NSF/CEE-82073 PB83-165209

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

THEORETICAL SEISMIC ANALYSIS OF CURVED BOX GIRDER BRIDGES

C. P. Heins, Professor
N. H. Falados, Graduate Research Assistant
University of Maryland, USA

G. H. Li, Professor
D. Shi, Associate Professor
Tongji University, PRC

to

National Science Foundation
Washington, D. C.

November 1982

UNIVERSITY OF MARYLAND

THEORETICAL SEISMIC ANALYSIS

OF

CURVED BOX GIRDER BRIDGES

C. P. Heins, Professor
N. H. Galados, Graduate Research Assistant
University of Maryland, USA

G. H. Li, Professor
D. Shi, Associate Professor
Tongji University, PRC

to

National Science Foundation
Washington, D.C.

TABLE OF CONTENTS

ABSTRACT

The seismic response of curved box girder bridges, incorporating interaction between the support columns and the girder, has been predicted by several schemes. The methods incorporate the influence of warping torsion and torsional-bending interaction.

The results from these various methods have been compared to other techniques, giving excellent correlation. These techniques will be applied in order to evaluate the influence of various parameters and the development of design formulations.

ACKNOWLEDGEMENT

The work covered in this report has been supported by the National Science Foundation Under Grant No. PFR-80-18729 covering the period December, 1980 through May 31, 1983. This grant is a cooperative US/PRC project between the University of Maryland and Tongji University.

The cooperation and encouragement of the NSF and the Hazards and Mitigation Section is gratefully acknowledged.

List of Tables

V

List of Figures

CHAPTER 1
INTRODUCTION

With the advent of the 1964 Alaskan Earthquake, the 1971 San Fernando Earthquake, the 1976 Tunsun Earthquake (1), the 1978 Santa Barbara Earthquake (2), and the Miyagi Earthquake (3), bridge structures throughout the world have undergone considerable destructive forces.

These earthquakes have caused the bridge professionals to reasses the design techniques that have been applied, up until that time, for seismic design.

A prime force in such modifications has been the Highway Department for the State of California (CALTRANS) and the California based professional organization Applied Technology Council (ATC).

The present 1977 AASHTO bridge code (4), as related to seismic design, was greatly influenced by the work developed by CALTRANS. This code suggests an equivalent static force method for simple structures and when the structure is complex, for example curved bridges, a computer based response spectrum or dynamic analysis should be considered.

In this present 1977 AASHTO code, most engineers would utilize the seismic coefficient method (SCM), because computer oriented dynamic programs may not be available nor are they amenable for direct design. However, the utilization of the S.C.M., may give erroneous results when designing under seismic conditions (5), as experienced by CALTRANS. CALTRANS in fact has utilized the response spectrum technique for the design of many structures.

Because of these conditions and from experience gained from recent earthquakes, the FHWA decided to reasses the 1975 AASHTO code and in 1977 sponsored a research program directed by ATC (6). The work of this council, in part, was to prepare a new specification, resulting in the ATC - 6 design guideline (7).

Although this code will be an improvement over the past criteria, there are major areas of research still requiring investigation. These areas, as suggested recently by the delegates attending the "Workshop on Earthquake REsistance of Highway Bridges", (7) conducted by ATC, include:

1. "Conduct Parametric Studies for the Seismic Response of Common Types of Bridges to Determine the Effects of Geometry and Constraint on Overall Seismic Response"

   Parameters should include:

   a) Span length

   b) Curvature

   c) Column height and stiffness

   d) Material, etc.

2. "Perform Appropriate Dynamic Analysis on Curved Bridges", (8,9) and develop a simple procedure for the design of curved bridges.

3. "Develop a Practical and Accurate Method to Estimate the Fundamental Period of Bridges".

4. "Correlate Vibrational Characteristics of Existing Bridges with Theory".

5. "Prepare Summary of Dynamic Behavior and Characteristics".

These various areas of research are presently being studied and will encom pass curved steel and concrete box girder bridges.

The techniques to be employed in the dynamic analysis of such structures can consist of:

i) response spectrum technique

ii) multi-mode method-response spectrum

iii) multi-mode time history analysis

These various methods are presently being used in this research. The method that was employed in the work described herein involves matrix formulations considering interactions between a single column bent or multi-column bent and an energy solution using the Rayleigh Ritz technique. These results have been computerized and compared to previously developed techniques (10), as well be presented herein.

These general theories are now being applied and the influence of various parameters studied. These results will then permit formulation of a proposed design criteria for curved steel and concrete bridges, which will be presented in a later report.

CHAPTER 2

THEORY - SINGLE COLUMN BENT

2.1 General

In this chapter, the formulation of the matrix method for the response of curved box girders, modeled as one dimensional beam, will be presented. Consider first the original curved box girder bridge system, beam and columns, as in Fig. 1(a), this system will be modeled as a combination of beam element connected with the pier element to form the entire structure, as shown in Fig. 1(b). With this basic modeling, the dynamic response of the entire structure will now be examined.

In developing the response of the curved bridge it is convenient to use curvilinear coordinates, shown in Fig. 2, where,

Y is the vertical direction

X is the radial direction

Z is the longitudinal direction and

ξ, η, ψ are displacements functions corresponding to X, Y, Z directions, are ϕ is the transverse rotation function.

Assuming the box girder structure has rigid internal diaphragm, and thus maintains its shape, the following displacement parameters $\delta(z)$ can be used to describe the displacement model of the element;

$$\delta = \begin{Bmatrix} \xi \\ \eta \\ \psi \\ \phi \end{Bmatrix} \tag{1}$$

2.2 Stress-Strain

The stresses induced on the curved element are represented by five actions or internal forces and can be described as follows;

$$\sigma = \begin{Bmatrix} My \\ Mx \\ N_Z \\ M_T \\ M_B \end{Bmatrix} \tag{2}$$

where My, and Mx, are the primary bending moments, N_Z is the axial force and M_T = pure torsional moment and M_B = warping torsional moment.

The strains are related to deformation according to the following;

$$\{\varepsilon\} = \begin{Bmatrix} \xi'' + \frac{\psi'}{R} \\ -\eta'' + \frac{\phi}{R} \\ -\frac{\xi}{R} + \psi' \\ \frac{\eta'}{R} + \phi' \\ -\frac{\eta''}{R} - \phi'' \end{Bmatrix} = \begin{bmatrix} p^2 & o & \frac{p}{R} & o \\ o & -p^2 & o & \frac{1}{R} \\ -\frac{1}{R} & o & p & o \\ o & \frac{p}{R} & o & p \\ o & -\frac{p^2}{R} & o & -p^2 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \\ \psi \\ \phi \end{Bmatrix} = [p]\,\{\delta\} \tag{3}$$

where R is radius curvature and $p = \frac{\partial}{\partial Z}$, $p^2 = \frac{\partial}{\partial Z^2}$. According to Hook's law, the stress-strain relationship for which the material is assumed linear, isotropic and elastic for one-dimensional problem is given by,

$$\{\sigma\} = [D]\,\{\varepsilon\} \tag{4}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ represent the induced internal forces and the corresponding strains, respectively; and [D] is material characteristic matrix, given as follows;

$$[D] = \begin{Bmatrix} EI_y & o & o & o & o \\ & EI_x & o & o & o \\ & & EA & o & o \\ (\text{Sym.}) & & & GK_t & o \\ & & & & EI_w \end{Bmatrix} \tag{5}$$

where; E = Young's modulus, G = shear elastic modulus, Iy, Ix are the moments of inertia in the Y and X direction; A = cross area, K_T = pure torsional constant, and $I\omega$ = warping torsional constant.

2.3 Displacement Functions

Normally a beam element in space can be represented by six degrees of freedom, three translation displacements and three rotational displacements. However, additiona degrees of freedom are required for curved bridge elements, such as warping, which may be important and must be considered. For example, consider the open cross section seen in Fig. (3a), or the separated box section shown in Fig. (3b) and a three pier system shown in Fig. (3c). If the super and sub-structures are properly modeled, then the warping torsional influence should be included. If the warping torsional resistance is not considered then the torsional resistance may be underestimated, giving erroneous results.

The expression for warping torsion, of the nodal point displacements, will include the parameter of twist angle along the longitundinal direction of the beam, thus increasing the nodal point displacements to seven. Also if the twist angle between the connection of the beam and pier is considered, then another displacement or digree of freedom is required thus giving eight displacements at a given nodal point. Therefore in the following development displacements δ_1, δ_2, . . . δ_8 will be used to express the joint i nodal point displacement parameters, and δ_9, δ_{10}, . . . δ_{16} will be used to express joint j nodal point displacements parameters.

The nodal point displacements of the element can be given by the following;

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{6}$$

where;

$$\{\delta_i\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_8 \end{Bmatrix} \quad \{\delta_j\} = \begin{Bmatrix} \delta_9 \\ \delta_{10} \\ \vdots \\ \delta_{16} \end{Bmatrix}$$

where;

δ_1, δ_2, δ_3, δ_9, δ_{10}, and δ_{11} are translational displacements;

δ_4, δ_5, δ_6, δ_{12}, δ_{13} and δ_{14} are rotational displacements;

δ_8 and δ_{16} are the rate of change of δ_6 and δ_{14} rotations respectively, along the longitudinal direction of the beam, which accounts for warping torsional influence.

δ_7 and δ_{15} are the continuity condition between the curved beam and pier. These nodal displacements are shown in Fig. 4.

Assuming polynomials for the one-dimensional displacement model; consequently, a third and a fifth order polynomials for the displacement functions for ϕ and ξ , and a linear displacement function for ψ are assumed respectively, thus the following displacement matrix is obtained;

$$\{\delta\} = [N]\ \{\delta\}^e \tag{7}$$

where;

$$[N] = \begin{bmatrix} N_7 & o & o & o & N_8 & o & N_9 & o & N_{10} & o & o & o & N_{11} & o & N_{12} & o \\ o & N_1 & o & N_2 & o & o & o & o & o & N_3 & o & N_4 & o & o & o & o \\ o & o & N_5 & o & o & o & o & o & o & o & N_6 & o & o & o & o & o \\ o & o & o & o & o & N_1 & o & N_2 & o & o & o & o & o & N_3 & o & N_4 \end{bmatrix}$$

and;

$$\left.\begin{aligned}
N_1 &= 1 - 3z^2 + 2z^3 \\
N_2 &= (z - 2z^2 + z^3)\ell \\
N_3 &= (3z^2 - 2z^3) \\
N_4 &= (-z^2 + z^3)\ell \\
N_5 &= 1 - z \\
N_6 &= z \\
N_7 &= 1 - 10z^3 + 15z^4 - 6z^5 \\
N_8 &= (z - 6z^3 + 8z^4 - 3z^5)\ell \\
N_9 &= \frac{1}{2}(z^2 - 3z^3 + 3z^4 - z^5)\ell^2 \\
N_{10} &= 10z^3 - 15z^4 + 6z^5 \\
N_{11} &= (-4z^3 + 7z^4 - 3z^5)\ell \\
N_{12} &= \frac{1}{2}(z^3 - 2z^4 + z^5)\ell^2
\end{aligned}\right\} \qquad (8)$$

where $z = Z/\ell$

2.4 Nodal Forces

Assuming a uniform load q acts on the elements;

$$q = \begin{Bmatrix} q_x \\ q_y \\ q_z \\ q_\phi \end{Bmatrix} \tag{9}$$

where q_x, q_y, and q_z are the uniform loads in the x, y, and z direction and q_ϕ is the uniform torque. Assuming now a virtual displacement $\{\delta^*\}$, and a corresponding virtual strain $\{\varepsilon\}$, the change in the total work caused by the stress (internal work) and uniform load (external work) is computed as:

$$W = \{\varepsilon *\}^T \{\sigma\}\, dz - \{\delta *\}^T \{q\}\, dz \tag{10}$$

Using the nodal point displacements to express the above, gives;

$\{\delta^*\} = [N]\ \{\delta^*\}^e$, and substituting in Eqs. (3) and (7) gives;

$$\{\varepsilon\} = [p]\ \{\delta\} = [p]\ [N]\ \{\delta\}^e = [B]\ \{\delta\}^e \tag{11}$$

or $\{\varepsilon^*\} = [B]\ \{\delta^*\}^e$

where $[B] = [p]\ [N]$ or

$$\{B\} = \begin{bmatrix} N_7'' & o & \frac{N_5'}{R} & o & N_8'' & o & N_9'' & o & N_{10}'' & o & \frac{N_6'}{R} & o & N_{11}'' & o & N_{12}'' & o \\ o & -N_1'' & o & -N_2'' & o & \frac{N_1}{R} & o & \frac{N_2}{R} & o & -N_3'' & o & -N_4'' & o & \frac{N_3}{R} & o & \frac{N_4}{R} \\ \frac{-N_7}{R} & o & N_5' & o & \frac{-N_8}{R} & o & \frac{-N_9}{R} & o & \frac{-N_{10}}{R} & o & N_6' & o & \frac{-N_{11}}{R} & o & \frac{-N_{12}}{R} & o \\ o & \frac{N_1'}{R} & o & \frac{N_2''}{R} & o & N_1' & o & N_2' & o & \frac{N_3'}{R} & o & \frac{N_4'}{R} & o & N_3' & o & N_4' \\ o & \frac{-N_1''}{R} & o & \frac{-N_2}{R} & o & -N_1'' & o & -N_2'' & o & \frac{-N_3''}{R} & o & \frac{-N_4''}{R} & o & -N_3'' & o & -N_4'' \end{bmatrix} \tag{12}$$

substituting (11) into (4) gives;

$$\{\sigma\} = [D]\ [B]\ \{\delta\}^e \tag{13}$$

substituting (13), (7) and (11) into (10) gives;

$$\{\delta^*\}^{eT} \int_\ell [B]^T [D]\ [B]\ dZ \{\delta\}^e = \{\delta^*\}^{eT} \int_\ell [N]^T \{q\}\ dZ \tag{14}$$

the element stiffness matrix and equivalent nodal point load matrix are therefore;

$$\{F\}^e = [k]\ \{\delta\}^e - \{Q\}\ \text{, where;}$$

$$[k] = \int_\ell\ [B]^T\ [D]\ [B]\ dZ \tag{15}$$

and

$$\{Q\} = \int_\ell\ [N]^T\ \{q\}\ dZ \tag{16}$$

2.5 Stiffness Matrix

Substituting [B] and [D] into Equation (15) and integrating, results in the following matrix;

$$[k] = [k]_1 + [k]_2 + [k]_3 + [k]_4 + [k]_5 \tag{17}$$

where $[k]_i$ are defined as follows;

$[k]_1 = EI_y \cdot$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	$\frac{120}{7\ell^3}$	0	0	0	$\frac{60}{7\ell^2}$	0	$\frac{3}{7\ell}$	0	$\frac{-120}{7\ell^3}$	0	0	0	$\frac{60}{7\ell^2}$	0	$-\frac{3}{7\ell}$	0
2		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3			$\frac{1}{\ell R^2}$	0	$\frac{1}{\ell R}$	0	0	0	0	0	$-\frac{1}{\ell R^2}$	0	$-\frac{1}{\ell R}$	0	0	0
4				0	0	0	0	0	0	0	0	0	0	0	0	0
5					$\frac{192}{35\ell}$	0	$\frac{11}{35}$	0	$-\frac{60}{7\ell^2}$	0	$-\frac{1}{\ell R}$	0	$\frac{108}{35\ell}$	0	$-\frac{4}{35}$	0
6						0	0	0	0	0	0	0	0	0	0	0
7							$\frac{3\ell}{35}$	0	$-\frac{3}{7\ell}$	0	0	0	$\frac{4}{35}$	0	$\frac{\ell}{70}$	0
8								0	0	0	0	0	0	0	0	0
9									$\frac{120}{7\ell^3}$	0	0	0	$-\frac{60}{7\ell^2}$	0	$\frac{3}{7\ell}$	0
10	Sym.									0	0	0	0	0	0	0
11											$\frac{1}{\ell R^2}$	0	$\frac{1}{\ell R}$	0	0	0
12												0	0	0	0	0
13													$\frac{192}{35\ell}$	0	$-\frac{11}{35}$	0
14														0	0	0
15															$\frac{3\ell}{35}$	0
16																0

$$
[k]_2 = IE_x \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & \frac{12}{\ell^3} & 0 & \frac{6}{\ell^2} & 0 & \frac{6}{5\ell R} & 0 & \frac{1}{10R} & 0 & -\frac{12}{\ell^3} & 0 & \frac{6}{\ell^2} & 0 & -\frac{6}{5R\ell} & 0 & \frac{1}{10R} & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & \frac{4}{\ell} & 0 & \frac{11}{10R} & 0 & \frac{2}{15}\frac{\ell}{R} & 0 & -\frac{6}{\ell^2} & 0 & \frac{2}{\ell} & 0 & -\frac{1}{10R} & 0 & -\frac{\ell}{30R} & 4 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
 & & & & & \frac{13\ell}{35R^2} & 0 & \frac{11\ell^2}{210R^2} & 0 & -\frac{6}{5R\ell} & 0 & \frac{1}{10R} & 0 & \frac{9\ell}{70R^2} & 0 & -\frac{13\ell^2}{420R^2} & 6 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
 & & & & & & & \frac{\ell^3}{105R^2} & 0 & -\frac{1}{10R} & 0 & -\frac{\ell}{30R} & 0 & \frac{13\ell^2}{420R^2} & 0 & -\frac{\ell^3}{140R^2} & 8 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
 & & & \text{Sym.} & & & & & & \frac{12}{\ell^3} & 0 & -\frac{6}{\ell^2} & 0 & \frac{6}{5R\ell} & 0 & -\frac{1}{10R} & 10 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\
 & & & & & & & & & & & \frac{4}{\ell} & 0 & -\frac{11}{10R} & 0 & \frac{2\ell}{15R} & 12 \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 & 13 \\
 & & & & & & & & & & & & & \frac{13\ell}{35R^2} & 0 & -\frac{11\ell^2}{210R^2} & 14 \\
 & & & & & & & & & & & & & & 0 & 0 & 15 \\
 & & & & & & & & & & & & & & & \frac{\ell^3}{105R^2} & 16
\end{array}
$$

$$
[k]_3 = EA \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
\frac{181\ell}{462R^2} & 0 & \frac{1}{2R} & 0 & \frac{311\ell^2}{4620R^2} & 0 & \frac{281\ell^3}{55440R^2} & 0 & \frac{25\ell}{231R^2} & 0 & -\frac{1}{2R} & 0 & -\frac{151\ell^2}{4620R^2} & 0 & \frac{181\ell^3}{55440R^2} & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
 & & \frac{1}{\ell} & 0 & \frac{\ell}{10R} & 0 & \frac{\ell^2}{120R} & 0 & \frac{\ell}{2R} & 0 & -\frac{1}{\ell} & 0 & -\frac{\ell}{10R} & 0 & \frac{\ell^2}{120R} & 0 & 3 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
 & & & & \frac{52\ell^3}{3465R^2} & 0 & \frac{69\ell^4}{55440R^2} & 0 & \frac{151\ell^2}{4620R^2} & 0 & -\frac{\ell}{10R} & 0 & -\frac{133\ell^3}{13860R^2} & 0 & \frac{13\ell^4}{13860R^2} & 0 & 5 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
 & & & & & & \frac{\ell^5}{9240R^2} & 0 & \frac{181\ell^3}{55440R^2} & 0 & -\frac{\ell^2}{120R} & 0 & -\frac{13\ell^4}{13860R^2} & 0 & \frac{\ell^5}{11088R^2} & 0 & 7 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\
 & & & & & & & & \frac{181\ell}{462R^2} & 0 & -\frac{1}{2R} & 0 & -\frac{311\ell^2}{4620R^2} & 0 & \frac{281\ell^3}{55440R^2} & 0 & 9 \\
 & & \text{Sym.} & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
 & & & & & & & & & & \frac{1}{\ell} & 0 & \frac{\ell}{10R} & 0 & -\frac{\ell^3}{120R} & 0 & 11 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 12 \\
 & & & & & & & & & & & & \frac{52\ell^3}{3465R^2} & 0 & -\frac{23\ell^4}{18480R^2} & 0 & 13 \\
 & & & & & & & & & & & & & 0 & 0 & 0 & 14 \\
 & & & & & & & & & & & & & & \frac{\ell^5}{9240R^2} & 0 & 15 \\
 & & & & & & & & & & & & & & & 0 & 16
\end{array}
$$

$$
[k]_4 = GK_T \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & \frac{6}{5\ell R^2} & 0 & \frac{1}{10R^2} & 0 & \frac{6}{5R\ell} & 0 & \frac{1}{10R} & 0 & -\frac{6}{5\ell R^2} & 0 & \frac{1}{10R^2} & 0 & -\frac{6}{5\ell R} & 0 & \frac{1}{10R} & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & \frac{2\ell}{15R^2} & 0 & \frac{1}{10R} & 0 & \frac{2\ell}{15R} & 0 & -\frac{1}{10R^2} & 0 & -\frac{\ell}{30R^2} & 0 & -\frac{1}{10R} & 0 & -\frac{\ell}{30R} & 4 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
 & & & & & \frac{6}{5\ell} & 0 & \frac{1}{10} & 0 & -\frac{6}{5\ell R} & 0 & \frac{1}{10R} & 0 & -\frac{6}{5\ell} & 0 & \frac{1}{10} & 6 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
 & & & & & & & \frac{2\ell}{15} & 0 & -\frac{1}{10R} & 0 & -\frac{\ell}{30R} & 0 & -\frac{1}{10} & 0 & -\frac{\ell}{30} & 8 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
 & & \text{Sym.} & & & & & & & \frac{6}{5\ell R^2} & 0 & -\frac{1}{10R^2} & 0 & \frac{6}{5\ell R} & 0 & -\frac{1}{10R} & 10 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\
 & & & & & & & & & & & \frac{2\ell}{15R^2} & 0 & -\frac{1}{10R} & 0 & \frac{2\ell}{15R} & 12 \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 & 13 \\
 & & & & & & & & & & & & & \frac{6}{5\ell} & 0 & -\frac{1}{10} & 14 \\
 & & & & & & & & & & & & & & 0 & 0 & 15 \\
 & & & & & & & & & & & & & & & \frac{2\ell}{15} & 16
\end{array}
$$

$$
[k]_5 = EI_\omega \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & \frac{12}{\ell^3 R^2} & 0 & \frac{6}{\ell^2 R^2} & 0 & \frac{12}{\ell^3 R} & 0 & \frac{6}{\ell^2 R} & 0 & -\frac{12}{\ell^3 R^2} & 0 & \frac{6}{\ell^2 R^2} & 0 & -\frac{12}{\ell^3 R} & 0 & \frac{6}{\ell^2 R} & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & \frac{4}{\ell R^2} & 0 & \frac{6}{\ell^2 R} & 0 & \frac{4}{\ell R} & 0 & -\frac{6}{\ell^2 R^2} & 0 & \frac{2}{\ell R^2} & 0 & -\frac{6}{\ell^2 R} & 0 & \frac{2}{\ell R} & 4 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
 & & & & & \frac{12}{\ell^3} & 0 & \frac{6}{\ell^2} & 0 & -\frac{12}{\ell^3 R} & 0 & \frac{6}{\ell^2 R} & 0 & -\frac{12}{\ell^3} & 0 & \frac{6}{\ell^2} & 6 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
 & & & & & & & \frac{4}{\ell} & 0 & -\frac{6}{\ell^2 R} & 0 & \frac{2}{\ell R} & 0 & -\frac{6}{\ell^2} & 0 & \frac{2}{\ell} & 8 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
 & & & & & & & & & \frac{12}{\ell^3 R^2} & 0 & -\frac{6}{\ell^2 R^2} & 0 & \frac{12}{\ell^3 R} & 0 & -\frac{6}{\ell^2 R} & 10 \\
 & & \text{Sym.} & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\
 & & & & & & & & & & & \frac{4}{\ell R^2} & 0 & -\frac{6}{\ell^2 R} & 0 & \frac{4}{\ell R} & 12 \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 & 13 \\
 & & & & & & & & & & & & & \frac{12}{\ell^3} & 0 & -\frac{6}{\ell^2} & 14 \\
 & & & & & & & & & & & & & & 0 & 0 & 15 \\
 & & & & & & & & & & & & & & & \frac{4}{\ell} & 16
\end{array}
$$

2.6 Element Mass Matrix

The influence of earthquake loading, using Newton's second law and considering the eccentricity of centroidal axes and shear center, will now be examined. These inertia loads include element vibration acceleration (q_v) and earthquake acceleration (q_g) loads and can be expressed as;

$$\{q\} = \{q_v\} + \{q_g\} \tag{18}$$

Consider first the element vibrations (q_v). Examination of Fig. 5 shows the eccentricity e between the centroidal axes and shear center, where the radial acceleration is given by; $\frac{\delta^2 \xi_1}{\delta t^2} = \frac{\delta^2 \xi}{\delta t^2} + \frac{\delta^2 \phi}{\delta t^2} \cdot e$. The radial horizontal inertia force is therefore;

$$q_{vx} = -\rho\ Am\ \ddot{\xi}_1 = -\delta Am\ (\ddot{\xi} + \ddot{\phi}\ e)$$

Similarly q_{vx} produces a torsion of $-q_{vx} \cdot e$, therefore the torsional inertia force is;

$$q_{v\phi} = -(\rho I \ddot{\phi} + q_{vx}\ e)$$

The total element vibration q_v is therefore;

$$\{q_v\} = \begin{Bmatrix} q_{vx} \\ q_{vy} \\ q_{vz} \\ q_{v\phi} \end{Bmatrix} = -\begin{bmatrix} \rho A_m(\ddot{\xi} + \ddot{\phi} e) \\ \rho Am\ \ddot{\eta} \\ \rho A_m \ddot{\psi} \\ \rho I_m \ddot{\phi} + \rho Ae(\ddot{\xi} + \ddot{\phi} \cdot e) \end{bmatrix} = -\rho \begin{bmatrix} A_m & o & o & A_m e \\ o & A_m & o & o \\ o & o & A_m & o \\ A_m e & o & o & I_m + A_m e^2 \end{bmatrix} \{\ddot{\delta}\}$$

$$= -\rho\ [A]\ \{\ddot{\delta}\} = -\rho\ [A]\ [N]\ \{\ddot{\delta}\}^e \tag{19}$$

where ρ = unit volume mass,

A_m = cross section in calculating mass,

I_m = mass moment of inertia

Substituting q_v into (14), the equivalent inertia force as a nodal point load is described as;

$$\{Q\}_v = -[m]\{\ddot{\delta}\}^e \tag{20}$$

where:

$$[m] = \int_{\ell} [N]^T \rho [\Lambda] [N] \, dZ \tag{21}$$

and m is element mass matrix. Integrating the above equation, gives;

$$[m] = [m]_1 + [m]_2 + [m]_3 \tag{22}$$

where m_1, m_2 and m_3 are given in the following;

$$
[m]_1 = \rho A_m \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
\frac{181\ell}{462} & 0 & 0 & 0 & \frac{311\ell^2}{4620} & 0 & \frac{281\ell^3}{55440} & 0 & \frac{25\ell}{231} & 0 & 0 & 0 & -\frac{151\ell^2}{4620} & 0 & \frac{181\ell^3}{55440} & 0 & 1 \\
 & \frac{13\ell}{35} & 0 & \frac{11\ell^3}{210} & 0 & 0 & 0 & 0 & 0 & \frac{9\ell}{70} & 0 & -\frac{13\ell^2}{420} & 0 & 0 & 0 & 0 & 2 \\
 & & \frac{\ell}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\ell}{6} & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & \frac{\ell^3}{105} & 0 & 0 & 0 & 0 & 0 & \frac{13\ell^2}{420} & 0 & -\frac{\ell^3}{140} & 0 & 0 & 0 & 0 & 4 \\
 & & & & \frac{52\ell^3}{3465} & 0 & \frac{69\ell^4}{55440} & 0 & \frac{151\ell^2}{4620} & 0 & 0 & 0 & -\frac{133\ell^3}{13860} & 0 & \frac{13\ell^4}{13860} & 0 & 5 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
 & & & & & & \frac{\ell^5}{9240} & 0 & \frac{181\ell^3}{55440} & 0 & 0 & 0 & -\frac{13\ell^4}{13860} & 0 & \frac{\ell^5}{11088} & 0 & 7 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\
 & & & & & & & & \frac{181\ell}{462} & 0 & 0 & 0 & -\frac{311\ell^2}{4620} & 0 & \frac{281\ell^3}{55440} & 0 & 9 \\
 & & & & & & & & & \frac{13\ell}{35} & 0 & -\frac{11\ell^2}{210} & 0 & 0 & 0 & 0 & 10 \\
 & \text{(Sym.)} & & & & & & & & & \frac{\ell}{3} & 0 & 0 & 0 & 0 & 0 & 11 \\
 & & & & & & & & & & & \frac{\ell^3}{105} & 0 & 0 & 0 & 0 & 12 \\
 & & & & & & & & & & & & \frac{52\ell^3}{3465} & 0 & -\frac{23\ell^4}{18480} & 0 & 13 \\
 & & & & & & & & & & & & & 0 & 0 & 0 & 14 \\
 & & & & & & & & & & & & & & \frac{\ell^5}{9240} & 0 & 15 \\
 & & & & & & & & & & & & & & & 0 & 16
\end{array}
$$

$$
[m]_2 = \rho I_m \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
 & & & & & \frac{13\ell}{35} & 0 & \frac{11\ell^2}{210} & 0 & 0 & 0 & 0 & 0 & \frac{9\ell}{70} & 0 & -\frac{13\ell^2}{420} & 6 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
 & & & & & & & \frac{\ell^3}{105} & 0 & 0 & 0 & 0 & 0 & \frac{13\ell^2}{420} & 0 & -\frac{\ell^3}{140} & 8 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\
 & \text{(Sym.)} & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 12 \\
 & & & & & & & & & & & & 0 & 0 & 0 & 0 & 13 \\
 & & & & & & & & & & & & & \frac{13\ell}{35} & 0 & -\frac{11\ell^2}{210} & 14 \\
 & & & & & & & & & & & & & & 0 & 0 & 15 \\
 & & & & & & & & & & & & & & & \frac{\ell^3}{105} & 16
\end{array}
$$

$$
[m]_3 = \rho A_m e \cdot
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\
\hline
0 & 0 & 0 & 0 & 0 & -\frac{8\ell}{21} & 0 & -\frac{3\ell^2}{56} & 0 & 0 & 0 & 0 & 0 & -\frac{5\ell}{42} & 0 & \frac{5\ell^2}{168} & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
 & & & & 0 & -\frac{11\ell^2}{168} & 0 & -\frac{\ell^3}{84} & 0 & 0 & 0 & 0 & 0 & -\frac{29\ell^2}{840} & 0 & \frac{7\ell^3}{840} & 5 \\
 & & & & & \frac{13e\ell}{35} & \frac{5\ell^3}{1008} & \frac{11e\ell^2}{210} & -\frac{5\ell}{42} & 0 & 0 & 0 & \frac{29\ell^2}{840} & \frac{9e\ell}{70} & -\frac{17\ell^3}{5040} & -\frac{13e\ell^2}{420} & 6 \\
 & & & & & & 0 & -\frac{\ell^4}{1008} & 0 & 0 & 0 & 0 & 0 & -\frac{17\ell^3}{5040} & 0 & \frac{\ell^4}{1260} & 7 \\
 & & & & & & & \frac{e\ell^3}{105} & -\frac{5\ell^2}{168} & 0 & 0 & 0 & \frac{\ell^3}{120} & \frac{13e\ell^2}{420} & -\frac{\ell^4}{1260} & -\frac{e\ell^3}{140} & 8 \\
 & & & & & & & & 0 & 0 & 0 & 0 & 0 & -\frac{8\ell}{21} & 0 & \frac{3\ell^2}{56} & 9 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
 & (\text{Sym.}) & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 11 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 12 \\
 & & & & & & & & & & & & 0 & \frac{11\ell^2}{168} & 0 & -\frac{\ell^3}{84} & 13 \\
 & & & & & & & & & & & & & \frac{13e\ell}{35} & -\frac{5\ell^3}{1008} & -\frac{11e\ell^2}{210} & 14 \\
 & & & & & & & & & & & & & & 0 & \frac{\ell^4}{1008} & 15 \\
 & & & & & & & & & & & & & & & \frac{e\ell^3}{105} & 16
\end{array}
$$

2.7 Seismic Mass Matrix

In developing the seismic mass matrix, curvilinear coordinates will be used. In general the earthquake acceleration, for the cartisian coordinate system, will be oriented in three-direction. Therefore the X and Z accelerations must be transformed into radial and tangent directions. As shown in Fig. 6a, the geometrical relationships can be written as;

$$\ddot{\xi}_g = \ddot{g}_x \cos\theta + \ddot{g}_z \sin\theta$$
$$\ddot{\eta}_g = g_y$$
$$\ddot{\psi}_g = -\ddot{g}_x \sin\theta + \ddot{g}_z \cos\theta$$
$$\ddot{\phi} = 0$$

As given by Eq. (19), the ground acceleration load matrix q_g can similarly be written as;

$$\{q_g\} = \begin{Bmatrix} q_{gx} \\ q_{gy} \\ q_{gy} \\ q_{g\phi} \end{Bmatrix} = - \begin{bmatrix} \rho A_m(\ddot{\xi}_g + \ddot{\phi}e) \\ \rho A_m \ddot{\eta}_g \\ \rho A_m \ddot{\psi} \\ \rho I\ddot{\phi} + \rho A_m e(\ddot{\xi} + \ddot{\phi}e) \end{bmatrix} = -\rho A_m \begin{bmatrix} \cos\theta & o & \sin\theta \\ o & 1 & o \\ -\sin\theta & o & \cos\theta \\ e\cos\theta & o & e\sin\theta \end{bmatrix} \begin{Bmatrix} \ddot{g}_x \\ \ddot{g}_y \\ \ddot{g}_z \end{Bmatrix}$$

$$= -\rho A\ [T]\ \{\ddot{g}\}$$

$$\{q_g\} = -\rho A[T]\{\ddot{g}\} \qquad (23)$$

Now substituting Eq. (23) into Eq. (16), gives the equivalent earthquake inertia load;

$$\{F\}g = -\rho A_m \int [N]^T [T] dZ \ \{\ddot{g}\} = -[g_m] \ \{\ddot{g}\} \tag{24}$$

where,

$$[gm] = \rho A_m \int_\ell [N]^T [T] dZ$$

Now substituting [N] and [T] into the above equation and integrating, gives the following element seismic mass matrix $[g_m]$, where the $\sin\theta$ and $\cos\theta$ terms, using Taylor series, can be written as;

$$\text{Sin } \theta = (1 - \frac{\ell^2}{2R^2} z^2) \text{ Sin } \theta_i - (\frac{\ell}{R} z - \frac{\ell^3 z^3}{6 R^3}) \text{ Cos } \theta_i$$

and

$$\text{Cos } \theta = (1 - \frac{\ell^2}{2R^2} z^2) \text{ Cos } \theta_i + (\frac{\ell}{R} z - \frac{\ell^3}{6R^3} z^3) \text{ Sin } \theta_i$$

$$[g_m] = \rho A_m \int_\ell [N]^T [T] \, dZ \text{ for girder}$$

$$
\begin{bmatrix}
[\ell/2 - 5\ell^3/168R^2]\cos\theta_i + [\ell^2/7R - 5\ell^4/1008R^3]\sin\theta_i & 0 & [\ell/2 - 5\ell^3/168R^2]\sin\theta_i - [\ell^2/7R - 5\ell^4/1008R^3]\cos\theta_i \\
0 & \ell/2 & 0 \\
-[\ell/2 - \ell^3/24R^2]\sin\theta_i + [\ell^2/6R - \ell^4/120R^3]\cos\theta_i & 0 & (\ell/2 - \ell^3/24R^2)\cos\theta_i + (\ell^2/6R - \ell^4/120R^3)\sin\theta_i \\
0 & \ell^2/12 & 0 \\
(\ell^2/10 - \ell^4/112R^3)\cos\theta_i + (4\ell^3/105R - \ell^5/630R^3)\sin\theta_i & 0 & (\ell^2/10 - \ell^4/112R^2)\sin\theta_i - (4\ell^3/105R - \ell^5/630R^3)\cos\theta_i \\
\{(\ell/2 - \ell^3/30R^2)\cos\theta_i + (3\ell^2/20R - \ell^4/168R^3)\sin\theta_i\}e & 0 & \{(\ell/2 - \ell^3/30R^2)\sin\theta_i - (3\ell^2/20R - \ell^4/168R^3)\cos\theta_i\}e \\
(\ell^3/120 - \ell^5/1120R^2)\cos\theta_i + (\ell^4/250R - \ell^6/6048R^3)\sin\theta_i & 0 & (\ell^3/120 - \ell^5/1120R^2)\sin\theta_i - (\ell^4/280R - \ell^6/6048R^3)\cos\theta_i \\
\{(\ell^2/12 - \ell^4/120R^2)\cos\theta_i + (\ell^3/30R - \ell^5/630R^3)\sin\theta_i\}e & 0 & \{(\ell^2/12 - \ell^2/120R^2)\sin\theta_i - (\ell^3/30R - \ell^5/630R^3)\cos\theta_i\}e \\
(\ell/2 - 23\ell^3/168R^2)\cos\theta_i + (5\ell^2/14R - 111\ell^4/3024R^3)\sin\theta_i & 0 & (\ell/2 - 23\ell^3/168R^2)\sin\theta_i - (5\ell^2/14R - 111\ell^4/3024R^3)\cos\theta_i \\
0 & \ell/2 & 0 \\
-(\ell/2 - \ell^3/8R^2)\sin\theta_i + (\ell^2/3R - \ell^4/30R^3)\cos\theta_i & 0 & (\ell/2 - \ell^3/8R^2)\cos\theta_i + (\ell^2/3R - \ell^4/30R^3)\sin\theta_i \\
0 & -\ell^2/12 & \\
(-\ell^2/10 + \ell^4/48R^2)\cos\theta_i + (-13\ell^3/210R + 5\ell^5/1008R^3)\sin\theta_i & 0 & (-\ell^2/10 + \ell^4/48R^2)\sin\theta_i - (-13\ell^3/210R + 5\ell^5/1008R^3)\cos\theta_i \\
\{(\ell/2 - 2\ell^3/15R^2)\cos\theta_i + (7\ell^2/20R - \ell^4/28R^3)\sin\theta_i\}e & 0 & \{(\ell/2 - 2\ell^3/15R^2)\sin\theta_i - (7\ell^2/20R - \ell^4/28R^3)\cos\theta_i\}e \\
(\ell^3/120 - \ell^5/672R^2)\cos\theta_i + (\ell^4/210R - \ell^6/3024R^3)\sin\theta_i & 0 & (\ell^3/120 - \ell^5/672R^2)\sin\theta_i - (\ell^4/210R - \ell^6/3024R^3)\cos\theta_i \\
\{(-\ell^2/12 + \ell^4/60R^2)\cos\theta_i + (-\ell^3/20R + \ell^5/252R^3)\sin\theta_i\}e & 0 & \{(-\ell^2/12 + \ell^4/60R^2)\sin\theta_i - (-\ell^3/20R + \ell^5/252R^3)\cos\theta_i\}e
\end{bmatrix}
$$

2.8 Pier Element and Coordinate Transformation

The basic pier element is similar to the curved-beam element except that pier is assumed as a straight element. Therefore the basic curved beam element matrix can be used to develop the local stiffness matrix (K_p) and mass matrix (m_p) for the Pier element, by letting $R \to \infty$. The curved beam element does not have to be transformed from the local coordinate into the global coordinate, however the pier element, must be transformed by using the following formula;

$$\{\delta_p\}^e = [L] \ \{\delta\}^e \tag{25}$$

where L is the coordinate transformation matrix:

$$[L] = \begin{bmatrix} L_i & 0 \\ 0 & L_j \end{bmatrix} \tag{26}$$

$$[L_i] = [L_j] = \left[\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & & & & & \\ 0 & 0 & -1 & & 0 & & & 0 \\ 0 & 1 & 0 & & & & & \\ & & & 1 & 0 & 0 & & \\ & 0 & & 0 & 0 & -1 & & 0 \\ & & & 0 & 1 & 0 & & \\ & 0 & & & 0 & & 0 & -1 \\ & & & & & & 1 & 0 \end{array}\right]$$

The stiffness and mass matrix for the global coordinate system, for the pier element, are therefore;

$$[k_p]_g = [L]^T \ [k_p] \ [L] \tag{27}$$

$$[m_p]_g = [L]^T \ [m_p] \ [L] \tag{28}$$

The intertia force components of the pier due to earthquake acceleration, can be calculated by using the following local coordinate system formula;

$$\{q\} = -\rho A_m \begin{bmatrix} \cos\theta_1 & 0 & \mathrm{Sin}\,\theta_1 \\ \mathrm{Sin}\,\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \{\ddot{g}\} = -\rho A[T_p]\{\ddot{g}\} \qquad (29)$$

where θ_1 is a constant, as shown in Fig. 6b.

Substituting Eq. (29) into Eq. (16) and integrating, gives the following seismic mass matrix for the pier element;

$$[g_{mp}] = \rho Am \begin{bmatrix} \frac{1}{2}\cos\theta_1 & 0 & \frac{1}{2}\mathrm{Sin}\,\theta_1 \\ \frac{1}{2}\mathrm{Sin}\,\theta_1 & 0 & -\frac{1}{2}\cos\theta_1 \\ 0 & \frac{1}{2} & 0 \\ \frac{\ell}{12}\mathrm{Sin}\,\theta_1 & 0 & \frac{-\ell}{12}\cos\theta_1 \\ \frac{\ell}{10}\cos\theta_1 & 0 & \frac{\ell}{10}\mathrm{Sin}\,\theta_1 \\ 0 & 0 & 0 \\ \frac{\ell^2}{120}\cos\theta_1 & 0 & \frac{\ell^2}{120}\mathrm{Sin}\,\theta_1 \\ 0 & 0 & 0 \\ \frac{1}{2}\cos\theta_1 & 0 & \frac{1}{2}\mathrm{Sin}\,\theta_1 \\ \frac{1}{2}\mathrm{Sin}\,\theta_1 & 0 & -\frac{1}{2}\cos\theta_1 \\ 0 & \frac{1}{2} & 0 \\ -\frac{\ell}{12}\mathrm{Sin}\,\theta_1 & 0 & \frac{\ell}{12}\cos\theta \\ -\frac{\ell}{10}\cos\theta_1 & 0 & -\frac{\ell}{10}\mathrm{Sin}\,\theta_1 \\ 0 & 0 & 0 \\ \frac{\ell^2}{120}\cos\theta_1 & 0 & \frac{\ell^2}{120}\mathrm{Sin}\,\theta_1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (30)$$

The local coordinate transformation, into the global coordinate system, is obtained from;

$$[g_{mp}]g = [L]^T [g_{mp}] \tag{31}$$

2.9 Dynamic Global Equations and Earthquake Response

Using the general dynamic equilibrium equation ground acceleration due to seismic loading, assembling the stiffness, mass and seismic matrix, and introducing the damping [C], the following global equation is obtained:

$$[M] \{\ddot{\delta}\} + [C] \{\dot{\delta}\} + [K] \{\delta\} = -[G_m] \{\ddot{g}(t)\} \tag{32}$$

where [M], [C] and [K] are mass, damping and stiffness matrix in global coordinate respectively and [G] is the seismic mass matrix represented in the global coordinate system and $\{\ddot{g}(t)\}$ represents the three directional ground accelerations.

<u>Boundary Conditions</u>:

In solving the bridge problem, the following set of boundary conditions will be assumed;

1) beam: hinge: $\delta_1 = 0$, $\delta_2 = 0$, $\delta_3 = 0$, $\delta_6 = 0$ and $\delta_7 = 0$

roller: $\delta_1 = 0$, $\delta_2 = 0$, $\delta_6 = 0$, and $\delta_7 = 0$

2) beam and pier:

<u>fixed end</u>: $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$

Using now the characteristic vector $[\Phi]$, as obtained from the free vibration equations and transformed into global coordinates, gives

$$\{\delta\} = [\Phi] \{Y\} = [\phi_1, \phi_2 \, \ldots\ldots\ldots\ldots \phi_n] \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{Bmatrix} \tag{33}$$

where Y_1, Y_2, ...Y_n are coordinate values and a function of time. ϕ_1, ϕ_2, ...ϕ_n are calculated from the free vibration equations, and must satisfy the following equations;

$$[K] [\Phi] - [\Omega^2] [M] [\Phi] = 0 \tag{34}$$

and from orthorgonality;

$$[\Phi]^T [K] [\Phi] = [\Omega^2]$$
$$[\Phi]^T [M] [\Phi] = [I] \tag{35}$$

where $[\Omega]$ is the frequency diagonal matrix. Substituting Eq. (33) into Eq. (32) and multiplying by $[\Phi]^T$ gives n uncoupled differential equations:

$$\{\ddot{Y}\} + 2\xi[\Omega] \{\dot{Y}\} + [\Omega^2] \{Y\} = - [\Phi]^T [G_m] \{\ddot{g}(t)\}$$
$$\text{or} \quad \{\ddot{Y}\} + 2\xi[\Omega] \{\dot{Y}\} + [\Omega^2] \{Y\} = -[\gamma] \{\ddot{g}(t)\} \tag{36}$$

where ξ is the structural damping ratio and $[\gamma]$ is the mode shape reference coefficients. If the ground acceleration $\{\ddot{g}(t)\}$ is known, then $\{Y\}$ can be determined. Substituting $\{Y\}$ into (33), the displacement response value at every nodal point can be evaluated. If the response spectrum analysis is used, the corresponding frequency spectrum displacement $\{S_d\}$, can then be determined. Therefore the displacement coefficients are:

$$\{Y\} \max = [\gamma] \{S_d\} \tag{37}$$

If the acceleration response spectrum $\{S_a\}$ is used, transformation into displacement response $\{S_d\}$ is obtained by applying $\{S_a\} = \omega^2\{S_d\}$.

The value of S_d from the response spectrum is the maximum, however these maximum values do not occur at the same time, therefore the following formula is used to calculate the node point displacements;

$$\delta = \sqrt{\delta_1{}^2 + \delta_2{}^2 + ..} \tag{38}$$

The element forces are given in two parts, (1) the force S_A due to element deformation, and (2) the force S_2 due to the element inertia force. The forces S_A is given as,

$$\{S_A\} = [k]\{\delta\}^e = [k]^e[\Phi]^e\{Y\} \tag{39}$$

The inertia force is produced by the element vibration acceleration, therefore the equivalent acceleration of the element vibration is;

$$\{A\}^e = [\Phi]^e \, [\omega^2] \, \{Y\}$$

the corresponding element end forces are therefore;

$$\{S_B\} = [m]^e\{A\}^e = [m]^e[\Phi]^e \, [\omega^2] \, \{Y\} \tag{40}$$

the total end forces are therefore;

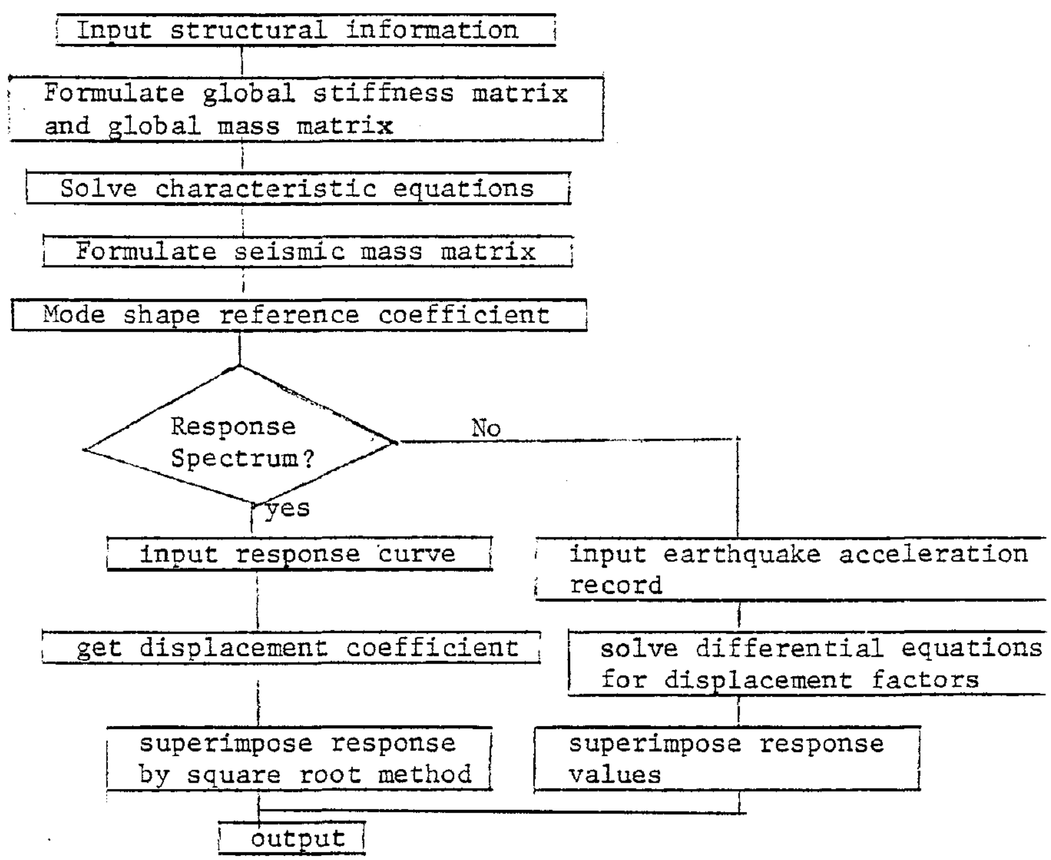
$$\{S\} = \{S_A\} + \{S_B\} \tag{41}$$

where according to the response spectrum method,

$$S = \sqrt{S_1{}^2 + S_2{}^2 + \ldots\ldots} \tag{42}$$

2.10 Computer Program Flow Chart

Using the matrix formulation, a computer program was developed to calculate the structural natural frequency and mode shape. These results, in conjunction with the response spectrum, were then used to determine the displacements and internal dynamic forces. The general flow chart for the computer program that was developed is as follows;



2.11 Example

Using the matrix method described herein and the resulting computer program, a three span continuous bridge, shown in Fig. 7(a), was analyzed and the results compared to the SAP IV computer results.

The bridge consists of a three-span continuous curved-beam bridge, with the left end hinged with all the other supports on rollers. The span lengths are 100', 100', 100', the radius is 600' and the pier height is 30'. The material properties are;

Young's modulus $E = 432000\ K/ft^2$

shear modulus $G = 183050\ K/ft^2$

density $\rho = 0.004658\ Kslug/ft^3$

Assuming a constant cross section, the sectional properties are;

	$A(ft^2)$	$I_x(ft^4)$	$I_Y(ft^4)$	$K_T(ft^4)$	$I_\omega(ft^6)$	$Am(ft^2)$	$I_m(K/ft\ sec^2)$
beam	61.18	425.9	5696.0	417.0	0.1	61.18	0.001
pier	35.93	300.0	138.3	178.7	0.1	35.93	0.001

The bridge was modeled with 16 nodal points and 13 elements, as shown in Fig. 7(b). The frequencies, from the dynamic analysis, are listed in the following table. The results obtained from a space frame system (six degrees of freedom at each nodal point), using the SAP IV program are also listed.

FREQUENCIES (CPS)

frequency NO.	curved-beam method	SAP IV results
1	20.22	19.96
2	25.00	25.12
3	31.73	31.99
4	44.40	44.14
5	45.33	45.15
6	53.53	53.48
7	91.34	91.78
8	101.40	97.39

As can be seen the frequencies and corresponding mode shapes, as shown in Fig. 8, agree well with the SAP IV data. The following mode shape figures, shown in Fig. 8, illustrate the mode shapes that occur in the radial direction (1,4 and 7), 4 mode shapes that occur in the vertical direction (2,3,5,8), and the axial direction mode shape as given by frequency 6.

Using the response spectrum method, the internal forces are calculated by both the curved beam method and SAP 4 program, giving the following results. The response spectrum curve that was used is the 1940 El Centro Earthquake record (1g), with a damping ratio of 5%.

TRANSVERSE EARTHQUAKE FORCE

		curved-beam method	SAP 1V
x direction displacement (ft)	node 7	$.577 \times 10^{-1}$	$.580 \times 10^{-1}$
	node 8	$.660 \times 10^{-1}$	$.669 \times 10^{-1}$
internal (K-ft) forces	element 4 My	$.139 \times 10^{5}$	$.139 \times 10^{5}$
	element 3 My	$.126 \times 10^{5}$	$.133 \times 10^{5}$
	element 7 My	$.193 \times 10^{5}$	$.204 \times 10^{5}$

VERTICAL EARTHQUAKE FORCE

		curved-beam method	SAP 1V
Y direction displacement(ft)	node 3	$.177 \times 10^{-1}$	$.161 \times 10^{-1}$
	node 8	$.231 \times 10^{-1}$	$.216 \times 10^{-1}$
internal (Kft) force or (K)	element 2 Mx	$.428 \times 10^{4}$	$.456 \times 10^{4}$
	element 6 Mx	$.605 \times 10^{4}$	$.589 \times 10^{4}$
	element 4 N	$.536 \times 10^{3}$	$.516 \times 10^{3}$

LONGITUDINAL EARHTQUAKE FORCE

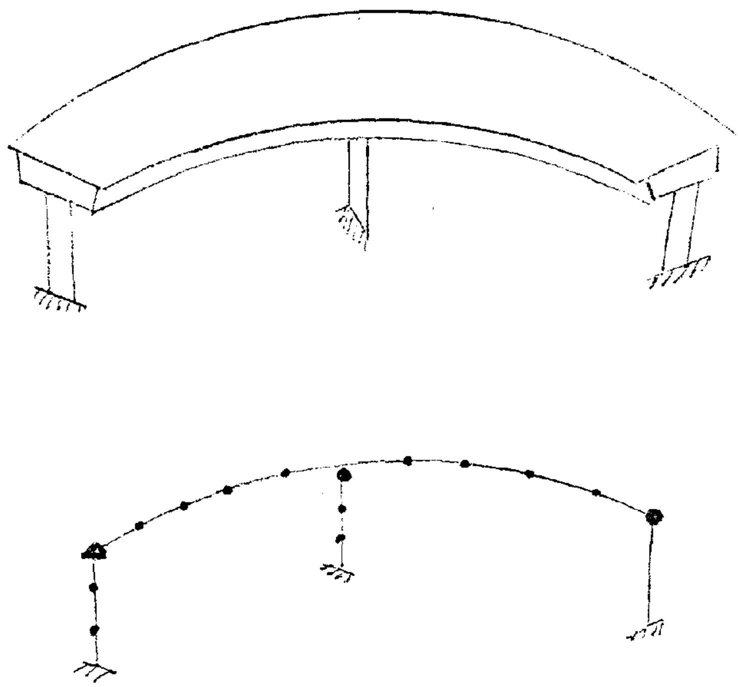
		curved-beam method	SAP IV
Z direction displacement (ft)	node 16	$.988 \times 10^{-2}$	$.942 \times 10^{-2}$
	node 7	$.415 \times 10^{-2}$	$.482 \times 10^{-2}$
internal force (Kft) or (K)	element 1 N	$.107 \times 10^{4}$	$.107 \times 10^{4}$
	element 3 My	$.793 \times 10^{4}$	$.804 \times 10^{4}$
	element 7 My	$.603 \times 10^{4}$	$.625 \times 10^{4}$

The preceding results represent the design values due to earthquake forces in the three directions. These results are in close agreement to the SAP IV results, but the method developed herein considers warping and the radial influence on the curved-element. Also of importance is fewer nodal points are required to obtain the same accuracy, as shown in the following table.

CURVED BEAM SOLUTION FREQUENCIES (CPS)

Mode	4 elements	3 elements	2 elements
1	12.71	12.72	12.77
2	46.11	46.23	46.56
3	51.29	51.70	56.76

As can be seen, using 4, 3, or 2 elements results in essentially the same frequency, thus a minimal number of elements are required for the same accuracy.



(a)

(b)

Fig. 1

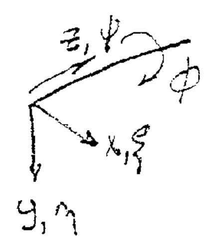
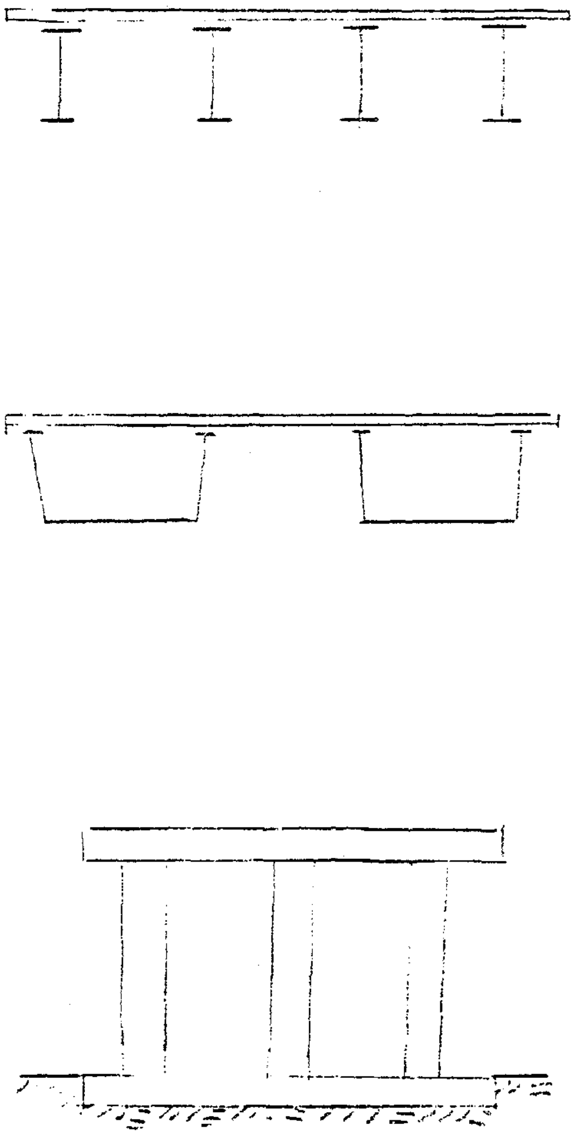


Fig. 2



(a)

(b)

(c)

Fig. 3

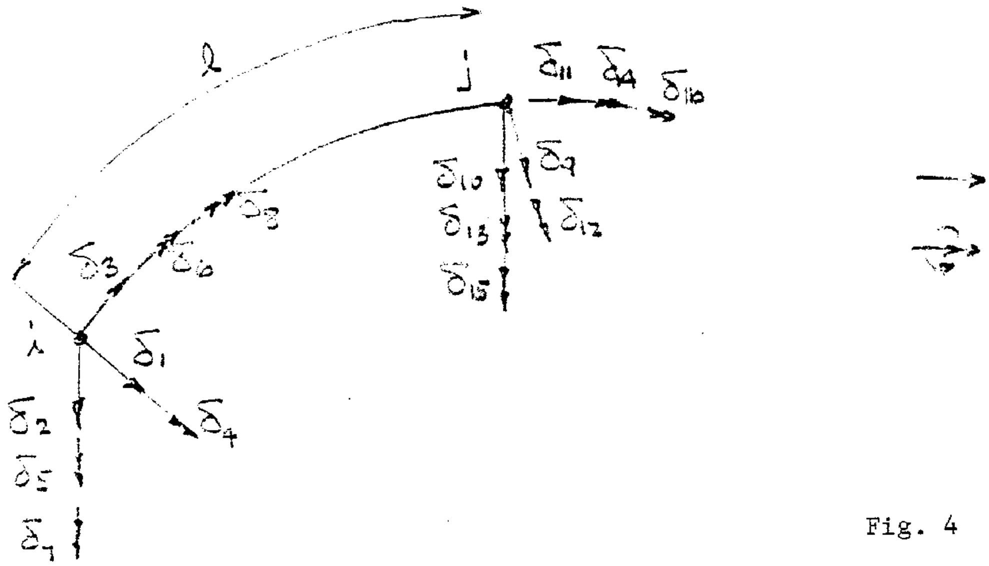


Fig. 4

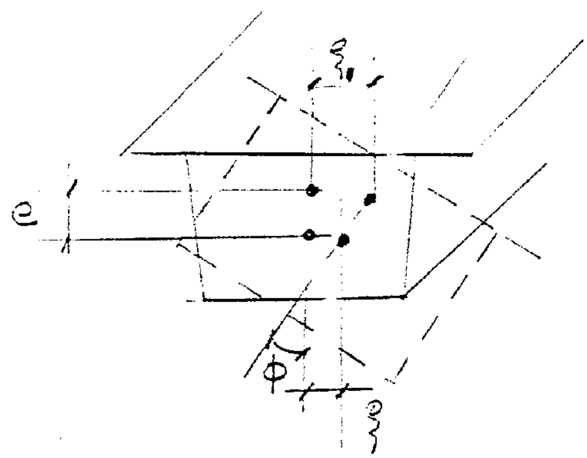


Fig. 5

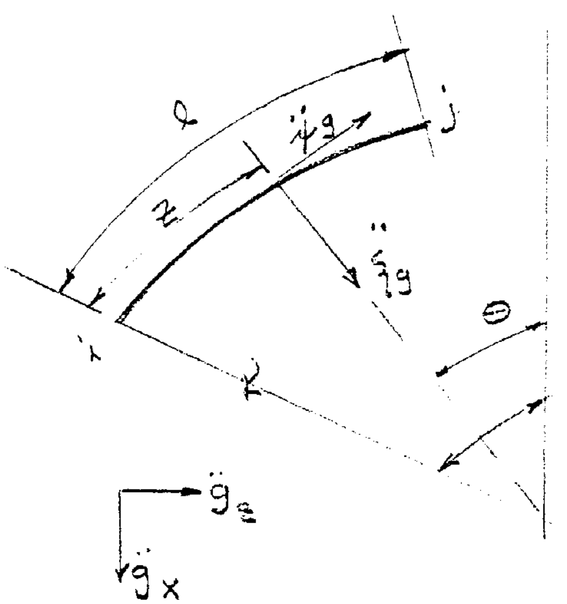
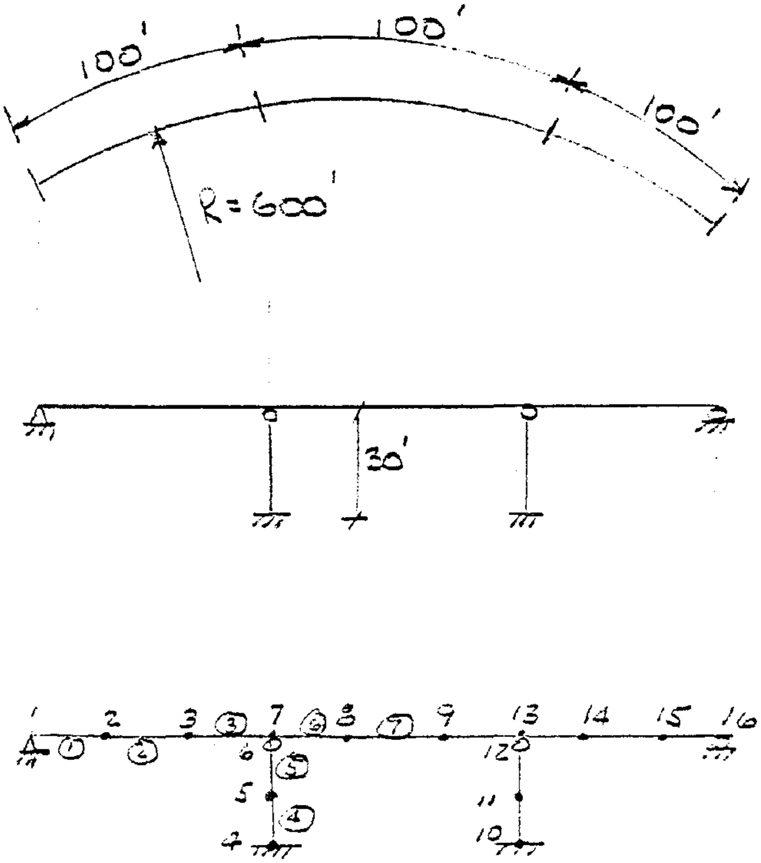


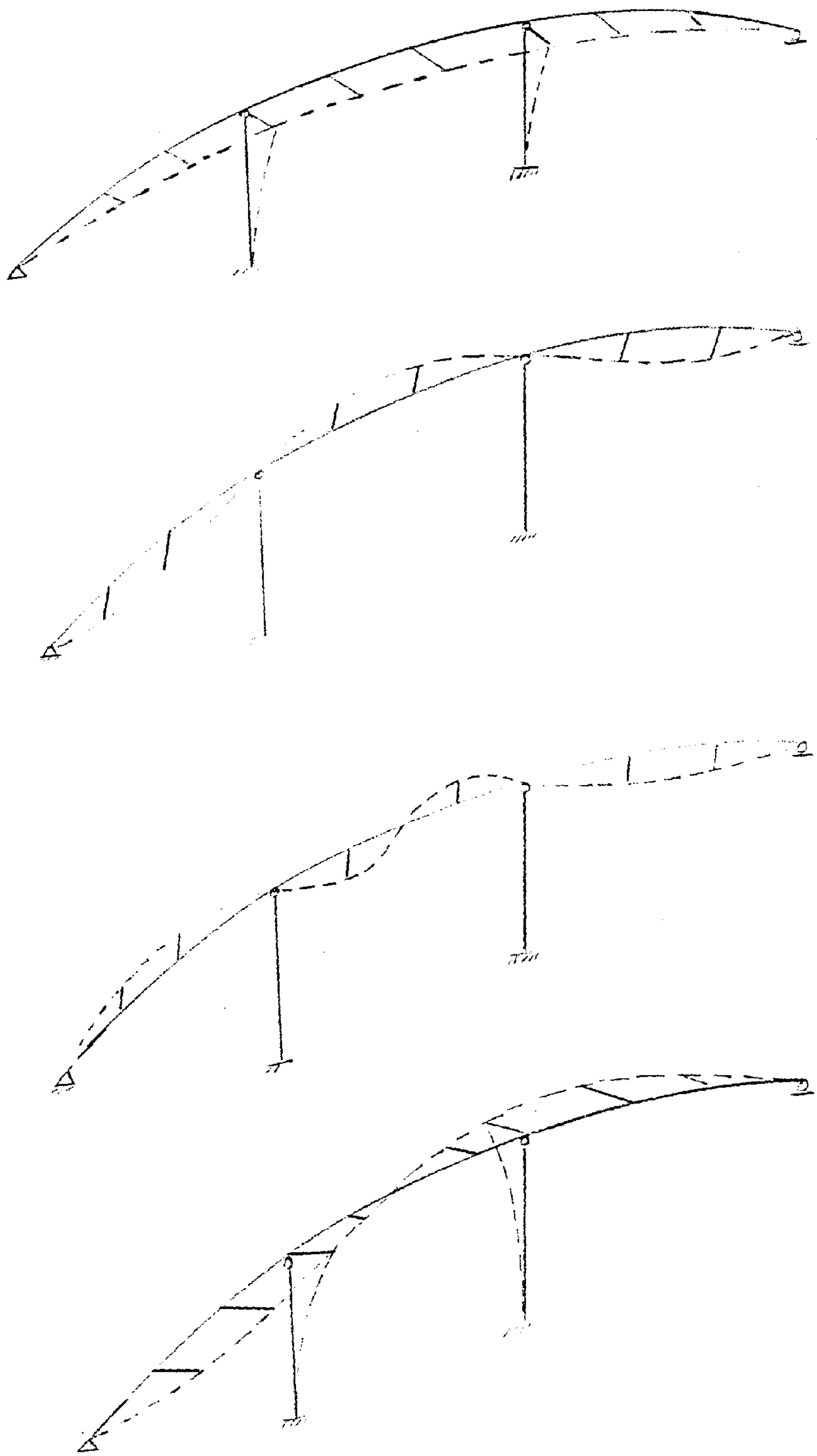
Fig. 6



(a)

(b)

Fig. 7 .



ω_1 = 20.22 TPS

ω_2 = 25.00

ω_3 = 31.73

ω_4 = 44.40

Fig. 8

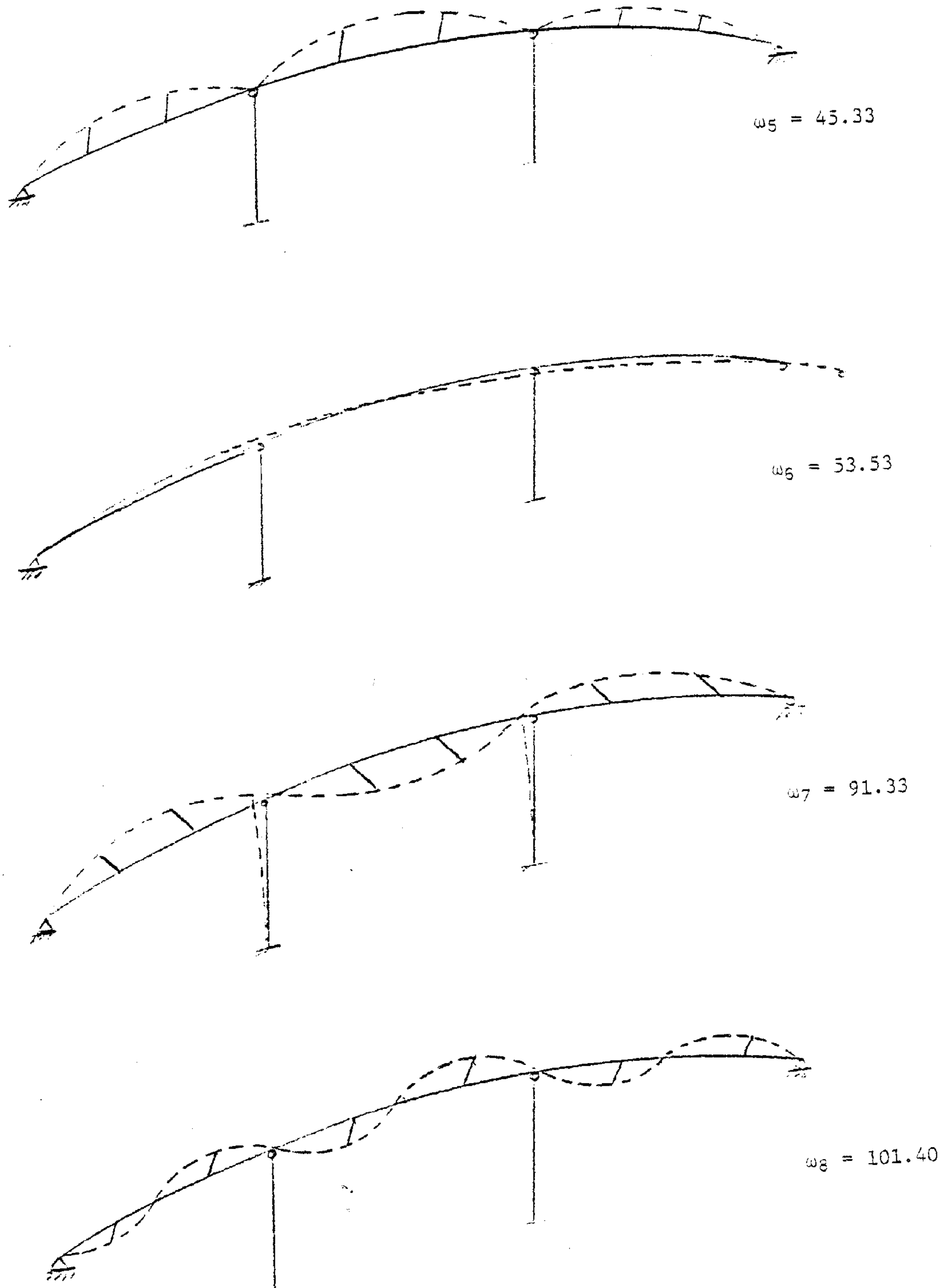


Fig. 8

CHAPTER 3

THEORY - MULTI COLUMN BENT

3.1 General

In the previous Chapter 2, the response of continuous curved beams, supported on single column bents was examined. This analysis required certain specified boundary conditions between the columns and beams and their complete interaction.

If the pier bent consists of multi columns, as shown in Fig. 9a, the rigid pier cap will restrain the piers such that the rotational and vertical influences from the curved beam are negligible. Therefore the vertical and rotational interaction between the beam and pier does not exist, and thus can be separated, in solving the curved beam response, Therefore, the primary displacements for the pier will be radial, longitudinal and rotational, and for the curved beam the primary displacements will be radial and longitudinal.

As shown in Fig. 9a, the radial displacement is assumed as ξ_p. As shown in Fig. 9b, the displacement in the longitudinal direction is ψ_p.

It is also known that the most critical element for these types of bridges is the pier, therefore in the following discussion, emphasis will be given to the transverse vibration of the pier.

As shown in Chapter 2, curvilinear coordinates will be used, where X is the radial direction, Y is the vertical direction and Z is the axial direction. The terms ξ, η, and ϕ are displacements corresponding to x, y, and z directions. The displacement of pier elements in the x and z directions are represented by displacement functions ξ_p, ψ_p, and torsional angle ϕ_p. For the curved-beam element the displacements considered are the radial displacement ξ and tangent displacement ψ.

3.2 Pier Element

The pier element describing the radial direction, uses a shear type, but in tangent direction the beam element will be used.

The displacement parameters for the pier are;

$$\{\delta\}_p = \begin{Bmatrix} \xi_p \\ \psi_p \\ \phi_p \end{Bmatrix} \tag{1}$$

The bridge pier element nodal point displacements parameters as described in Fig. 10, are;

$$\{\delta_p\}^e = \begin{Bmatrix} \delta_{pi} \\ \delta_{pj} \end{Bmatrix} \tag{2}$$

$$\{\delta_{pi}\} = \begin{Bmatrix} \xi_i \\ \psi_i \\ \psi_i' \\ \phi_i \\ \phi_i' \end{Bmatrix}, \quad \{\delta_{pj}\} = \begin{Bmatrix} \xi_j \\ \psi_j \\ \psi_j' \\ \phi_j \\ \phi_j' \end{Bmatrix} \tag{3}$$

Assuming the displacement function and evaluating using known nodal values then it is obtained,

$$\{\delta_p\} = [N]_p \{\delta_p\}^e$$

and $$[N_p] = \begin{bmatrix} N_5 & o & o & o & o & N_6 & o & o & o & o \\ o & N_1 & N_2 & o & o & o & N_3 & N_4 & o & o \\ o & o & o & N_1 & N_2 & o & o & o & N_3 & N_4 \end{bmatrix} \qquad (4)$$

where;

$$\left.\begin{aligned} N_1 &= 1 - 3y + 2\,y^3 \\ N_2 &= (y - 2y^2 + y^3)\,\ell \\ N_3 &= (3y^2 - 2y^3) \\ N_4 &= (-y^2 + y^3)\,\ell \\ N_5 &= 1 - y \\ N_6 &= y \end{aligned}\right\} \qquad (5)$$

the induced force vector is;

$$\{\sigma\}_p = \begin{Bmatrix} Q_x \\ M_x \\ M_T \\ M_B \end{Bmatrix} \qquad (6)$$

where Q_x : radial shear

M_x : radial bending moment

M_T : pure torsional moment

M_B : warping torsional moment

the strain matrix is;

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial \xi_p}{\partial y} \\ \frac{\partial^2 \psi_p}{\partial y^2} \\ \frac{\partial \phi_p}{\partial y} \\ \frac{\partial^2 \phi_p}{\partial y^2} \end{Bmatrix} = \begin{bmatrix} p & o & o \\ o & p^2 & o \\ o & o & p \\ o & o & -p^2 \end{bmatrix} \begin{Bmatrix} \xi_p \\ \psi_p \\ \phi_p \end{Bmatrix} = [P]_p \{\delta\}_p$$

$$\{\varepsilon\} = [P]_p [N]_p \{\delta\}_p^e = [B]_p \{\delta\}_p^e \tag{7}$$

where

$p = \frac{\partial}{\partial Y}$ and

$$[B]_p = \begin{bmatrix} N_5' & o & o & o & o & N_6' & o & o & o & o \\ o & N_1'' & N_2'' & o & o & o & N_3'' & N_4'' & o & o \\ o & o & o & N_1' & N_2' & o & o & o & N_3' & N_4' \\ o & o & o & -N_1'' & -N_2'' & o & o & o & -N_3'' & -N_4'' \end{bmatrix} \tag{8}$$

the Elastic matrix is;

$$[D]_p = \begin{bmatrix} \frac{12EI_z}{\ell^3} & & & 0 \\ & EI_x & & \\ & & Gk_T & \\ & & & EI\omega \end{bmatrix} \tag{9}$$

where;

$\frac{12EI_z}{\ell^3}$ is the equivalent shear stiffness of the pier element.

Using Equation (2-15), the pier element stiffness matrix can be developed;

$$[k] = \int_{\ell} [B]^T \; [D] \; [B] \; dY \tag{10}$$

Substituting [B] and [D] into Eq. (10) and integrating gives the following;

$$[k] = [k_1] + [k_2] \tag{11}$$

where k_1 and k_2 are defined as;

$$[k]_1 = \begin{bmatrix}
\frac{12EI_z}{\ell^4} & 0 & 0 & 0 & 0 & -\frac{12EI_z}{\ell^4} & 0 & 0 & 0 & 0 \\
 & \frac{12EI_x}{\ell^3} & \frac{6EI_x}{\ell^2} & 0 & 0 & 0 & -\frac{12EI_x}{\ell^3} & \frac{6EI_x}{\ell^2} & 0 & 0 \\
 & & \frac{4EI_x}{\ell} & 0 & 0 & 0 & -\frac{6EI_x}{\ell^2} & \frac{2EI_x}{\ell} & 0 & 0 \\
 & & & \frac{6GK_T}{5\ell} & \frac{GK_T}{10} & & 0 & 0 & -\frac{6GK_T}{5\ell} & \frac{GK_T}{10} \\
 & & & & \frac{5GK_T\ell}{12} & 0 & 0 & 0 & -\frac{GK_T}{10} & -\frac{GK_T\ell}{30} \\
 & \text{Symm.} & & & & \frac{12EI_z}{\ell^4} & 0 & 0 & 0 & 0 \\
 & & & & & & \frac{12EI_x}{\ell^3} & -\frac{6EI_x}{\ell^2} & 0 & 0 \\
 & & & & & & & \frac{4EI_x}{\ell} & 0 & 0 \\
 & & & & & & & & \frac{6GK_T}{5\ell} & -\frac{GK_T}{10} \\
 & & & & & & & & & \frac{5GK_T}{12}
\end{bmatrix} \tag{12}$$

$$[k]_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & \frac{12EI\omega}{\ell^3} & \frac{6EI\omega}{\ell^2} & 0 & 0 & 0 & \frac{-12EI\omega}{\ell^3} & \frac{6EI\omega}{\ell^2} \\
 & & & & \frac{4EI\omega}{\ell} & 0 & 0 & 0 & -\frac{6EI\omega}{\ell^2} & \frac{2EI\omega}{\ell} \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & \text{Symm.} & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & \frac{12EI\omega}{\ell^3} & \frac{-6EI\omega}{\ell^2} \\
 & & & & & & & & & \frac{4EI\omega}{\ell}
\end{bmatrix} \tag{13}$$

Using the following equation as given in Chapter 2, the element mass matrix can be developed as follows:

$$[m] = \int_{\ell} [N]^T \rho \, [\Lambda] \, [N] \, dY$$

where ρ is the unit volume mass and for pier element $[\Lambda]$ is defined as;

$$[\Lambda] = \begin{bmatrix} \frac{A_{pm}\ell}{2} & o & o \\ o & A_{pm} & o \\ o & o & I_{pm} \end{bmatrix}$$

where A_{pm} is the cross section area used in calculating the mass. Now substituting Eq. (4) into the [m] equation and integrating, the pier element mass matrix is developed as follows;

$$[m] = \rho A_{pm} \begin{bmatrix}
\frac{\ell^2}{6} & o & o & o & o & \frac{\ell^2}{12} & o & o & o & o \\
 & \frac{13}{35}\ell & \frac{11\ell^2}{210} & o & o & o & \frac{9}{70}\ell & -\frac{13}{420}\ell^2 & o & o \\
 & & \frac{\ell^3}{105} & o & o & o & \frac{13}{420}\ell^2 & -\frac{1}{140}\ell^3 & o & o \\
 & & & \frac{13\ell}{35}\frac{Ipm}{Apm} & \frac{11\ell^2}{210}\frac{Ipm}{Apm} & o & o & o & \frac{9\ell Ipm}{70Apm} & -\frac{13\ell^2}{420}\frac{Ipm}{Apm} \\
 & & & & \frac{\ell^3}{105}\frac{Ipm}{Apm} & o & o & o & \frac{13\ell^2 Ipm}{420\,Apm} & -\frac{\ell^3 Ipm}{140\,Apm} \\
 & & & & & \frac{\ell^2}{6} & o & o & o & o \\
 & & & & & & \frac{13}{35}\ell & -\frac{11}{210}\ell^2 & o & o \\
 & \text{Symm.} & & & & & & \frac{\ell^3}{105} & o & o \\
 & & & & & & & & \frac{13\ell Ipm}{35\,Apm} & -\frac{11\ell^2 Ipm}{210\,Apm} \\
 & & & & & & & & & \frac{\ell^3}{105}\frac{Ipm}{Apm}
\end{bmatrix}$$

3.3 Curved-Beam Element

As described previously, the vertical and rotational vibrations will be neglected, and therefore the curved-beam displacement functions are;

$$\{\delta\} = \{ {\xi \atop \psi} \} \tag{14}$$

where the element nodal point displacements parameters, shown in Fig. 11, are therefore;

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad \{\delta_i\} = \begin{Bmatrix} \xi_i \\ \psi_i \\ \xi_i' \\ \xi''_i \end{Bmatrix} \qquad \{\delta_j\} = \begin{Bmatrix} \xi_j \\ \psi_j \\ \xi_j' \\ \xi_j'' \end{Bmatrix} \tag{15}$$

Now equating the beam ξ'' function to the function ϕ', and using a fifth order polynomial to express the ξ displacement function, gives

$$\{\delta\} = [N]\ \{\delta\}^e \tag{16}$$

where;

$$[N] = \begin{bmatrix} N_7 & o & N_8 & N_9 & N_{10} & o & N_{11} & N_{12} \\ o & N_5 & o & o & o & N_6 & o & o \end{bmatrix} \tag{17}$$

and;

$$\begin{aligned}
N_7 &= 1 - 10z^3 + 15z^4 - 6z^5 \\
N_8 &= (z - 6z^3 + 8z^4 - 3z^5)\ell \\
N_9 &= 2(z^2 - 3z^3 + 3z^4 - z^5)\ell^2 \\
N_{10} &= 10z^3 - 15z^4 + 6z^4 \\
N_{11} &= (-4z^3 + 7z^4 - 3z^5)\ell \\
N_{12} &= 2\ (z^3 - 0.5z^4 + z^5)\ell^2 \\
N_5 &= 1 - z \\
N_6 &= z
\end{aligned}$$

where $z = Z/\ell$

Considering now the internal bending moments and axial forces, the force vector can be written as follows;

$$\{\sigma\} = \begin{Bmatrix} M_Y \\ N_Z \end{Bmatrix} \tag{18}$$

and the corresponding strains are;

$$\{\varepsilon\} = \begin{Bmatrix} \xi'' + \frac{\psi'}{R} \\ -\frac{\xi}{R} + \psi' \end{Bmatrix} = \begin{bmatrix} p^2 & \frac{p}{R} \\ -\frac{1}{R} & p \end{bmatrix} \{\delta\} = [P]\{\delta\}$$

$$\{\varepsilon\} = [P]\ [N]\ \{\delta\}^e = [B]\ \{\delta\}^e \tag{19}$$

where;

$$[B] = \begin{bmatrix} N_7'' & \frac{1}{R}N_5' & N_8'' & N_9'' & N_{10}'' & \frac{1}{R}N_6' & N_{11}'' & N_{12}'' \\ -\frac{1}{R}N_7 & N_5' & -\frac{1}{R}N_8 & -\frac{1}{R}N_9 & -\frac{1}{R}N_{10} & N_6' & -\frac{1}{R}N_{11} & -\frac{1}{R}N_{12} \end{bmatrix} \tag{20}$$

the Elastic matrix is;

$$[D] = \begin{bmatrix} EI_Y & o \\ o & EA \end{bmatrix}$$

Substituting [B] and [D] into Eq. (10), gives the curved-beam element stiffness matrix and follows;

$$[k] = [k]_1 + [k]_2$$

where $[k]_1$ equals;

$$[k]_1 = EI_Y \begin{bmatrix} \frac{120}{7\ell^3} & 0 & \frac{60}{7\ell^2} & \frac{3}{7\ell} & -\frac{120}{7\ell^3} & 0 & \frac{60}{7\ell^2} & -\frac{3}{7\ell} \\ & \frac{1}{\ell R^2} & \frac{1}{\ell R} & 0 & 0 & -\frac{1}{\ell R^2} & -\frac{1}{\ell R} & 0 \\ & & \frac{192}{35\ell} & \frac{11}{35} & -\frac{60}{7\ell^2} & -\frac{1}{\ell R} & \frac{108}{35\ell} & -\frac{4}{35} \\ & & & \frac{3\ell}{35} & -\frac{3}{7\ell} & 0 & \frac{4}{35} & \frac{1}{70}\ell \\ & & \text{Symm.} & & \frac{120}{7\ell^3} & 0 & -\frac{60}{7\ell^2} & \frac{3}{7\ell} \\ & & & & & \frac{1}{\ell R^2} & \frac{1}{\ell R} & 0 \\ & & & & & & \frac{192}{35\ell} & -\frac{11}{35} \\ & & & & & & & \frac{3}{35}\ell \end{bmatrix}$$

and $[k]_2$ equals;

$$[k]_2 = EA \begin{bmatrix} \frac{181\ell}{462R} & \frac{1}{2R} & \frac{311}{4620}\frac{\ell^2}{R^2} & \frac{281}{55440}\frac{\ell^3}{R^2} & \frac{25}{231}\frac{\ell^3}{R^2} & -\frac{1}{2R} & -\frac{151}{4620}\frac{\ell^2}{R^2} & \frac{181}{55440}\frac{\ell^3}{R^2} \\ & \frac{1}{\ell} & \frac{1}{10}\frac{\ell}{R} & \frac{1}{120}\frac{\ell^2}{R} & \frac{1}{2}\frac{1}{R} & -\frac{1}{\ell} & -\frac{1}{10}\frac{\ell}{R} & \frac{1}{120}\frac{\ell^2}{R} \\ & & \frac{52}{3465}\frac{\ell^3}{R^2} & \frac{69}{55440}\frac{\ell^4}{R^2} & \frac{151}{4620}\frac{\ell^2}{R^2} & -\frac{1}{10}\frac{\ell}{R} & -\frac{133}{13860}\frac{\ell^3}{R^2} & \frac{13}{13860}\frac{\ell^4}{R^2} \\ & & & \frac{1}{9240}\frac{\ell^5}{R^2} & \frac{181}{55440}\frac{\ell^3}{R^2} & -\frac{1}{120}\frac{\ell^2}{R} & -\frac{13}{13860}\frac{\ell^4}{R^2} & \frac{1}{11088}\frac{\ell^5}{R^2} \\ & \text{Symm.} & & & \frac{181\ell}{462R^2} & -\frac{1}{2R} & -\frac{311}{4620}\frac{\ell^2}{R^2} & \frac{281}{55440}\frac{\ell^3}{R^2} \\ & & & & & \frac{1}{\ell} & \frac{1}{10}\frac{\ell}{R} & -\frac{1}{120}\frac{\ell^2}{R} \\ & & & & & & \frac{52}{3465}\frac{\ell^3}{R^2} & -\frac{23}{18480}\frac{\ell^4}{R^2} \\ & & & & & & & \frac{1}{9240}\frac{\ell^5}{R^2} \end{bmatrix}$$

Using Eq. (2-21) the element mass matrix is developed as follows:

$$
= \rho A_m \begin{bmatrix}
\frac{181}{462}\ell & 0 & \frac{311}{4620}\ell^2 & \frac{281}{55440}\ell^3 & \frac{25}{231}\ell & 0 & -\frac{151}{4620}\ell^2 & \frac{181}{55440}\ell^3 \\
 & \frac{1}{3}\ell & 0 & 0 & 0 & \frac{1}{6}\ell & 0 & 0 \\
 & & \frac{52}{3465}\ell^3 & \frac{69}{55440}\ell^4 & \frac{151}{4620}\ell^2 & 0 & -\frac{133}{13860}\ell^3 & \frac{13}{13860}\ell^4 \\
 & & & \frac{1}{9240}\ell^5 & \frac{181}{55440}\ell^3 & 0 & -\frac{13}{13860}\ell^4 & \frac{1}{11088}\ell^5 \\
 & & \text{Symm.} & & \frac{181}{462}\ell & 0 & -\frac{311}{4620}\ell^2 & \frac{281}{55440}\ell^3 \\
 & & & & & \frac{1}{3}\ell & 0 & 0 \\
 & & & & & & \frac{52}{3465}\ell^3 & -\frac{23\ell^4}{18480} \\
 & & & & & & & \frac{1}{9240}\ell^5
\end{bmatrix}
$$

The pier element matrix is written in a local coordinate system which is the same as the global coordinate system, therefore it is not necessary to use the transformation matrix. However, it is important to recognize the differences in the nodal point parameters of the beam and pier, when developing the global matrix. This is necessary because of the different continuity conditions, between the connection of the beam and pier, which is given as follows:

<u>pin</u>: $\xi = \xi_p \quad \psi = \psi_p \quad \xi' = \phi_p \quad \xi_m'' = \phi_p'$

<u>roller</u>: $\xi = \xi_p$

Using the following global equation for free vibration and the appropriate bonding conditions, the frequency and eigen vector, are then determined.

$$[k]\{\delta\} - \omega^2[M]\{\delta\} = 0 \qquad (21)$$

3.4 Seismic Mass Matrix

As discussed in Chapter 2, curvelinear coordinates do not coincide with the earthquake cartisian coordinates, therefore a transformation of the earthquake accelerations into structural curvilinear coordiantes are necessary. The acceleration components, as shown in Fig, 12, for the curved-beam element are;

$$\{\ddot{q}\} = \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \ddot{g}_x \\ \ddot{g}_z \end{Bmatrix}$$

$$\{\ddot{q}\} = [T] \{\ddot{g}\} \qquad (22) \tag{22}$$

The seismic mass matrix for the beam element is given by;

$$[g_m] = \int_{\ell} [N]^T \rho A_m [T] \, dZ \tag{23}$$

It is convenient to integrate this equation using the sin θ and cos θ terms, listed as follows:

$$\sin\theta = \sin\theta_i - \frac{\ell}{R} z \cos\theta_i$$
$$\cos\theta = \cos\theta i - \frac{\ell}{R} z \sin\theta_i$$

These sin and cos equations represent a condition when the ratio (ℓ/R) is less than 1/10, and are obtained using Taylor's series expression.

Substituting these sin and cos relationships into Eq. (23) and integrating, gives the following seismic mass matrix for the beam element;

$$[g_m] = \begin{bmatrix} \frac{\ell^2}{7R}S + \frac{\ell}{2}C & \frac{\ell}{2}S - \frac{\ell^2}{7R}C \\ -\frac{\ell}{2}S + \frac{\ell^2}{6R}C & \frac{\ell^2}{6R}S + \frac{\ell}{2}C \\ \frac{4\ell^3}{105R}S + \frac{\ell^2}{10}C & \frac{\ell^2}{10}S - \frac{4\ell^3}{105R}C \\ \frac{\ell^4}{280R}S + \frac{\ell^3}{120}C & \frac{\ell^3}{120}S - \frac{\ell^4}{280R}C \\ \frac{5\ell^2}{14R}S + \frac{\ell}{2}C & \frac{\ell}{2}S - \frac{5\ell^2}{14}C \\ -\frac{\ell}{2}S + \frac{\ell^2}{3R}C & \frac{\ell^2}{3R}S + \frac{\ell}{2}C \\ -\frac{13\ell^3}{210R}S - \frac{\ell^2}{10}C & -\frac{\ell^2}{10}S + \frac{13\ell^3}{210R}C \\ \frac{\ell^4}{210R}S + \frac{\ell^3}{120}C & \frac{\ell^3}{120}C - \frac{\ell^4}{210R}C \end{bmatrix} \qquad (24)$$

where $S = \sin\theta_i$ and $C = \cos\theta_i$.

The components of the earthquake acceleration for the pier element are;

$$\{\ddot{q}\}_p = \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_z \\ \ddot{q}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \\ 0 & 0 \end{bmatrix} \{\ddot{g}\} = [T_p]\{\ddot{g}\} \qquad (25)$$

Substituting Eq. (5), [Np] and [Tp] into Eq. (23) and integrating, gives the seismic mass matrix of pier element as follows;

$$[g_m]_p = \begin{bmatrix} \frac{1}{2}C & \frac{1}{2}S \\ -\frac{1}{2}S & \frac{1}{2}C \\ \frac{\ell}{12}S & -\frac{\ell}{12}C \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2}C & \frac{1}{2}S \\ -\frac{1}{2}S & \frac{1}{2}C \\ -\frac{\ell}{12}S & \frac{\ell}{12}C \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (26)$$

3.5 Dynamic Global Equations and Earthquake Response

After assembling the stiffness, mass and seismic mass matrix, and applying the viscos damping matrix [C], the dynamic equilibrium equations under seismic loading can be obtained as follows;

$$[M]\ \{\ddot{\delta}\} + [C]\ \{\dot{\delta}\} + [K]\ \{\delta\} = -[Gm]\{\ddot{g}(t)\} \qquad (27)$$

Using the characteristic matrix [Φ], as obtained from the free vibration, and using a transformation matrix;

$$\{\delta\} = [\Phi]\ \{Y\} \qquad (28)$$

and substituting Eq (28) into Eq. (27) and multiplying by $[\Phi]^T$, gives n uncoupled differential equations of the following form;

$$\{\ddot{Y}\} + 2\xi\,[\Omega]\,\{\dot{y}\} + [\Omega^2]\,\{Y\} = -[\Phi]^T\,[Gm]\,\{\ddot{g}(t)\} \qquad (29)$$

The solution gives the value of {Y}, which is then substituted into Eq. (28) in order to obtain the resulting earthquake induced displacements. The resulting internal forces are then determined from the following Equation

$$\{S\} = ([k]^e\,[\Phi]^e + [m]^e\,[\Phi]^e\,[\,\omega^2\,])\,\{Y\} \qquad (30)$$

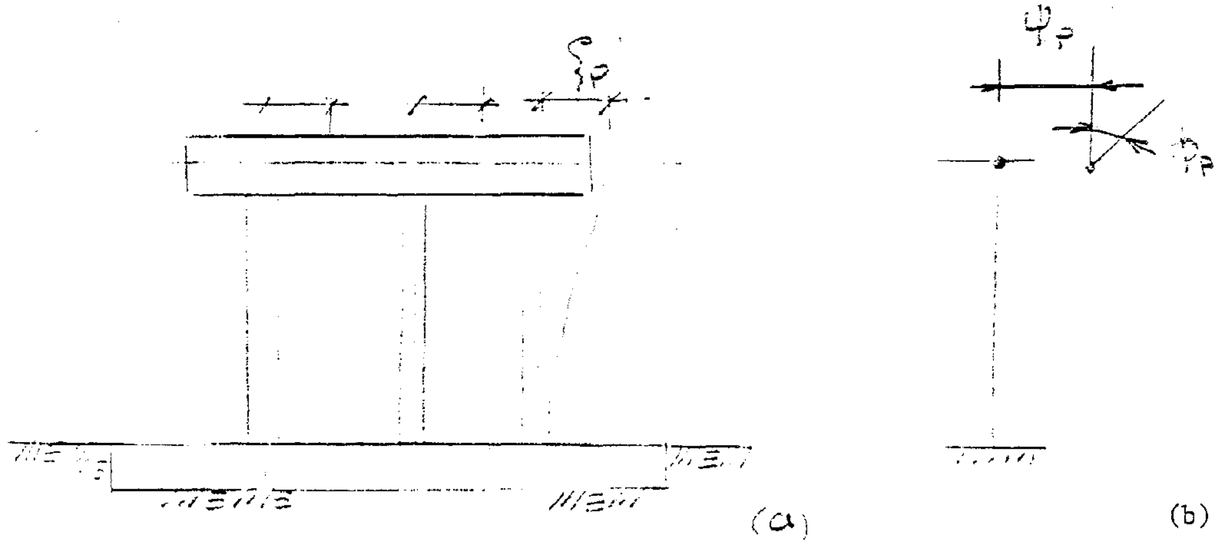


Fig. 9

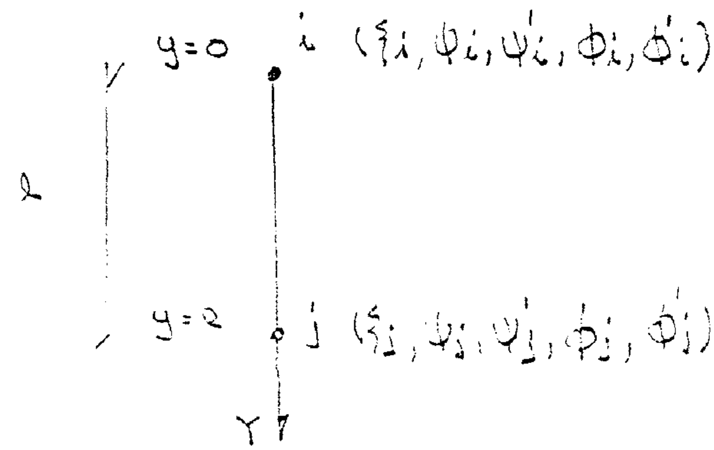


Fig. 10

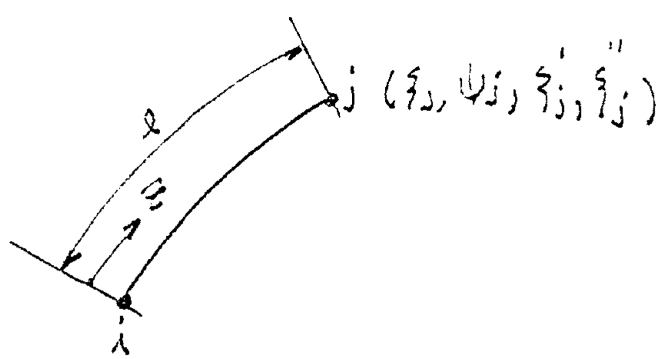


Fig. 11

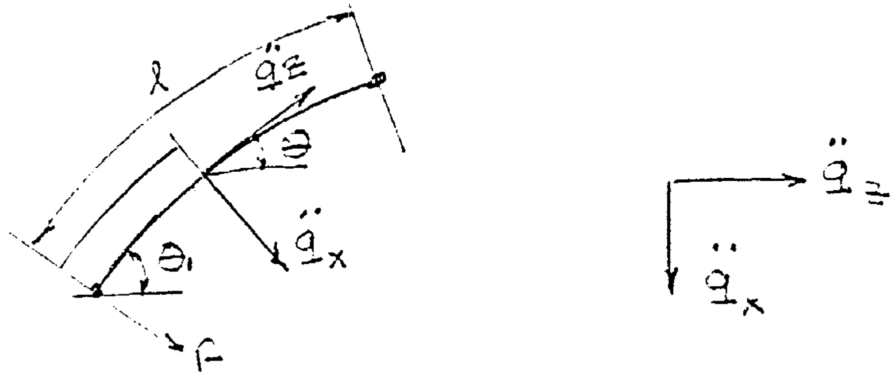


Fig. 12

CHAPTER 4

THEORY - RAYLEIGH RITZ METHOD

4.1 General

The Matrix Method, as described in Chapter 2, provides a method that is very accurate for dynamic analysis. Furthermore, it is very effective in calculating the earthquake response of curved bridges. However, the computation time required by the computer is excessive and thus is very uneconomical. It is therefore the purpose of this chapter to present an alternate method in calculating the dynamic frequency and mode shapes. Then the spectrum method is used to compute the earthquake response of curved bridges.

In order to determine the actual vibration conditions of a curved beam, the energy effects due to bending, mass eccentricity, axial force, torsion and warping must be determined. Also one primary requirement in utilizing the Rayleigh-Ritz Method is the determination of the mode shape function. As it will be described herein, the deflections due to various static loads will be selected as the mode shape function and linear combination will give the final mode shapes. Using therefore the deflection under static load as the mode shape function, the internal forces can then be calculated directly from the curvature of the defected shape.

4.2 Basic Equations

For convenience, the curvilinear coordinate system as shown in Fig. 2 of Chapter 2, will be adopted to describe the curved beam.

The displacement $\{\Delta\ (s,t)\}$ is a function of the coordinate and time, which is given as follows;

$$\{\Delta\} = \{\delta(s)\} \cdot Y(t) \tag{1}$$

where Y(t) is a time function.

$\{\delta\}$ is the vector of the displacement functions of the elements of the bridge structure and can be represented as;

$$\{\delta\} = \begin{Bmatrix} \xi\ (z) \\ \eta\ (z) \\ \psi\ (z) \\ \phi\ (z) \\ \xi_p(i,Y) \\ \psi_p(i,Y) \end{Bmatrix} \tag{2}$$

where ξ, η, ψ are the displacement functions in the X, Y, Z direction for the element dZ and ϕ is the twisting angle of the beam. The function $\xi_p(i,Y)$ is the displacement function in X direction of element dY of the ith pier, and $\psi(i,Y)$ is the displacement in Z direction of the ith pier.

The stress can be represented as a function of actions, and are as follows;

$$\{\sigma\} = \begin{Bmatrix} M_Y \\ M_x \\ N_z \\ M_T \\ M_B \\ M_{xp} \\ M_{zp} \end{Bmatrix} \tag{3}$$

where M_Y, M_X, are the bending moments of the beam in X and Y direction; N_Z is the axial force of the beam and M_T and M_B are torsion and warping of the beam respectively. The actions M_{xp} and M_{zp} are the bending moments in the pier in the X and Z direction respectively.

The corresponding strain can be represented by the following Eq.;

$$\{\varepsilon\} = \begin{Bmatrix} \xi'' + \frac{\psi'}{R} \\ \eta'' - \frac{\phi}{R} \\ \psi' - \frac{\xi}{R} \\ \phi' + \frac{\eta'}{R} \\ \phi'' + \frac{\eta''}{R} \\ \xi_p'' \\ \psi_p'' \end{Bmatrix} = \begin{bmatrix} \frac{\partial^2}{\partial Z^2} & 0 & \frac{1}{R}\frac{\partial}{\partial Z} & 0 & 0 & 0 \\ 0 & \frac{\partial^2}{\partial Z^2} & 0 & -\frac{1}{R} & 0 & 0 \\ -\frac{1}{R} & 0 & \frac{\partial}{\partial Z} & 0 & 0 & 0 \\ 0 & \frac{1}{R}\frac{\partial}{\partial Z} & 0 & \frac{\partial}{\partial Z} & 0 & 0 \\ 0 & \frac{1}{R}\frac{\partial}{\partial Z^2} & 0 & \frac{\partial^2}{\partial Z^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial^2}{\partial Y^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial^2}{\partial Y^2} \end{bmatrix} \begin{Bmatrix} \delta \end{Bmatrix}$$

or

$$\{\varepsilon\} = [P]\ \{\delta\} \tag{4}$$

Using Hooke's Law for induced internal forces;

$$\{\sigma\} = [D]\ \{\varepsilon\} \tag{5}$$

where

$$\begin{bmatrix} EI_Y & & & & & & \\ & EI_X & & & & 0 & \\ & & EA & & & & \\ & & & GK_T & & & \\ & 0 & & & EI_\omega & & \\ & & & & & EI_{pX} & \\ & & & & & & EI_{pZ} \end{bmatrix} \tag{6}$$

When the earhtquake occurs, the maximum potential energy of the structure is given as follows;

$$U = \frac{1}{2} \int \left([P]\ \{\Delta\}_{max}\right)^T [D] \left([P]\ \{\Delta\}_{max}\right) dS = \frac{Y^2}{2} \int \left([P]\ \{\Delta\}\right)^T [D]\ [P]\ \{\delta\}\, dS$$

$$U = U_{max} \cdot \frac{Y^2}{2} \tag{7}$$

The maximum kinetic energy of the structure is given by;

$$T = \frac{1}{2} \int \{\dot{\Delta} + \dot{q}\}^T_{max}\ [m]\ \{\dot{\Delta} + \dot{q}\}_{max}\ dS \tag{8}$$

where q is the displacement of the structure due to the ground motion, and the differentiation given is with respect to time.

The mass [m] given in Eq. (8) is given by;

$$[m] = \begin{bmatrix} \rho A & 0 & 0 & \rho Ae & & \\ 0 & \rho A & 0 & 0 & & \\ 0 & 0 & \rho A & 0 & & 0 \\ \rho Ae & 0 & 0 & \rho I + \rho Ae^2 & & \\ & & & & \rho_p A_p & \\ & & 0 & & & \\ & & & & & \rho_p A_p \end{bmatrix} \tag{9}$$

where e is the eccentricity between the centroidal axes and the shear center. Substituting Eq. (1) into Eq. (8), gives,

$$T = \frac{\dot{Y}^2}{2} \int \{\delta\}^T [m]\ \{\delta\}\ ds + \dot{Y} \int \{\delta\}^T [m]\ \{\dot{q}\}\ ds + \frac{1}{2} \int \{\dot{q}\}^T [m]\ \{\dot{q}\}\ ds$$

$$= T_{max} \frac{\dot{Y}^2}{2} + \dot{Y} \int \{\delta\}^T [m]\ \{\dot{q}\}\ ds + T_g$$

According to the Lagrange Equation;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) + \frac{\partial U}{\partial Y} = 0$$

therefore the following equation can be obtained;

$$\ddot{Y} + \frac{U_{max}}{T_{max}} Y = -\frac{\int \{\delta\}^T [m] \{\ddot{q}\} ds}{T_{max}} \qquad (10)$$

which is the dynamic response differential equation of the curved beam where damping is neglected.

4.3 Eigenvalue Solutions

If the displacement function $\{\delta\}$ is accurately determined, then based on the Rayleigh-Ritz Method, the frequency ω of structure is given by;

$$\omega^2 = \frac{U_{max}}{T_{max}} = \frac{\int ([P] \{\delta\})^T [D] ([P] \{\delta\}) ds}{\int \{\delta\}^T [m]\{\delta\} ds} \qquad (11)$$

However, it is difficult to obtain an accurate displacement function $\{\delta\}$, therefore the linear combination of several functions $\{F_1\}$, $\{F_2\}$, ... will be used to represent $\{\delta\}$;

Let

$$\{\delta\} = [F_1 F_2 \ . \ . \ .] \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \end{Bmatrix} = [F] \{A\} \qquad (12)$$

The parameter $\{A\}$ should be such to make ω^2 minimum or;

$$\frac{\partial \omega^2}{\partial \{A\}} = 0 \qquad (13)$$

Substituting Eq. (12) into Eq. (11), gives

$$\omega^2 = \frac{\{A\}^T [K] \{A\}}{\{A\}^T [M] \{A\}}$$

where the generalized stiffness matrix is;

$$[K] = \int [B]^T [D] [B] ds \qquad (14)$$

and

$$[B] = [P]\ [F] \tag{15}$$

the generalized mass matrix;

$$[M] = \int [F]^T\ [m]\ [F]\ dS \tag{16}$$

Using Eq. (13) gives;

$$[K]\ \{A\} - \omega^2\ [M]\ \{A\} = 0 \tag{17}$$

which is a homogeneous equation of $\{A\}$ whose solution gives the frequency ω and eigenvector $\{A\}$. Substituting $\{A\}$ into Eq. (12), gives the corresponding mode shape.

4.4 Displacement Functions - Three Span Structure

The primary requirement in using the approximation method, is to select the mode shape function. As described herein the deflection curve as obtained from the static loading, will be used as the mode shape function for one through four span continuous curved bridge. The resulting mode shape functions for 1, 2 and 4 spans are given in Tables 6, 7 and 9. The techniques required in developing the functions and matrices will be illustrated for the three span bridge.

A 3-span continuous bridge is shown in Fig. 13, with a total length S and an end span length of αS. The height of the end pier is h_1 and the height of the interior pier is h_2. The connection between beam and pier on the left end is hinged, with rollers at all the other supports. Considering these support connections, 10 displacement functions are required which include; 3ϕ, 3ξ, 2η and 2ψ.

(1) <u>Displacements in the radial direction</u>

The displacement will consist of 3 functions and are given as follows;

ξ_1 is the deflection due to uniform load as shown in Fig. 14(a) or

$$\xi_1(z) = S \cdot F_1(z) = S \cdot (z^4 - 2z^3 + z) \tag{18}$$

where $z = \frac{Z}{S}$

ξ_2 is the deflection due to a unit displacement at the interior pier, as shown in Fig. 14(b) or

$$\xi_2(z) = S \cdot F_2\ (\alpha, z) = S \cdot \begin{cases} z^3 + 3\alpha\ (\alpha - 1)\ z & ;\ 0 \le z < \alpha \\ 3\alpha z^2 - 3\alpha z + \alpha^3 & ;\ \alpha \le z \le 0.5 \end{cases} \tag{19}$$

when $z > 0.5$, the same displacement function can be used if the structure is symmetric. ξ_3 is the deflection due to a unit displacement at the end pier, shown in Fig. 14(c), or

$$\xi_3\ (z) = S \cdot F_3\ (\alpha, z) = S \cdot\ [F_2(\alpha, \alpha) - F_2\ (\alpha, z)] \tag{20}$$

If the support of the structure is on an abutment, this function can be neglected.

The pier displacement, as shown in Fig. 15, is given as;

$$F_{p1} = \frac{1}{2}\, y^2\ \ (3 - y) \tag{21}$$

where:

$$y = \frac{Y}{h}$$

Because the connection between the beam and the top of pier is hinge, the deflection of pier will be determined by the displacement of the beam. The deflection of the interior pier corresponding to the functions ξ_1, ξ_2, ξ_3, are therefore;

$$\left.\begin{aligned} \xi_{p4}\ (y) &= \xi_1\ (\alpha) \cdot F_{p1} \\ \xi_{p5}\ (y) &= \xi_2\ (\alpha) \cdot F_{p1} \\ \xi_{p6}\ (y) &= \xi_3\ (\alpha) \cdot F_{p1} \end{aligned}\right\} \tag{22}$$

(2.) Rotation ϕ

The rotation ϕ of the curved girder will consists of two parts, first, the rotation due to the vibration of the beam, as shown in Fig. 16 It should be noted that the functions for rotation are similar to the function for radial displacement and are;

$$\left.\begin{aligned} \phi_1 (z) &= F_1(z) \\ \phi_2 (z) &= F_2(\alpha, z) \\ \phi_3 (z) &= F_3(\alpha, z) \end{aligned}\right\} \quad (23)$$

Second, the beam rotation instituted by the displacement of the pier is considered as shown in Fig. 17. Using Eq. (22a), the rotation is given by;

$$\phi_p = \xi_{p\xi}' (1.0) = \xi_1 (\alpha) F_{p1}' (1.0)$$

The function of rotation can then be computed as;

$$\phi_4 = \frac{3S}{2h_2} \phi_1$$

Similarly, the rotation function corresponding to the functions ξ_2 and ξ_3 are given by;

$$\phi_5 = \frac{3}{2} \frac{S}{h_2} \phi_2 \qquad \phi_6 = \frac{3}{2} \frac{S}{h_1} \phi_3 \quad (24)$$

(3.) Vertical Displacement η

The vertical displacement will consists of two functions, as shown in Fig. 18, where;

$$\eta_1 = \begin{cases} -(\bar{z}^4 - 2\bar{z}^3 + \bar{z}) \cdot S, & \bar{z} = \dfrac{z}{\alpha} \quad 0 \leq z < \alpha \\ (\bar{z}^4 - 2\bar{z}^3 + \bar{z}) \dfrac{1 - 2\alpha}{\alpha} S, & \bar{z} = \dfrac{z - \alpha}{1 - 2\alpha} \quad \alpha \leq z \leq 0.5 \end{cases} \tag{25}$$

and

$$\eta_2 = \begin{cases} (2\bar{z}^4 - 3\bar{z}^3 + \bar{z}) \cdot S, & \bar{z} = \dfrac{z}{\alpha} \quad 0 \leq z < \alpha \\ (\bar{z}^4 - 2\bar{z}^3 + \bar{z}^2) \dfrac{3(1-2\alpha)^2}{\alpha^2} S, & \bar{z} = \dfrac{z - \alpha}{1 - 2\alpha}, \quad \alpha \leq z \leq 0.5 \end{cases}$$

(4.) Tangential Displacement ψ

The left end support is hinged and the other supports are on rollers, therefore, two linear functions are chosen. First, the rigid displacement of the beam is considered and if there is no displacement at the end pier, then this term can be ignored. The second displacement consist of a linear variable, as shown in Fig. 19. The displacement functions can be written as;

$$\left.\begin{aligned} \psi_1 &= S \\ \psi_2 &= S \cdot z \end{aligned}\right\} \tag{26}$$

(5) Radial displacement of the interior piers

The displacement ξ_p due to ξ is given by Eq. (21), however, the ξ_p due to rotation ϕ has not been developed and will be given herein as illustrated in Fig. 20.

Applying a unit rotation at the top of the column, the curve can be described as;

$$F_{p2} = h(y^3 - y^2)$$

The rotation angles at the top of the piers are ϕ_1, ϕ_2, ϕ_3; the corresponding displacements of the piers are therefore;

$$\begin{aligned} \xi_{p1} &= \phi_1(\alpha)\ h\ (y^3 - y^2) \\ \xi_{p2} &= \phi_2(\alpha)\ h\ (y^3 - y^2) \\ \xi_{p3} &= \phi_3(\alpha)\ h\ (y^3 - y^2) \end{aligned} \tag{27}$$

(6) Longitudinal displacement of the end pier

If there is a longitudinal displacement of the pier at the left support, Eq. (21) and ψ_1 can be combined to obtain ψ_p, or ;

$$\psi_p = S \cdot \frac{1}{2}y^2 \ (3-y) \tag{28}$$

Substituting the functions described above into [F] gives

$$[F] = \begin{array}{c} \begin{array}{cccccccccc} & \phi & & & \xi & & \eta & & \psi & \end{array} \\ \left[\begin{array}{cccccccccc} 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta_1 & \eta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_1 & \psi_2 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & 0 & 0 & 0 & 0 \\ \Sigma\xi_{p1} & \Sigma\xi_{p2} & \Sigma\xi_{p3} & \Sigma\xi_{p4} & \Sigma\xi_{p5} & \Sigma\xi_{p6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_p & 0 \end{array}\right] \end{array} \tag{29}$$

Substituting Eq. (29) into Eq. (15), gives;

$$[B] = \begin{bmatrix} 0 & 0 & 0 & \xi_1'' & \xi_2'' & \xi_3'' & 0 & 0 & \frac{1}{R}\psi_1' & \frac{1}{R}\psi_2' \\ -\frac{1}{R}\phi_1 & -\frac{1}{R}\phi_2 & -\frac{1}{R}\phi_3 & -\frac{1}{R}\phi_4 & -\frac{1}{R}\phi_5 & -\frac{1}{R}\phi_6 & \eta_1'' & \eta_2'' & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R}\xi_1 & -\frac{1}{R}\xi_2 & -\frac{1}{R}\xi_3 & 0 & 0 & \psi' & \psi_2' \\ \phi_1' & \phi_2' & \phi_3' & \phi_4' & \phi_5' & \phi_6' & \frac{1}{R}\eta_1' & \frac{1}{R}\eta_2' & 0 & 0 \\ \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' & \phi_5'' & \phi_6'' & \frac{1}{R}\eta_1'' & \frac{1}{R}\eta_2'' & 0 & 0 \\ \Sigma\xi_{p1}'' & \Sigma\xi_{p2}'' & \Sigma\xi_{p3}'' & \Sigma\xi_{p4}'' & \Sigma\xi_{p5}'' & \Sigma\xi_{p6}'' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_p'' & 0 \end{bmatrix} \qquad (30)$$

Substitute [B] into Eq. (14), the Generalized Stiffness Matrix can be obtained as;

$$[K] = [K_1] + [K_2] + [K_3] + [K_4] + [K_5] + [K_6] + [K_7] \qquad (31)$$

where;

$$[K_1] = EI_yS\ [\ ]$$

and [ ] is;

$$\begin{array}{c}
\begin{array}{cccccccccc}
\overbrace{\qquad\qquad\qquad\qquad}^{\phi} & & & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{\xi} & & & \overbrace{\qquad\qquad}^{\eta} & & \overbrace{\qquad\qquad}^{\psi} &
\end{array}\\
\left[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & \int_0^1 \xi_1''^2 dz & \int \xi_1''\xi_2'' dz & \int \xi_1''\xi_3'' dz & 0 & 0 & 0 & \frac{1}{R}\int \xi_1''\psi_2'\ dz\\
 & & & & \int \xi_2''^2 dz & \int \xi_2''\xi_3'' dz & 0 & 0 & 0 & \frac{1}{R}\int \xi_2''\psi_2'\ dz\\
 & & & & & \int \xi_3''^2 dz & 0 & 0 & 0 & \frac{1}{R}\int \xi_3''\psi_2'\ dz\\
 & & & & & & 0 & 0 & 0 & 0\\
 & \text{(Sym.)} & & & & & & 0 & 0 & 0\\
 & & & & & & & & 0 & 0\\
 & & & & & & & & & \frac{1}{R^2}\int \psi_2'^2 dz
\end{array}\right]
\end{array}$$

$$[K_2] = EI_sS\ [\ \ ]$$

where [ ] is;

$$
\begin{array}{cccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \\
\left[\begin{array}{cccccc}
\frac{1}{R^2}\int\phi_1^2dz & \frac{1}{R^2}\int\phi_1\phi_2dz & - & - & - & \frac{1}{R^2}\int\phi_1\phi_6dz \\
 & \frac{1}{R^2}\int\phi_2^2dz & & & & \frac{1}{R^2}\int\phi_2\phi_6dz \\
 & & \ddots & \frac{1}{R^2}\int\phi_i\phi_jdz & & \\
 & & & i=1\sim6 & & \\
 & & & j=1\sim6 & \ddots & \\
 & & & & & \frac{1}{R^2}\int\phi_6^2dz \\
 & \text{(Sym.)} & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & &
\end{array}\right. &&&&&
\begin{array}{cc}
 & \\
 & \\
\multicolumn{2}{c}{-\frac{1}{R}\int\phi_i\eta_jdz} \\
\multicolumn{2}{c}{i=1\sim6} \\
\multicolumn{2}{c}{J=1\sim2} \\
 & \\
\int\eta_1''^2dz & \int\eta_1''\eta_2''dz \\
 & \int\eta_2''dz \\
 & \\
 &
\end{array} &&
\left.\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
 & 0
\end{array}\right] &
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10
\end{array}
\end{array}
$$

$[K_3] = EAS\ [\ ]$

where [ ] is;

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & \\
\left[\begin{array}{c} \\ \end{array}\right. & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 2 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 3 \\
 & & & \frac{1}{R^2}\int_0^1 \xi_1^2 dz & \frac{1}{R^2}\int \xi_i \xi_j dz & & 0 & 0 & 0 & -\frac{1}{R}\int \xi_i \psi_2' dz & & 4 \\
 & & & & i = 1 \sim 3 & & & & & i = 1 \sim 3 & & \\
 & & & & j = 1 \sim 3 & & 0 & 0 & 0 & & & 5 \\
 & (Sym.) & & & & & 0 & 0 & 0 & & & 6 \\
 & & & & & & 0 & 0 & 0 & 0 & & 7 \\
 & & & & & & & 0 & 0 & 0 & & 8 \\
 & & & & & & & & 0 & 0 & & 9 \\
 & & & & & & & & & \int \psi_2'^2 dz & & 10
\end{array}
$$

$$[K_4] = GK_T S\,[\ \ ]$$

where [ ] is;

$$
\begin{array}{cc}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} & \\
\left[\begin{array}{cccccc|cc|cc}
 & & & & & & & & 0 & 0 \\
 & & & \int_0^1 \phi'_i \phi'_j dz & & & \frac{1}{R}\int \phi'_i \eta'_j dz & & 0 & 0 \\
 & & & i = 1 \sim 6,\ j = i \sim 6 & & & & & 0 & 0 \\
 & & & & & & i = 1 \sim 6,\ j = 1 \sim 2 & & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
\hline
 & \text{(Sym.)} & & & & & \frac{1}{R^2}\int \eta'_i \eta'_j dz & & 0 & 0 \\
 & & & & & & i = 1 \sim 2,\ j = i \sim 2 & & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}\right] &
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ - \end{array}
\end{array}
$$

$$[K_5] = EI_{\omega}S\ [\ \ ]$$

where [ ] is;

$$
\begin{array}{c c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} & \\
\left[\begin{array}{c|c|cc}
\begin{array}{c} \displaystyle\int_0^1 \phi_i''\phi_j''\,dz \\ i = 1 \sim 6 \\ j = i \sim 6 \end{array} & \begin{array}{c} \displaystyle\int \phi_i''\eta_j''\,dz \\ i = 1 \sim 6 \\ j = 1 \sim 2 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\
\hline
 & \begin{array}{c} \displaystyle\frac{1}{R^2}\int \eta_i''\eta_j''\,dz \\ i = 1 \sim 2 \\ j = i \sim 2 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\
\text{(Sym.)} & & 0 & 0 \\
 & & & 0
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\end{array}
$$

$$[K_6] = E_p I_{px}\ [\ \ ]$$

where [ ] is;

$$
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left[\begin{array}{cccccc|cccc}
\multicolumn{6}{c|}{\Sigma \int_0^h \xi''_{pi}\xi''_{pj}\, dY \quad i = 1 \sim 6,\ j = i \sim 6} & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & (\mathrm{Sym.}) & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
$$

$$[K_7] = E_p I_z \, [\ ]$$

where [ ] is;

$$\begin{array}{cc}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} & \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 \\
 & & (\text{Sym.}) & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & \int_0^{h_1} \psi_p''^2 \, dY & 0 \\
 & & & & & & & & & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix}
\end{array}$$

Substituting the functions described by Eq. (18) through Eq. (28) into the matrices just given and completing the integration, will yield the values for each element in the matrix. Obviously these formulas are very long and tedious, however, the integration can be performed by a computer program.

Substituting Eqs. (29) and (9) into Eq. (16), the Generalized Mass Matrix can be obtained as follows;

$$[M] = [M_1] + [M_2] + [M_3] + [M_4] \tag{32}$$

where,

$$
[M_1]=\rho AS\;
\begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}\\
\left[\begin{array}{ccc|ccc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
 & & & \int_0^1 \xi_i\xi_j\,dz & & & 0 & 0 & 0 & 0\\
 & & & i=1\sim 3 & & & 0 & 0 & 0 & 0\\
 & & & j=i\sim 3 & & & 0 & 0 & 0 & 0\\
\hline
 & \text{(Sym.)} & & & & & \int \eta_i\eta_j\,dz & & 0 & 0\\
 & & & & & & i=1\sim 2,\; j=i\sim 2 & & 0 & 0\\
\hline
 & & & & & & & & \int \psi_i\psi_j\,dz & \\
 & & & & & & & & i=1\sim 2,\; j=i\sim 2 &
\end{array}\right]
\end{array}
\begin{array}{c}
\\ 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10
\end{array}
$$

$$
[M_2]=\rho\,(I+Ae^2)S\;
\begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}\\
\left[\begin{array}{cccccc|cccc}
 & & & \int \phi_i\phi_j\,dz & & & 0 & 0 & 0 & 0\\
 & & & & & & 0 & 0 & 0 & 0\\
 & & & i=1\sim 6 & & & 0 & 0 & 0 & 0\\
 & & & j=i\sim 6 & & & 0 & 0 & 0 & 0\\
 & & & & & & 0 & 0 & 0 & 0\\
 & & & & & & 0 & 0 & 0 & 0\\
 & \text{(Sym.)} & & & & & 0 & 0 & 0 & 0\\
 & & & & & & & 0 & 0 & 0\\
 & & & & & & & & 0 & 0\\
 & & & & & & & & & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\\ 1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10
\end{array}
$$

$$
[M_3] = \rho A e S
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left[\begin{array}{ccc|ccc|cc|cc}
0 & 0 & 0 & \multicolumn{3}{c|}{\int \phi_i \xi_j dz} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & \multicolumn{3}{c|}{i = 1 \sim 3} & 0 & 0 & 0 & 0 \\
 & & 0 & \multicolumn{3}{c|}{j = 1 \sim 3} & 0 & 0 & 0 & 0 \\
\hline
 & & & 2\int \xi_1 \phi_4 & \int \xi_1 \phi_5 + \int \xi_2 \phi_4 & \int \xi_1 \phi_6 + \int \xi_3 \phi_4 & 0 & 0 & 0 & 0 \\
 & & & & 2\int \xi_2 \phi_5 & \int \xi_2 \phi_6 + \int \xi_3 \phi_5 & 0 & 0 & 0 & 0 \\
 & & & & & 2\int \xi_3 \phi_6 & 0 & 0 & 0 & 0 \\
 & (\text{Sym.}) & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
$$

$$
[M_4] = \rho_p A_p
\begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array} \\
\left[\begin{array}{cccccccccc}
 & & & \Sigma \int_0^h \xi_{pi}\xi_{pj}\,dy & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & \begin{matrix} i = 1 \sim 6 \\ j = i \sim 6 \end{matrix} & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & \text{(Sym.)} & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & \int_0^h \psi_p^2\,dY & 0 \\
 & & & & & & & & & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10
\end{array}
$$

4.5 Earthquake Response Solution

In order to solve Eq. (10), Eq. (11) is first substituted into Eq. (17) and ω^2 and {A} are obtained. Then, Eq. (11) is substituted into Eq. (10) giving the following;

$$\ddot{Y} + \omega^2 Y = - \frac{\int \{\delta\}^T [m] \{\ddot{q}\}\, ds}{T_{max}}$$

If damping is considered, then the dynamic equation is;

$$\ddot{Y} + 2\xi\omega\dot{Y} + \omega^2 Y = - \frac{\int \{\delta\}^T [m] \{\ddot{q}\}}{T_{max}} \tag{33}$$

where ξ is the damping ratio.

Normalizing the mode shape $\{\delta\}$ by

$$\{\bar{\delta}\} = \{\delta\} \Big/ \sqrt{T_{max}} \tag{34}$$

and

$$\bar{y} = \sqrt{T_{max}}\, Y, \qquad \text{and}$$

substituting $\{\bar{\delta}\}$ into Eq. (33), gives

$$\ddot{Y} + 2\xi\omega\dot{Y} + \omega^2 Y = - \int \{\bar{\delta}\}^T [m] \{\ddot{q}\} ds \tag{35}$$

where $\{\bar{\delta}\}$ and $\{\ddot{q}\}$ are described in curvilinear coordinate system. However, the earthquake acceleration record is presented using a Cartesian Coordinate System, therefore for consistency, the earthquake acceleration will be transformed from $\{\ddot{g}\}$ to $\{\ddot{q}\}$.

Because the vertical directions for the two coordinate systems are the same, transformation is not necessary, however, the acceleration in the plane of the curved beam during earthquake differs from the ground acceleration as shown in Fig. 21. As shown in Fig. 21, $\{\ddot{g}\}$ is the acceleration of the ground in three directions, and $\{\ddot{q}\}$ is the component of ground acceleration $\{\ddot{g}\}$ in the direction of structural displacements. Using Eq. (2-23), and the transformation for columns, $\{\ddot{q}\}$ can be obtained in terms of $\{\ddot{g}\}$ as follows;

$$\{\ddot{q}\} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \\ e\ \cos\theta & 0 & e\ \sin\theta \\ \cos\theta_1 & 0 & \sin\theta_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \ddot{g}_x \\ \ddot{g}_y \\ \ddot{g}_z \end{Bmatrix} \tag{36}$$

$$\{\ddot{g}\} = [T]\ \{\ddot{g}\}$$

Substituting Eq. (36) into Eq. (35) gives

$$\ddot{\overline{Y}}_1 + 2\xi\omega\dot{\overline{Y}}_1 + \omega^2\overline{Y}_1 = -\int\{\overline{\delta}\}^T[m]\ [T]\ ds\ \{\ddot{g}\} = -\ [\gamma]\ \{\ddot{g}\} \tag{37}$$

where;

$$[\gamma] = \int [\overline{\delta}]^T\ [m]\ [T]\ ds \tag{38}$$

Using the Response Spectrum Method, the record of the earthquake acceleration $\{\ddot{g}\}$ can be readily obtained, the displacement of the structure can then be obtained from the following equation;

$$\{\overline{\Delta}\}_{max} = \{\delta\}\ Y = \{\overline{\delta}\}\ \overline{Y} = \{\overline{\delta}\}\ [\gamma]\ \overline{Y} \tag{39}$$

When $\{\overline{\delta}\}$ is not sufficiently accurate, then $\omega^2\{\overline{\Delta}\}$ may be used as the static load, to calculate the final value of the response. However, the functions used in this work are the deflections due to the load, therefore Eq. (39) can be substituted into Eqs. (4) and (5), to calculate the internal actions which gives;

$$\{\sigma\}_{max} = [D]\ [P]\ \{\overline{\delta}\}\ [\gamma]\ \overline{Y} \tag{40}$$

4.6 Example Solutions

Using the method just described, the following structure was analyzed. As shown in Fig. 22, the bridge is 3-span continuous curved beam of equal span length (77.7'), with a hinge at the left-end support and rollers at the other supports. The geometry of the bridge, as shown in Fig. 22, has the following properties;

$R = 700\text{ft}$

$E = 432000\ \text{K/ft}^2$ $\qquad$ $G = 183000\ \text{k/ft}^2$

$\rho = 0.004658\ \text{Kslug/ft}^3$ $\qquad$ $e = 0$

$A = 61.18\ \text{ft}^2$ $\qquad$ $A_p = 35.93\ \text{ft}^2$

$I_X = 425.9\ \text{ft}^4$ $\qquad$ $I_{xp} = 94.87\ \text{ft}^4$

$I_y = 5676.03\ \text{ft}^4$ $\qquad$ $I_{zp} = 138.3\ \text{ft}^4$

$K_T = 917.3\ \text{ft}^4$ $\qquad$ $K_{Tp} = 178.7\ \text{ft}^4$

$I_\omega = 0$ $\qquad$ $I_{\omega p} = 0$

Using these properties and the R-R technique the first three frequencies have been obtained as shown in Table 1. Also shown in this table are the results obtained from the Finite Element Method (SAP 4) using the space frame element. In using the SAP 4 program 32 nodal points, were considered, thus requiring solution of 192 equations.

TABLE 1 Frequencies(CPS)

frequency order	R-R Method	SAP 4	principal direction of vibration
1	22.17	22.18	(radial)
2	31.41	32.99	(tangential)
3	41.40	41.43	(vertical)

These results indicate that the first three frequencies calculated by the R-R method are in good agreement with those obtained from the SAP 4 program.

The earthquake responses has also been computed by the R-R method, and compared with the results caculated by the SAP 4 program. The earthquake accelerations selected were $\ddot{g}_x = 1.0g^{ft/sec^2}$. The displacements were examined at the supports (1,2) and mid span (3) of the bridge, and the actions were [illegible] at the bottom of the pier and the mid span of the bridge. The Response Spectrum curve that was used was the PRC Specification for the Earthquake Design of Bridges.

Table 2

Radial Displacement in (ft) for (ω = 22.17 CPS)
($\ddot{g}_x$ = 1.0g ft/sec^2)

	1	2	3	rotation at point 3
R-R Method	0.208	2.33	2.65	0.889 P
SAP 4	0.212	2.34	2.69	0.886 R

Table 3

Bending Moment (kft) for (ω = 22.17 CPS)
($\ddot{g}_x$ = 1.0g ft/sec^2)

	M a-a	M b-b	M c-c
R-R Method	244×10^3	453×10^3	105×10^4
SAP 4	306×10^3	445×10^3	113×10^4

Table 4

Tangential Displacement (ft) for (ω = 31.41CPS)
($\ddot{g}_z$ = 1.0g ft/sec^2)

	1	3
R-R Method	1.95	2.16
SAP 4	1.67	2.13

TABLE 5

Actions for (ω = 31.41CPS) ($\ddot{g}_z$ = 1.0g ft/sec^2)

	a-a bending moment (K-ft)	c-c axial force (k)
R - R Method	192×10^4	540×10^2
SAP 4	167×10^4	725×10^2

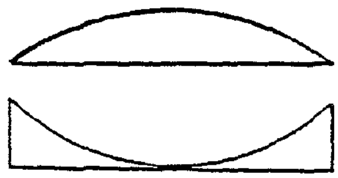
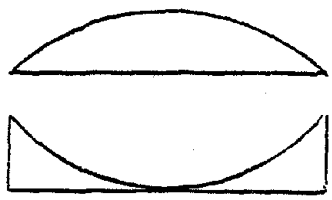
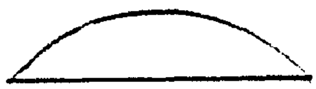
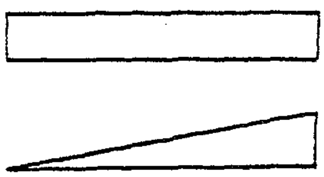
The results presented herein show that the radial displacement, for the first mode, is similar to the data obtained from SAP 4, with a maximum difference of 15%. The resulting stresses indicate that the bending moment at the interior pier controls the design and is 2% greater than that calculated by the SAP 4 program. The induced lateral bending moment at the end pier, results in a difference of 20%. The maximum tangential bending moment of the pier occurs when the acceleration $\ddot{g}_z$ is applied and occurs at the second frequency.

4.7 Displacement Functions-One, Two, Four Span Bridges

As given in Section 4.4, the displacement functions for a three span continuous bridge were described in detail. The beam displacement functions for a single, two and four span structure are given in the following Tables 6 through 9, where the functions for piers are the same as developed for the three span continuous bridge.

TABLE 6

SINGLE SPAN

ϕ		$\phi_1 = F_1(z)$ $\phi_2 = F_2(0.5,\ 0.5) - F_2(0.5,\ z)$
ξ		$\xi_1 = S \cdot \phi_1$ $\xi_2 = S \cdot \phi_2$
η		$\eta_1 = F_1(z)$
ψ		$\psi_1 = S$ $\psi_1 = S \cdot z$

F_1 and F_2 are functions as given in Eqs. (18) and (19).

TABLE 7

TWO SPAN

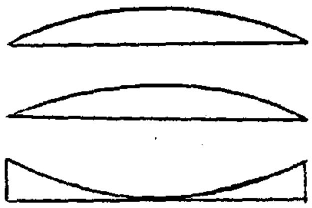
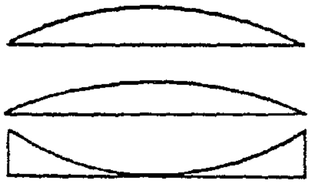
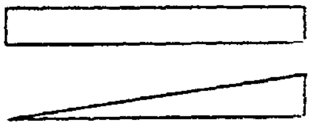
ϕ		$\phi_1 = F_1(z)$ $\phi_2 = F_2(0.5\ ,\ z)$ $\phi_3 = F_2(0.5,\ 0.5) - F_2(0.5\ ,\ z)$
ξ		$\xi_1 = S\,\phi_1$ $\xi_2 = S\,\phi_2$ $\xi_3 = S\,\phi_3$
η		$\eta_1 = (2\bar{z}^4 - 3\bar{z}^3 + \bar{z})\ .\ S$ $\bar{z} = 2z \qquad z \leq 0.5$
ψ		$\psi_1 = S$ $\psi_2 = S\ .\ z$

TABLE 8

THREE SPAN

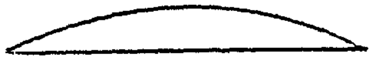
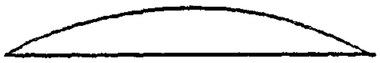
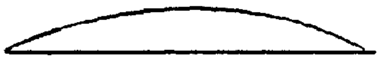
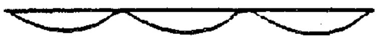
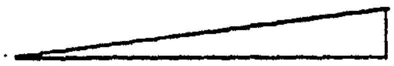
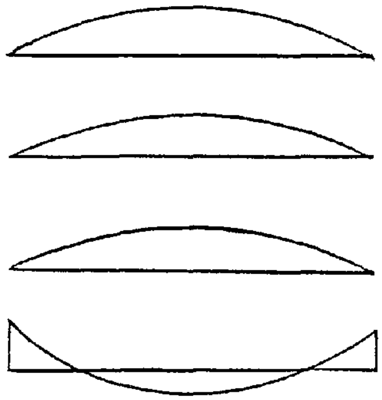
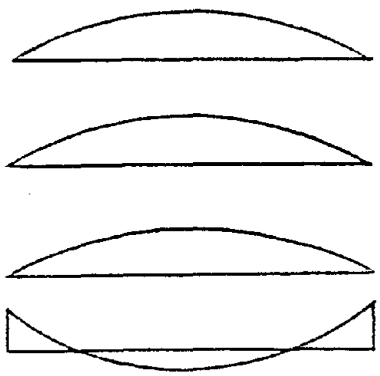
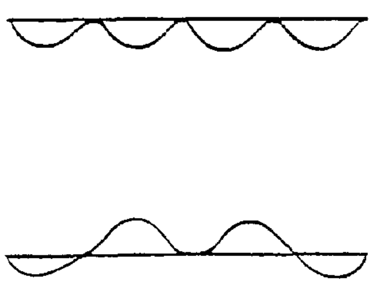
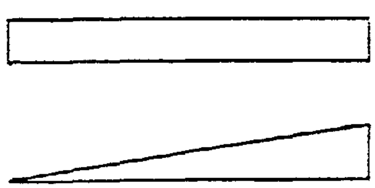
		$\phi_1 = F_1(z)$
ϕ		$\phi_2 = F_2(\alpha\ ,\ z)$
		$\phi_3 = F_2(\alpha,\alpha) - F_2\ (\alpha,z)$
		$\xi_1 = S\ \phi_1$
ξ		$\xi_2 = S\,\phi_2$
		$\xi_3 = S\,\phi_3$
η		η_1 and η_2 are shown in eq. (25)
		
ψ		$\psi_1 = S$
		$\psi_2 = S\ .\ z$

TABLE 9

FOUR SPAN

ϕ		$\phi_1 = F_1(z)$ $\phi_2 = F_2(0.5,\ z)$ $\phi_3 = F_2(\alpha\ ,\ z)$ $\phi_4 = F_2(\alpha,\ \alpha) - F_2(\alpha\ ,\ z)$
ξ		$\xi_1 = S\phi_1$ $\xi_2 = S\phi_2$ $\xi_3 = S\phi_3$ $\xi_4 = S\phi_4$
η		$\eta_1 = \begin{cases} -(\tilde{z}^4 - 2\tilde{z}^3 + \tilde{z}), & \tilde{z} = \frac{z}{\alpha},\ 0 \le z \le \alpha \\ \frac{0.5 - \alpha}{\alpha}(2\tilde{z}^4 - 3\tilde{z}^3 + \tilde{z}), & \tilde{z} = \frac{z - \alpha}{0.5 - \alpha},\ \alpha < z \le 0.5 \end{cases}$ $\eta_2 = \begin{cases} 2\tilde{z}^4 - 3\tilde{z}^3 + \tilde{z}, & \tilde{z} = \frac{z}{\alpha},\ 0 \le z \le \alpha \\ \frac{3(0.5 - \alpha)^2}{\alpha^2}(\tilde{z}^4 - 2\tilde{z}^3 + \tilde{z}^2), & \tilde{z} = \frac{z-\alpha}{0.5-\alpha},\ \alpha < z \le 0.5 \end{cases}$
ψ		$\psi_1 = S$ $\psi_2 = S \cdot z$

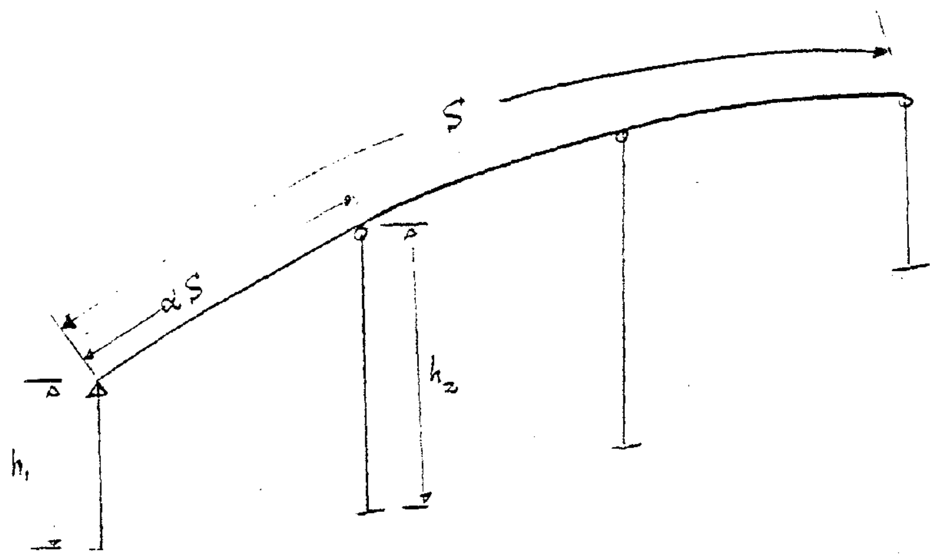


Fig. 13

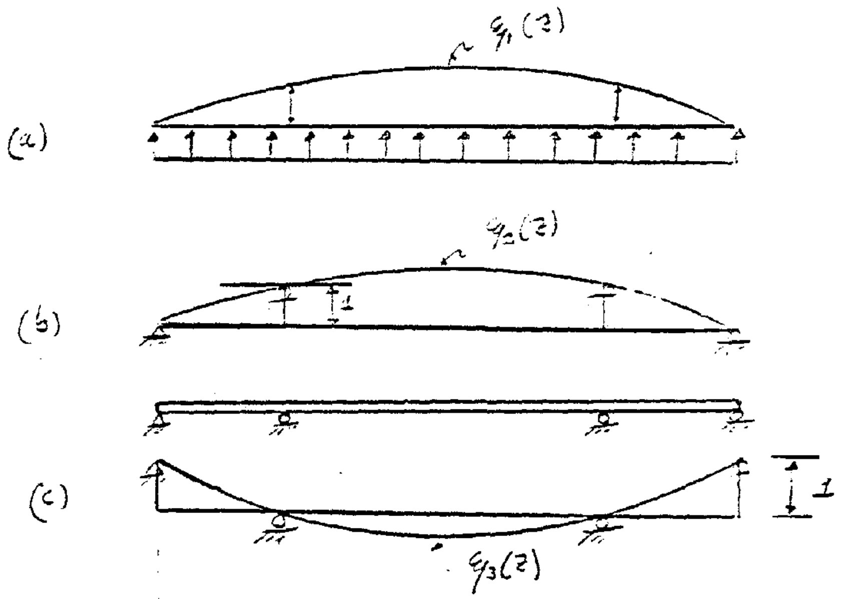


Fig. 14

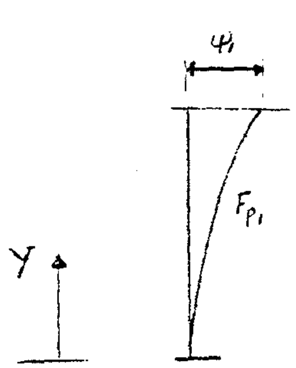


Fig. 15

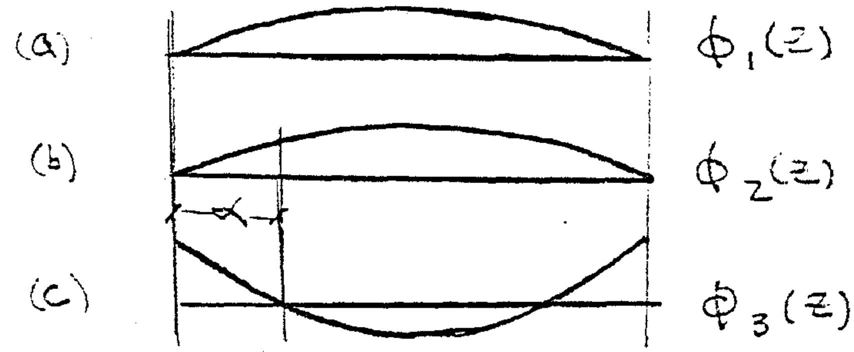


Fig. 16

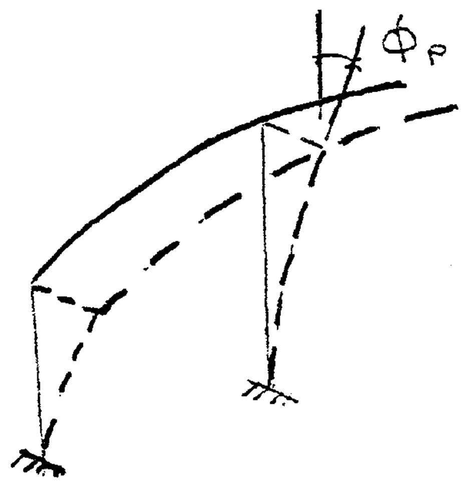


Fig. 17

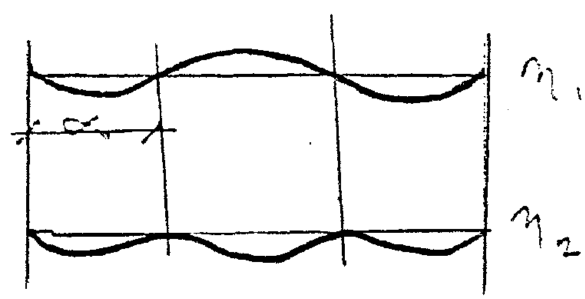


Fig. 18

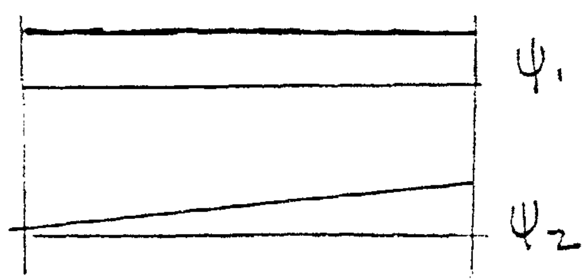


Fig. 19

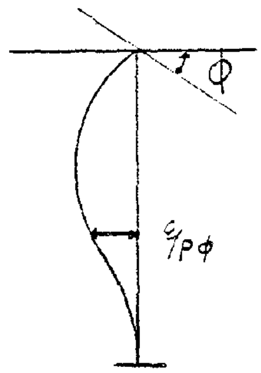


Fig. 20

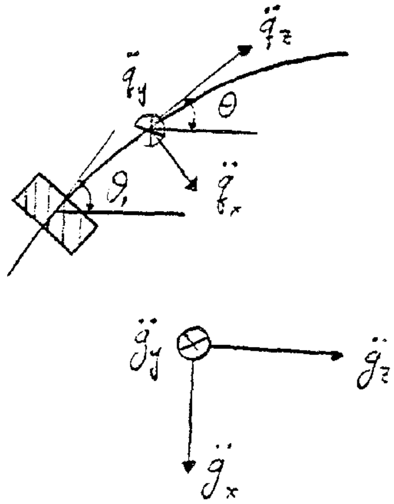


Fig. 21

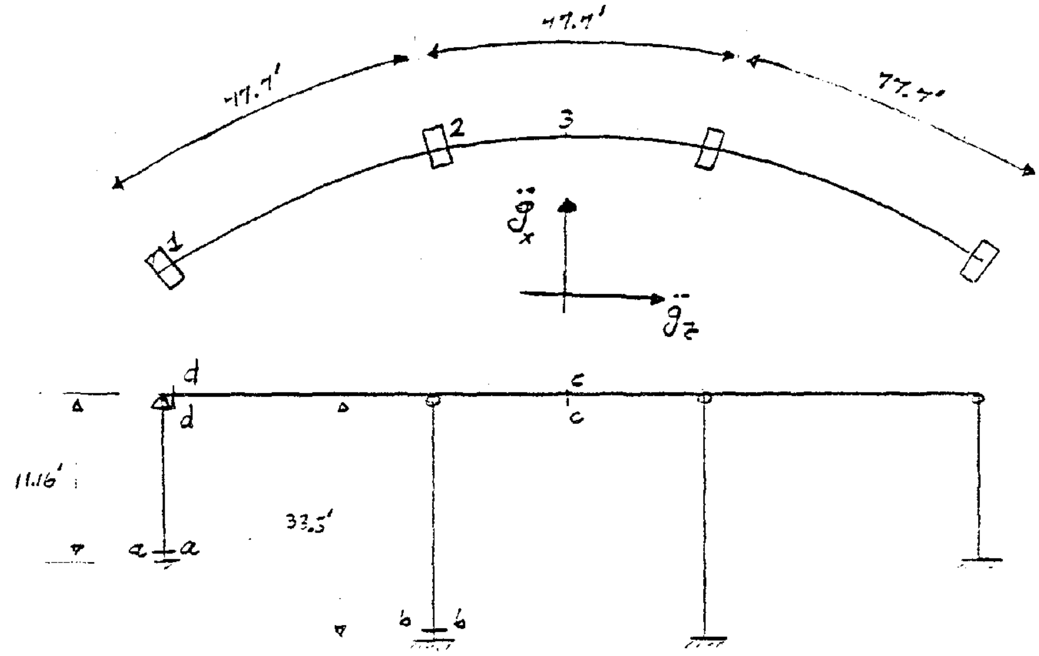


Fig. 22

CHAPTER 5

THEORY-RAYLEIGH METHOD INCLUDING MULTI COLUMN BENT

5.1 General

As described in Chapter 3, a curved girder, when interacting with a rigid pier bent (Fig. 23), the transverse and longitudinal torsional vibrations become uncoupled when the rotation of top of the piers are neglected. Consideration of this type of structural system can therefore simplify the previously developed equations given in Chapter 4.

As detailed in Chapter 4, the vibration functions developed are for a single column bent bridge structure, and thus transverse and rotational displacement develope at the top of piers. Therefore, the transverse and rotational displacements are coupled between the piers and girders. In this chapter however, the rotation angle of pier, and eccentricity between centriod and shear center axis are neglected and therefore and the vibration of the girder can be divided into two parts, (1) transverse and (2) longitudinal. In general in curved multicolumn bent structures the longitudinal vibration is coupled with the torsional vibration, however by neglecting the longitudinal deformation of the piers, the longitudinal vibration is not influenced by pier interaction and therefore the bridge can be analyzed as a continuous beam on rigid supports.

5.2 Basic Equations

As shown in Fig. 24, the curvilinear coordinate system will be used;

where;

X = radial direction

Y = vertical direction

Z = tangential direction

The displacement parameters, where Y is time function, are chosen as;

$$\{\Delta\ (s,t)\} = \{\delta(s)\}\ Y\ (t) \tag{1}$$

Using the Ritz method, a linear combination of $[F_i]$, may be chosen to represent the displacements;

$$\{\delta\} = \begin{Bmatrix} \xi(Z) \\ \psi(Z) \\ \xi_p(i,Y) \\ \psi_p(i,Y) \end{Bmatrix} = [F_1, F_2 - - -] \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_i \end{Bmatrix} = [F]\ \{A\} \tag{2}$$

where: $[F]$ matrix of chosen functions

$\{A\}$ unknown parameters.

The parameters ξ , ψ are displacements functions in the X and Z directions for an element dZ , the parameters $\xi_p(i,Y)$ and $\psi_p(i,Y)$ are displacements functions in X and Z directions for an element dY .

The internal forces can be written as;

$$\{\sigma\} = \begin{Bmatrix} M_y \\ N_z \\ M_{pz} \\ M_{px} \end{Bmatrix} \tag{3}$$

where;

M_y = bending moment about the Y - axis

N_z = axial force

M_{px}, M_{pz} bending moments of piers on X and Z directions, respectively.

The corresponding generalized displacements are:

$$\{\varepsilon\} = \begin{Bmatrix} \xi'' + \frac{\psi'}{R} \\ \psi' - \frac{\xi}{R} \\ \xi''_p \\ \psi''_p \end{Bmatrix} = \begin{bmatrix} \frac{\partial^2}{\partial z^2} & \frac{1}{R}\frac{\partial}{\partial z} & 0 & 0 \\ -\frac{1}{R} & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial y^2} & 0 \\ 0 & 0 & 0 & \frac{\partial^2}{\partial y^2} \end{bmatrix} \{\delta\}$$

$$\{\varepsilon\} = [P]\{\delta\} = [P]\ [F]\ \{A\} = [B]\ \{A\} \quad (4)$$

where;

$$[B] = [P]\ [F] \quad (5)$$

From Hook's Law: $[\sigma] = [D]\{\varepsilon\} = [D]\ [B]\ \{A\}$ (6)

where;

$$[D] = \begin{bmatrix} EI_y & 0 & 0 & 0 \\ & EA & 0 & 0 \\ & & E_pI_{pz} & 0 \\ \text{Sym} & & & E_pI_{px} \end{bmatrix}$$

The natural frequency can now be obtained by Rayleigh quotient;

$$\omega^2 = \frac{\int \{\varepsilon\}\ \{\sigma\}\ ds}{\int \{\delta\}^T\ [m]\ \{\delta\}\ ds} \quad (7)$$

where;

$$[m] = \text{mass matrix} = \begin{bmatrix} \rho A_m & 0 & 0 & 0 \\ & \rho A_m & 0 & 0 \\ & & \rho_p A_{pm} & 0 \\ \text{Sym.} & & & \rho_p A_{pm} \end{bmatrix}$$

Substituting Eqs. (2), (4) and (6) into Eq. (7) yields;

$$\omega^2 = \frac{\{A\}^T\ [K]\ \{A\}}{\{A\}^T\ [M]\ \{A\}} \tag{8}$$

Because $\{A\}$ should be selected such that ω^2 is minimum, gives;

i.e. $\frac{\partial \omega^2}{\partial \{A\}} = 0$, therefore;

$$[K]\ \{A\} - \omega^2\ [M]\ \{A\} = 0 \tag{9}$$

where;

$$[K] = \int [B]^T\ [D]\ [B]\ ds \tag{10}$$

$$[M] = \int [F]^T\ [M]\ [F]\ ds \tag{11}$$

Solving Eq. (9) gives the natural frequency and characteristic vector $\{A\}$, and substituting $\{A\}$ into Eq. (2), yields the mode shape.

5.3 Displacement Functions.

The deflection curve, as obtained considering the dead load response, will be used. The following is an example describing the functions selected for a three continuous span bridge.

(1) Radial displacement; ξ

ξ is composed of 3 functions;

$$\xi_1\ (z) = S.\ F_1(z) = S\ .\ (z^4 - 2z^3 + z) \quad \text{where } z = \frac{Z}{s} \tag{12}$$

$$\xi_2\ (z) = S\ .\ F_2\ (\alpha,\ z) = S\ .\begin{cases} z^3 + 3\alpha(\alpha-1)z & 0 \leq z < \alpha \\ 3\alpha z^2 - 3\alpha z + \alpha^3 & \alpha \leq z \leq 0.5 \end{cases} \tag{13}$$

$$\xi_3\ (z) = S\ .\ F_3\ (\alpha,\ z) = S\ [F_2\ (\alpha,\alpha) - F_2\ (\alpha,\ z)] \tag{14}$$

(2) Tangential displacement

$$\psi_1 = S \tag{15}$$

$$\psi_2 = S\ .\ z \tag{16}$$

(3) Radial displacement of piers; ξ_p

The radial displacement function is taken as;

$$\xi_p(y) = y^2\ (3 - 2y)\ \xi\ (\alpha) \qquad y = \frac{Y}{h}$$

where the top displacement of piers is the same as that of beam, as shown in Fig. 25.

The corresponding displacement functions of columns are

$$\left.\begin{aligned} \xi_{p1} &= y^2(3 - y)\ \xi_1(\alpha) \\ \xi_{p2} &= y^2(3 - y)\ \xi_2(\alpha) \\ \xi_{p3} &= y^2(3 - y)\ \xi_3(\alpha) \end{aligned}\right\} \tag{17}$$

(4) Tangential displacement of pier; ψ_p

$$\psi_p = S\ \frac{1}{2}\ y^2(3 - y) \tag{18}$$

The [F] matrix can now be written as;

$$[F] = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 \\ 0 & 0 & 0 & \psi_1 & \psi_2 \\ \Sigma\xi_{p1} & \Sigma\xi_{p2} & \Sigma\xi_{p3} & 0 & 0 \\ 0 & 0 & 0 & \psi_p & 0 \end{bmatrix} \qquad (19)$$

5.4 Stiffness and Mass Matrices

Substituting Eq. (19) into (5) gives the [B] matrix as;

$$[B] = \begin{bmatrix} \xi''_1 & \xi''_2 & \xi''_3 & \frac{1}{R}\psi'_1 & \frac{1}{R}\psi'_2 \\ -\frac{1}{R}\xi_1 & -\frac{1}{R}\xi_2 & \frac{1}{R}\xi_3 & \psi'_1 & \psi'_2 \\ \Sigma\xi''_{p1} & \Sigma\xi''_{p2} & \Sigma\xi''_{p3} & 0 & 0 \\ 0 & 0 & 0 & \psi''_p & 0 \end{bmatrix} \qquad (20)$$

Substituting Eq. (20) into Eq. (10), gives generalized stiffness matrix;

$$[K] = [K_1] + [K_2] + [K_3] \qquad (21)$$

where: $[K_1]$, $[K_2]$ and $[K_3]$ are defined as follows:

$$[K_1] = EI_yS \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{22} & a_{23} & a_{24} & a_{25} \\ & & a_{33} & a_{34} & a_{35} \\ & \text{(sym.)} & & a_{44} & a_{45} \\ & & & & a_{55} \end{bmatrix}$$

where;

$$a_{11} = 2.4/S^2$$

$$a_{12} = \frac{1}{S^2}\,(0.2\alpha^6 - 0.6\alpha^5 + \alpha^3 - 0.6\,\alpha)$$

$$a_{13} = -a_{12}$$

$$a_{14} = 0$$

$$a_{15} = -\frac{1}{SR}$$

$$a_{22} = \frac{1}{S^2}\,(-24\alpha^3 + 18\alpha^2)$$

$$a_{23} = \frac{1}{S^2}\,(-4.8\alpha^5 + 6\alpha^4 - 1.5\alpha^2)$$

$$a_{24} = 0$$

$$a_{25} = \frac{1}{SR}\,(-3\alpha^2 + 3\alpha)$$

$$a_{33} = a_{22}$$

$$a_{34} = 0$$

$$a_{35} = \frac{1}{SR}\,(3\alpha^2 - 3\alpha)$$

$$a_{44} = 0$$

$$a_{45} = 0$$

$$a_{55} = \frac{1}{R^2}$$

$$[K_2] = EAS \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ & b_{22} & b_{23} & b_{24} & b_{25} \\ & & b_{33} & b_{34} & b_{35} \\ \text{(sym.)} & & & b_{44} & b_{45} \\ & & & & b_{55} \end{bmatrix}$$

where;

$$b_{11} = \left(\frac{S}{R}\right)^2 \times 0.02795$$

$$b_{12}' = \left(\frac{S}{R}\right)^2 (-\alpha^4 + 1.6\alpha^3 - 0.31\alpha)$$

$$b_{13} = \left(\frac{S}{R}\right)^2 (\alpha^4 - 1.2\alpha^3 - 0.3\alpha^2 + 0.31\alpha)$$

$$b_{14} = 0$$

$$b_{15} = -\frac{S}{R} \times 0.1$$

$$b_{22} = \left(\frac{S}{R}\right)^2 (-0.5\alpha^4 + 0.15\alpha^2)$$

$$b_{23} = \left(\frac{S}{R}\right)^2 (-0.5\alpha^4 + 0.75\alpha^3 - 0.1633\alpha^2)$$

$$b_{24} = 0$$

$$b_{25} = -\frac{S}{R} (-0.25\alpha^4 + 0.5\alpha^3 - 0.25\alpha)$$

$$b_{33} = \left(\frac{S}{R}\right)^2 (5.75\alpha^4 - 1.25\alpha^3 + 0.15\alpha^2)$$

$$b_{34} = 0$$

$$b_{35} = -\frac{S}{R} (0.25\alpha^4 + 1.5\alpha^3 - 1.5\alpha^2 + 0.25\alpha)$$

$$b_{44} = 0$$

$$b_{45} = 0$$

$$b_{55} = 1$$

$$[K_3] = E_p \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ & c_{22} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 \\ & \text{(sym.)} & & c_{44} & 0 \\ & & & & 0 \end{bmatrix}$$

where;

$$c_{11} = I_{px}\xi_1^2(\alpha) \cdot \frac{12}{h_2^3}$$

$$c_{12} = I_{px}\xi_1(\alpha) \cdot \xi_2(\alpha) \frac{12}{h_2^3}$$

$$c_{22} = I_{px}\xi_2^2(\alpha) \frac{12}{h_2^3}$$

$$c_{33} = I_{px}\xi_3^2(0) \frac{12}{h_1^3}$$

$$c_{44} = I_{pz} S^2 \frac{3}{h_1^3}$$

Substituting Eq. (19) into Eq. (11), the generalized mass matrix is

$$[M] = [M_1] + [M_2] \tag{22}$$

where;

$$[M_1] = \rho A_m \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 \\ & & D_{33} & 0 & 0 \\ & & & D_{44} & D_{45} \\ & \text{symmetric} & & & D_{55} \end{bmatrix}$$

and;

$$D_{11} = 0.02795 \quad S^2$$

$$D_{12} = S^2(-\alpha^4 + 1.6\alpha^3 - 0.31\alpha)$$

$$D_{13} = S^2(\alpha^4 - 1.2\alpha^3 - 0.3\alpha^2 + 0.31\alpha)$$

$$D_{22} = S^2(-0.5\alpha^4 + 0.15\alpha^2)$$

$$D_{23} = S^2(-0.5\alpha^4 + 0.75\alpha^3 - 0.1633\alpha^2)$$

$$D_{44} = S^2$$

$$D_{45} = \frac{S^2}{2}$$

$$D_{55} = \frac{S^2}{3}$$

$$[M_2] = \rho'_p A_{pm} \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ & E_{22} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 \\ & \text{symmetric} & & E_{44} & 0 \\ & & & & 0 \end{bmatrix}$$

and;

$$E_{11} = 0.371 \; h_2 \; \xi_1^2(\alpha)$$

$$E_{12} = 0.371 \; h_2 \; \xi_1(\alpha) \; \xi_2(\alpha)$$

$$E_{22} = 0.371 \; h_2 \; \xi_2^2 \; (\alpha)$$

$$E_{33} = 0.371 \; h_1 \; \xi_3^2 \; (\alpha)$$

$$E_{44} = 0.942 \; h_1 S^2$$

5.5 Earthquake Response

As given in the previous cha½ters, the ground accelerations must be converted into a curvilinear coordinate system. As shown in Fig. 26, the following relationship can be obtained;

$$[\ddot{q}] = \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_z \\ \ddot{q}_{px} \\ \ddot{q}_{pz} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \\ \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \ddot{g}_x \\ \ddot{g}_z \end{Bmatrix} = [T] \{ \ddot{g} \} \quad (22)$$

where; θ = tangential angle for the beam element dz

θ_1 = tangential angle of pier

Therefore the mode parameters are;

$$\{\gamma\} = \{\bar{\delta}\}^T [m] [T] ds \quad (23)$$

where;

$$\{\bar{\delta}\} = \{\delta\} / \left(\int \{\delta\}^T [m] \{\delta\} ds \right)^{\frac{1}{2}} \quad (24)$$

= normalized mode shape

The final earthquake response for displacements at the structure is given by:

$$\{\Delta\}_{max} = \{\bar{\delta}\} Y = \{\bar{\delta}\} [\gamma] \bar{Y} \quad (25)$$

where Y can be obtained from spectrum diagram, and the actions can be obtained from Eq. (3); or

$$\{\sigma\}_{max} = [D] [P] \{\bar{\delta}\} [\gamma] \bar{Y} \quad (26)$$

These results can be computerized to obtain the dynamic response of pier bent supported curved bridges.

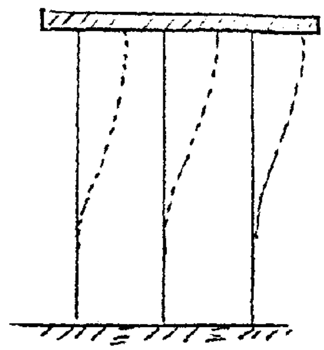


Fig. 23

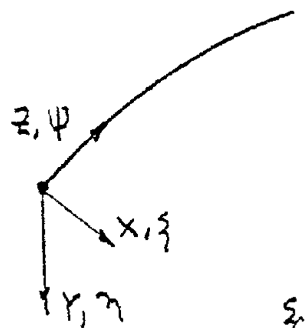


Fig. 24

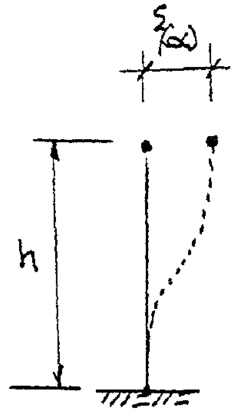


Fig. 25

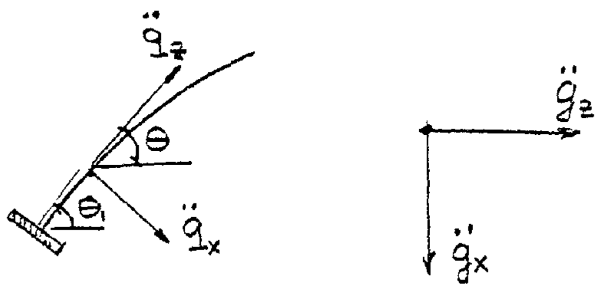


Fig. 26

CHAPTER 6

Results and Conclusions

The theoretical response of single and continuous curved box girder bridges, subjected to induced earthquake accelerations has been predicted by several analytical techniques. These methods consider various girder-pier bent interaction, which may be simplified depending on the type of bent, i.e. single or multi column bent.

Several problems have been solved by these methods indicating good correlation with previously developed techniques.

Utilization of the methods described herein will permit development of simplified design techniques, to be presented in a later report.

Chapter 7

References

1. Jennings, P. C., "Earthquake Engineering and Hazards Reduction in China", National Academy of Sciences, Washington, D. C. 1980.

2. Miller, R. K., Felszegly, S. F., "Engineering Features of the Santa Barbara Earthquake of Aug. 1978", EERI, Berkeley, Calif., Dec. 1978.

3. Yaner, P. I., "Miyagi-Ken-Oki, Japan Earthquake", EERI, Berkeley, Calif., Dec. 1978.

4. "Eleventh Edition-AASHTO Bridge Design Specification", Washington, D. C.

5. Imbsen, R. A., Nutt, R. V., Penzien, J., "Evaluation of Analytical Procedures Used in Bridge Seismic Design Practice", Proceedings: Earthquake Resistance of Highway Bridges", ATC, Jan. 29/31, 1979.

6. Sharpe, R. L., Mayes, R. L., "Development of Highway Bridge Seismic Design Criteria for the U. S."; Proceedings: "Earthquake Resistance of Highway Bridges", ATC, Jan. 29/31, 1979.

7. "Seismic Design Guidelines for Highway Bridges", Proceedings: "Earthquake Resistance of Highway Bridges", ATC, Jan. 29/31, 1979.

8. Kawashima, K.; Penzien, J: "An Investigation of the Effectiveness of Existing Bridge Design Methodology in Providing Adequate Structural Resistance to Seismic Disturbances", Phase V, FHWA RD-77-57, Washington, D.C., July 1976.

9. Williams, D.; Godden, W.: "Effectiveness of Existing Bridge Design Methodology in Resisting Earthquakes", Phase IV, FHWA RD-77-91, Washington, D.C., June 1976.

10. Bathe, K.; Wilson, E.; Peterson, F.: "SAPIV, A Structural Analysis Program for Static and Dynamic Response of Linear Systems", C.E. Dept., Univ. of Calif., Berkeley, Calif., June 1973.

11. Bathe, K.J.; Wilson, E.L.; Iding, R.H.: "NONSAP-A Structural Analysis Program for Static and Dynamic Response of Nonlinear Systems", Rept. No. VC SESM 74-3, C.E. Dept., Univ. of Calif., Berkeley, Calif., 1974.

12. Heins, C.P.; Sahin, M.A.: "Natural Frequency of Curved Box Girder Bridges", ASCE Structural Journal vol. 105, no. ST12, Dec., 1979.

13. Chapman, H.E. : "An Overview of the State of Practice in Earthquake Resistant Design of Bridges in New Zealand", Proceedings: "Earthquake Resistance of Highway Bridges", ATC, Jan. 29/31, 1979.

14. Robinson, R.R.; Longinow, A.; Chu, K.H.: "Seismic Retrofit Measures for Highway Bridges", Vol. 1, FHWA TS-79-216, Washington, D.C., April 1979.

15. Heins, C.P.; Humphreys, R.S.: "Bending and Torsion Interaction of Box Girders", ASCE Structural Journal vol. 105, no. ST5, May 1979.

16. Yoo, C.H.; Heins, C.P.: "Plastic Collapse of Horizontally Curved Bridge Girders", ASCE Structural Journal vol. 98, No. ST4, April 1972.

17. Hall, D.H.; Heins, C.P. et al: "Curved Steel Box Girder Bridges, State of the Art", ASCE Structural Journal vol. 104, No. ST11, Nov. 1978.

18. Heins, C.P.: "Box Girder Design, State of the Art", AISC Engr.Journal, vol. 15, no. 4, 1978.

19. Yoshimura, J.; Maeda, H.: "Vibration of Continuous Curved Box Girder Bridge on High Piers", Hokkaido University, Civil Engineering Dept., Sapporo, Japan, 1974.

20. Iwasaki, T.: "Earthquake Resistant Design of Bridges in Japan", vol. 29, PWRI, Ministry of Construction, Tokyo, Japan, May 1973.

21. Robinson, R.R. et al: "Structural Analysis and Retrofitting of Existing Highway Bridges Subjected to Strong Motion Seismic Loading", FHWA, May 1975.

22. Ohashi, M.; Kuribayashi, E.; Iwasaki, T.; Kawashima, K.: "An Overview of the State of Practices in Earthquake Resistant Design of Highway Bridges in Japan", Workshop on Research Needs of Seismic Problems Related to Bridges, San Diego, Calif., Jan. 1979.

23. Yamadera, N.; Oyema, Y.: "Special Considerations and Requirements for Seismic Design of Bridges in Japan", Metropolitan Expressway Public Corp., Tokyo, Japan.

24. "Earthquake Engineering and Hazards Recuction in China", CSCPRC Rept. No. 8 National Academy of Sciences, 1980.

25. "Specification for Earthquake Design of Highway Projects", People's Republic of China, Bejing, 1978.

26. Heins, C.P.; Firmage, D.A.: "Design of Modern Steel Highway Bridges", J. Wiley Interscience, New York, 1979.

27. Heins, C. P.: "Bending and Torsional Design in Structural Members", Lexington Books, D.C. Heath Co., Mass., 1975.

28. Heins, C.P.: "Applied Plate Theory for the Engineer", Lexington Books, D.C. Heath Co., Mass., 1976.

29. Heins, C.P.; Kuo, J.T.C.: "Ultimate Load Distribution Factor", ASCE Structural Journal, vol. 101, no. ST5, July 1975.

30. Heins, C.P.; Fan, H.M.: "Effective Composite Beam Width at Ultimate ASCE Structural Journal, vol. 102, no. ST11, Nov. 1976.

31. Heins, C.P.; Kurzweil, A.P.: "Load Factor Design of Continuous Span Bridges", ASCE Structural Journal, vol. 102, no. ST6, June 1976.

32. "Earthquake Resistant Design for C.E. Structures, Earth Structures and Fnds. in Japan", JSCE, 1977.

33. Reconnaissance Report, "Miyagi-ken-oki, Japan Earthquake - June 12, 1978", BERI, Berkeley, Calif., 1978.

34. Newmark, N.M.; Rosenblueth, E.: "Fundamentals of Earthquake Engineering", Prentice Hall, New Jersey, 1971.

35. Dowrick, D.J.: "Earthquake Resistant Design", J.Wiley, New York, 1977.

36. Clough, R.W.; Penzien, J.: "Dynamics of Structures", McGray Hill Book Co., New York, 1975.

37. Wiegel, R.L.: "Earthquake Engineering", Prentice Hall, New Jersey, 1970.

38. Vlasov, V. Z., "Thin-Walled Elastic Beams", NSF, Washington, D. C., 1961.

39. Timoshenko, S., "Vibration Problems in Engineering", D. Van Nostrand Co., New Jersey, 1956.

40. Zienkiewicz, O. C. "The Finite Element Method in Structural and Continuum Mechanics", McGraw-Hill Co., London, 1970.

41. Desai, C. S., Abel, J. F., "Introduction to the Finite Element Method",

42. Bath, K. J., Wilson, E. L. "Numerical Methods in Finite Element Analysis", Prentice-Hall Inc., New Jersey, 1976.

43. Mierovitch, L., "Analytical Methods in Vibration", the Macmillan Co., New York, 1967.

44. Popov, E. P., "Introduction to Mechanics of Solids", Prentice-Hall, Inc., New Jersey, 1968.

45. Timoshenko, S., "Strength of Materials", D. Van Nostrand Co., New Jersey, 1955.

46. Timoshenko, S. P., Goodier, J. N., "Theory of Elasticity", McGraw-Hill Co., New York, 1970.