

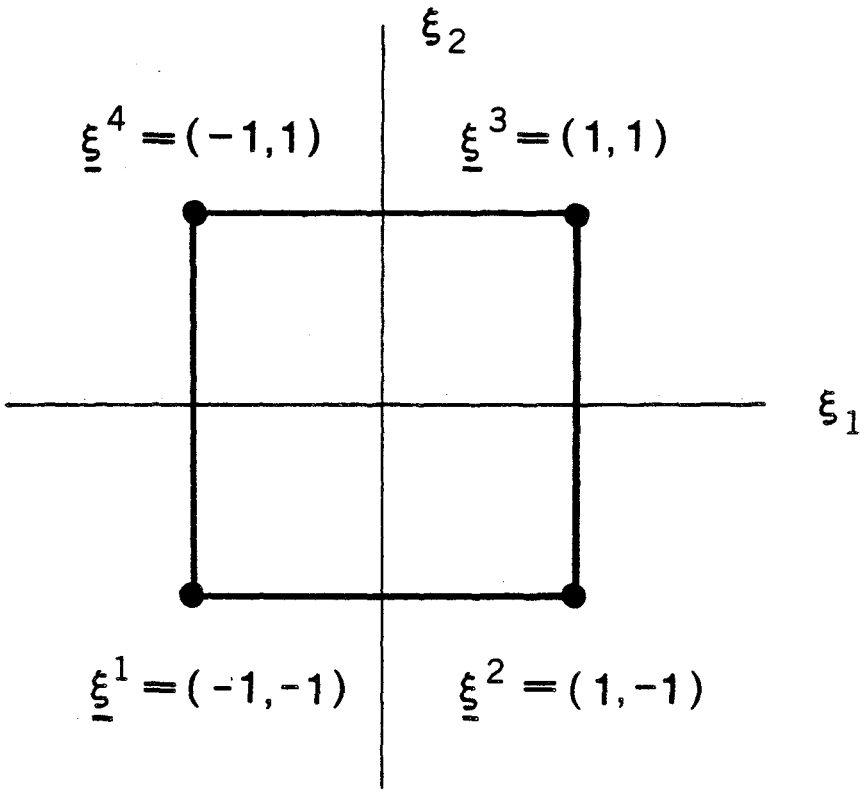


Figure 5.2.1 Domain of Plate Element in Local Coordinates

a	ψ_a
1	$\hat{\psi}_1(\xi_1)\,\hat{\psi}_1(\xi_2)$
2	$\hat{\psi}_3(\xi_1)\,\hat{\psi}_1(\xi_2)$
3	$\hat{\psi}_1(\xi_1)\,\hat{\psi}_3(\xi_2)$
4	$\hat{\psi}_3(\xi_1)\,\hat{\psi}_3(\xi_2)$
5	$\hat{\psi}_2(\xi_1)\,\hat{\psi}_1(\xi_2)$
6	$\hat{\psi}_4(\xi_1)\,\hat{\psi}_1(\xi_2)$
7	$\hat{\psi}_2(\xi_1)\,\hat{\psi}_3(\xi_2)$
8	$\hat{\psi}_4(\xi_1)\,\hat{\psi}_3(\xi_2)$
9	$\hat{\psi}_2(\xi_1)\,\hat{\psi}_2(\xi_2)$
10	$\hat{\psi}_4(\xi_1)\,\hat{\psi}_2(\xi_2)$
11	$\hat{\psi}_2(\xi_1)\,\hat{\psi}_4(\xi_2)$
12	$\hat{\psi}_4(\xi_1)\,\hat{\psi}_4(\xi_2)$
13	$\hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\xi_2)$
14	$\hat{\psi}_3(\xi_1)\,\hat{\psi}_2(\xi_2)$
15	$\hat{\psi}_1(\xi_1)\,\hat{\psi}_4(\xi_2)$
16	$\hat{\psi}_3(\xi_1)\,\hat{\psi}_4(\xi_2)$

where

$\hat{\psi}_1(z) = (z-1)^2\,(2+z)/4$

$\hat{\psi}_2(z) = (z+1)^2\,(2-z)/4$

$\hat{\psi}_3(z) = h_z\,(z+1)\,(z-1)^2/8$

$\hat{\psi}_4(z) = h_z\,(z+1)^2(z-1)/8$

Table 5.2.1. The Bogner-Fox-Schmidt Shape Functions

The finite element discretization of plates governed by the Kirchoff theory has an inherent difficulty in modeling curved boundaries. As mentioned in reference [19], such modeling of a simply-supported, curved boundary boundary will impose a "clamped" boundary condition as the mesh is refined. Consequently, analysis will be restricted to plates having straight edges and , in particular, only rectangular elements will be used.

5.2.4 Convergence considerations for equivalent linear parameters

For the previous example of the one-dimensional shear member, the convergence of the equivalent linear parameters with mesh refinement was readily established. In this example of the nonlinear response of thin plates, complexities arise which do not permit such a simple analsysis. Recall from the last section that the finite element analysis of the auxiliary linear equation was performed using bi-cubic shape functions in which their first derivatives were continuous across the element boundaries. But from the minimization relations of formulation III, for example, in equations (5.2.16), (5.2.31) and (5.2.32), it can be seen that continuity of the second derivatives of the displacements are required. Otherwise the terms containing three spatial derivatives such as $M^{(i)}_{\gamma\alpha,\delta}$ will possess singularities at the element boundaries which are not square integrable. The technique used herein to avoid this difficulty is to compute the integrals within each separate element, and then simply add together the results, thus ignoring the contribution of the interelement discontinuities.

Using this type of construction of the minimization equations, one should be concerned with how the omission of the singularities on the element boundaries affects the solution. This is also a concern in the finite element area when "nonconforming" elements are used (reference [48]). In this latter case, a "patch test" of elements can be used to determine the completeness convergence cri-

terion of the element. If the element passes such a test, it is then concluded that the discontinuities in the "non-conforming" elements do not contribute to the strain energy. For the minimization equations, it is not clear whether such a test would be applicable, and therefore the convergence of such a scheme has not been resolved.

In spite of these uncertaintities, the minimization relations will be formed at the element level, as mentioned above, for all three problem formulations. Investigation of how these considerations affect the convergence of the method will be deferred to further investigations.

5.2.5 Numerical results

This section will deal with the symmetric, steady-state, harmonic response of a simply supported rectangular plate governed by the nonlinear theory introduced in section 5.2.1. The numerical results from the equivalent linearization technique will be compared with those from Galerkin's method described in section 2.4.1.

Let the domain of the rectangular plate $\mathcal{D}(\mathbf{x})$ be defined as

$$\mathcal{D}(\mathbf{x}) = (-a \leq x_1 \leq a,\ -b \leq x_2 \leq b)$$

The simply supported boundary conditions are therefore given by:

$$\left.\begin{aligned} & w(-a, x_2, t) = w(a, x_2, t) = 0 \\ & w(x_1, -b, t) = w(x_1, b, t) = 0 \\ & \frac{\partial^2 w}{\partial x_1^2}(-a, x_2, t) = \frac{\partial^2 w}{\partial x_1^2}(a, x_2, t) = 0 \\ & \frac{\partial^2 w}{\partial x_2^2}(x_1, -b, t) = \frac{\partial^2 w}{\partial x_2^2}(x_1, b, t) = 0 \end{aligned}\right\} \qquad (5.2.43)$$

That is, the displacements and tangential moments must vanish on the boundary of the plate. If the material properties are constant throughout the plate, the symmetric eigenfunctions $\Phi_k(\mathbf{x})$ and corresponding eigenvalues λ_k for the linearized problem with the boundary conditions of (5.2.43) are

$$\Phi_k(\mathbf{x}) = \cos[\pi(2k-1)\frac{x_1}{2a}]\cos[\pi(2k-1)\frac{x_2}{2b}] \tag{5.2.44}$$

$$\lambda_k = \frac{\pi^2}{4}(2k-1)^2[\frac{1}{a^2}+\frac{1}{b^2}]\sqrt{\frac{D}{\rho}} \tag{5.2.45}$$

Let the applied load to the plate be due to a spatially constant pressure P_0, and define the following dimensionless parameters

$$\Omega \equiv \frac{\omega}{\lambda_1}$$

$$\bar{\lambda}_i \equiv \frac{\lambda_i}{\lambda_1} = (2i-1)^2$$

$$\overline{w} \equiv \frac{w}{h}$$

If the nonlinear plate solution is expanded in terms of the first N eigenfunctions of the linearized problem, the non-dimensinal form of the general Galerkin equations (2.18) is:

$$\begin{bmatrix} \Lambda & \Gamma \\ -\Gamma & \Lambda \end{bmatrix}\begin{Bmatrix} \overline{\mathbf{u}} \\ \overline{\mathbf{v}} \end{Bmatrix} + \begin{Bmatrix} \mathfrak{F}_C^G \\ \mathfrak{F}_S^G \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^G \\ \mathbf{0} \end{Bmatrix} \tag{5.2.46}$$

where for i,j=1,2,...,N

$$\Lambda_{ij} = \begin{cases} \overline{\lambda_i}^2 - \Omega^2 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Gamma_{ij} = \begin{cases} 2\zeta_i \bar{\lambda}_i \Omega & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$f_i^G = \frac{3072(1-\nu^2)}{\pi^6(2i-1)^2}\bar{P}$$

$$\bar{P} = \frac{P_0 b^4}{Eh^4[1+(\frac{b}{a})^2]^2}$$

$$\mathcal{F}_{Ci}^G = \frac{3}{8}\bar{\lambda}_i \sum_{l=1}^{N} \bar{\lambda}_l [(3\bar{u}_l^2 + \bar{v}_l^2)\bar{u}_i + \bar{u}_i \bar{v}_l \bar{u}_l]$$

$$\mathcal{F}_{Si}^G = -\frac{3}{8}\bar{\lambda}_i \sum_{l=1}^{N} \bar{\lambda}_l [2\bar{u}_l \bar{v}_l \bar{v}_i + (\bar{u}_l^2 + 3\bar{v}_l^2)\bar{u}_i]$$

$$\bar{w} = \sum_{i=1}^{N} \Phi_i [\bar{u}_i \cos\omega t + \bar{v}_i \sin\omega t]$$

and ζ_i is the damping ratio in the i^{th} linear mode. E and ν are the elastic modulus and Poisson's ratio of the material , respectively.

Therefore, the steady-state Galerkin response is governed by a set of non-linear algebraic equations in the "in-phase" and "quadrature" (with respect to the forcing term) components of displacement, $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$, respectively. It is interesting to note that for the equations written in the normalized form of (5.2.46) that the plate dimensions a, b and h enter in only through the forcing term. Consequently, the normalized free vibration response of the plate can be presented independent of its aspect ratio or thickness.

5.2.5.1 Free vibration response

If the undamped, unloaded nonlinear plate is given an initial prescribed displacement, it should be expected that the frequency of response will depend on the amplitude of the resulting motion. This contrasts with the linear theory in

which the frequency is independent of the amplitude of response.

For free vibration response that can be modeled sufficiently well by the first mode of the linearized problem, the normalized frequency of response, Ω, given by Galerkin's method takes on the rather simple form of

$$\Omega = [1 + \frac{9}{8}\bar{u}_1^2]^{\frac{1}{2}} \tag{5.2.47}$$

where $\bar{u}_1$ is the normalized amplitude at the center of the plate. This one-mode Galerkin response is shown in figure 5.2.2.

The equivalent linear free vibration response is governed by the free vibration form of the discretized auxiliary equation (5.2.35) and the minimization relations of (5.2.27) - (5.2.32) for the three problem formulations. Solutions to these equations have been effected through the use of the matrix iteration method detailed in section 4.3.1. Results for the the first nonlinear mode of vibration are presented in figure 5.2.2 for a finite element mesh with 9 degrees-of-freedom (DOF's) and in figure 5.2.3 for a 49 DOF mesh.

In order to determine the validity of the equivalent linear solutions, a direct numerical integration procedure was applied to the discretized form of the original nonlinear equation (5.2.10). The discretization was accomplished by using the finite element bi-cubic shape functions in section 5.2.3, which produces a set of nonlinear, ordinary differential equations in terms of the nodal displacements. With prescribed initial displacements, these differential equations were solved by a Runge-Kutta time integration algorithm. The frequency-amplitude results are given in figures 5.2.2 and 5.2.3 for finite element meshes with 9 and 49 DOF's.

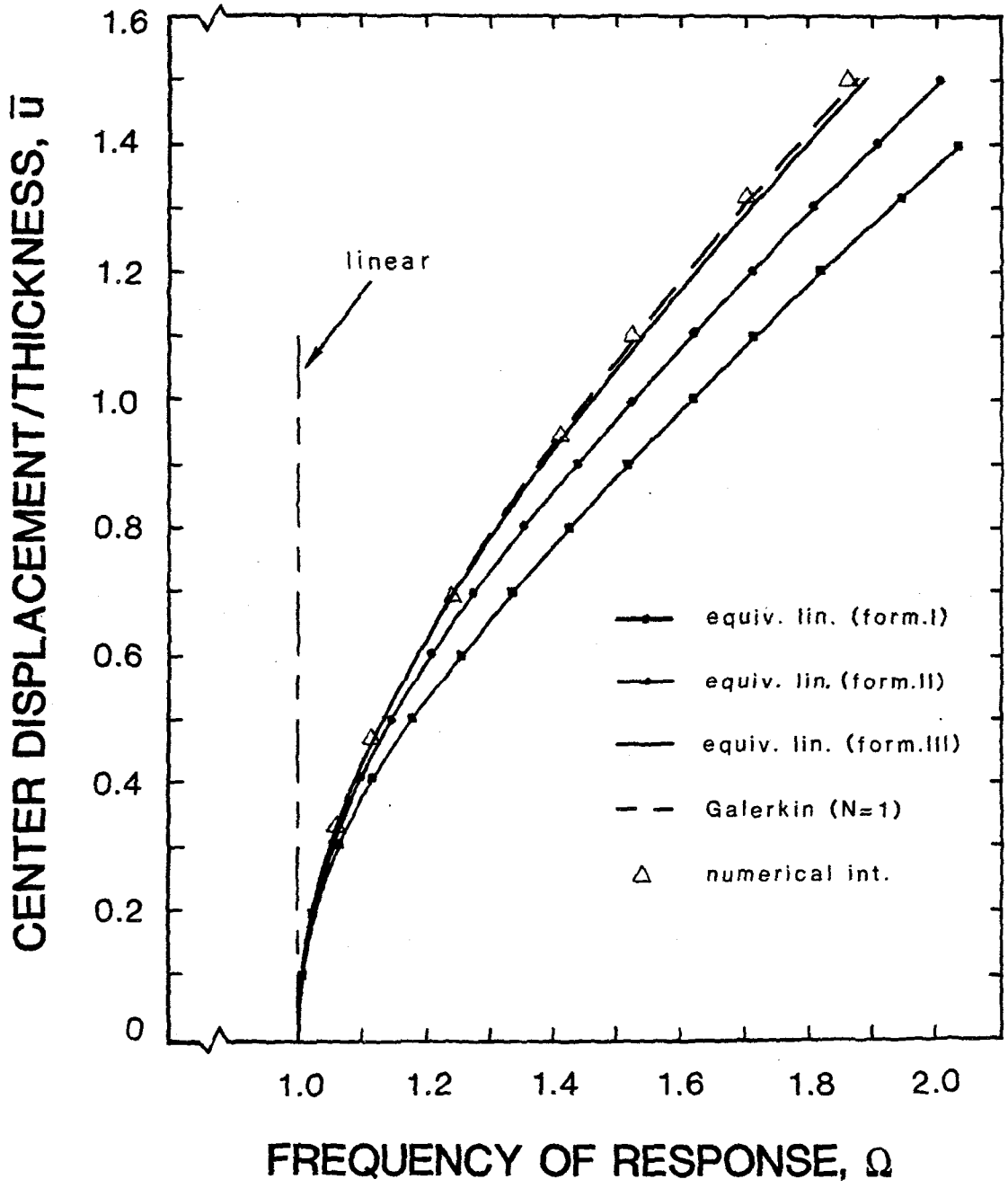


Figure 5.2.2 Free Vibration Response of Simply Supported Rectangular Plate (for 9 DOF finite element mesh)

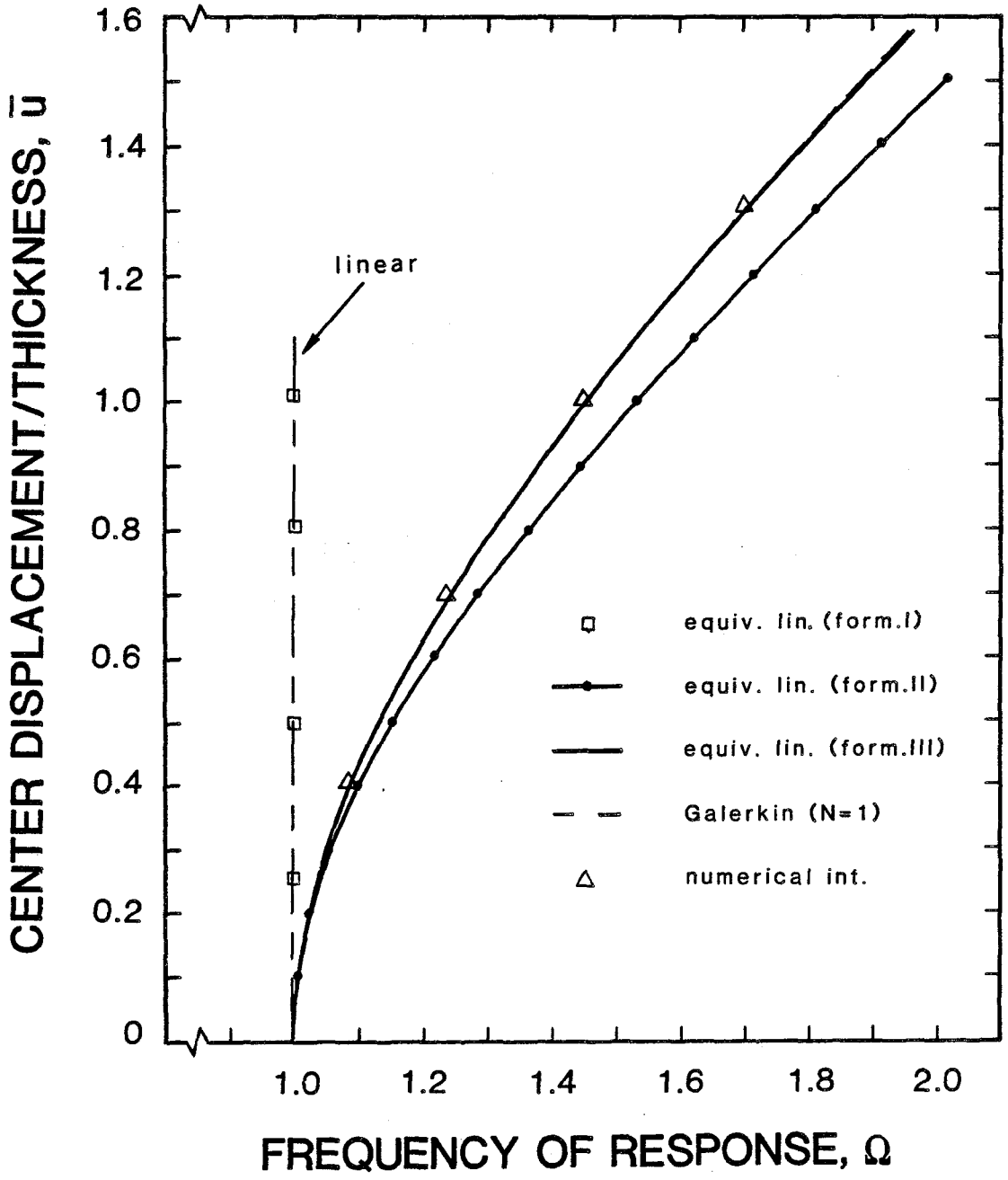


Figure 5.2.3 Free Vibration Response of Simply Supported Rectangular Plate (for 49 DOF finite element mesh)

5.2.5.2 Forced response

The forced response to the one mode Galerkin equations are given by

$$[(1-\Omega^2)^2\bar{A}_1+\frac{9}{8}\bar{A}_1^3]^2+[2\zeta_1\Omega\bar{A}_1]^2=f_1^2 \tag{5.2.48}$$

where

$$\bar{A}_1=[\bar{u}_1^2+\bar{v}_1^2]^{\frac{1}{2}}$$

$$f_1=48(\frac{2}{\pi})^6(1-\nu^2)\bar{P}$$

and the normalized pressure, $\bar{P}$ is as given following (5.2.46). A plot of Ω versus $\bar{A}$ in (5.2.48) is given in figure (5.2.4) for $\bar{P}=\frac{1}{4}$ and $\frac{3}{4}$. Also in figure (5.2.4) is the formulation III forced response.

5.2.6 Discussion

From figure 5.2.2, it can be seen that solutions from both Galerkin's method and formulation III (stress minimization) of equivalent linearization compare well with the results of direct integration, even for a somewhat "crude" finite element mesh having 9 DOF's. The difference between formulation III and numerical integration becomes somewhat more pronounced at larger amplitudes of vibration. The other two problem formulations, I and II (differential stress and energy difference minimization, respectively), on the other hand, predict a "stiffer" response. That is, for a given amplitude, the frequency of oscillation given by the two methods is significantly higher than that given by numerical integration.

The results from using a more refined finite element mesh are as presented in figure 5.2.3. The difference between formulation III equivalent linearization and numerical integration solutions are less pronounced at larger amplitudes than in figure 5.2.2. Formulation II solutions were not significantly affected by

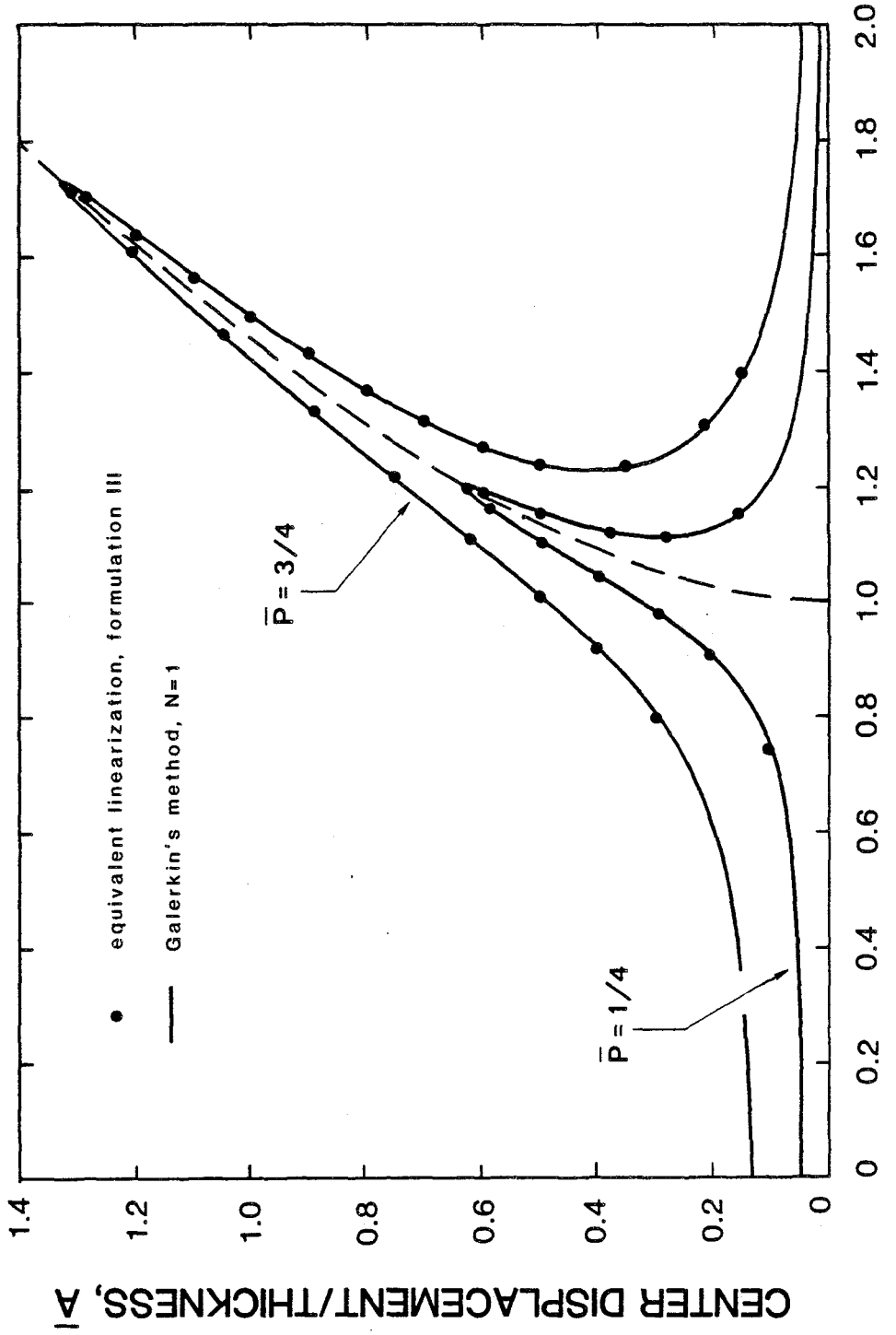


Figure 5.2.4 Forced Response of Simply Supported Rectangular Plate, $\nu = 0.3$, $\xi = 0.03$

the change in finite element meshes.

On the other hand, the formulation I approach to the equivalent linearization method behaves rather erratically for the 49 DOF model. The results show that the equivalent linear parameters become small, which in effect makes the free vibration results nearly linear, as can be seen in figure 5.2.3. The reason for such an erratic response could lie in the shape functions used in the finite element model. Recall from section 5.2.2 that the integrands in the equations for the formulation I equivalent linear stiffness parameters are singular at the element boundaries for the bi-cubic shape functions. These singularities are more severe than those encountered by the other two formulations. Therefore the poor performance of formulation I is not particularly surprising.

In conclusion, formulation III appears to be the most attractive of the three approaches to equivalent linearization. The results from formulation III agree with those form direct integration and compare well with free and forced solutions of Galerkin's method. The fact that the one term Galerkin expansion of this simple example produced simpler expressions for the solutions than that given for equivalent linearization is noteworthy. In the next example, however, it can be seen that the simpler Galerkin approach (and the similar Ritz method) can not handle a problem with somewhat more complicated boundary conditions as well as the finite element-equivalent linearization technique.

5.3 Steady-State Harmonic Response of a Nonlinear Plate with a Hole

In this section, the nonlinear response of a plate with somewhat more complex boundary conditions will be investigated. The simply supported rectangular plate of the last section will again be dealt with, but here the effect of a rectangular cutout centered in the plate will be of interest.

The geometry of the problem is as shown in figure 5.3.1. Let $\partial\ \mathcal{S}_O(\mathbf{x})$ be the outer boundary of the plate and $\partial\ \mathcal{S}_I(\mathbf{x})$ be on the perimeter of the cutout. Simply supported conditions are to be imposed on $\partial\ \mathcal{S}_O$, and on $\partial\ \mathcal{S}_I$, moment and shear free conditions exist. It should be noted , however, that in order for the nonlinear theory to be applicable to this problem, the *inplane* displacements of the plate must be constrained at the hole. Therefore a membrane stress will exist on $\partial\ \mathcal{S}_I$ although $\partial\ \mathcal{S}_I$ is free of bending stresses.

5.3.1 Ritz method

For the continuous plate (that is, a plate without cutouts) a Galerkin method has been used in the analysis. This was possible because of the availability of trial functions which satisfied all of the boundary conditions, i.e. comparison functions. However, the presence of the hole, and its associated boundary conditions, create difficulties in finding usuable comparison functions. As can be recalled from section 2.4.1, the class of trial functions that can be used with the Ritz method need only satisfy the geometric boundary conditions. Thus the field of candidates for use with the Ritz method is larger than that which can be used with Galerkin's method.

The use of the linear mode shapes of the continuous plate for analysis of a plate with a cutout is suggested in reference [29]. Reference [26] used a set of trial functions which modeled the appropriate singularities within the hole. For a square plate, the trial functions used in [26] are

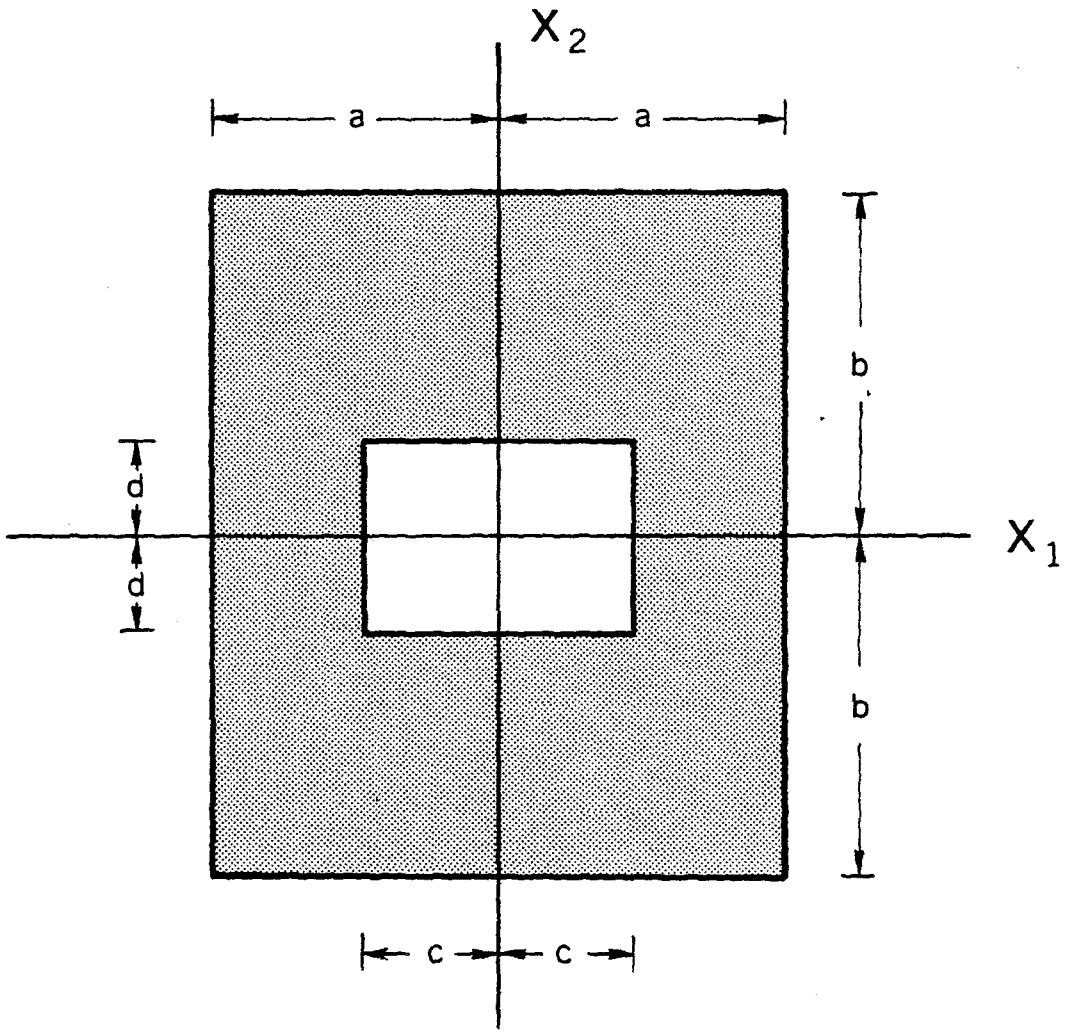


Figure 5.3.1 Geometry of Rectangular Plate with Centered Rectangular Cutout

$$\Phi_1(\mathbf{x}) = f(\mathbf{x}) \tag{5.3.1}$$

$$\Phi_2(\mathbf{x}) = f(\mathbf{x})\frac{x_1^2 + x_2^2}{a^2} \tag{5.3.2}$$

$$\Phi_3(\mathbf{x}) = f(\mathbf{x})\log[\frac{\sqrt{x_1^2 + x_2^2}}{a}] \tag{5.3.3}$$

where

$$f(\mathbf{x}) = [1-(\frac{x_1}{a})^2][1-(\frac{x_2}{a})^2] \tag{5.3.4}$$

Both types of trial functions will be used in later analysis.

Substitution of the energy and virtual work expressions for the undamped, nonlinear plate into the Ritz equations of section 2.4.2 gives the following normalized relations

$$[-\omega^2 \mathcal{M} + \mathcal{K}]\mathbf{u} + \mathfrak{F}(\mathbf{u}) = \mathbf{f} \tag{5.3.5}$$

where

$$\mathcal{M}_{ij} = \int_{\mathcal{D}} \Phi_i \Phi_j \, d\mathcal{D}(\mathbf{x})$$

$$\mathcal{K}_{ij} = \frac{D}{\rho}\int_{\mathcal{D}} [(1-\nu)\Phi_{i,\gamma\delta}\Phi_{j,\gamma\delta} + \nu\Phi_{i,\gamma\gamma}\Phi_{j,\delta\delta}] \, d\mathcal{D}(\mathbf{x})$$

$$\mathfrak{F}_i = \frac{9}{8}\frac{D}{\rho h^2(ab-cd)}\sum_{j,k,l=1}^{N}[\int_{\mathcal{D}}\Phi_{i,\gamma}\Phi_{j,\gamma}\, d\mathcal{D}(\mathbf{x})][\int_{\mathcal{D}}\Phi_{k,\delta}\Phi_{l,\delta}\, d\mathcal{D}(\mathbf{x})]u_j u_k u_l$$

$$f_i = \frac{P_0}{\rho}\int_{\mathcal{D}}\Phi_i \, d\mathcal{D}(\mathbf{x})$$

$$\Phi_{i,\gamma} = \frac{\partial \Phi_i}{\partial x_\gamma}$$

$$\int(\cdot)\, d\mathcal{D}(\mathbf{x}) = \int_{-a}^{a}\int_{-b}^{b}(\cdot)\, dx_1 dx_2 - \int_{-c}^{c}\int_{-d}^{d}(\cdot)\, dx_1 dx_2$$

and repeated Greek subscripts imply summation of indices from 1 to 2.

5.3.2 Fundamental natural frequency

For low levels of amplitude, the response of the nonlinear plate should be expected to approach that of the linearized response. Before investigating the nonlinear, free vibration of the plate with a hole, its linearized response should be understood. The satisfaction of the free boundaries at the hole creates difficulties in obtaining exact solutions for the linearized natural frequencies. In this section, the fundamental frequency of a square plate governed by the linear theory will be investigated by three methods: (1) Rayleigh-Ritz method with the mode shapes for the plate without a hole, (2) Rayleigh-Ritz method with the singular trial functions of (5.3.1) - (5.3.4) and (3) the finite element element method.

Recall from section 2.4.2 that the Ritz method reduces to the Rayleigh-Ritz eigenvalue formulation, equation (2.28), for a linear conservative system. The mass and linearized stiffness matrices, $\mathcal{M}$ and $\mathcal{K}$, for the Ritz method are given in equation (5.3.5). Substitution of the first eigenvalue and eigenfunction of the continuous plate into the expressions for $\mathcal{M}$ and $\mathcal{K}$ give the following relationship for the non-dimensional natural frequency , $\bar{\lambda}$,

$$\bar{\lambda}^2 = 1 + \frac{2(1-\nu)\left[F\left(\frac{c}{a},\frac{d}{b}\right) - 1\right]\left(\frac{a}{b}\right)^2}{\left[1 + \left(\frac{a}{b}\right)^2\right]^2} \tag{5.3.6}$$

where

$$\bar{\lambda} = \frac{\lambda_H}{\lambda_{NH}}$$

$$F = \frac{1 - [\frac{c}{a} - \frac{1}{\pi}\sin\pi\frac{c}{a}][\frac{d}{b} - \frac{1}{\pi}\sin\pi\frac{d}{b}]}{1 - [\frac{c}{a} + \frac{1}{\pi}\sin\pi\frac{c}{a}][\frac{d}{b} + \frac{1}{\pi}\sin\pi\frac{d}{b}]}$$

and λ_H and λ_{NH} are the fundamental natural frequencies of the linear plate with and without a hole, respectively. The use of the three singular trial functions of (5.3.1) - (5.3.4) in the Rayleigh-Ritz equations produces a 3x3 eigenvalue problem. The solution for the lowest eigenvalue for a square plate with a square cutout is shown in figure 5.3.2 for the full range of hole sizes. Equations (5.3.6) with a = b and c = d is also plotted in figure 5.3.2. The fundamental eigenvalue for the finite element formulation of the linear plate has also been determined using typical element meshes of figure 5.3.3 are also shown in figure 5.3.2.

It can be seen from figure 5.3.2 that the Rayleigh-Ritz frequencies are always as large as those obtained via the finite element method. In fact, for hole-width-to-plate-width ratios less than ≈ 0.34 , the finite element frequencies are actually lower than those of the plate without a hole. Reference [48] provides a proof that eigenvalues from the finite element method are an upper bound for the actual eigenvalues. Consequently, it should be assumed from figure 5.3.2 that the finite element frequencies are better estimates of the actual frequencies than those of either of the two Rayleigh-Ritz formulations.

5.3.3 Nonlinear free vibration response

Substitution of the first eigenfunction of the linear, continuous plate from equation (5.2.44) into the Ritz equations (5.3.5) for a square plate with a square cutout gives the following amplitude-frequency relationship:

$$\Omega = [\tilde{\lambda}^2 + \frac{9}{8}\tilde{\mu}\tilde{u}^2]^{\frac{1}{2}} \tag{5.3.7}$$

where

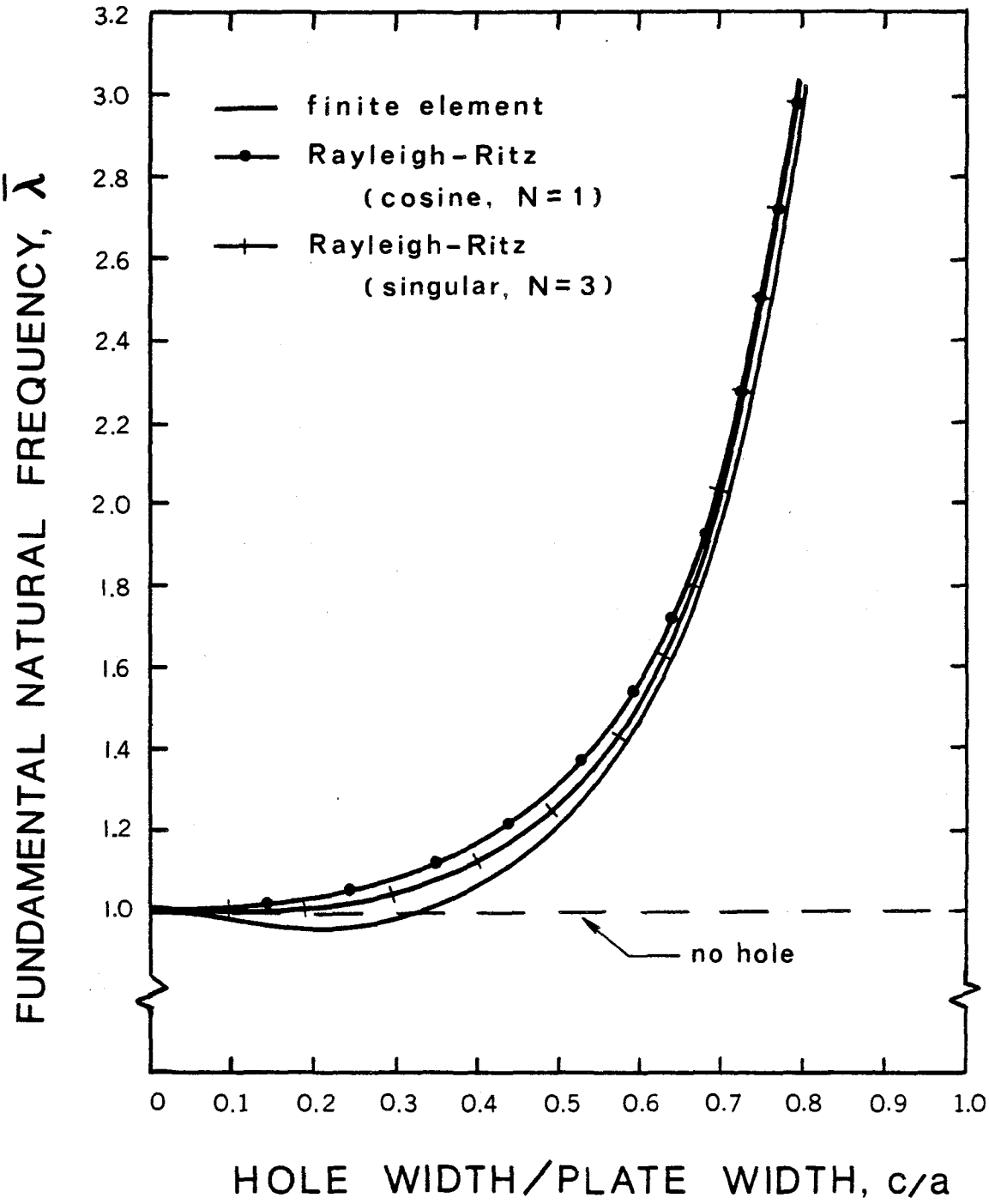


Figure 5.3.2 Linear Fundamental Natural Frequency of Simply Supported Square Plate with Square Hole for $\nu = 0.3$

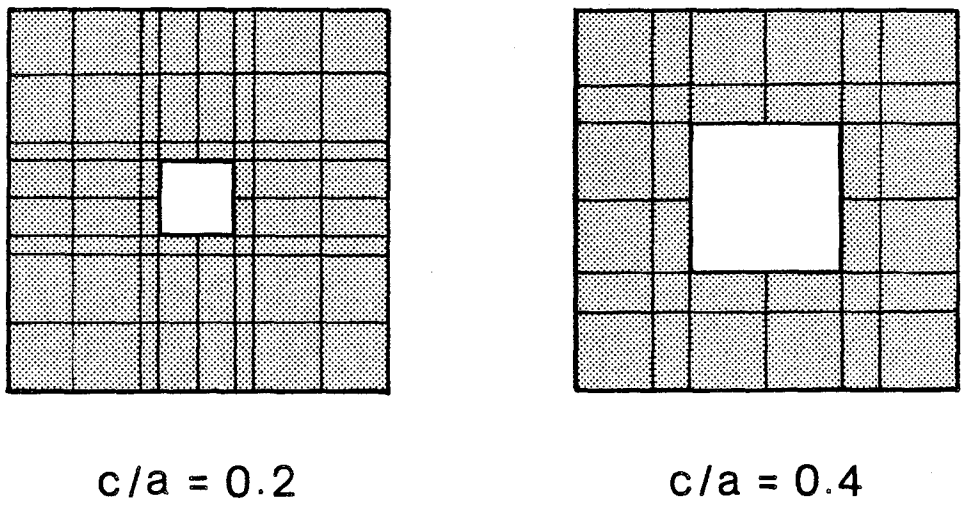


Figure 5.3.3 Finite Element Meshes for Eigenvalue Analysis of Simply Supported Plate with Hole

$$\Omega = \frac{\omega}{\lambda_{NH}}$$

$$\overline{u} = \frac{w_{max}}{h}$$

$$\overline{\mu} = \frac{[1-(\frac{c}{a})^2+\frac{1}{\pi^2}\sin^2\pi\frac{c}{a}]^2}{[1-(\frac{c}{a})^2][1-(\frac{c}{a}+\frac{1}{\pi}\sin\pi\frac{c}{a})^2]\cos^2\frac{\pi}{2}\frac{c}{a}}$$

and w_{max} is the maximum displacement in the plate. $\overline{\lambda}$ is the Rayleigh-Ritz fundamental frequency of the plate given in (5.3.6) for a = b and c = d. If c = 0 , i.e. no hole, both $\overline{\lambda}$ in (5.3.6) and $\overline{\mu}$ in (5.3.7) are equal to one , and the amplitude frequency relationship (5.3.7) reduces to that derived by Galerkin's method, equation (5.2.47). This is consistent with the fact that the Ritz method is equivalent to Galerkin's method when comparison functions are used (reference [34]).

In the same manner as for the continuous plate, the free vibration response using the formulation III equivalent linearization technique has been determined for a range of moderately sized holes. The results are presented in figure 5.3.4 along with the Ritz relationship (5.3.7). Also shown in figure 5.3.4 is the free vibration response fo a hole size of $\frac{c}{a} = 0.5$ obtained by numerical integration technique described in section 5.2.5.1.

The effect of hole size on the nonlinear resonant frequency shift of the plate can be seen by looking at a given level of strain, say $\max(\frac{\partial w}{\partial x})$, rather than for a given level of displacement, as before. Recall that the nonlinear stiffness in the plate is related to the total midplane stretch, which is in itself a functional of the strain and not the displacement. In figure 5.3.5, the difference in the linear and nonlinear resonant frequencies , $\Delta\Omega$,is plotted versus hole size for

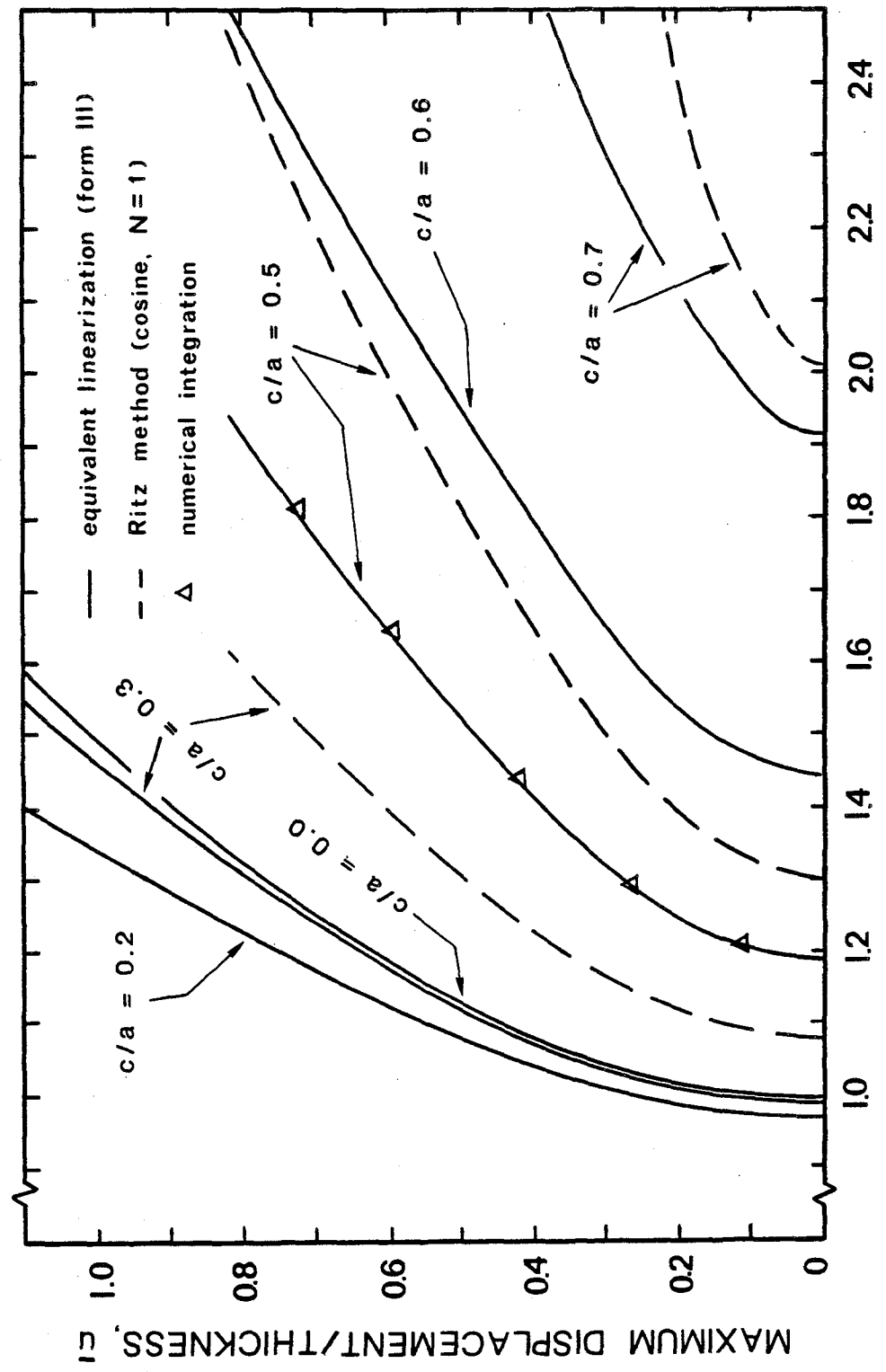


Figure 5.3.4 Free Vibration Response of Simply Supported Square Plate with Square Hole for $\nu = 0.3$

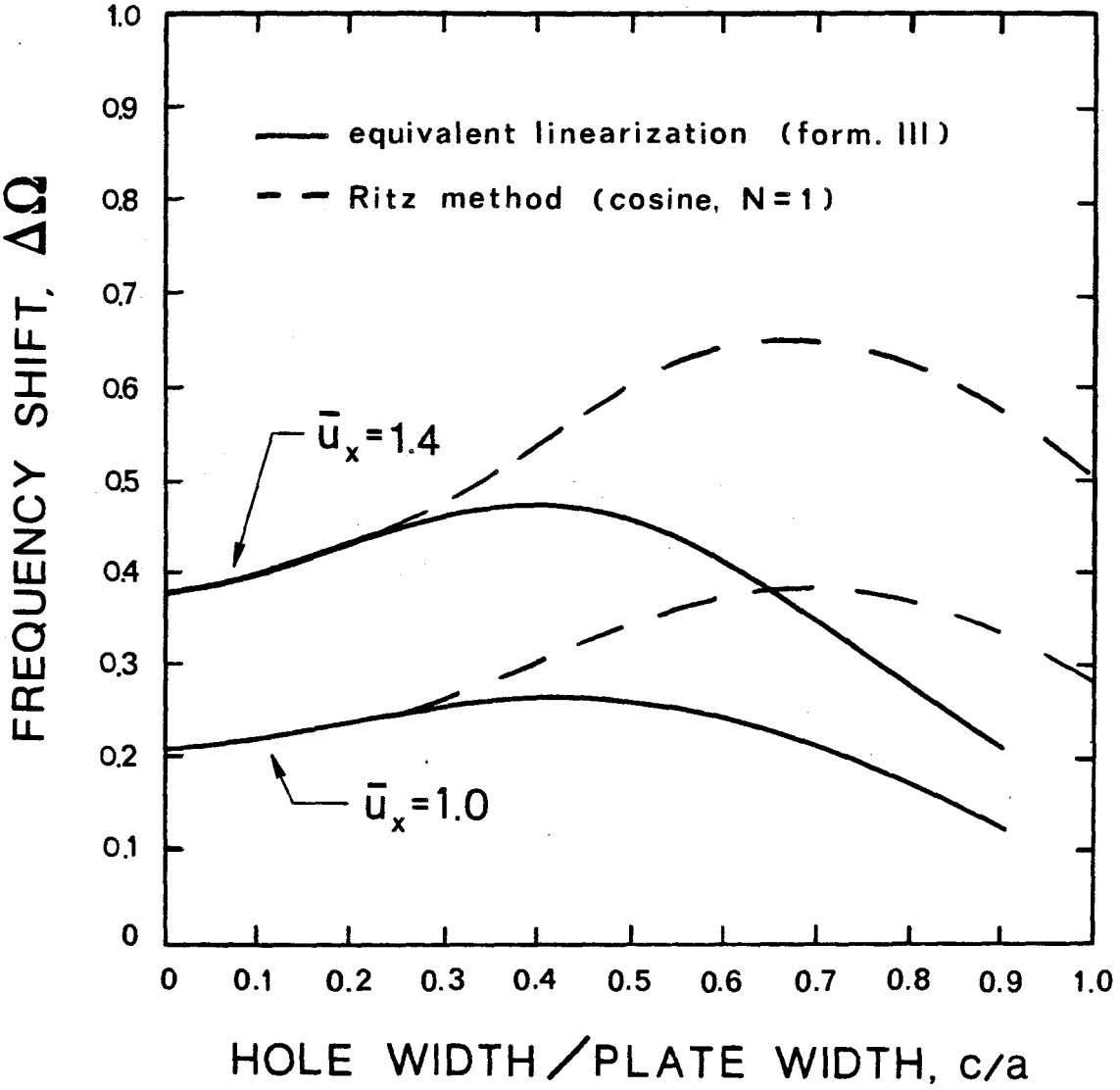


Figure 5.3.5 Effect of Hole Size on Resonant Frequency Shift for Simply Supported Square Plate with Square Hole, $\nu = 0.3$

two normalized levels of strain , $\overline{u}_{,x}$, where

$$\Delta\Omega \equiv \frac{\omega - \lambda_H}{\lambda_H}$$

$$\overline{u}_{,x} = \frac{a}{h} \max\left(\frac{\partial w}{\partial x}\right)$$

and λ_H , as before , is the linear, fundamental natural frequency of the plate without a hole.

5.3.4 Nonlinear forced response

As with the analysis of the continous plate, the nonlinear response of the cutout plate to a temporally harmonic and spatially constant pressure load will be investigated. Here the emphasis will be on establishing a relation between the size of the hole and the resonant response experienced by the plate.

In section 4.3.2, a method was introduced to determine the relationship between applied pressure and the resonant amplitude of response of a lightly damped system via the equivalent linearization technique. The method established a rather simple relationship, equation (4.59), between the applied load and resonant response, once the free vibration "backbone" curve had been calculated.

Results of the analysis are shown in figure 5.3.6 for a damping ratio of $\zeta = 0.03$ and for a range of moderately sized holes. If the linear theory for the vibration of plates were used, a linear relationship would exist between the applied load and the resonant amplitude. This linear relation is also shown in figure 5.3.6 and, as consistent with the usual hypothesis implied by the use of the linear theory, is tangent to the nonlinear curve at small amplitudes.

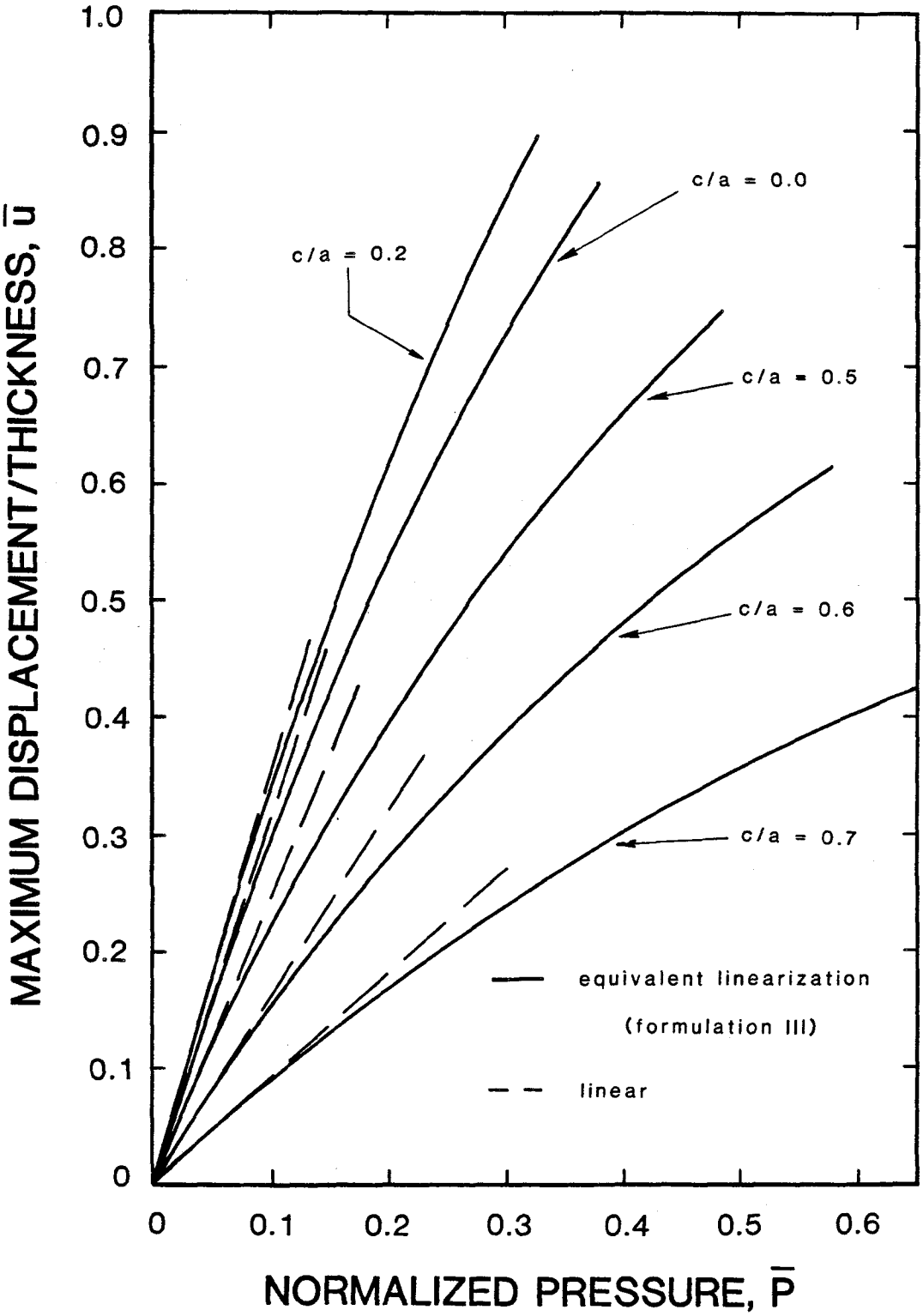


Figure 5.3.6 Effect of Applied Pressure on Resonant Response of Simply Supported Square Plate with Square Hole, $\nu = 0.3$, $\xi = 0.03$

5.3.5 Discussion

As shown in section 5.3.2, the finite element method provided a means of obtaining a better estimate of the eigenvalues than by the Rayleigh-Ritz approach. In particular, the finite element method was able to model an initial decrease in the fundamental frequency with an increase in hole size. This phenomena is qualitatively similar to that experienced by a circular, simply supported plate with a concentric circular hole , as reported in [26], where, for hole-width-to-plate-width ratios up to 0.4, the fundamental frequency was less that that of the same plate without a hole. The inability of the Ritz procedure to accurately determine the linear eigenvalues will introduce an artificial frequency shift when considering nonlinear analysis for even small amplitudes of response.

In section 5.3.3, the nonlinear free vibration response has been reported in two different ways. The physically appealing maximum displacement-frequency response has been shown in figure 5.3.4 while in figure 5.3.5, a mathematically appealing maximum strain amplitude-frequency response has been presented. It can be seen from figure 5.3.4 that the equivalent linearization predicted a frequency shift in displacements due to the nonlinearity for hole sizes of $c/a = 0.2$ and 0.3 to be less than the frequency shift for the plate without a hole. For all hole sizes, the nonlinear frequency shift predicted for displacement response by the Ritz approach is always larger than that given by equivalent linearization.

In figure 5.3.5, the frequency shift in the maximum strain response has been normalized by the fundamental natural frequency of the cutout plate. Here both methods show that the frequency shift monotonically increases up to a maximum value with increasing hole size, and then decreases. The difference between the two methods is that the Ritz frequency shift for the cutout plate is

always larger than that for the continuous plate, whereas with the equivalent linearization method, the frequency shift for large holes is less than for the continuous plate. Hence the two methods predict significantly different type of behavior in the vibration of a plate with a hole.

Figure 5.3.6 shows that for small holes, say $c/a = 0.2$, the nonlinear forced response remains nearly linear for larger amplitudes than for the same plate without a hole. However, for hole sizes of at least $c/a = 0.5$, the need for nonlinear analysis becomes more apparent, even for lower amplitudes of response.

VI. CONCLUDING REMARKS

The analysis presented in this dissertation has been focused on the type of nonlinear dynamic system introduced in Chapter II. The system described there was one for which the limiting behavior for low level response can be represented by a linear model. This linear model was prescribed to be one of which the restoring stress operation was positive definite and self-adjoint.

The linearizability of the nonlinear system has been the foundation of the new continuum equivalent linearization method presented in Chapter III. That is, it was felt that the behavior of the nonlinear system could be modeled sufficiently well by a replacement, auxiliary linear system. The auxiliary system was determined three ways by the minimization of differences between three distinct quantities of the nonlinear and auxiliary linear systems. These difference quantities were:

1. differential stress difference (formulation I)
2. energy difference (formulation II)
3. stress difference (formulation III)

At this point, the new continuum equivalent equivalent linearization method became significantly different from its well-established equivalent linearization counterpart for discrete systems. Since the auxiliary system was defined at the continuum level, the system difference was not only a function of time, as with the discrete method, but also a function of the spatial coordinates. Hence the equivalent linear parameters were expanded in terms of trial functions (of the spatial coordinates), and the minimization between the system difference terms was performed with respect to the equivalent linear parameters.

It was shown in section 3.8 that, through the use of the positive definiteness

and self-adjointness of the stress operation in the auxiliary system, the minimization relationships for the equivalent linear stiffness and damping parameters were always invertible for formulations I and III. For formulation II, the invertibility was established for special systems. This contrasts with the minimization relations for discrete equivalent linearization. As pointed out in section 2.4.3, if the nonlinear system is discretized first followed by linearization of the resulting ordinary differential equations, the invertibility of the minimization relations is guaranteed for only a small number of degrees-of-freedom. Therefore, an advantage has been established for the continuum approach to the equivalent linearization over its discrete counterpart.

The above minimization procedure delineates one set of relations between the equivalent linear parameters and solutions to the auxiliary equations. Another set of relationships arise from the spatial discretization of the auxiliary equations. The finite element method to be used in the spatial discretization has been described in section 3.7 in terms of its convergence requirements. Of importance was the fact that continuity requirements on the displacement shape functions are generally more restrictive for the minimization relations than for the discretization of the auxiliary equations. Therefore consideration of the linearization technique must be made in choosing the displacement shape functions.

In Chapter IV, the general equivalent linear relationships from Chapter III were applied to the specific analysis of steady-state response to harmonic input and stationary response to stochastic excitation. For harmonic input, the minimization relations have been written in terms of the in-phase and quadrature components of the auxiliary equation solutions, whereas for random response the relations are written in terms of the covariance matrix of the discretized auxiliary system.

The continuum equivalent linearization technique was applied to the steady-state, harmonic analysis of three specific types of structural elements governed by nonlinear models. The first, the one-dimensional shear beam, was included to clarify the concepts and mechanics of one formulation of the method. The second and third examples, both pertaining to the nonlinear vibration of thin plates, were used to compare the results from the three problem formulations to solutions from existing methods and to demonstrate the applicability of the new method to problems to which standard techniques are not well suited.

In section 5.1, the response of a one-dimensional shear beam composed of a hysteretic material was investigated. Convergence of the formulation III approach was established. Single mode solutions were obtained for the equivalent linearization method and compared with results obtained from Galerkin's method. It was found that, in the limit as the mesh for the equivalent linear parameters is further subdivided, the formulation III solutions were precisely those of Galerkin's method. For finite subdivisons of the mesh, it was seen that the peak response underestimated the Galerkin solution.

The second example, found in section 5.2, was the vibration of a thin plate in which nonlinear coupling exists between the membrane and bending stresses. Use of the Poisson-Kirchhoff theory of thin plates dictated that the displacement shape functions have continuous first derivatives. Therefore Hermite bicubic shape functions were chosen. However, the minimization relations in section 5.2.2 required the use of displacements that had continuous second derivatives (formulations II and III) and third derivatives (formulation I). In lieu of using displacement fields with higher levels of continuity, it was decided that the resulting inter-element singularities would be discarded in the formation of the minimization equations.

Results from section 5.2 showed that the problem formulation I did not perform well in the analysis of nonlinear plates, and that the poor performance could possibly be attributed to the previously mentioned singularities in the minimization relations. The formulation III results compared well with the numerical integration solutions while formulation II consistently predicted a stiffer response. Formulaion III was therefore chosen to be the best approach for further investigation.

Section 5.3 dealt with the further examination of the nonlinear plate from section 5.2, whereas in section 5.3 the plate was permitted to have a concentrically located cutout. The Ritz procedure described in section 2.4.2 was not capable of satisfying the stress free boundary conditions at the hole. This resulted in an artificial frequency shift in the nonlinear response.

A finite element linear eigenvalue analysis of the cutout plate showed that for a range of small hole sizes, the fundamental natural frequency is less than that of the plate with no cutout. The use of the finite element-equivalent linearization technique predicted a similar, more flexible response of the nonlinear displacements for small holes than for the same plate without a hole.

In conclusion, it is felt that the continuum approach to equivalent linearization holds promise for use in the nonlinear analysis of structural members. The formulation of the method is such that the method can easily be incorporated into existing finite element computer codes. The construction of the method has been such that an understanding of the solution properties has been possible in this dissertation. Most importantly, the stress difference formulation has been successfully applied to structural elements in which both stiffness and energy dissipation properties are response dependent.

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Strong-Motion Earthquake Accelerograms
Digitized and Plotted Data

Uncorrected Accelerograms

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Strong-Motion Earthquake Accelerograms
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