Analysis of the Spatial Behavior and Size of Modern Buildings for Earthquake-proof Design

M A Mardzhanishvili

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The book is aimed at researchers, post-graduates and design engineers.

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STATE COMMITTEE ON TOWN PLANNING AND ARCHITECTURE UNDER THE USSR STATE COMMITTEE FOR CONSTRUCTION
Zonal Scientific Research and Design Institute for Standard and Experimental Design of Residential and Public Buildings, Tbilisi (TbilZNIIEP)

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Analysis of the Spatial Behavior and Size of Modern Buildings for Earthquake-proof Design

[*Metodika Ucheta Prostranstvennoi Raboty i Protyazhennosti Sovremennykh Zdanii pri Raschete ikh na Seismicheskie Vozdeistviya*]

M.A. MARDZHANISHVILI

Edited by
SH.G. NAPETVARIDZE
Corresponding Member, Academy of Science Georgian SSR

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Introduction

The XXV Congress of the Communist Party of the Soviet Union presented civil engineers with the task of constructing buildings, quickly and economically, based on modern technology. To achieve this objective, "... the building materials and construction industries would have to, firstly, expand the in-plant production of building components and joints necessary for the complex and mechanized assembly of residential and public buildings in an uninterrupted manner."*

Extensive use of modern methods of constructing buildings and the increasing number of stories in urban housing systems in seismic regions have led to a qualitative change in building construction as well as new problems in engineering. The discrepancies between some assumptions used in methods of analysis and the actual behavior of buildings under seismic activity should be carefully noted.

One of the principal discrepancies arises when a complex three-dimensional structure is represented by a simplified analytical model in the form of a cantilever bar. Such a model cannot define the behavior of a building under seismic activity with a sufficient degree of reliability. In current practice, the three-dimensional analytical model of buildings is represented by the action of a grid system subjected to horizontal forces. However, this represents the reaction of the structure to only one type of force, for example, wind. However, during an earthquake, when the propagation of seismic waves results in building vibrations in all directions in space, the given analytical model cannot fully represent the true nature of structural behavior.

Research on this subject aims at the creation of a three-dimensional analytical model of frame-panel buildings which will represent real conditions during seismic activity and will be convenient for practical methods of analysis. The following problems have been examined: the theoretical basis of a three-dimensional analytical model of buildings considering the mutual behavior of horizontal (floors) and vertical (frames and diaphragms) members

*Proceedings of XXV Congress of the Communist Party of the Soviet Union, Moscow. Politizdat, 1976, p. 142.

when the model is subjected to forces in all directions; substitution of vertical and horizontal members of buildings with equivalent bars with equivalent stiffness characteristics; determination of the dynamic characteristics of the building; examination of seismic wave effect and estimates of seismic loads on a building and design forces in its members.

CHAPTER 1

Special Features of Analysis of Frame-panel Buildings for Seismic Effects

1. ANALYTICAL MODELS OF BUILDINGS

A building is represented by two analytic models: the dynamic and the static. The dynamic model gives the distribution of mass and structural deformability. It shows those properties of a building which determine its main dynamic characteristics: periods, wave forms of natural oscillations and damping characteristics. In the spectral method of analysis, the dynamic model helps to establish the seismic forces in buildings. The static model, which shows only the deformable characteristics of a building, permits determination of the state of stress in the structure. In practical design, the static model is used to redistribute seismic loads and determine forces in the building members. Many theoretical investigations are devoted to the selection of a suitable analytical model and have in turn helped to develop and improve design methods.

The frame-panel building system, in which the load-bearing and space enclosing functions are different, is a widely used building method for seismic construction. The external partitioning panels, mainly made of light weight materials, share so small a part of the load that their contribution can be ignored in practical design. Hence the design model of frame-panel buildings is represented by a grid system of vertical and horizontal members (Fig. 1). Depending on its deformation characteristics, a building design model may be two or three dimensional. In a two-dimensional model the floor is considered as an absolutely rigid disk in its own plane. In such models the load-bearing structures, which are parallel to each other in an actual building, are shown standing side by side in one plane and joined together by hinged braces at the level of each story. These braces, which simulate the role of a rigid floor, provide simultaneous displacement of vertical members (frames and diaphragms) due to the action of horizontal loads (S) (Fig. 2). The use of the method of forces results in laborious computations in the design of multistory buildings; therefore, simplified design models are often used. The simplification consists of the replacement of braces of the two upper floors and the third at the level of the ground floor (Fig. 3). The other simplification in the model consists of changeover from the discrete interlinking members to a continuous system of braces. In this case, the system of canonical equations with many unknown design parameters with one vertical row of braces may be replaced by a linear differential equation using the theory of composite bars. The solution of this linear differential equation provides readymade formulae for a cantilever bar. If there are many vertical rows, a system of differential equations must be solved.

The simplified design model of a building may be represented by a double layer cantilever beam with continuously distributed braces between them, where one layer (diaphragm) undergoes bending strain and the other (frame) shear strain (Fig. 4).

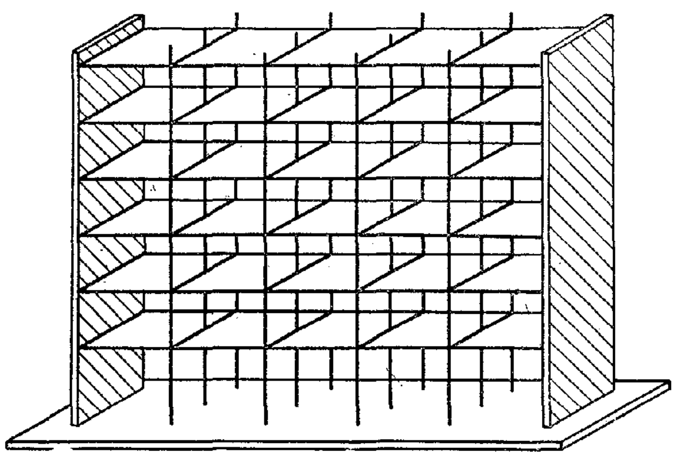


Fig. 1. General view of a frame-panel building.

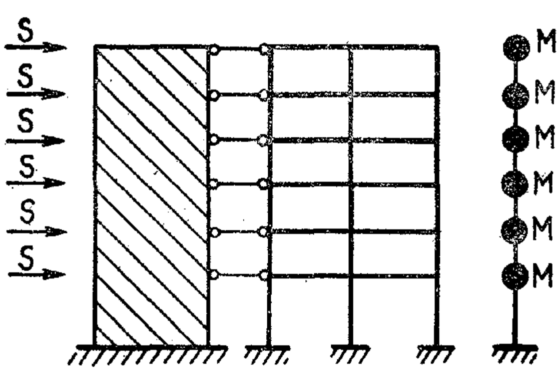


Fig. 2. A two-dimensional analytical model.

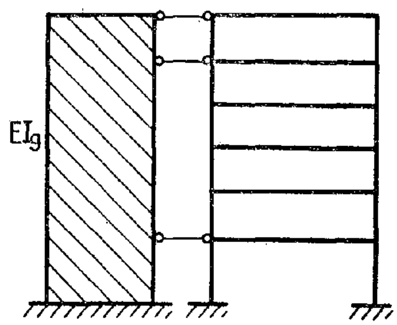


Fig. 3. A simplified two-dimensional discrete model.

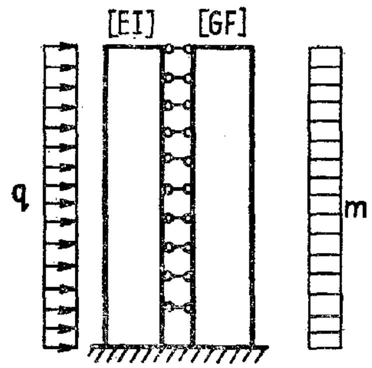


Fig. 4. A two-dimensional continuous model.

In many cases we must consider the deformation of floors because this is comparable to the deformation of vertical components. Consideration of this factor leads to a three-dimensional model in the form of a thin walled cantilever bar or a plate-like system. The three-dimensional design model in which the floor undergoes shear and bending strain may be represented by multicolumn vertical diaphragms connected at the floor level with elastic horizontal braces (Fig. 5).

A three-dimensional analytical model of a building, with floors which resist torsion, is represented by a double layer composite cantilever bar with continuous braces which yield to torsion and lateral displacement (Fig. 6). A discrete analytical model in the form of a grid system is solved, without considering the mutual torsion of floors and vertical members (Figs. 7, 8), by using the method of forces or the method of grouping the floors and frames, which, while reducing the number of unknowns and simplifying computation, determines the three-dimensional behavior of the building with sufficient accuracy.

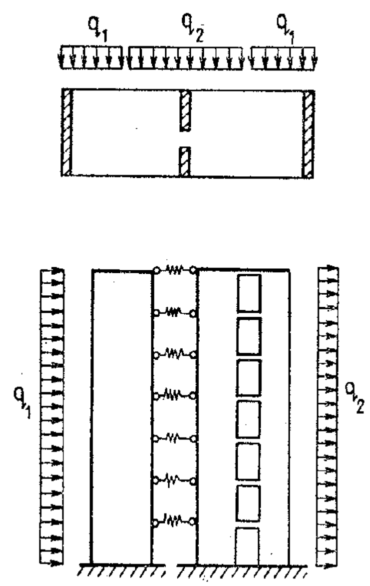


Fig. 5. A discretely continuous three-dimensional model.

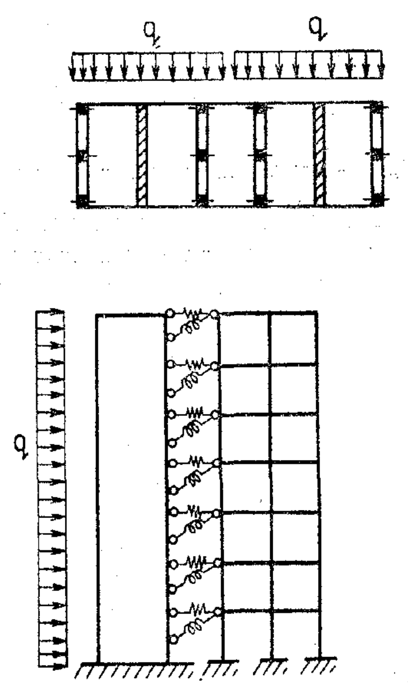


Fig. 6. A three-dimensional continuous model of a building.

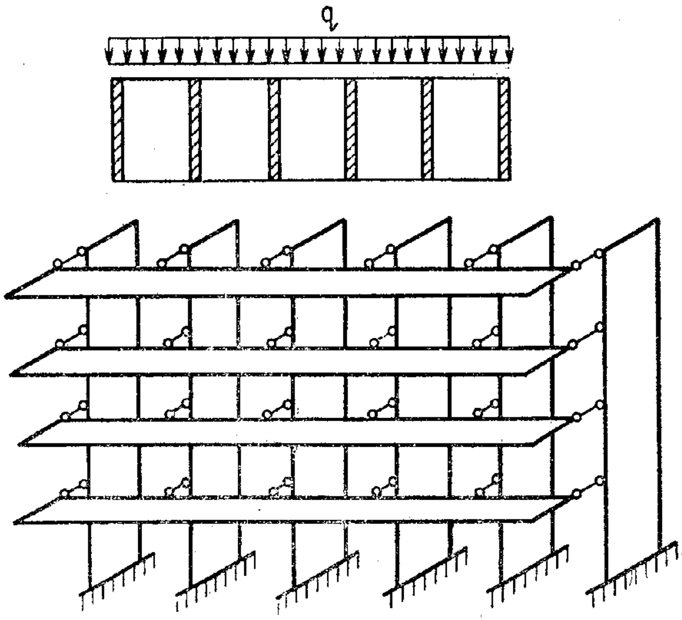


Fig. 7. A three-dimensional discrete model of a building.

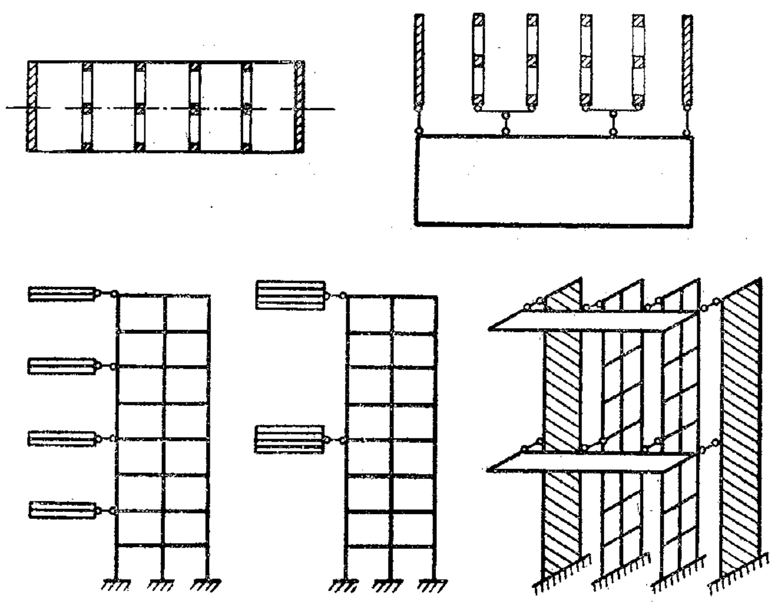


Fig. 8. A simplified discrete three-dimensional analytical model of a building.

In a discretely continuous three-dimensional model (Fig. 9) in which the rigidity of vertical frame members is uniformly distributed in the horizontal direction, we use the method of dividing the three-dimensional system into two-dimensional uniformly spaced elements. The interaction between these members is modeled by elastic supports. The design problem of a three-dimensional system leads to the design of a beam on an elastic foundation with elastic supports and the determination of displacement of vertical members and rigidities of the thrust carrying supports.

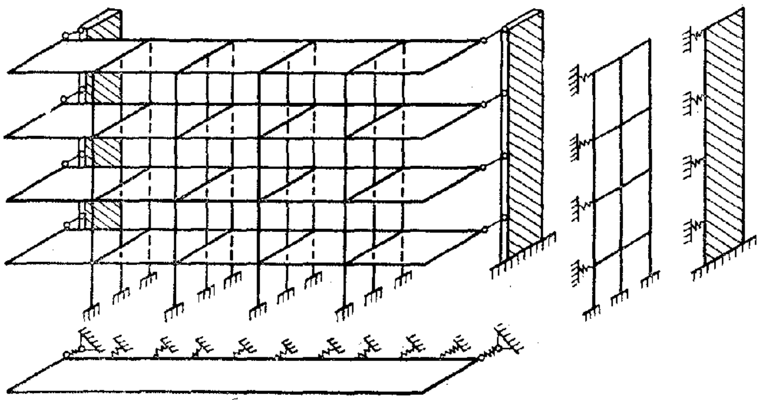


Fig. 9. A three-dimensional analytical model of a building divided into two-dimensional elements.

The three-dimensional analytical model of a building may also be represented by a prismatic shell consisting of a finite number of rectangular plates, with a cross section defined by an arbitrary broken line.

The analytical model of a building in the vertical direction also deserves attention (Fig. 10). Modern buildings are not absolutely rigid because of their precast construction and dimensions in the plan.

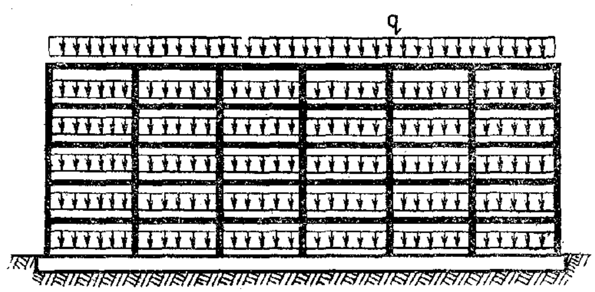


Fig. 10. General view of a building including action by vertical loads.

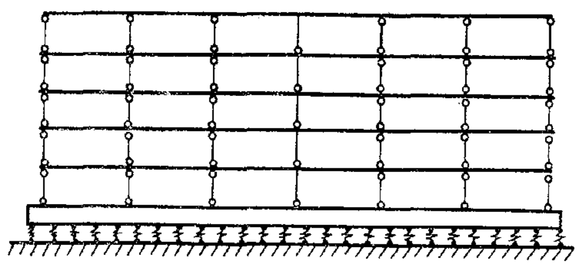


Fig. 11. A simplified analytical model of a building in the vertical direction.

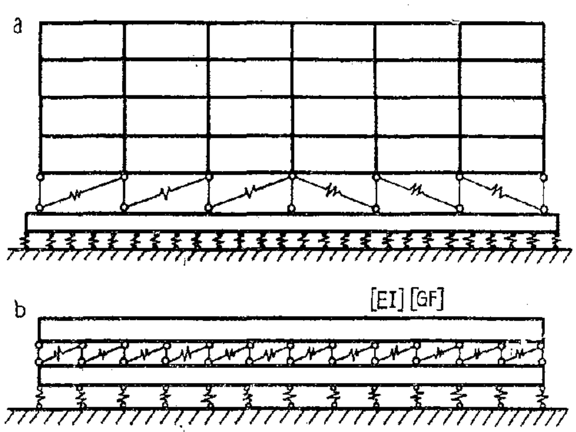


Fig. 12. A continuous design model of a building with foundation. a—foundation with superstructure; b—foundation with equivalent superstructure.

While designing framed buildings for non-uniform sinking of foundation, the analytical model of a building in the vertical direction may be represented by separate floors joined with the foundation by hinged braces in the form of columns (Fig. 11), ignoring the effect of bending moments in them due to the sinking of the foundation. Another model is in the form of a composite system consisting of two members (foundation and superstructure) joined with distributed vertical and elastic horizontal shear braces. Their role is played by the columns of the ground floor frame (Fig. 12).

2. STIFFNESS CHARACTERISTICS OF BUILDINGS

An important aspect of the seismic design of buildings is the determination of the stiffness characteristics of their load-bearing components. Such characteristics define the physical significance of a building design model. Their correct determination helps to establish the seismic loads and their distribution between the various components. Examples include the universal concepts of *axial, flexural, shear and torsional rigidities*. The concepts of *equivalent or apparent rigidities* correspond to apparent deformations of a building.

Separate bars and their joints with each other, which may be elastic, affect the magnitude of equivalent stiffness characteristics. The latter is determined by the laboratory method, that is, by engineering design as well as by static and dynamic tests. The static test consists of application of a horizontal force on the building. The dynamic test involves determination of the respective parameters in the process of oscillatory motion of the building. The vibration tests help to determine the equivalent shear and bending stiffnesses of vertical members of a building as well as floor stiffness.

3. BUILDING OSCILLATIONS

For the dynamic design model of a building in the form of a cantilever bar the following types of natural oscillations are significant: transverse, torsional and vertical. Depending on the stiffness characteristics of a bar, the transverse oscillations may be flexural, shear or flexural-cum-shear. The transverse and torsional oscillations occur separately as well as simultaneously when the centers of mass and rigidity are not coincident.

In buildings with large dimensions in the plan, for which the dynamic model is a plate fixed to the foundation, transverse oscillations occur in the horizontal and vertical planes. There are three types of natural transverse oscillations representing the nature of floor displacement in space: translatory, torsional and flexural.

During seismic activity, soil movement is conveyed to the building through the foundation which suffers horizontal and vertical displacements. In this process the building is subjected to two types of oscillations: horizontal (translatory and torsional in the lateral and longitudinal directions) and vertical. Seismic oscillations of buildings with large dimensions in plan are special. Here the seismic effect is reduced according to the spectral method of the theory of seismic stability.

4. MODERN METHODS AND TECHNIQUES OF ANALYSIS

For greater accuracy, the analysis of the building should be based on a three-dimensional model; this better reflects the behavior of the building than a one dimensional cantilever model. The computations become very laborious as the design model becomes more complex. Modern computational techniques have significantly changed the methods of solving many problems, accelerated computations and improved their accuracy. The use of computers has shown that linear algebra is more adaptable to computations. This explains the extensive use of the theory of matrices in structural mechanics. The number of equations to be solved for a real, modern building may run into a few thousand. It is therefore necessary to use special techniques to reduce the quantum of operations and the memory required to store this information in computers. For example, obtaining the differential equations of equilibrium in terms of displacements on the basis of a discrete-base system is an effective method. This method permits derivation of differential equations in much the same way as canonical equations of the algebraic type used in structural mechanics and helps to reduce two-dimensional and three-dimensional problems to one-dimensional ones. This method of obtaining a system of differential equations makes it easier to solve bar systems which have a high degree of statical indeterminancy. The changeover from an original complex system to a simpler one, that is, analysis of complex systems by subdivision into parts, is also effective.

Indeed, in practice, the analysis of multistory buildings uses approximate methods. This is justified because the prerequisites, which form the basis of analysis, are largely arbitrary. The design models of buildings are idealized and the stiffness characteristics are entirely approximate because their values significantly change due to the formation of cracks, creeping of concrete and so on. The criterion for assessing the accuracy of approximate methods of analysis is the experimental study of a real building and its model in a state of stress. The methods described in this book are based on the methods investigated by the author.

CHAPTER 2

Plane Orthogonal Bar Systems

The dimensions of a body in the form of a beam are characterized by three dimensions: two are of the same order while the third is much larger relative to the first two. From this basic feature, the size of this body, we can make many geometric hypotheses:

1. Hypothesis of plane sections, according to which the cross sections of a beam remain plane and normal to the elastic line of the beam after it bends. The bending strain of the beam is examined independently of the shear strain which distorts the plane of beam cross sections.

2. It is assumed that the distance between the longitudinal layers of a beam does not change and they do not interact with each other.

3. Only relatively rigid beams are examined in which the bending is slight, relative to the height of beam cross section and the angles of rotation of cross sections are small relative to unity.

To study the stress-strain state of a plane orthogonal system of bars under the action of coplanar forces we must study the stiffness characteristics of its constituents made up of individual bars.

1. STIFFNESS CHARACTERISTICS OF BARS

When examining the construction of a building, an individual bar may be a column, cross bar, lintel, partition and so on. As the construction is precast or monolithic, their resistance to different forms of deformations varies. Let us introduce the following concepts of stiffness characteristics of a bar:

1. The rotational end stiffnesses of bars are denoted by α^l, β, α^r where α is the reaction developing at the bar end when this end is turned through a unit angle. β is the reaction at the opposite end of the bar (Fig. 13). From such a concept of stiffness we can include the bending and shear strains simultaneously, omit defining the reactions per unit displacement of the bar end and consider different types of end fixities.

In general α^l and α^r are unequal because of different end resistances to turning caused by unsymmetric geometric and physical factors.

2. Longitudinal end stiffness of a bar is denoted by γ and is equal to the

reaction developing in the bar under unit longitudinal displacement of its end (Fig. 14).

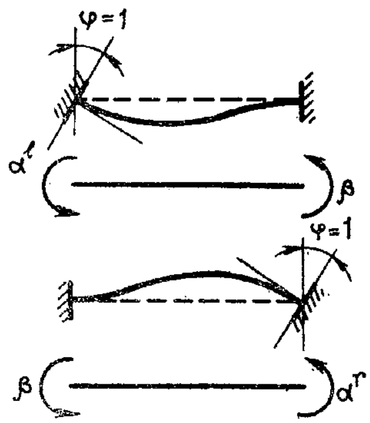


Fig. 13. Rotational end stiffness.

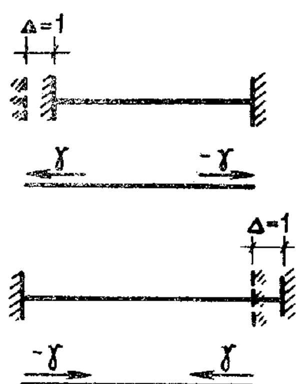


Fig. 14. Longitudinal end stiffness.

3. Torsional end stiffness of a bar is denoted by Θ. It is equal to the reaction to torsion developing in the bar when its cross section is twisted by a unit angle (Fig. 15). The reactions of a bar, fixed at the ends are given in Fig. 16 for known values of rotational end stiffnesses.

Knowing the end stiffnesses of a rod, a stiffness matrix may be formed by which the forces at both ends can be determined for given end displace-

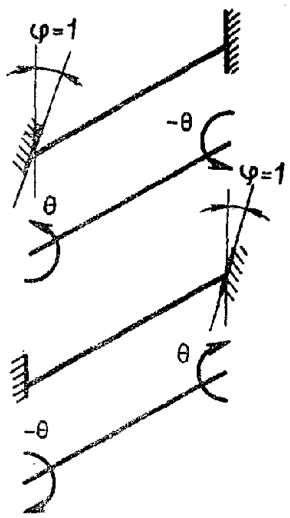


Fig. 15. Torsional end stiffness.

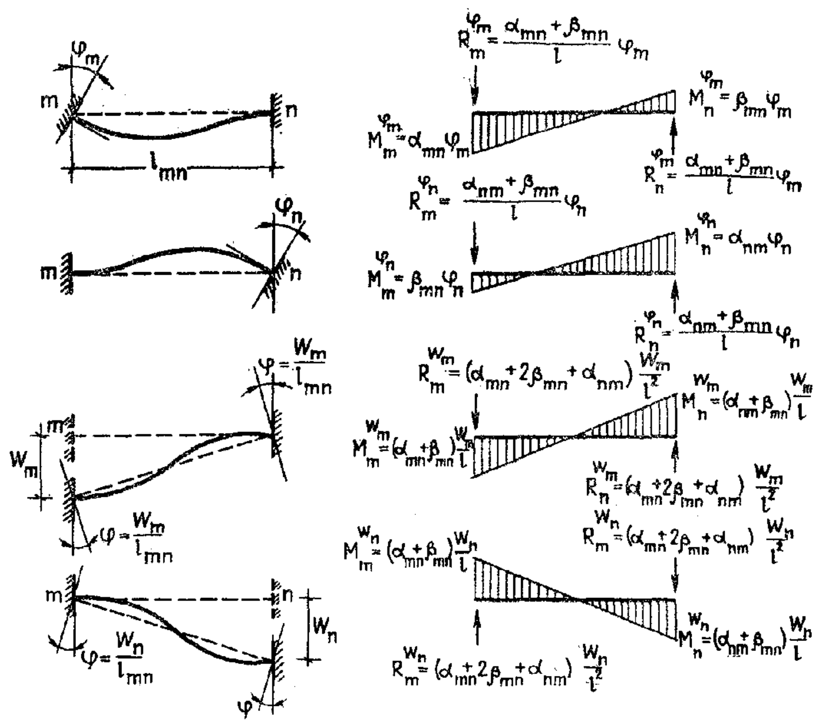


Fig. 16. End reactions on bars due to displacements.

ments. In the expanded matrix form this relationship will be:

$$\begin{Vmatrix} M_{mn} \\ M^{\text{tor.}}_{mn} \\ R_{mn} \\ N_{mn} \\ M_{nm} \\ M^{\text{tor.}}_{nm} \\ R_{nm} \\ N_{nm} \end{Vmatrix} = \begin{Vmatrix} \alpha_{mn} & A_{mn} & & & \beta_{mn} & -A_{mn} & & \\ & & \Theta_{mn} & & & & & -\Theta_{nm} \\ A_{mn} & B_{mn} & & & A_{nm} & -B_{nm} & & \\ & & & -\gamma_{mn} & & & & \gamma_{nm} \\ \beta_{mn} & A_{nm} & & & \alpha_{nm} & -A_{nm} & & \\ & & -\Theta_{mn} & & & & \Theta_{nm} & \\ -A_{mn} & -B_{nm} & & & -A_{nm} & B_{nm} & & \\ & & & \gamma_{mn} & & & & -\gamma_{nm} \end{Vmatrix} \times \begin{Vmatrix} \varphi_m \\ V_m \\ W_m \\ \varphi^{\text{tor.}}_m \\ \varphi_n \\ V_n \\ W_n \\ \varphi^{\text{tor.}}_n \end{Vmatrix}, \quad (2.1)$$

where

$$A_{mn} = \frac{\alpha_{mn} + \beta_{mn}}{l_{mn}}\,; \; A_{nm} = \frac{\alpha_{nm} + \beta_{mn}}{l_{mn}};$$

$$B_{mn} = B_{nm} = \frac{\alpha_{nm} + 2\beta_{mn} + \alpha_{nm}}{l^2_{mn}}.$$

The first column matrix is the matrix of forces, the second square matrix is the stiffness matrix for the bar while the third column matrix is the displacement matrix. Considering these matrices as block matrices, we can divide them into submatrices. Matrix (2.1) will then become:

$$\begin{Vmatrix} S_{mn} \\ S_{nm} \end{Vmatrix} = \begin{Vmatrix} k_{mn} & t_{nm} \\ t_{mn} & k_{nm} \end{Vmatrix} \times \begin{Vmatrix} z_m \\ z_n \end{Vmatrix}. \quad (2.2)$$

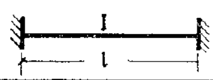
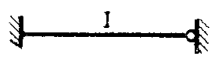
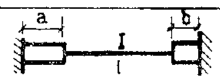
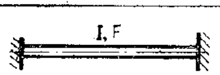
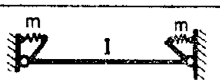
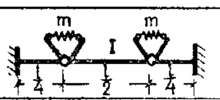
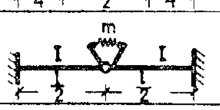
Individual bars may be of these types: bars with constant stiffness and infinitely rigid end connections, undergoing flexural and flexural-cum-shear deformations, and bars with flexible joints which yield under bending moment and longitudinal force, as in precast reinforced concrete frames.

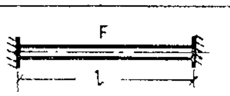
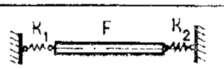
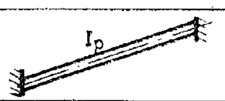
We determine the end stiffness characteristics of bars by the method of forces. Their magnitudes for various types of bars are given in Table 1.

2. ANALYSIS OF PLANE ORTHOGONAL BAR SYSTEMS

A plane orthogonal system of bars is one in which the active loads lie in one plane (Fig. 17). Let us examine a plane orthogonal system of bars in which the joints are theoretically squares of zero dimensions to which the bars are connected. The joints of the system are assumed to be elastically connected to points which are fixed in terms of horizontal, vertical and angular displacements. The connections of the bars with the joints may be rigid or elastic. These joints are subjected to the actions of external loads (Fig. 18) in addition to the support reactions transmitted to them by the loaded bars connected to them.

Table 1.

Type of rod	Value of end stiffness α^{ℓ}	Value of end stiffness β	Value of end stiffness α^{r}
	$\frac{4EI}{l}$	$\frac{2EI}{l}$	$\alpha^{r} = \alpha^{\ell}$
	$\frac{3EI}{l}$	0	0
	$\frac{4EI}{l}\left(1+\frac{3a}{l}+\frac{3a^2}{l^2}\right)$	$\frac{2EI}{l}\left(1+3\frac{a+b}{l}+\frac{6ab}{l^2}\right)$	$\frac{4EI}{l}\left(1+\frac{3b}{l}+\frac{3b^2}{l^2}\right)$
	$\frac{4EI}{l}\cdot\frac{GFl^2+3EI}{GFl^2+12EI}$	$\frac{2EI}{l}\cdot\frac{GFl^2-6EI}{GFl^2+12EI}$	$\alpha^{r} = \alpha^{\ell}$
	$\frac{3EI}{l}\cdot\frac{GFl^2}{GFl^2+3EI}$	0	0
	$\frac{4EI}{l}\cdot\frac{1+\frac{3EI}{lm}}{1+\frac{8EI}{lm}\left(1+\frac{3EI}{2lm}\right)}$	$\frac{2EI}{l}\cdot\frac{1}{1+\frac{8EI}{lm}\left(1+\frac{3EI}{2lm}\right)}$	$\alpha^{r} = \alpha^{\ell}$
	$\frac{4EI}{l}\cdot\frac{1+\frac{15EI}{8lm}}{1+\frac{7EI}{2lm}\left(1+\frac{6EI}{7lm}\right)}$	$\frac{2EI}{l}\cdot\frac{1+\frac{9EI}{4lm}}{1+\frac{7EI}{2lm}\left(1+\frac{6EI}{7lm}\right)}$	$\alpha^{r} = \alpha^{\ell}$
	$\frac{4EI}{l}\cdot\frac{1+\frac{3EI}{4lm}}{1+\frac{EI}{ml}}$	$\frac{2EI}{l}\cdot\frac{1+\frac{3EI}{2lm}}{1+\frac{EI}{lm}}$	$\alpha^{r} = \alpha^{\ell}$

	γ	θ
	$\frac{EF}{l}$	—
	$\frac{1}{\frac{l}{EF}+\frac{1}{K_1}+\frac{1}{K_2}}$	—
	—	$\frac{GI_p}{l}$

The support connections are the usual joints of the system with specific stiffness characteristics. For example, a rigid support connection has $C_\varphi = \infty$, $C_W = \infty$, $C_V = \infty$. A support which allows only horizontal displacement has $C_\varphi = \infty$, $C_W = 0$, $C_V = \infty$.

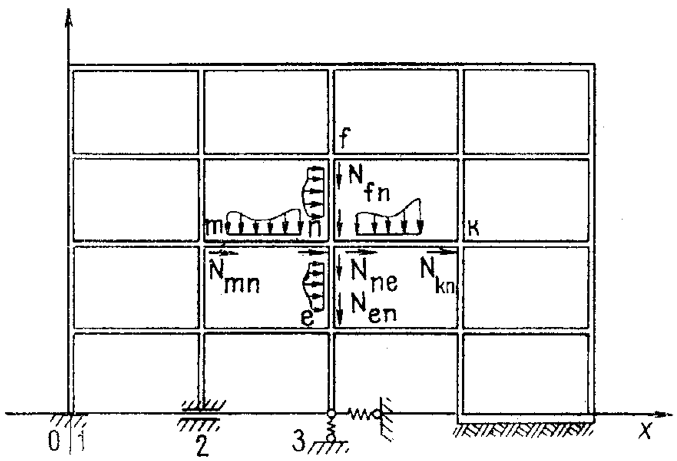


Fig. 17. Plane orthogonal system of bars.

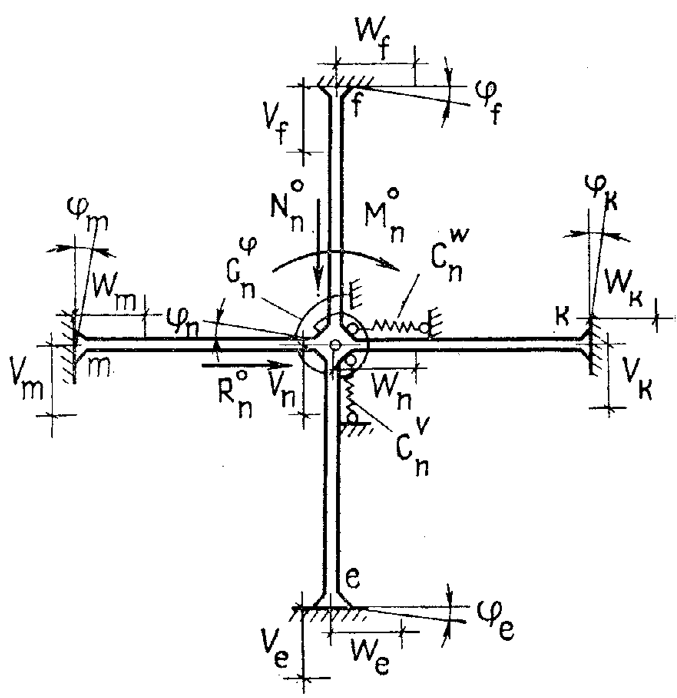


Fig. 18. A unit of the bar system.

A support which allows free rotation and elastic displacement in the horizontal and vertical directions has $C_\varphi = 0$, $C_W \neq 0$, $C_V \neq 0$. The foundation support of the system may be provided by a bar on an elastic base.

Because the system of bars under consideration follows the linear elastic law, the forces and displacements have a linear relationship. For a horizontal bar this relationship is expressed in the following matrix form:

$$\begin{Vmatrix} M_{mn} \\ R_{mn} \\ N_{mn} \\ M_{nm} \\ R_{nm} \\ N_{nm} \end{Vmatrix} = \begin{Vmatrix} \alpha_{mn} & A_{mn} & & \beta_{mn} & -A_{mn} & \\ A_{mn} & B_{mn} & & A_{nm} & -B_{nm} & \\ & & -\gamma_{mn} & & & \gamma_{nm} \\ \beta_{mn} & A_{nm} & & \alpha_{nm} & -A_{nm} & \\ -A_{mn} & -B_{mn} & & -A_{nm} & B_{nm} & \\ & & \gamma_{mn} & & & -\gamma_{nm} \end{Vmatrix} \times \begin{Vmatrix} \varphi_m \\ V_m \\ W_m \\ \varphi_n \\ V_n \\ W_n \end{Vmatrix} + \begin{Vmatrix} M^0_{mn} \\ R^0_{mn} \\ N^0_{mn} \\ M^0_{nm} \\ R^0_{nm} \\ N^0_{nm} \end{Vmatrix}. \quad (2.3)$$

For a vertical bar it is:

$$\begin{Vmatrix} M_{en} \\ N_{en} \\ R_{en} \\ M_{ne} \\ N_{ne} \\ R_{ne} \end{Vmatrix} = \begin{Vmatrix} -\alpha_{en} & & A_{en} & \beta_{en} & & -A_{en} \\ & \gamma_{en} & & & -\gamma_{ne} & \\ -A_{en} & & -B_{en} & -A_{ne} & & B_{ne} \\ \beta_{en} & & A_{ne} & \alpha_{ne} & & -A_{ne} \\ & -\gamma_{en} & & & \gamma_{ne} & \\ A_{en} & & B_{en} & A_{ne} & & B_{ne} \end{Vmatrix}$$

$$\times \begin{Vmatrix} \varphi_e \\ V_e \\ W_e \\ \varphi_n \\ V_n \\ W_n \end{Vmatrix} + \begin{Vmatrix} M^0_{en} \\ N^0_{en} \\ R^0_{en} \\ M^0_{ne} \\ N^0_{ne} \\ R^0_{ne} \end{Vmatrix}. \quad (2.4)$$

The matrices of forces and stiffnesses for horizontal and vertical bars differ due to the rearrangement of some elements. The forces in the elastic links of a joint may be expressed in the following matrix form:

$$\begin{Vmatrix} M_n \\ N_n \\ R_n \end{Vmatrix} = \begin{Vmatrix} C^\varphi_n & 0 & 0 \\ 0 & C^V_n & 0 \\ 0 & 0 & C^W_n \end{Vmatrix} \times \begin{Vmatrix} \varphi_n \\ V_n \\ W_n \end{Vmatrix} + \begin{Vmatrix} M^0_n \\ N^0_n \\ R^n_0 \end{Vmatrix}. \quad (2.5)$$

The last columns of the matrices in these three relationships are the matrices of external loads. These relationships, written for individual bars and joints, are considered the elements of a block matrix which we shall call a complete matrix.

Let us form complete matrices of forces for the bars and joints of the system, separately, by successive increments. In this case the respective complete matrices of stiffnesses of the bars and joints of the system will become quasidiagonal. The complete column matrix of displacements, for the bars constituting the system, will have repetitive elements and the joints of the system will have no repetitive elements.

Hence the relationship between the forces and displacements for a fragment of the system will be expressed by two matrices of the form:

for bars

$$\begin{Vmatrix} S_{mn} \\ S_{nm} \\ S_{nk} \\ S_{kn} \\ S_{en} \\ S_{ne} \\ S_{nf} \\ S_{fn} \end{Vmatrix} = \begin{Vmatrix} k_{mn}\, t_{nm} & & & & & & & \\ t_{nm}\, k_{nm} & & & & & & & \\ & & k_{nk}\, t_{kn} & & & & & \\ & & t_{nk}\, k_{kn} & & & & & \\ & & & & k_{en}\, t_{ne} & & & \\ & & & & t_{ne}\, k_{en} & & & \\ & & & & & & k_{nf}\, t_{fn} & \\ & & & & & & t_{nf}\, k_{fn} & \end{Vmatrix} \times \begin{Vmatrix} z_m \\ z_n \\ z_n \\ z_k \\ z_e \\ z_n \\ z_n \\ z_f \end{Vmatrix} + \begin{Vmatrix} S^0_{mn} \\ S^0_{nm} \\ S^0_{nk} \\ S^0_{kn} \\ S^0_{en} \\ S^0_{ne} \\ S^0_{nf} \\ S^0_{fn} \end{Vmatrix}; \qquad (2.6)$$

for joints

$$\begin{Vmatrix} S_m \\ S_n \\ S_k \\ S_e \\ S_f \end{Vmatrix} = \begin{Vmatrix} C_m & & & & \\ & C_n & & & \\ & & C_k & & \\ & & & C_e & \\ & & & & C_f \end{Vmatrix} \times \begin{Vmatrix} z_m \\ z_n \\ z_k \\ z_e \\ z_f \end{Vmatrix} + \begin{Vmatrix} S^0_m \\ S^0_n \\ S^0_k \\ S^0_e \\ S^0_f \end{Vmatrix}. \qquad (2.7)$$

Let us combine the complete column matrices for displacements of the bars by grouping identical elements and shifting the lower elements into the vacancies. The matrix then acquires the form of a complete column matrix for the displacement of the joints. The complete matrix of stiffnesses of bars must correspondingly change. For this its left side is multiplied by matrix $\|a\|$. Matrix $\|a\|$ is obtained from the columns in which all unknowns successively become unity.

$$\begin{Vmatrix} z_m \\ z_n \\ z_n \\ z_k \\ z_e \\ z_n \\ z_n \\ z_f \end{Vmatrix} \quad \|a\| = \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{2.8}$$

To formulate the equilibrium equations it is necessary to sum up the respective forces in all the bars framed in each joint of the system. To do so we shall use the matrix $\|D\|$ for the bar connections and unit matrix $\|E\|$ for the joint forces.

Matrix $\|D\|$ of bar connections is:

$$\|D\| = \begin{Vmatrix} \multicolumn{2}{c}{m\text{–}n} & \multicolumn{2}{c}{n\text{–}k} & \multicolumn{2}{c}{e\text{–}n} & \multicolumn{2}{c}{n\text{–}f} \\ \hline 000 & 100 & 100 & 000 & 000 & 100 & 100 & 000 \\ 000 & 010 & 010 & 000 & 000 & 010 & 010 & 000 \\ 000 & 001 & 001 & 000 & 000 & 001 & 001 & 000 \end{Vmatrix}. \tag{2.9}$$

The equilibrium equation of the system is written as:

$$\|D\|\,(\|k_{\text{bars}}\| \times \|a\| \times \|z\| + \|S^0_{\text{bars}}\|) + \|E\|\,(\|k_{\text{joints}}\| \times \|z\| + \|S^0_{\text{joints}}\|) = 0. \tag{2.10}$$

The matrix of displacements is determined from equation 2.10.

$$\|z\| = (\|a\|^{-1} \times \|k_{\text{bars}}\|^{-1} \times \|D\|^{-1} + \|k_{\text{joints}}\|^{-1} \times \|E\|) \times (\|D\| \times \|S^0_{\text{bars}}\| + \|E\| \times \|S^0_{\text{joints}}\|). \tag{2.11}$$

The complete system of analysis consists of finding all the forces in the end sections of the bars and joint connections. These are determined by the following formulae:

$$\|S_{\text{bars}}\| = \|k_{\text{bars}}\| \times \|a\| \times \|z\| + \|S^0_{\text{bars}}\|; \tag{2.12}$$

$$\|S_{\text{joints}}\| = \|k_{\text{joints}}\| \times \|z\| + \|S^0_{\text{joints}}\|. \tag{2.13}$$

The proposed computational method leads to simpler equations which simultaneously consider the conditions of fixity. As a result, all types of deformations in a bar and elastic restraining forces at its ends may be considered. The problem of frames in the form of bars on an elastic foundation may also be solved by this method. The given method introduces substantial simplification in the preparation of basic data, thus permitting automatic computation for all stages.

To derive the equilibrium equation in the expanded form for a bar system, by the displacement method, let us use matrices (2.6) and (2.7) given for a fragment of the system of the nth joint. By using the connecting matrix $\|D\|$ we can obtain the equilibrium equation of the system in the form of algebraic equations:

$$\begin{gathered}
\beta_{mn}\varphi_m + (\alpha_{nm} + \alpha_{nk} + \alpha_{ne} + \alpha_{nf} + C_n^{\varphi})\,\varphi_n + \beta_{nk}\varphi_k + \beta_{en}\varphi_e \\
+ \beta_{nf}\varphi_f + A_{nm}V_m + (-A_{nm} + A_{nk})\,V_n - A_{nk}V_k + A_{ne}W_e \\
+ (-A_{ne} + A_{nf})\,W_n - A_{nf}W_f + M_n^0 = 0; \\
A_{mn}\varphi_m + (A_{nm} - A_{nk})\,\varphi_n - A_{kn}\varphi_k + B_{mn}V_m - (B_{mn} + B_{nk} \\
+ \gamma_{en} + \gamma_{nf} + C_n^{v})\,V_n + B_{nk}V_k - \gamma_{en}V_e + \gamma_{nf}V_f + N_n^0 = 0; \\
A_{en}\varphi_e + (A_{ne} - A_{nf})\,\varphi_n - A_{fe}\varphi_f + B_{en}W_e - (\gamma_{mn} + \gamma_{nh} \\
+ B_{en} + B_{nf} + C_n^{w})\,W_n + B_{ef}W_f + \gamma_{mn}W_m + \gamma_{nk}W_k + P_n^0 = 0.
\end{gathered} \tag{2.14}$$

The first expresses the equilibrium condition by equating to zero all external moments and moments in the end sections of the bars and joints. The remaining express the equilibrium conditions by equating to zero the sum of reactions caused by transverse and longitudinal forces in the end sections of the bars and joints as well as by external forces acting along axes x and y.

3. DIVISION OF A PLANE ORTHOGONAL BAR SYSTEM INTO CONSTITUENT ELEMENTS

Let us examine a joint of a plane bar system which, as described above, is a theoretical square of negligible dimensions to which the bars are attached (Fig. 19). We assume that this square consists of two layers and we assume that the two bars joined with it in one direction also consist of two layers. By dividing the square into layers we obtain a separate joint of bars with it in the other direction. To satisfy the condition of continuity of deformation in the joint the layers of the theoretical square are assumed to be connected with braces that provide rigidity against angular rotation and horizontal and vertical displacements (Fig. 20).

The equivalent scheme (Fig. 21) may be obtained by joining the two schemes, of which the main one has braces for horizontal and vertical displacements. In the second scheme of the joint, the individual bars framing into that joint have the same stiffness as in the main scheme. Such a concept enables us to divide the joint, which results in separate groups of bar systems.

If the entire bar system is cut by a horizontal plane passing through the nodal points (points connecting the joints), an equivalent system is obtained

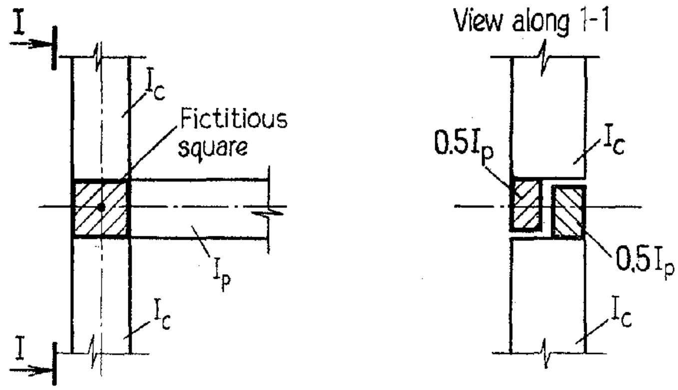


Fig. 19. A fragment of a joint.

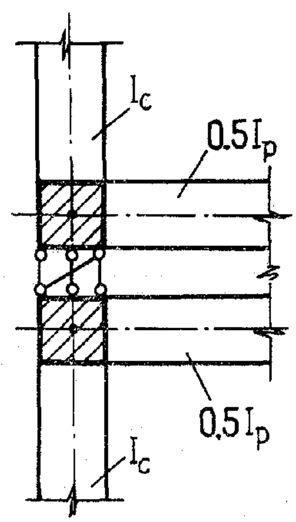


Fig. 20. A joint split up into its constituent components.

connecting all separated joints and linking to it all the vertical rods (Fig. 22).

Introduction of a rigid plane makes it possible to satisfy the hypothesis of plane sections for all the nodal points of the system lying on the section. Individually each vertical bar attached to the system of a row also satisfies the hypothesis of plane sections.

When an orthogonal bar system is cut by horizontal planes at each floor level, the system divides into separate floors and columns. Each floor is in the form of a single story closed bar system in which the horizontal bars have

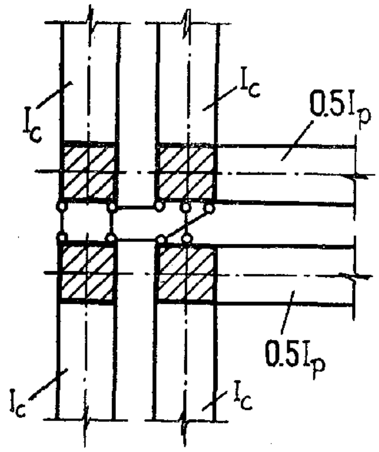


Fig. 21. Equivalent scheme of the joint.

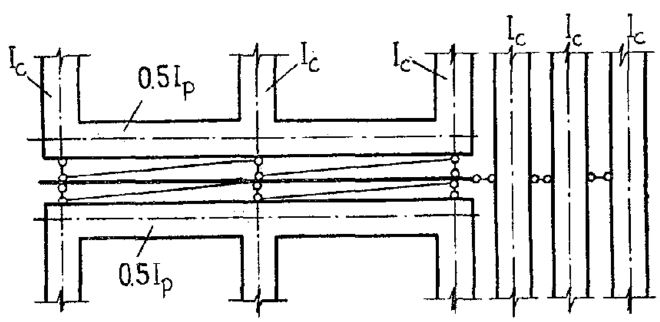


Fig. 22. Separated system.

half the stiffness while the vertical bars have the full stiffness. The columns form a joined row of a system of vertical bars (Figs. 23, 24).

Similarly the system may be cut by vertical planes passing through the joints. In this case the system breaks up into bays and beams. The bays of the system also form a single story bar system in which the vertical bars have half the stiffness. The beams form a joined row of a system of horizontal bars (Fig. 25).

4. STIFFNESS CHARACTERISTICS OF INTERFLOOR ELEMENTS AND COLUMNS

Let us examine a story located between two rigid planes to which it is connected (Fig. 26). We assume the story is a bar of uniform section which has equivalent shear and bending stiffnesses. To determine the effective shear

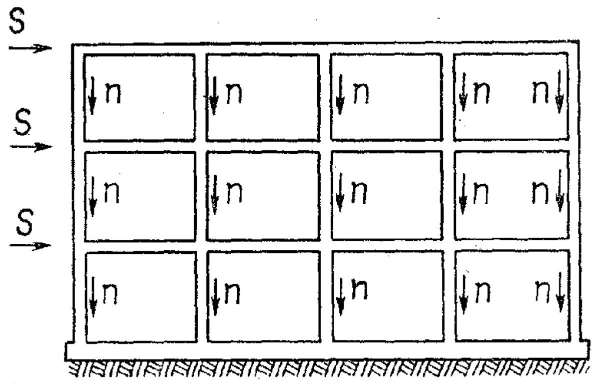


Fig. 23. A given bar system.

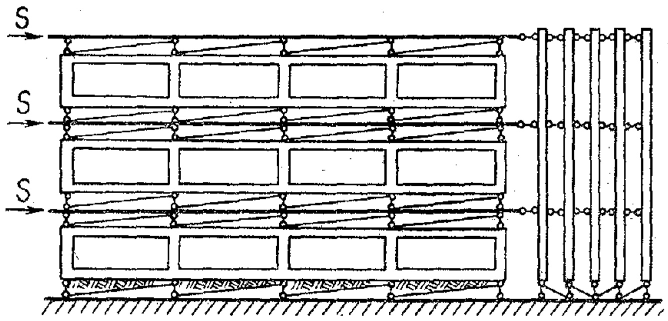


Fig. 24. A bar system divided by a horizontal plane.

stiffness of the story let us shift the upper rigid plane by $W = 1$. All joints of this story also shift by 1, thereby turning the joints (Fig. 27). A reaction $Q = \Sigma R_i$ develops in the story and its equivalent shear stiffness is determined by the formula:

$$[GF] = \frac{Qh}{W}, \tag{2.15}$$

where h is the story height.

The equivalent bending stiffness is determined as follows. Let us turn both planes around the center of rigidity of the story in opposite directions so that an angle $\varphi = 1$ is formed between them (Fig. 28). All units of the story turn and shift vertically by a known magnitude. A reactive moment of magnitude $M = \Sigma M_i + \Sigma (Nl)_i$ develops in the story and the equivalent

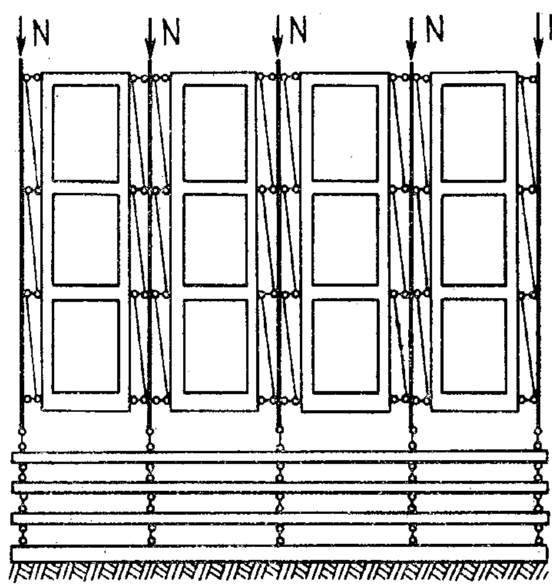


Fig. 25. A bar system divided by a vertical plane.

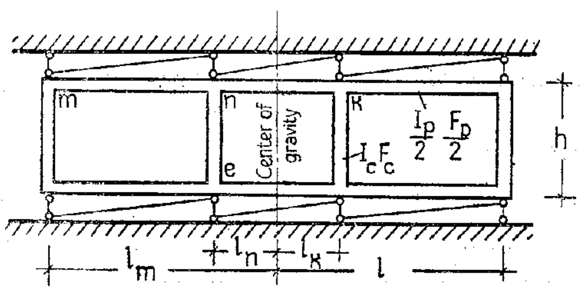


Fig. 26. A story.

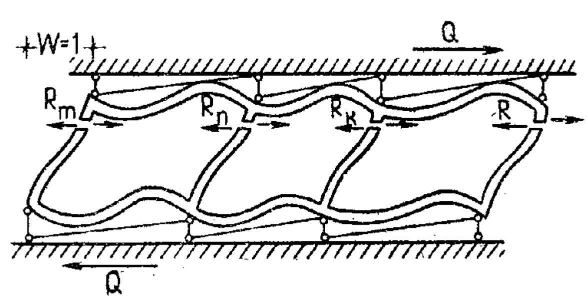


Fig. 27. Determination of shear stiffness of the story.

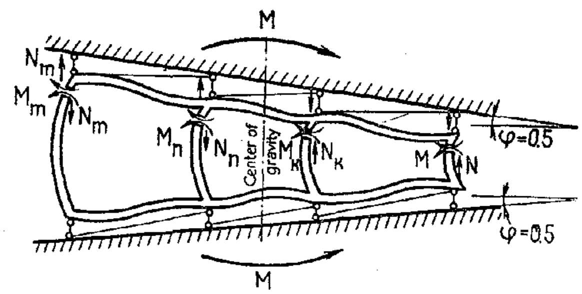


Fig. 28. Determination of bending stiffness of the story.

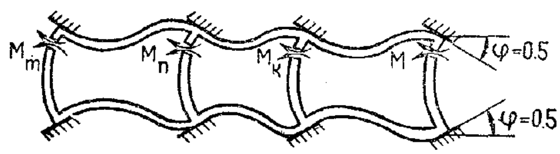


Fig. 29. Determination of bending stiffness of columns.

bending stiffness will be equal to:

$$[EI] = \frac{Mh}{\varphi}. \tag{2.16}$$

The second member of the system, the column, has bending stiffness. A column of height h may be taken to represent the story (Fig. 29). When the system becomes deformed we assume that all columns bend through the same angle. Hence they may be represented by one column, with an equivalent stiffness equal to the sum of the stiffnesses of all of them. To determine the latter, let us turn all units of the story so that an angle $\varphi = 1$ is formed between the opposite joints. Reactive moments M_i will develop at the ends of the vertical bars of the story. The total moment will be equal to $M = \Sigma M_i$ and the equivalent bending stiffness of the column can be determined by the formula:

$$[EI]_0 = \frac{h \Sigma M_i}{\varphi}. \tag{2.17}$$

Let us determine the stiffness characteristics with the help of the matrices:

$$[GF] = h\|Q\| \times \|k\| \times \|a\| \times \|z_Q\|; \tag{2.18}$$

$$[EI] = h\|M\| \times \|k\| \times \|a\| \times \|z_M\|; \tag{2.19}$$

$$[EI]_0 = h\|M_0\| \times \|k\| \times \|a\| \times \|z_{M_0}\|; \tag{2.20}$$

where $\|k\|$ is the complete stiffness matrix of the story, $\|a\|$ is the matrix corresponding to the group of displacements, $\|Q\|$ is the matrix of interconnected posts to determine the reactions in the story, $\|M\|$ is the matrix of interconnected posts to determine the moments and longitudinal forces in the story, $\|M_0\|$ is the matrix of interconnected posts to determine the moments of the column, $\|z_Q\|$ and $\|z_M\|$ are the matrices of unknown angular displacements to determine the shear and bending stiffnesses of the story, $\|z_{M_0}\|$ is the matrix of given angular rotations to determine the bending stiffness of the column.

The following equations are used to determine the matrices $\|z_Q\|$ and $\|z_M\|$:

$$\|D\| \times \|k\| \times \|a\| \times \|z_Q\| = \|D\| \times \|k\| \times \|a\| \times \|z_W\|; \tag{2.21}$$

$$\|D\| \times \|k\| \times \|a\| \times \|z_M\| = \|D\| \times \|k\| \times \|a\| \times \|z_V\|; \tag{2.22}$$

where $\|z_W\|$ is a matrix of unit horizontal displacements and $\|z_V\|$ is a displacement matrix of unit vertical displacements.

The required unknowns $\|z_Q\|$ and $\|z_M\|$ are determined from the following formulae:

$$\begin{aligned} \|z_Q\| &= (\|a\|^{-1} \times \|k\|^{-1} \times \|D\|^{-1}) \times (\|D\| \times \|k\| \times \|a\| \times \|z_W\|); \\ \|z_M\| &= (\|a\|^{-1} \times \|k\|^{-1} \times \|D\|^{-1}) \times (\|D\| \times \|k\| \times \|a\| \times \|z_V\|). \end{aligned} \tag{2.23}$$

To determine the stiffness characteristics of a story in the given system, the equilibrium equations are used in the expanded form:

$$\begin{aligned} &\beta_{mn}\varphi_m + (\alpha_{nm} + \alpha_{nk} + \alpha_{ne})\varphi_n + \beta_{nk}\varphi_k + \beta_{en}\varphi_e + A_{nm}V_m \\ &\quad - (A_{nm} - A_{nk})V_n - A_{nk}V_k + A_{ne}W_e - A_{ne}W_n = 0; \\ &A_{mn}\varphi_m + (A_{nm} - A_{nk})\varphi_n - A_{kn}\varphi_k + B_{mn}V_m - (B_{mn} + B_{nk} \\ &\quad + \gamma_{en})V_n + B_{nk}V_k + \gamma_{en}V_e = 0; \\ &\sum_{i=1}^{n} (A^{\mathrm{H}}_{\mathrm{bar}}\varphi_{\mathrm{H}})_i + \sum_{i=1}^{n} (A^{b}_{\mathrm{bar}}\varphi_b)_i + \sum_{i=1}^{n} (B_{\mathrm{bar}})_i\, W_{\mathrm{H}} - \sum_{i=1}^{n} (B_{\mathrm{bar}})_i\, W^b = 0, \end{aligned} \tag{2.24}$$

where α, β, γ are the end stiffnesses of the bars.

5. DERIVATION OF DIFFERENTIAL EQUATION FOR A TWO-LAYER BAR

An orthogonal plane bar system (Fig. 30) may be represented by a two-layer bar. The first layer of this bar is built up along its height by the interfloor elements while the second is a column consisting of individual vertical members (Fig. 31). The layers are joined by discrete rigid horizontal braces. Such a concept significantly simplifies the analysis of an orthogonal bar system.

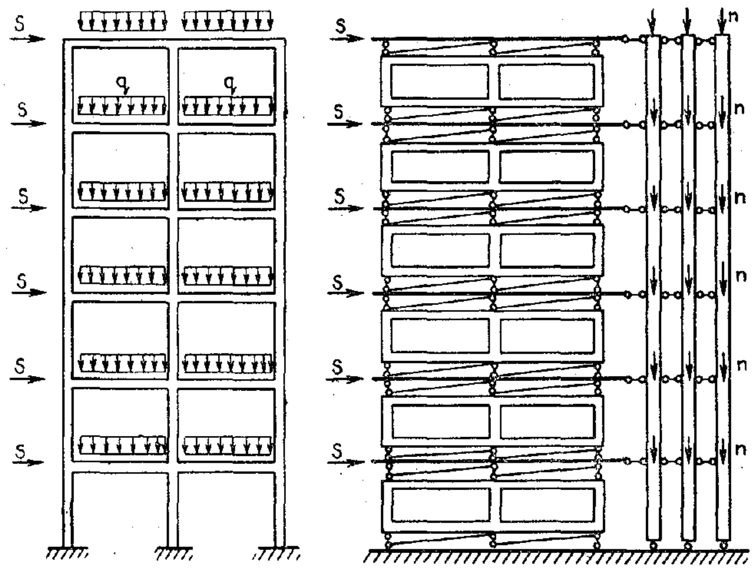


Fig. 30. A given framed system and the equivalent bar system separated into stories and columns.

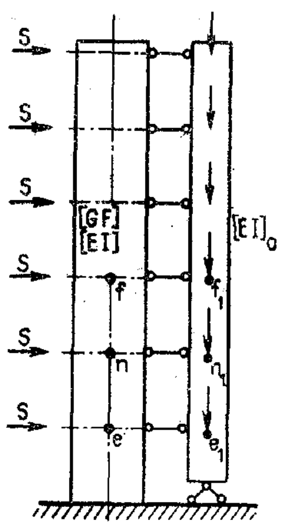


Fig. 31. A two-layer bar equivalent to the framed system.

When studying plane orthogonal systems with many horizontal and vertical members, the number of equations becomes very large in practical situations. If the number of horizontal members is represented by $\ll n \gg$ and the number of vertical members by $\ll m \gg$ then the number of equations is $2nm + n$, without the longitudinal deformations in the horizontal members. If the system is considered in the form of a two-layer bar, the number of equations will be $3n$. However, in this case, m equations are formed separately to determine the equivalent stiffness characteristics of a story. In many practical situations the algebraic equations of a two-layer bar become differential equations when $\frac{1}{n} \to 0$.

Let us examine a two-layer bar with discrete braces, acted upon by horizontal forces applied at its joints. In the matrix form the equilibrium equation for the joints and their neighborhood will be of the type:

$$\|D\|(\|k\| \times \|a\| \times \|z\|) + (\|k_{\text{joints}}\| \times \|E\| \times \|z\|) = \|E\| \times \|S^0_{\text{joints}}\|, \tag{2.25}$$

where the matrix of connections in the neighborhood of the joints under consideration will be

$$\|D\| = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{Vmatrix}, \tag{2.26}$$

and the matrix of forces will be of the type:

$$\|S\| = \|k\| \times \|a\| \times \|z\| = \begin{Vmatrix} M_{ne} \\ R_{ne} \\ M_{nf} \\ R_{nf} \\ M_{n_1e_1} \\ R_{n_1e_1} \\ M_{n_1f_1} \\ R_{n_1f_1} \end{Vmatrix}. \tag{2.27}$$

The matrix of displacements will be

$$\|z\| = \begin{Vmatrix} z_e \\ z_n \\ z_f \end{Vmatrix} = \begin{Vmatrix} \varphi_e \\ W_e \\ \varphi_{e1} \\ \varphi_n \\ W_n \\ \varphi_{n1} \\ \varphi_f \\ W_f \\ \varphi_{f_1} \end{Vmatrix}, \tag{2.28}$$

where $\|k\|$ is the matrix of stiffnesses of the bars in which the rotational-end stiffnesses for the first layer will be equal to:

$$\alpha = \frac{4\,[EI]}{h}\,\frac{[GF]\,h^2 + 3\,[EI]}{[GF]\,h^2 + 12\,[EI]};\quad \beta = \frac{2\,[EI]}{h}\,\frac{[GF]\,h^2 - 6\,[EI]}{[GF]\,h^2 + 12\,[EI]}.$$

For the second layer:

$$\alpha = \frac{4\,[EI]_0}{h};\quad \beta = \frac{2\,[EI]_0}{h},$$

where $[GF]$, $[EI]$, $[EI]_0$ are the equivalent shear and bending stiffnesses of a story and the bending stiffness of the column; h is the height of the story.

Let us express the displacement of point n by the function $Z_n(z)$. The respective displacements of points $\ll e \gg$ and $\ll f \gg$ will be expressed as $Z_e(z-h)$, $Z_f(z+h)$. Let us expand the functions $Z_e(z-h)$ and $Z_f(z+h)$ in the Taylor series in the neighborhood of point n. Then the matrix of displacement $\|z\|$ will become:

$$\|z\| = \begin{Vmatrix} Z_e(z-h) \\ Z_n(z) \\ Z_f(z+h) \end{Vmatrix} = \left(\|a_1\| + \|a_2\| \times \begin{Vmatrix} h, & \frac{h^3}{3!}, & \frac{h^5}{5!} & \ldots \end{Vmatrix} \right.$$

$$\left. \times \begin{Vmatrix} \frac{d}{dz} \\ \frac{d^3}{dz^3} \\ \frac{d^5}{dz^5} \\ \vdots \end{Vmatrix} + \|a_3\| \times \begin{Vmatrix} \frac{h^2}{2!}, & \frac{h^4}{4!} & \ldots \end{Vmatrix} \times \begin{Vmatrix} \frac{d^2}{dz^2} \\ \frac{d^4}{dz^4} \\ \frac{d^6}{dz^6} \\ \vdots \end{Vmatrix} \right) \times \begin{Vmatrix} \varphi_n \\ W_n \\ \varphi_{n1} \end{Vmatrix}, \tag{2.29}$$

where matrices $\|a\|$ have the following values:

$$\|a_1\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix};\ \|a_2\| = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix};\ \|a_3\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{2.30}$$

To obtain the system of differential equations of equilibrium let us substitute displacements $\|z\|$ in (2.25) and examine the limit $\frac{h}{H} \to 0$:

$$\lim_{\frac{h}{H}\to 0} [\|D\| \times (\|k\| \times \|a\|) + (\|k_{\text{joints}}\| \times \|E\|)] \times \Bigg(\|a_1\| + \|a_2\| \times \left\| h, \frac{h^3}{3!}, \frac{h^5}{5!} \cdots \right\|$$

$$\times \begin{Vmatrix} \frac{d}{dz} \\ \frac{d^3}{dz^3} \\ \frac{d^5}{dz^5} \\ \vdots \end{Vmatrix} + \|a_3\| \times \left\| \frac{h^2}{2!}, \frac{h^4}{5!}, \frac{h^6}{6!} \cdots \right\| \times \begin{Vmatrix} \frac{d^2}{dz^2} \\ \frac{d^4}{dz^4} \\ \frac{d^6}{dz^6} \\ \vdots \end{Vmatrix} \Bigg) \begin{Vmatrix} \varphi_n \\ W_n \\ \varphi_{n_1} \end{Vmatrix}$$

$$\times \frac{1}{H} = -\|E\| \times \|S^0_{\text{joints}}\| \times \frac{1}{H}, \qquad (2.31)$$

where H is the height of a two-layer bar, the matrix of stiffnesses of joints $\ll n \gg$ and $\ll n_1 \gg$ $\|k_{\text{joints}}\| = 0$ and the matrix of external loads $\|S^0_{\text{joints}}\| = \begin{Vmatrix} 0 \\ P \\ 0 \end{Vmatrix}$ where P is the horizontal nodal force.

The system of differential equations for a two-layer bar is as follows:

$$\varphi - \frac{dW}{dz} - \frac{[EI]}{[GF]} \frac{d^2\varphi}{dz^2} = 0;$$

$$-[GF]\frac{d\varphi}{dz} - 2[EI]_0 \frac{d^3\varphi_1}{dz^3} + [GF]\frac{d^2W}{dz^2} + [EI]_0 \frac{d^4W}{dz^4} = -q;$$

$$\varphi_1 - \frac{dW}{dz} = 0, \qquad (2.32)$$

where $q = \frac{P}{h}$ is a distributed load.

To determine the forces in the layers of the bar, let us use the matrix of member forces $\ll en \gg$:

$$\begin{Vmatrix} M_{en} \\ R_{en} \\ M_{ne} \\ R_{ne} \end{Vmatrix} = \begin{Vmatrix} \alpha_{en} & A_{en} & \beta_{en} & -A_{en} \\ -A_{en} & -B_{en} & -A_{ne} & B_{ne} \\ \beta_{en} & A_{ne} & \alpha_{ne} & -A_{ne} \\ A_{en} & B_{en} & A_{ne} & -B_{ne} \end{Vmatrix} \times \begin{Vmatrix} \varphi_e \\ W_e \\ \varphi_n \\ W_n \end{Vmatrix}. \qquad (2.33)$$

Substituting displacements in (2.33) we obtain the forces in the bar at point $\ll n \gg$ in the limit $h \to 0$:

$$\left\| \begin{matrix} M_n \\ R_n \end{matrix} \right\| = \lim_{h \to 0} \|D_1\| \times \|k\| \times \left(\|a_1\| + \|a_2\| \times \left\| h, \frac{h^3}{3!}, \frac{h^5}{5!} \cdots \right\| \right.$$

$$\times \left\| \begin{matrix} \frac{d}{dz} \\ \frac{d_3}{dz^3} \\ \frac{d^5}{dz^5} \\ \vdots \end{matrix} \right\| + \|a_3\| \times \left\| \frac{h^2}{2!}, \frac{h^4}{4!} \cdots \right\| \times \left. \left\| \begin{matrix} \frac{d^2}{dz^2} \\ \frac{d^4}{dz^4} \\ \frac{d^6}{dz^6} \\ \vdots \end{matrix} \right\| \right) \times \left\| \begin{matrix} \varphi_n \\ W_n \end{matrix} \right\|, \tag{2.34}$$

where the matrices have the following values:

$$\|D_1\| = 0.5 \left\| \begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{matrix} \right\|; \|a_1\| = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix} \right\|; \|a_2\| = \left\| \begin{matrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \right\|;$$

$$\|a_3\| = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \right\|.$$

In the first layer of the bar, the forces are determined by the formulae:

$$M_1 = -[EI] \frac{d\varphi}{dz}; \; R_1 = -[GF]\varphi + [GF] \frac{dW}{dz}; \tag{2.35}$$

and in the second layer by:

$$M_2 = -[EI]_0 \frac{d^2 W}{dz^2}; \; R_2 = -[EI]_0 \frac{d^3 W}{dz^3}. \tag{2.36}$$

The magnitude of φ is determined from the condition:

$$\frac{[EI]}{[GF]} \frac{d^2\varphi}{dz^2} - \varphi = \frac{dW}{dz}.$$

The boundary conditions are expressed by a system of differential equations of equilibrium for the end points obtained from (2.31), in which the matrix of joint stiffnesses $\|k_{\text{joints}}\|$ has a specific value. For the lower point of the two-layer bar it will be:

$$\|k_{\text{joints}}\| = \begin{Vmatrix} C_\varphi & 0 & 0 \\ 0 & C_W & 0 \\ 0 & 0 & C_{\varphi_1} \end{Vmatrix}, \tag{2.37}$$

where C_φ, $C_{\varphi 1}$ and C_W are the characteristic stiffnesses of the elastic connection against rotation of the first and second layers of the bar and against horizontal displacements of both layers.

The boundary conditions for the lower point of the bar are of the following type. For $z = 0$:

$$\varphi C_\varphi - [EI]\left.\frac{d\varphi}{dz}\right|_{z=0} = 0;$$

$$-[GF]\varphi + [GF]\frac{dW}{dz} - [EI]_0\frac{d^3W}{dz^3} + C_W W|_{z=0} = 0;$$

$$\varphi_1 C_{\varphi 1} - [EI]_0\left.\frac{d^2W}{dz^2}\right|_{z=0} = 0. \tag{2.38}$$

When the lower point is rigidly fixed, the stiffness characteristics will be $C_\varphi = \infty$, $C_{\varphi_1} = \infty$, $C_W = \infty$ and the boundary conditions at $z = 0$ will be

$$\varphi = 0;\ W = 0,\ \varphi_1 = 0. \tag{2.39}$$

For the upper point of the bar the stiffness characteristics will be $C_\varphi = 0$, $C_{\varphi 1} = 0$, $C_W = 0$ while the boundary conditions will be as follows:

for $z = H$:

$$\frac{d\varphi}{dz} = 0;$$

$$-[GF]\varphi + [GF]\frac{dW}{dz} - [EI]_0\frac{d^3W}{dz^3} = 0;$$

$$\frac{d^2W}{dz^2} = 0. \tag{2.40}$$

To solve the system of differential equations (2.32) let us represent the displacements by the following sum: $W = W_{\text{bend}} + W_{\text{sh}}$ where W_{bend} is the displacement due to bending strain and its differential is equal to $\varphi = \frac{dW_{\text{bend}}}{dz}$; while W_{sh} is the displacement due to shear strain.

Substituting these values in 2.32 we get two independent equations:

$$[EI]_0\frac{d^4W_{\text{sh}}}{dz^4} - \left(1 + \frac{[EI]_0}{[EI]}\right)[GF]\frac{d^2W_{\text{sh}}}{dz^2} = q; \tag{2.41}$$

$$\frac{[EI]_0[EI]}{[GF]}\frac{d^6W_{\text{bend}}}{dz^6} - ([EI]_0 + [EI])\frac{d^4W_{\text{bend}}}{dz^4} = -q. \tag{2.42}$$

Hence a two-layer bar is represented by two differential equations, one of which gives displacements caused by shear strains and the other by bending strains. The forces in the first layer of the bar are determined by the following formulae:

$$M_1 = -[EI]\frac{d^2W_{\text{bend}}}{dz^2}; \quad R_1 = [GF]\frac{dW_{\text{sh}}}{dz};$$

or

$$R_1 = -[EI]\frac{d^3W_{\text{bend}}}{dz^3}. \tag{2.43}$$

and in the second layer by:

$$M_2 = -[EI]_0\,\frac{d^2\,(W_{\text{bend}} + W_{\text{sh}})}{dz^2};$$

$$R_2 = -[EI]_0\,\frac{d^3\,(W_{\text{bend}} + W_{\text{sh}})}{dz^3}. \tag{2.44}$$

Differentiating equation (2.41) twice and adding to it (2.42) based on the condition $W = W_{\text{bend}} + W_{\text{sh}}$, we get one differential equation which can be solved as:

$$\frac{[EI]_0\,[EI]}{[GF]}\,\frac{d^6W}{dz^6} - ([EI] + [EI]_0)\,\frac{d^4W}{dz^4} - \frac{[EI]}{[GF]}\,\frac{d^2q}{dz^2} + q = 0. \tag{2.45}$$

To solve a plane orthogonal bar system it is best to use equations (2.41) and (2.42). When an orthogonal system is cut horizontally, the action of a longitudinal force produces an additional transverse force in the bar given by

$$Q = n\,(H-z)\,\frac{dW}{dz}; \tag{2.46}$$

$$q_{\text{add}} = \frac{dQ}{dz} = -n\,(H-z)\,\frac{d^2W}{dz^2} + n\,\frac{dW}{dz}, \tag{2.47}$$

where n is the uniformly distributed longitudinal load.

If the longitudinal distributed load is replaced by an equivalent concentrated force N applied at the upper end of the bar, then

$$q_{\text{add}} = -N\,\frac{d^2W}{dz^2}. \tag{2.48}$$

The total external load will be equal to:

$$q - N\,\frac{d^2W}{dz^2}. \tag{2.49}$$

Substituting this corrected value of load in equation (2.45) we obtain the differential equation for a two-layer bar as:

$$\frac{[EI][EI]_0}{[GF]}\frac{d^6W}{dz^6}-\left([EI]+[EI]_0-\frac{[EI]}{[GF]}N\right)\frac{d^4W}{dz^4}-N\frac{d^2W}{dz^4}$$

$$-\frac{[EI]}{[GF]}\frac{d^2q}{dz^2}+q=0. \tag{2.50}$$

When an orthogonal system is cut vertically we get a two-layer bar resting on an elastic Winkler foundation. In this case the total external load will be:

$$q-C_VW. \tag{2.51}$$

If (2.51) is substituted in (2.45) we get the differential equation of a two-layer bar as:

$$\frac{[EI][EI]_0}{[GF]}\frac{d^6W}{dz^6}-([EI]+[EI]_0)\frac{d^4W}{dz^4}+\frac{[EI]}{[GF]}C_V\frac{d^2W}{dz^2}$$

$$-C_VW-\frac{[EI]}{[GF]}\frac{d^2q}{dz^2}+q=0. \tag{2.52}$$

Differential equation (2.45) may be obtained in a different way by assuming that the braces between the layers of the bar are continuously distributed along the height (Fig. 32) for a system with many stories.

The first layer of the bar has bending and shear stiffnesses while the second layer has only bending stiffness. The latter may be considered a bend-

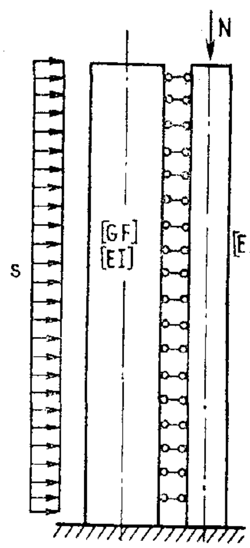


Fig. 32. Two-layer bar with continuously distributed horizontal braces.

ing base for the first layer. The horizontal load, concentrated at the joints, is transformed into a distributed load along the height of the two-layer bar. The curvature of the first layer due to shear and bending deformations will be:

$$W^{\mathrm{II}}_{\mathrm{sh}} = -\frac{1}{[GF]}\, q_0;\ W^{\mathrm{II}}_{\mathrm{bend}} = \frac{1}{[EI]}\, M. \tag{2.53}$$

The total curvature, considering both types of deformations, is expressed by their sum:

$$W^{\mathrm{II}} = -\frac{1}{[GF]}\, q_0 + \frac{1}{[EI]}\, M. \tag{2.54}$$

Differentiating this expression twice, we get

$$[EI]\, W^{\mathrm{IV}} + \frac{[EI]}{[GF]}\, q_0^{\mathrm{II}} - q_0 = 0, \tag{2.55}$$

where q_0 is the load acting on the first layer.

If the second layer is considered as a bending base for the first layer, then under the action of the external load, the value of q_0 will be:

$$q_0 = q - [EI]_0 \frac{d^4W}{dz^4}. \tag{2.56}$$

By differentiating this equation twice and substituting it in (2.55) we get differential equation (2.45).

6. EXAMPLES OF ANALYSIS OF A PLANE ORTHOGONAL BAR SYSTEM ON A COMPUTER

A two-bay six story framed building was analyzed. The cross section of columns and beams was taken as 50×50 cm. The frame span was taken as $6 + 6$ m while the floor height was taken as 3 m. The concrete grade was 200. The horizontal joint load was $S = 3$ ton. The mass of each story was $m = 5$ ton. The seismic intensity was 8 points. The analytical model of the frame with the load is shown in Fig. 33. Static and dynamic analyses for the frame were made for a common orthogonal bar system as well as for separate stories and columns.

Figure 34 shows the bending moment diagrams, displacements of story due to horizontal loads, seismic loads developed, wave form of first harmonic of natural oscillations and time periods based on the results of the frame analysis.

Figure 35 shows the analytical model of the same frame separated into stories and columns. The results are shown in Fig. 36.

A comparative analysis showed that the frame and its separated system of bars have almost identical stiffnesses and states of stress. For example, the

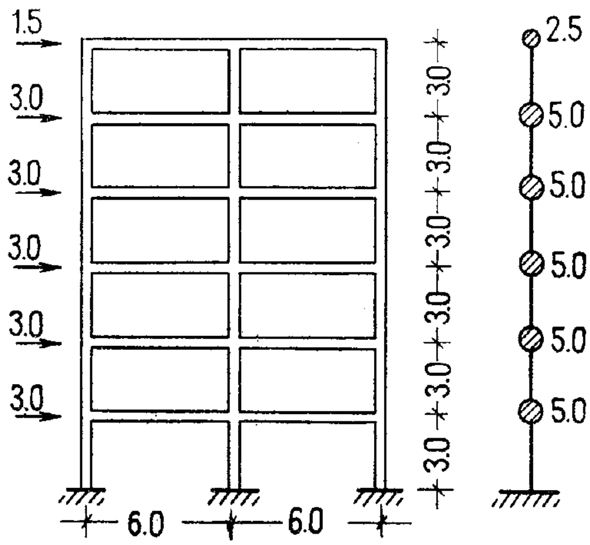


Fig. 33. Static and dynamic model of frame.

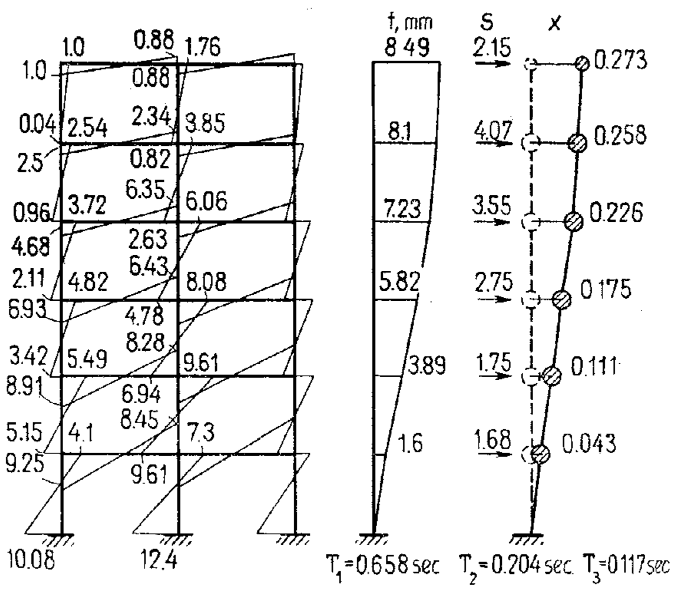


Fig. 34. Analytical results obtained on a computer.

displacement of the top of the frame is 8.49 mm while for the separated system it is 8.4 mm.

The bending moments in the transverse beam of the ground floor frame

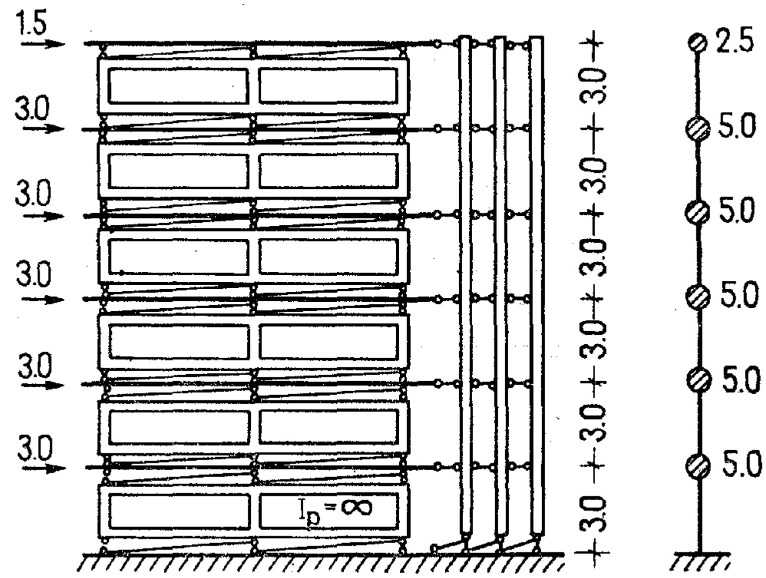


Fig. 35. Separated system of bars.

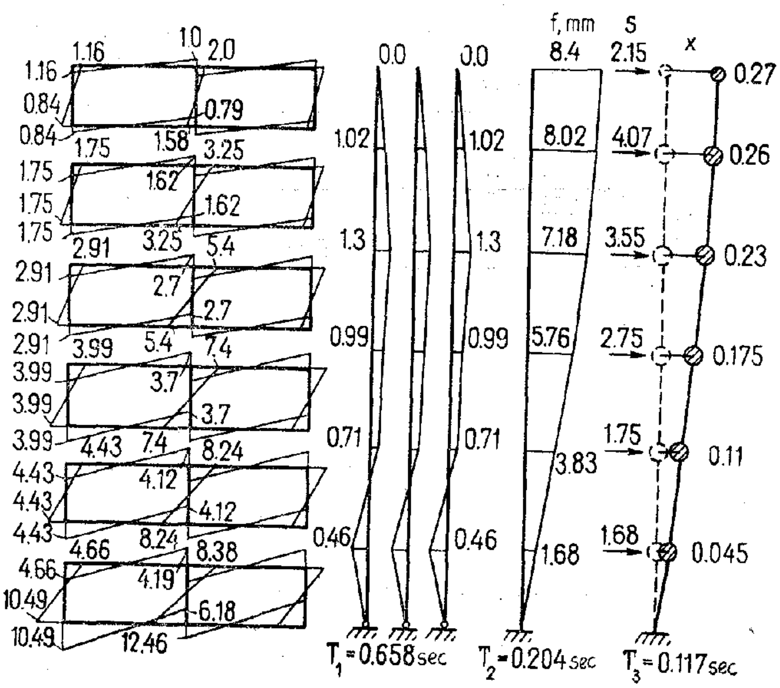


Fig. 36. Analytical results obtained on a computer.

are 9.25 and 8.45 ton-meter at its ends. For the separated system they are $4.66 + 4.43 = 9.09$ ton-meter and $4.19 + 4.12 = 8.31$ ton-meter, respectively. In the transverse beam of the third story frame they are equal to 6.93 and 6.43 ton-meter while in the separated system they are $3.99 + 2.91 = 6.9$ ton-meter and $3.7 + 2.7 = 6.4$ ton-meter, respectively.

The bending moments in the extreme column of the ground floor frame are equal to 10.08 ton-meter and 4.1 ton-meter at its ends. In the separated system they are 10.49 ton-meter and $4.66 - 0.46 = 4.2$ ton-meter, respectively; in the central column of the ground floor frame they are 12.40 and 7.30 ton-meter. In the separated system these values are 12.36 ton-meter and $8.38 - 0.46 = 7.92$ ton-meter. In the extreme column of the third floor frame they are 3.42 and 4.82 ton-meter, whereas in the separated system they are $3.99 - 0.71 = 3.28$ ton-meter and $3.99 + 0.99 = 4.98$ ton-meter; in the central column of the frame they are 6.94 and 8.08 ton-meter, while in the separated system the corresponding values are $7.4 - 0.71 = 6.7$ ton-meter and $7.4 + 0.99 = 8.39$ ton-meter. The dynamic parameters of the frame and the separated system of bars, periods and wave forms of oscillations were identical.

For the second analysis, the same frame was examined with a wider section, 200×50 cm, of the central column (Fig. 37). The increased section of the central column was used to determine the active participation of the column. The results of frame analysis are given in Fig. 38. The analytical model for the separated system of this frame and the results of its analysis are given in Figs. 39 and 40.

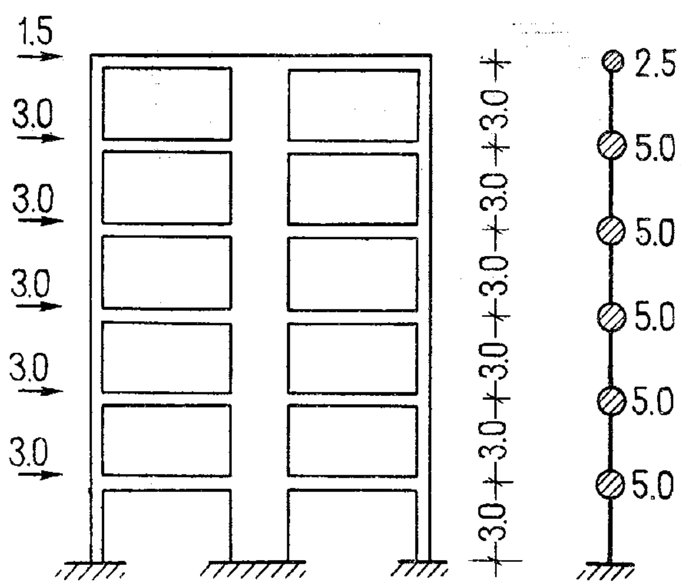


Fig. 37. Analytical model of a frame with wider central column.

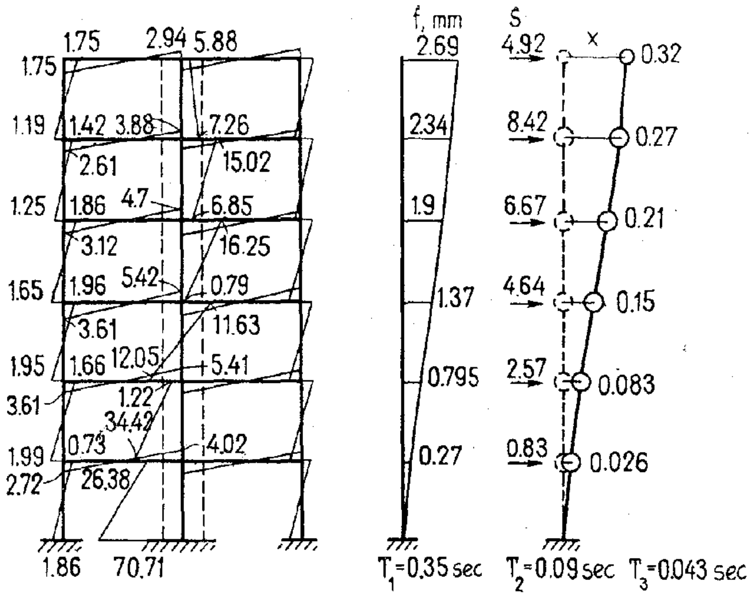


Fig. 38. Analytical results on a computer for a model with wider central column.

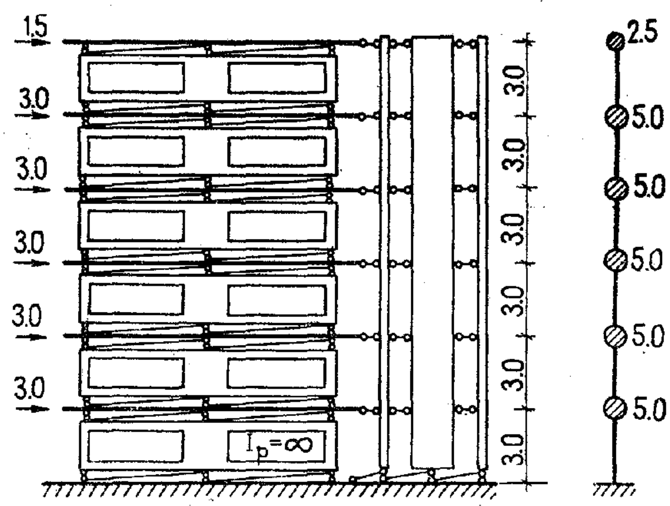


Fig. 39. Separated system of bars.

A comparative study showed that the stiffnesses and forces of the frame and its separated system of bars were identical. For example, the displacement of the upper end of the frame and its separated system of bars was identical and equal to 2.69 mm.

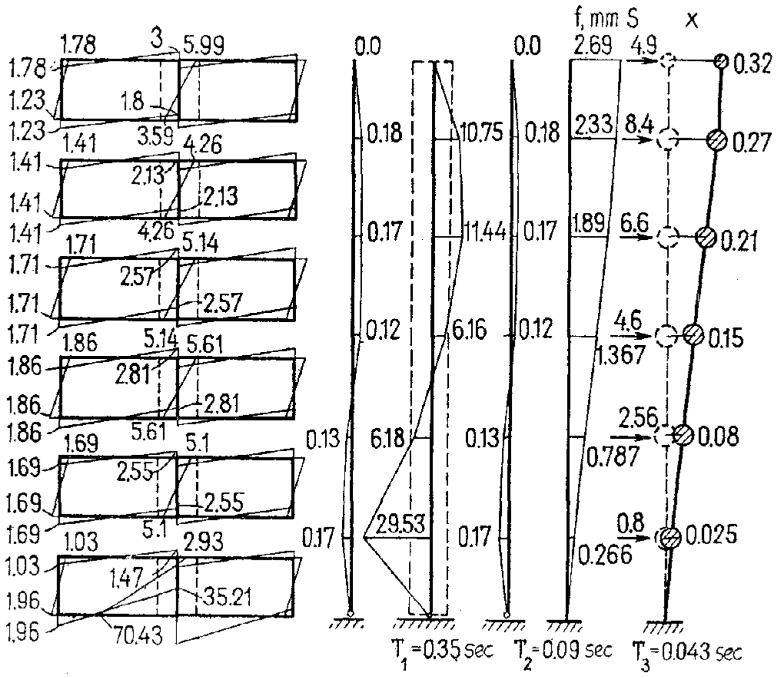


Fig. 40. Analytical results obtained on a computer for a model with the separated system of bars.

The bending moments in the cross beam of the ground floor frame are 2.72 and 4.02 ton-meter at the ends. In the separated system they are, respectively, 1.03 + 1.69 = 2.72 ton-meter and 1.47+2.55 = 4.02 ton-meter. In the cross beam of the third floor frame they are 3.61 and 5.42 ton-meter, whereas in the separated system they are 1.86 + 1.71 = 3.57 ton-meter and 2.81 + 2.57 = 5.38 ton-meter. The bending moments in the extreme column of the ground floor are 1.86 and 0.73 ton-meter at the ends. In the separated system they are 1.96 ton-meter and 1.03 − 0.17 = 0.86 ton-meter. In the central column of the ground floor frame they are 70.71 ton-meter and 26.38 ton-meter and in the separated system 70.43 ton-meter and 29.53 − 2.93 = 26.6 ton-meter. In the extreme column of the third floor frame the bending moments are 1.95 and 1.96 ton-meter while in the separated system they are 1.86 + 0.13 = 1.99 ton-meter and 1.86 + 0.12 = 1.96 ton-meter. In the central column of the third floor frame the bending moments are 12.05 and 11.63 ton-meter while in the separated system they are 5.61 + 6.18 = 11.8 ton-meter and 5.61 + 6.16 = 11.77 ton-meter. The dynamic parameters, periods and wave forms of oscillations were also identical.

Hence, the separated system of bars is equivalent to the orthogonal frame and the use of the method of separating a system into equivalent members enables us to examine an orthogonal bar system as a two-layer bar.

CHAPTER 3

Stiffness Characteristics of Load Bearing Members of Frame-panel Buildings

1. MAIN LOAD BEARING MEMBERS OF FRAME-PANEL BUILDINGS

A frame-panel building is a complex three-dimensional system. It consists of plane vertical and horizontal members and rests on a compacted foundation. The vertical members are arranged in the plan of the building in the transverse as well as longitudinal directions. They are slabs in the form of walls or a system of bars in the form of frames. Their functional purpose in the building is to take the vertical and horizontal loads and transfer them to the foundation. The common types of vertical members of buildings are multiple bay and multiple story frames, frames with filter walls and with stiffness, frame diaphragms and walls in the form of rigid diaphragms (Fig. 41).

Interstory floors are the horizontal members of a building. They are parallel to each other along the building height and are of plane solid construction with a small number of apertures for staircases and elevators. Their function in a building is to carry the effective vertical load, interconnect the vertical members of the building and transmit the horizontal loads to them. The spatial behavior of a building is only due to floors. The horizontal members of a building may be monolithic floors of the beams and slab type or built-up floors of the framework type with parallel rigid cross bars whose apertures are filled with separate slabs (Fig. 42). We must analyze the stiffness characteristics of the main load bearing vertical and horizontal members of a building to study the stress-strain state of the building and to determine its dynamic parameters.

2. DETERMINATION OF STIFFNESS CHARACTERISTICS OF FRAMES

A frame is an orthogonal column system whose geometric shape remains unchanged through the rigid connection of its members at the joints. The frames may be multiple bay or multiple story types. Determining the equivalent

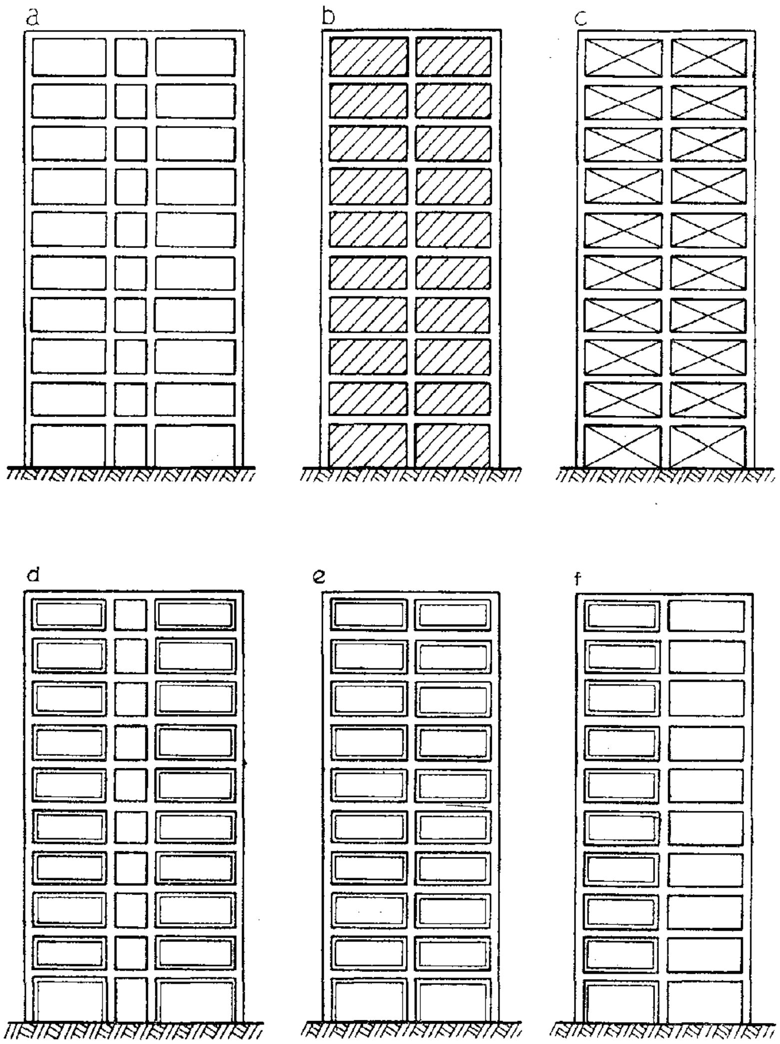


Fig. 41. Vertical members of a building.
a—frame; b—frame with filler walls; c—braced frame; d—diaphragm with apertures; e—blind diaphragm; f—frame diaphragm.

shear stiffness of a story $[GF] = \dfrac{Qh}{\Delta}$ leads to find the reaction Q per unit deflection of the joints in each story (Figs. 43 and 44). For this, equation 2.24 is written in the following form:

$$Q = 2\sum_{i=1}^{n}\left(\frac{\alpha+\beta}{h^2}\right)_i^{\text{col}} - \sum_{i=1}^{n}\left(\frac{\alpha+\beta}{h}\right)_i^{\text{col}}(\varphi^{\text{U}} + \varphi^{\text{L}}). \tag{3.1}$$

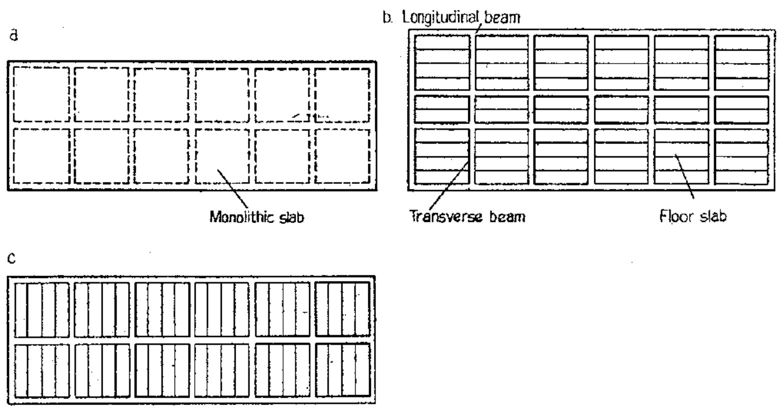


Fig. 42. Horizontal members of building.
a—monolithic floor; b—built-up floor with transverse load bearing system; c—built-up floor with longitudinal load bearing system.

The equivalent shear stiffness of a story will be

$$[GF] = \sum_{i=1}^{n} (\alpha + \beta)_i^{\text{col}} \left[\frac{2}{h} - (\varphi^{\text{U}} + \varphi^{\text{L}}) \right]. \tag{3.2}$$

The unknown quantities in this expression are the angles of rotation φ^{U} and φ^{L} at the top and the bottom of the story. They are determined from equation 2.24, which will be of the type given below for the ith upper joint:

$$\beta_{i-1,\,i}^{\text{beam.U}}\ \varphi_{i-1}^{\text{U}} + \left(\alpha_{i-1,\,i}^{\text{beam.U}} + \alpha_{i,\,i+1}^{\text{beam.U}} + \alpha_i^{\text{col}}\right) \varphi_i^{\text{U}} + \beta_{i,\,i+1}^{\text{beam.U}}\ \varphi_{i+1}^{\text{beam.U}} + \beta_i^{\text{col}} \varphi_i^{\text{L}} - \left(\frac{\alpha + \beta}{h}\right)_i^{\text{col}} = 0; \tag{3.3}$$

from which the unknown φ_i^{U} will be equal to:

$$\varphi_i^{\text{U}} = \frac{\left(\dfrac{\alpha + \beta}{h}\right)_i^{\text{col}} - \beta_{i-1,\,i}^{\text{beam.U}}\ \varphi_{i-1}^{\text{U}} + \beta_{i,\,i+1}^{\text{beam.U}}\ \varphi_{i+1}^{\text{U}} + \beta_i^{\text{col}} \varphi_i^{\text{L}}}{\alpha_{i-1,\,i}^{\text{beam.U}} + \alpha_{i,\,i+1}^{\text{beam.U}} + \alpha_i^{\text{col}}}. \tag{3.4}$$

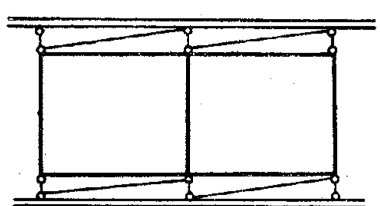


Fig. 43. A typical story.

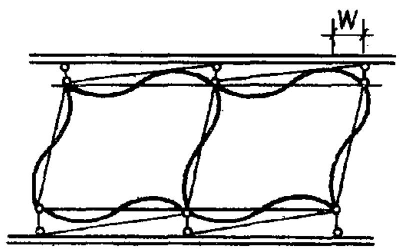


Fig. 44. Determination of the shear stiffness of a story.

Using the method of successive approximations, let us assume:

$$\varphi_i^{\mathrm{U}} = \varphi_{i-1}^{\mathrm{U}} = \varphi_{i+1}^{\mathrm{U}} = \varphi_i^{\mathrm{L}}.$$

The magnitude of φ_i^{U} is determined to the first degree of approximation using the following formula:

$$(\varphi_i^{\mathrm{U}})_{\mathrm{I}} = \frac{I}{h} \frac{(\alpha+\beta)_i^{\mathrm{col}}}{(\alpha+\beta)_{i-1,\,i}^{\mathrm{beam.U}} + (\alpha+\beta)_{i,\,i+1}^{\mathrm{beam.U}} + (\alpha+\beta)_i^{\mathrm{col}}}, \quad (3.5)$$

where $(\alpha+\beta)_{i-1,\,i}^{\mathrm{beam.U}}$ and $(\alpha+\beta)_{i,\,i+1}^{\mathrm{beam.U}}$ are stiffnesses of the upper beams of the story that meet at the ith joint.

The magnitude of φ_i^{U} to the second degree of approximation is determined by substituting the value $(\varphi_i^{\mathrm{U}})_{\mathrm{I}}$ in (3.4):

$$(\varphi_i^{\mathrm{U}})_{\mathrm{II}} = \frac{\left(\frac{\alpha+\beta}{h}\right)_i^{\mathrm{col}} - \beta_{i-1,\,i}^{\mathrm{beam.U}}(\varphi_{i-1}^{\mathrm{U}})_{\mathrm{I}} + \beta_{i,\,i+1}^{\mathrm{beam.U}}(\varphi_{i+1}^{\mathrm{U}})_{\mathrm{I}} + \beta_i^{\mathrm{col}}(\varphi_i^{\mathrm{L}})_{\mathrm{I}}}{\alpha_{i-1,\,i}^{\mathrm{beam.U}} + \alpha_{i,\,i+1}^{\mathrm{beam.U}} + \alpha_i^{\mathrm{col}}}. \quad (3.6)$$

For practical purposes the first approximation may be considered acceptable. This permits the determination of shear stiffness with ready-made formulae.

After substituting the values of $(\varphi_i^{\mathrm{U}})_{\mathrm{I}}$ and $(\varphi_i^{\mathrm{L}})_{\mathrm{I}}$ in 3.2 and doing the necessary transformations we get:

$$[GF] = \frac{1}{h}\sum_{i=1}^{n}\left\{2(\alpha+\beta)_i^{\mathrm{col}} - \frac{[(\alpha+\beta)_i^{\mathrm{col}}]^2}{(\alpha+\beta)_{i-1,\,i}^{\mathrm{beam.U}} + (\alpha+\beta)_{i,\,i+1}^{\mathrm{beam.U}} + (\alpha+\beta)_i^{\mathrm{col}}} - \frac{[(\alpha+\beta)_i^{\mathrm{col}}]^2}{(\alpha+\beta)_{i-1,\,i}^{\mathrm{beam.L}} + (\alpha+\beta)_{i,\,i+1}^{\mathrm{beam.L}} + (\alpha+\beta)_i^{\mathrm{col}}}\right\}. \quad (3.7)$$

In a monolithic frame there are three main types of stories: ground floor, a typical intermediate one and the top floor. They differ from each other in their stiffness characteristics. Because a typical story has the same stiffness characteristics as the upper and lower beams, formula (3.7) becomes:

$$[GF] = \frac{2}{h}\sum_{i=1}^{n}\frac{1}{\frac{1}{(\alpha+\beta)_i^{\mathrm{col}}} + \frac{1}{(\alpha+\beta)_{i-1,\,i}^{\mathrm{beam}} + (\alpha+\beta)_{i,\,i+1}^{\mathrm{beam}}}}. \quad (3.8)$$

The equivalent shear stiffness of a ground floor members differs from a typical story mainly because it has an infinitely rigid lower beam, a different height and different constructional features. The shear stiffness of a ground floor member is determined by the formula:

$$[GF]_1 = \frac{1}{h_1}\sum_{i=1}^{n}(\alpha+\beta)_i^{\text{col}}\left[2 - \frac{(\alpha+\beta)_i^{\text{col}}}{(\alpha+\beta)_{i-1,\,i}^{\text{beam}} + (\alpha+\beta)_{i,\,i+1}^{\text{beam}} + \alpha_i^{\text{col}}}\right]. \quad (3.9)$$

The upper story member of the frame differs from the other typical ones in that it has double the stiffness of the upper cross beam. Its shear stiffness will be:

$$[GF] = \frac{1}{2h}\sum_{i=1}^{n}(\alpha+\beta)_i^{\text{col}}\left[4 - 3\frac{(\alpha+\beta)_i^{\text{col}}}{(\alpha+\beta)_{i-1,\,i}^{\text{beam}} + (\alpha+\beta)_{i,\,i+1}^{\text{beam}} + (\alpha+\beta)_i^{\text{col}}}\right]. \quad (3.10)$$

If many types of story members with different shear stiffness characteristics exist along the height of the frame they may be reduced to a single stiffness characteristic:

$$h_i^{\text{r}} = \frac{[GF]_{\text{T}}}{[GF]_i} h_i. \quad (3.11)$$

If only the bending deformations of bars are considered in the story then formula (3.8) is written as follows:

$$[GF] = \frac{12}{h}\sum_{i=1}^{n}\frac{1}{\dfrac{1}{S_i} + \dfrac{1}{r_{i-1,\,i} + r_{i,\,i+1}}}, \quad (3.12)$$

where $S_i = \left(\dfrac{EI}{h}\right)_i^{\text{col}}$ is the transverse bending stiffness of the ith column;

$$r_{i-1,\,i} + r_{i,\,i+1} = \left(\frac{EI}{l}\right)_{i-1,\,i}^{\text{beam}} + \left(\frac{EI}{l}\right)_{i,\,i+1}^{\text{beam}}$$

is the sum of the transverse bending stiffnesses of the cross beams of the story meeting at the ith joint.

Assuming the angles of rotation of all joints are equal to a story during the deflection of the latter, from (3.2) we obtain:

$$[GF] = 2\left(\frac{1}{h} - \varphi\right)\sum_{i=1}^{n}(\alpha+\beta)_i^{\text{col}}, \quad (3.13)$$

where φ is the average angle of rotation and is equal to:

$$\varphi = \frac{1}{h}\,\frac{\sum_{i=1}^{n}(\alpha+\beta)_i^{\text{col}}}{2\sum_{i=1}^{n}(\alpha+\beta)_{i-1,\,i}^{\text{beam}} + \sum_{i=1}^{n}(\alpha+\beta)_i^{\text{col}}}. \quad (3.14)$$

By substituting the value of φ from (3.14) in (3.13) we get the equivalent shear stiffness as:

$$[GF] = \frac{2}{h} \frac{1}{\dfrac{1}{\sum\limits_{i=1}^{n} (\alpha + \beta)_i^{\text{col}}} + \dfrac{1}{2\sum\limits_{i=1}^{n} (\alpha + \beta)_{i-1,\,i}^{\text{beam}}}}. \tag{3.15}$$

While considering the bending strain of bars in a story, formula (3.15) becomes:

$$[GF] = \frac{12}{h} \frac{1}{\dfrac{1}{S} + \dfrac{1}{r}}, \tag{3.16}$$

where $S = \sum\limits_{i=1}^{n} \left(\dfrac{EI}{h}\right)_i^{\text{col}}$ —is the total transverse bending stiffness of struts and $r = \sum\limits_{i=1}^{n} 2\left(\dfrac{EI}{l}\right)_{i-1,\,i}^{\text{beam}}$ —is the total transverse bending stiffness of the upper and lower cross beams of a story.

Expression (3.16), the accepted formula of E.E. Sigalov [3], is very effective in a multiple bay frame in which there is little difference between the stiffnesses of struts and cross beams. Formula (3.12) is derived for different angles of rotation of the joints of a deflected story and gives more accurate results.

To find the equivalent bending stiffness of a story, let us determine the position of the center of rigidity, that is, the neutral axis of the story relative to the extreme strut, by the formula:

$$x_0 = \frac{\sum\limits_{i=1}^{n} \gamma_i b_i}{\sum\limits_{i=1}^{n} \gamma_i}, \tag{3.17}$$

where γ_i is the longitudinal end stiffness of the ith strut; b_i is the distance of the ith strut measured from the extreme strut.

Determination of the equivalent bending stiffness $[EI] = \dfrac{Mh}{\psi}$ leads to the determination of the reactive moment when the end planes of the story turn around the center of rigidity through unit angle (Fig. 45). Vertical displacement of the joints, forming the story, occurs during this turning. For the ith joint it will be:

$$V_i = 0.5\,(x_0 - b_i). \tag{3.18}$$

The joints in the story turn through angles φ_i. These angles are determined

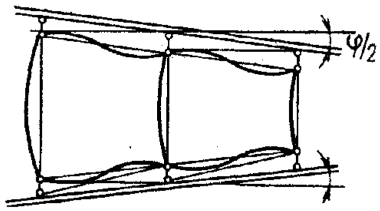


Fig. 45. Determination of bending stiffness of a story.

by using equation (2.24) in the following form:

$$\beta^{\text{beam}}_{i-1,\,i}\,\varphi^{\text{beam}}_{i-1}+(\alpha^{\text{beam}}_{i-1,\,i}+\alpha^{\text{beam}}_{i,\,i+1}+\alpha^{\text{col}}_{i})\,\varphi^{\text{U}}_{i}+\beta^{\text{beam}}_{i,\,i+1}\,\varphi^{\text{U}}_{i+1}-\beta^{\text{col}}_{i}\,\varphi^{\text{L}}_{i}$$
$$+\left(\frac{\alpha+\beta}{l}\right)^{\text{beam}}_{i+1,\,i}(V_{i-1}-V_i)+\left(\frac{\alpha+\beta}{l}\right)^{\text{beam}}_{i,\,i+1}(V_i-V_{i+1})=0. \qquad (3.19)$$

Angle φ^{U}_i is determined from this equation by substituting the value of vertical displacement of joints (3.18):

$$\varphi^{\text{U}}_i=\frac{(\alpha+\beta)^{\text{beam}}_{i-1,\,i}+(\alpha+\beta)^{\text{beam}}_{i,\,i+1}}{\alpha^{\text{beam}}_{i-1,\,i}+\alpha^{\text{beam}}_{i,\,i+1}+\alpha^{\text{col}}_{i}}-\frac{\beta^{\text{beam}}_{i-1,\,i}\,\varphi^{\text{U}}_{i-1}+\beta^{\text{beam}}_{i,\,i+1}\,\varphi^{\text{beam}}_{i+1}-\beta^{\text{col}}_{i}\,\varphi^{\text{L}}_{i}}{\alpha^{\text{beam}}_{i-1,\,i}+\alpha^{\text{beam}}_{i,\,i+1}+\alpha^{\text{col}}_{i}}. \qquad (3.20)$$

The value of φ^{U}_i in the first approximation is found by the following formula assuming $\varphi^{\text{U}}_i=\varphi^{\text{U}}_{i-1}=\varphi^{\text{U}}_{i+1}=\varphi^{\text{L}}_i$:

$$(\varphi^{\text{U}}_i)_{\text{I}}=0.5\,\frac{(\alpha+\beta)^{\text{beam}}_{i-1,\,i}+(\alpha+\beta)^{\text{beam}}_{i,\,i+1}}{(\alpha+\beta)^{\text{beam}}_{i-1,\,i}+(\alpha+\beta)^{\text{beam}}_{i,\,i+1}+(\alpha+\beta)^{\text{col}}_{i}}. \qquad (3.21)$$

By substituting the value of $(\varphi^{\text{U}}_i)_{\text{I}}$ in (3.20) we get φ^{U}_i to the second degree of approximation.

The reactive moment in the ith strut of the story relative to the centre of rigidity will be equal to:

$$M_i=\gamma_i\,(x_0-b_i)^2+(\alpha-\beta)^{\text{col}}_i\,\varphi_i, \qquad (3.22)$$

where $\gamma_i\,(x_0-b_i)$ is the longitudinal reaction in a strut and $(\alpha-\beta)^{\text{col}}_i\,\varphi_i$ is the reactive moment at the end of the strut. The total reactive moment for the story is determined as follows:

$$M=\sum_{i=1}^{n}\gamma_i\,(x_0-b_i)^2+\sum_{i=1}^{n}(\alpha-\beta)^{\text{col}}_i\,\varphi_i. \qquad (3.23)$$

The equivalent bending stiffness of the story will be:

$$[EI]=h\left[\sum_{i=1}^{n}\gamma_i\,(x_0-b_i)^2+\sum_{i=1}^{n}(\alpha-\beta)^{\text{col}}_i\,\varphi_i\right], \qquad (3.24)$$

where α, β, γ are the end stiffnesses (see Table 1).

For the bars in which both bending and longitudinal deformations are considered, formula (3.24) becomes:

$$[EI] = \sum_{i=1}^{n} EF_i\,(x_0 - b_i)^2 + 2\sum_{i=1}^{n} EI_i\,\varphi_i. \tag{3.25}$$

Depending on the stiffness of struts and cross beams the angle of rotation has the values $0 \leqslant \varphi_i \leqslant 0.5$; the expression $\sum_{i=1}^{n} EI_i$ is small compared to the first term of (3.25) and hence it may be neglected. The equivalent bending stiffness of the story is determined by the formula:

$$[EI] = \sum_{i=1}^{n} EF_i\,(x_0 - b_i)^2. \tag{3.26}$$

The equivalent bending stiffness of columns which constitute the second layer of the equivalent bar for the frame is determined as follows. Let us represent the columns in the story by struts and turn all its joints so that an angle $\varphi = 1$ (Fig. 46) is formed between the upper and lower ends of a column. The sum of the reactive moments at the ends of the columns is determined by the formula:

$$[EI]_0 = \frac{h}{2}\sum_{i=1}^{n} (\alpha - \beta)_i^{\text{col}}, \tag{3.27}$$

which, when bending deformations in the struts are considered (3.27), becomes:

$$[EI]_0 = \sum_{i=1}^{n} (EI)_i. \tag{3.28}$$

To determine the shear stiffness, Fig. 47 shows a ground floor story frame

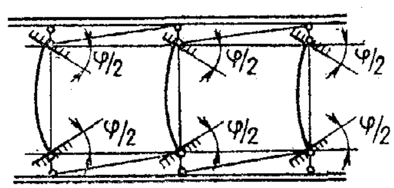


Fig. 46. Determination of bending stiffness of columns.

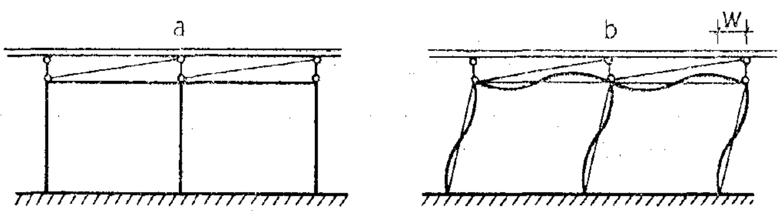


Fig. 47. Analytical models.
a—a ground floor story; b—determination of shear stiffness.

and its deformed state under a unit deflection.

In designing high rise frames, the bending stiffness of individual columns is small and hence may be neglected.

3. DETERMINATION OF STIFFNESS CHARACTERISTICS OF IN-FILLED FRAMES

A frame may be in-filled with individual small blocks, stiffening walls, trusses and other members which are strained when the frame is deformed. Considering the compatibility of deformations, the equivalent shear stiffness of a story with filled up members (Fig. 48) is determined by the sum:

$$[GF] = [GF]_{\text{frame}} + [GF]_{\text{filler}}. \tag{3.29}$$

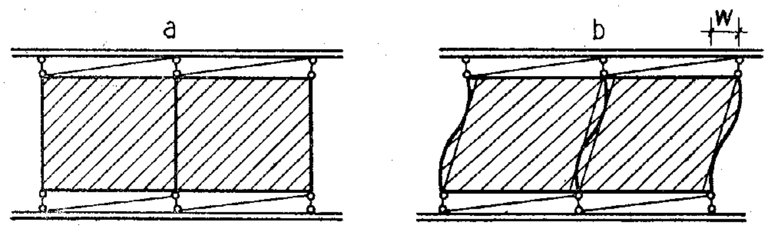


Fig. 48. Analytical models.
a—a story with filled frame; b—determination of shear stiffness.

When the frame is filled up with masonry, the stiffness will be:

$$[GF]_{\text{filler}} = FG\gamma_{\text{long}}, \tag{3.30}$$

where F, G, γ_{long} are respectively the area in plan, shear modulus and filling coefficient for the filler material.

If the masonry is compactly laid in the upper part of the story, the stiffness of the cross beam may be considered equal to infinity. Then the frame stiffness is determined by the formula:

$$[GF]_{\text{frame}} = \frac{2\sum_{i+1}^{n}(\alpha + \beta)_i^{\text{col}}}{h} = \frac{12}{h^2}\sum_{i=1}^{n}(EI)_i^{\text{col}}. \tag{3.31}$$

The equivalent shear stiffness of the story, with filled-in wall, will be determined by the following expression:

$$[GF] = \frac{12}{h^2}\sum_{i=1}^{n}(EI)_i^{\text{co}} + FG\gamma_{\text{long}}. \tag{3.32}$$

If the braces are arranged diagonally (Fig. 49), tensile and compressive deformations will occur in them under a unit deflection of the story. The equivalent shear stiffness of the braces will be:

$$[GF]_{\text{filler}} = \sum_{i=1}^{n} (EF \sin\alpha \cos^2\alpha)_i, \tag{3.33}$$

where F, E, α are respectively the cross sectional area of the braces, the elastic modulus of the material and the angle of inclination of the brace to the horizontal.

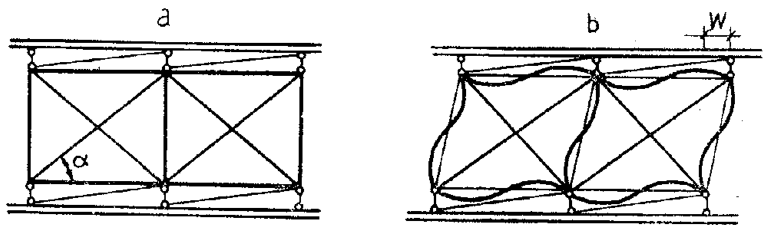


Fig. 49. Frame models.
a—a story with frame and trusses; b—determination of shear stiffness.

The equivalent shear stiffness of a story with braces is determined by the following formula:

$$[GF] = \frac{12}{h}\,\frac{1}{\dfrac{1}{\sum\limits_{i=1}^{n}\left(\dfrac{EI}{h}\right)_i^{\text{col}}} + \dfrac{1}{2\sum\limits_{i=1}^{n-1}\left(\dfrac{EI}{l}\right)_i^{\text{beam}}}} + \sum_{i=1}^{k} (EF \sin\alpha \cos^2\alpha)_i. \tag{3.34}$$

Filling the frame with walls does not affect the magnitude of the equivalent bending stiffness of a story because the filling does not undergo the same bending deformation of the frame nor provides any significant resistance to it.

Considering the compatibility of deformations, the equivalent bending stiffness of the story with braces (Fig. 50) is determined by the sum:

$$[EI] = [EI]_{\text{frame}} + [EI]_{\text{brace}}. \tag{3.35}$$

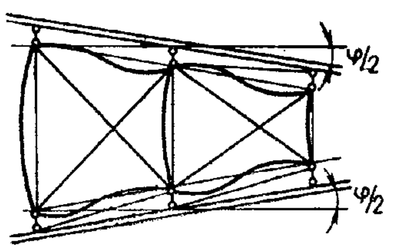


Fig. 50. Determination of bending stiffness.

The reactive moment of the ith end of a brace taken about the center of stiffness of the story will be equal to:

$$M_i = 0.25 \frac{EF_{\text{brace}}}{h} \sin^3 \alpha \, (2x_0 - b_i - b_{i-1}), \qquad (3.36)$$

where b_i and b_{i-1} are the distances of the upper and lower ends of the braces from the extreme strut.

The total reactive moment will be:

$$M = \sum_{i=1}^{k} 0.25 \left(\frac{EF_{\text{brace}}}{h} \right)_i \sin^3 \alpha \, (2x_0 - b_i - b_{i-1})^2. \qquad (3.37)$$

The equivalent bending stiffness of the story with braces is determined by the following expression:

$$[EI] = \sum_{i=1}^{n} EF_i^{\text{col}} (x_0 - b_i)^2 + \sum_{i=1}^{k} 0.25 \, EF_i^{\text{brace}} \sin^3 \alpha \, (2x_0 - b_i - b_{i-1})^2. \qquad (3.38)$$

Figure 51 shows the distribution of forces in the braces when their ends are displaced. Consequently, for the practical purpose of analysis under the action of horizontal loads, the filled up frame may be represented by an equivalent single layer bar, in which the filling increases the equivalent shear stiffness but does not affect the bending stiffness. However, both the equivalent shear and the bending stiffnesses increase in the presence of braces.

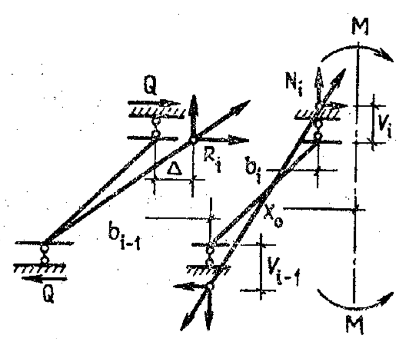


Fig. 51. Distribution of forces.

4. DETERMINATION OF STIFFNESS CHARACTERISTICS OF VERTICAL DIAPHRAGMS

Stiffening diaphragms are vertical members of buildings in the form of walls; they may be blind or have apertures. Walls with apertures divide into separate partitions joined with cross pieces or lintels. Blind diaphragms may be represented by a cantilever bar. For a high rise building the effect of shear strain may be neglected and hence its stiffness characteristic will be the equivalent bending stiffness only which is calculated as for a solid section. A diaphragm with apertures may be represented by a frame with wide struts and cross bars with rigid ends. By the method of dividing a frame into stories and columns, a diaphragm may be represented by a two-layer bar. The equivalent shear stiffness of the first layer is determined by formula (3.8) in which the stiffness of struts in the story is omitted, since it is a large quantity. The formula then becomes:

$$[GF] = \frac{4}{h} \sum_{i=1}^{n} (\alpha + \beta)_i^{\text{beam}}, \tag{3.39}$$

where α and β are given in Table 1. The shear and bending strains in the lintel as well as their flexible connection with partitions can be considered with the help of formula (3.39). The equivalent bending stiffness of the first layer is determined by formula (3.26).

The representative column will be in the form of interconnected individual partitions, the latter having a large bending stiffness, determined by formula (3.28).

Wide multi-aperture diaphragms may be considered as two-layer bars in which the first layer has shear stiffness and the second bending stiffness. The bending deformations of the first layer are neglected because, here, the bending stiffness is a large quantity.

Figures 52-55 shows stories with a diaphragm with an aperture. The load bearing structures consisting of wide partitions and columns, joined by cross beams, are called frame-diaphragms. A frame-diaphragm may be represented by a two-layer bar in which the second layer is a wide partition. The equivalent shear stiffness of the first layer is determined by the following formula:

$$[GF] = \frac{2}{h} \frac{1}{\dfrac{1}{\sum\limits_{i=1}^{k} (\alpha + \beta)_i^{\text{col}}} + \dfrac{1}{2\sum\limits_{i=1}^{n-1} (\alpha + \beta)_i^{\text{beam}}}}, \tag{3.40}$$

where k is the number of joints connecting the strut.

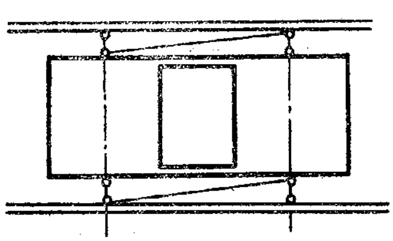


Fig. 52. A diaphragm with aperture in a story.

The equivalent bending stiffness of the first layer is determined by formula (3.26) and of the second layer by the following formula:

$$[EI]_0 = \sum_{i=1}^{n-k} (EI)_i^{\text{partition}}. \tag{3.41}$$

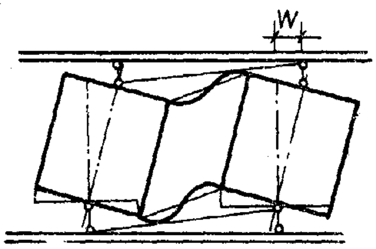


Fig. 53. Determination of shear stiffness.

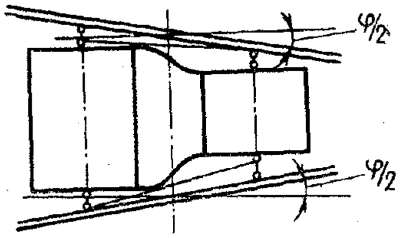


Fig. 54. Determination of bending stiffness of a story.

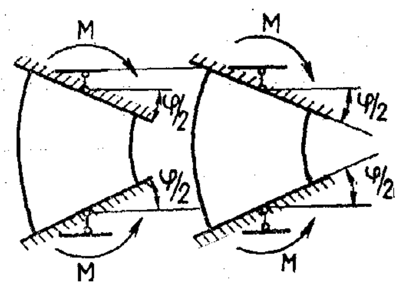


Fig. 55. Determination of bending stiffness of columns.

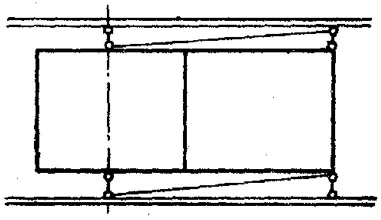


Fig. 56. A frame-diaphragm in a story.

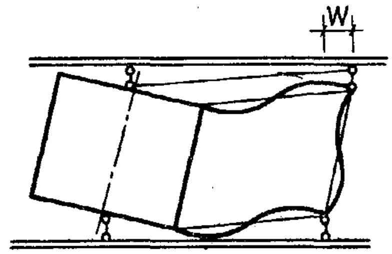


Fig. 57. Determination of shear stiffness of a frame-diaphragm.

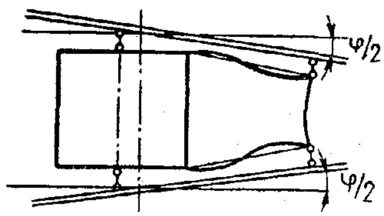


Fig. 58. Determination of bending stiffness of a frame-diaphragm.

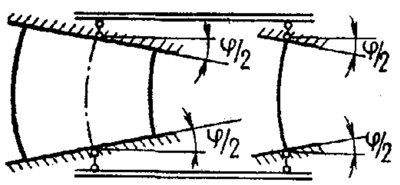


Fig. 59. Determination of bending stiffness of frame-diaphragm columns.

Figures 56-59 show a frame-diaphragm in a story, the determination of shear and bending stiffnesses for the story and the bending stiffness of a column.

5. DETERMINATION OF STIFFNESS CHARACTERISTICS OF HORIZONTAL MEMBERS OF A BUILDING (INTER-STORY FLOORS)

A monolithic floor in its plane is a beam with a large section which undergoes shear and bending deformations. The equivalent shear and bending stiffnesses of a floor are determined as for a full section.

A built-up floor may be considered a brace consisting of rigid joints, whose apertures are filled with individual slabs. This includes cases of slabs with weak and rigid joints among themselves as well as with the brace contour.

The chords of the brace are the cross beams of the longitudinal frame and the struts are the cross beams of the transverse frame of the building. The stiffness characteristics of a built-up floor are determined in the same way as for a filled frame. To do so, let us divide the floor into individual bays (Fig. 60). The equivalent shear stiffness of a bay is expressed by formula (3.32). When the slabs are loosely connected among themselves and with the cross beams, the equivalent shear stiffness is determined by the formula:

$$[GF] = \frac{2}{h} \frac{1}{\dfrac{1}{\sum\limits_{i=1}^{n} (\alpha + \beta)_i^{\text{long}}} + \dfrac{1}{2\sum\limits_{i=1}^{n} (\alpha + \beta)_{i-1,\,i}^{\text{trans}}}}, \tag{3.42}$$

where $(\alpha + \beta)^{\text{long}}$ and $(\alpha + \beta)^{\text{trans}}$ are the end stiffnesses of the longitudinal and transverse cross beams of the building frame in the plane of the floor. The equivalent bending stiffness of the floor is determined by formula (3.26).

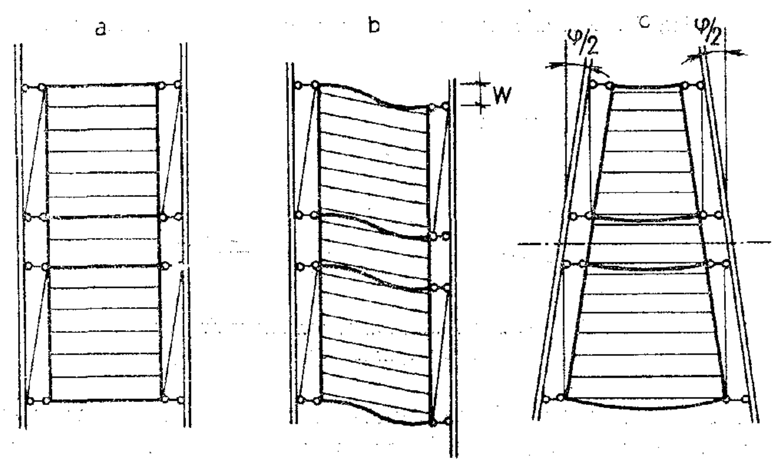


Fig. 60. A fragment of floor.
a—a floor bay; b—determination of shear stiffness; c—determination of bending stiffness.

When the slabs are rigidly connected at their ends in a transverse load bearing system, the equivalent shear stiffness of the floor will be:

$$[GF] = \frac{2}{h}\left[\sum_{i=1}^{n} (\alpha + \beta)_i^{\text{long}} + \sum_{i=1}^{k} (\alpha + \beta)_i^{\text{fl}}\right], \tag{3.43}$$

where $(\alpha + \beta)^{\text{fl}}$ is the end stiffness of floor slabs in which the shear and bending deformations of slabs are considered; k is the number of slabs laid in a bay.

For a longitudinal load bearing system the equivalent shear stiffness of the floor is determined by the following formula:

$$[GF] = \frac{2}{h}\left[2\sum_{i=1}^{n}(\alpha+\beta)_i^{\text{trans}} + \sum_{i=1}^{k}(\alpha+\beta)_i^{\text{fl}}\right]. \tag{3.44}$$

When the floor slabs are rigidly connected among themselves, the equivalent shear stiffness of the floor will be:

$$[GF] = \frac{2}{h}\,\frac{1}{\dfrac{1}{\sum\limits_{i=1}^{n}(\alpha+\beta)_i^{\text{long}}} + \dfrac{1}{2\sum\limits_{i=1}^{n}(\alpha+\beta)_i^{\text{trans}}}} + F^{\text{fl}}G, \tag{3.45}$$

where F^{fl} is the cross sectional area of all slabs of the floor, G is the shear modulus of the floor slab material.

Thus, a floor in its own plane may be represented by an equivalent beam with free ends, with shear and bending stiffnesses. The beam rests on an elastic foundation constituted by the vertical load bearing members of the building.

6. STIFFNESS CHARACTERISTICS OF AN EQUIVALENT BAR

Vertical members of buildings, represented by orthogonal bar systems, may be considered an equivalent two-layer cantilever bar system defined by differential equations (2.45). Depending on the magnitude of the stiffness characteristics $[EI]$, $[GF]$ and $[EI_0]$ of the first and second layers, which may vary from zero to infinity, equation (2.45) is likely to change. Let us examine separately the first layer of the bar undergoing shear and bending strains and determine its potential energy (Fig. 61).

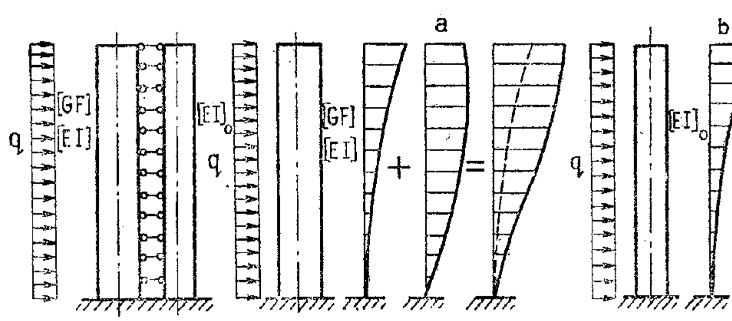


Fig. 61. A two-layer bar.
a—bending and shear strains of the first layer; b—bending strain of the second layer.

The energy capacity of the first layer will be:

$$V_{\mathrm{I}} = V_{\mathrm{I}}^{[GF]} + V_{\mathrm{I}}^{[EI]}. \tag{3.46}$$

The energy capacity due to a uniformly distributed unit load along the height of the bar is determined by the formula:

$$V_{\mathrm{I}} = \frac{H^3}{6\,[GF]} + \frac{H^5}{40\,[EI]}. \tag{3.47}$$

To determine the predominant types of strains, shear bending, in a bar, let us examine the ratio of total potential energy to the energy due to the respective types of strains.

$$\frac{V_{\mathrm{I}}}{V_{\mathrm{I}}^{[GF]}} = \frac{3}{20}\,\lambda^2 + 1; \tag{3.48}$$

$$\frac{V_{\mathrm{I}}}{V_{\mathrm{I}}^{[EI]}} = \frac{20}{3\lambda^2} + 1, \tag{3.49}$$

where λ is the stiffness characteristics of the first layer and is equal to:

$$\lambda = H\sqrt{\frac{[GF]}{[EI]}}. \tag{3.50}$$

Let us assume that the rod undergoes only bending strain if $\frac{V_{\mathrm{I}}}{V_{\mathrm{I}}^{[GF]}} = 10$ and shear strain if $\frac{V_{\mathrm{I}}}{V_{\mathrm{I}}^{[EI]}} = 10$. Considering these ratios, the following conditions may be written: for $\lambda \leqslant 0.8$—the bar undergoes only shear strain; and for $\lambda \geqslant 8$—only bending strain; for $0.8 < \lambda < 8$ both shear and bending (Fig. 62).

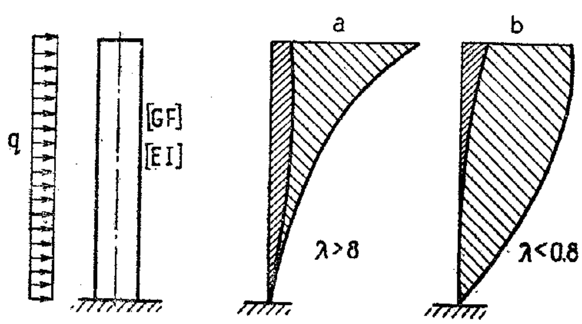


Fig. 62. A single layer bar.
a—bar undergoing bending strain; b—bar undergoing shear strain.

The following relationship for potential energy may be used for a two-layer rod:

$$\frac{1}{V} = \frac{1}{V_{\rm I}} + \frac{1}{V_{\rm II}}, \tag{3.51}$$

where V, $V_{\rm I}$ and $V_{\rm II}$ are total potential energy and potential energies of the first and second layers, respectively.

The energy capacity of the first layer is determined by (3.47) and of the second layer, which undergoes only bending, by formula:

$$V_{\rm II} = \frac{H^5}{40\,[EI]_0}. \tag{3.52}$$

Let us examine the following ratios to determine the type of predominant load resistance offered by the first and second layers of the bar:

$$\frac{V_{\rm I}}{V} = \frac{V_{\rm I} + V_{\rm II}}{V_{\rm II}} = 1 + \frac{20}{3k^2}; \tag{3.53}$$

$$\frac{V_{\rm II}}{V} = \frac{V_{\rm I} + V_{\rm II}}{V_{\rm I}} = 1 + \frac{3k^2}{20}, \tag{3.54}$$

where k is the stiffness characteristic of the two-layer bar and is equal to:

$$k = H\sqrt{\frac{[GF]}{\left(1 + \frac{3}{20}\lambda^2\right)[EI]_0}}. \tag{3.55}$$

We shall assume that the resistance of the first layer may be neglected if $\frac{V_{\rm I}}{V} = 10$ and of the second if $\frac{V_{\rm II}}{V} = 10$. Then, the following conditions may be written: for $k \leqslant 0.8$—all the load is taken by the second layer; for $k \geqslant 8$—by the first layer; for $0.8 < k < 8$—both layers of the bar undergo strain (Fig. 63).

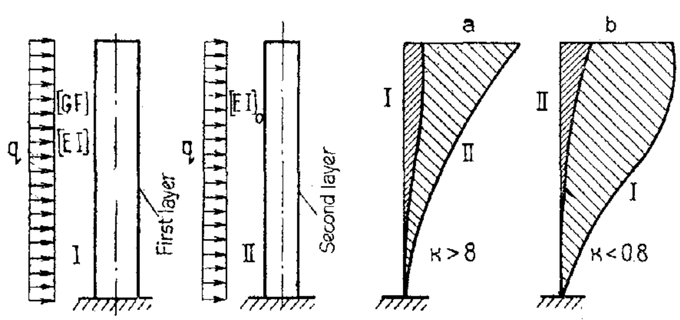


Fig. 63. Bar with two separated layers.
a—bar undergoing shear; b—bar undergoing bending.

Hence, the stiffness characteristic λ of the first layer of the bar enables us to determine the predominant types of strain in the bar and this stiffness characteristic indicates whether the bar should be taken as a single or two-layer one.

7. DIFFERENTIAL EQUATION OF EQUILIBRIUM FOR VERTICAL MEMBERS OF A BUILDING

Depending on the stiffness characteristics λ and k differential equation (2.50) may be simplified and particular equations obtained for different types of vertical members of a building.

For a multistory, multiple bay frame $k > 8$, assuming $[EI]_0 = 0$. The frame may be represented by a single layer bar for which the differential equation will be:

$$\left([EI]- \frac{[EI]}{[GF]} N\right)\frac{d^4W}{dz^4} + N\frac{d^2W}{dz^2} + \frac{[EI]}{[GF]}\frac{d^2q}{dz^2} - q = 0. \tag{3.56}$$

The longitudinal deformations in columns are insignificant, in the case of multiple bay frames with a smaller number of stories with $\lambda < 0.8$. Assuming $[EI] = \infty$ these frames may be represented by the following differential equation:

$$([GF]-N)\frac{d^2W}{dz^2} + q = 0. \tag{3.57}$$

For $q = \text{const}$, the solution of equation (3.57) will be:

$$W = \frac{q}{[GF]-N} z\left(H - \frac{z}{2}\right). \tag{3.58}$$

The deflection of the top of frame is determined by the formula:

$$f = \frac{qH^2}{2([GF]-N)}. \tag{3.59}$$

Narrow multistory frames with rigid cross beams for which $\lambda > 8$ are defined by the following formula without considering the shear strains in the equivalent bar and assuming $[GF] = \infty$,

$$[EI]\frac{d^4W}{dz^4} + N\frac{d^2W}{dz^2} - q = 0. \tag{3.60}$$

Solution of equation (3.60) will be:

$$W = \frac{q}{Na^2}\frac{1 - Ha\sin Ha}{\cos Ha}(\cos az - 1) + \frac{qH}{Na}\sin az + \frac{qHz}{N}\left(\frac{z}{2H} - 1\right). \tag{3.61}$$

The deflection of the frame top will be equal to:

$$f = \frac{q}{Na^2}\left(1 - \frac{1}{\cos Ha}\right) + \frac{qH}{N}\left(\frac{\tan Ha}{a} - \frac{H}{2}\right), \tag{3.62}$$

where

$$a^2 = \frac{N}{[EI]}.$$

Without considering the action of longitudinal force N applied to the top of the equivalent bar, the equation for deflections will be:

$$W = \frac{qz^4}{24\,[EI]}\left(1 - 4\frac{H}{z} + 6\frac{H^2}{z^2}\right). \tag{3.63}$$

If $0.8 < \lambda < 8$, the frame is considered an equivalent bar which undergoes shear and bending. Without considering the action of longitudinal force N, the equation for deflections will be:

$$W = \frac{qz}{[GF]}\left(H - \frac{z}{2}\right) + \frac{qz^4}{24\,[EI]}\left(1 - 4\frac{H}{z} + 6\frac{H^2}{z^2}\right). \tag{3.64}$$

The deflection of the top of the bar is determined by the formula:

$$f = \frac{qH^2}{2\,[GF]}\left(1 + \frac{1}{4}\lambda^2\right). \tag{3.65}$$

Let us examine the vertical diaphragms which may be represented by frames with wide struts. The stiffness characteristic of the equivalent bar will always be $k < 8$. Hence, we shall have a two-layer bar for which the differential equation will be (2.50).

In a diaphragm with slender cross pieces for which $\lambda < 0.8$, assuming $[GF] = 0$, we have a single layer bar undergoing bending strain, which is defined by the equation:

$$[EI]_0 \frac{d^4W}{dz^4} + N\frac{d^2W}{dz^2} - q = 0. \tag{3.66}$$

Its solution will be similar to equation (3.60).

For a multiple unit diaphragm we have the condition $0.8 < k < 8$ and $\lambda < 0.8$; assuming $[EI] = \infty$, we have a two-layer bar in which the first layer, consisting of stories, undergoes shear strain and the second, consisting of the columns, undergoes bending strain. In this case the longitudinal strains of the partitions are ignored because they are small. The differential equation will be:

$$[EI]_0 \frac{d^4W}{dz^4} - ([GF] - N)\frac{d^2W}{dz^2} - q = 0. \tag{3.67}$$

Its solution will be:

$$W = C_1 + C_2 z + C_3 \operatorname{ch} bz + C_4 \operatorname{sh} bz - \frac{qz^2}{2([GF]-N)}, \tag{3.68}$$

where

$$b^2 = \frac{[GF]-N}{[EI]_0}.$$

The boundary conditions for the two-layer cantilever bar will be:

$$\text{at} \quad z = 0;\ W = 0;\ \frac{dW}{dz} = 0;$$

and

$$\text{at } z = H;\ \frac{d^2W}{dz^2} = 0;\ [EI]_0 = \frac{d^3W}{dz^3} - ([GF]-N)\frac{dW}{dz} = 0.$$

After substituting the values of constants of integration in 3.68 we get:

$$W = \frac{q}{([GF]-N)\,b^2}\,\frac{1 + Hb \operatorname{sh} bH}{\operatorname{ch} bH}\,(\operatorname{ch} bz - 1)$$
$$- \frac{qH}{([GF]-N)\,b} \operatorname{sh} bz + \frac{qHz}{[GF]-N}\left(1 - \frac{z}{2H}\right). \tag{3.69}$$

The deflection of the top of the bar will be equal to:

$$f = \frac{q}{([GF]-N)\,b^2}\,\frac{1 + Hb \operatorname{sh} bH}{\operatorname{ch} bH}\,(\operatorname{ch} bH - 1)$$
$$- \frac{qH}{([GF]-N)\,b} \operatorname{sh} bH + \frac{0.5\,qH^2}{[GF]-N}. \tag{3.70}$$

In a narrow diaphragm with rigid connectors, for $0.8 < k < 8$; $\lambda > 8$; $[GF] = \infty$ we have a two-layer bar in which both layers undergo bending strain. The differential equation will be:

$$([EI] + [EI])_0 \frac{d^4W}{dz^4} + N\frac{d^2W}{dz^2} - q = 0. \tag{3.71}$$

Its solution will be similar to the solution of (3.60).

Depending on the stiffness characteristics λ and k the frame-diaphragms are represented by an equivalent two-layer bar for which the differential equation is similar to the case of vertical diaphragms.

8. DIFFERENTIAL EQUATION OF EQUILIBRIUM FOR HORIZONTAL MEMBERS OF A BUILDING

Horizontal members of a building, floors, are always represented by a single layer bar, with free ends, undergoing shear and bending strain and resting on an elastic base. The differential equation will be:

$$[EI]\frac{d^4W}{dx^4}-\frac{[EI]}{[GF]}C\frac{d^2W}{dx^2}+CW+\frac{[EI]}{[GF]}\frac{d^2q}{dx^2}-q=0, \tag{3.72}$$

where C is the modulus of subgrade reaction of the base constituted by the vertical members of the building.

The stiffness characteristic λ of the bar is determined by the formula:

$$\lambda=\frac{L}{2}\sqrt{\frac{[GF]}{[EI]}}. \tag{3.73}$$

For $\lambda \geqslant 8$ only the bending strain in the floor is considered, for $\lambda \leqslant 0.8$ only the shear strain and for $0.8 < \lambda < 8$ both forms of strains are considered for the floor.

In a built-up floor in narrow buildings $\lambda < 0.8$. Assuming $[EI]=\infty$ we have the following differential equation:

$$[GF]\frac{d^2W}{dx^2}-CW+q=0. \tag{3.74}$$

In a monolithic floor for a building with large dimensions in plane $\lambda > 8$. Assuming $[GF]=\infty$ we have the following type of equation:

$$[EI]\frac{d^4W}{dx^4}+CW-q=0. \tag{3.75}$$

Differential equations (3.74) and (3.75) will be used in the future to form general equations for a building.

9. DETERMINATION OF FORCES IN THE BARS OF INTERFLOOR ELEMENTS AND COLUMNS

Let us determine the combined forces in the layers of an equivalent two-layer bar. For the first layer, consisting of stories, the combined bending moment M and the combined transverse force Q are found from the following formulae:

$$M_k=-[EI]\frac{d^2W_{\text{bend}}}{dz^2}; \tag{3.76}$$

$$Q_k=-[GF]\frac{dW_{\text{sh}}}{dz}. \tag{3.77}$$

For the second layer of the bar, which is a column, the combined bending moment M_0 and the combined transverse force Q_0 are determined by the formulae:

$$M_{0k}=-[EI]\frac{d^2(W_{\text{bend}}+W_{\text{sh}})}{dz^2}; \tag{3.78}$$

$$Q_{0k}=[EI]\frac{d^3(W_{\text{bend}}+W_{\text{sh}})}{dz^3}. \tag{3.79}$$

The bending moment, acting in the first layer of the bar, causes rotation and vertical displacement of the joints in the stories while the transverse force produces rotation and horizontal displacements. The magnitude of displacement of joints in the story is proportional to the combined forces. The angles of rotation of the joints are determined by the following expressions:

$$(\varphi_i^M)_k = \pm\, 0.5 \frac{(\alpha+\beta)^{\text{beam}}_{i-1,\,i} + (\alpha+\beta)^{\text{beam}}_{i,\,i+1}}{(\alpha+\beta)^{\text{beam}}_{i-1,\,i} + (\alpha+\beta)^{\text{beam}}_{i,\,i+1} + (\alpha+\beta)^{\text{col}}_i} \frac{h}{[EI]} M_k. \quad (3.80)$$

$$(\varphi_i^Q)_k = \frac{(\alpha+\beta)^{\text{col}}_i}{(\alpha+\beta)^{\text{beam}}_{i-1,\,i} + (\alpha+\beta)^{\text{beam}}_{i,\,i+1} + (\alpha+\beta)^{\text{col}}_i} \frac{1}{[GF]} Q_k. \quad (3.81)$$

The relative horizontal displacement between the lower and upper sections of the story are determined by the formula:

$$\delta_k^Q = \frac{Qh}{[GF]}. \quad (3.82)$$

The vertical displacement of the lower and upper sections of the story will be:

$$(\Delta_i^M)_k = \frac{M_k h}{[EI]} (x_0 - b_i). \quad (3.83)$$

The bending moment M_0 acting in the second layer of the equivalent bar, causes the column ends to rotate within the domain of each story by angle ψ which is equal to:

$$(\psi)_k = \pm\, 0.5 \frac{M_0 h}{[EI]_0}. \quad (3.84)$$

If we know these deflections, the bending moment on the lower and upper ends of the struts in the kth story can be determined as

$$(M_i^{\text{col}})_k = (\alpha+\beta)^{\text{col}}_i \varphi_i^Q - \left(\frac{\alpha+\beta}{h}\right)^{\text{col}}_i \delta_k^Q \pm (\alpha-\beta)^{\text{col}}_i \varphi_i^M. \quad (3.85)$$

Substituting the magnitudes of deflections from (3.80), (3.81) and (3.82) in (3.85) we get the magnitude of bending moment at the ends of the strut as

$$(M_i^{\text{col}})_k = -A_i^{\text{col}} Q_k \pm B_i^{\text{col}} M_k, \quad (3.86)$$

where

$$A_i^{\text{col}} = \frac{1}{[GF]} \frac{1}{\dfrac{1}{(\alpha+\beta)^{\text{beam}}_{i-1,\,i} + (\alpha+\beta)^{\text{beam}}_{i,\,i+1}} + \dfrac{1}{(\alpha+\beta)^{\text{col}}_i}}; \quad (3.87)$$

$$B_i^{\text{col}} = \frac{h}{2\,[EI]} \frac{1}{\dfrac{1}{(\alpha+\beta)^{\text{beam}}_{i-1,\,i} + (\alpha+\beta)^{\text{beam}}_{i,\,i+1}} + \dfrac{1}{(\alpha+\beta)^{\text{col}}_i}}. \quad (3.88)$$

By placing the respective pillar on the story strut and restoring the latter to its initial shape, we determine the total bending moment in the frame strut by the following expression:

$$(M_i^{\text{col}})_k = -A_i^{\text{col}} Q_k \pm B_i^{\text{col}} M_k \pm D_i^{\text{pillar}} M_{0k}, \tag{3.89}$$

where

$$D_i^{\text{pillar}} = (\alpha - \beta)_i^{\text{col}} \frac{h}{2\,[EI]_0}. \tag{3.90}$$

Normal force, acting in the strut of the kth story will be:

$$(N_i^{\text{col}})_k = \varDelta_i \gamma_i^{\text{col}} = \frac{h\gamma_i^{\text{col}}}{[EI]} (x_0 - b_i) M_k. \tag{3.91}$$

The transverse force is determined by the formula:

$$(Q_i^{\text{col}})_k = \frac{2\,A_i^{\text{col}}}{h}. \tag{3.92}$$

The forces in cross beams are similarly determined.

CHAPTER 4

Grid Systems

1. SPECIAL FEATURES OF THE ANALYSIS

In a grid or plane orthogonal bar system the bars are arranged in one plane and the forces act out of this plane. We shall represent the joints of the grid system by theoretical squares of zero dimensions, elastically connected to points which are relatively fixed with respect to mutually perpendicular angular rotations and horizontal displacements (Fig. 64). Let us examine the behavior of this grid system under an external load applied in the form of concentrated forces at the joints. The supporting connections may be provided in the form of normal joints with different stiffness characteristics: for example, a rigid support connection will have $C_W = \infty$, $C_\varphi^x = \infty$, $C_\varphi^z = \infty$; a connection which permits only horizontal displacements will have $C_W = 0$, $C_\varphi^x = \infty$, $C_\varphi^z = \infty$; a connection which allows free bending along z axis and elastic displacement in the horizontal direction will have $C_W \neq 0$, $C_\varphi^x = \infty$, $C_\varphi^z = 0$ etc.

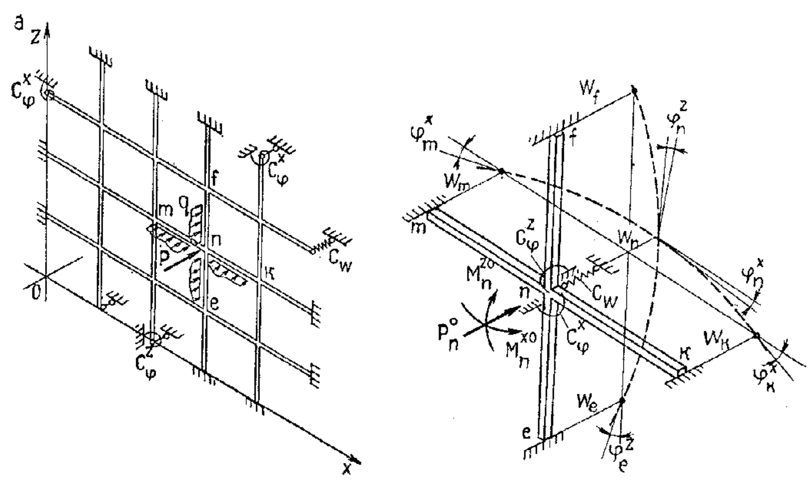


Fig. 64. A grid system of bars.
a—general view; b—an elastically fixed joint.

The relationship between forces and displacements is expressed by the following matrix for the horizontal bar

$$\left\|\begin{matrix} M_{mn} \\ M^{\mathrm{tor}}_{mn} \\ R_{mn} \\ M_{nm} \\ M^{\mathrm{tor}}_{nm} \\ R_{nm} \end{matrix}\right\| = \left\|\begin{matrix} \alpha_{mn}A_{mn} & \beta_{mn}-A_{mn} \\ \Theta_{mn} & -\Theta_{mn} \\ -A_{mn}-B_{mn} & -A_{nm}B_{mn} \\ \beta_{mn}A_{mn} & \alpha_{nm}-A_{nm} \\ -\Theta_{nm} & \Theta_{nm} \\ A_{mn}B_{nm} & A_{nm}-B_{mn} \end{matrix}\right\| \times \left\|\begin{matrix} \varphi^x_m \\ W_m \\ \varphi^z_m \\ \varphi^x_n \\ W_n \\ \varphi^z_n \end{matrix}\right\|. \tag{4.1}$$

For the vertical bar it is given by

$$\left\|\begin{matrix} M^{\mathrm{tor}}_{en} \\ M_{en} \\ R_{en} \\ M^{\mathrm{tor}}_{ne} \\ M_{ne} \\ R_{ne} \end{matrix}\right\| = \left\|\begin{matrix} \Theta_{en} & -\Theta_{en} \\ A_{en}\alpha_{en} & -A_{en}\beta_{en} \\ -B_{en}-A_{en} & B_{en}-A_{ne} \\ -\Theta_{en} & \Theta_{en} \\ A_{en}\beta_{en} & -A_{ne}\alpha_{ne} \\ B_{en}A_{en} & -B_{en}A_{ne} \end{matrix}\right\| \times \left\|\begin{matrix} \varphi^x_e \\ W_e \\ \varphi^z_e \\ \varphi^x_n \\ W_n \\ \varphi^z_e \end{matrix}\right\|. \tag{4.2}$$

The forces in the elastic connections with the joints are determined by the following formula:

$$\left\|\begin{matrix} M^x_n \\ M^z_n \\ R_n \end{matrix}\right\| = \left\|\begin{matrix} C^x_\varphi & 0 & 0 \\ 0 & 0 & C^z_\varphi \\ 0 & C_W & 0 \end{matrix}\right\| \times \left\|\begin{matrix} \varphi^x_n \\ W_n \\ \varphi^z_n \end{matrix}\right\| + \left\|\begin{matrix} M^{x0}_n \\ M^{z0}_n \\ P^0_n \end{matrix}\right\|, \tag{4.3}$$

where α, β, θ are the end stiffness characteristics of the bars as given in Table 1.

Let us form complete force matrices, separately for the bars and the joints of the grid system by the method of successive increments. The corresponding displacement matrices for the bars will contain repetitive elements. Let us use the transformation matrix $\|a\|$ to refine them in the same way as for a plane orthogonal bar system.

The condition for static equilibrium of the joints is expressed by the following formula by using the connection matrix $\|D\|$ for all displacements.

$$\|D\|(\|k_{\mathrm{col}}\| \times \|a\| \times \|z\|) + \|E\|(\|k_{\mathrm{joint}}\| \times \|z\| + \|S^0_{\mathrm{joint}}\|) = 0. \tag{4.4}$$

The unknown displacements $\|z\|$ are found from equation (4.4):

$$\|z\| = (\|a\|^{-1} \times \|k_{\mathrm{col}}\|^{-1} \times \|D\|^{-1} + \|k_{\mathrm{joint}}\|^{-1} \times \|E\|)(-\|E\| \times \|S^0_{\mathrm{joint}}\|). \tag{4.5}$$

The forces in the bars and the elastic connections with the joints of the grid system are determined by the following formulae:

$$\|S_{\text{col}}\| = \|k_{\text{col}}\| \times \|a\| \times \|z\|. \qquad (4.6)$$

$$\|S_{\text{joint}}\| = \|k_{\text{joint}}\| \times \|z\| + \|S^0_{\text{joint}}\|. \qquad (4.7)$$

The algebraic equation of equilibrium of the nth joint of the cross bar system, obtained by using the above matrix method, will be:

$$\begin{aligned}
&\beta_{mn}\varphi^x_m + (\alpha_{nk} + \alpha_{nm} + \theta_{en} + \theta_{nf} + C^x_\varphi)\,\varphi^x_n + \beta_{nk}\varphi^x_k + A_{mn}W_m \\
&\quad + (-A_{mn} + A_{nk})\,W_n - A_{nk}W_k - \theta_{en}\varphi^x_e - \theta_{nf}\,\varphi^x_f + M^{x0}_n = 0; \\
&\beta_{en}\varphi^z_e + (\alpha_{ne} + \alpha_{nf} + \theta_{nm} + \theta_{nk} + C^z_\varphi)\,\varphi^z_n + \beta_{nf}\varphi^z_f + A_{en}W_e \\
&\quad + (-A_{ne} + A_{nf})\,W_n - A_{nf}W_f - \theta_{mn}\varphi^z_m - \theta_{nk}\,\varphi^z_k + M^{z0}_n = 0; \\
&A_{mn}\varphi^x_m + (A_{nm} - A_{nk})\,\varphi^x_n - A_{kn}\varphi^x_k + A_{en}\varphi^z_e + (A_{ne} - A_{nf})\varphi^z_n \\
&\quad + A_{fn}\varphi^z_f + B_{mn}W_m - (B_{mn} + B_{nk} + B_{en} + B_{nf} + C_W)W_n \\
&\quad + B_{nk}W_k + B_{en}W_e + B_{nf}W_f + P_n = 0. \qquad (4.8)
\end{aligned}$$

The first and second equations of system (4.8) show that the sum of all moments acting along axes x and z is zero. The third equation shows that the sum of reactions acting along y axis is zero.

2. PARTICULAR CASES OF A GRID SYSTEM

Let us examine a grid system with short bars with large sections. Let us consider the shear and bending strains but neglect the torsional strength. Considering joints which are not fixed (Fig. 65), the equilibrium equation will assume the form:

$$\begin{aligned}
&\beta_{mn}\varphi^x_m + (\alpha_{nk} + \alpha_{nm})\,\varphi^x_n + \beta_{nk}\varphi^x_k + A_{mn}W_m + (-A_{mn} + A_{nk})W_n \\
&\quad - A_{nk}W_k + M^{x0}_n = 0; \\
&\beta_{en}\varphi^z_e + (\alpha_{ne} + \alpha_{nf})\,\varphi^z_n + \beta_{nf}\varphi^z_f + A_{en}W_e + (A_{ne} + A_{nf})W_n \\
&\quad - A_{nf}W_f + M^{z0}_n = 0; \\
&A_{mn}\varphi^x_m + (A_{nm} - A_{nk})\,\varphi^x_n - A_{kn}\,\varphi^x_k + A_{en}\varphi^z_e + (A_{ne} - A_{nf})\,\varphi^z_n \\
&\quad + A^z_{fn}\varphi^z_f + B_{mn}W_m - (B_{mn} + B_{nk} + B_{en} + B_{nf})\,W_n \\
&\quad + B_{nk}W_k + B_{en}W_e + B_{nf}W_f + P_n = 0, \qquad (4.9)
\end{aligned}$$

where

$$\alpha = \frac{4[EI]}{h}\,\frac{[GF]\,h^2 + 3\,[EI]}{[GF]\,h^2 + 12\,[EI]}; \quad \beta = \frac{2\,[EI]}{h}\,\frac{[GF]\,h^2 - 6\,[EI]}{[GF]h^2 + 12\,[EI]};$$

$$A = \frac{6\,[EI]\,[GF]}{[GF]\,h^2 + 12\,[EI]}; \quad B = \frac{12\,[EI]\,[GF]}{h\,[GF]\,h^2 + 12\,[EI]}. \qquad (4.10)$$

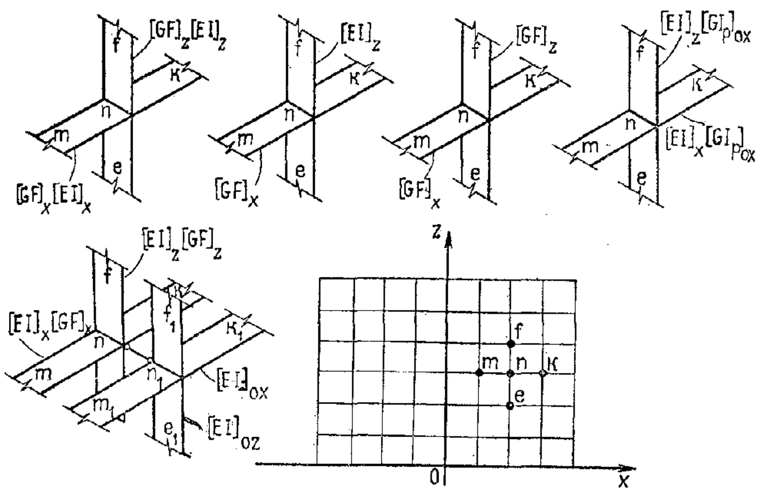


Fig. 65. A grid system with different stiffness characteristics of the bars.

Let us examine a case in which the horizontal bars of the cross system undergo only shear strain and the verticals only bending. The equilibrium condition is expressed by two algebraic equations:

$$2i_{en}\varphi_e^z + 4(i_{ne}+i_{nf})\varphi_n^z + 2i_{nf}\varphi_f^z + \frac{6i_{en}}{h_{en}}W_e + \left(-\frac{6i_{ne}}{h_{ne}} + \frac{6i_{nf}}{h_{nf}}\right)W_n - \frac{6i_{nf}}{h_{nf}}W_f + M_n^{z0} = 0;$$

$$\frac{6i_{en}}{h_{en}}\varphi_e^z + \left(\frac{6i_{ne}}{h_{ne}} - \frac{6i_{nf}}{h_{nf}}\right)\varphi_n^z + \frac{6i_{fn}}{h_{fn}} + g_{mn}W_m - \left(g_{mn} + g_{nk} + \frac{12i_{en}}{h_{en}^2} + \frac{12i_{nf}}{h_{nf}^2}\right)W_n + g_{nk}W_k + \frac{12i_{en}}{h_{en}^2}W_e + \frac{12i_{nk}}{h_{nk}^2}W_f + P_n = 0, \qquad (4.11)$$

where $i = \frac{[EI]}{h}$ and $g = \frac{[GF]}{l}$ are the running bending and shear stiffnesses per unit length of the bar.

If the horizontal and vertical bars of the grid system undergo only shear strain, then the equilibrium condition is expressed by one algebraic equation which is similar to the suspension cable-girder lattice system.

$$g_{mn}W_m - (g_{mn} + g_{nk} + g_{en} + g_{nf})W_n + g_{nk}W_k + g_{en}W_e + g_{nf}W_f + P_n = 0. \qquad (4.12)$$

When the horizontal and vertical bars of a grid system undergo only bending strain, the equilibrium condition is expressed by these three well known equations:

$$2i_{mn}\varphi_m^x + (4i_{nk} + 4i_{nm} + \Theta_{en} + \Theta_{nf})\varphi_n^x + 2i_{nk}\varphi_k^x$$
$$+ \frac{6i_{mn}}{h_{mn}} W_m + \left(-\frac{6i_{mn}}{h_{mn}} + \frac{6i_{nk}}{h_{nk}}\right) W_n - \frac{6i_{nk}}{h_{nk}} W_k - \Theta_{en}\varphi_e^x - \Theta_{nf}\varphi_f^x$$
$$+ M_n^{x0} = 0;$$

$$2i_{en}\varphi_e^z + (4i_{ne} + 4i_{nf} + \Theta_{nm} + \Theta_{nk})\varphi_n^z + 2i_{nf}\varphi_f^z$$
$$+ \frac{6i_{en}}{h_{en}} W_e + \left(-\frac{6i_{ne}}{h_{ne}} + \frac{6i_{nf}}{h_{nf}}\right) W_n$$
$$- \frac{6i_{nf}}{h_{nf}} W_f - \Theta_{mn}\varphi_m^z - \Theta_{nk}\varphi_k^z + M_n^{z0} = 0;$$

$$\left(\frac{6i_{nm}}{h_{nm}} - \frac{6i_{nk}}{h_{nk}}\right)\varphi_n^x + \frac{6i_{mn}}{h_{mn}}\varphi_m^x - \frac{6i_{nk}}{h_{nk}}\varphi_k^x + \left(\frac{6i_{ne}}{h_{ne}} - \frac{6i_{nf}}{h_{nf}}\right)\varphi_n^z$$
$$+ \frac{6i_{fn}}{h_{fn}}\varphi_f^z + \frac{12i_{mn}}{h_{mn}^2} W_m - 12\left(\frac{i_{mn}}{h_{mn}^2} + \frac{i_{nk}}{h_{nk}^2} + \frac{i_{en}}{h_{en}^2} + \frac{i_{nf}}{h_{nf}^2}\right) W_n$$
$$+ \frac{12i_{nk}}{h_{nk}^2} W_k + \frac{12i_{en}}{h_{en}^2} W_e + \frac{12i_{nf}}{h_{nf}^2} W_f + P_n = 0. \qquad (4.13)$$

By eliminating the torsional stiffness of bars in these equations we get the equations for cross beams.

3. DIFFERENTIAL EQUATIONS OF EQUILIBRIUM FOR A GRID SYSTEM

Let us examine a grid bar system consisting of n horizontal and m vertical bars. The number of algebraic equations required to solve it is $3\,nm$. Assuming $\frac{1}{n} \to 0$ and $\frac{1}{m} \to 0$, we assume the system under study consists of uniform horizontal and vertical strips and not separate discrete members.

To obtain the differential equations of equilibrium for the grid system let us substitute displacements $\|z\|$ and $\|x\|$ in the joint equilibrium equation 4.4 and examine the limits for $\frac{h}{H} \to 0$ and $\frac{h}{L} \to 0$.

When the vertical and horizontal members of the grid system undergo shear strain, the differential equation of equilibrium will be:

$$C_x \frac{\partial^2 W}{\partial x^2} + C_z \frac{\partial^2 W}{\partial z^2} + q = 0, \qquad (4.14)$$

where $C_x = \frac{[GF]^x}{h}$ and $C_z = \frac{[GF]^z}{l}$ are the shear stiffnesses of strips along the x and z axes, respectively, and $q = \frac{P}{lh}$ is the load per unit area.

Equation (4.14) is similar to the equation of an anistropic membrane.

When the vertical and horizontal members of a grid system are rigidly connected with each other and undergo bending strain, the differential equation of equilibrium will be:

$$B_x \frac{\partial^4 W}{\partial x^4} + (K_x + K_z) \frac{\partial^4 W}{\partial x^2 \partial z^2} + B_z \frac{\partial^4 W}{\partial z^4} - q = 0, \tag{4.15}$$

where $B_x = \frac{[EI]^x}{h}$ and $B_z = \frac{[EI]^z}{l}$ are the bending stiffnesses of the strips while $K_x = \frac{[GI_0]^x}{h}$ and $K_z = \frac{[GI_0]^z}{l}$ are the torsional stiffnesses of the strips along the x and z axes, respectively.

Equation (4.15) is similar to the equation of an anistropic plate considering the torsion of strips. This is the equation of a beam lattice. Considering the shear and bending strains in the vertical and horizontal members of a grid system and not considering torsion, the equilibrium equation will be:

$$B_x \frac{\partial^4 W}{\partial x^4} + B_z \frac{\partial^4 W}{\partial z^4} - \frac{B_x}{C_x} \frac{\partial^2 q}{\partial x^2} - \frac{B_z}{C_z} \frac{\partial^2 q}{\partial z^2} - q = 0. \tag{4.16}$$

This equation is similar to the equation of a thick anistropic plate.

If the horizontal members undergo shear and the vertical ones bending strain, the differential equation of equilibrium for the grid system is expressed by the following formula:

$$C_x \frac{\partial^2 W}{\partial x^2} - K_x \frac{\partial^4 W}{\partial z^2 \partial x^2} - B_z \frac{\partial^4 W}{\partial z^4} + q = 0. \tag{4.17}$$

Let us examine a grid system consisting of two-layer horizontal and vertical members in which the first layer undergoes shear and bending strain and the second undergoes bending strain. Here the differential equation of equilibrium for the grid system is expressed by the formula:

$$\frac{B_x B_{x0}}{C_x} \frac{\partial^6 W}{\partial x^6} + \frac{B_z B_{z0}}{C_z} - (B_x + B_{x0}) \frac{\partial^4 W}{\partial x^4} - (B_z + B_{z0}) \frac{\partial^4 W}{\partial z^4}$$
$$- \frac{B_x}{C_x} \frac{\partial^2 q}{\partial x^2} - \frac{B_z}{C_z} \frac{\partial^2 q}{\partial z^2} + q = 0, \tag{4.18}$$

where $B_{x0} = \frac{[EI]_0^x}{h}$ and $B_{z0} = \frac{[EI]_0^z}{l}$ are the bending stiffnesses of the second layer of members in the grid.

When the first layer of members undergoes shear strain and the second ones bending, the differential equation of the grid system becomes:

$$B_{x0}\frac{\partial^4 W}{\partial x^4} + B_{z0}\frac{\partial^4 W}{\partial z^4} - C_x\frac{\partial^2 W}{\partial x^2} - C_z\frac{\partial^2 W}{\partial z^2} - q = 0. \tag{4.19}$$

Equation (4.19) is similar to that for a two-layer membrane.

When the members of a grid system are multi-layered, in which the layers undergo shear and the joints between layers are connected by elastic braces, the differential equation of equilibrium will be:

$$C'_x\frac{\partial^2 W}{\partial x^2} + C'_z\frac{\partial^2 W}{\partial z^2} + F_y\frac{\partial^2 W}{\partial y^2} + q' = 0, \tag{4.20}$$

where $C'_x = \frac{[GF]^x}{ah}$ and $C'_z = \frac{[GF]^z}{al}$ are the shear stiffnesses of strips, $F_y = \frac{[EF]^y}{lh}$ is the longitudinal stiffness along y axis, $q' = \frac{P}{alh}$ is the force per unit volume of the grid system and a is the distance between the layers of the grid system.

Equation (4.20) is the equation of a multi-layer membrane. If the horizontal members of the grid system have different shear, bending, shear-bending stiffness characteristics, then the grid system may be considered a discretely-continuous analytical model (Fig. 66) for which each vertical strip will have its own differential equation. For example, using the given method, for the ith strip, which undergoes shear strain, we obtain the following differential equation of equilibrium:

$$[GF]^z_i\frac{d^2W_i}{dz^2} + C_x\left[\frac{W_{i-1}}{l_{i-1,\,i}} + \frac{W_{i+1}}{l_{i,\,i+1}} - \left(\frac{1}{l_{i-1,\,i}} + \frac{1}{l_{i,\,i+1}}\right)W_i\right] + q_i = 0. \tag{4.21}$$

When the distances between the vertical strips are equal, equation (4.21) becomes:

$$[GF]^z_i\frac{d^2W_i}{dz^2} + \frac{C_x}{l}(W_{i-1} - 2W_i + W_{i+1}) + q_i = 0. \tag{4.22}$$

Assuming that the adjacent $(i + 1)$th vertical strip undergoes bending strain, the differential equation will be:

$$[EI]^z_{i+1}\frac{d^4W_{i+1}}{dz^4} - \frac{C_x}{l}(W_i - 2W_{i+1} + W_{i+2}) - q_{i+1} = 0. \tag{4.23}$$

If $(i + 2)$th vertical strip undergoes shear-cum-bending strain, its differential equation will be

$$[EI]^z_{i+2}\,\frac{d^4W_{i+2}}{dz^4}-\frac{E[I]^z_{i+2}}{[GF]^z_{i+2}}\,\frac{d^2q_{i+2}}{dz^2}-\frac{C_x}{l}\,(W_{i+1}-2W_{i+2}+W_{i+3})$$
$$-q_{i+2}=0. \qquad (4.24)$$

Simultaneous solution of the differential equations of equilibrium for vertical strips will give the stress strain state of the grid system.

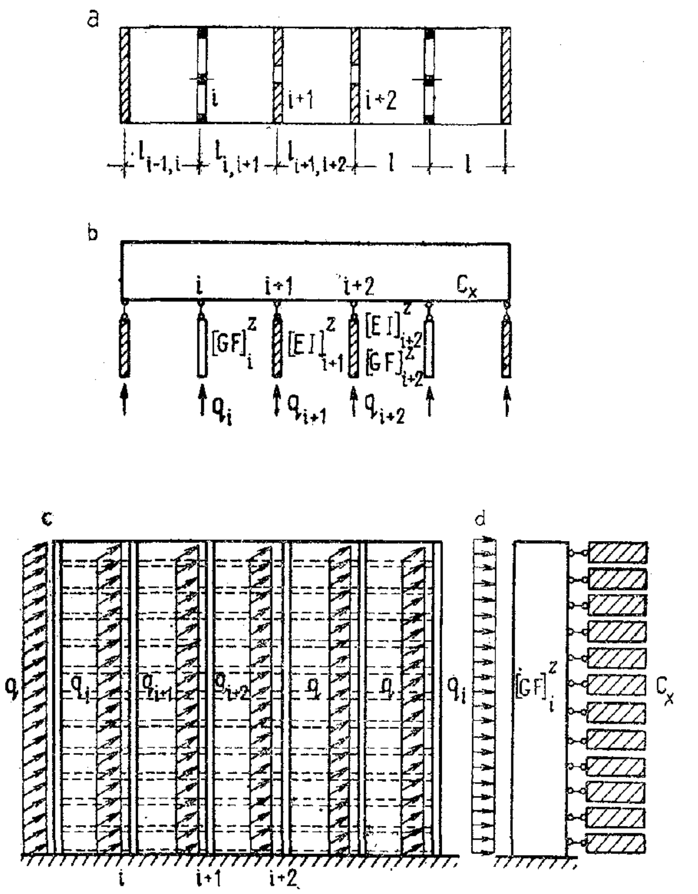


Fig. 66. Discretely continuous analytical model of a grid system. a—plan of building; b—plan of model; c—elevation of model; d—end view of model.

4. USE OF DIFFERENTIAL EQUATIONS OF A GRID SYSTEM FOR DIFFERENT SCHEMATIC MODELS OF BUILDINGS

Under the action of horizontal forces a panel-type building may be represented by a grid system consisting of horizontal and vertical members. The horizontal force is applied on the joints in the form of concentrated forces and the vertical in the form of concentrated floor loads transmitted to the vertical members. In the process of bending, the latter creates an additional horizontal force distributed along the height and determined by (2.47). Considering this force the equilibrium equation becomes non-linear. To obtain a linear equation the distributed vertical force n is considered as an equivalent concentrated force N applied to the upper end of the vertical member. This force is determined from the condition that the work done by forces n and N, when the bar is displaced due to the action of uniformly distributed horizontal force, is equal:

$$\frac{1}{2} N \int_0^H \left(\frac{dW}{dz}\right)^2 dz = \frac{1}{2} \int_0^H \left(\frac{dW}{dz}\right)^2 n\,(H-z)\,dz. \tag{4.25}$$

The magnitude of displacement W is taken from (3.64) and by substituting in it λ from (3.50) we obtain the differential expression for W:

$$\frac{dW}{dz} = \frac{q}{[GF]} \left(H - z + \frac{z^3\lambda^2}{6H^2} - \frac{z^2\lambda^2}{2H} + \frac{z\lambda^2}{2}\right). \tag{4.26}$$

By substituting this expression in (4.25) and integrating the expression, we obtain the magnitude of the equivalent force N:

$$N = nH\xi, \tag{4.27}$$

where ξ is the equivalence coefficient. It is equal to:

$$\xi = \frac{0.625\lambda^4 + 5.55\lambda^2 + 25}{1.785\lambda^4 + 10\lambda^2 + 33.3}. \tag{4.28}$$

For a bar undergoing shear for $\lambda < 0.8$, the equivalence coefficient $\xi = 0.75$ while for a bar undergoing bending for $\lambda > 8$, $\xi = 0.35$.

Let us examine a multistory frame panel building which has large dimensions in plan and has built-up floors where horizontal (floors) and vertical (frames) members undergo shear strain. The differential equation of equilibrium for such a building will be:

$$C_x \frac{\partial^2 W}{\partial x^2} + (C_z - N)\frac{\partial^2 W}{\partial z^2} + q = 0, \tag{4.29}$$

where N is the equivalent longitudinal force per running meter length along the top of the building.

For a high rise building or with narrow frames the equilibrium equation of the building will be:

$$C_x \frac{\partial^2 W}{\partial x^2} - B_z \left(1 - \frac{N}{C_z}\right) \frac{\partial^4 W}{\partial z^4} - N \frac{\partial^2 W}{\partial z^2} - \frac{B_z}{C_z} \frac{\partial^2 q}{\partial z^2} + q = 0. \quad (4.30)$$

For a braced building in which vertical diaphragms undergo bending strain and the torsional strength of the floor is also considered, the following equation is obtained:

$$C_x \frac{\partial^2 W}{\partial x^2} - K_x \frac{\partial^4 W}{\partial z^2 \partial x^2} - B_z \frac{\partial^4 W}{\partial z^4} - N \frac{\partial^2 W}{\partial z^2} + q = 0; t. \quad (4.31)$$

A building with monolithic floor and vertical diaphragms undergoing bending strain will have the following equation:

$$B_x \frac{\partial^4 W}{\partial x^4} + (K_x + K_z) \frac{\partial^4 W}{\partial x^2 \partial z^2} + B_z \frac{\partial^4 W}{\partial z^4} + N \frac{\partial^2 W}{\partial z^2} - q = 0. \quad (4.32)$$

In panel buildings of the frame brace type the differential equation of equilibrium may be written for each vertical strip, considering the design model of the building to be discretely continuous.

If the floor undergoes shear strain, the diaphragm of the ith strip is defined by equation (4.23).

Equation (4.29) may be used for the frame brace building. In this case the diaphragms are considered stiffeners for which the boundary condition for compatibility of deformation will be:

$$[EI]_q^z \frac{\partial^4 W}{\partial z^4} - C_x \left.\frac{\partial W}{\partial x}\right|_{x=a} = 0. \quad (4.33)$$

In the frame brace structure with many vertical diaphragms undergoing bending strain, the building is considered two-layered in the vertical direction and single-layered in the horizontal direction. For this the equation will be:

$$C_x \frac{\partial^2 W}{\partial x^2} + (C_z - N) \frac{\partial^2 W}{\partial z^2} - B_{z0} \frac{\partial^4 W}{\partial z^4} + q = 0. \quad (4.34)$$

For braced construction, that is, when the frame stiffness is small, the equilibrium equation will be:

$$C_x \frac{\partial^2 W}{\partial x^2} - N \frac{\partial^2 W}{\partial z^2} - B_{z0} \frac{\partial^4 W}{\partial z^4} + q = 0. \quad (4.35)$$

For the vertical direction of a building, if it is considered multilayered and the struts between the layers are represented by elastic braces that resist bending, the differential equation of equilibrium will be of the type (4.20).

5. GENERAL DIFFERENTIAL EQUATION OF EQUILIBRIUM FOR A BUILDING

If we consider the building as a three-dimensional spatial system, we can simplify its analytical model. Such a model may be represented by intersecting plates and in the vertical direction as a collection of multilayer plates with elastic braces. From such a model we can analyze the spatial behavior of different schemes of buildings for transverse, longitudinal and vertical forces.

Let us examine a frame-panel building in which the vertical members are frames and diaphragms. Frames are represented as a single layer equivalent bar with equivalent shear and bending stiffnesses. The diaphragms are similarly represented by a single layer equivalent bar with equivalent bending stiffness. Where the frames and diaphragms are simultaneously present in a building, the vertical members are represented by two layers: the first layer is a frame and the second a diaphragm. Floors form the horizontal members. A floor is represented by a single layer equivalent bar with shear and bending stiffnesses. The torsional strains of horizontal and vertical members are neglected, as they are small.

A view of the building with the x, y, z coordinate axes is shown in Fig. 67. The displacements along the respective axes are considered functions of two variables $W(x, z)$ and $V(x, y)$ and in the vertical direction, as a function of three variables $U(x, y, z)$. The condition of building equilibrium may be represented by differential equations in terms of the forces acting on the building and the displacements.

In its general form the differential equation of equilibrium for the building will be:

$$\frac{B_z^{\text{trans}} B_{z0}^{\text{trans}}}{C_z^{\text{trans}}} \frac{\partial^6 W}{\partial z^6} - \left(B_z^{\text{trans}} + B_{z0}^{\text{trans}} - \frac{B_z^{\text{trans}}}{C_z^{\text{trans}}} N \right) \frac{\partial^4 W}{\partial z^4} - B_x^{\text{trans}} \frac{\partial^4 W}{\partial x^4}$$

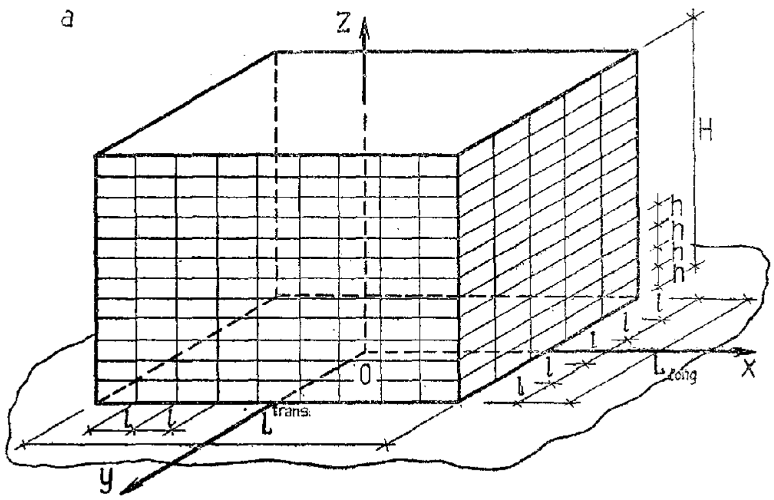
$$- N^{\text{trans}} \frac{\partial^2 W}{\partial z^2} - \frac{B_z^{\text{trans}}}{C_z^{\text{trans}}} \frac{\partial^2 q^{\text{trans}}}{\partial z^2} - \frac{B_x^{\text{trans}}}{C_x^{\text{trans}}} \frac{\partial^2 q^{\text{trans}}}{\partial x^2} + q^{\text{trans}} = 0;$$

$$\frac{B_z^{\text{long}} B_{z0}^{\text{long}}}{C_z^{\text{long}}} \frac{\partial^6 V}{\partial z^6} - \left(B_z^{\text{long}} + B_{z0}^{\text{long}} - \frac{B_z^{\text{long}}}{C_z^{\text{long}}} N^{\text{long}} \right) \frac{\partial^4 V}{\partial z^4} - B_y^{\text{long}} \frac{\partial^4 V}{\partial y^4}$$

$$- N^{\text{long}} \frac{\partial^2 V}{\partial z^2} - \frac{B_z^{\text{long}}}{C_z^{\text{long}}} \frac{\partial^2 q^{\text{long}}}{\partial z^2} - \frac{B_y^{\text{long}}}{C_y^{\text{long}}} \frac{\partial^2 q^{\text{long}}}{\partial y^2} + q^{\text{long}} = 0;$$

$$C_x^{\text{ver}} \frac{\partial^2 U}{\partial x^2} + C_y^{\text{ver}} \frac{\partial^2 U}{\partial y^2} + F_z^{\text{ver}} \frac{\partial^2 u}{\partial z^2} + q^{\text{ver}} = 0. \qquad (4.36)$$

The first two equations of (4.36) express the building equilibrium under



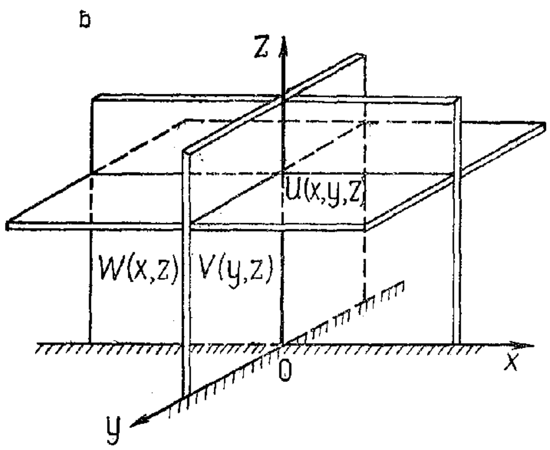


Fig. 67. Building model.
a—general view; b—three-dimensional analytical model.

the action of transverse and longitudinal forces and the third that under vertical forces.

Let

$$B_x^{trans} = \frac{\Sigma[EI]_x^{trans}}{H} \text{ and } B_y^{long} = \frac{\Sigma[EI]_y^{long}}{H}$$

be the equivalent bending stiffnesses of horizontal members in the transverse and longitudinal directions and

$$B_z^{trans} = \frac{\Sigma[EI]_z^{trans}}{L^{trans}} \text{ and } B_z^{long} = \frac{\Sigma[EI]_z^{long}}{L^{long}}$$

be the equivalent bending stiffnesses of frames in the transverse and longitudinal directions. Similarly,

$$C_z^{\text{trans}} = \frac{\Sigma[GF]_z^{\text{trans}}}{L^{\text{trans}}} \quad \text{and} \quad C_z^{\text{long}} = \frac{\Sigma[GF]_z^{\text{long}}}{L^{\text{long}}}$$

will be the equivalent shear stiffnesses of frames in the transverse and longitudinal directions and

$$B_{z0}^{\text{trans}} = \frac{\Sigma[EI]_{z0}^{\text{trans}}}{L^{\text{trans}}} \quad \text{and} \quad B_{z0}^{\text{long}} = \frac{\Sigma[EI]_{z0}^{\text{long}}}{L^{\text{long}}}$$

will be the equivalent bending stiffnesses of diaphragms in the transverse and longitudinal directions. Similarly,

$$C_x^{\text{ver}} = \frac{\Sigma[GF]_x^{\text{ver}}}{HL^{\text{long}}} \quad \text{and} \quad C_y^{\text{ver}} = \frac{\Sigma[GF]_y^{\text{ver}}}{HL^{\text{trans}}}$$

will be the shear stiffnesses of horizontal members in the vertical plane while

$$F_z^{\text{ver}} = \frac{\Sigma[EF]_z^{\text{ver}}}{L^{\text{trans}}L^{\text{long}}}$$

will be the equivalent longitudinal stiffness of vertical members;
q^{trans}, q^{long} will be the distributed forces acting on the area of transverse and longitudinal facades of the building and

$$q^{\text{ver}} = \frac{\Sigma Q_{\text{story}}}{HL^{\text{trans}}L^{\text{long}}}$$

will be the force per unit volume of the building.

$$N^{\text{trans}} = \xi \frac{\Sigma Q_{\text{story}}}{HL^{\text{trans}}} \text{ and } N^{\text{long}} = \xi \frac{\Sigma Q_{\text{story}}}{HL^{\text{long}}} - \lambda_z^{\text{trans}} = H\sqrt{\frac{C_z^{\text{trans}}}{B_z^{\text{trans}}}}; \tag{4.37}$$

will be the vertical distributed forces applied on top of the building in the transverse and longitudinal directions; ξ is the equivalence coefficient.

Figure 68 shows the analytical model of the building. Depending on the magnitudes of equivalent stiffnesses, the differential equations of equilibrium (4.36) may define different structural models of frame-panel buildings. By introducing the concept of stiffness characteristic of a building and its members, we can define the corresponding building model. To determine the predominant types of strains due to horizontal forces, let us write the stiffness characteristics λ^{trans} and λ^{long} which are equal for the vertical members:

$$\lambda_z^{\text{long}} = H\sqrt{\frac{C_z^{\text{long}}}{B_z^{\text{long}}}}. \tag{4.38}$$

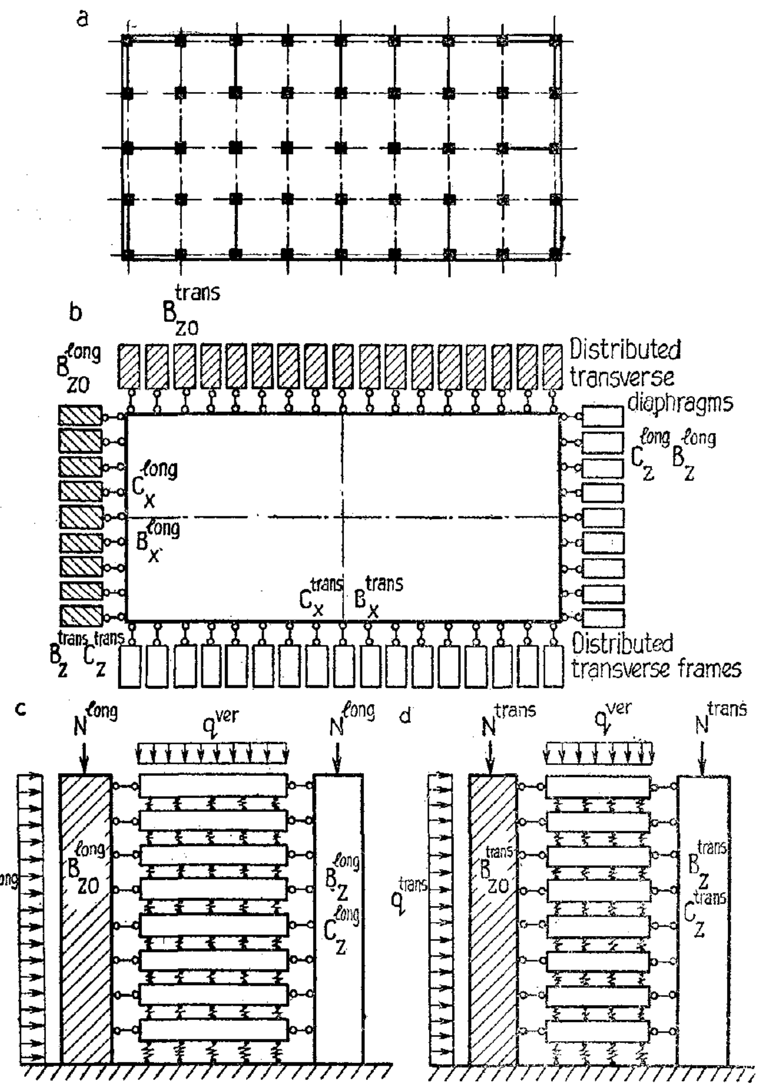


Fig. 68. Analytical model of a building with floor undergoing strain. a—building plan; b—plan of model; c—elevation of model; d—end view of model.

For horizontal members:

$$\lambda_x^{\text{trans}} = \frac{L^{\text{trans}}}{2}\sqrt{\frac{C_x^{\text{trans}}}{B_x^{\text{trans}}}}; \tag{4.39}$$

$$\lambda_y^{\text{long}} = \frac{L^{\text{long}}}{2}\sqrt{\frac{C_y^{\text{long}}}{B_y^{\text{long}}}}. \tag{4.40}$$

In vertical members due to the action of longitudinal forces:

$$\mu = H \frac{C_{\text{hor}}}{F_z^{\text{ver}}}, \tag{4.41}$$

where C_{hor} is the modulus of subgrade reaction for soil foundation.

We may write the following conditions: for $\lambda \leqslant 0.8$—since the building member undergoes only shear strain we can assume that all values of $B = \infty$; for $\lambda \geqslant 8$ only bending strain occurs and hence we can assume all values of $C = \infty$. For $0.8 < \lambda < 8$ both shear and bending strains occur. In such a case, for $\mu \leqslant 30$ longitudinal deformations are considered in a vertical member and for $\mu > 30$ they are not.

The stiffness characteristic of the building k_{build} determines the nature of the building design model defined by the predominant stiffness of a frame or a diaphragm. It is expressed by the formulae:

$$k_{\text{build}}^{\text{trans}} = H \sqrt{\frac{1}{1 + 0.15\,(\lambda_z^{\text{trans}})^2} \frac{C_z^{\text{trans}}}{B_{z0}^{\text{trans}}}}; \tag{4.42}$$

$$k_{\text{build}}^{\text{trans}} = H \sqrt{\frac{1}{1 + 0.15(\lambda_z^{\text{long}})^2} \frac{C_z^{\text{long}}}{B_{z0}^{\text{long}}}}\,. \tag{4.43}$$

For $k_{\text{build}} \leqslant 0.8$ we have the brace model for the building where all load is taken by vertical diaphragms and hence we may assume $C_z = 0$. For $k_{\text{build}} \geqslant 8$ the building has a frame model in which all load is taken by frames and hence we assume $B_{z0} = 0$. For $0.8 < k_{\text{build}} < 8$ we have the frame brace model, in which the load is redistributed between frames and diaphragms.

Let us examine particular cases of a frame-panel building with built-up floors for which $\lambda_x^{\text{trans}} < 0.8$ and $\lambda_y^{\text{long}} < 0.8$.

If we have frames in the transverse direction for which $\lambda_z^{\text{trans}} < 0.8$ and $k_{\text{build}}^{\text{trans}} > 8$, in the longitudinal direction, frames and diaphragms for which $\lambda_{\text{build}}^{\text{long}} < 0.8$ and $k_{\text{build}}^{\text{long}} < 0.8$ and for the vertical direction $\mu > 30$, then the differential equations of the building will be of the following types:

$$C_z^{\text{trans}} \frac{\partial^2 W}{\partial z^2} + C_x^{\text{trans}} \frac{\partial^2 W}{\partial x^2} - N^{\text{trans}} \frac{\partial^2 W}{\partial z^2} + q^{\text{trans}} = 0;$$

$$B_{z0}^{\text{long}} \frac{\partial^4 V}{\partial z^4} - C_y^{\text{long}} \frac{\partial^2 V}{\partial y^2} + N^{\text{long}} \frac{\partial^2 V}{\partial z^2} - q^{\text{long}} = 0;$$

$$C_x^{\text{ver}} \frac{\partial^2 u}{\partial x^2} + C_y^{\text{ver}} \frac{\partial^2 u}{\partial y^2} + q^{\text{ver}} = 0. \tag{4.44}$$

If we have frames and diaphragms in the transverse and longitudinal directions for which $\lambda_z^{\text{trans}} < 0.8$; $0.8 < k_{\text{build}}^{\text{trans}} < 8$; $\lambda_z^{\text{long}} < 0.8$; $0.8 < k_{\text{build}}^{\text{long}} < 8$ and for the vertical direction $\mu < 30$, then we have the frame brace model in the transverse and longitudinal directions and hence the differential equations of equilibrium for the building will be:

$$B_{z0}^{\text{trans}} \frac{\partial^4 W}{\partial z^4} - (C_z^{\text{trans}} - N^{\text{trans}}) \frac{\partial^2 W}{\partial z^2} - C_x^{\text{trans}} \frac{\partial^2 W}{\partial x^2} - q^{\text{trans}} = 0;$$

$$B_{z0}^{\text{long}} \frac{\partial^4 V}{\partial z^4} - (C_z^{\text{long}} - N^{\text{long}}) \frac{\partial^2 V}{\partial z^2} - C_y^{\text{long}} \frac{\partial^2 V}{\partial y^2} - q^{\text{long}} = 0;$$

$$C_x^{\text{ver}} \frac{\partial^2 u}{\partial x^2} + C_y^{\text{ver}} \frac{\partial^2 u}{\partial y^2} + F_z^{\text{ver}} \frac{\partial^2 u}{\partial z^2} + q^{\text{ver}} = 0. \qquad (4.45)$$

6. DIFFERENTIAL EQUATION OF EQUILIBRIUM FOR A BUILDING WITH RIGID FLOOR

In practical building construction one often finds point-type high rise buildings with finite dimensions in plan, which are however small compared to the height. In such cases the floor may be considered a rigid disk. The design model of such buildings is discretely continuous and is obtained as follows: all vertical, transverse and longitudinal members of the building are moved out in the respective directions and joined with the rigid disk (floor) (Fig. 69). The frames and diaphragms will be treated as the vertical members. Reactions due to vertical members act on the rigid disk. The reaction from the ith vertical transverse strip which happens to be a frame, will be equal to:

$$([GF]_i^{\text{trans}} - N_i^{\text{trans}}) \frac{d^2 W_i}{dz^2} + q_i^{\text{trans}} = -R_i^{\text{trans}}. \qquad (4.46)$$

The reaction from the jth strip, which happens to be a diaphragm, will be:

$$-[EI]_{0j}^{\text{trans}} \frac{d^4 W_j}{dz^4} - N_j^{\text{trans}} \frac{d^2 W_j}{dz^2} + q_j^{\text{trans}} = -R_j^{\text{trans}}. \qquad (4.47)$$

The turning of vertical members causes torsional reactions to act on the rigid disk. These reactions are equal to:

$$[GI_0]_i^{\text{trans}} \frac{d^2 \varphi}{dz^2} = -m_i^{\text{trans}}; \qquad (4.48)$$

$$[GI_0]_j^{\text{trans}} \frac{d^2 \varphi}{dz^2} = -m_j^{\text{trans}}. \qquad (4.49)$$

Similarly we obtain the reactions due to vertical members in the longitudinal direction. Considering the floor disk as a solid body we can write the following static equilibrium equations for it:

$$\sum X = \sum R_i^{\text{long}} + \sum R_j^{\text{long}} = 0;$$

$$\sum Y = \sum R_i^{\text{trans}} + \sum R_i^{\text{trans}} = 0;$$

$$\sum M = \sum R_i^{\text{trans}} a_i^{\text{trans}} + \sum R_j^{\text{trans}} b_j^{\text{trans}} + \sum R_i^{\text{long}} a_i^{\text{long}} + \sum R_j^{\text{long}} b_j^{\text{long}} + \sum m_i^{\text{trans}} + \sum m_j^{\text{trans}} + \sum m_i^{\text{long}} + \sum m_j^{\text{long}} = 0. \qquad (4.50)$$

Let point O of the rigid disk be displaced by W_0 and V_0 along the y and x axes, respectively, and let the entire disk at this point turn through an

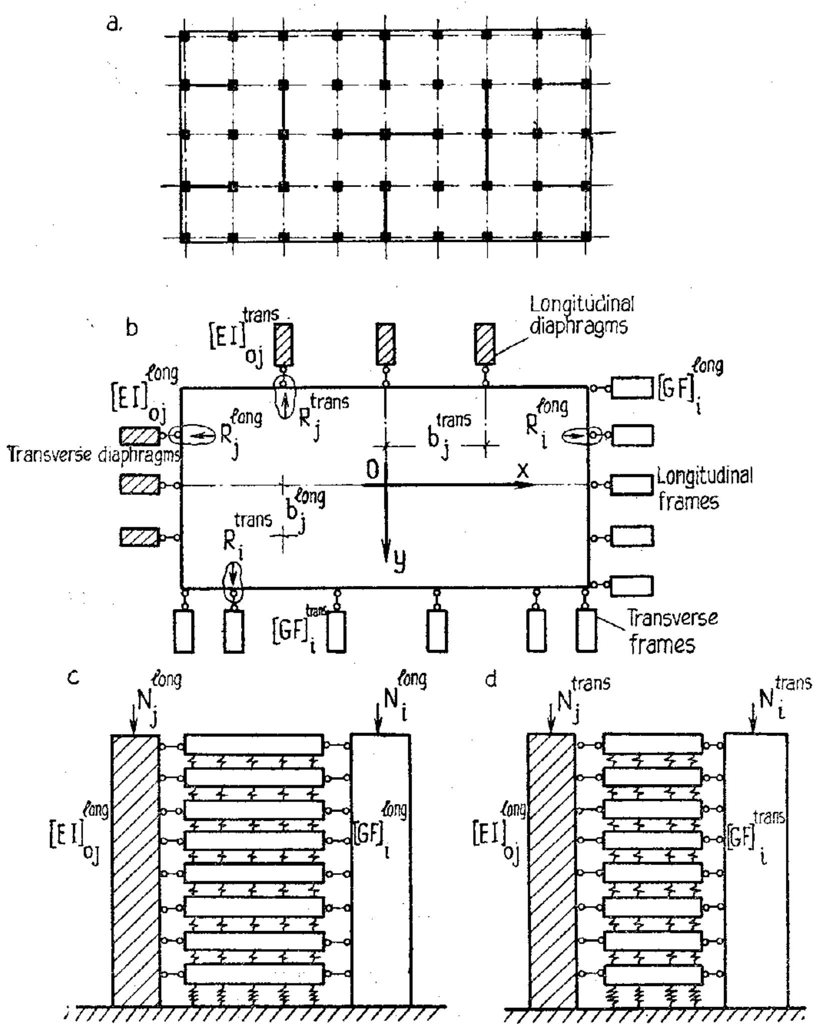


Fig. 69. Analytical model of a building with rigid floor.
a—plan of building; b—plan of model; c—elevation of model; d—end view of model.

angle φ. Points i and j of the rigid disk will undergo the following displacements along the y and x axes:

$$
\begin{aligned}
W_i &= W_0 + \varphi\, a_i^{\text{trans}};\\
W_j &= W_0 + \varphi\, b_j^{\text{trans}};\\
V_i &= V_0 + \varphi\, a_i^{\text{long}};\\
V_j &= V_0 + \varphi\, b_j^{\text{long}}.
\end{aligned} \tag{4.51}
$$

By substituting the values of reactions and displacements (4.51) in the system of equations (4.50) we obtain the equilibrium equations for the building in the following form:

$$
\begin{aligned}
&\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} \frac{d^4 W_0}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{trans}} - \sum_{i=1}^{n} ([GF]-N)_i^{\text{trans}}\right] \frac{d^2 W_0}{dz^2}\\
&+ \sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} \frac{d^4 \varphi}{dz^4} b_j^{\text{trans}} + \left[\sum_{j=1}^{k} N_j^{\text{trans}} b_j^{\text{trans}} - \sum_{i=1}^{n} ([GF]-N)_i^{\text{trans}} a_i^{\text{trans}}\right]\\
&\times \frac{d^2 \varphi}{dz^2} - \sum_{i=1}^{n} q_i^{\text{trans}} - \sum_{j=1}^{k} q_j^{\text{trans}} = 0;\\
&\sum_{j=1}^{k} [EI]_{0j}^{\text{long}} \frac{d^4 V_0}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{long}} - \sum_{i=1}^{n} [GF]-N_i^{\text{long}}\right] \frac{d^2 V_0}{dz^2} + \sum_{j=1}^{k} [EI]_{0j}^{\text{long}} b_j^{\text{long}}\\
&\times \frac{d^4 \varphi}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{long}} b_j^{\text{long}} - \sum_{i=1}^{n} ([GF]-N)_j^{\text{long}} a_i^{\text{long}}\right] \frac{d^2 \varphi}{dz^2} - \sum_{i=1}^{n} q_i^{\text{long}}\\
&- \sum_{j=1}^{k} q_j^{\text{long}} = 0;\\
&\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} b_j^{\text{trans}} \frac{d^4 W_0}{dz^4} + \sum_{j=1}^{k} [EI]_{0j}^{\text{long}} b_j^{\text{long}} \frac{d^4 V_0}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{trans}} b_j^{\text{trans}}\right.\\
&\left. - \sum_{i=1}^{n} ([GF]-N)_i^{\text{trans}} a_i^{\text{trans}}\right] \frac{d^2 W_0}{dz^2} + \left[\sum_{j=1}^{k} N_j^{\text{long}} b_j^{\text{long}} - \sum ([GF]-N)_j^{\text{long}}\right.\\
&\left.\times a_i^{\text{long}}\right] \frac{d^2 V_0}{dz^2} + \left[\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} (b_j^{\text{trans}})^2 + \sum_{i=1}^{k} [EI]_{0j}^{\text{long}} (b_j^{\text{long}})^2\right] \frac{d^4 \varphi}{dz^4}\\
&+ \left[\sum_{j=1}^{k} N_j^{\text{trans}} (b_j^{\text{trans}})^2 - \sum_{j=1}^{n} ([GF]-N)_i^{\text{trans}} (a_i^{\text{trans}})^2 + \sum_{i=1}^{k} N_j^{\text{long}} (b_j^{\text{long}})^2\right.\\
&\left. - \sum_{i=1}^{k} ([GF]-N)_i^{\text{long}} (a_i^{\text{long}})^2\right] \frac{d^2 \varphi}{dz^2} - \sum_{i=1}^{n} q_i^{\text{trans}} a_i^{\text{trans}} - \sum_{j=1}^{k} q_j^{\text{trans}} b_j^{\text{trans}}\\
&- \sum_{i=1}^{n} q_i^{\text{long}} a_i^{\text{long}} - \sum_{j=1}^{k} q_j^{\text{long}} a_j^{\text{long}} - \left(\sum_{i=1}^{n} [GI_0]_i^{\text{trans}} + \sum_{j=1}^{k} [GI_0]_j^{\text{trans}}\right.\\
&\left. + \sum_{i=1}^{n} [GI_0]_i^{\text{trans}} + \sum_{j=1}^{k} [GI_0]_j^{\text{long}}\right) \frac{d^2 \varphi}{dz^2} = 0.
\end{aligned} \tag{4.52}
$$

To obtain the differential equations of equilibrium for the building, add the equation which expresses the building equilibrium in the vertical direction to the system of equations (4.52):

$$\sum_{i=1}^{n} [EF]_i^{\text{ver}} \frac{d^2 U}{dz^2} + \sum_{i=1}^{n} q_i^{\text{ver}} = 0.$$

When considering different structural models of buildings these equations may be simplified depending on the stiffness characteristic k_{build} of a building. When the building twists $k_{\text{build}}^{\text{tor}}$ will be equal to:

$$k_{\text{build}}^{\text{tor}} =$$

$$H\sqrt{\frac{\sum [GF]_i^{\text{trans}} (a_i^{\text{trans}})^2 + \sum [GF]_i^{\text{long}} (a_i^{\text{long}})^2 + \sum [GI_0]_i^{\text{trans}} + \sum [GI_0]_i^{\text{long}}}{\sum [EI]_j^{\text{trans}} (b_j^{\text{trans}})^2 + \sum [EI]_j^{\text{long}} (b_j^{\text{long}})^2}}. \tag{4.53}$$

If the building has only plane diaphragms and there are no centers of rigidity then the torsional stiffness of vertical members may be neglected.

If point O of the rigid floor disk is a center of rigidity of the vertical members and center of gravity for the longitudinal forces, then the following expressions will be equal to zero:

$$\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} b_j^{\text{trans}} = 0; \quad \sum_{j=1}^{k} [EI]_{0j}^{\text{long}} b_j^{\text{long}} = 0; \quad \sum_{j=1}^{k} N_j^{\text{trans}} b_j^{\text{trans}} = 0;$$

$$\sum_{j=1}^{k} N_j^{\text{long}} b_j^{\text{long}} = 0; \quad \sum_{i=1}^{n} ([GF] - N)_i^{\text{trans}} a_i^{\text{trans}} = 0; \quad \sum_{i=1}^{n} ([GF] - N)_i^{\text{long}} a_i^{\text{long}} = 0.$$

System 4.52 is then separated into three independent equations:

$$\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} \frac{d^4 W_0}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{trans}} - \sum_{i=1}^{n} ([GF] - N)_i^{\text{trans}} \right]$$

$$\times \frac{d^2 W}{dz^2} - \sum_{i=1}^{n} q_i^{\text{trans}} - \sum_{j=1}^{k} q_j^{\text{trans}} = 0;$$

$$\sum_{j=1}^{k} [EI]_{0j}^{\text{long}} \frac{d^4 V_0}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{long}} - \sum_{i=1}^{n} ([GF] - N)_i^{\text{long}} \frac{d^2 V_0}{dz^2} \right.$$

$$- \sum_{i=1}^{k} q_i^{\text{long}} - \sum_{j=1}^{k} q_j^{\text{long}} = 0;$$

$$\left[\sum_{j=1}^{k} [EI]_{0j}^{\text{trans}} (b_j^{\text{trans}})^2 + \sum_{j=1}^{k} [EI]_{0j}^{\text{long}} (b_j^{\text{long}})^2 \right] \frac{d^4 \varphi}{dz^4} + \left[\sum_{j=1}^{k} N_j^{\text{trans}} (b_j^{\text{trans}})^2 \right.$$

$$- \sum_{i=1}^{n} ([GF] - N)_i^{\text{trans}} (a_i^{\text{trans}})^2 + \sum_{j=1}^{k} N_j^{\text{long}} (b_j^{\text{long}})^2$$

$$\left. - \sum_{i=1}^{n} ([GF] - N)_i^{\text{long}} (a_i^{\text{long}})^2 \right] \frac{d^2 \varphi}{dz^2} - \left(\sum_{i=1}^{n} [GF_0]_i^{\text{trans}} + \sum_{j=1}^{k} [GI_0]_j^{\text{trans}} \right.$$

$$+\sum_{i=1}^{n}[GI_0]_i^{\text{long}}+\sum_{j=1}^{k}[GI_0]_j^{\text{long}}\Big)\frac{d^2\varphi}{dz^2}-\sum_{i=1}^{n}q_i^{\text{trans}}a_i^{\text{trans}}-\sum_{j=1}^{k}q_j^{\text{trans}}b_j^{\text{trans}}$$

$$-\sum_{i=1}^{n}q_i^{\text{long}}a_i^{\text{long}}-\sum_{j=1}^{k}q_j^{\text{long}}b_j^{\text{long}}=0. \tag{4.54}$$

The first and second equations of (4.54) express the building equilibrium condition for transverse and longitudinal forces; this gives rise to corresponding independent displacements. The third equation expresses the equilibrium condition due to the combined action of transverse and longitudinal forces which cause the building to turn.

If the load is symmetrical with respect to the center of rigidity of the building, then the third equation loses its significance because the building will not undergo torsion. Due to different shear and bending strains of vertical members, the center of rigidity of the building will change along the height. Therefore, we shall use an approximate method to determine it. According to this method, the coordinates of the center of rigidity will be:

$$x_0^{\text{rigid}}=\frac{\sum([GF]-N)_i^{\text{trans}}a_i^{\text{trans}}+\sum([EI]+N)_j^{\text{trans}}b_j^{\text{trans}}}{\sum([GF]-N)_i^{\text{trans}}+\sum([EI]+N)_j^{\text{trans}}};$$

$$y_0^{\text{rigid}}=\frac{\sum([GF]-N)_i^{\text{long}}a_i^{\text{long}}+\sum([EI]+N)_j^{\text{long}}b_j^{\text{long}}}{\sum([GF]-N)_i^{\text{long}}+\sum([EI]+N)_j^{\text{long}}}, \tag{4.55}$$

where a and b are the distances from the edge of the building to the corresponding vertical member.

7. BOUNDARY CONDITIONS FOR THE DIFFERENTIAL EQUATIONS OF EQUILIBRIUM OF BUILDINGS

To consider the action of horizontal forces, the three dimensional design model of a building is represented by a plate-like system fixed at one end. The base of the system is a beam which is elastically fixed against vertical and horizontal displacements and torsion. Under the action of vertical forces the design model of the building is represented by a multi-layer membrane with elastic braces between the layers. The lower layer of the membrane is a slab on an elastic foundation.

The building foundation may be designed in the form of either a monolithic slab or as grid work continuous beams or as isolated footings, connected by wall beams. Its solution leads to a horizontal beam with equivalent shear and bending stiffnesses, resting on an elastic foundation. It will have free ends in the horizontal direction. In shear deformation of the floor the boundary conditions will be:

$$\text{for } x = -\frac{L}{2}: \quad \frac{\partial W}{\partial x} = 0;$$

$$\text{for } x = \frac{L}{2}: \quad \frac{\partial W}{\partial x} = 0. \tag{4.56}$$

In bending deformation of the floor, the boundary conditions will be:

$$\text{for } x = -\frac{L}{2}: \quad \frac{\partial^2 W}{\partial x^2} = 0; \ \frac{\partial^3 W}{\partial x^3} = 0;$$

$$\text{for } x = \frac{L}{2}: \quad \frac{\partial^2 W}{\partial x^2} = 0; \ \frac{\partial^3 W}{\partial x^3} = 0. \tag{4.57}$$

For the vertical direction, in a frame-type building in which the foundation beam is rigidly fixed to the foundation, the boundary conditions will be:

$$\text{for } z = 0: \quad W = 0;$$

$$\text{for } z = H: \quad \frac{\partial W}{\partial z} = 0. \tag{4.58}$$

If the foundation is a monolithic slab, with equivalent bending stiffness $[EI]_F^x$ resting on elastic foundation, the boundary conditions will be:

$$\text{for } z = 0: \quad [EI]_F^x \frac{\partial^4 W}{\partial x^4} - C_z \frac{\partial W}{\partial z} + C^w W = 0;$$

$$\text{for } z = H: \quad \frac{\partial W}{\partial z} = 0. \tag{4.59}$$

In a grid work of strip foundations with equivalent shear stiffness $[GF]_F^x$ resting on a rigid foundation, the boundary conditions will be:

$$\text{for } z = 0: \quad [GF]_F^x \frac{\partial^2 W}{\partial x^2} + C_z \frac{\partial W}{\partial z} C^w W = 0;$$

$$\text{for } z = H: \quad \frac{\partial W}{\partial z} = 0. \tag{4.60}$$

Isolated foundations, for which the equivalent rigidity is zero, shall be considered as a foundation beam with elastic restraint against horizontal displacements. It will then have the following boundary conditions:

$$z = 0: \quad C_z \frac{\partial W}{\partial z} = C^w W;$$

$$z = H: \quad \frac{\partial W}{\partial z} = 0; \tag{4.61}$$

$$C^w = C_{\text{soil}}^{\text{hor}} F_F, \tag{4.62}$$

where F_F is the area of foundation and $C_{\text{soil}}^{\text{hor}}$ is the modulus of subgrade reaction of the foundation under horizontal displacement.

For the vertical direction of a frame brace model of a building with uniformly spaced diaphragms along its length, the boundary conditions will be as follows when the foundation beam is rigidly fixed to the foundation:

$$\text{for } z = 0: \quad W = 0; \quad \frac{\partial W}{\partial z} = 0;$$

$$\text{for } z = H: \quad \frac{\partial^2 W}{\partial z^2} = 0;$$

$$B_{z0} \frac{\partial^3 W}{\partial z^3} - C_z \frac{\partial W}{\partial z} = 0. \tag{4.63}$$

When the foundation is elastically fixed against horizontal displacements and against turning in the case of a foundation slab, the boundary conditions will be:

$$\text{for } z = 0: \quad [GI_0]_F^x \frac{\partial^3 W}{\partial x^2 \partial z} + B_{z0} \frac{\partial^2 W}{\partial z^2} - C^{\varphi} \frac{\partial W}{\partial z} = 0;$$

$$[EI]_F^x \frac{\partial^4 W}{\partial x^4} + B_{z0} \frac{\partial^3 W}{\partial z^3} - C_z \frac{\partial W}{\partial z} - C^w W = 0;$$

$$\text{for } z = H: \quad \frac{\partial^2 W}{\partial z^2} = 0;$$

$$B_{z0} \frac{\partial^3 W}{\partial z^3} - C_z \frac{\partial W}{\partial z} = 0. \tag{4.64}$$

In a grid work of strip foundations:

$$\text{for } z = 0: \quad [GI_0]_F^x \frac{\partial^3 W}{\partial x^2 \partial z} + B_{z0} \frac{\partial^2 W}{\partial z^2} - C^{\varphi} \frac{\partial W}{\partial z} = 0;$$

$$[GF]_F^x \frac{\partial^2 W}{\partial x^2} - B_{z0} \frac{\partial^3 W}{\partial z^3} + C_z \frac{\partial W}{\partial z} + C^w W = 0;$$

$$\text{for } z = H: \quad \frac{\partial^2 W}{\partial z^2} = 0;$$

$$B_{z0} \frac{\partial^3 W}{\partial z^3} - C_z \frac{\partial W}{\partial z} = 0. \tag{4.65}$$

For isolated foundations the boundary conditions will be:

$$\text{for } z = 0: \quad B_{z0} \frac{\partial^2 W}{\partial z^2} - C^{\varphi} \frac{\partial W}{\partial z} = 0;$$

$$[GF]_F^x \frac{\partial^2 W}{\partial x^2} - B_{z0} \frac{\partial^3 W}{\partial z^3} + C_z \frac{\partial W}{\partial z} + C^w W = 0;$$

$$\text{for } z = H: \quad \frac{\partial^2 W}{\partial z^2} = 0;$$

$$B_{z0} \frac{\partial^3 W}{\partial z^3} - C_z \frac{\partial W}{\partial z} = 0; \tag{4.66}$$

$$C^{\varphi} = \frac{I_F C_{soil}}{L}, \qquad (4.67)$$

where I_F is the moment of inertia of the foundation and C_{soil} is the foundation coefficient for non-uniform ground compression.

In diaphragms concentrated along the length of a building, the boundary conditions will be (4.59), (4.60) and (4.61). Where the diaphragms are located, we shall have additional boundary conditions as for a slab with stiffeners:

for $x = a$:

$$B_{z0} \frac{\partial^4 W}{\partial z^4} - C_x \frac{\partial W}{\partial x} = 0. \qquad (4.68)$$

When the building is under the action of vertical forces, the boundary conditions will be:

$$\text{for } x = -\frac{L^{trans}}{2}: \quad \frac{\partial U}{\partial x} = 0;$$

$$\text{for } x = \frac{L^{trans}}{2}: \quad \frac{\partial U}{\partial x} = 0;$$

$$\text{for } y = -\frac{L^{long}}{2}: \quad \frac{\partial U}{\partial y} = 0;$$

$$\text{for } y = \frac{L^{long}}{2}: \quad \frac{\partial U}{\partial y} = 0;$$

$$\text{for } z = H: \quad \frac{\partial U}{\partial z} = 0. \qquad (4.69)$$

In isolated foundations:

for $z = 0$:

$$F_z^{ver} \frac{\partial U}{\partial z} = C_{soil} U. \qquad (4.70)$$

In a foundation slab:

for $z = 0$:

$$B_F^x \frac{\partial^4 U}{\partial x^4} + B_F^y \frac{\partial^4 U}{\partial y^4} + (K_F^x + K_F^y) \frac{\partial^4 U}{\partial x^2 \partial y^2} - F_z^{ver} \frac{\partial U}{\partial z} + C_{soil} U = 0. \qquad (4.71)$$

If the foundation is absolutely rigid we shall have:

for $z = 0$:

$$U = 0. \qquad (4.72)$$

8. DETERMINATION OF FORCES IN THE VARIOUS MEMBERS OF THE BUILDING

The system of differential equations of equilibrium for a building (4.36) and the particular cases (4.44) and (4.55) are solved separately for transverse and longitudinal horizontal forces as well as for vertical forces by the method of single, double and triple trigonometric series. After determining the unknown displacements W, V and U in the horizontal and vertical members of buildings, we find the combined resultant forces. Under the action of horizontal forces the combined bending moments and transverse forces in a diaphragm will be:

$$M_{z0}^{\text{trans}} = -[EI]_{z0}^{\text{trans}} \frac{\partial^2 W}{\partial z^2};$$

$$M_{z0}^{\text{long}} = -[EI]_{z0}^{\text{long}} \frac{\partial^2 V}{\partial z^2};$$

$$Q_{z0}^{\text{trans}} = -[EI]_{z0}^{\text{trans}} \frac{\partial^3 W}{\partial z^3};$$

$$Q_{z0}^{\text{long}} = -[EI]_{z0}^{\text{long}} \frac{\partial^3 V}{\partial z^3}, \tag{4.73}$$

where $[EI]_{z0} = B_{z0}\, l$ and l is the distance between the diaphragms.

The combined transverse force in a frame will be:

$$Q_{z}^{\text{trans}} = [GF]_{z}^{\text{trans}} \frac{\partial W}{\partial z};$$

$$Q_{z}^{\text{long}} = [GF]_{z}^{\text{long}} \frac{\partial V}{\partial z}. \tag{4.74}$$

When the floor undergoes shear deformation the forces in it will be:

$$Q_{x}^{\text{trans}} = [GF]_{x}^{\text{trans}} \frac{\partial W}{\partial x};$$

$$Q_{y}^{\text{long}} = [GF]_{y}^{\text{long}} \frac{\partial V}{\partial y}; \tag{4.75}$$

and for bending:

$$M_{x}^{\text{trans}} = -[EI]_{x}^{\text{trans}} \frac{\partial^2 W}{\partial x^2};$$

$$M_{y}^{\text{long}} = -[EI]_{y}^{\text{long}} \frac{\partial^2 V}{\partial y^2};$$

$$Q_{x}^{\text{trans}} = -[EI]_{x}^{\text{trans}} \frac{\partial^3 W}{\partial x^3};$$

$$Q_{y}^{\text{long}} = -[EI]_{y}^{\text{long}} \frac{\partial^3 V}{\partial y^3}. \tag{4.76}$$

When the building is loaded vertically, the transverse and longitudinal forces for the vertical direction will be:

$$Q_x^{\text{ver}} = [GF]_x^{\text{ver}} \frac{\partial U}{\partial x};$$

$$Q_y^{\text{ver}} = [GF]_y^{\text{ver}} \frac{\partial U}{\partial y};$$

$$N_z^{\text{ver}} = [EF]_z^{\text{ver}} \frac{\partial U}{\partial z}. \tag{4.77}$$

The values of these forces may be substituted in differential equations (4.44) and (4.45) after which their order will diminish and the unknown in these equations will be the forces which are to be solved. The values obtained are applied to individual stories and bays of the building and then the forces in their constituent elements are determined as in (3.9).

CHAPTER 5

Natural Oscillations of Buildings

1. GENERAL DIFFERENTIAL EQUATIONS OF NATURAL OSCILLATIONS OF BUILDINGS

To solve the problem of building oscillation we use d'Alembert's principle, which treats the dynamic problem as a static one by adding inertia forces to elastic forces. Here the displacements are taken to be functions of time $W(x, z, t)$, $V(z, y, t)$ and $U(x, y, z, t)$.

By multiplying the second differential of displacement with respect to time by the respective mass, we obtain the inertia forces which become the external forces in the case of natural oscillations of a building:

$$q^{\text{trans}} = -m^{\text{trans}} \frac{\partial^2 W}{\partial t^2};$$

$$q^{\text{long}} = -m^{\text{long}} \frac{\partial^2 V}{\partial t^2};$$

$$q^{\text{ver}} = -m^{\text{ver}} \frac{\partial^2 U}{\partial t^2}. \tag{5.1}$$

The uniformly distributed masses in the respective directions of the building are equal to:

$$m^{\text{trans}} = \frac{\Sigma Q_{\text{story}}}{g L^{\text{trans}} H};$$

$$m^{\text{long}} = \frac{\Sigma Q_{\text{story}}}{g L^{\text{long}} H};$$

$$m^{\text{ver}} = \frac{\Sigma Q_{\text{story}}}{g L^{\text{trans}} L^{\text{long}} H}, \tag{5.2}$$

where Q_{story} is the weight of a story and g is the acceleration due to gravity.

By substituting the values of inertia forces (5.1) in equation (4.36) we obtain the general differential equations of natural oscillations of a building:

$$\frac{B_z^{\text{trans}} B_{z0}^{\text{trans}}}{C_z^{\text{trans}}} \frac{\partial^6 W}{\partial z^6} - \left(B_z^{\text{trans}} + B_{z0}^{\text{trans}} - \frac{B_z^{\text{trans}}}{C_z^{\text{trans}}} N^{\text{trans}} \right) \frac{\partial^4 W}{\partial z^4} -$$

$$-B_x^{\text{trans}}\frac{\partial^4 W}{\partial x^4}-N^{\text{trans}}\frac{\partial^2 W}{\partial z^2}+m^{\text{trans}}\frac{B_z^{\text{trans}}}{C_z^{\text{trans}}}\frac{\partial^4 W}{\partial z^2\,dt^2}$$

$$+m^{\text{trans}}\frac{Bx^{\text{trans}}}{C_x^{\text{trans}}}\cdot\frac{\partial^4 W}{\partial x^2\,dt^2}-m^{\text{trans}}\frac{\partial^2 W}{\partial t^2}=0;$$

$$\frac{B_z^{\text{long}}B_{z0}^{\text{long}}}{C_z^{\text{long}}}\frac{\partial^6 V}{\partial z^6}-\left(B_z^{\text{long}}+B_{z0}^{\text{long}}-\frac{B_z^{\text{long}}}{C_z^{\text{long}}}N^{\text{long}}\right)\frac{\partial^4 V}{\partial z^4}-B_y^{\text{long}}\frac{\partial^4 V}{\partial y^4}$$

$$-N^{\text{long}}\frac{\partial^2 V}{\partial z^2}+m^{\text{long}}\frac{B_z^{\text{long}}}{C_z^{\text{long}}}\frac{\partial^4 V}{\partial z^2\,dt^2}+m^{\text{long}}\frac{B_y^{\text{long}}}{C_y^{\text{long}}}\frac{\partial^4 V}{\partial y^2\,dt^2}-m^{\text{long}}\frac{\partial^2 V}{\partial t^2}=0;$$

$$C_x^{\text{ver}}\frac{\partial^2 U}{\partial x^2}+C_y^{\text{ver}}\frac{\partial^2 U}{\partial y^2}+F_z\frac{\partial^2 U}{\partial z^2}-m^{\text{ver}}\frac{\partial^2 U}{\partial t^2}=0. \tag{5.3}$$

The first, second and third equations (5.3) refer respectively to transverse, longitudinal and vertical oscillations of a building.

Let us study a building which is frame type in the transverse direction and brace type in the longitudinal direction. We assume the building is undergoing shear strain. The differential equations of natural oscillations of the building will be:

$$(C_z^{\text{trans}}-N^{\text{trans}})\frac{\partial^2 W}{\partial z^2}+C_x^{\text{trans}}\frac{\partial^2 W}{\partial x^2}-m^{\text{trans}}\frac{\partial^2 W}{\partial t^2}=0;$$

$$B_{z0}^{\text{long}}\frac{\partial^4 V}{\partial z^4}-C_y^{\text{long}}\frac{\partial^2 V}{\partial y^2}+N^{\text{long}}\frac{\partial^2 V}{\partial z^2}+m^{\text{long}}\frac{\partial^2 V}{\partial t^2}=0;$$

$$C_x^{\text{ver}}\frac{\partial^2 U}{\partial x^2}+C_y^{\text{ver}}\frac{\partial^2 U}{\partial y^2}+F_z^{\text{ver}}\frac{\partial^2 U}{\partial z^2}-m^{\text{ver}}\frac{\partial^2 U}{\partial t^2}=0. \tag{5.4}$$

Let us examine a frame brace type building in the transverse and longitudinal directions with the floor undergoing bending strain in the transverse and shear strain in the longitudinal directions. The building rests on a soft foundation. The differential equations of natural oscillations of the building will be:

$$B_{z0}^{\text{trans}}\frac{\partial^4 W}{\partial z^4}-(C_z^{\text{trans}}-N^{\text{trans}})\frac{\partial^2 W}{\partial z^2}+B_x^{\text{trans}}\frac{\partial^4 W}{\partial x^4}+m^{\text{trans}}\frac{\partial^2 W}{\partial t^2}=0;$$

$$B_{z0}^{\text{long}}\frac{\partial^4 V}{\partial z^4}-(C_z^{\text{long}}-N^{\text{long}})\frac{\partial^2 V}{\partial z^2}-C_y^{\text{long}}\frac{\partial^2 V}{\partial y^2}+m^{\text{long}}\frac{\partial^2 V}{\partial t^2}=0;$$

$$C_x^{\text{ver}}\frac{\partial^2 U}{\partial x^2}+C_y^{\text{ver}}\frac{\partial^2 U}{\partial y^2}-m^{\text{ver}}\frac{\partial^2 U}{\partial t^2}=0. \tag{5.5}$$

In point type high rise buildings represented by the frame brace scheme in which the floors are absolutely rigid disks, the differential equations of natural oscillations of the building will be:

$$\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}\frac{\partial^4 W_0}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{trans}}-\sum_{i=1}^{n}([GF]-N)_i^{\text{trans}}\right]\frac{\partial^2 W_0}{\partial z^2}$$

$$+\left(\sum_{i=1}^{n}m_i^{\text{trans}}-\sum_{j=1}^{k}m_j^{\text{trans}}\right)\frac{\partial^2 W_0}{\partial t^2}+\left(\sum_{i=1}^{n}m_i^{\text{trans}}a_i^{\text{trans}}+\sum_{j=1}^{k}m_j^{\text{trans}}b_j^{\text{trans}}\right)$$

$$\times\frac{\partial^2\varphi}{\partial t^2}=0;$$

$$\sum_{j=1}^{k}[EI]_{0j}^{\text{long}}\frac{\partial^4 V_0}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{long}}-\sum_{i=1}^{n}([GF]-N)_i^{\text{long}}\right]\frac{\partial^2 V_0}{\partial z^2}$$

$$+\left(\sum_{i=1}^{n}m_i^{\text{long}}+\sum_{j=1}^{k}m_j^{\text{long}}\right)\frac{\partial^2 V_0}{\partial t^2}+\left(\sum_{i=1}^{n}m_i^{\text{long}}a_i^{\text{long}}+\sum_{j=1}^{k}m_j^{\text{long}}b_j^{\text{long}}\right)$$

$$\times\frac{\partial^2\varphi}{\partial t^2}=0;$$

$$\left[\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}(b_j^{\text{trans}})^2+\sum_{j=1}^{k}[EI]_{0j}^{\text{long}}(b_j^{\text{long}})^2\right]\frac{\partial^4\varphi}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{trans}}(b_j^{\text{trans}})^2\right.$$

$$-\sum_{i=1}^{n}([GF]-N)_i^{\text{trans}}(a_i^{\text{trans}})^2+\sum_{j=1}^{k}N_j^{\text{long}}(b_j^{\text{long}})^2-\sum_{i=1}^{n}([GF]-N)_i^{\text{long}}$$

$$\left.\times(a_i^{\text{long}})^2\right]\frac{\partial^2\varphi}{\partial z^2}-\left(\sum_{i=1}^{n}[GI_0]_i^{\text{trans}}+\sum_{j=1}^{k}[GI_0]_j^{\text{trans}}+\sum_{i=1}^{n}[GI_0]_i^{\text{long}}+\sum_{j=1}^{k}\right.$$

$$\left.\times[GI_0]_j^{\text{long}}\right)\frac{\partial^2\varphi}{\partial z^2}+\left(\sum_{i=1}^{n}m_i^{\text{trans}}a_i^{\text{trans}}+\sum_{j=1}^{k}m_j^{\text{trans}}b_j^{\text{trans}}\right)\frac{\partial^2 W_0}{\partial t^2}$$

$$+\left(\sum_{i=1}^{n}m_i^{\text{long}}a_i^{\text{long}}+\sum_{j=1}^{k}m_j^{\text{long}}b_j^{\text{long}}\right)\frac{\partial^2 V_0}{\partial t^2}+\left[\sum_{i=1}^{n}m_i^{\text{trans}}(a_i^{\text{trans}})^2\right.$$

$$\left.+\sum_{j=1}^{k}m_j^{\text{trans}}(b_j^{\text{trans}})^2+\sum_{i=1}^{n}m_i^{\text{long}}(a_i^{\text{long}})^2+\sum_{j=1}^{k}m_j^{\text{long}}(b_j^{\text{long}})^2\right]\frac{\partial^2\varphi}{\partial t^2}=0;$$

$$\sum_{i=1}^{n}[EF]_i^{\text{ver}}\frac{\partial^2 U}{\partial z^2}-\sum_{i=1}^{n}m_i^{\text{ver}}\frac{\partial^2 U}{\partial t^2}=0, \tag{5.6}$$

where m_i and m_j are the masses per unit length of the vertical members of frames and diaphragms.

The first three equations of (5.6) constitute a total system giving the expressions for the transverse, longitudinal and torsional oscillations of a building. The fourth equation is independent of the first three and expresses the vertical oscillations of the building.

If the center of rigidity of the building coincides with the center of mass, then the following expressions are equal to zero:

$$\sum_{i=1}^{n}m_i^{\text{trans}}a_i^{\text{trans}}+\sum_{j=1}^{k}m_j^{\text{trans}}b_j^{\text{trans}}=0;$$

$$\sum_{i=1}^{n}m_i^{\text{long}}a_i^{\text{long}}+\sum_{j=1}^{k}m_j^{\text{long}}b_j^{\text{long}}=0.$$

In this case the first three equations of (5.6) become independent and separately express the transverse, longitudinal and torsional oscillations of the building. In that case differential equations of natural oscillations acquire the following form:

$$\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}\frac{\partial^4 W_0}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{trans}}-\sum_{i=1}^{n}([GF]-N)_i^{\text{trans}}\right]\frac{\partial^2 W_0}{\partial z^2}$$
$$+\left(\sum_{i=1}^{n}m_i^{\text{trans}}+\sum_{j=1}^{k}(m_j^{\text{trans}}\right)\frac{\partial^2 W_0}{\partial t^2}=0;$$
$$\sum_{j=1}^{k}[EI]_{0j}^{\text{long}}\frac{\partial^4 V_0}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{long}}-\sum_{i=1}^{n}([GF]-N)_i^{\text{long}}\right]\frac{\partial^2 V_0}{\partial z^2}$$
$$+\left(\sum_{i=1}^{n}m_i^{\text{long}}+\sum_{j=1}^{k}m_j^{\text{long}}\right)\frac{\partial^2 V_0}{\partial t^2}=0;$$
$$\left[\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}(b_j^{\text{trans}})^2+\sum_{j=1}^{k}[EI]_{0j}^{\text{long}}(b_j^{\text{trans}})^2\right]\frac{\partial^4\varphi}{\partial z^4}+\left[\sum_{j=1}^{k}N_j^{\text{trans}}(b_j^{\text{trans}})^2\right.$$
$$-\sum_{i=1}^{n}([GF]-N_i)^{\text{trans}}(a_i^{\text{trans}})^2+\sum_{j=1}^{k}N_j^{\text{long}}(b_j^{\text{long}})^2-\sum_{i=1}^{n}([GF)-N)_i^{\text{long}}$$
$$\left.\times(a_i^{\text{long}})^2\right]\frac{\partial^2\varphi}{\partial t^2}-\left(\sum_{i=1}^{n}[GI_0]_i^{\text{trans}}+\sum_{j=1}^{k}[GI_0]_j^{\text{trans}}+\sum_{i=1}^{n}[GI_0]_i^{\text{long}}\right.$$
$$\left.+\sum_{j=1}^{k}[GI_0]_j^{\text{long}}\right)\frac{\partial^2\varphi}{\partial z^2}+\left[\sum_{i=1}^{n}m_i^{\text{trans}}(a_i^{\text{trans}})^2+\sum_{j=1}^{k}m_j^{\text{trans}}(b_j^{\text{trans}})^2\right.$$
$$\left.+\sum_{i=1}^{n}m_i^{\text{long}}(a_i^{\text{long}})^2+\sum_{j=1}^{k}m_j^{\text{long}}(b_j^{\text{long}})^2\right]\frac{\partial^2\varphi}{\partial t^2}=0;$$
$$\sum_{i=1}^{n}[EI]_i^{\text{ver}}\frac{\partial^2 U_0}{\partial z^2}-\sum_{i=1}^{n}m_i^{\text{ver}}\frac{\partial^2 U_0}{\partial t^2}=0. \tag{5.7}$$

The position of center of mass on the floor is determined by the following coordinates:

$$x_0^m=\frac{\Sigma m_i^{\text{trans}}a_i^{\text{trans}}+\Sigma m_j^{\text{trans}}b_j^{\text{trans}}}{\Sigma m_i^{\text{trans}}+\Sigma m_j^{\text{trans}}};$$
$$y_0^m=\frac{\Sigma m_i^{\text{long}}a_i^{\text{long}}+\Sigma m_j^{\text{long}}b_j^{\text{long}}}{\Sigma m_i^{\text{long}}+\Sigma m_j^{\text{long}}}, \tag{5.8}$$

where a and b are the distances between the edge of the building and the respective center of gravity of the vertical member.

To solve the differential equations of natural oscillations of a building let us write the displacements in the following form using the method of separation of variables:

$$W(x,z,t)=W_A(x,z)\sin(\omega^{\text{trans}}t+\epsilon_1);$$
$$V(y,z,t)=V_A(y,z)\sin(\omega^{\text{long}}t+\epsilon_2);$$
$$U(x,y,z,t)=U_A(x,y,z)\sin(\omega^{\text{ver}}t+\epsilon_3). \tag{5.9}$$

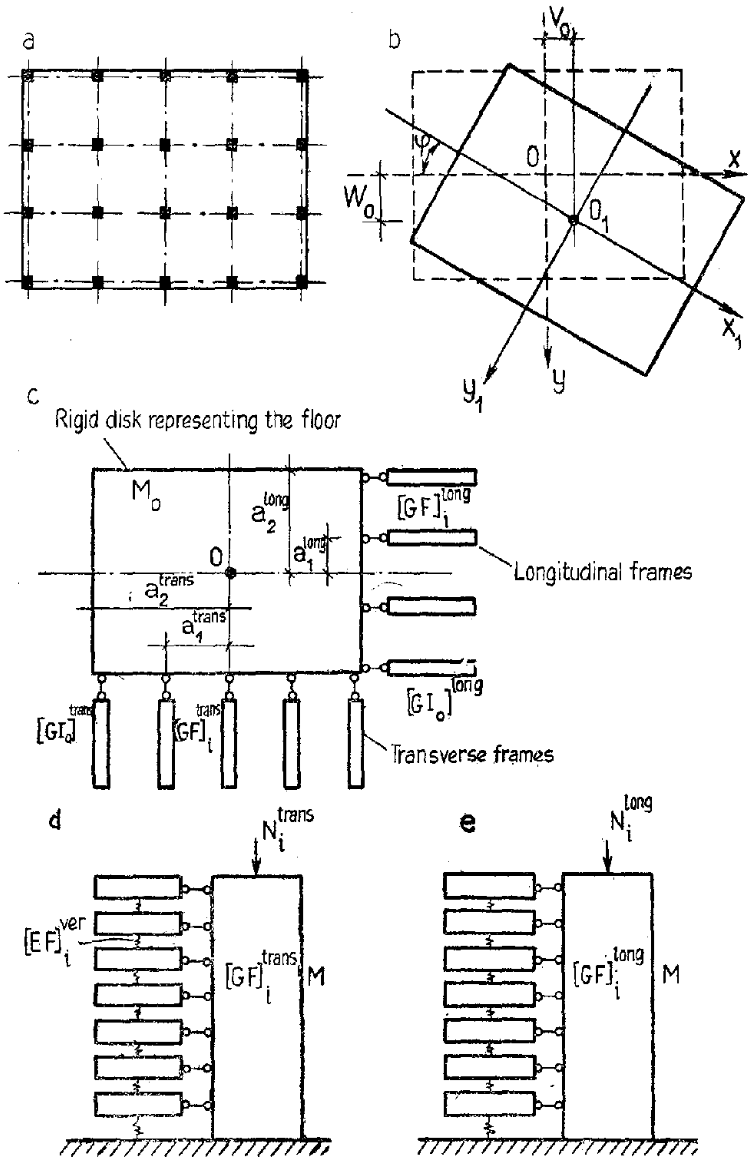


Fig. 70. Frame type building.

a—plan; b—movement of floor during oscillations; c—analytical model of building; d—cross section of the model; e—longitudinal section of the model.

For torsional oscillations of point type buildings the angular rotation will be:

$$\varphi(z, t) = \varphi_A(z) \sin(\omega^{tor} t + \epsilon_4) \tag{5.10}$$

By substituting the values of displacements from (5.9) in equation (5.3) and using the corresponding boundary conditions we obtain the frequency equations. The solution of these equations gives the values of frequencies of natural oscillations ω^{trans}, ω^{long} and ω^{ver}. From these values the relative magnitude of amplitudes of natural oscillations W_{A}, V_{A} and U_{A} may be determined.

2. DETERMINATION OF PERIODS AND WAVE FORMS OF NATURAL OSCILLATIONS OF BUILDINGS WITH AN IDEALIZED POINT SIZED AREA IN PLAN

Buildings with rigid floors and non-deforming contours may have different structural schemes depending on their stiffness characteristics. These are determined by the following formulae:

$$k_{\text{build}}^{\text{trans}} = H\sqrt{\frac{\sum\limits_{i=1}^{n}[GF]_i^{\text{trans}}}{\sum\limits_{j=1}^{k}[EI]_{0j}^{\text{trans}}}}; \tag{5.11}$$

$$k_{\text{build}}^{\text{long}} = H\sqrt{\frac{\sum\limits_{i=1}^{n}[GF]_i^{\text{long}}}{\sum\limits_{i=1}^{n}[EI]_{0j}^{\text{long}}}}. \tag{5.12}$$

We shall use formula (4.35) to determine the torsional stiffness characteristic of building $k_{\text{build}}^{\text{tor}}$. We shall assume that the center of rigidity and the center of mass of the building coincide. For $k_{\text{build}}^{\text{trans}} > 8$, $k_{\text{build}}^{\text{long}} > 8$ and $k_{\text{build}}^{\text{tor}} > 8$ we have the frame model of the building in the transverse and longitudinal directions (Fig. 70). For this the equations of natural oscillations (5.7) will become:

$$\sum_{i=1}^{n}([GF]-N)_i^{\text{trans}}\frac{\partial^2 W_0}{\partial z^2} - M\frac{\partial^2 W_0}{\partial z^2} = 0;$$

$$\sum_{i=1}^{n}([GF]-N)_i^{\text{long}}\frac{\partial^2 V_0}{\partial z^2} - M\frac{\partial^2 V_0}{\partial t^2} = 0;$$

$$\left[\sum_{i=1}^{n}([GF]-N)_i^{\text{trans}}(a_i^{\text{trans}})^2 + \sum_{i=1}^{n}([GF]-N)_i^{\text{long}}(a_i^{\text{long}}) + \sum_{i=1}^{n}[GI_0]_i^{\text{trans}} + \sum_{i=1}^{n}[GI_0]_i^{\text{long}}\right]\frac{\partial^2\varphi}{\partial z^2} - M_0\frac{\partial^2\varphi}{\partial t^2} = 0;$$

$$\sum_{i=1}^{n}[GF]_i^{\text{ver}}\frac{\partial^2 U_0}{\partial z^2} - M\frac{\partial^2 U_0}{\partial t^2} = 0,$$

where

$$M = \frac{\Sigma Q_{\text{story}}}{Hg};$$

$$M_0 = \frac{M}{12}[(L^{\text{trans}})^2 + (L^{\text{long}})^2]. \tag{5.13}$$

To solve the first equation of (5.13), let us write the values of displacements as:

$$W_0(z, t) = W_A(z) \sin(\omega^{\text{trans}} t + \epsilon_1). \tag{5.14}$$

Let us substitute the value of the second differential of displacement in the first equation of system (5.13). Thereafter we obtain the following expression:

$$\frac{\partial^2 W_A}{\partial z^2} + k^2 W_A = 0, \tag{5.15}$$

where

$$k^2 = \frac{(\omega^{\text{trans}})^2 M}{\sum_{i=1}^{n} ([GF] - N)_i^{\text{trans}}}. \tag{5.16}$$

The solution of equation (5.15) will be:

$$W_A = A_1 \cos kz + A_2 \sin kz. \tag{5.17}$$

For a cantilever undergoing shear deformations, we will have the following boundary conditions:

$$\text{for } z = 0: \quad W_A = 0;$$

$$\text{for } z = H: \quad \frac{dW_A}{dz} = 0. \tag{5.18}$$

We obtain the following periodic equation:

$$\cos kH = 0, \tag{5.19}$$

from which $kH = \dfrac{\pi(2n-1)}{2}$, where $n = 1, 2, 3, \ldots$

By substituting the value of k in (5.16) we obtain the frequency of natural oscillations of the building as

$$\omega_n^{\text{trans}} = \frac{\pi(2n-1)}{2H} \sqrt{\frac{\sum([GF] - N)_i^{\text{trans}}}{M}}. \tag{5.20}$$

The period of natural oscillation of the building in the transverse direction will be:

$$T_n^{\text{trans}} = \frac{4H}{2n-1} \sqrt{\frac{M}{\Sigma([GF] - N)_i^{\text{trans}}}}. \tag{5.21}$$

Similarly the period of natural oscillations in the longitudinal direction will be:

$$T_n^{\text{long}} = \frac{4H}{2n-1} \sqrt{\frac{M}{\sum([GF] - N)_i^{\text{long}}}}. \tag{5.22}$$

For torsional oscillations of the building:

$$T_n^{\text{tor}} = \frac{4H}{2n-1}$$

$$\times \sqrt{\frac{M_0}{\sum([GF]-N)_i^{\text{trans}}(a_i^{\text{trans}})^2 + \sum([GF]-N)_i^{\text{long}}(a_i^{\text{long}})^2 + \sum[GI_0]_i^{\text{trans}} + \Sigma[GI_0]_i^{\text{long}}}}. \quad (5.23)$$

For vertical oscillations:

$$T_n^{\text{ver}} = \frac{4H}{2n-1}\sqrt{\frac{M}{\sum[EF]_i^{\text{ver}}}}. \quad (5.24)$$

The wave form of natural oscillations in the transverse direction of the building will be:

$$(W_A)_n = A \sin \frac{\pi(2n-1)z}{2H}. \quad (5.25)$$

The wave forms of all remaining cases should be as in (5.25).

For $k_{\text{build}}^{\text{trans}} < 0.8$; $k_{\text{build}}^{\text{long}} < 0.8$ and $k_{\text{build}}^{\text{tor}} < 0.8$, we have the braced scheme of building in the transverse and longitudinal directions. Let us examine the deformations in the diaphragms without considering the effect of longitudinal forces. The equation (5.7) of natural oscillations of the building will then become:

$$\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}\frac{\partial^4 W_0}{\partial z^4} + M\frac{\partial^2 W_0}{\partial t^2} = 0;$$

$$\sum_{j=1}^{k}[EI]_{0j}^{\text{long}}\frac{\partial^4 V_0}{\partial z^4} + M\frac{\partial^2 V_0}{\partial t^2} = 0;$$

$$\left[\sum_{j=1}^{k}[EI]_{0j}^{\text{trans}}(b_j^{\text{trans}})^2 + \sum_{j=1}^{k}[EI]_{0j}^{\text{long}}(b_j^{\text{long}})^2\right] \times \frac{\partial^4 \varphi}{\partial z^4} + M_0\frac{\partial^2 \varphi}{\partial t^2} = 0;$$

$$\sum_{i=1}^{n}[EF]_i^{\text{ver}}\frac{\partial^2 U_0}{\partial z^2} - M\frac{\partial^2 U_0}{\partial t^2} = 0. \quad (5.26)$$

Let us use (5.14) to solve the first equation of (5.26). This gives the following differential equation:

$$\frac{-d^4 W_A}{dz^4} - k^4 W_A = 0, \quad (5.27)$$

where

$$k^4 = \frac{(\omega^{\text{trans}})^2 M}{\sum[EI]_{0j}^{\text{trans}}}. \quad (5.28)$$

Solution of equation (5.27) will be:

$$W_A = A_1 \sin kz + A_2 \cos kz + A_3 \operatorname{sh} kz + A_4 \operatorname{ch} kz. \tag{5.29}$$

Let us examine the boundary conditions:

$$\text{for } z = 0: \quad W_A = 0; \frac{dW_A}{dz} = 0;$$

$$\text{for } z = H: \quad \frac{d^2 W_A}{dz^2} = 0; \frac{d^3 W_A}{dz^3} = 0. \tag{5.30}$$

The transcendental equation for frequency will be:

$$\cos kH \times \operatorname{ch} kH = -1. \tag{5.31}$$

The roots of this equation are $Hk = 1.875; 4.69; 7.86 \ldots$

The frequency of natural oscillations of the building is obtained from (5.28):

$$\omega^{\text{trans}} = k^2 \sqrt{\frac{\sum[EI]_{0j}}{M}}. \tag{5.32}$$

The periods of natural oscillations of the building in the transverse and longitudinal directions will be:

$$T_n^{\text{trans}} = \frac{2\pi}{\omega} = \frac{2\pi H^2}{C_i^2} \sqrt{\frac{M}{\sum[EI]_{0j}^{\text{trans}}}} = \frac{7.15H^2}{(3n-1)^2} \sqrt{\frac{M}{\sum[EI]_{0j}^{\text{trans}}}}; \tag{5.33}$$

$$T_n^{\text{long}} = \frac{7.15H^2}{(3n-1)^2} \sqrt{\frac{M}{\sum[EI]_{0j}^{\text{long}}}}. \tag{5.34}$$

For torsional oscillations of the building:

$$T_n^{\text{tor}} = \frac{7.15H^2}{(3n-1)^2} \sqrt{\frac{M}{\sum[EI]_{0j}^{\text{trans}} (b_j^{\text{trans}})^2 + \sum[EI]_{0j}^{\text{long}} (b_j^{\text{long}})^2}}, \tag{5.35}$$

where $C_1 = 1.875$; $C_2 = 4.69$; $C_3 = 7.86 \ldots n = 1, 2, 3, \ldots$

The period of vertical oscillations of the building is determined from (5.24).

Based on the solution (5.29) and the boundary conditions (5.30), the wave form of natural oscillations of the building in the transverse direction will be:

$$(W_A)_i = (\cos C_i H + \operatorname{ch} C_i H)(\sin C_i z - \operatorname{sh} C_i z) - (\sin C_i H + \operatorname{sh} C_i H)(\cos C_i z - \operatorname{ch} C_i z). \tag{5.36}$$

The wave forms of oscillations for all remaining cases should be considered according to (5.36); however, they can be simplified and expressed in the form of the following trigonometric function:

$$(W_A)_n = A\left[1-\cos\frac{\pi z}{2H}-(-1)^n \sin\frac{\pi(n-1)z}{H}\right]. \tag{5.37}$$

The wave forms of all three harmonics, plotted on the basis of (5.36) and (5.37) are given in Fig. 71.

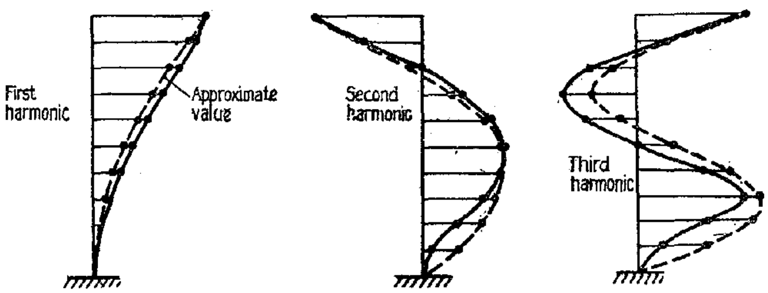


Fig. 71. Wave forms of building oscillations for a braced scheme.

For $0.8 < k_{\text{build}}^{\text{trans}} < 8$, $0.8 < k_{\text{build}}^{\text{long}} < 8$, $0.8 < k_{\text{build}}^{\text{tor}} < 8$ we have the braced frame model of a building in the transverse and longitudinal directions (Fig. 72) for which the differential equations of natural oscillations (5.7) become:

$$\sum [EI]_{0j}^{\text{trans}} \frac{\partial^4 W_0}{\partial z^4} + [\sum N_j^{\text{trans}} - \sum ([GF]-N)_i^{\text{trans}}] \frac{\partial^2 W_0}{\partial z^2} + M\frac{\partial^2 W_0}{\partial t^2} = 0;$$

$$\sum [EI]_{0j}^{\text{long}} \frac{\partial^4 V_0}{\partial z^4} + [\sum N_j^{\text{long}} - \sum ([GF]-N)_i^{\text{long}}] \frac{\partial^2 V_0}{\partial z^2} + M\frac{\partial^2 V_0}{\partial t^2} = 0;$$

$$[\sum [EI]_{0j}^{\text{trans}} (b_j^{\text{trans}})^2 + \sum [EI]_{0j}^{\text{long}} (b_j^{\text{long}})] \frac{\partial^4 \varphi}{\partial z^4} + [\sum N_j^{\text{trans}} (b_j^{\text{trans}})^2$$

$$-\sum ([GF]-N)_i^{\text{trans}} (a_i^{\text{trans}})^2 + \sum N_j^{\text{long}} (b_j^{\text{long}})^2 - \sum ([GF]-N)_i^{\text{long}} (a_i^{\text{long}})^2]$$

$$\times \frac{\partial^2 \varphi}{\partial z^2} + M_0 \frac{\partial^2 \varphi}{\partial t^2} = 0;$$

$$\sum [EF]_i^{\text{ver}} \frac{\partial^2 U_0}{\partial z^2} - M\frac{\partial^2 U_0}{\partial t^2} = 0. \tag{5.38}$$

The first equation of (5.38) expresses the natural oscillations of a two-layer bar in which the first layer (diaphragm) undergoes bending strain and the second (frame) shear strain.

Let us isolate the two layers of the bar from each other and separately examine their natural oscillations. Then the first equation of (5.38) may be

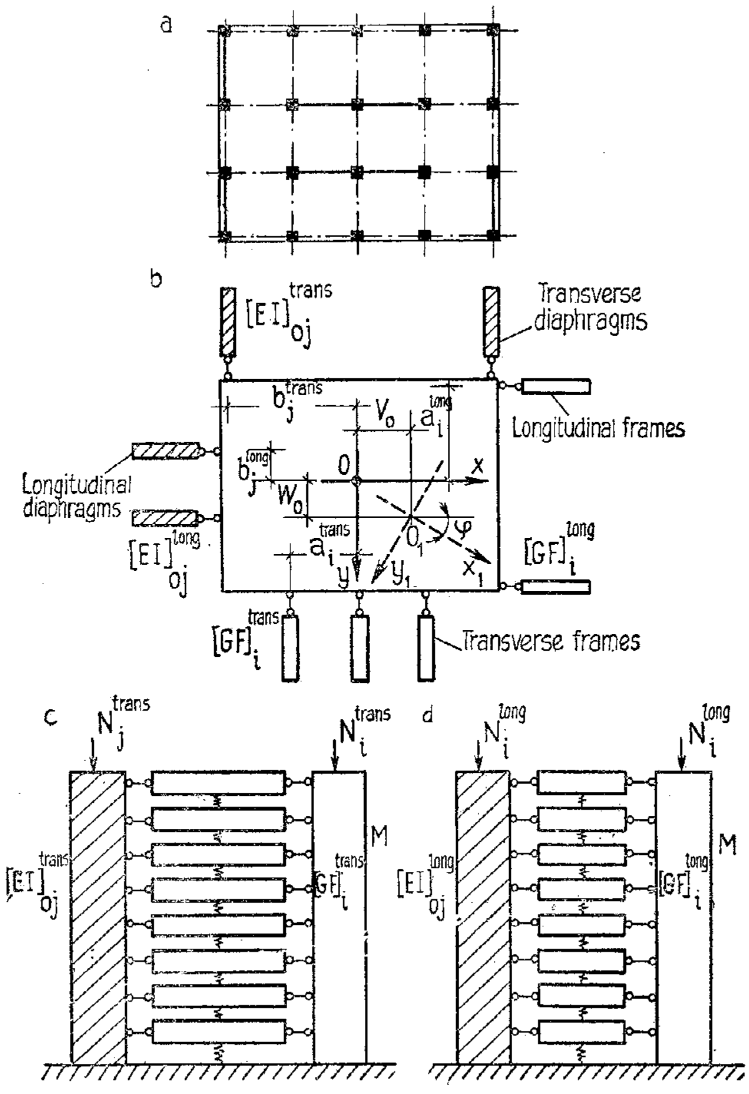


Fig. 72. A braced frame building model.
a—plan; b—analytical model of building; c—cross section of model; d—longitudinal section of model.

represented by two equations:

$$\sum [EI]_{0j} \frac{\partial^4 W_4}{\partial z^4} + M_1 \frac{\partial^2 W_0}{\partial t^2} = 0; \tag{5.39}$$

$$[-\sum(-N)_j - \sum([GF]-N)_i] \frac{\partial^2 W_0}{\partial z^2} + M_2 \frac{\partial^2 W_0}{\partial t^2} = \mathbf{0}, \tag{5.40}$$

where M_1 and M_2 are the masses of the first and second layers, respectively, satisfying the condition:

$$M_1 + M_2 = M. \tag{5.41}$$

The period of natural oscillations of diaphragms is determined by (5.33) in the following form:

$$T_n = \frac{7.15H^2}{(3n-1)^2}\sqrt{\frac{M_1}{\sum [EI]_{0j}}}. \tag{5.42}$$

The period of natural oscillations of frames is determined by (5.21). It will be as follows:

$$T_n = \frac{4H}{2n-1}\sqrt{\frac{M_2}{\sum(-N)_j + \sum([GF]-N)_i}}. \tag{5.43}$$

The two layers of the bar should have identical periods of natural oscillations because, in reality, they are continuously connected to each other. By equating their periods and using condition (5.41) we determine the masses M_1 and M_2 of the respective layers. Then by substituting the value of mass M_1 in (5.42) we obtain the periods of natural oscillations of the braced frame system in the transverse direction as:

$$T_n^{\mathrm{trans}} = 7.15H^2\sqrt{\frac{M}{\sum[EI]_{0j}^{\mathrm{trans}}[3.2(2n-1)^2(k_{\mathrm{build\ 1}}^{\mathrm{trans}})^2 + (3n-1)^4]}}. \tag{5.44}$$

For the longitudinal direction we obtain the period in a similar way:

$$T_n^{\mathrm{long}} = 7.15H^2\sqrt{\frac{M}{\sum[EI]_{0j}^{\mathrm{long}}[3.2(2n-1)^2(k_{\mathrm{build\ 1}}^{\mathrm{long}})^2 + (3n-1)^4]}}, \tag{5.45}$$

where $k_{\mathrm{build}}^{\mathrm{long}}$ is the stiffness characteristic of the building taking into account the effect of longitudinal forces and is equal to:

$$k_{\mathrm{build\ 1}} = H\sqrt{\frac{\sum(-N)_j + \sum([GF]-N)_i}{\sum[EI]_{0j}}}. \tag{5.46}$$

For torsional oscillations the period will be:

$$T_n^{\mathrm{tor}} = 7.15\,H^2$$

$$\times\sqrt{\frac{M_0}{[\sum[EI]_{0j}^{\mathrm{trans}}(b_j^{\mathrm{trans}})^2 + \sum[EI]_{0j}^{\mathrm{long}}(b_j^{\mathrm{long}})^2][3.2(2n-1)^2(k_{\mathrm{build\ 1}}^{\mathrm{tor}})^2 + (3n-1)^4]}} \tag{5.47}$$

where $k_{\mathrm{build}}^{\mathrm{tor}}$ is the torsional stiffness characteristic of the building, determined by taking into account the effect of the longitudinal forces.

The wave form of natural oscillations of the braced frame system is determined approximately by summing the wave forms of the bars undergoing bending and shear. These wave forms are taken to be proportional to the masses with equal frequencies of natural oscillations:

$$(W_A)_n = A\left\{\left[(-1)^n \sin\frac{\pi(n-1)}{H} + \cos\frac{\pi z}{2H} - 1\right]\xi_1 - \left[\sin\frac{\pi(2n-1)}{2H}\right]\xi_2\right\}, \tag{5.48}$$

where

$$\xi_1 = \frac{M_1}{M} = \frac{1}{1 + \dfrac{3.2\,k^2_{\text{build}}\,(2n-1)^2}{(3n-1)^4}}; \tag{5.49}$$

$$\xi_2 = \frac{M_2}{M} = \frac{1}{1 + \dfrac{3\,(n-1)^4}{3.2\,k^2_{\text{build}}\,(2n-1)^2}}. \tag{5.50}$$

For the first harmonic of oscillations we shall have

$$(W_A)_1 = A\left[\xi_1\left(\cos\frac{\pi z}{2H} - 1\right) - \xi_2 \sin\frac{\pi z}{2H}\right], \tag{5.51}$$

where

$$\xi_1 = \frac{1}{1 + 0.2\,k^2_{\text{build}}};$$

$$\xi_2 = \frac{1}{1 + \dfrac{5}{k^2_{\text{build}}}}.$$

There may also be cases of point type buildings in which the vertical members undergo shear and bending strains. The stiffness characteristic of such a building is $0.8 < \lambda < 8$.

The differential equation of natural oscillations of such a building, considering the effect of shear and bending strains in vertical members, will be:

$$\sum[EI]_i \frac{\partial^4 W}{\partial z^4} - \frac{\sum[EI]_i}{\sum[GF]_i} M \frac{\partial^4 W}{\partial z^2\, \partial t^2} + M\frac{\partial^2 W}{\partial t^2} = 0. \tag{5.52}$$

To solve equation (5.52) let us examine separately the natural bending and shearing oscillations of the building, expressing the deflection as in (5.14). Based on the result of (5.52), we shall have two independent equations:

$$\sum[EI]\frac{d^4 W_A^{\text{bend}}}{dz^4} - M(\omega^{\text{bend}})^2 W_A^{\text{bend}} = 0; \tag{5.53}$$

$$\sum[GF]\frac{d^2 W_A^{\text{sh}}}{dz^2} + M(\omega_A^{\text{sh}})^2 W_A^{\text{sh}} = 0. \tag{5.54}$$

The frequency of bending oscillations of the building is determined from (5.32) and shearing oscillations from (5.20). The frequency of combined natural shearing-bending oscillations of the building is determined as for a bar with additional mass undergoing shear deformation:

$$\omega_n = \frac{\pi(2n-1)}{2H}\sqrt{\frac{\sum[GF]}{M + M_{\text{add}}}}. \tag{5.55}$$

The additional mass is determined from the condition that it executes natural shearing oscillations with the frequency of bending oscillations:

$$M_{\text{add}} = \frac{1}{(\omega^{\text{bend}})^2}\,\frac{[GF]\,\pi^2(2n-1)^2}{4H^2} = 3.2\lambda^2\,\frac{(2n-1)^2}{(3n-1)^4}M. \tag{5.56}$$

The period of natural shearing-bending oscillations of the building is determined by the formula:

$$T_n = \frac{4H}{2n-1}\sqrt{\frac{M}{\sum[GF]}\left[1 + 3.2\lambda^2\,\frac{(2n-1)^2}{(3n-1)^4}\right]}. \tag{5.57}$$

The wave form of natural shearing-bending oscillations of buildings is determined by (5.48) where coefficients ξ_1 and ξ_2 have the values:

$$\xi_1 = 3.2\lambda^2\,\frac{(2n-1)}{(3n-1)^4};\ \ \xi_2 = 1. \tag{5.58}$$

The stiffness characteristic of the building member is equal to:

$$\lambda = H\sqrt{\frac{\sum[GF]_i}{\sum[EI]_i}}. \tag{5.59}$$

3. DETERMINATION OF PERIODS AND WAVE FORMS OF NATURAL OSCILLATIONS OF BUILDINGS WITH LARGE DIMENSIONS IN PLAN

Let us examine buildings which have large dimensions in plan in both directions. Let them have deformable floors which can be represented by frame, braced frame and braced structural schemes.

1. A frame type building in transverse and longitudinal directions for $k^{\text{trans}}_{\text{build}} > 8$ and $k^{\text{long}}_{\text{build}} > 8$ with floors undergoing shear for $\lambda^{\text{trans}}_x < 0.8$ and $\lambda^{\text{long}}_y < 0.8$ and the stiffness characteristic in the vertical direction $\mu > 30$ (which, in turn, means consideration of longitudinal deformation in columns) will have the following differential equations of natural oscillations:

$$\begin{aligned}
&(C^{\text{trans}}_z - N^{\text{trans}})\,\frac{\partial^2 W}{\partial z^2} + C^{\text{trans}}_x\,\frac{\partial^2 W}{\partial x^2} - m^{\text{trans}}\,\frac{\partial^2 W}{\partial t^2} = 0;\\
&(C^{\text{long}}_z - N^{\text{long}})\,\frac{\partial^2 V}{\partial z^2} - C^{\text{long}}_y\,\frac{\partial^2 V}{\partial y^2} - m^{\text{long}}\,\frac{\partial^2 V}{\partial t^2} = 0;\\
&C^{\text{ver}}_x\,\frac{\partial U}{\partial x^2} + C^{\text{ver}}_y\,\frac{\partial^2 U}{\partial y^2} + F^{\text{ver}}_z\,\frac{\partial^2 U}{\partial z^2} - m^{\text{ver}}\,\frac{\partial^2 U}{\partial t^2} = 0.
\end{aligned} \tag{5.60}$$

To solve the first equation of (5.60) let us write the deflections in the following form by using the method of separation of variables:

$$W(x, z, t) = X(x)\, Z(z) \sin(\omega^{\mathrm{trans}} t + \epsilon_1). \tag{5.61}$$

By substituting the value of differential coefficients of deflections in the first equation we get:

$$(C_z - N) \frac{\partial^2 Z}{\partial z^2} + C_x Z \frac{\partial^2 N}{\partial x^2} + m\omega^2 XZ = 0. \tag{5.62}$$

The frequency of natural oscillations is written as:

$$\omega^2 = \omega_z^2 + \omega_x^2. \tag{5.63}$$

Then equation (5.62) becomes:

$$\frac{C_z - N}{Z} \frac{\partial^2 Z}{\partial z^2} + m\omega_z^2 + \frac{C_x}{X} \frac{\partial^2 X}{\partial x^2} + m\,\omega_x^2 = 0. \tag{5.64}$$

Because Z and X are independent variables, equation (5.64) divides into two equations:

$$(C_z - N) \frac{d^2 Z}{dz^2} + Zm\,\omega_z^2 = 0; \tag{5.65}$$

$$C_x \frac{d^2 X}{dx^2}\, Xm\,\omega_x^2 = 0. \tag{5.66}$$

Equation (5.65) is similar to (5.15) with the boundary conditions given in (5.18). The frequency and wave form of natural oscillations are determined by formulae:

$$(\omega_z)_n = \frac{\pi(2n-1)}{2H} \sqrt{\frac{C_z - N}{m}}; \tag{5.67}$$

$$Z_n = A \sin \frac{\pi(2n-1)\, z}{2H}, \tag{5.68}$$

where $n = 1, 2, 3, \ldots$

The solution of equation (5.66) will be:

$$X = A_1 \cos \sqrt{\frac{m}{C_x}}\, \omega_x x + A_2 \sin \sqrt{\frac{m}{C_x}}\, \omega_x x. \tag{5.69}$$

The boundary conditions for the symmetrical wave form of oscillations are:

$$\text{for } z = 0: \quad \frac{dX}{dx} = 0; \tag{5.70}$$

$$\text{for } x = \frac{L}{2}: \quad \frac{dX}{dx} = 0.$$

We get

$$A_2 = 0 \text{ and } \sin\sqrt{\frac{m}{C_x}}\,\omega_x\frac{L}{2} = 0.$$

The frequency and wave form of natural oscillations are determined by formulae:

$$(\omega_x)_k^{\text{sym}} = \frac{2\pi k}{L}\sqrt{\frac{C_x}{m}}\,; \tag{5.71}$$

$$X_k^{\text{sym}}\, A\cos\frac{2\pi k}{L}x, \tag{5.72}$$

where $k = 0, 1, 2, 3, \ldots$

For skew-symmetric forms of oscillations the boundary conditions will be:

for $x = 0,\ X = 0$

$$x = \frac{L}{2}\cdot\frac{dX}{dx} = 0. \tag{5.73}$$

we get

$$A_1 = 0 \text{ and } \cos\sqrt{\frac{m}{C_x}}\,\omega_x\frac{L}{2} = 0;$$

The frequency and wave form of natural oscillations are determined by formulae:

$$(\omega_x)^{\text{sk.sym}} = \frac{\pi k}{L}\sqrt{\frac{C_x}{m}}. \tag{5.74}$$

$$X_k^{\text{sk.sym}} = A\sin\frac{\pi k}{L}x, \tag{5.75}$$

where $k = 1, 3, 5, \ldots$

By comparing frequencies (5.71) and (5.74) we can write the formula for general frequency of natural oscillations of symmetric and skew-symmetric wave forms as:

$$(\omega_x)_k = \frac{\pi\,(k-1)}{L}\sqrt{\frac{C_x}{m}}. \tag{5.76}$$

Similarly it is possible to write the formula for general wave form of natural oscillations as:

$$X_k = A\left[\sin\frac{\pi\,(k-1)}{2}\sin\frac{\pi\,(k-1)x}{L} + \cos\frac{\pi\,(k-1)}{2}\cos\frac{\pi\,(k-1)\,x}{L}\right], \tag{5.77}$$

where $k = 1, 2, 3, 4, 5, \ldots$

For $k = 1$ we have translational deflections; for $k = 2, 4, 6, \ldots$ skew-symmetric forms of oscillations; for $k = 3, 5, 7, \ldots$ symmetric forms of oscillations.

From condition (5.63) we get the frequency of natural oscillations of the building in the transverse directions as:

$$\omega_{nk} = \frac{\pi}{2\,HL}\sqrt{\frac{1}{m}[L^2\,(2n-1)^2\,(C_z - N) + 4H^2\,(k-1)^2\,C_x]}. \quad (5.78)$$

The period and wave form of natural oscillations of the building in the transverse direction will be:

$$T_{nk}^{\text{trans}} = 4HL^{\text{trans}}\sqrt{\frac{m^{\text{trans}}}{(L^{\text{trans}})^2\,(2n-1)^2\,(C_z - N) + 4H^2\,(k-1)^2 C_x}}; \quad (5.79)$$

$$(W_A)_{nk} = A\sin\frac{\pi\,(2n-1)\,z}{2H}\left[\sin\frac{\pi\,(k-1)\,x}{L^{\text{trans}}} + \cos\frac{\pi\,(k-1)}{2}\cos\frac{\pi\,(k-1)\,x}{L^{\text{trans}}}\right], \quad (5.80)$$

where $n = 1, 2, 3, 4, 5, \ldots, k = 1, 2, 3, 4, 5, \ldots$

Similar formulae for periods and wave forms of natural oscillations of a building are obtained for the longitudinal direction by substituting the respective geometrical dimensions, stiffness characteristics and loads in (5.79) and (5.80).

To determine the frequency and wave form of natural oscillations of a building in the vertical direction, let us solve the third equation of (5.60) by writing the deflections in the following form:

$$U\,(x, y, z, t) = X\,(x)\,Z\,(z)\,I\,(y)\sin\,(\omega^{\text{ver}}\,t + \epsilon). \quad (5.81)$$

Considering (5.63) and (5.81) the third equation is broken down to three independent equations:

$$C_x^{\text{ver}}\frac{d^2X}{dx^2} + Xm^{\text{ver}}\,(\omega_x^{\text{ver}})^2 = 0; \quad (5.82)$$

$$C_y^{\text{ver}}\frac{d^2I}{dy^2} + Im^{\text{ver}}\,(\omega_y^{\text{ver}})^2 = 0; \quad (5.83)$$

$$F_z^{\text{ver}}\frac{d^2Z}{dz^2}\,Zm^{\text{ver}}\,(\omega_z^{\text{ver}})^2 = 0. \quad (5.84)$$

Equations (5.82) and (5.83) are similar to (5.66) while (5.84) is similar to (5.65). On this basis we obtain a formula to determine the period of natural oscillations of the building in the vertical direction:

$$T_{nkr}^{\text{ver}} = \sqrt{\frac{4m^{\text{ver}}}{\frac{(k-1)^2\,C_x^{\text{ver}}}{(L^{\text{trans}})^2} + \frac{(r-1)^2\,C_y^{\text{ver}}}{(L^{\text{long}})^2} + \frac{(2n-1)^2\,F_z^{\text{ver}}}{4H^2}}}. \quad (5.85)$$

The wave form of natural oscillations of the building in the vertical direction is obtained in a similar way as (5.80):

$$(U_A)_{nkr} = A \sin \frac{\pi (2n-1) z}{2H} \left[\sin \frac{\pi (k-1)}{2} \sin \frac{\pi (k-1) x}{L^{\text{trans}}} + \cos \frac{\pi (k-1)}{2} \cos \frac{\pi (k-1) x}{L^{\text{trans}}} \right] \left[\sin \frac{\pi (r-1)}{2} \sin \frac{\pi (r-1) y}{L^{\text{long}}} + \cos \frac{\pi (r-1)}{2} \cos \frac{\pi (r-1) y}{L^{\text{long}}} \right], \tag{5.86}$$

where $n = 1, 2, 3, \ldots, k = 1, 2, 3, \ldots, r = 1, 2, 3, \ldots$

If $L^{\text{trans}} \gg L^{\text{long}}$ (Fig. 73) the floor may be considered rigid in the longitudinal direction and the general differential equation will be:

$$(C_z - N)^{\text{trans}} \frac{\partial^2 W}{\partial x^2} + C_x^{\text{trans}} \frac{\partial^2 W}{\partial x^2} - m^{\text{trans}} \frac{\partial^2 W}{\partial t^2} = 0;$$

$$\sum_{i=1}^{n} ([GF] - N)_i^{\text{long}} \frac{\partial^2 V_0}{\partial z^2} - M \frac{\partial^2 V_0}{\partial t^2} = 0;$$

$$C_x^{\text{ver}} \frac{\partial^2 U}{\partial x^2} + F_z^{\text{ver}} \frac{\partial^2 U}{\partial z^2} - m^{\text{ver}} \frac{\partial^2 U}{\partial t^2} = 0. \tag{5.87}$$

The period and wave form of natural oscillations of such a building in the transverse direction are determined by (5.79) and (5.80) and in the longitudinal direction by (5.22) and (5.25). The period of natural oscillations in the vertical direction of the building will be:

$$T_{nk}^{\text{ver}} = \sqrt{\frac{4\, m^{\text{ver}}}{\frac{(k-1)^2 C_x^{\text{ver}}}{(L^{\text{trans}})^2} + \frac{(2n-1)^2 F_z^{\text{ver}}}{4 H^2}}}. \tag{5.88}$$

Let us examine a frame building with a floor which undergoes bending strain for $\lambda_x^{\text{trans}} < 8$. The differential equation of natural oscillations of the building in the transverse direction will be:

$$(C_z - N)^{\text{trans}} \frac{\partial^2 W}{\partial z^2} - B_y^{\text{trans}} \frac{\partial^4 W}{\partial x^4} - m^{\text{trans}} \frac{\partial^2 W}{\partial t^2} = 0. \tag{5.89}$$

By using the method of separation of variables, equation (5.89) is separated into two equations, of which the first will be (5.65) while the second is written in the form:

$$B_x^{\text{trans}} \frac{d^4 x}{dx^4} - X m^{\text{trans}} \omega_x^2 = 0. \tag{5.90}$$

This equation has a known solution based on formula 130 given in [65]

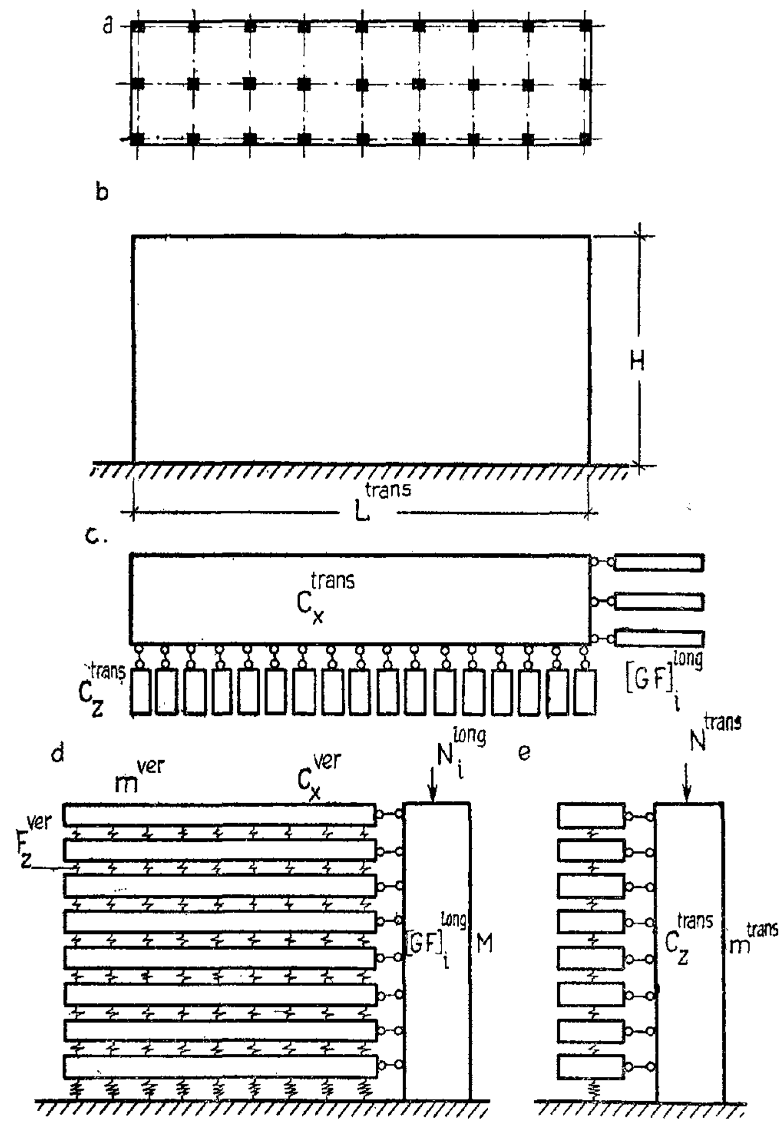


Fig. 73. Frame building which is very long in the transverse direction.
a—plan; b—facade; c—analytical model; d—longitudinal section of model; e—cross section of model.

which is written in the following way:

$$X = A_1 (\cos \alpha x + \operatorname{ch} \alpha x) + A_2 (\cos \alpha x - \operatorname{ch} \alpha x) + A_3 (\sin \alpha x + \operatorname{sh} \alpha x) + A_4 (\sin \alpha x - \operatorname{sh} \alpha x). \tag{5.91}$$

The coordinates are measured from the left end. The boundary conditions will be:

$$\text{for } x = 0: \quad \frac{d^2 X}{dx^2} = 0; \quad \frac{d^3 X}{dx^3} = 0;$$

$$\text{for } x = L^{\text{trans}}: \quad \frac{d^2 X}{dx^2} = 0; \quad \frac{d^3 X}{dx^3} = 0. \tag{5.92}$$

Finally the transcendental equation for frequency based on formula (141) of [65] will be of the following form:

$$\cos \alpha L \cdot \operatorname{ch} \alpha L = 1, \tag{5.93}$$

where

$$\alpha^4 = \frac{m\omega_x^2}{B_x}. \tag{5.94}$$

The roots of equation (5.93) are as follows:

$$\alpha_1 L = 0; \ \alpha_2 L = 0; \ \alpha_3 L = 4.73; \ \alpha_4 L = 7.853; \ \alpha_5 L = 10.99.$$

From where

$$\alpha_i = \frac{b_i}{L}.$$

Let us represent b_i in the form:

$$b_i^2 = 11\ (k^2 - 3k + 2); \tag{5.95}$$

where $k = 1, 2, 3, 4, 5, \ldots$

By substituting the values of α in (5.94) we obtain the formula for the frequency of natural oscillations of the horizontal component:

$$\omega_x^2 = \frac{11\ (k^2 - 3k + 2)^2}{(L^{\text{trans}})^4} \ \frac{B_x^{\text{trans}}}{m^{\text{trans}}}. \tag{5.96}$$

Using (5.63), (5.67) and (5.96) we obtain the period of natural oscillations of the building in the following form:

$$T_{nk}^{\text{trans}} = 4H\ (L^{\text{trans}})^2$$

$$\times \sqrt{\frac{m^{\text{trans}}}{(2n-1)^2 (L^{\text{trans}})^4\ (C_z - N)^{\text{trans}} + 49\ (k^2 - 3k + 2)^2 H^2 B_x^{\text{trans}}}}. \tag{5.97}$$

where $n = 1, 2, 3, \ldots, k = 1, 2, 3, \ldots$

The wave form of natural oscillations of the horizontal component will be:

$$X_i = A_1 \left[\cos\left(x - \frac{L}{2} \right) b_i + \operatorname{ch}\left(x - \frac{L}{2} \right) b_i \right]$$

$$+ A_3 \left[\sin\left(x - \frac{L}{2} \right) b_i + \operatorname{sh}\left(x - \frac{L}{2} \right) b_i \right]. \tag{5.98}$$

For symmetrical forms of oscillations we have the following additional conditions:

$$\text{for } x = 0: \quad \frac{dX}{dx} = 0; \tag{5.99}$$

from which we get:

$$\frac{A_1}{A_3} = \frac{\cos\frac{L}{2} b_i + \operatorname{ch}\frac{L}{2} b_i}{-\sin\frac{L}{2} b_i + \operatorname{sh}\frac{L}{2} b_i}. \tag{5.100}$$

For skew-symmetric forms of oscillations we have the following additional conditions:

$$\text{for } x = 0;\ X = 0 \tag{5.101}$$

from which we get:

$$\frac{A_1}{A_3} = \frac{\sin\frac{L}{2} b_i + \operatorname{sh}\frac{L}{2} b_i}{\cos\frac{L}{2} b_i + \operatorname{ch}\frac{L}{2} b_i}. \tag{5.102}$$

Oscillations given by (5.98) may be expressed by the following approximate function:

$$X_k = A\left[(3-k)\,x^{k-1} + \left(\cos\frac{\pi x}{L} - 0.6\right)(k-1)(k-2)\right], \tag{5.103}$$

where $k = 1, 2, 3$.

The wave form of natural oscillations of a building is expressed by the following equation:

$$(W_A)_{nk} = A \sin\frac{\pi(2n-1)z}{2H}\left[(3-k)\,x^{k-1} + \left(\cos\frac{\pi x}{L} - 0.6\right)(k-1)(k-2)\right], \tag{5.104}$$

where $n = 1, 2, 3, \ldots,\ k = 1, 2, 3$.

Let us examine a building which is braced in the transverse direction for $k_{\text{build}}^{\text{trans}} < 0.8$ and has a braced frame in the longitudinal direction for $0.8 < k_{\text{build}}^{\text{long}} < 8$ with the floor undergoing shear strain for $\lambda^{\text{trans}} < 0.8$ and $\lambda^{\text{long}} < 0.8$. The differential equations of natural oscillations will be:

$$B_{z0}^{\text{trans}} \frac{\partial W}{\partial z^4} - C_x^{\text{trans}} \frac{\partial^2 W}{\partial x^2} + N^{\text{trans}} \frac{\partial^2 W}{\partial z^2} + m^{\text{trans}} \frac{\partial^2 W}{\partial t^2} = 0;$$

$$B_{z0}^{\text{long}} \frac{\partial^4 V}{\partial z^4} - C_y^{\text{long}} \frac{\partial^2 V}{\partial y^2} - (C_z - N)^{\text{long}} \frac{\partial^2 V}{\partial z^2} + m^{\text{long}} \frac{\partial^2 V}{\partial t^2} = 0. \tag{5.105}$$

By using the method of separation of variables, we obtain two independent equations each for (5.105) in the following form:

$$B_{z0}^{\mathrm{trans}} \frac{d^4 Z}{dz^4} - (-N^{\mathrm{trans}}) \frac{d^2 Z}{dz^2} - Zm^{\mathrm{trans}} (\omega_z^{\mathrm{trans}})^2 = 0; \quad (5.106)$$

$$C_x^{\mathrm{trans}} \frac{d^2 X}{dx^2} + Xm^{\mathrm{trans}} (\omega_x^{\mathrm{trans}})^2 = 0; \quad (5.107)$$

$$B_{z0}^{\mathrm{long}} \frac{d^4 Z}{dz^4} - (C_z^{\mathrm{long}} - N^{\mathrm{long}}) \frac{d^2 Z}{dz^2} - Zm^{\mathrm{long}} (\omega_z^{\mathrm{long}})^2 = 0; \quad (5.108)$$

$$C_y^{\mathrm{long}} \frac{d^2 I}{dy^2} + Im^{\mathrm{long}} (\omega_y^{\mathrm{long}})^2 = 0. \quad (5.109)$$

Equations (5.106) and (5.108) are similar to the first and second equations of (5.38) while equations (5.107) and (5.109) are similar to (5.66). By using the solutions of the above equations we obtain the period and wave form of natural oscillations of the building in the transverse direction:

$$T_{nk}^{\mathrm{trans}} = 7.15 H^2 L^{\mathrm{trans}} \times \sqrt{\frac{m^{\mathrm{trans}}}{3.2\,(2n-1)^2 H^2 (L^{\mathrm{trans}})^2 (-N^{\mathrm{trans}}) + (3n-1)^4 (L^{\mathrm{trans}})^2 B_{z0} + 12.75\,(k-1)^2 H^4 C_x^{\mathrm{trans}}}}; \quad (5.110)$$

$$(W_A)_{nk} = A \left\{ \left[(-1)^n \sin \frac{\pi (n-1) x}{H} + \cos \frac{\pi z}{2H} - 1 \right] \times \left[\sin \frac{\pi (k-1)}{2} \sin \frac{\pi (k-1) x}{L^{\mathrm{trans}}} + \cos \frac{\pi (k-1)}{2} \cos \frac{\pi (k-1) x}{L^{\mathrm{trans}}} \right] \right\}; \quad (5.111)$$

and in the longitudinal direction:

$$T_{nk}^{\mathrm{long}} = 7.15 H^2 L^{\mathrm{long}} \sqrt{\frac{m^{\mathrm{long}}}{3.2\,(2n-1)^2 H^2 (L^{\mathrm{long}})^2 (C_z - N)^{\mathrm{long}} + (3n-1)^4 (L^{\mathrm{long}})^2 B_{z0} + 12.75\,(k-1)^2 H^4 C_y^{\mathrm{long}}}}; \quad (5.112)$$

$$(V_A)_{nk} = A \left\{ \left[(-1)^n \sin \frac{\pi (n-1) z}{H} + \cos \frac{\pi z}{2H} - 1 \right] = \xi_1 - \left[\sin \frac{\pi (2n-1) z}{2H} \right] \xi_2 \right\} \left[\sin \frac{\pi (k-1)}{2} \sin \frac{\pi (k-1) y}{L^{\mathrm{long}}} + \cos \frac{\pi (k-1)}{2} \cos \frac{\pi (k-1) y}{L^{\mathrm{long}}} \right], \quad (5.113)$$

where $n = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots$; ξ_1 and ξ_2 are determined by formulae (5.49) and (5.50).

The deduced formulae to determine the periods of natural oscillations of buildings consider the longitudinal force applied to the top of the building. This longitudinal force is computed with the equivalence coefficient as in (4.28). These formulae may be used in the dynamic method to analyze the static stability of a building.

4. SPATIAL WAVE FORMS OF NATURAL OSCILLATIONS OF BUILDINGS

For horizontal oscillations the three dimensional analytical model of a building is considered a cantilever plate system with one fixed edge and three free edges. In the expression for periods and wave forms of natural oscillations T_{nk} and W_{nk}, the values of indices n and k will be considered equal to the number of sections between the edges and the nodal lines into which the plate is divided. The values of n and k appear in the expressions for periods and wave forms of oscillations and permit their complete determination.

For frame buildings, with floors undergoing shear strain, the period and wave form of oscillations are determined by (5.79) and (5.80). In the first harmonic of oscillations the period and wave form of oscillations are represented by T_{11} and W_{11}. Here the floor executes symmetrical translational deflections as a rigid body. For $k = 2$ we shall have skew-symmetric wave forms of floor oscillations resembling its rotation. For the second harmonic of oscillations the period may be T_{12} or T_{21} depending on the building height and length and the stiffness characteristics of the frames and floors. For example, if the floor stiffness C_x is ten times greater than the stiffness of frames C_z, then for ratio $\frac{L}{H} > 2.5$, for buildings which are not very long, we shall have the second harmonic of oscillations with periods T_{12}, that is, rotation of the floor. If $\frac{L}{H} < 2.5$ for high buildings with short length, the period of second harmonic of oscillations will be T_{21}, that is, we shall have translational deflection of the floor.

The ratio between indices $\ll n \gg$ and $\ll k \gg$ is represented in the form of nk diagrams in Fig. 74 which shows the spatial wave forms of oscillations. For a frame building, with the floor undergoing bending strain, the period and wave form of oscillations are determined by (5.97) and (5.104). For the first harmonic of oscillations the period and wave form will be T_{11} and W_{11}. In this case the floor executes symmetrical translational deflections. For $k = 2$ we shall have pure torsion of the floor as a rigid body and periods $T_{n1} = T_{n2}$. Then, in accordance with the principle of superposition, in these cases it is possible to superimpose two wave forms with random amplitudes. The composite wave form so obtained is the natural oscillation of the given plate. For $k = 3$ we have the symmetrical wave form of floor oscillations resembling its bending.

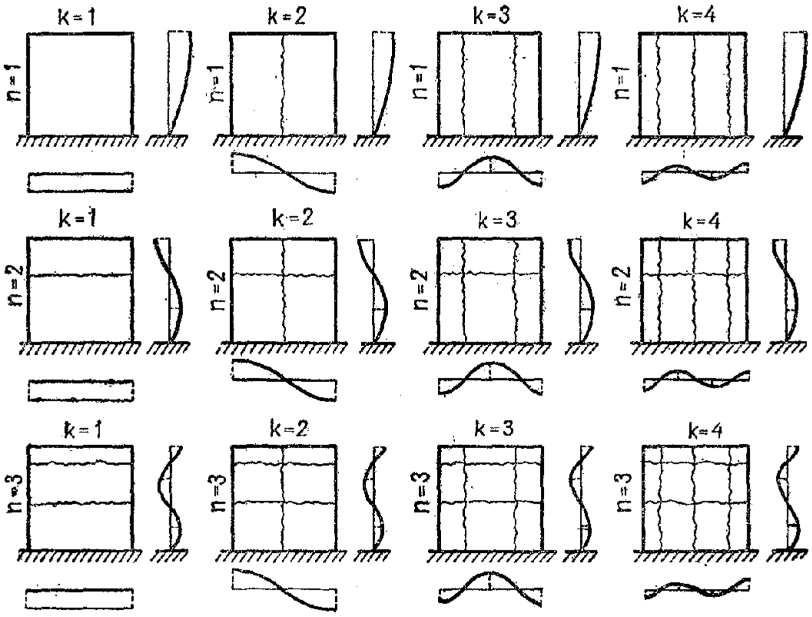


Fig. 74. Spatial wave forms of oscillations of a frame with floor undergoing shear strain.

The spectrum of wave forms of oscillations are represented by nk-diagrams in Fig. 75. We see that torsional oscillations of a building are one of the components of the spatial wave form of oscillations. Buildings with rigid and elastic floors have identical translational and torsional oscillations for low initial harmonics, so we can obtain the spatial wave forms of oscillations for point-type buildings by combining these separate types of oscillations. Finally, using (5.21) and (5.23), the formula for transverse direction of a building will be:

$$T_{nk}^{\text{trans}} = \frac{4H}{2n-1}\sqrt{\frac{(2-k)\,M}{\sum([GF]-N)_i^{\text{trans}}} + \frac{(k-1)\,M_0}{\sum([GF]-N)_i^{\text{long}}\,(a_i^{\text{long}})^2 + \sum([GF]-N)_i^{\text{long}}\,(a_i^{\text{long}})^2}}; \tag{5.114}$$

and for longitudinal direction:

$$T_{nk}^{\text{long}} = \frac{4H}{2n-1}\sqrt{\frac{(2-k)\,M}{\sum([GF]-N)_i^{\text{long}}} + \frac{(k-1)\,M_0}{\sum([GF]-N)_i^{\text{trans}}\,(a_i^{\text{trans}})^2 + \sum([GF]-N)_i^{\text{long}}\,(a_i^{\text{long}})^2}}. \tag{5.115}$$

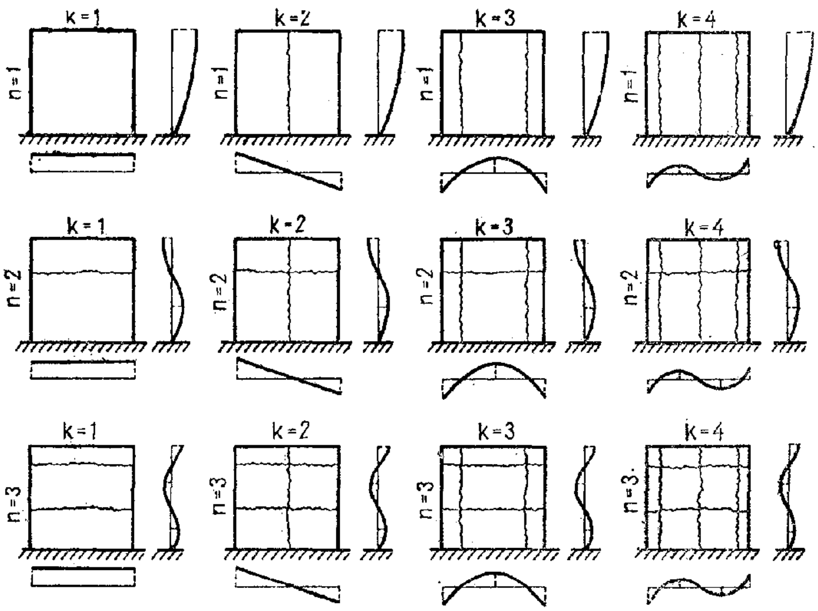


Fig. 75. Spatial wave forms of oscillations of a frame with floor undergoing bending strain.

The wave forms of oscillations along the floor length are determined by the following expressions:

$$(W_A)_{nk} = A\,[(2-k) + x\,(k-1)] \sin \frac{\pi\,(2n-1)\,z}{2H};$$

$$(V_A)_{nk} = A\,[(2-k) + y\,(k-1)] \sin \frac{\pi\,(2n-1)\,z}{2H}, \qquad (5.116)$$

where $n = 1, 2, 3, \ldots, k = 1, 2$.

For $k = 1$ we have translational oscillations, for $k = 2$ torsional. In torsional oscillations the periods for the transverse and longitudinal directions are equal, that is, $T_{n2}^{\text{trans}} = T_{n2}^{\text{long}}$ and the wave forms of oscillations will be:

$$(W_A)_{n2} = Ax \sin \frac{\pi\,(2n-1)\,z}{2H};$$

$$(V_A)_{n2} = Ay \sin \frac{\pi\,(2n-1)\,z}{2H}. \qquad (5.117)$$

Formulae to determine the periods and wave forms of three-dimensional natural oscillations of point type buildings with rigid floors, which can be represented by braced frame or braced schemes, are similarly deduced.

For vertical oscillations of buildings with large dimensions in plan, the first harmonic of oscillations similarly represents translational vertical displacements of the floor as a rigid body while the second harmonic represents rotation of the floor in the vertical plane. The periods and wave forms of vertical oscillations are determined by (5.85) and (5.86).

CHAPTER 6

Seismic Effects on Buildings

1. SEISMIC WAVES

We shall assume that each point on the surface of the earth executes translational deflection. This deflection may be resolved into a vertical and two mutually perpendicular horizontal components. Their respective accelerations may be represented by $\ddot{U}_0(t)$, $\ddot{W}_0(t)$ and $\ddot{V}_0(t)$. These accelerations may be considered amplitudes of wave functions in the following form:

$$\ddot{W}_0^* = \ddot{W}_0(t)\,\Phi_x(\lambda_x, x, c, t);$$

$$\ddot{V}_0^* = \ddot{V}_0(t)\,\Phi_y(\lambda_y, y, c, t);$$

$$\ddot{U}_0^* = \ddot{U}_0(t)\,\Phi_{zx}(\lambda_{zx}, x, c, t)\,\Phi_{zy}(\lambda_{zy}, y, c, t), \tag{6.1}$$

where Φ_x and Φ_y are the wave functions in the horizontal plane along the x and y axes; Φ_{zx} and Φ_{zy} are the wave functions in the vertical plane along the x and y axes.

The wave functions may be represented by a series of harmonic curves:

$$\Phi_x = \sum_{i=1}^{n} \sin \frac{2\pi}{(\lambda_x)_i}(x - ct);$$

$$\Phi_y = \sum_{i=1}^{n} \sin \frac{2\pi}{(\lambda_y)_i}(y - ct);$$

$$\Phi_{zx} = \sum_{i=1}^{n} \sin \frac{2\pi}{(\lambda_{zx})_i}(x - ct);$$

$$\Phi_{zy} = \sum_{i=1}^{n} \sin \frac{2\pi}{(\lambda_{zy})_i}(y - ct), \tag{6.2}$$

where λ_i is the wave length

$$\lambda_i = cT_i; \tag{6.3}$$

c and T_i are the velocity of propagation and time period of seismic waves.

Expressions (6.2) represent a complex oscillatory process containing many waves of varying wave lengths. The component which has a frequency close to the frequency of natural oscillation of the building will have significant effect on the building. Hence we shall consider seismic accelerations depending on the period of natural oscillations of the building:

$$\ddot{W}_0^* = \ddot{W}_0(t)\sin\frac{2\pi}{T_{\text{build}}^{\text{trans}}}\left(\frac{x}{c}-t\right);$$

$$\ddot{V}_0^* = \ddot{V}_0(t)\sin\frac{2\pi}{T_{\text{build}}^{\text{long}}}\left(\frac{y}{c}-t\right);$$

$$\ddot{U}_0^* = \ddot{U}_0(t)\sin\frac{2\pi}{T_{\text{build}}^{\text{ver}}}\left(\frac{x}{c}-t\right)$$

$$\times\sin\frac{2\pi}{T_{\text{build}}^{\text{ver}}}\left(\frac{y}{c}-t\right). \qquad (6.4)$$

The effect of seismic waves on a building, represented by a three-dimensional analytical model, is shown in Fig. 76.

Let us examine a standing seismic wave when the point of inflexion of

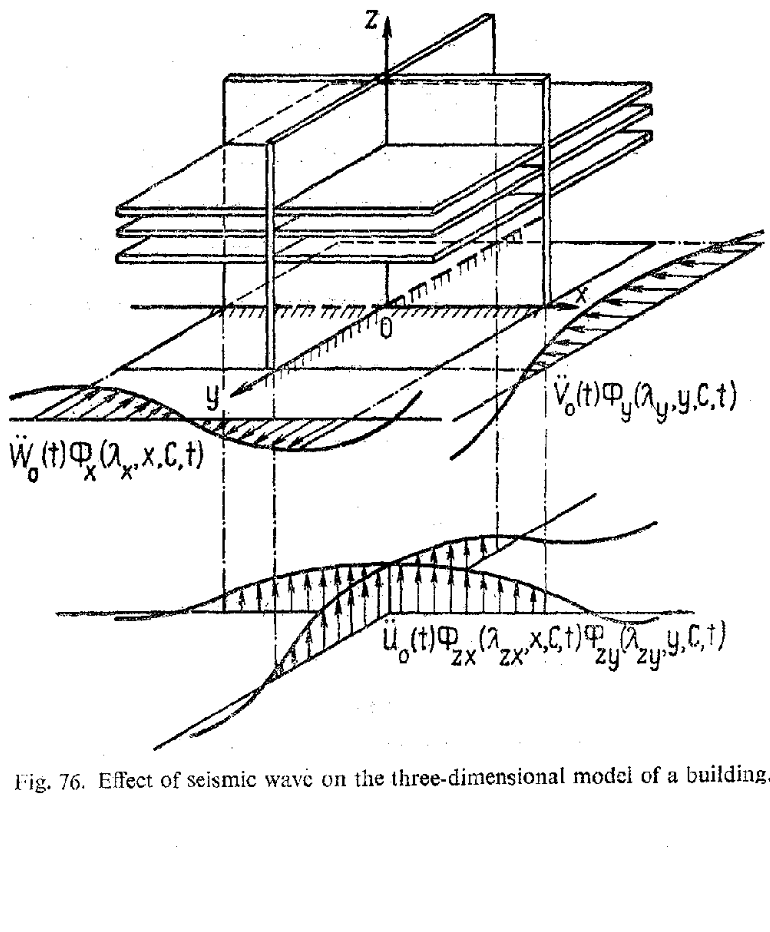


Fig. 76. Effect of seismic wave on the three-dimensional model of a building.

the wave coincides with the center of rigidity of the building. Acceleration acting in the transverse horizontal direction of the building will be:

$$\ddot{W}_0^* = \ddot{W}_0(t) \sin \frac{2\pi x}{\lambda_x}. \tag{6.5}$$

Coordinate ≪ x ≫ will be measured from the center of rigidity of the building. In this case the building will experience torsional seismic effect. If the center of rigidity of the building is located on the crest of the standing wave (Figs. 77, 78) the acceleration will be:

$$\ddot{W}_0^* = \ddot{W}_0(t) \cos \frac{2\pi x}{\lambda_x}. \tag{6.6}$$

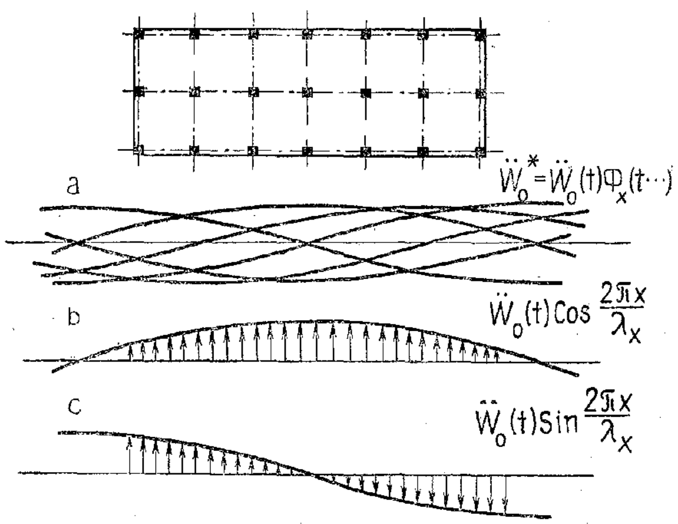


Fig. 77. Building model.
a—effect of running seismic waves; b, c—horizontal standing seismic waves.

In this case the building experiences translational seismic effect. Let us examine two standing seismic waves which cause torsional and translational effects. These two effects on the building (Fig. 77) may be represented as follows:

$$\ddot{W}_0^* = \ddot{W}_0 \Phi_x = \ddot{W}_0 \left[\sin \frac{\pi (k-1)}{2} \sin \frac{2\pi x}{\lambda_x} + \cos \frac{\pi (k-1)}{2} \cos \frac{2\pi x}{\lambda_x} \right]. \tag{6.7}$$

For $k = 1$ we shall have translational seismic effect and for $k = 2$ torsional.

Accelerations in the other directions may be similarly obtained.

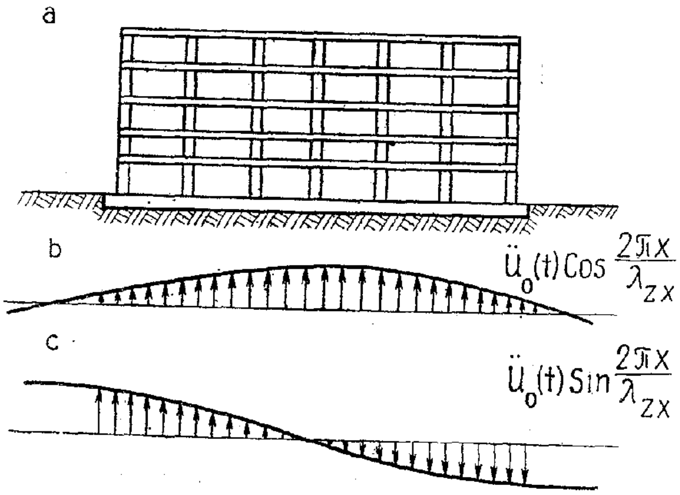


Fig. 78. Building model.
a—section; b, c—vertical standing seismic waves.

2. DETERMINATION OF SEISMIC LOADS ON A BUILDING BY THE SPECTRAL METHOD

During earthquakes the ground oscillates. The ground movement and the respective accelerations cause inertia forces to appear in the upper part of a building. The oscillations caused by the foundation displacement may be considered forced and the foundation fixed.

Oscillations of foundation in an elastic medium, with the ground mass and building superstructure connected to it, are a complex process. Let us examine two cases. The first is one in which we assume that forced oscillations of a building occur as a result of foundation deformation due to the passage of a seismic wave, fully reproducing the shape of these waves. This assumes that the ground suffers little deformation. In the second case the foundation is considered an absolutely rigid body and the ground is assumed to be pliable. Then the shape of the seismic wave will be distorted and the foundation will deflect as a rigid platform.

In other cases for purposes of calculation, a coefficient may be introduced which reduces the magnitude of acceleration imparted by the ground to the foundation due to the pliability of the ground.

Let us consider the simultaneous effect of all the components of acceleration (6.1) on a three-dimensional analytical model of a building and equate equations (5.3) to the respective disturbing seismic forces:

$$R^W = -m^{\text{trans}}\,\ddot{W}_0(t)\,\Phi_x;$$

$$R^V = -m^{\text{long}}\, \ddot{V}_0(t)\, \Phi_y;$$

$$R^U = -m^{\text{ver}}\, \ddot{U}_0(t)\, \Phi_{zx} \Phi_{zy}. \tag{6.8}$$

For frame buildings the differential equations of seismic oscillations will be:

$$(C_z^{\text{trans}} - N^{\text{trans}}) \frac{\partial^2 W}{\partial z^2} + C_x^{\text{trans}} \frac{\partial^2 W}{\partial x^2} - m^{\text{trans}} \frac{\partial^2 W}{\partial t^2} = -m^{\text{trans}}\, \ddot{W}_0(t)\, \Phi_x;$$

$$(C_z^{\text{long}} - N^{\text{long}}) \frac{\partial^2 V}{\partial z^2} - C_y^{\text{long}} \frac{\partial^2 V}{\partial y^2} - m^{\text{long}} \frac{\partial^2 V}{\partial t^2} = -m^{\text{long}}\, \ddot{V}_0(t)\, \Phi_y;$$

$$C_x^{\text{ver}} \frac{\partial^2 U}{\partial x^2} + C_y^{\text{ver}} \frac{\partial^2 U}{\partial y^2} + F_z^{\text{ver}} \frac{\partial^2 U}{\partial z^2} - m^{\text{ver}} \frac{\partial^2 U}{\partial t^2}$$

$$= -m^{\text{ver}}\, \ddot{U}_0(t)\, \Phi_{zx} \Phi_{zy}. \tag{6.9}$$

Let us examine the first equation and use the method of resolution along the main directions. In this case the external force may be resolved into a series of components of the following form:

$$R^W = m\ddot{W}_0(t)\, \Phi_x = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} r_{nk}. \tag{6.10}$$

Components r_{nk} deform the system according to the wave form of natural oscillations and should be distributed on an area proportional to the expression $X_k(x)\, Z_n(z)\, m(x, z)$:

$$r_{nk} = \alpha_{nk} X_k Z_n m, \tag{6.11}$$

where α_{nk} is a coefficient depending on the wave form of oscillations of the system.

For this the following condition should be satisfied:

$$R^W = \sum_n^{\infty} \sum_k^{\infty} \alpha_{nk}\, X_k(x)\, Z_n(z)\, m(x, z). \tag{6.12}$$

Let us multiply both sides of equation (6.12) by $X_i Z_j$ and integrate on the area XOZ:

$$\iint_S R^W = X_i Z_j\, dxdz = \iint_S \sum\sum \alpha_{nk}\, X_k\, Z_n\, m X_i\, Z_j\, dxdz. \tag{6.13}$$

For $i \neq k$ and $j \neq n$ we have

$$\iint_S m X_k\, Z_n\, X_i Z_j\, dxdz = 0.$$

Coefficient α_{ij} will be equal to:

$$\alpha_{ij} = \frac{\iint_S R^W\, X_i\, Z_j\, dxdz}{\iint_S X_i^2\, Z_j^2\, m dxdz}. \tag{6.14}$$

Considering (6.10) and changing indices i, j we get

$$\alpha_{nk} = \ddot{W}_0(t) \frac{\iint\limits_S m\Phi_x X_k Z_n \, dxdz}{\iint\limits_S mX_k^2 Z_n^2 \, dxdz}. \tag{6.15}$$

Components r_{nk} will be equal to:

$$r_{nk} = m\ddot{W}_0(t) \eta_{nk}, \tag{6.16}$$

where η_{nk} is a coefficient for the given wave form of oscillations:

$$\eta_{nk}(x, z) = X_k(x) Z_n(z) \frac{\iint\limits_S m\Phi_x X_k(x) Z_n(z) \, dxdz}{\iint\limits_S mX_k^2(x) Z_n^2(z) \, dxdz}. \tag{6.17}$$

We know that the principle of resolution permits examination of oscillations as a system with one degree of freedom in each direction. Based on this we may write the differential equation for a point by considering the dissipation of energy:

$$\frac{d^2 W}{dt^2} + \frac{\epsilon dW}{mdt} + \omega_{nk}^2 W \frac{r_{nk}}{m}, \tag{6.18}$$

where ϵ_{nk} is the dispersion factor of energy for nkth wave form of oscillations.

The general solution of this equation for zero initial boundary conditions is known to be:

$$W_{nk} = \frac{1}{m\omega_{nk}^2} \omega_{nk} \int r_{nk}(\xi) \exp\left[-\frac{\delta_{nk}}{2\pi} \omega_{nk}(t-\xi)\right] \sin \omega_{nk}(t-\xi) \, d\xi, \tag{6.19}$$

where δ_{nk} is the logarithmic decrement of natural oscillations of nkth wave form in the building.

By substituting the value of r_{nk} from (6.16) in (6.19) we get:

$$W_{nk} = \frac{1}{\omega_{nk}^2} \omega_{nk} \int \ddot{W}_0(\xi) \eta_{nk} \exp\left[-\frac{\delta_{nk}}{2\pi} \omega_{nk}(t-\xi)\right] \times \sin \omega_{nk}(t-\xi) \, d\xi. \tag{6.20}$$

Considering the standing wave and taking the acceleration to be $\ddot{W}_0(t) = \ddot{W}_{\max} f(t)$, equation (6.20) becomes:

$$W_{nk} = \frac{k_c g}{\omega_{nk}^2} \eta_{nk} \omega_{nk} \int t(\xi) \exp\left[-\frac{\delta_{nk}}{2\pi} \omega_{nk}(t-\xi)\right] \times \sin \omega_{nk}(t-\xi) \, d\xi. \tag{6.21}$$

where $k_c g = \ddot{W}_{\max}$, k_c is the coefficient of seismic stability and g is acceleration due to gravity.

Expression (6.21) may be written in the standard form:

$$W_{nk} = \frac{k_c g}{\omega_{nk}^2} \beta_{nk} \eta_{nk}, \tag{6.22}$$

where β_{nk} is the dynamic response factor:

$$\beta_{nk} = \omega_{nk} \int f(\xi) \exp\left[-\frac{\delta_{nk}}{2\pi} \omega_{nk} (t-\xi)\right] \sin \omega_{nk} (t-\xi) d\xi. \tag{6.23}$$

On the basis of the spectral curve of the dynamic response coefficient, the seismic load, distributed on the facade of the building, is expressed by the formula analogous to the prevailing standards:

$$S_{nk} = m\omega_{nk}^2 W_{nk} = k_c g m \beta_{nk} \eta_{nk}. \tag{6.24}$$

Consequently, the seismic loads acting in the transverse, longitudinal and vertical directions on a building for different wave forms of oscillations are determined by the following formulae:

$$S_{nk}^{\text{trans}} = k_c\, g m^{\text{trans}}\, \beta_{nk}^{\text{trans}}\, \eta_{nk}^{\text{trans}};$$

$$S_{nk}^{\text{long}} = k_c\, g m^{\text{long}}\, \beta_{nk}^{\text{long}}\, \eta_{nk}^{\text{long}};$$

$$S_{nkr}^{\text{ver}} = k_c\, g m^{\text{ver}}\, \beta_{nkr}^{\text{ver}}\, \eta_{nkr}^{\text{ver}}, \tag{6.25}$$

where $\beta = \frac{1}{T}$ and $0.8 \leqslant \beta \leqslant 3$;

$$\eta_{nk}^{\text{trans}} = \frac{(W_A)_{nk} \iint\limits_S m^{\text{trans}} \Phi_x (W_A)_{nk}\, dxdz}{\iint\limits_S m^{\text{trans}} (W_A)_{nk}^2\, dxdz};$$

$$\eta_{nk}^{\text{long}} = \frac{(V_A)_{nk} \iint\limits_S m^{\text{long}} \Phi_y (V_A)_{nk}\, dydz}{\iint\limits_S m^{\text{long}} (V_A)_{nk}^2\, dydz};$$

$$\eta_{nkr}^{\text{ver}} = \frac{(U_A)^{nkr} \iiint\limits_V m^{\text{ver}} \Phi_{zx} \Phi_{zy} (U_A)_{nkr}\, dxdzdy}{\iiint\limits_V m^{\text{ver}} (U_A)_{nkr}^2\, dxdydz}. \tag{6.26}$$

The total seismic load at any point of a building, for example in the transverse direction, is determined as the sum of forces acting for all main wave forms of oscillations:

$$S^{\text{trans}} (t) = \sum S_{nk}^{\text{trans}} (t) = k_c\, g m^{\text{trans}} \sum \beta_{nk} (t) \eta_{nk}. \tag{6.27}$$

Formula (6.27) is rewritten in the following form:

$$S^{\text{trans}}(t) = k_c\, g m^{\text{trans}} [\beta_{1k}(t)\,\eta_{1k} + \beta_{2k}(t)\,\eta_{2k} + \ldots + \beta_{n1}(t)\,\eta_{n1}$$
$$+ \beta_{n2}(t)\,\eta_{n2} + \ldots + \beta_{nk}(t)\,\eta_{nk}] = k_c\, g m^{\text{trans}} (\alpha_{1k}\,\beta_{1k}\,\eta_{nk} + \alpha_{2k}\,\beta_{2k}\,\eta_{2k}$$
$$+ \ldots + \alpha_{n1}\,\beta_{n1}\,\eta_{n1} + \alpha_{n2}\,\beta_{n2}\,\eta_{n2} + \ldots + \alpha_{nk}\,\beta_{nk}\,\eta_{nk}), \qquad (6.28)$$

where β_{1k}, β_{2k} are the maximum values of the dynamic response coefficient, α_{1k}, α_{2k} the coefficients depending on time which vary from -1 to $+1$ and attain their maximum values at a specific time.

We must determine the most likely magnitude of force $S^{\text{trans}}(t)$ during the entire period of an earthquake . . . that force which causes maximum force N_s (bending moment, transverse force, longitudinal force) in the section under consideration, that is, we must determine the coefficient α_{nk} when force N_s will be maximum.

There are several propositions according to which magnitudes of maximum seismic load are found for each wave form of oscillation after which diagrams of force N_s are constructed separately. Combinations of these diagrams give complete design force. This condition is expressed in standard form as:

$$N_p = \sqrt{N_{\max}^2 + 0.5 \sum_{i=1}^{n} N_i^2}. \qquad (6.29)$$

3. DETERMINATION OF OSCILLATION ANALYSIS FACTOR

Seismic load (6.25) may be written in the following form:

$$S_{nk} = k_c\, g m\, \beta_{nk}\, D_{nk}\, (W_A)_{nk}; \qquad (6.30)$$

$$D_{nk} = \frac{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} m \Phi_x\, (W_A)_{nk}\, dx dz}{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} m\, (W_A)_{nk}^2\, dx dz}, \qquad (6.31)$$

where D_{nk} is the oscillation analysis factor. It is a specific number depending on the ratio of building length to the length of the seismic wave $D_{nk}\,(n, k, \frac{L}{\lambda})$.

Using (5.80) and (6.7), the analysis factor for frame buildings and floors undergoing shear strain under translational seismic action will be:

$$D_{n1} = \frac{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} m \cos \frac{2\pi x}{\lambda_x} \sin \frac{\pi (2n-1) z}{2H} dxdz}{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} m \sin^2 \frac{\pi (2n-1) z}{2H} dxdz}. \tag{6.32}$$

By evaluating the specific integrals we get:

$$D_{n1} = \frac{0.4}{(2n-1)\chi} \sin \pi\chi; \tag{6.33}$$

$$\chi = \frac{L}{\lambda} = \frac{L}{cT_{\text{wave}}}, \tag{6.34}$$

where T_{wave} is the period of seismic wave.

When the length of a building is very much smaller than the wave length, that is, $\frac{L}{\lambda} \to 0$, the analysis factor will have the maximum value:

$$D_{n1} = \frac{1.27}{2n-1}.$$

For torsional seismic waves the value of analysis factor D_{n2} will be

$$D_{n2} = \frac{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin \frac{\pi (2n-1) z}{2H} \sin \frac{2\pi x}{\lambda x} \sin \frac{\pi x}{L} dxdz}{\int_0^H \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2 \frac{\pi (2n-1) z}{2H} \sin^2 \frac{\pi x}{L} dxdz}. \tag{6.35}$$

By determining the integral we get:

$$D_{n2} = \frac{1.27}{2n-1} \left[\frac{2 \sin \frac{\pi}{2} (2\chi - 1)}{\pi (2\chi - 1)} - \frac{2 \sin \frac{\pi}{2} (2\chi + 1)}{\pi (2\chi + 1)} \right]. \tag{6.36}$$

For $\chi = 0.5$, that is, $L = 0.5\lambda$, D_{n2} will have the maximum value $D_{n2} = \frac{1.27}{2n-1}$.

For higher harmonic translational seismic oscillations the analysis factor will be:

$$D_{n3}=\frac{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\sin\frac{\pi(2n-1)z}{2H}\cos\frac{2\pi x}{\lambda_x}\cos\frac{2\pi x}{L}\,dxdz}{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\sin^2\frac{\pi(2n-1)z}{2H}\cos^2\frac{2\pi x}{L}\,dxdz}. \tag{6.37}$$

After determining the integrals we have:

$$D_{n3}=\frac{1.27}{2n-1}\left[\frac{\sin\pi(X-1)}{\pi(X-1)}+\frac{\sin\pi(X+1)}{\pi(X+1)}\right]. \tag{6.38}$$

For $X=1$, that is, $L=\lambda$, D_{n3} attains the maximum value to $D_{n3}=\frac{1.27}{2n-1}$ (see Fig. 79).

If the floor undergoes bending strain, then by using (5.104) we obtain the following analysis factors: D_{n1} is determined by (6.33).

$$D_{n2}=\frac{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}x\sin\frac{\pi(2n-1)z}{2H}\sin\frac{2\pi x}{\lambda x}\,dxdz}{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}x^2\sin^2\frac{\pi(2n-1)z}{2H}\,dxdz}; \tag{6.39}$$

$$D_{n3}=\frac{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\sin\frac{\pi(2n-1)z}{2H}\cos\frac{2\pi x}{\lambda x}\left(\cos\frac{\pi x}{L}-0.6\right)dxdz}{\int\limits_0^H\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\sin^2\frac{\pi(2n-1)z}{2H}\left(\cos\frac{\pi x}{L}-0.6\right)^2dxdz}. \tag{6.40}$$

Analysis factors D_{nk} may be determined similarly for buildings with different structural schemes.

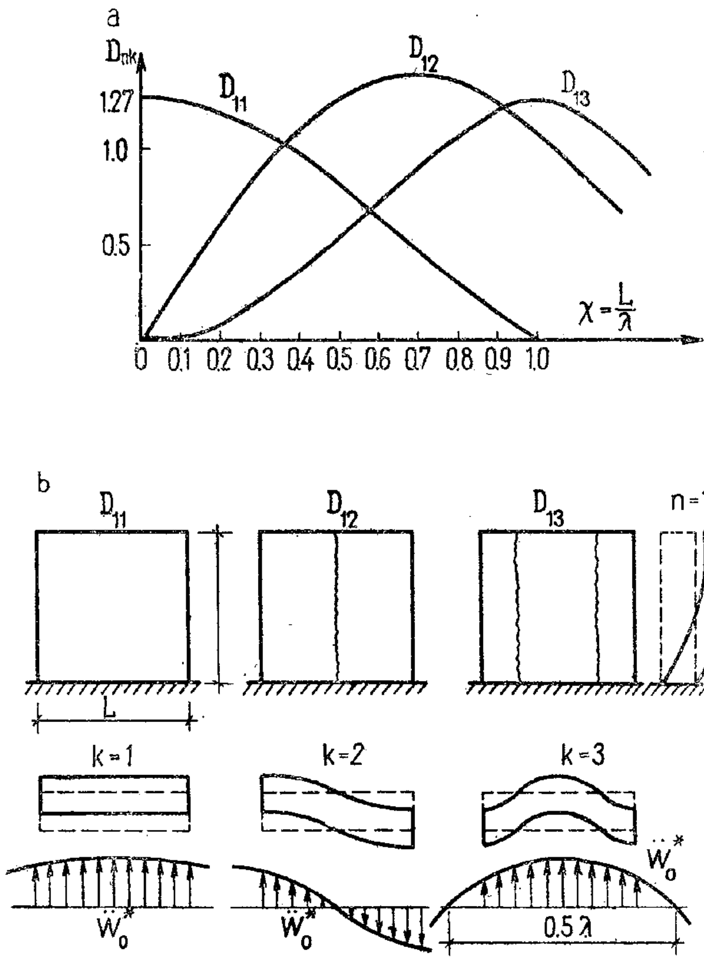


Fig. 79. Types of building oscillations.
a—graph of analysis factors of oscillations; b—seismic effects and wave forms of building oscillations.

4. DETERMINATION OF SEISMIC FORCES

The differential equations of equilibrium for a frame type building will be of the following form under the action of seismic loading:

$$(C_z^{\text{trans}} - N^{\text{trans}})\frac{\partial^2 W_{nk}}{\partial z^2} + C_x^{\text{trans}}\frac{\partial^2 W_{nk}}{\partial x^2} + S_{nk}^{\text{trans}} = 0;$$

$$(C_z^{\text{long}} - N^{\text{long}})\frac{\partial^2 V_{nk}}{\partial z^2} + C_y^{\text{long}}\frac{\partial^2 V_{nk}}{\partial y^2} + S_{nk}^{\text{long}} = 0;$$

$$C_x^{\text{ver}} \frac{\partial^2 U_{nkr}}{\partial x^2} + C_y^{\text{ver}} \frac{\partial^2 U_{nkr}}{\partial y^2} + F_z^{\text{ver}} \frac{\partial^2 U_{nkr}}{\partial z^2} + S_{nkr}^{\text{ver}} = 0. \quad (6.41)$$

Displacements W_{nk}, V_{nk} and U_{nkr} for the respective wave forms of oscillations due to seismic loads are determined by solving (6.41). We then determine the overall separate maximum seismic forces for frames in the form of normal components:

(a) in the transverse direction:

$$(Q_z^{\text{trans}})_{nk} = C_z \frac{\partial W_{nk}}{\partial z}; \; (Q_x^{\text{trans}})_{nk} = C_x \frac{\partial W_{nk}}{\partial x}; \quad (6.42)$$

and (b) in the longitudinal direction:

$$(Q_z^{\text{long}})_{nk} = C_z \frac{\partial V_{nk}}{\partial z}; \; (Q_y^{\text{long}})_{nk} = C_y \frac{\partial V_{nk}}{\partial y}; \quad (6.43)$$

and (c) in the vertical direction:

$$N_{nkr}^{\text{ver}} = F_z^{\text{ver}} \frac{\partial U_{nkr}}{\partial z};$$

$$(Q_x^{\text{ver}})_{nkr} = C_x^{\text{ver}} \frac{\partial U_{nkr}}{\partial x};$$

$$(Q_y^{\text{ver}})_{nkr} C_y^{\text{ver}} \frac{\partial U_{nkr}}{\partial y}. \quad (6.44)$$

We can determine these forces in a different manner without determining the seismic loading and by solving equations (6.41). For this we use expression (6.22) which is rewritten in the following form:

$$W_{nk} = \frac{k_c g}{4\pi^2} D_{nk}^{\text{trans}} (T_{nk}^{\text{trans}})^2 \beta_{nk}^{\text{trans}} (W_A)_{nk}; \quad (6.45)$$

$$V_{nk} = \frac{k_c g}{4\pi^2} D_{nk}^{\text{long}} (T_{nk}^{\text{long}})^2 \beta_{nk}^{\text{long}} (V_A)_{nk}; \quad (6.46)$$

$$U_{nkr} = \frac{k_c g}{4\pi^2} D_{nkr}^{\text{ver}} (T_{nkr}^{\text{ver}})^2 \beta_{nkr}^{\text{ver}} (U_A)_{nkr}. \quad (6.47)$$

By substituting the values of (6.45), (6.46) and (6.47) in the expressions (6.42), (6.43) and (6.44), respectively, we obtain the desired forces in the transverse direction as:

$$(Q_z^{\text{trans}})_{nk} = C_z^{\text{trans}} \frac{k_c g}{4\pi^2} D_{nk}^{\text{trans}} (T_{nk}^{\text{trans}})^2 \beta_{nk}^{\text{trans}} \frac{\partial (W_A)_{nk}}{\partial z}; \quad (6.48)$$

$$(Q_z^{\text{trans}}) = C_x \frac{k_c g}{4\pi^2} D_{nk}^{\text{long}} (T_{nk}^{\text{long}})^2 \beta_{nk}^{\text{long}} \frac{\partial (W_A)_{nk}}{\partial z}. \quad (6.49)$$

In the vertical direction the compressive force will be:

$$N_{nkr}^{\text{ver}} = F_z^{\text{ver}} \frac{k_c g}{4\pi^2} D_{nkr}^{\text{ver}} (T_{nkr}^{\text{ver}})^2 \beta_{nkr}^{\text{ver}} \frac{\partial (U_A)_{nkr}}{\partial z}. \tag{6.50}$$

For a braced building the bending moment and shear force will be:

$$(M_z^{\text{trans}})_{nk} = -B_{z0}^{\text{trans}} \frac{k_c g}{4\pi^2} D_{nk}^{\text{trans}} (T_{nk}^{\text{trans}})^2 \beta_{nk}^{\text{trans}} \frac{\partial (W_A)_{nk}}{\partial z^2}; \tag{6.51}$$

$$(Q_z^{\text{trans}})_{nk} = -B_{z0}^{\text{trans}} \frac{k_c g}{4\pi^2} D_{nk}^{\text{trans}} (T_{nk}^{\text{trans}})^2 \beta_{nk}^{\text{trans}} \frac{\partial^3 (W_A)_{nk}}{\partial z^3}. \tag{6.52}$$

The distribution of forces in individual components of building members is determined by (3.9) and (4.8). The seismic forces for design purposes are determined by summation according to formula (6.29).

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