MAXIMUM RESPONSE OF NONPROPORTIONALLY AND PROPORTIONALLY DAMPED STRUCTURAL SYSTEMS UNDER MULTICOMPONENT EARTHQUAKES

by

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16. Abstracts

Earthquake motions as felt by structures along three orthogonal directions are, in general, statistically correlated. Herein the effect of this correlation on structural response is studied. Assuming the existence of principal axes of excitation along which the components of motion are uncorrelated, the mean square and design response are obtained in terms of spectral characteristics of the uncorrelated components. These responses depend upon the relative orientation of the structure with respect to the principal excitation components. For each response quantity, there exists a set of three orthogonal direction along which if principal excitations are applied, they will cause a maximum response. Identification of these directions is necessary to obtain the worst case response for design purposes. A methodology to obtain such a maximum response of a quantity of design interest is outlined.

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NOTATION

The following symbols are used in this report:

$[A]$ = real symmetric matrix of order 2n, defined by Eq. 16

A', B', C', D' = elements of vector $\{A'\}$, defined by Eq. I.6

A'', B'', C'', D'' = elements of vector $\{A''\}$, defined by Eq. I.21

A''', B''', C''', D''' = elements of vector $\{A'''\}$, defined by Eq. IV.26

$\overline{A}_j$ = constant defined by Eq. 14

A_{mj}, a_{mj} = elements of vector $\{A_j\}$ and $\{a_j\}$, defined in Eq. 18

$a_{\ell j}, b_{\ell j}$ = the real and imaginary parts of $q_{\ell j}$

$[B]$ = real symmetric matrix of order 2n, defined by Eq. 16

$[C]$ = damping matrix

C_s, C_j = peaks factor for a response quantity S and psuedo acceleration response of an oscillator with frequency ω_j and damping β_j

C_0, C_1, C_2, C_3 = elements of vector $\{w'''\}$, defined by Eq. IV.22

$[D]$ = matrix of direction cosine of principle excitation components measured with respect to structural axes

$\{d\}$ = vector of direction cosines of an excitation with respect to structural axes

d_{mm} = the direction cosine of the m^{th} unprimed axis with respect to n^{th} primed axis, with m,n = 1,2,3

$\{E\}$ = the vector of principle excitation components $\ddot{X}_1, \ddot{X}_2$ and $\ddot{X}_3$ specified along the principle axes of the ground motion

$\{E'\}$ = the vector of correlated ground motion components $\ddot{X}_1'$, $\ddot{X}_2'$ and $\ddot{X}_3'$ along structural axes

$E_g'(\tau)$ = excitation defined in Eq. III.3

$Ex[\bullet]$ = the expected value of $[\bullet]$

e = eccentricity parameter, distance between mass and stiffness centers of a floor slab

$F_\ell(\omega_j)$ = square of the ratio of the relative velocity to psuedo response spectrum values at frequency ω_j and damping β_j

G_{mj} = an element of the vector $\{G_j\}$ defined by Eq. 18

g_j = modal response value of the response quantity S(t)

$H_j(\omega)$ = frequency response function of Eq. 5

$h_j(t)$ = impulse response function

$[K]$ = stiffness matrix

$L(\lambda)$ = auxiliary function, defined by Eq. 31

$[M]$ = mass matrix

m_j = j^{th} modal mass obtained as $\{\phi_j\}^T[M]\{\phi_j\}$

N = number of degrees of freedom

p_j = j^{th} complex eigenvalue

$\{q_j\}$ = modal response vector defined in Eq. 14, a complex quantity

$[\bar{R}]$ = response matrix defined by Eq. 26

$\bar{R}_{mn}$ = element of response matrix defined by Eq. 27

$R_{\ell a}(\omega_j)$ = psuedo acceleration response spectrum value at frequency ω_j and damping β_j for the ℓth component of excitation

$R_{\ell d}(\omega_j)$ = relative displacement response spectrum value at frequency ω_j and damping β_j for the ℓth component of excitation

$R_{\ell d}(\omega_j)$ = relative velocity response spectrum value at frequency ω_j and damping β_j for the ℓth component of excitation

$R_{\ell jk}$ = constant factor defined by Eq. 9

r = radius of gyration of the floor slab

$\{r_1\}, \{r_2\}, \{r_3\}$ = ground displacement influence vectors for the excitation in the three directions

S(t) = a response quantity of interest

S^2_{max} = maximum mean square response value

S_i = the parameter of spectral density function in Eq. 36

$\{T\}$ = vector which transforms the relative displacement eigenvector $\{\phi_j\}$ into modal response quantity g_j

$\{u\}$ = relative displacement response vector of the system with respect to the three ground excitations

$U_{\ell jk}, V_{\ell jk}, W_{\ell jk}$ = factors defined in Eq. I.15

$\ddot{X}_1, \ddot{X}_2, \ddot{X}_3$ = principal components of ground acceleration

$\ddot{X}_1', \ddot{X}_2', \ddot{X}_3'$ = components of ground acceleration along the structural axes

α = angle of orientation of the maximum response measured from the major principle axis

β_j = j^{th} modal damping ratio

$\Gamma_{\ell jk}$ = constant factor defined by Eq. 19

$\gamma_{\ell j}$ = element of the vector $\{\gamma_j\}$, $\ell = 1,2,3$

ζ_j = j^{th} modal response of the response quantity S(t) for proportionally damped system

λ = Lagrange Multiplier

λ_i = eigenvalue of Eq. 29, also the mean square value of the principal response

$[\rho']$ = matrix defined in Eq. 11

$[\rho'']$ = matrix used in Eq. 23 with diagonal elements $= \sqrt{(1-\rho_i)/\rho_i}$, i=2,3 and all other elements being zero

ρ_{mnjk} = a term defined in Eq. 28 and 29

ρ_1, ρ_2 = variance ratios for intermediate and minor principal excitations. Also defined as $\Phi_2(\omega) = \rho_1 \Phi_1(\omega)$ and $\Phi_3(\omega) = \rho_2 \Phi_1(\omega)$

$[\Phi]$ = matrix of spectral density function, Eq. 7

$\Phi_1(\omega), \Phi_2(\omega), \Phi_3(\omega)$ = spectral density functions of major, intermediate and minor principal excitation components

$\{\phi_j\}$ = jth undmaped mode shape or eigenvector of Eq. 1 for proportional damping system and the lower half part of the jth complex eigenvector for nonproportional damping system

ω_i, β_i = frequency and damping parameters for the spectral density function in Eq. 36

ω_j = j^{th} modal frequency

$\delta(t)$ = delta dirac function

1. INTRODUCTION

1.1 GENERAL

Earthquake induced ground motions as felt by structural systems are usually multidimensional, with six components: three translations and three rotations. In seismic design practice, however, only the translation components in three orthogonal directions are usually considered. Often these are assumed to be statistically uncorrelated and as such a design response is obtained as the square root of the sum of the squares of the responses due to each of the components (3,11). Other response combination methods which are primarily concerned with the evaluation of a design response satisfying a limit state have also been proposed in the literature (5,6,14). In this report, however, a related but different aspect of this problem is examined.

The three components of ground motion along an arbitrary set of orthogonal directions will usually be statistically correlated. However, as observed by Penzien et al in a series of papers (9,12,13), a special set of three orthogonal axes called the principal axes exist along which the ground motion components are not correlated. It has also been observed that the intensities of the three principal components are usually different. The orientation of the major principal axis has been observed to be along the epicentral direction, although the correlation between these two directions has not been found to be very strong (9). Herein these principal axes will be called as the principal <u>excitation</u> axes.

As a structure may not be oriented such that its own geometric axes align with the principal excitation axes, the motions felt by the structure along its own axes will, in general, be correlated. The degree of this correlation will depend upon the relative orientation of the structural axes with respect to the principal excitation axes. The objective of this investigation is to analytically study the effect of this correlation on a calculated structural response. The conditions under which a largest structural response will be obtained are investigated for proportionally and nonproportionally damped structural systems. A set of three orthogonal directions, herein called as the <u>principal response directions</u>, have been observed to exist for each response quantity such that if the principal excitations are applied to the structure along these direction they will induce the maximum response. The existence of such principal directions is of practical interest in as much as it helps in the evaluation of the worst case response. A step-by-step methodology to evaluate the worst cases response is outlined. Numerical results, illustrating the effect of structural orientation and input correlation on the structural response are presented.

1.2 REPORT ORGANIZATION

Section 2 of the report describes the equations of motion for three correlated ground components. The expressions for the mean square and design response of structural systems with proportional and nonproportional damping matrices are developed. The term "nonproportional" is used in a more broad sense than what it literally means; any viscously damped system whose damping matrix cannot be decoupled by undamped modes is considered to be a nonproportionally damped system. The input for design response is assumed to be defined in terms of response spectra, a commonly used form of seismic design input.

Section 3 explores the condition on the input which will give the maximum mean square response irrespective of the orientation of the structure. It is shown that if all the components of excitation are assumed to be equal to the major principal component, the mean square response will be maximum and the same for all orientation.

The evaluation of maximum mean square response when the excitation components are of unequal intensity is presented in Section 4. Here, the existence of principal response direction is identified.

Numerical results of a torsional structural system are presented in Section 5. The effect of change in the angle of orientation and variance ratio on the calculated response is shown. The response is shown to attain a maximum value at certain angular positions. In Section 6 a step-by-step procedure is outlined to obtain the worst case response for multicomponent earthquake inputs. Numerical results for another illustrative example are presented. Summary and Conclusions are provided in Section 7.

To make the report self contained, some necessary details of algebraic manipulation are provided in Appendices I to IV.

2. ANALYTICAL FORMULATION

2.1 EQUATIONS OF MOTION

The equations of motion of a multi-degree-of-freedom structural system excited at its base by three components of ground motion with two lying in, say, the horizontal plane and the other one in the vertical direction can be written as follows:

$$[M][\ddot{u}\} + [C]\{\ddot{u}\} + [K]\{u\} = -[M]\{\{r_1\}\ddot{X}'_1 + \{r_2\}\ddot{X}'_2 + \{r_3\}\ddot{X}'_3\} \qquad (1)$$

where [M], [C] and [K] are, respectively, the mass, damping and stiffness matrices of the system; $\{u\}$ = relative displacement response vector of the system with respect to the three ground excitations; $\ddot{X}'_1(t)$, $\ddot{X}'_2(t)$ and $\ddot{X}'_3(t)$ are the components of the ground acceleration along the structural axes, represented by the primed set x'_1, x'_2, and x'_3 in Fig. 1; and $\{r_1\}$, $\{r_2\}$ and $\{r_3\}$ are the ground displacement influence vectors (4) for the excitations in the three directions. The dot over a time varying quantity represents its time derivative.

For an arbitrary structural orientation, the excitation components $\ddot{X}'_1$, $\ddot{X}'_2$ and $\ddot{X}'_3$ will be correlated. Let the principal axes of the ground motion represented by the unprimed set x_1, x_2, and x_3 be as shown in Fig. 1. In terms of the ground motion components along the principal axes, the components along the structural axes can be written as follows:

$$\{E'\} = [D]^T \{E\} \tag{2}$$

where

$$\{E'\} = \begin{Bmatrix} \ddot{X}'_1 \\ \ddot{X}'_2 \\ \ddot{X}'_3 \end{Bmatrix}, \quad \{E\} = \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix}, \quad [D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \tag{3}$$

where d_{mn} = the direction cosine of m^{th} unprimed axis with respect to the n^{th} primed axis, with m,n = 1,2,3, and $\{E\}$ = the vector of uncorrelated components $\ddot{X}_1$, $\ddot{X}_2$ and $\ddot{X}_3$, specified along the principal axes of ground motion.

With Eq. 2, the solution of Eq. 1 now can be obtained in terms of the uncorrelated components using any standard technique. Here the main interest is in the evaluation of seismic design response. Thus $\ddot{X}_i(t)$'s, i = 1,2,3 are considered as random processes so that the ensemble of ground motions are included in the evaluation of response.

Furthermore, only linear systems are considered and the modal analysis approach is used so that seismic design inputs commonly defined in terms of ground response spectra (7) can be directly used. For analytical convenience the ground motions will be assumed to be characterized by stationary random processes.

The formulation of the modal analysis approach will depend upon the constitution of the damping matrix [C] in Eq. 1. If this matrix is a so-called "proportional" or "classical" (2,10), the normal mode approach can be used where the undamped eigenvectors of the system can be employed in diagonalizing [C]. This forms the basis of the commonly used square-root-of-the-sum-of-the-squares (SRSS) approach for proportionally damped systems (1,15). However, if [C] is not proportional or classical it becomes essential to employ the 2n-state vector approach (10) in which complex eigenvectors are used to affect the decoupling of equations of motion. In this latter case, though there are no normal modes, it is still possible to devise an SRSS type of approach for the calculation of seismic design response. The details of such an approach are provided by the senior writer in Ref. 16. Herein, the systems with proportional as well as nonproportional damping matrix have been considered, and the expressions for the mean square and design responses have been developed for multicomponent excitations. The terms proportional or nonproportional is used in a broader sense and is meant to include all nonclassically damped systems (2,10).

2.2 PROPORTIONALLY DAMPED SYSTEMS

Here undamped normal modes are used to decouple Eq. 1. For a linearly behaving structural system a response quantity, $S(t)$, can be written in terms of modal parameters as follows:

$$S(t) = \sum_{j}^{N} \zeta_j \int_0^t \{\gamma_j\}^T \{E'\} h_j(t-\tau) d\tau \tag{4}$$

in which N = number of degrees-of-freedom; ζ_j = j^{th} modal response of the response quantity S(t); $\{\gamma_j\} = (\gamma_{1j}, \gamma_{2j}, \gamma_{3j})^T$ = vector of participation factors for excitations along the structural axes, with its elements defined as $\gamma_{\ell j} = \{\phi_j\}^T[M]\{r_\ell\}/m_j$, $\ell = 1,2,3$; $\{\phi_j\}$ = j^{th} undamped mode shape or eigenvector of Eq. 1; $m_j = \{\phi_j\}^T[M]\{\phi_j\}$; T over a vector indicates its transpose; and $h_j(t)$ is the impulse response function of the following decoupled equation:

$$\ddot{v}_j + 2\beta_j \omega_j \dot{v}_j + \omega_j^2 v_j = \delta(t) \tag{5}$$

wherein, β_j = j^{th} modal damping ratio defined as = $\{\phi_j\}^T[C]\{\phi_j\}/2\omega_j m_j$ and ω_j = j^{th} natural frequency of the system.

For stochastic excitations, Eq. 4 can be used to obtain the mean square, and also the design, response. Substituting for $\{E\}'$ in terms of $\{E\}$ from Eq. 2 and after some standard algebraic manipulations of random vibration analysis, the mean square value of the stationary response for zero mean stationary excitations can be written in terms of the spectral density function of the excitation components as follows:

$$Ex[S^2] = \sum_{j}^{N} \sum_{k}^{N} \zeta_j \zeta_k \int_{-\infty}^{\infty} (\{\gamma_j\}^T[D]^T[\Phi][D]\{\gamma_k\}) H_j(\omega) H_k^*(\omega) d\omega \tag{6}$$

where Ex[•] denotes the expected value of [•] and

$$[\Phi] = \begin{bmatrix} \Phi_1(\omega) & & 0 \\ & \Phi_2(\omega) & \\ 0 & & \Phi_2(\omega) \end{bmatrix} \tag{7}$$

is the matrix of the spectral density functions of the three components along the principal excitation axes, and $H_j(\omega) = 1/(\omega_j^2 - \omega^2 + 2i\beta_j\omega_j\omega)$ is the frequency response function of Eq. 5. The asterisk over a quantity denotes its complex conjugate. For development of $[\Phi]$ see Appendix II, Eqs. II.3-II.6.

Employing some commonly made, and fairly reasonable simplying assumptions (15,16), Eq. 6 can be cast into an SRSS type approach for the evaluation of seismic design response as follows:

$$S_d^2 = \sum_{j=1}^{N} \zeta_j^2 \{ \sum_{\ell=1}^{3} \Gamma_{\ell jj} R_{\ell d}^2(\omega_j) \} + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \{\zeta_j \zeta_k \sum_{\ell=1}^{3} \Gamma_{\ell jk} R_{\ell jk}\} \quad (8)$$

where

$$\Gamma_{\ell jk} = \sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} \gamma_{mj} \gamma_{nk}; \quad R_{\ell d}(\omega_j) = R_{\ell a}(\omega_j)/\omega_j^2 ;$$

$$R_{\ell jk} = [A' + \omega_j^2 F_\ell(\omega_j) B'] R_{\ell a}^2(\omega_j)/\omega_j^4 + [C' + \omega_k^2 F_\ell(\omega_k) D'] R_{\ell a}^2(\omega_k)/\omega_k^4 ; \quad (9)$$

And, for frequency ω_j and damping ratio β_j, $R_{\ell a}(\omega_j)$ = psuedo acceleration spectrum for the ℓ^{th} component of excitation and $F_\ell(\omega_j)$ = the square of the ratio of the relative velocity to psuedo velocity response spectrum values. A', B', C' and D' are defined in Appendix I. Eq. 8 can be used to obtain design response for an arbitrary orientation of the structure defined with respect to the principal directions of the excitation, if they are known.

It has been observed (9,12,13) that the variances of the ground accelerations along the principal directions are unequal. Here it is assumed that they are in the ratio of $1:\rho_1:\rho_2$. Later a parametric variation study is conducted where numerical values have been assigned to these variance ratios. For simplification it is also assumed that the frequency characteristics of the input motions, as defined by their spectral density functions are also same. Thus, $\Phi_2(\omega) = \rho_1 \Phi_1(\omega)$ and $\Phi_3(\omega) = \rho_2 \Phi_1(\omega)$ with $\Phi_1(\omega)$ being the spectral density function of the major principal component. Substituting these in equation (6) and realizing that $[D]^T[D] = [I]$, an identity matrix, the following is obtained:

$$Ex[S^2] = \sum_j^N \sum_k^N \zeta_j \zeta_k (\{\gamma_j\}^T\{\gamma_k\} - \{\gamma_j\}^T[D]^T[\rho'][D]\{\gamma_k\}) \int_{-\infty}^{\infty} \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \tag{10}$$

in which

$$[\rho'] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\rho_1 & 0 \\ 0 & 0 & 1-\rho_2 \end{bmatrix} \tag{11}$$

The first term in Eq. 10 represents the mean square response due to three independent but equal intensity excitations applied along the structural axes. The remaining term represents the effect of correlation between the components which, of course, depends upon the orientation of the structure with respect to the principal axes. The effect of orientation and variance ratio on the response is examined numerically later.

2.3 Nonproportionally Damped Systems

Equations parallel to Eq. 8 and 10 for the calculation of stationary mean square response and the design response can also be derived for nonclassically damped structural systems excited by correlated excitation components. Using the 2n-dimensional state vector approach (10,16), the solution of Eq. 1 for response quantity, S(t), can be written as

$$S(t) = \sum_{j}^{2N} \int_0^t \{q_j\}^T \{E'\} e^{p_j(t-\tau)} d\tau \tag{12}$$

Using Eq. 2

$$S(t) = \sum_{j}^{2N} \int_0^t \{q_j\}^T [D]^T \{E\} e^{p_j(t-\tau)} d\tau \tag{13}$$

in which,

$$\{q_j\}^T = (q_{1j}, q_{2j}, q_{3j})$$

$$q_{\ell j} = -\frac{g_j}{\overline{A}_1} \{\phi_j\}^T [M][r_\ell]; \quad \ell = 1,2,3 \tag{14}$$

$$\overline{A}_j = \{\phi_j\}^T (2p_j[M] + [C]) \{\phi_j\}; \; g_j = \{T\}\{\phi_j\}$$

and p_j = j^{th} complex eigenvalue and $\{\phi_j\}$ = the lower half part of the j^{th} eigenvector of the following 2n-dimension eigenvalue problem:

$$p_j[A]\{\phi\} + [B]\{\phi\} = \{0\} \tag{15}$$

wherein

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}; \quad [B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix} \tag{16}$$

and $\{T\}$ is a vector which transforms the relative displacement eigenvector $\{\phi_j\}$ into the modal response value g_j of the response quantity $S(t)$.

Realizing that the components of $\{E\}$ are uncorrelated random processes, the stationary mean response of $S(t)$ can be written in the following form (see Appendix II):

$$Ex[S^2] = \sum_{j}^{N} \sum_{j}^{N} \int_{-\infty}^{\infty} \{G_j\}^T[D]^T[\Phi][D]\{G_k^*\}H_j(\omega)H_k^*(\omega)d\omega \tag{17}$$

where

$$\{G_j\} = 2i\,\omega\{a_j\}+2\{A_j\} = <G_{1j},\ G_{2j},\ G_{3j}>^T$$

$$\{a_j\} = \begin{Bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{Bmatrix} \quad ; \quad \{A_j\} = \omega_j \begin{Bmatrix} a_{1j}\beta_j - b_{1j}\sqrt{1-\beta_j^2} \\ a_{2j}\beta_j - b_{2j}\sqrt{1-\beta_j^2} \\ a_{3j}\beta_j - b_{3j}\sqrt{1-\beta_j^2} \end{Bmatrix} \tag{18}$$

in which $a_{\ell j}$, $b_{\ell j}$ are the real and imaginary parts of $q_{\ell j}$ respectively; and ω_j and β_j are analogous to the modal frequency and damping ratio, defined in terms of the real and imaginary parts of the p_j as follows:

$$\omega_j = |p_j| \quad ; \quad \beta_j = -\frac{\text{Real}(p_j)}{\omega_j} \tag{19}$$

Eq. 17 can be further extended to develop the following SRSS type of approach for the calculation of design response for nonclassically damped systems:

$$S_d^2 = 4 \sum_{j=1}^{N} \sum_{\ell=1}^{3} \{U_{\ell jj} + \omega_j^2 F_\ell(\omega_j) V_{\ell jj}\} R_{\ell d}^2(\omega_j)$$
$$+ 8 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \sum_{\ell=1}^{3} [\{A'' + \omega_j^2 F_\ell(\omega_j) B''\} R_{\ell d}^2(\omega_j) + \{C'' + \omega_k^2 F_\ell(\omega_k) D''\} R_{\ell d}^2(\omega_k)] \tag{20}$$

where, $U_{\ell jk}$, $V_{\ell jk}$ and A", B", C" and D" are defined in Appendix I. Eq. 20, applicable to nonproportionally damped systems, is similar to Eq. 8 and can be used with prescribed ground spectra to obtain design response, again only if the elements of [D] are known.

For the special case of excitations, with variance ratios of ρ_1 and ρ_2, Eq. 17 can also be rewritten as follows:

$$Ex(S^2) = \sum_j^N \sum_k^N \int_{-\infty}^{\infty} (\{G_j\}^T\{G_k^*\} - \{G_j\}^T[D]^T[\rho'][D]\{G_k^*\})\Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \tag{21}$$

Here, like Eq. 10, the response terms which are affected by the input correlation and structural orientation have also been separated.

3. MAXIMUM MEAN SQUARE RESPONSE

Since the principal excitation axes are not known in advance, a question is immediately raised: what orientaton of a structure with respect to ground excitations will result in the highest response? An examination of Eqs. 10 and 21 indicates that when all three excitations are of equal intensity, $[\rho']$ is a null matrix and thus the total response is given by the first term. This response will be maximum if and only if the second terms in Eqs. 10 and 21 are always positive. That it is, indeed, the case can be seen from the fact that the second term in Eq. 10 represents the stationary mean square value (a positive quantity) of the following response quantity:

$$S'(t) = \sum_j^N \zeta_j \int_0^t E_g'(\tau)h_j(t-\tau)d\tau \tag{22}$$

where the excitation term is defined as

$$E_g'(t) = \{\gamma_j\}^T[D]^T[\rho'']\{E(t)\} \tag{23}$$

where $[\rho'']$ is a diagonal matrix like $[\rho']$ except that the nonzero diagonal terms are defined as $\sqrt{(1-\rho_i)/\rho_i}$, with i = 2 and 3. Similarly it can be shown that the second terms associated with $[\rho']$ in Eq. 21 for nonproportionally damped systems also represents a mean square value, always a positive quantity (see Appendix III for details). This shows that a maximum response will be obtained if the intensities of the excitations along the intermediate and minor principal axes of the ground motion are assumed to be the same as that of the excitation component along the major principal axis. In such a case, the maximum response is independent of the orientaton of the structure; that is, all orientations of the structure will result into the same maximum response. This is similar to the uniform state of stress in solid mechanics where stresses are equal and maximum in all directions. This is, probably, implicitly understood by seismic analysts and has practical implications. It shows that a designer should chose his inputs in all three orthogonal directions to be of intensity equal to that of the major principal component to ensure that the calculated forces will not be exceeded no matter what the orientation of the structure. However, if the ground motions of unequal intensity are to be used as iputs, it is essential to locate the worst orientation which will induce the maximum structural response. The following approach defines such an orientation for a particular response quantity of design interest.

4. PRINCIPAL RESPONSE DIRECTIONS

It is desired to obtain the elements of matrix [D] such that the response in Eqs. 6 or 17 is maximum for a given set of principal excitation components. First, the worst direction causing maximum response for a single excitation is obtained. The mean square response for a proportionally damped system due to, say, the major principal excitation component applied along an axis with direction cosines d_1, d_2 and d_3, defined in the primed coordinate axes, can be written as

$$Ex(S^2) = \sum_{j}^{N}\sum_{k}^{N} \zeta_j \zeta_k (\gamma_{1j}d_1+\gamma_{2j}d_2+\gamma_{3j}d_3)(\gamma_{1k}d_1+\gamma_{2k}d_2+\gamma_{3k}d_3) \int_{-\infty}^{\infty} \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \tag{24}$$

This, and a similar quadratic for a nonproportionally damped system, can be written in matrix notations as follows:

$$Ex(S^2) = \{d\}^T[\bar{R}]\{d\} \tag{25}$$

where $\{d\}$ is a vector of direction cosines d_1, d_2 and d_3 and the response matrix $[\bar{R}]$ is defined as follows:

$$[\bar{R}] = \begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} \\ \bar{R}_{21} & \bar{R}_{22} & \bar{R}_{23} \\ \bar{R}_{31} & \bar{R}_{32} & \bar{R}_{33} \end{bmatrix} \tag{26}$$

$[\bar{R}]$ is a Hermitian matrix. The elements of this matrix are defined as follows:

$$\bar{R}_{mn} = \sum_{j}^{N}\sum_{k}^{N} \rho_{mnjk} \quad ; \quad m,n = 1,2,3 \tag{27}$$

where ρ_{mnjk} are defined for proportionally damped systems as follows:

$$\rho_{mnjk} = \zeta_j \zeta_k \gamma_{mj} \gamma_{nk} \int_{-\infty}^{\infty} \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \tag{28}$$

and for nonproportionally damped systems as follows:

$$\rho_{mnjk} = \int_{-\infty}^{\infty} G_{mj}G_{nk}^* \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \tag{29}$$

in which G_{mj} is an element of the vector defined in Eq. 18.

To obtain the excitation direction which will extremize $Ex[S^2]$ with the following constraint on its direction cosines,

$$\{d\}^T\{d\} = 1 \tag{30}$$

the Lagrange multiplier approach can be used. The extremization of the auxiliary function with Lagrange multiplier, λ,

$$L(\lambda) = \{d\}^T[\bar{R}]\{d\} - \lambda(\{d^T\}\ \{d\}-1) \tag{31}$$

leads to the following eigenvalue problem

$$[\bar{R}]\{d\} - \lambda\{d\} = \{0\} \tag{32}$$

It is seen that the Lagrange multiplier is also the eigenvalue. A non-trivial solution of Eq. 32 requires that the determinant of the characteristic matrix be zero. This gives a third degree characteristic equation in λ, like the equation of the principal stresses in continuum mechanics. It can be shown that all roots of this equation are real and positive. For each root, λ_i, i = 1,2,3, there is a corresponding vector of direction cosines $\{d\}_i$. For different λ_i's these direction cosines are orthogonal; that is, $\{d\}_i^T\{d\}_j = 0$ for $i \neq j$. One of these three direction corresponds with maximum value of response whereas the others correspond with the intermediate and minimum values of response. For an i^{th} eigenpair, Eq. 32 can be written as

$$[\bar{R}]\{d\}_i = \lambda_i\{d\}_i \tag{33}$$

Premultiplying Eq. 33 with $\{d\}_i^T$ and using Eq. 30, one obtains

$$\lambda_i = \{d\}_i^T[\bar{R}]\{d\}_i \tag{34}$$

Comparing this with Eq. 25, it is seen that the Lagrange multiplier, λ_i, which is the eigenvalue of Eq. 29 is also the mean square value of the response when the excitation is applied along the direction specified by $\{d\}_i$. The largest eigenvalue gives the largest mean square response.

It is seen that these principal directions could be different for each response quantity because ζ_j or G_{mj} are different. Here these principal directions associated with a response quantity are called as the principal response directions to differentiate these from the principal excitation axes. Also since ρ_{mnjk} depends upon the excitation spectral density function, the principal response directions will, in general, also depend upon the frequency characteristics of the excitation. However, if the frequency characteristics of the input components are not significantly different, their principal response directions for a response will not be much different either. Thus, for the three orthogonal components of input it will, probably, be accurate enough to assume that these principal response directions are input independent. This simplifies the identification of the worst orientation of excitation for the calculation of worst case design response. To obtain the worst case response the major, intermediate and minor principal excitation axes should, respectively, coincide with the major intermediate and minor principal response directions which are identified by the eigenvalue problem in Eq. 32. The total response in such a case is given by Eq. 6 for proportionally damped and by Eq. 17 for nonproportionally damped structural system, with the matrix [D] containing the direction cosines of principal response directions. This total response, as per Eqs. 6 or 17, is seen to be the square-root-of-the-sum-of-the-squares of the response due to each principal excitation component.

5. NUMERICAL RESULTS

To study the effect of structural orientation and variance ratios on the structural response, the numerical results have been obtained for the structural system shown in Fig. 1. This system is similar to the one considered in Ref. 8. The system represents a multi story building with rigid floors interconnected by columns. Each floor has three degrees of freedom in its own plane. The mass and stiffness centers of the system are eccentrically placed by an amount e with the purpose of creating torsional effect and coupling in modal responses. The torsional and bending stiffnesses of the system and the eccentricity between the mass and stiffness centers can be adjusted to introduce varying degrees of modal correlation, and interaction of this correlation with the correlation in the input components is reflected in the calculated response. A nonproportional damping matrix is created by assigning different damping constants in the two horizontal directions. Parallel numerical results have also been obtained in which this damping matrix has been assumed to be proportional. This implies that the off diagonal terms of the matrix $[\Phi]^T[C][\Phi]$ are negligible. Such an assumption is often made in practice to simplify the analysis. The comparison of the two similar results shows the error introduced by the assumption of proportionality of a damping matrix.

To simplify the study of orientation effect on the response, the major and intermediate principal excitation directions are assumed to lie in the horizontal plane and the minor principal excitation is aligned along the vertical direction. In this situation, the orientation is completely defined by one angle. Assuming this angle to be α, measured in the counter-clockwise direction from the major principal axis, the matrix of direction cosines can be written as follows:

$$[D] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

The effect of changing orientation parameter α on the response is studied.

The input motions used in this study are defined in terms of a spectral density function. As mentioned earlier, the spectral characteristics of the three principal components have been assumed to be the same. Their intensities as measured by their variance are, however, different. The spectral density function for the major principal excitation is assumed to be of Kanai-Tajimi form as follows:

$$\Phi_1(\omega) = \sum_{i=1}^{3} S_i \frac{\omega_i^4 + 4\beta_i^2\omega_i^2\omega^2}{(\omega_i^2 - \omega^2)^2 + 4\beta_i^2\omega_i^2\omega^2} \tag{36}$$

The parameters S_i, ω_i and β_i of this density function are given in Table 1. This density function represents a broad-band seismic input, suitable for design purposes. The effect of change in the variance ratio ρ on the response has been evaluated. For two principal excitation components, the variance ratio values of 1.0, 0.75 and 0.5, have been considered to obtain numerical results of various response quantities.

The numerical results of the mean square response values have been obtained for the story shear, story torsion and bending moment in a column using Eq. 17 for nonproportional and Eq. 6 for proportional damping matrices. As these response quantities are not primarily affected by the vertical component of excitation, the terms corresponding to this excitation are deleted from these equations.

The effect of the structural orientation and the variance ratio on the root mean square (RMS) value of the base shear in x_2'-direction is shown in Fig. 2 for two values of the eccentricity ratio, e/r, equal to .01 and .1; r is the radius of gyration of floor slabs. Fig. 3 shows similar results for the base shear in x_1'-direction. The results have been normalized by the maximum base shear in x_2' direction obtained for nonproportionally damped system. For comparison, the results obtained by Eqs. 6 and 17 both are plotted. The differences in these two results show the magnitude of error which is caused by the assumption of proportionality of [C] when it is not. From these figures it is seen that this error is large when e/r is small. For small e/r values in this case, the frequencies were closely spaced causing a strong modal interaction effect. In such a case, the off-diagonal terms in $[\Phi]^T[C][\Phi]$ attain a special significance and neglecting them can cause large errors in the calculated response. On the other hand, if the frequencies are widely separated, the modal correlation will be weaker and the error due to the assumption of proportionality may not be significant. Similar trend in the magnitude of the error were also observed for other response quantities like story torsional moment and column bending moments.

Figs. 4 and 5 show the RMS value of the base torsional moment for proportional and nonproportional damping case for e/r values of .01 and .10. Figs. 5 and 7 show the RMS bending moment in one of the columns of the lower story for proportional damping case whereas Figs. 8 and 9 show similar results for nonproportional damping case. It is noted that maximum values of these quantities are obtained when the variance ratio is 1.0. For other variance ratios the response changes with the orientation and reaches a maximum value which is almost equal to the value for variance ratio of 1. The orientation for the maximum response can be obtained as discussed in the previous section. For the present two dimensional case this angle is given by the following expression

$$\tan 2\alpha = -\frac{2\bar{R}_{12}}{\bar{R}_{11}-\bar{R}_{22}} \tag{37}$$

where $\bar{R}_{mn}$ is defind in Eq. 27. Eq. 37 provides two values of α which are 90° apart which correspond to the maximum and minimum values of the RMS response. For base shear, Figs. 2 and 3, the directions for maximum response are almost in the x_1' and x_2' directions. For the structure in Fig. 1 these directions are also the axes of symmetry for small e/r. For larger e/r, a slight shift in the directions for maximum from the x_1' and x_2' is noted, Figs. 2b and 3b. Also for the case of torsional moment, Figs. 4 and 5, the maximum and minimum response values do not occur when the excitations are along the x_1' and y_1'-directions. Furthermore, the excitation directions for the largest response are different for the proportional and nonproportional systems. As the RMS value of the bending moment in a column is affected by shear and torsional moment, the angles for maximum response are also affected by both. It is seen from Figs. 6-9 where the excitation directions for the maximum column moment response is clearly seen to be somewhere between the directions for the maximum base shear, Figs. 2 and 3, and maximum torsional moment, Figs. 4 and 5. These results also show that the principal response directions could be different for different response quantities.

In seismic structural analysis, it is a common practice to take two horizontal components of excitation to be equal in intensity and the intensity of the vertical excitation to be equal to 2/3 of the intensity of the horizontal excitations. For structures in which horizontal and vertical responses are uncoupled, this practice ensures that a calculated response is always maximum and is independent of structural orientation. However, structures in which vertical and horizontal responses are coupled, orienting the smallest excitation in the vertical direction may not induce the worst case response. In such cases, it is necessary to orient the principal excitation components along the major, minor and intermediate principal response directions to obtain the worst case response. This, however, does not require a reanalysis of structural system after the principal response directions have been identified. Rather, a step-by-step procedure as outlined in the following sections can be used to calculate a worst case design response.

6. EVALUATION OF WORST CASE RESPONSE

6.1 STEP-BY-STEP PROCEDURE

To obtain a worst case response, that is a response which will be maximum for all possible directions of impinging ground motions, the following step-by-step procedure can be used:

1. Select a set of coordinate axes to model the structural system. Though not necessary, it may be preferrable to orient one or more of these axes along the axes of symmetry if these can be identified.

2. Obtain the dynamic characteristics of the structure such as frequencies, ω_j, mode shapes, $\{\phi_j\}$, and participation factors, γ_{mj}. The participation factors will depend upon the choice of axes made in Step 1.

3. Obtain the elements of response matrix, Eq. 26, for all three components of ground motion. It is noted that the prescribed response spectra can be directly used in the calculation of these elements. For example, for a proportional damping case, an element $\bar{R}_{mn}$ can be written as follows:

$$\bar{R}_{mn} = \sum_{j=1}^{N} \zeta_j^2 \gamma_{mj} \gamma_{nj} R_{\ell d}^2(\omega_j) + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \zeta_j \zeta_k \gamma_{mj} \gamma_{nj} R_{\ell jk} \quad (38)$$

where $R_{\ell jk}$ is defined in Eq. 9.

4. Solve the eigenvalue problem defined by Eq. 32 for one of the response matrices $[\bar{R}]$. The response matrix obtained for the largest principal component of excitation should, preferably, be used. For such a response matrix, the largest eigenvalue in Eq. 32 then also provides the worst case response due to the largest principal component.

5. In Eq. 25, use the intermediate and minor eigenvectors obtained in step (4) with the response matrices $[\bar{R}]$ for the intermediate and minor principal components to obtain the contributions of these to the total response.

6. As per Eqs. 8 or 20, the square root of the sum of the squares of the response due to the three components of excitation, as obtained in steps (4) and (5), gives the worst case design response.

6.2 ILLUSTRATIVE EXAMPLES

The numerical results presented in the previous section were obtained for various values of angle α and variance ratio ρ. It was observed that for each response quantity there is a special direction to cause the maximum response. To identify this direction and to obtain the maximum response, the eigenvalue approach outlined in Section 4 will now be used.

Tables 2-5 show the results obtained by this approach for the structural system shown in Fig. 1. In Tables 2 and 3, the results are for the e/r ratio of .01, and in Tables 4 and 5 for e/r = .10. Tables 2 and 4 present the results for proportional damping case, whereas Tables 3 and 5 for nonproportional damping case.

Columns (2) in the Tables show the root mean square values which were obtained when the stronger of the two excitation components was aligned along the x_1' axis. For excitations with similar spectral characteristics and with the variance ratio of ρ, the mean square response is simply obtained as:

$$Ex[S^2] = \bar{R}_{11} + \rho \bar{R}_{22} \tag{39}$$

where $\bar{R}_{11}$ and $\bar{R}_{22}$ are the elements of $[\bar{R}]$ matrix obtained with the spectral density function of the stronger excitation component. Here, for the two dimensional cases, $[\bar{R}]$ is a 2x2 matrix.

The response values in Columns (3) are the maximum values obtained by the approach established in Section 4. That is, it is the square root of the following mean square value:

$$S^2_{max} = \lambda_1 + \rho\lambda_2 \tag{40}$$

where λ_1 and λ_2 are the maximum and minimum eigenvalues, respectively, of $[\bar{R}]$.

The angle, measured in degrees from the x_1'-axis, at which the major excitation component should be applied to cause the maximum response is shown in Column 5. These values agree very well with the results plotted in Figs. 2-5.

Columns (4) show the ratio of the values in Columns (2) and (3). These values indicate the magnitude of possible underestimation of response if the excitation directions are fixed along the structural axes and no search for the maximum response is made.

For some response quantities, the orientation of excitation for the worst effect is obvious. For example, if the maximum story shear in x_2' direction is being sought, the stronger excitation component should be nearly along the x_2'-direction. For other response quantities, the worst direction may not be that obvious. Even if the worst directions are known in advance, it may not be practically feasible to keep changing the orientation of excitation to obtain the maximum values of various response quantities. These trials on re-orientation and re-analyses can obviously be avoided by adopting the method outlined in Section 6.

In practice the search for the maximum response is avoided by assuming that the two horizontal excitation components are exactly the same. In such a situation, any two orthogonal directions in the horizontal plane are principal response directions. This obviously ensures a conservative evaluation of response, no matter what orientation of inputs is assumed. The method outlined in Section 6.1, however, can be used to assess the degree of conservation involved in such a practice if the two horizontal excitation components are not equal.

In practice it is customary to assume the vertical excitation to be 2/3 of the horizontal components. Thus in situations where a response quantity is affected by both horizontal and vertical excitations, that is when there is a coupling between the effects of two excitations, the approach presented in Sections 4 and 6.1 can be effectively used to obtain the worst case response. This will usually happen where there is an asymmetry in the structure. To illustrate this, here the problem of a vertical asymmetric frame is used. The frame is shown in Fig. 10.

The mass and stiffness properties of the frame are given in Table 6. The natural frequencies of the system and participation factors are shown in Table 7. In Tables 8 and 9 are shown the results for axial force, shear force bending moment and maximum bending stress in various members of the frame. The latter quantity is of design interest where a maximum effect of two response quantities (moment and axial force) is sought. Tables 7 and 8 have parallel columns as in Tables 2 through 5. Again, it is seen that the response is likely to be severely underestimated if the excitation directions are fixed along geometric X_1' and X_2' axes and no search for the maximum is made.

7. SUMMARY AND CONCLUSIONS

The response of linearly behaving structures subjected to three translational components of earthquake has been examined. The effect on dynamic response of the orientation of a structure with respect to the ground excitation components is considered. The formulation for the evaluation of design response, employing response spectra as inputs, is presented for proportionally as well as nonproportionally damped structures. Assuming the existence of the so-called principal component of excitation, the response which considers correlation of input along structural axes can be expressed in terms of the uncorrelated principal components.

The existence of a set of input directions, herein called as the principal response directions, has been identified. These principal directions depend upon the type of response and the frequency characteristics of the input. If an applied excitation is directed along the major principal direction the induced response will be maximum. This is of practical significance as it helps in the evaluation of the worst case response for a given set of principal excitations of unequal intensities. Although each response quantity has its own principal response directions, the worst case can be directly obtained without conducting any parametric variation on possible angular orientations of the structure.

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TABLE 1: PARAMETERS OF SPECTRAL DENSITY FUNCTION $\Phi_g(\omega)$, Eq. 36

i	S_i ft^2-sec/rad	ω_i rad/sec	β_i
1	.006	13.5	.3925
2	.00198	23.5	.3600
3	.0015	39.0	.3350

TABLE 2: COMPARISON OF BASE SHEAR, TORSIONAL MOMENT AND COLUMN BENDING MOMENT RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE, Eq. 39, FOR STRUCTURE IN FIG. 1. PROPORTIONAL DAMPING; e/r = 0.01

Response Quantity		Response By Eq. 38	Maximum Response By Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
Base Shear in - x_1'		1.006	1.006	1.00	-0.08
Base Shear in - x_2'		0.711	1.006	0.71	90.08
Torsional Moment		0.069	0.080	0.87	45.00
Bending Moment in Col. 1	x_1'-Direc.	1.743	1.743	1.00	-0.64
	x_2'-Direc.	1.234	1.743	0.71	90.64
Bending Moment in Col. 2	x_1'-Direc.	1.782	1.782	1.00	0.63
	x_2'-Direc.	1.262	1.782	0.71	89.37

TABLE 3: COMPARISON OF BASE SHEAR, TORSIONAL MOMENT AND COLUMN BENDING MOMENT RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE, Eq. 39, FOR STRUCTURE IN FIG. 1. NONPROPORTIONAL DAMPING; e/r = 0.01.

Response Quantity		Response By Eq. 38	Maximum Response By Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
Base Shear in - x_1'		0.874	0.874	1.00	-.05
Base Shear in - x_2'		0.989	1.399	0.71	90.16
Torsional Moment		0.084	0.112	0.76	72.94
Bending Moment in Col. 1	x_1'-Direc.	1.515	1.515	1.00	-.77
	x_2'-Direc.	1.722	1.434	0.71	90.45
Bending Moment in Col. 2	x_1'-Direc.	1.540	1.540	1.00	0.81
	x_2'-Direc.	1.747	2.470	0.71	89.95

TABLE 4: COMPARISON OF BASE SHEAR, TORSIONAL MOMENT AND COLUMN BENDING MOMENT RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE, Eq. 39, FOR STRUCTURE IN FIG. 1 PROPORTIONAL DAMPING; e/r = 0.10.

Response Quantity		Response By Eq. 38	Maximum Response By Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
Base Shear in - x_1'		0.945	0.945	1.00	-3.31
Base Shear in - x_2'		0.699	.946	0.74	93.31
Torsional Moment		0.407	0.470	0.65	45.00
Bending Moment in Col. 1	x_1'-Direc.	1.577	1.563	1.00	-7.57
	x_2'-Direc.	1.196	1.563	0.76	97.57
Bending Moment in Col. 2	x_1'-Direc.	1.911	1.918	1.00	+7.33
	x_2'-Direc.	1.474	1.918	0.77	82.67

TABLE 5: COMPARISON OF BASE SHEAR, TORSIONAL MOMENT AND COLUMN BENDING MOMENT RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE, Eq. 39, FOR STRUCTURE IN FIG. 1. NONPROPORTIONAL DAMPING; e/r = 0.10.

Response Quantity		Response By Eq. 38	Maximum Response By Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
Base Shear in - x_1'		0.865	0.866	1.00	-2.35
Base Shear in - x_2'		0.812	1.106	0.73	95.28
Torsional Moment		0.430	0.538	0.80	63.84
Bending Moment in Col. 1	x_1'-Direc.	1.441	1.447	1.00	97.87
	x_2'-Direc.	1.395	1.881	0.74	-7.43
Bending Moment in Col. 2	x_1'-Direc.	2.398	2.402	1.00	7.22
	x_2'-Direc.	1.688	2.218	0.76	86.26

TABLE 6: MASS AND STIFFNESS PROPERTIES OF FRAME IN FIG. 10

MEMBER PROPERTIES

Member Number	Cross Sectional Area (sq. in.)	Moment of Inertia ($in.^4$)	Modulus of Elasticity (KSI)
1	5.88	42.4	30,000
2	10.30	147.0	30,000
3	5.88	42.4	30,000
4	10.30	147.0	30,000

NODAL COORDINATES AND MASS

NODE NO.	MASS Kips-Slugs	COORDINATE	
		X, in	Y, in
1	-	0.	0.
2	3.0	-30.	120.
3	5.0	36.	228.
4	3.0	120.	228.
5	-	180.	0.

TABLE 7: NATURAL FREQUENCIES AND PARTICIPATION FACTORS FOR EXCITATIONS ALONG THE STRUCTURAL AXES (x_1' and x_2'-DIRECTION) OF STRUCTURE IN FIG. 10.

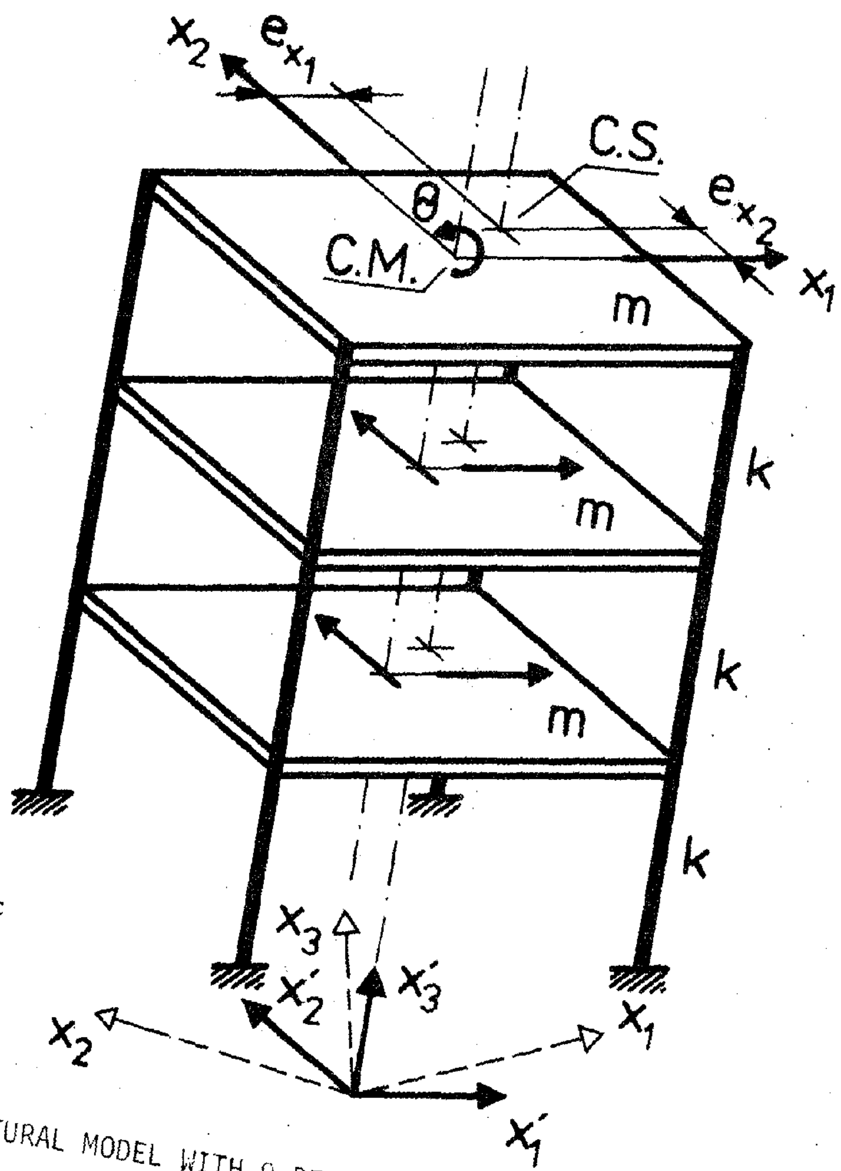
DEGREE OF FREEDOM	FREQUENCY (CPS)	PARTICIPATION FACTOR IN x_1'-DIRECTION	PARTICIPATION FACTOR IN x_2'-DIRECTION
1	0.165	3.231	0.737
2	0.527	-.305	1.840
3	2.667	0.565	-2.218
4	3.260	-.357	1.349
5	5.179	0.133	-0.518
6	6.357	-.061	0.255

TABLE 8: COMPARISON OF MEMBER AXIAL AND SHEAR FORCE RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE BY Eq. 39, FOR STRUCTURE IN FIG. 10. PROPORTIONAL DAMPING.

Response	Member No.	Response By Eq. 38	Maximum Response Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
MEMBER	1	15.690	21.490	0.73	75.79
AXIAL	2	10.067	13.162	0.76	75.58
FORCE	3	4.408	5.924	0.74	75.57
	4	8.844	11.982	0.74	75.43
MEMBER	1	6.232	6.310	0.99	-12.89
SHEAR	2	30.948	38.449	0.80	81.37
FORCE	3	23.586	28.513	0.83	81.77
	4	33.462	33.882	0.99	-12.85

TABLE 9: COMPARISON OF MEMBER BENDING MOMENT AND STRESS RESPONSE OBTAINED BY CONVENTIONAL PROCEDURE, Eq. 38, AND MAXIMUM RESPONSE BY Eq. 39, FOR STRUCTURE IN FIG. 10. PROPORTIONAL DAMPING.

Response	Member No.	Response By Eq. 38	Maximum Response By Eq. 39	Ratio	Angle in Degree
(1)		(2)	(3)	(4)	(5)
MEMBER	1	385.42	390.22	0.99	-12.89
BENDING	2	195.85	243.32	0.80	81.37
MOMENT	3	90.06	119.76	0.83	81.77
	4	394.44	399.40	0.99	-12.85
COMBINED	1	31.678	32.064	0.99	-12.88
STRESS	2	6.962	9.520	0.82	81.17
	3	8.132	9.735	0.84	81.87
	4	13.284	13.449	0.99	-12.83



$\sqrt{k/m} = 62.8$ r/sec

Figure 1: STRUCTURAL MODEL WITH 9 DEGREES OF FREEDOM.

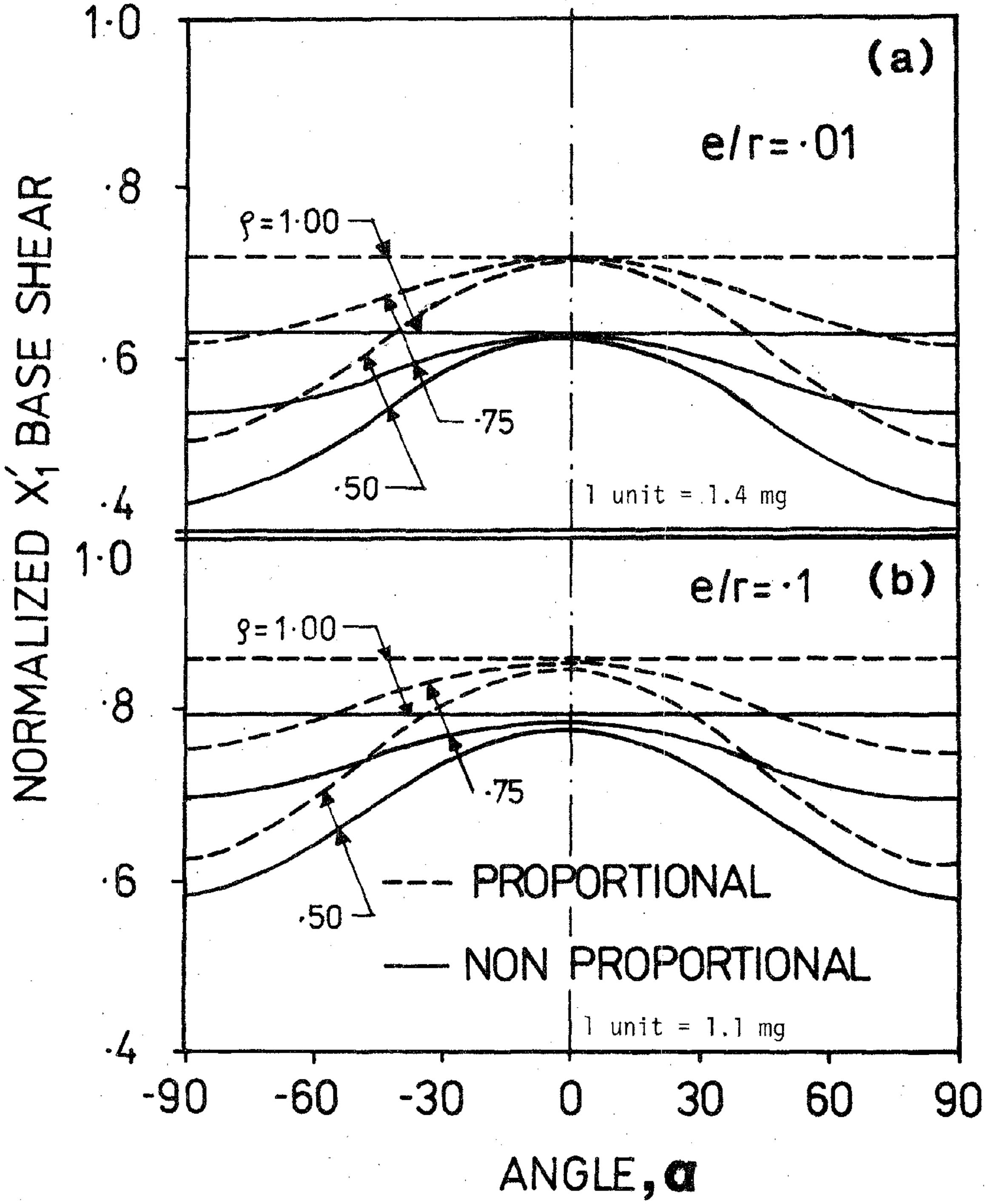


Figure 2: VARIATION OF NORMALIZED BASE SHEAR ALONG X_1'-DIRECTION FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS.

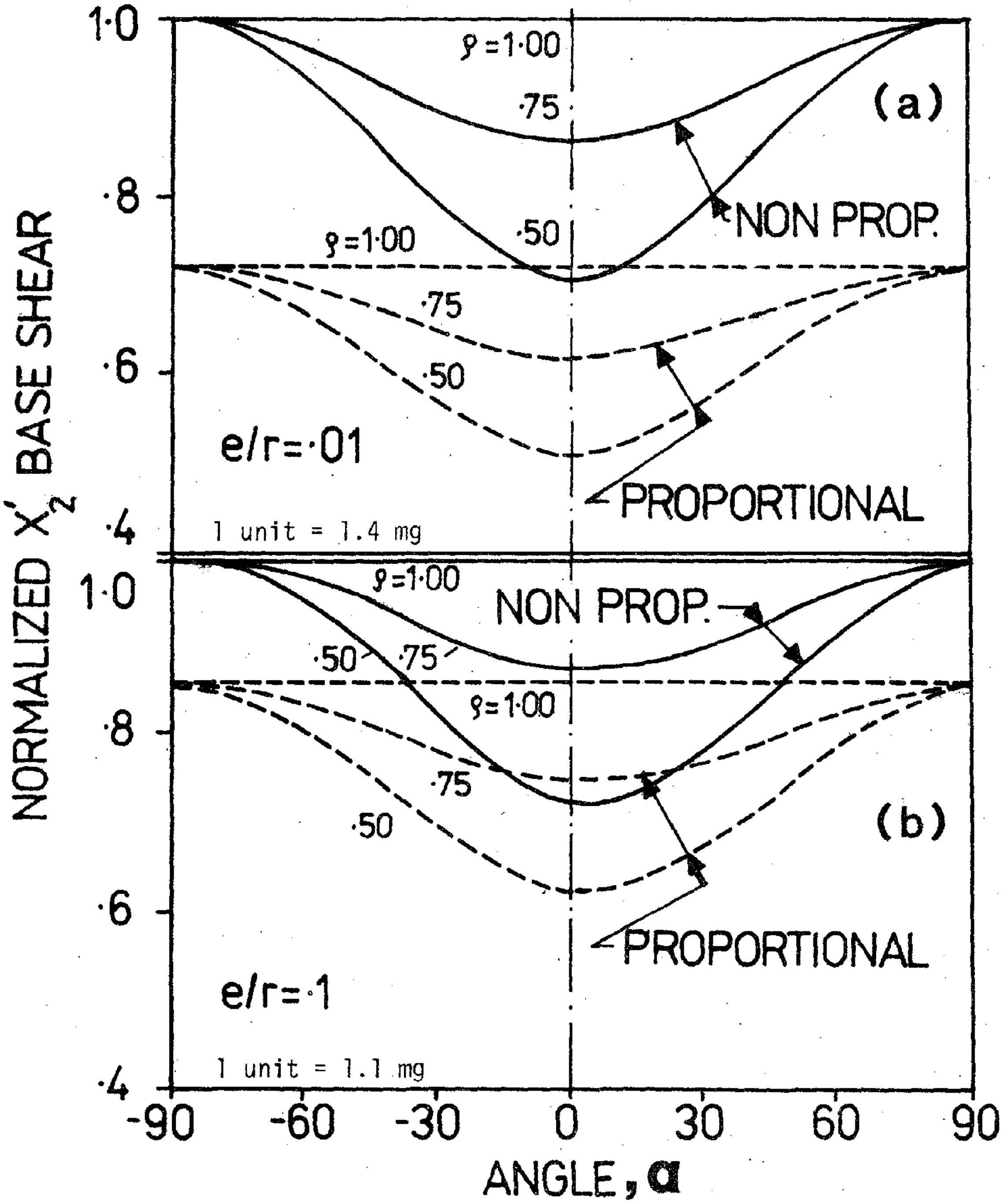


Figure 3: VARIATION OF NORMALIZED BASE SHEAR ALONG X_2'-DIRECTION FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS.

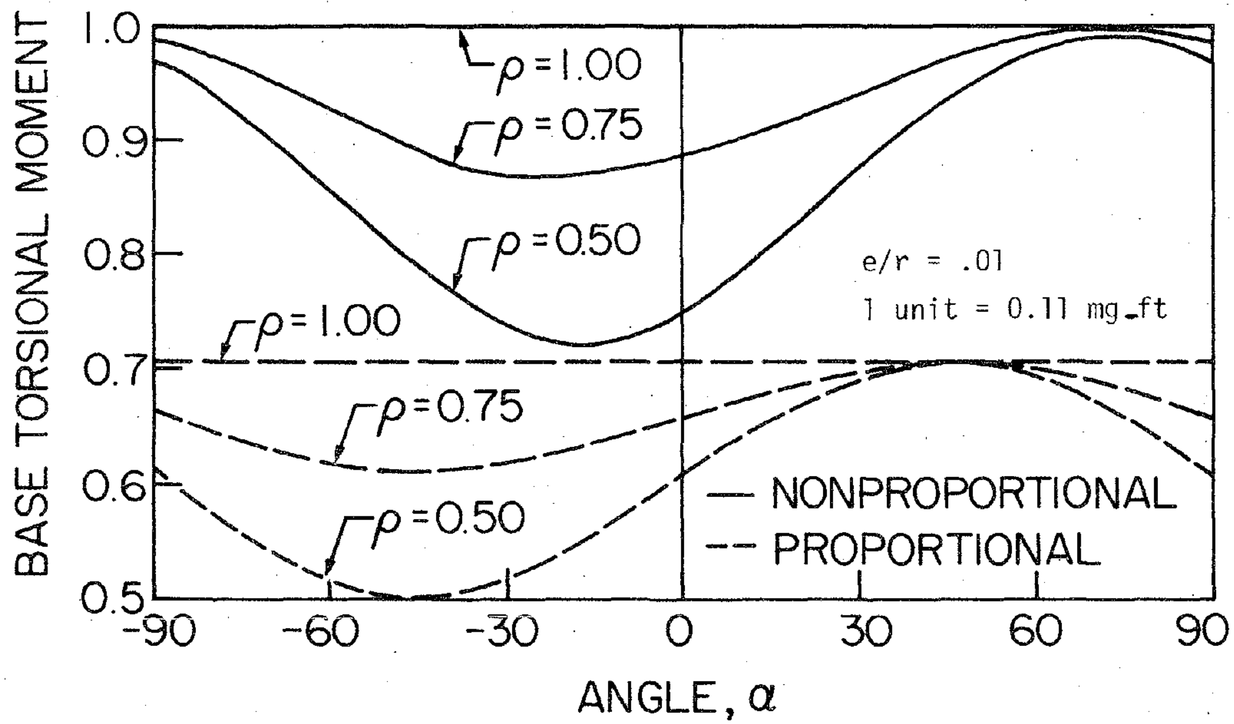


Figure 4: VARIATION OF NORMALIZED BASE TORSIONAL MOMENT FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS; e/r = .01.

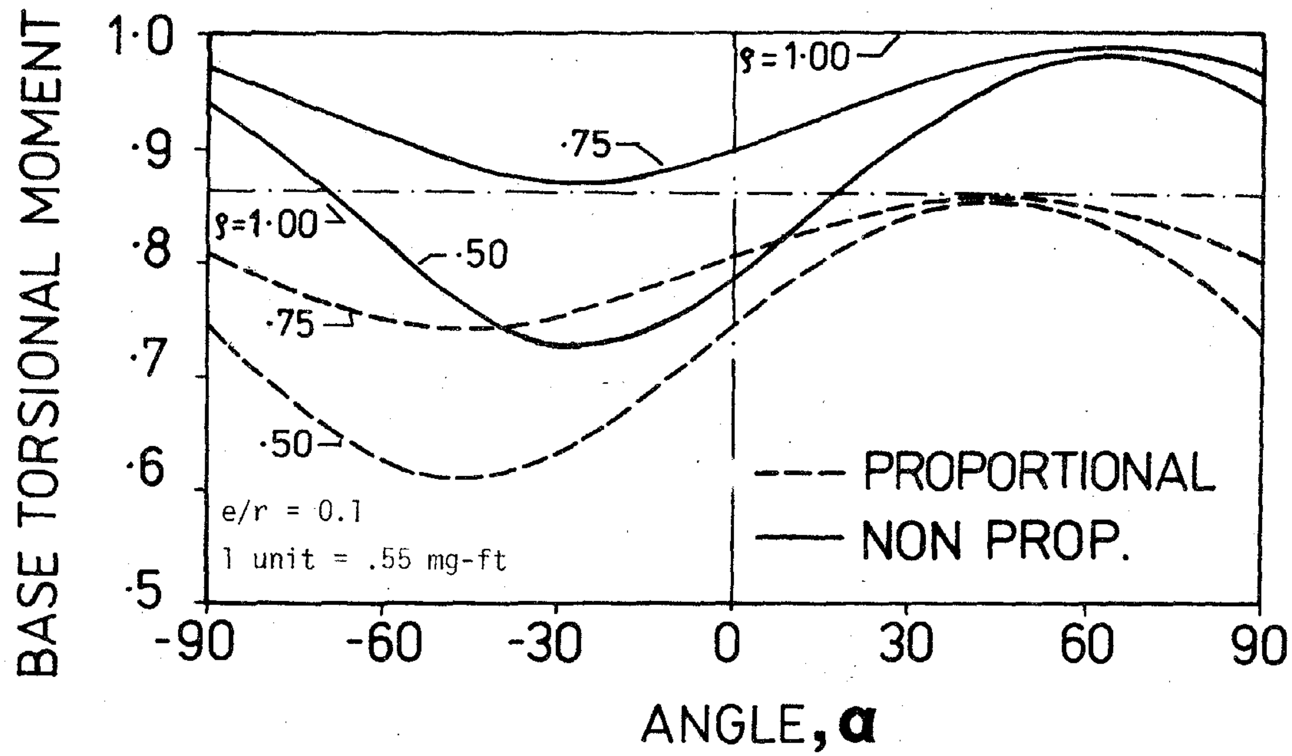


Figure 5: VARIATION OF NORMALIZED BASE TORSIONAL MOMENT FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS; e/r = 0.10.

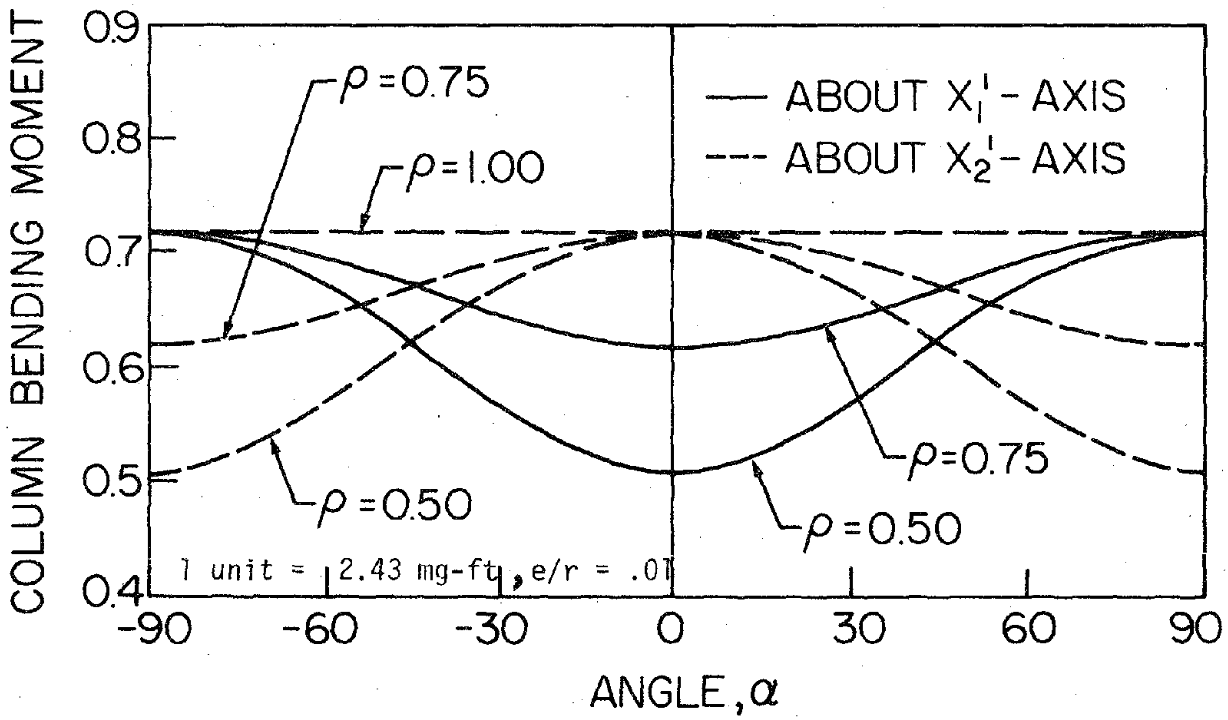


Figure 6: VARIATION OF COLUMN BENDING MOMENTS FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS FOR PROPORTIONAL DAMPING: e/r = .01.

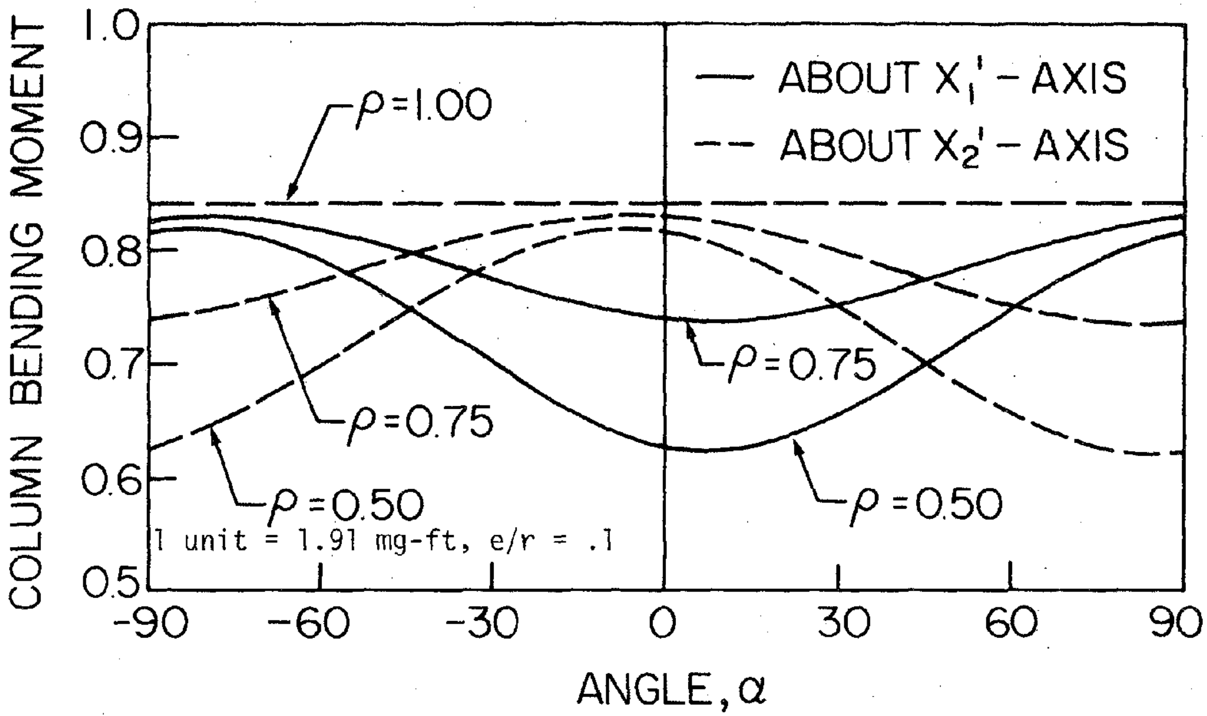


Figure 7: VARIATION OF COLUMN BENDING MOMENTS FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS FOR PROPORTIONAL DAMPING; e/r = .10.

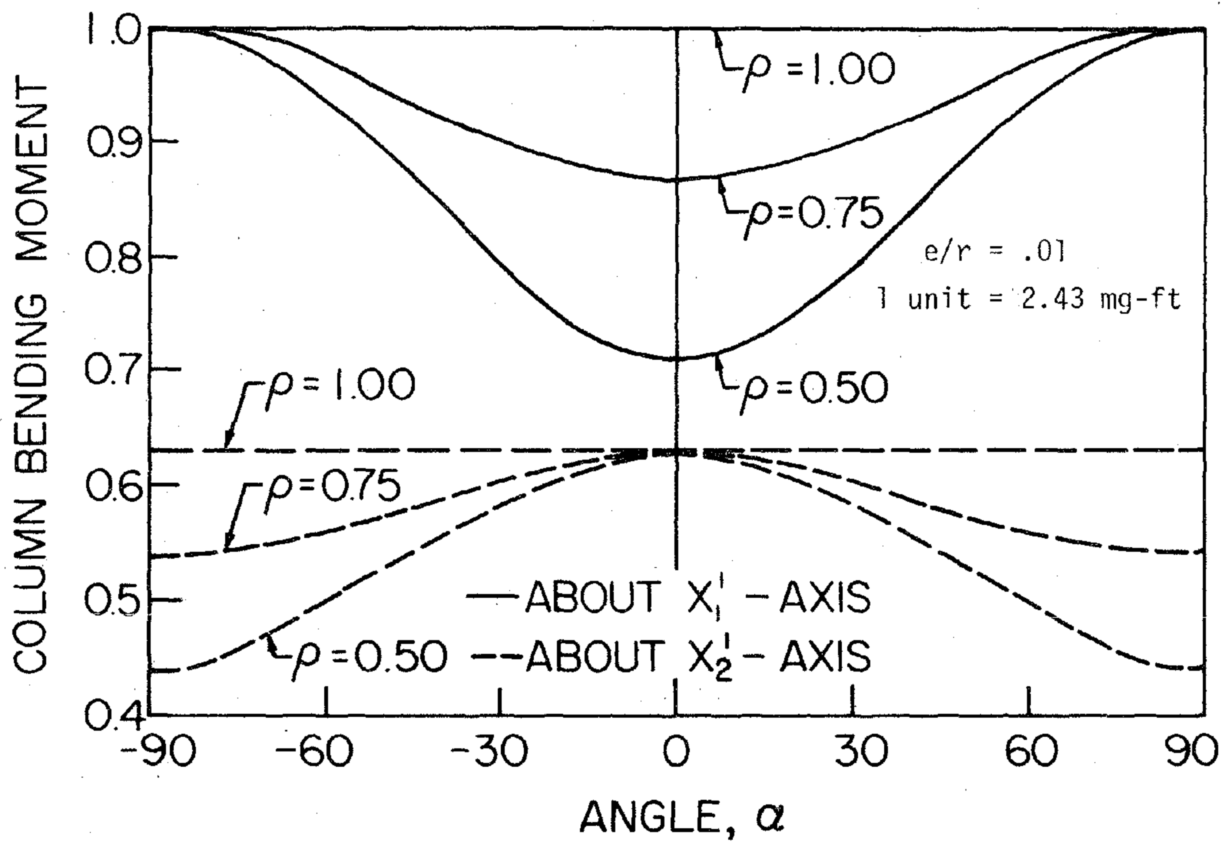


Figure 8: VARIATION OF COLUMN BENDING MOMENTS FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS FOR NONPROPORTIONAL DAMPING; e/r = .01.

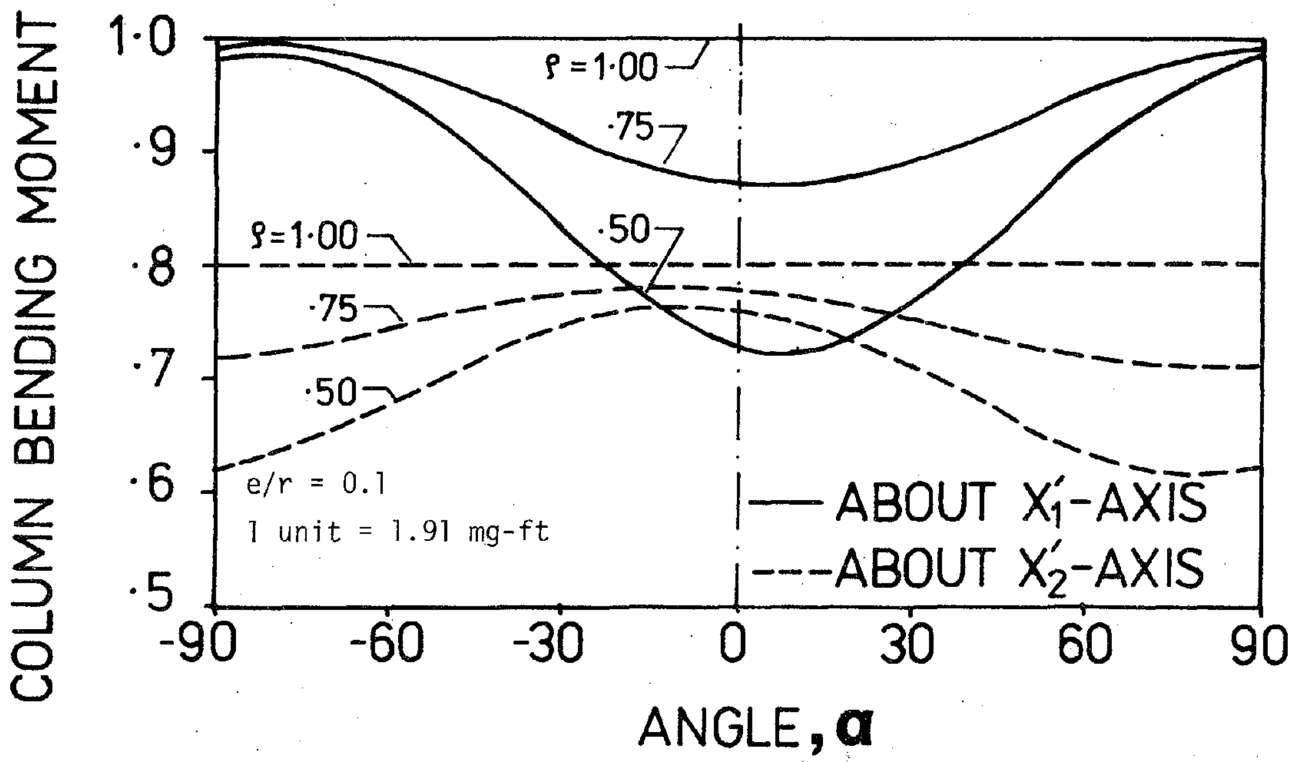


Figure 9: VARIATION OF COLUMN BENDING MOMENTS FOR VARIOUS ANGULAR ORIENTATIONS OF STRUCTURE WITH RESPECT TO EXCITATION COMPONENTS FOR NONPROPORTIONAL DAMPING; $e/r = .10$.

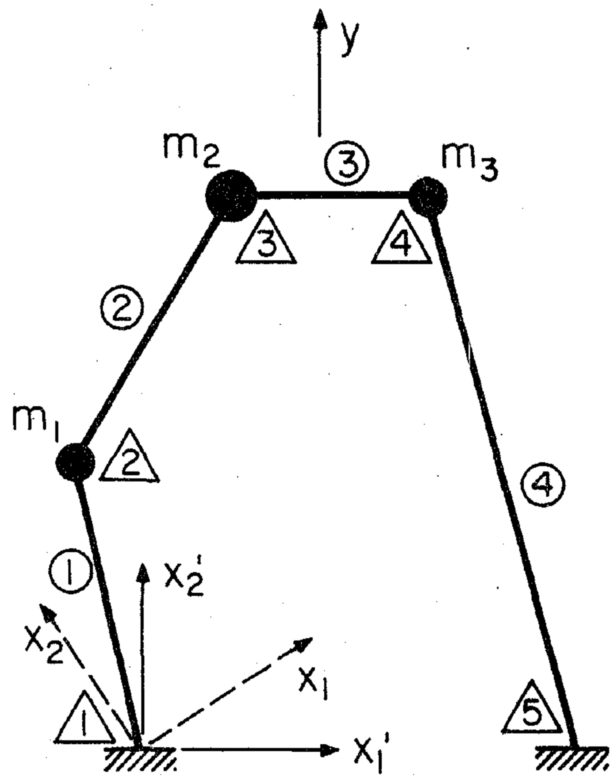


Figure 10: ANALYTICAL MODEL OF A PLANE FRAME WITH SIX DEGREES-OF-FREEDOM.

APPENDIX I

Evaluation of Design Response - The Response Spectrum Approach

To obtain the expressions for the design response the root mean square values, obtained from either Eq. 6 for proportional damping or Eq. 17 for nonproportional damping, must be multiplied by an appropriate value of the peak factor. The peak factor will of course depend upon the characteristics of random response S(t) and the desired probability of exceedence. Several approximate but reasonable methods are available to evaluate these. See for example Vanmarcke (20).

To obtain the expressions for design response in terms of design inputs defined as ground response spectra for the three components, the approach used in References 17 can be used. This approach forms the basis for the derivation of Eqs. 8 and 20. For the sake of completeness and ready reference, these derivations are given as follows for proportional and nonproportional damping cases:

Design Response for Proportionally Damped Systems: Expanding Eq. 6, the mean square value of S can be written as follows:

$$Ex[S^2] = \sum_{j}^{N} \sum_{k}^{N} \zeta_j \zeta_k \sum_{\ell=1}^{3} \left(\sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} \gamma_{mj} \gamma_{nk} \int_{-\infty}^{\infty} \Phi_{\ell}(\omega) H_j(\omega) H_k^*(\omega) d\omega \right) \qquad (I.1)$$

Further separating terms with j=k and j≠k and using the notation

$$\Gamma_{\ell jk} = \sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} \gamma_{mn} \gamma_{nk} \qquad (I.2)$$

The following is obtained:

$$Ex[S^2] = \sum_{j=1}^{N} \zeta_j^2 \sum_{\ell=1}^{3} \Gamma_{\ell jj} \int_{-\infty}^{\infty} \Phi_\ell(\omega)|H_j(\omega)|^2 d\omega$$

$$+ 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \zeta_j \zeta_k \sum_{\ell=1}^{3} \Gamma_{\ell jk} \int_{-\infty}^{\infty} \Phi_\ell(\omega) N(\omega)|H_j(\omega)|^2 |H_k(\omega)|^2 d\omega \tag{I.3}$$

where

$$N(\omega) = \omega^4 - \omega^2(\omega_j^2 + \omega_k^2 - 4\beta_j \beta_k \omega_j \omega_k) + \omega_j^2 \omega_k^2 \tag{I.4}$$

The double summation terms in I.3 represent the interaction of modes, and must be considered when the modes are closely spaced as well as when the effect of modes with frequency higher than forcing function frequency are required to be included correctly (18,19). The input independent expressions for the modal correlation coefficients have been obtained by assuming that $\Phi_\ell(\omega)$ is a white noise. The use of such correlation coefficient in the evaluation of modal correlation in expressions such as Eq. I.3, though, may be acceptable in some cases, can also cause serious errors especially when the high frequency modes are significant (19). It is thus a best practice to obtain the correlation coefficient as a function of the input. This can be done as in Ref. 15, 16 for single excitation component. For the sake of ready reference, this approach is repeated here for the case of multicomponent excitations.

The frequency integrand in the second term of Eq. I.3 can be expressed in terms of partial fractions as follows:

$$N(\omega)|H_j(\omega)|^2|H_k(\omega)|^2 = (A'+B'\omega^2)|H_j(\omega)|^2 + (C'+D'\omega^2)|H_k(\omega)|^2 \tag{I.5}$$

where the coefficients of partial fractions A', B', C' and D' are obtained from the solution of the following simultaneous equations:

$$[Z][A'\} = \{W'\} \tag{I.6}$$

where the elements of [Z], {A'} and {W'} are

$$
\begin{aligned}
&z(1,1) = z(1,3) = z(4,2) = z(4,4) = 0.;\\
&z(1,2) = z(1,4) = z(2,1) = z(2,3) = 1.0.;\\
&z(2,2) = z(3,1) = -2w_k^2(1-2\beta_k^2);\\
&z(2,4) = z(3,3) = -2w_j^2(1-2\beta_j^2);\\
&z(3,2) = z(4,1) = w_k^4;\\
&z(3,4) = z(4,3) = w_j^4 \qquad\qquad (I.7)\\
&\{A'\}^T = (A',B',C',D')\\
&w'(1) = 0., \ w'(2) = 1.\\
&w'(3) = -w_j^2 - w_k^2 + 4\beta_j\beta_k\omega_j\omega_k; \ \omega'(4) = \omega_j^2\omega_k^2
\end{aligned}
$$

To express the frequency integrals in Eq. I.3 in terms of response spectra, the following expressions are used

$$C_j^2\omega_j^4 \int_{-\infty}^{\infty} \Phi_\ell(\omega)|H(\omega_j)|^2 d\omega = R_{\ell a}^2(\omega_j) \tag{I.8}$$

$$C_j^2 \int_{-\infty}^{\infty} \omega^2\Phi_\ell(\omega)|H(\omega_j)|^2 d\omega = R_{\ell v}^2(\omega_j) \tag{I.9}$$

where $R_{\ell a}(\omega_j)$ and $R_{\ell v}(\omega_j)$ are the psuedo acceleration and relative velocity response spectrum values. The integrals in Eqs. I.8 and I.9 represent the mean square values of the relative displacement and relative velocity response of an oscillator of frequency ω_j and damping β_j when it is excited by the ground motion represented by the spectral density $\Phi_\ell(\omega)$. These values when amplified by their respective peak factors give the respective response spectrum values. In Eq. I.8, these peak factors for the relative displacement (or psuedo velocity or psuedo acceleration) and relative velocity responses have been assumed to be the same, although they are likely to be slightly different from each other. Using Eqs. I.5, I.8 and I.9 in Eq. I.3, the design response S_d can be obtained as follows:

$$S_d^2 = C_s^2 \sum_{j=1}^{N} \zeta_j^2 \sum_{\ell=1}^{3} \Gamma_{\ell jj} \frac{R_{\ell d}^2(\omega_j)}{C_j^2}$$

$$+ 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \zeta_j \zeta_k \sum_{\ell=1}^{3} \Gamma_{\ell jk} \{[A'+\omega_j^2 F_\ell(\omega_j)B'] \frac{R_{\ell d}^2(\omega_j)}{C_j^2} + [C'+\omega_k^2 F_\ell(\omega_k)] \frac{R_{\ell d}^2(\omega_k)}{C_k^2}\}$$

(I.10)

where C_s = the peak factor for the response S, $R_{\ell d}(\omega_j) = R_{\ell a}(\omega_j)/\omega_j^2$ = relative displacement response and

$$F_\ell(\omega_j) = \frac{\omega_j^2 R_{\ell v}^2(\omega_j)}{R_{\ell a}^2(\omega_j)} \qquad (I.11)$$

If we make an assumption that C_j are not widely different for each mode and are also the same as C_s, Eq. I.10 becomes the same as Eq. 8 in the text. That is

$$S_d^2 = \sum_{j=1}^{N} \zeta_j^2 \sum_{\ell=1}^{3} \Gamma_{\ell jj} R_{\ell d}^2(\omega_j) + 2 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \zeta_j \zeta_k \sum_{\ell=1}^{3} \Gamma_{\ell jk} R_{\ell jk} \qquad (I.12)$$

where $\Gamma_{\ell jj}$ and $R_{\ell jk}$ are as defined by Eq. 9 in the text.

The assumption of equality of peak factor is really not very critical and is quite acceptable. Usually there are just a few modes in a cluster which contribute most to a response. If the frequencies of these modes are not very different, their peak factors wil also not be very different unless they have very different damping ratios. Likewise the effective frequency of the response S(t) will also be in the range of the frequencies of these modes. A peak factor magnitude is also affected by bandwidth and some other characteristics of a response. Usually these characteristics will also not be very different for the dominant modes and quantity S(t) in Eq. I.10, thus the equality of peak factors has mostly found to be acceptable. This assumption also forms a basis for the commonly used SRSS rule. The mode response combination rule in Eq. I.12 is a modified form of the SRSS rule, as applicable to multicomponent excitations.

<u>Design Response of Nonproportionally Damped Systems</u>: The derivation of Eq. 20 from Eq. 17 is similar as for the proportionally damped system. Expanding Eq. 17,

$$Ex[S^2] = \sum_{j}^{N} \sum_{k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega) \sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} G_{mj} G^*_{nk} H_j(\omega) H^*_k(\omega) d\omega \quad (I.13)$$

Substituting for G_{mj} and G^*_{nk} from Eq. 18

$$Ex(S^2) = 4 \sum_{j}^{N} \sum_{k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega) \sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} [A_{mj} A_{nk} + \omega^2 a_{mj} a_{nk}$$

$$+ i\omega(a_{mj} A_{nk} - a_{nk} A_{mj})] H_j(\omega) H^*_k(\omega) d\omega$$

$$= 4 \sum_{j}^{N} \sum_{k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega) \{U_{\ell jk} + \omega^2 V_{\ell jk} + i\omega W_{\ell jk}\} H_j(\omega) H_k(\omega) d\omega \quad (I.14)$$

where

$$U_{\ell jk} = \sum_{m}^{3} \sum_{n}^{3} d_{\ell m} d_{\ell n} A_{mj} A_{nk}$$

$$V_{\ell jk} = \sum_{m}^{3}\sum_{n}^{3} d_{\ell m} d_{\ell n} a_{mj} a_{nk} \tag{I.15}$$

$$W_{\ell jk} = \sum_{m}^{3}\sum_{n}^{3} d_{\ell m} d_{\ell n} (a_{mj}A_{nk} - a_{nk}A_{mj})$$

in which a_{mj} and A_{mj} are the elements of vectors $\{a_j\}$ and $\{A_j\}$, defined in Eq. 18 in the text.

Separating terms with $j=k$ and $j \neq k$,

$$Ex(S^2) = 4 \sum_{j}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega)(U_{\ell jj} + \omega^2 V_{\ell jj})|H_j(\omega)|^2 d\omega$$

$$+ 4 \sum_{j}^{N} \sum_{j \neq k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega) \{U_{\ell jk} + \omega^2 V_{\ell jk} + i\omega W_{\ell jk}\} H_j(\omega) H_k^*(\omega) d\omega \tag{I.16}$$

Now realizing that

$$U_{\ell jk} = U_{\ell kj}, \quad V_{\ell jk} = V_{\ell kj}$$
$$W_{\ell jk} = -W_{\ell kj} \tag{I.17}$$

The double summation terms can be written as

$$4 \sum_{j}^{N} \sum_{k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega)(U_{\ell jk} + \omega^2 V_{\ell jk} + i\omega W_{\ell jk}) H_j(\omega) H_k^*(\omega) d\omega$$

$$= 4 \sum_{j}^{N} \sum_{k}^{N} \sum_{\ell}^{3} \int_{-\infty}^{\infty} \Phi_\ell(\omega) [(U_{\ell jk} + \omega^2 V_{\ell jk}) \{H_j(\omega) H_k^*(\omega) + H_j^*(\omega) H_k(\omega)\}$$

$$+ i\omega W_{\ell jk}\{H_j(\omega)H_k^*(\omega) - H_j^*(\omega)H_k(\omega)\}]d\omega \qquad (I.18)$$

which after some simplification can be written as

$$= 8\sum_{j}^{N}\sum_{k=j+1}^{N}\sum_{\ell=1}^{3}\int_{-\infty}^{\infty}\Phi_\ell(\omega)\ N'(\omega)|H_j(\omega)|^2|H_k(\omega)|^2 d\omega \qquad (I.19)$$

where

$$N'(\omega) = V_{\ell jk}\omega^6 + \omega^4\{U_{\ell jk} - (\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k)V_{\ell jk} + 2(\beta_k\omega_k-\beta_j\omega_j)W_{\ell jk}\}$$

$$+ \omega^2\{-U_{\ell jk}(\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k) + \omega_j^2\omega_k^2V_{\ell jk} + 2W_{\ell jk}\omega_j\omega_k(\beta_j\omega_k-\beta_k\omega_j)\}$$

$$+ \omega_j^2\omega_k^2U_{\ell jk} \qquad (I.20)$$

The integrand of Eq. I.19 can be further broken into partial fractions as follows:

$$N'(\omega)|H_j(\omega)|^2|H_k(\omega)|^2 = [A''+B''\omega^2]|H_j(\omega)|^2 + [C''+D''\omega^2]|H_k(\omega)|^2 \qquad (I.21)$$

where A", B", C" and D" are obtained as a solution of the following simultaneous equations.

$$[Z]\{A''\} = \{w''\} \qquad (I.21)$$

where

$$\{A''\}^T = (A'', B'', C'', D'')$$

$$w''(1) = V_{\ell jk};$$

$$w''(2) = U_{\ell jk} - V_{\ell jk}(\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k) + 2W_{\ell jk}(\beta_k\omega_k-\beta_j\omega_j); \qquad (I.22)$$

$$w''(3) = -U_{\ell jk}(\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k) + V_{\ell jk}\omega_j^2\omega_k^2 + 2\omega_j\omega_k(\beta_j\omega_k-\beta_k\omega_j)W_{\ell jk};$$

$$w''(4) = \omega_j^2 \omega_k^2 U_{\ell k}$$

Substitution of Eqs. I.19 and 21 into I.16 and using Eqs. I.8 and I.9, the expression for the design response S_d can be written as follows:

$$S_d^2 = 4 \sum_{j=1}^{N} \sum_{\ell=1}^{3} [U_{\ell jk} + \omega_j^2 F(\omega_j) V_{\ell jj}] R_{\ell d}^2(\omega_j) \frac{C_s^2}{C_j^2}$$

$$+ 8 \sum_{j=1}^{N} \sum_{k=j+1}^{N} \sum_{\ell=1}^{3} [\{A'' + \omega_j^2 F(\omega_j) B''\} R_{\ell d}^2(\omega_j) \frac{C_s^2}{C_j^2}$$

$$+ \{C'' + \omega_j^2 F(\omega_j) D''\} R_{\ell d}^2(\omega_k) \frac{C_s^2}{C_k^2}] \qquad (I.23)$$

And if peak factors C_s and C_j are assumed to be the same,

$$S_d^2 = 4 \sum_{j=1}^{N} \sum_{\ell=1}^{3} \{U_{\ell jj} + \omega_j^2 F(\omega_j) V_{\ell jj}\} R_{\ell d}^2(\omega_j)$$

$$+ 8 \sum_{j=1}^{N} \sum_{k=j+1}^{N} [\{A'' + \omega_j^2 F(\omega_j) B''\} R_{\ell d}^2(\omega_j) + \{C'' + \omega_k^2 F(\omega_k) B''\} R_{\ell d}^2(\omega_1)] \qquad (I.24)$$

which is the same as Eq. 20 in the text.

APPENDIX II

STATIONARY MEAN SQUARE RESPONSE OF NONPROPORTIONALLY DAMPED SYSTEMS

This appendix describes the development of Eq. 17. With slight modifications, which help to simplify the algebraic manipulation involved in the derivation, the following formulation is essentially the same as described by the senior writer in Reference 16 for a single component excitation.

Considering the complex and conjugate terms as a pair, Eq. 13 of the text can be written as a summation over N terms as follows:

$$S(t) = \sum_{j=1}^{N} \int_0^t \left(\{q_j\}^T [D]^T \{E\} e^{p_j(t-\tau)} + \{q_j^*\}^T [D]^T \{E\} e^{p_j^*(t-\tau)} \right) d\tau \quad (II.1)$$

The autocorrelation function can then be written as

$$Ex[S(t_1)S(t_2)] = \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{t_1} \int_0^{t_2} \left(\{q_j\}^T e^{p_j(t_1-\tau_1)} + \{q_j^*\}^T e^{p_j^*(t_1-\tau_1)} \right)$$

$$[D]^T Ex(\{E\}\{E\}^T)[D]\left(\{q_k\} e^{p_k(t_2-\tau_2)} + \{q_k^*\} e^{p_k^*(t_2-\tau_2)}\right) d\tau_1 d\tau_2 \quad (II.2)$$

in which

$Ex(\{E\}\{E\}^T) =$

$$\begin{bmatrix} Ex[\ddot{x}_1(\tau_1)\ddot{x}_1(\tau_2)] & Ex[\ddot{x}_1(\tau_1)\ddot{x}_2(\tau_2)] & Ex[\ddot{x}_1(\tau_1)\ddot{x}_3(\tau_2)] \\ Ex[\ddot{x}_2(\tau_1)\ddot{x}_1(\tau_2)] & Ex[\ddot{x}_2(\tau_1)\ddot{x}_2(\tau_2)] & Ex[\ddot{x}_2(\tau_1)\ddot{x}_3(\tau_2)] \\ Ex[\ddot{x}_3(\tau_1)\ddot{x}_1(\tau_2)] & Ex[\ddot{x}_3(\tau_1)\ddot{x}_2(\tau_2)] & Ex[\ddot{x}_3(\tau_1)(\ddot{x}_3(\tau_2)] \end{bmatrix} \quad (II.3)$$

For stationary excitation components, the autocorrelation function terms in Eq. II.3 can be written in terms of the power spectral density functions of the excitations as follows:

$$\mathrm{Ex}[\ddot{x}_\ell(\tau_1)\ddot{x}_\ell(\tau_2)] = \int_{-\infty}^{\infty} \Phi_\ell(\omega) e^{i\omega(\tau_1-\tau_2)} d\omega \tag{II.4}$$

Furthermore, since $\ddot{x}_1$, $\ddot{x}_2$ and $\ddot{x}_3$ are along the principal excitation axes they are statistically uncorrelated. Thus

$$\mathrm{Ex}[\ddot{x}_\ell(\tau_1)\ddot{x}_m(\tau_2)] = 0 \quad \text{for} \quad \ell \neq m \tag{II.5}$$

With these the correlation function matrix in Eq. II.3 can be written as follows:

$$\mathrm{Ex}[\{E\}\{E\}^T] = \int_{-\infty}^{\infty} [\Phi] e^{i\omega(\tau_1-\tau_2)} d\omega \tag{II.6}$$

where $[\Phi]$, the matrix of spectral density functions, is as defined by Eq. 7 in the text.

Substituting Eq. II.6 in Eq. II.2 with $t_1-\tau_1 = u$, $t_2-\tau_2 = v$, we obtain:

$\mathrm{Ex}[S(t_1)S(t_2)]$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N}\int_{-\infty}^{\infty} e^{i\omega(t_1-t_2)} \left[\int_0^{t_1} (\{q_j\}^T e^{(-i\omega+p_j)u} + \{q_j^*\}^T e^{(-i\omega+p_j^*)u})du\right]$$

$$[D]^T[\Phi][D]\left[\int_0^{t_2} (\{q_j\} e^{(i\omega+p_k)v} + \{q_k^*\} e^{(i\omega+p_k^*)v})dv\right]d\omega \tag{II.7}$$

The integral term over u (and similarly, the one over v) can be written as follows:

$$\int_0^{t_1} (\{q_j\}^T e^{(-i\omega+p_j)u} + \{q_j^*\}^T e^{(-i\omega+p_j^*)u})du$$

$$= \{q_j\}^T \frac{e^{(-i\omega+p_j)u}}{(-i\omega+p_j)} + [q_j^*\}^T \frac{e^{(-i\omega+p_j^*)u}}{-i\omega+p_j^*} \Bigg|_0^{t_1} \qquad \text{(II.8)}$$

Considering the situation when a sufficient time has elapsed after the application of excitation, i.e., $t_1 \to \infty$ and $t_2 \to \infty$, the response will become stationary. In such a case, the integral in Eq. II.8 evaluated at the upper limit of $t_1 = \infty$ becomes zero because of the negative real part of p_j. Then for $t_1 \to \infty$ Eq. II.8 becomes

$$\int_0^{\infty} (\{q_j\}^T e^{(-i\omega+p_j)u} + \{q_j^*\} e^{(-i\omega+p_j^*)u})du$$

$$= \frac{1}{i\omega-p_j} \{q_j\}^T + \frac{1}{i\omega-p_j^*} \{q_j^*\}^T \qquad \text{(II.9)}$$

With $p_j = -\beta\omega_j + i\omega_j\sqrt{1-\beta_j^2}$ and $\{q_j\} = \{a_j + ib_j\}^T$ Eq. II.9 can be written in the vector form as follows:

$$= (2i\omega\{a_j\}^T + 2\{A_j\}^T)H_j(\omega) = \{G_j\}^T H_j(\omega) \qquad \text{(II.10)}$$

where $\{a_j\}$ and $\{A_j\}$ are as defined by Eq. 18 in the text. Similarly it can be shown that the integral over v can be written as

$$\int_0^{\infty} (\{q_k\}e^{(i\omega+p_k)v} + \{q_k^*\}e^{(i\omega-p_k)v})dv$$

$$= (-2i\omega\{a_k\} + 2\{A_k\})H_k^*(\omega) = \{G_k^*\}H_k^*(\omega) \qquad \text{(II.11)}$$

Substituting these integral in Eq. II.7, the stationary value of the autocorrelation function of response S(t) can be written as follows:

$$Ex[S(t_1)S(t_2)] = \sum_{j=1}^{N}\sum_{k=1}^{N}\int_{-\infty}^{\infty} \{G_j\}^T[D]^T[\phi][D]\{G_k^*\}H_j(\omega)H_k^*(\omega)e^{i\omega(t_1-t_2)} \tag{II.12}$$

For $t_1 = t_2$, this equation defines the mean square response as in Eq. 17 of the text.

APPENDIX III

This appendix shows that the second terms in Eq. 10 and 21 in the main text are always positive as they, respectively, represent the mean square values of the following response quantity.

$$S'(t) = \sum_{j=1}^{N} \zeta_j \int_0^t E_g'(\tau)h_j(t-\tau)d\tau \qquad \text{(III.1)}$$

$$S''(t) = \sum_{j=1}^{N} \int_0^t (E_g''(\tau)e^{p_j(t-\tau)} + E_g^{*''}e^{p_j^*(t-\tau)})d\tau \qquad \text{(III.2)}$$

where

$$E_g'(\tau) = \{\gamma_j\}^T[D]^T[\rho''] \{E\} \qquad \text{(III.3)}$$

and

$$E_g''(\tau) = \{q_j\}^T[D][\rho''] \{E\} \qquad \text{(III.4)}$$

and $E_g^{*''}$ is the complex conjugate of E_g'' and ρ'' is defined as

$$[\rho''] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{(1-\rho_1)/\rho_1} & 0 \\ 0 & & \sqrt{(1-\rho_2)/\rho_2} \end{bmatrix}$$

The autocorrelation function of $S'(t)$ is

$$Ex[S'(t_1)S'(t_1)] = \sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \zeta_k \int_0^{t_1} \int_0^{t_2} Ex[E_g'(\tau_1)E_g'[\tau_2]h_j(t_1-\tau_2)$$

$$h_k(t_2-\tau_2)d\tau_1 d\tau_2 \qquad \text{(III.5)}$$

where

$$Ex[E_g'(\tau_1)E_g'(\tau_2)] = \{\gamma_j\}^T[D]^T[\rho''] Ex(\{E\}\{E\}^T)[\rho''][D]\{\gamma_k\} \qquad (III.6)$$

Substituting for the autocorrelation function in $Ex(\{E\}\{E\}^T)$ in terms of spectral density functions and considering stationary response when $t_1 \to \infty$, $t_2 \to \infty$ and t_1-t_2 remain finite, the Eq. III.5 becomes

$$Ex[S'(t_1)S'(t_2)] = \sum_{j=1}^{N}\sum_{k=1}^{N} \zeta_j\zeta_k \int_{-\infty}^{\infty} \{\gamma_j\}^T[D]^T[\rho''][\Phi][\rho''][D]\{\gamma_k\} H_j(\omega)H_k^*(\omega)e^{i\omega(t_1-t_2)}d\omega \qquad (III.7)$$

$$\text{Now} \quad [\rho''][\Phi][\rho''] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1-\rho_1}{\rho_1}\Phi_2 & 0 \\ 0 & 0 & \frac{1-\rho_2}{\rho_2}\Phi_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-\rho_1)\Phi_1 & 0 \\ 0 & 0 & (1-\rho_2)\Phi_1 \end{bmatrix} = [\rho']\Phi_1(\omega) \qquad (III.8)$$

Since

$$\Phi_2 = \rho_1\Phi_1 \quad \text{and} \quad \Phi_3 = \rho_2\Phi_1$$

Thus,

$$Ex[S'(t_1)S'(t_2)] = \sum_{j=1}^{N}\sum_{k=1}^{N} \zeta_j\zeta_k\{\gamma_j\}^T[D]^T[\rho'][D]\{v_k\} \int_{-\infty}^{\infty} \Phi_g(\omega)H_j(\omega)H_k^*(\omega)e^{i\omega(t_1-t_2)}d\omega \qquad (III.9)$$

which when $t_1 = t_2$ give the mean square value, same as the second term of Eq. 10.

Similarly the autocorrelation function of S"(t) can be written as:

$Ex[S''(t_1)S''(t_2)]$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \int_0^{t_1} \int_0^{t_2} (\{q_j\}^T e^{p_j(t_1-\tau_1)} + \{q_j^*\} e^{p_j^*(t_1-t_2)})[D]^T[\rho''] Ex(\{E\}\{E\}^T)[\rho''][D]$$

$$(\{q_k\} e^{p_x(t_2-\tau_2)} + \{q_k^*\} e^{p_k^*(t_2-\tau_2)}) d\tau_1 d\tau_2 \qquad (III.10)$$

Proceeding as in Appendix II, and considering the stationary response when $t_1 \to \infty$, $t_2 \to \infty$, this correlation function can be written as

$Ex[S''(t_1)S''(t_2)$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{-\infty}^{\infty} \{G_j\}^T[D]^T[\rho''][\Phi][\rho''][D]\{G_k\} e^{i\omega(t_1-t_2)} H_j(\omega) H_k^*(\omega) d\omega \qquad (III.11)$$

Employing Eq. III.5, we obtain

$Ex[S''(t_1)S''(t_2)]$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{-\infty}^{\infty} \{G_j\}^T[D]^T[\rho'][D]\{G_k\} e^{i\omega(t_1-t_2)} \Phi_1(\omega) H_j(\omega) H_1^*(\omega) d\omega \quad (III.12)$$

For $t_1 = t_2$, this gives the mean square value of S"(t) which is the same as the second term of Eq. 21.

APPENDIX IV

CHARACTERISTICS OF PRINCIPAL RESPONSE AND PRINCIPAL RESPONSE DIRECTIONS

Here a more complete development of Eqs. 25 through 34 is provided. Also some special characteristics of the prinicpal response quantities and principal response directions are examined.

PROPORTIONALLY DAMPED SYSTEMS:

For an excitation component applied along an axis with direction cosines $\{d\} = \{d_1, d_2, d_3\}^T$ defined in the coordinate system of the structure, a response quantity S(t) can be written as follows:

$$S(t) = \sum_{j=1}^{N} \zeta_j \int_0^t \{d\}^T \{\gamma_j\} \ddot{x}_1(\tau) h_j(t-\tau) d\tau \qquad \text{(IV.1)}$$

where $\{\gamma_j\}$ = vector of participation factors for the jth mode $= \{\gamma_{1j}, \gamma_{2j}, \gamma_{3j}\}^T$. The stationary mean square response can be then written as:

$$Ex[S^2(t)] = \sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \zeta_k \int_{-\infty}^{\infty} \{d\}^T \{\gamma_j\}\{\gamma_k\}^T \{d\} \Phi_1(\omega) H_j(\omega) H_k^*(\omega) d\omega \qquad \text{(IV.2)}$$

With,

$$\int_{-\infty}^{\infty} \Phi_1(\omega) H_j(\omega) H_k^*(\omega) d\omega = e_{jk} \qquad \text{(IV.3)}$$

Eq. IV.2 becomes

$$Ex[S^2(t)] = \{d\}^t \left(\sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \zeta_k e_{jk} \begin{bmatrix} \gamma_{1j}\gamma_{1k} & \gamma_{1j}\gamma_{2k} & \gamma_{1j}\gamma_{3k} \\ \gamma_{2j}\gamma_{1k} & \gamma_{2j}\gamma_{2k} & \gamma_{2j}\gamma_{3k} \\ \gamma_{3j}\gamma_{1k} & \gamma_{3j}\gamma_{2k} & \gamma_{3j}\gamma_{3k} \end{bmatrix} \right) \{d\}^T \qquad \text{(IV.4)}$$

where the matrix in paranthesis is $[\bar{R}]$ of Eq. 26,

$$[\bar{R}] = \begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} & \bar{R}_{13} \\ \bar{R}_{21} & \bar{R}_{22} & \bar{R}_{23} \\ \bar{R}_{31} & \bar{R}_{32} & \bar{R}_{33} \end{bmatrix} \tag{IV.5}$$

where for proportionally damped systems:

$$\bar{R}_{mn} = \sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \zeta_k \gamma_{mj} \gamma_{nk} \int_{-\infty}^{\infty} \Phi_1(\omega) H_j(\omega) H_k^*(\omega) d\omega. \tag{IV.6}$$

Similarly

$$\bar{R}_{nm} = \sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \zeta_k \gamma_{nj} \gamma_{mk} \int_{-\infty}^{\infty} \Phi_1(\omega) H_j(\omega) H_k^*(\omega) d\omega \tag{IV.7}$$

Interchanging the dummy indices j and k,

$$\bar{R}_{nm} = \sum_{k=1}^{N} \sum_{j=1}^{N} \zeta_j \zeta_k \gamma_{nk} \gamma_{mk} \int_{-\infty}^{\infty} \Phi_1(\omega) H_k(\omega) H_j^*(\omega) d\omega \tag{IV.8}$$

Comparing Eqs. IV.6 and IV.8, it is seen that

$$\bar{R}_{mn} = \bar{R}_{nm}^* \tag{IV.9}$$

Thus matrix $[\bar{R}]$ is a Hermetian matrix. A further simplification, however, also shows that the imaginary parts of $\bar{R}_{mn}$ are zero. Separating terms with $j=k$ and $j \neq k$ in Eq. IV.6,

$$\bar{R}_{mn} = \sum_{j}^{N} \zeta_j^2 \gamma_{mj} \gamma_{nj} \int_{-\infty}^{\infty} \Phi_1(\omega) |H_j(\omega)|^2 d\omega$$

$$+\sum_{j}^{N}\sum_{\substack{k \\ j\neq k}}^{N} \zeta_j \zeta_k \gamma_{mj}\gamma_{nk} \int_{-\infty}^{\infty} \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \qquad (IV.10)$$

Thus the single summation terms are real. Also the integral of a typical term with $j \neq k$ can be rewritten as

$$\int_{-\infty}^{\infty} \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega = \int_{-\infty}^{\infty} \Phi_1(\omega)[N'(\omega)+iN''(\omega)]|H_j(\omega)|^2|H_k(\omega)|^2 d\omega \qquad (IV.11)$$

where

$$N'(\omega) = (\omega_j^2-\omega^2)(\omega_k^2-\omega^2) + 4\beta_j\beta_k\omega_j\omega_k\omega^2$$

$$(IV.12)$$

$$N''(\omega) = \omega\,[(\omega_j^2-\omega^2)\omega_k\beta_k - (\omega_k^2-\omega^2)\omega_j\beta_j]$$

Now since $\Phi_1(\omega)$, $|H_j(\omega)|^2$ and $|H_k(\omega)|^2$ are even function of ω, the imaginary part of the integral associated with $N''(\omega)$, which is an odd function, would be equal to zero. Thus the entire expression in Eq. IV.10 is real and also symmetric with j and k and m and n. Therefore, the matrix $[\bar{R}]$ is a real symmetric matrix. To express, $\bar{R}_{mn}$ in terms of response spectrum values, Eqs. I.8 and I.9 in Appendix I are used in Eqs. IV.10 as follows:

$$\bar{R}_{mn} = \sum_{j=1}^{N} \zeta_j^2 \gamma_{mj}\gamma_{nj}R_{1d}^2(\omega_j)/C_j^2 + \sum_{j}^{N}\sum_{\neq k}^{N} \zeta_j\zeta_k\gamma_{mj}\gamma_{nk}$$

$$\{[A' + \omega_j^2F_1(\omega_j)B']\frac{R_{1d}^2(\omega_j)}{C_j^2} + [C'+\omega_k^2F_1(\omega_k)D']\frac{R_{1d}^2(\omega_k)}{C_1^2}\} \qquad (IV.13)$$

where A', B', etc., are defined in Appendix I. If $\bar{R}_{mn}$ is associated with the evaluation of design response, then C_j's can be dropped from

Eq. IV.13, as in Eq. 38 of the text.

Nonproportionally Damped Systems:

A quadratic similar to Eq. IV.4 can also be developed for nonproportionally damped systems. For a single excitation along an axis with direction cosines (d_1,d_2,d_3), the response $S(t)$ can be written as:

$$S(t) = \sum_{j=1}^{N} \int_0^t \{d\}^T(\{q_j\}e^{p_j(t-\tau)} + \{q_j^*\}e^{p_j^*(t-\tau)})\ddot{x}_1(\tau)d\tau \quad \text{(IV.14)}$$

The stationary value of the mean square response can then be written as:

$$Ex[S^2(t)] = \sum_{j=1}^{N}\sum_{k=1}^{N} \{d\}^T(\int_{-\infty}^{\infty} \{G_j\}\{G_k^*\}^T \Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega)\{d\} \quad \text{(IV.15)}$$

or

$$Ex[S^2(t)] = \{d\}^T[\bar{R}]\{d\} \quad \text{(IV.16)}$$

where, now the elements of $\bar{R}$ are defined as

$$\bar{R}_{mn} = \sum_{j=1}^{N}\sum_{k=1}^{N} \int_{-\infty}^{\infty} G_{mj}(\omega)G_{nk}^*(\omega)\Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega, \quad m,n=1,2,3 \quad \text{(IV.17)}$$

Also

$$\bar{R}_{nm} = \sum_{j=1}^{N}\sum_{k=1}^{N} \int_{-\infty}^{\infty} G_{nj}(\omega)G_{mk}^*(\omega)\Phi_1(\omega)H_j(\omega)H_k^*(\omega)d\omega \quad \text{(IV.18)}$$

Interchanging the dummy summation indices j and k,

$$\bar{R}_{nm} = \bar{R}_{mn}^* \quad \text{(IV.19)}$$

Thus matrix $[\bar{R}]$ is Hermetian. A further inspection of these terms shows that the imaginary parts of $\bar{R}_{mn}$ are again zero.

In Eq. IV.16 separating terms with j=k and j≠k we obtain

$$\bar{R}_{mn} = \sum_{j=1}^{N} \int_{-\infty}^{\infty} G_{mj}(\omega) G^*_{nj}(\omega) \Phi(\omega) |H_j(\omega)|^2 d\omega$$

$$+ \sum_{j}^{N} \sum_{\neq k}^{N} \int_{-\infty}^{\infty} G_{mj}(\omega) G^*_{nk}(\omega) \Phi_1(\omega) H_j(\omega) H^*_k(\omega) d\omega \qquad \text{(IV.20)}$$

Substituting for $G_{mj}(\omega)$ and $G^*_{nj}(\omega)$ from Eq. 18 in the text, a single summation term can be written as:

$$\int_{-\infty}^{\infty} G_{mj}(\omega) G^*_{nj}(\omega) \Phi_1(\omega) |H_j(\omega)|^2 d\omega$$

$$= 4 \int_{-\infty}^{\infty} \{A_{mj}A_{nj} + \omega^2 a_{mj}a_{nj} + i\omega(a_{mj}A_{nj} - a_{nj}A_{mj})\} \Phi_1(\omega) |H_j(\omega)|^2 d\omega \qquad \text{(IV.21)}$$

In which again, the imaginary term is zero being the integral of odd function over a symmetric range. Similarly a typical double summation term can be written as

$$\int_{-\infty}^{\infty} G_{mj}(\omega) G^*_{nk}(\omega) \Phi_1(\omega) H_j(\omega) H^*_k(\omega) d\omega$$

$$= 4 \int_{-\infty}^{\infty} \{A_{mj}A_{nk} + \omega^2 a_{mj}a_{nk} + i\omega(a_{mj}A_{nk} - a_{nj}a_{mk})\} \Phi_1(\omega) H_j(\omega) H^*_k(\omega) d\omega$$

$$= 4 \int_{-\infty}^{\infty} (C_0\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3) \Phi_1(\omega) |H_j(\omega)|^2 |H_k(\omega)|^2 d\omega \qquad \text{(IV.22)}$$

where

$C_0 = a_{mj}a_{nk}$

$$C_1 = A_{mj}A_{nk} - a_{mj}a_{nk}(\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k) - 2(\beta_j\omega_j-\beta_k\omega_k)(a_{mn}A_{nk}-a_{nk}A_{nj})$$

$$C_2 = a_{mj}a_{nk}\omega_j^2\omega_k^2 - A_{mj}A_{nk}(\omega_j^2+\omega_k^2-4\beta_j\beta_k\omega_j\omega_k) - 2\omega_j\omega_k(\beta_k\omega_j-\beta_j\omega_k)(a_{mj}A_{nk}-a_{nk}A_{mj})$$

$$C_3 = A_{mj}A_{nk}\omega_j^2\omega_k^2 \qquad \text{(IV.23)}$$

Here again the complex terms are zero because they represent an integration of odd function over the symmetric frequency domain.

Thus $[\bar{R}]$ is a real symmetric matrix even for nonproportionally damped systems. For the purpose of using response spectrum as input, Eq. IV.22 can be further split into its partial fraction as follows:

$$\int_{-\infty}^{\infty} G_{mj}(\omega)G^*_{nk}(\omega)\Phi_1(\omega)H_j(\omega)H^*_k(\omega)d\omega$$

$$= 4\int_{-\infty}^{\infty} [\{A'''+B'''\omega^2\}|H_j(\omega)|^2 + \{C'' + D''\omega^2\}|H_k(\omega)|^2]\Phi_1(\omega)d\omega \qquad \text{(IV.24)}$$

where A''', B''', etc., are obtained as solution of the following simultaneous equations:

$$[Z]\{A'''\} = \{W'''\} \qquad \text{(IV.25)}$$

in which the elements of Z are the same as defined in Appendix I and

$$\{A'''\}^T = (A''', B''', C''', D''')$$

$$W'''(1) = C_0;\; W'''(2) = C_1 \qquad \text{(IV.26)}$$

$$W'''(3) = C_2;\; W'''(4) = C_3$$

Employing Eqs. I.8 and I.9 in Eqs. IV.21 and IV.23 R_{mn} for nonproportional systems can also be expressed in terms of response spectrum values as follows:

$$\bar{R}_{mn} = 4 \sum_{j=1}^{N} \{A_{mj}A_{nj}+\omega_j^2F(\omega_j)a_{mj}a_{nj}\}R_{1d}^2(\omega_j)/C_j^2$$

$$+ 4 \sum_{j}^{N} \sum_{\neq k}^{N} \{[A'''+\omega_j^2F(\omega_j)B'''] \frac{R_{1d}^2(\omega_j)}{C_j^2} + [C'''+\omega_k^2F(\omega_k)D'''] \frac{R_{1d}^2(\omega_k)}{C_k^2}\} \tag{IV.27}$$

Again the peak factors can be dropped from Eq. IV.27 if $\bar{R}_{mn}$ is associated with the evaluation of maximum design response.

To obtain the direction cosines (d_1,d_2,d_3) for maximum response, the stationary value of the auxiliary function given by Eq. 31 in the text is obtained. This requires:

$$\frac{\partial L(\lambda)}{\partial d_\ell} = 0, \qquad \ell = 1,2,3 \tag{IV.28}$$

This gives the following three simultaneous equations:

$$[\bar{R}]\{d\} - \lambda\{d\} = \{0\} \tag{IV.29}$$

Solution of this eigenvalue equation with the constraint of Eq. 30 provides the direction cosines which will extremize the response.

$[\bar{R}]$ has been shown to be a symmetric matrix, both for proportional as well as nonproportional systems. It is also realized that it is a positive definte matrix as the quadratic form in Eqs. IV.4 and Eqs. IV.16 represents the mean square value of a response quantity, always a positive quantity. It is zero only in the case of no excitation on the system, a trivial case. The positive definiteness of $[\bar{R}]$ ensures that the eigenvalues of Eq. IV.29 will be real and positive. Furthermore, for each distinct eigenvalue, the eigenvectors will be orthogonal. For two equal eigenvalues, there will be several orthogonal sets out of which any two can be conveniently chosen. It was also shown, Eq. 34, that

$$\lambda_i = \{d_i\}^T[\bar{R}]\{d_i\} \qquad \text{IV.30}$$

That is, each eigenvalue also represents the mean square value of the response if the excitation is applied along the direction defined by the corresponding eigenvector.

Thus to obtain the maximum response, its only necessary to form $[\bar{R}]$ and obtain its eigenvalues. The largest eigenvalue gives the maximum mean square response.

Also for multicomponent excitations these eigenvalues can be used to obtain the maximum response if the frequency characteristics of the input components are similar. For three orthogonal principal excitation components with the variance ratios of $1:\rho_1:\rho_2$, the maximum mean square response can be written as:

$$Ex(S^2) = \lambda_1 + \rho_1 \lambda_2 + \rho_2 \lambda_3 \qquad (IV.31)$$

in which λ_1, λ_2 and λ_3 are the major, intermediate and minor eigenvalues, respectively, of $[\bar{R}]$.