EIGENPROPERTIES AND RESPONSE OF PRIMARY STRUCTURE AND EQUIPMENT SYSTEMS BY PERTURBATION APPROACH

by

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16. Abstracts

The report presents a systematic matrix perturbation analysis for calculating the eigenproperties of combined single degree-of-freedom equipment and structure systems in terms of the eigenproperties of the individual subsystems. For a comprehensive treatment of the problem, three cases are considered: (1) classically damped combined system, (2) classically damped primary system but nonclassically damped combined system and (3) nonclassically damped primary as well as combined system. The combined system properties, when calculated according to the procedures presented in the report, can be used to obtain the response like seismic floor response spectra which incorporate the effect of the dynamic interaction between the two sub-systems. The approach can be applied to most practical problems in which the supported equipment is light to moderately heavy compared to its supporting primary structure. The numerical results demonstrating the applicability of the approaches for calculating the combined eigenproperties and the equipment response are presented.

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CHAPTER 1

INTRODUCTION

1.1 GENERAL BACKGROUND

The proper seismic design of equipment and secondary subsystems supported on primary structures is of great practical interest. Usually these subsystems are very light compared to their primary structure. Thus, often in their analyses for seismic motions they are assumed to be decoupled from their supporting primary structures. That is, they receive the input from the primary structure, but being very light they are assumed not to affect the response of their supporting structure. However, when an equipment is not very light or when its natural frequency is tuned or nearly tuned to a dominant frequency of the supporting structure, the decoupled analysis may give inaccurate response. In such cases it is necessary to consider the effect of the dynamic interaction between the equipment and its supporting structure on the equipment response.

This interaction effect can be properly included in response calculations, if the dynamic characteristics such as the mode shapes, frequencies, modal damping ratios and participation factors of the combined equipment and the primary structure can be obtained somehow. The most straightforward, but impractical, approach would be to analyze the analytical model of the combined system. This approach is impractical for several reasons. Firstly, the matrices of the combined system will be ill-conditioned because of the large differences in the mass and stiffness characteristics of the two subsystems; this can cause numerical errors unless an eigenvalue routine with extended numerical precision is utilized. Secondly, even if one is willing to use extended precision in the analysis, the approach becomes impractical when several equipment with different characteristics are to be analysed. This is usually the case when the floor response spectra are required to be generated at several points of a primary structure. Such analytical procedures requiring a repeated analysis of the combined structure-equipment system are, therefore, not adopted.

Probably, the best approach is the one in which the modal properties of the individual systems are synthesized to obtain the eigenproperties of the combined system. One such mode synthesis approach has been developed by Suarez and Singh [23]. However, this approach requires a second eigenvalue analysis of a transformed system. Another approach is to use perturbation methods to obtain the perturbed eigenvalues of the combined system. Obviously, these approaches can be used only when the perturbation in the eigenvalues of the two subsystems are small. The perturbations will be small if the equipment attachments are light compared to the primary system. Yet with these limitations, some useful results of practical importance can still be obtained by the application of the perturbation approach.

The writers believe that Sackman and Kelly [11,16] were probably the first to apply the perturbation approach for this purpose. This approach was later utilized by Sackman et al [17], Hernried and Sackman [8] and Gupta [5-7] in their further studies related to the seismic response of light equipment.

This report also presents the application of yet another perturbation scheme for calculating the eigenproperties of the combined equipment-structure systems. This scheme is based on a systematic application of the matrix perturbation theory, developed and applied earlier by Lancaster [12], Franklin [4] and Meirovitch and Ryland [14] in a rather different context.

The matrix perturbation approach has been applied to three different cases of the equipment and structure systems here. The first case deals with the analysis of a classically damped [1,3] combined equipment- structure system. The second case deals with analysis of a classically damped primary structure supporting an equipment where the combined system can not be treated as a classically damped system. In the third case, the primary system itself is nonclassically damped and thus the combined system is also nonclassically damped. These three cases were sequentially developed, and reported in References 20, 25, and 26. This report presents a synthesis of these papers. A common link between these three cases is the matrix perturbation analysis which is presented in Chapter 2. Because of the special separation of the matrices required for the solution of the problem at hand, the analytical details of this perturbation analysis are quite different from those reported in References 4, 12, and 14. Also, the case of an equipment tuned to a primary structural frequency requires a quite different perturbation expansion scheme. The details of this scheme are also provided in Chapter 2. This is followed by the treatment of the aforementioned three cases in Chapters 3, 4, and 5. For each of these cases the closed-form expressions are obtained for the eigenvalues (frequencies) and eigenvectors of the combined system in terms of the eigenproperties of the subsystems. Both, the tuned and detuned cases are considered. The numerical results demonstrating the applicability and limitations of the approach for various cases are also given in these chapters.

CHAPTER 2

MATRIX PERTURBATION ANALYSIS

2.1 INTRODUCTION

In this chapter we describe the perturbation analysis of a generic eigenvalue problem. The problems to be discussed in the subsequent chapters are the special cases of this generalized eigenvalue problem. The case in which all the unperturbed eigenvalues are distinct as well as the case in which any two eigenvalues are closely spaced are considered. The first case pertains to the problem of a detuned equipment, whereas the second case is related to the problem of a tuned or nearly tuned equipment.

2.2 A GENERIC EIGENVALUE PROBLEM

The eigenvalue analyses of the three damping cases of the combined structure-equipment system, described in the later chapters of the report, show that they all can be dealt within the framework of the following generic eigenvalue problem:

$$[A_o + \varepsilon A_1 + \varepsilon^2 A_2] \underset{\sim}{\psi}_j = p_j [B_o + \varepsilon^2 B_2] \underset{\sim}{\psi}_j \; ; \quad j = 1, \ldots . n \quad (2.1)$$

where the matrices $[A_o]$ and $[B_o]$ of the original eigenvalue problem have now been perturbed by the addition of the matrices $[A_1]$, $[A_2]$ and $[B_2]$, which are of lower orders of magnitude. The parameter ε identifies the order of the matrices as well as helps in keeping track of various quantities of different orders of magnitude. Therefore, this parameter is also called as the "bookkeeping parameter." The elements of matrix $[A_1]$ are one order of magnitude smaller than the elements of $[A_o]$, while the elements of $[A_2]$ and $[B_2]$ are two orders of magnitude smaller than the elements of $[A_o]$. Therefore, we consider the elements of $[A_1]$ to be $O(\varepsilon)$ and the elements of $[A_2]$ and $[B_2]$ as $O(\varepsilon^2)$, where O(..) means the "order of (..)".

We will now analyze this eigenvalue problem to obtain the perturbed eigenproperties in terms of the original or unperturbed eigenvalues. We will consider two cases in which the eigenvalues of the original system are: (1) well separated and (2) closely spaced. These cases are often referred to as the detuned and tuned cases, respectively.

2.3 PERTURBATION ANALYSIS OF A DETUNED SYSTEM

The matrices in Eq. (1) are nxn real symmetric matrices. Furthermore, matrices $[B_o]$ and $[B_2]$ are positive definite whereas the other matrices are not necessarily positive definite. For this eigenvalue problem, we now seek the conventional second order expansions for the eigenvalues and eigenvectors of the following form:

$$p_j = p_{oj} + \varepsilon\, p_{ij} + \varepsilon^2\, p_{2j} + \ \dots \qquad (2.2)$$

$$; \quad j=1, \dots, n$$

$$\underset{\sim}{\psi}_j = \underset{\sim}{u}_{oj} + \varepsilon\, \underset{\sim}{u}_{ij} + \varepsilon^2\, \underset{\sim}{u}_{2j} + \ \dots \qquad (2.3)$$

Substituting p_j and $\underset{\sim}{\psi}_j$ from Eqs. (2.2) and(2.3) into Eq. (2.1) and equating the coefficients of equal power of ε we obtain the following hierarchy of equations:

$$O(\varepsilon^0): \quad A_o\, \underset{\sim}{u}_{oj} = p_{oj}\, B_o\, \underset{\sim}{u}_{oj} \qquad (2.4)$$

$$O(\varepsilon): \quad A_o\, \underset{\sim}{u}_{1j} + A_1\, \underset{\sim}{u}_{oj} = p_{oj}\, B_o \underset{\sim}{u}_{1j} + p_{1j}\, B_o\, \underset{\sim}{u}_{oj}\ ; \quad j=1,\dots,n \qquad (2.5)$$

$$O(\varepsilon^2): \quad A_o\, \underset{\sim}{u}_{2j} + A_1\, \underset{\sim}{u}_{1j} + A_2\, \underset{\sim}{u}_{oj} = p_{oj}\, B_o\, \underset{\sim}{u}_{2j} + p_{1j}\, B_o\, \underset{\sim}{u}_{1j} +$$

$$p_{2j} B_o \underset{\sim}{u}_{oj} + p_{oj} B_2 \underset{\sim}{u}_{oj} \tag{2.6}$$

The eigenvectors $\underset{\sim}{\psi}_j$ obtained with the assumed expansion have to satisfy the following orthonormality condition up to the second order terms

$$\underset{\sim}{\psi}_i^T [B_o + \varepsilon^2 B_2] \underset{\sim}{\psi}_j = \delta_{ij} \quad ; \quad i,j=1,\ldots,n \tag{2.7}$$

where δ_{ij} is the Kronecker delta. Substituting Eq. (2.3) in the above equation and comparing the coefficients of like powers of ε we arrive at

$$O(\varepsilon^o): \quad \underset{\sim}{u}_{oi}^T B_o \underset{\sim}{u}_{oj} = \delta_{ij} \tag{2.8}$$

$$O(\varepsilon): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{1j} + \underset{\sim}{u}_{1j}^T B_o \underset{\sim}{u}_{oj} = 0 \quad ; \quad i,j=1,\ldots,n \tag{2.9}$$

$$O(\varepsilon^2): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{2j} + \underset{\sim}{u}_{oj}^T B_2 \underset{\sim}{u}_{oj} + \underset{\sim}{u}_{1i}^T B_o \underset{\sim}{u}_{1j} + \underset{\sim}{u}_{2j}^T B_o \underset{\sim}{u}_{oj} = 0 \tag{2.10}$$

Following Meirovitch and Ryland [14] we expand the first order correction terms $\underset{\sim}{u}_{1j}$, using the eigenvectors of the unperturbed system $\underset{\sim}{u}_{oj}$ as the base vectors, as follows:

$$\underset{\sim}{u}_{1j} = \sum_{k=1}^{n} \theta_{jk} \underset{\sim}{u}_{ok} \quad ; \quad j=1,\ldots,n \tag{2.11}$$

where the coefficients θ_{jk} are yet to be determined. Premultiplying Eq. (2.5) by $\underset{\sim}{u}_{oj}^T$ and substituting $\underset{\sim}{u}_{1j}$ from Eq. (2.11) we obtain

$$p_{oj} \theta_{ji} + \underset{\sim}{u}_{oj}^T A_1 \underset{\sim}{u}_{oj} = p_{oj} \theta_{ji} + p_{1i} \delta_{ij} \tag{2.12}$$

where Eqs. (2.4) and (2.8) were used in arriving at Eq. (2.12). Considering first the case of i=j we obtain the first order correction terms to the eigenvalues as

$$p_{1j} = \underset{\sim}{u}_{oj}^T A_1 \underset{\sim}{u}_{oj} \quad ; \quad j=1,\ldots,n \tag{2.13}$$

If we consider $i \neq j$ we obtain

$$\theta_{ji} = \frac{\underset{\sim}{u}_{oi}^T A_1 \underset{\sim}{u}_{oj}}{p_{oj} - p_{oi}} \quad ; \quad i,j=1,\ldots,n, \; i \neq j \tag{2.14}$$

To obtain the coefficients θ_{jj} we must consider the orthogonality conditions given by eq. (2.7). By replacing Eq. (2.11) in Eq. (2.9) we obtain:

$$\theta_{ij} = - \theta_{ji} \tag{2.15}$$

and hence:

$$\theta_{jj} = 0 \quad ; \quad j=1,\ldots,n \tag{2.16}$$

To obtain the second order correction terms, we again expand $\underset{\sim}{u}_{2j}$ in terms of the base vectors $\underset{\sim}{u}_{ok}$ as:

$$\underset{\sim}{u}_{2j} = \sum_{k=1}^{n} \hat{\theta}_{jk} \underset{\sim}{u}_{ok} \quad ; \quad j=1,\ldots,n \tag{2.17}$$

Premultiplying Eq. (2.6) by $\underset{\sim}{u}_{oj}^T$, substituting $\underset{\sim}{u}_{1j}$ and $\underset{\sim}{u}_{2j}$ from Eqs. (2.11) and (2.17) respectively and considering the orthonormality properties of the unperturbed eigenproblem, we obtain:

$$p_{oi} \hat{\theta}_{ji} + \sum_{k=1}^{n} (p_{ok} - p_{oi}) \theta_{jk}\theta_{ki} + \underset{\sim}{u}_{oj}^T A_2 \underset{\sim}{u}_{oj} = p_{oj} \hat{\theta}_{ji} + p_{1j} \theta_{ji} + p_{oj} \underset{\sim}{u}_{oi}^T B_2 \underset{\sim}{u}_{oj} + p_{2j} \delta_{ij} \tag{2.18}$$

Letting i=j we obtain the second order correction terms for the eigenvalues:

$$p_{2j} = \sum_{k=1}^{n} (p_{oj} - p_{ok}) (\theta_{jk})^2 + \underset{\sim}{u}_{oj}^T [A_2 - p_{oj} B_2] \underset{\sim}{u}_{oj} \quad ; \quad j=1,\ldots,n \tag{2.19}$$

For $i \neq j$ the coefficients $\hat{\theta}_{ji}$ are directly obtained from equation (2.18):

$$\hat{\theta}_{ji} = \frac{1}{p_{oj} - p_{oi}} \{(p_{1j} - p_{1j}) \theta_{ji} + \sum_{k=1}^{n} (p_{ok} - p_{oi}) \theta_{jk} \theta_{ki} + \underset{\sim}{u}_{oi}^T [A_2 - p_{oj} B_2] \underset{\sim}{u}_{oj}\} \tag{2.20}$$

The above expression is not valid for i=j. To find $\hat{\theta}_{jj}$ we have to use the orthogonality conditions of equation (2.10). Substituting Eqs. (2.11) and (2.17) in Eq. (2.10) we obtain:

$$\hat{\theta}_{ji} + \hat{\theta}_{ij} + \sum_{k=1}^{n} \theta_{ik}\, \theta_{jk} + \underset{\sim}{u}_{oi}^{T}\, B_2\, \underset{\sim}{u}_{oj} = 0 \qquad (2.21)$$

and for i=j

$$\hat{\theta}_{jj} = -\frac{1}{2}\left[\underset{\sim}{u}_{oj}^{T}\, B_2\, \underset{\sim}{u}_{oj} + \sum_{k=1}^{n} (\theta_{jk})^2\right] \quad ; \quad j=1,\ldots,n \qquad (2.22)$$

2.4 PERTURBATION ANALYSIS OF A TUNED SYSTEM

It will be observed in later chapters that if the equipment is tuned to one of the supporting structure's frequencies, the eigenvalues of the original system will be equal. Such a case can not be treated by formulation developed in the previous section, primarily because of the numerical problem in evaluation of Eqs. (2.14) and (2.20) for a perfectly tuned case. Even in a nearly tuned case, the terms in these equation become very large thus invalidating the expansion of Eqs. (2.2) - (2.3) because the first and the second order terms, which were assumed to be of lower order in magnitude than the unperturbed terms, now become large. Therefore, to treat the problem of tuned system we need a different perturbation expansion.

We will assume that the ℓ^{th} and m^{th} unperturbed eigenvalues are equal or nearly equal. When the nonclassically damped cases are considered, the matrices in the eigenvalue problem (2.1) would be of dimension 2mx2m. In such a case we will also be concerned with the complex conjugate $(\ell+m)^{th}$ and $2m^{th}$ eigenvalues and corresponding eigenvectors of this eigenvalue problem. Herein, therefore a general eigenvalue problem of size 2mx2m with complex and conjugate eigenpairs will be analyzed.

The perturbed eigenvalues and eigenvectors will now be obtained assuming the following expansions:

$$p_i = p_{oi} + \varepsilon^{1/2} p_{1i} + \varepsilon p_{2i} + \varepsilon^{3/2} p_{3i} + \varepsilon^2 p_{4i} + \ldots \qquad (2.23)$$

$$; \; i=\ell,\ m,\ m\ell,\ 2m$$

$$\underset{\sim}{\psi}_i = \underset{\sim}{u}_{oi} + \varepsilon^{1/2} \underset{\sim}{u}_{1i} + \varepsilon \underset{\sim}{u}_{2i} + \varepsilon^{3/2} \underset{\sim}{u}_{3i} + \varepsilon^2 \underset{\sim}{u}_{4i} + \ldots \qquad (2.24)$$

where we introduced the notation $m\ell = m+\ell$. The justification for the above expansions lies in the fact that the eigenvalues of Eq. (2.1) for the tuned case can be obtained as the roots of a $(2m)^{th}$ degree polynomial with two equal or nearly equal roots and its complex conjugate values. It has been observed [15] that a proper expansion for finding the roots of an algebraic equation with two closely spaced roots should have the form of Eq. (2.23), that is, it must be expressed in terms of exponents of the parameter ε which are integer multiples of 1/2.

When the two assumed expansions are substituted into Eq. (2.1) and the coefficients of equal powers of ε are compared, we obtain

$$O(\varepsilon^0): \quad A_o \underset{\sim}{u}_{oi} = p_{oi} B_o \underset{\sim}{u}_{oi} \qquad (2.25)$$

$$O(\varepsilon^{1/2}): \quad A_o \underset{\sim}{u}_{1i} = p_{oi} B_o \underset{\sim}{u}_{1i} + p_{1i} B_o \underset{\sim}{u}_{oi} \qquad 2.26)$$

$$O(\varepsilon): \quad A_o \underset{\sim}{u}_{2i} + A_1 \underset{\sim}{u}_{oi} = p_{oi} B_o \underset{\sim}{u}_{2i} + p_{1i} B_o \underset{\sim}{u}_{1i} + p_{2i} B_o \underset{\sim}{u}_{oi} \qquad (2.27)$$

$$O(\varepsilon^{3/2}): \quad A_o \underset{\sim}{u}_{3i} + A_1 \underset{\sim}{u}_{1i} = p_{oi} B_o \underset{\sim}{u}_{3i} + p_{1i} B_o \underset{\sim}{u}_{2i} + p_{2i} B_o \underset{\sim}{u}_{1i} + p_{3i} B_o \underset{\sim}{u}_{oi} \qquad (2.28)$$

$$O(\varepsilon^2): \quad A_o \underset{\sim}{u}_{4i} + A_1 \underset{\sim}{u}_{2i} + A_2 \underset{\sim}{u}_{oi} = p_{oi} B_o \underset{\sim}{u}_{4i} + p_{1i} B_o \underset{\sim}{u}_{3i} + p_{2i} B_o \underset{\sim}{u}_{2i} + p_{3i} B_o \underset{\sim}{u}_{1i} + p_{4i} B_o \underset{\sim}{u}_{oi} + p_{oi} B_2 \underset{\sim}{u}_{oi} \qquad (2.29)$$

where the subscript i takes the values ℓ, m, $m\ell$ or 2m. A similar substitution for $\underset{\sim}{\psi}_i$ into the orthogonality condition, Eq. (2.7), yields:

$$O(\varepsilon^0): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{oi} = \delta_{ij} \tag{2.30}$$

$$O(\varepsilon^{1/2}): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{1j}^T B_o \underset{\sim}{u}_{oi} = 0 \tag{2.31}$$

$$O(\varepsilon): \quad \underset{\sim}{u}_{1j}^T B_o \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2j}^T B_o \underset{\sim}{u}_{oi} + \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{2i} = 0 \tag{2.32}$$

$$O(\varepsilon^{3/2}): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{3i} + \underset{\sim}{u}_{1j}^T B_o \underset{\sim}{u}_{2i} + \underset{\sim}{u}_{2j}^T B_o \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{3j}^T B_o \underset{\sim}{u}_{oi} = 0 \tag{2.33}$$

$$O(\varepsilon^2): \quad \underset{\sim}{u}_{oj}^T B_o \underset{\sim}{u}_{4i} + \underset{\sim}{u}_{1j}^T B_o \underset{\sim}{u}_{3i} + \underset{\sim}{u}_{2j}^T B_o \underset{\sim}{u}_{2i} + \underset{\sim}{u}_{3j}^T B_o \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{4j}^T B_o \underset{\sim}{u}_{oi} + \underset{\sim}{u}_{oj}^T B_2 \underset{\sim}{u}_{oi} = 0 \tag{2.34}$$

where again i,j take the values ℓ,m,mℓ, or 2m.

We examine first the terms $O(\varepsilon^{1/2})$. We again expand $\underset{\sim}{u}_{1i}$ in terms of the base vectors $\underset{\sim}{u}_{ok}$:

$$\underset{\sim}{u}_{1i} = \sum_{k=1}^{2m} \theta_{ik} \underset{\sim}{u}_{ok} \quad ; \quad i=\ell,m,m\ell,2m \tag{2.35}$$

Premultiplying Eq. (2.26) by $\underset{\sim}{u}_{oi}^T$, substituting $\underset{\sim}{u}_{1i}$ from Eq. (2.35) and invoking the orthogonality properties of the unperturbed eigenvectors, we obtain:

$$p_{1i}\,\delta_{ij} = (p_{oj} - p_{oi})\,\theta_{ij} \quad ; \quad \begin{array}{l} i=\ell,m,m\ell,2m \\ j=1,\ldots,2m \end{array} \tag{2.36}$$

With i=j we obtain:

$$p_{1i} = 0 \quad ; \quad i=\ell,m,m\ell,2m \tag{2.37}$$

and with i≠j we have:

$$\theta_{\ell j} = \theta_{mj} = 0 \quad ; \quad j=1,\ldots,2m \;;\; j\neq\ell,m \tag{2.38}$$

$$\theta_{m\ell,j} = \theta_{2m,j} = 0 \quad ; \quad j=1,\ldots,2m-1 \;;\; j\neq m\ell \tag{2.39}$$

Introducing Eq. (2.35) into the condition (2.31), it follows that:

$$\theta_{ij} = -\theta_{ji} \qquad ; \quad i,j=\ell,m,m\ell,2m \tag{2.40}$$

and hence:

$$\theta_{\ell\ell} = \theta_{mm} = \theta_{m\ell,m\ell} = \theta_{2m,2m} = 0 \tag{2.41}$$

And thus considering the four possible values of i, Eq. (2.35) becomes:

$$\begin{aligned} \underset{\sim}{u}_{1\ell} &= \theta_{\ell m}\, \underset{\sim}{u}_{om} \\ \underset{\sim}{u}_{1m} &= -\theta_{\ell m}\, \underset{\sim}{u}_{o\ell} \\ \underset{\sim}{u}_{1m\ell} &= \theta_{m\ell,2m}\, \underset{\sim}{u}_{o2m} \\ \underset{\sim}{u}_{12m} &= -\theta_{m\ell,2m}\, \underset{\sim}{u}_{om\ell} \end{aligned} \tag{2.42}$$

Two coefficients in the set $\hat{\theta}_{ik}$ still remain unknown, namely $\theta_{\ell m}$ and $\theta_{m\ell,2m}$. To obtain these we must consider the higher order hierarchical equations. We consider next the correction terms of order ε. We express $\underset{\sim}{u}_{2i}$ in terms of the base vectors as:

$$\underset{\sim}{u}_{2i} = \sum_{k=1}^{2m} \hat{\theta}_{ik}\, \underset{\sim}{u}_{ok} \qquad ; \quad i=\ell,m,m\ell,2m \tag{2.43}$$

Substituting this in Eq. (2.27), premultiplying by $\underset{\sim}{u}_{oj}^T$ and considering Eqs. (2.25) and (2.30) we obtain:

$$p_{2i}\,\delta_{ij} = (p_{oj} - p_{oi})\,\hat{\theta}_{ij} + \underset{\sim}{u}_{oj}^T A_1\, \underset{\sim}{u}_{oi} \quad ; \quad \begin{array}{l} i=\ell,m,m\ell,2m \\ j=1,\ldots,2m \end{array} \tag{2.44}$$

Letting i=j, we obtain the correction terms:

$$p_{2i} = \underset{\sim}{u}_{oi}^T A_1\, \underset{\sim}{u}_{oi} \qquad ; \quad i=\ell,m,m\ell,2m \tag{2.45}$$

If $i \neq j$ we obtain instead:

$$\hat{\theta}_{ij} = \frac{\underset{\sim}{u}_{oj}^T A_1\, \underset{\sim}{u}_{oi}}{p_{oi} - p_{oj}} \quad ; \quad \begin{array}{l} i=\ell,m \\ j=1,\ldots,2m\ ;\ j\neq\ell,m \\ \\ i=m\ell,2m \\ j=1,\ldots,2m-1;\ j\neq m\ell \end{array} \tag{2.46}$$

And with i=m, j=ℓ and i=2m, j=mℓ

$$\underset{\sim}{u}_{o\ell}^{T} A_1 \underset{\sim}{u}_{om} = \underset{\sim}{u}_{om\ell}^{T} A_1 \underset{\sim}{u}_{o2m} = 0 \qquad (2.47)$$

When the expansions of Eqs. (2.35) and (2.43) are substituted into the orthogonality conditions (2.32) we obtain:

$$\hat{\theta}_{ij} + \hat{\theta}_{ji} + \sum_{k=1}^{2m} \theta_{jk}\,\theta_{ik} = 0 \quad ; \quad \begin{matrix} i=\ell,m,m\ell,2m \\ j=1,\dots,2m \end{matrix} \qquad (2.48)$$

Equation (2.48) in turn leads to:

$$\hat{\theta}_{\ell\ell} = \hat{\theta}_{mm} = -\frac{1}{2}\,(\theta_{\ell m})^2 \qquad (2.49)$$

$$\hat{\theta}_{m\ell,m\ell} = \hat{\theta}_{2m,2m} = -\frac{1}{2}\,(\theta_{m\ell,2m})^2 \qquad (2.50)$$

And considering Eqs. (2.38), (2.39) and (2.41), from Eq. (2.48) we find:

$$\hat{\theta}_{\ell m} = -\hat{\theta}_{m\ell} \qquad (2.51)$$

$$\hat{\theta}_{m\ell,2m} = -\hat{\theta}_{2m,m\ell} \qquad (2.52)$$

We conclude here the analysis of the terms of $O(\varepsilon)$ with the coefficients $\hat{\theta}_{\ell m}$, $\hat{\theta}_{m\ell,2m}$, $\hat{\theta}_{\ell\ell}$, $\hat{\theta}_{mm}$, $\hat{\theta}_{2m,2m}$, and $\hat{\theta}_{m\ell,m\ell}$ still undefined. It turns out that they can be defined only when we consider the terms of $O(\varepsilon)$.

In order to examine the correction terms of $O(\varepsilon^{3/2})$ we assume that $\underset{\sim}{u}_{3i}$ can be expressed as a linear combination of the unperturbed eigenvectors $\underset{\sim}{u}_{ok}$:

$$\underset{\sim}{u}_{3i} = \sum_{k=1}^{2m} \overset{*}{\theta}_{ik}\,\underset{\sim}{u}_{ok} \quad ; \quad i = \ell,m,m\ell,2m \qquad (2.53)$$

Inserting the above expansion and Eqs. (2.35) and (2.43) into Eq. (2.28), premultiplying by $\underset{\sim}{u}_{oj}^{T}$ and considering Eqs. (2.25), (2.30) and

(2.37) we obtain:

$$p_{3i}\,\delta_{ij} = (p_{oj} - p_{oi})\,\overset{\star}{\theta}_{ij} + \sum_{k=1}^{2m} \theta_{ik}\,\underset{\sim}{u}_{oj}^{T}\,A_1\,\underset{\sim}{u}_{ok} - p_{2i}\,\theta_{ij} \qquad \begin{matrix} i=\ell,m,m\ell,2m \\ j=1,\ldots,2m \end{matrix} \tag{2.54}$$

Evaluating the above expression for i=j it follows that

$$p_{3i} = 0 \quad ; \quad i=\ell,m,m\ell,2m \tag{2.55}$$

If $i=\ell$, $j=m$, from Eqs. (2.47) and (2.54) we obtain

$$\theta_{\ell m} = 0 \tag{2.56}$$

Similarly for $i=m\ell$, $j=2m$ we obtain

$$\theta_{m\ell,2m} = 0 \tag{2.57}$$

And inserting the above results into Eqs. (2.42), (2.49) and (2.50) we conclude that

$$\underset{\sim}{u}_{1i} = 0 \quad ; \; i=\ell,m,m\ell,2m \tag{2.58}$$

$$\hat{\theta}_{\ell\ell} = \hat{\theta}_{mm} = \hat{\theta}_{m\ell,m\ell} = \hat{\theta}_{2m,2m} = 0 \tag{2.59}$$

Moreover, from Eq. (2.54) for $i\neq j$ we also obtain

$$\overset{\star}{\theta}_{\ell j} = \overset{\star}{\theta}_{mj} = 0 \quad ; \quad j=1,\ldots,2m; j\neq\ell,m \tag{2.60}$$

$$\overset{\star}{\theta}_{m\ell,j} = \overset{\star}{\theta}_{2m,j} = 0 \quad ; \quad j=1,\ldots,2m-1; j\neq m\ell \tag{2.61}$$

The orthogonality conditions, Eq. (2.33), with $\underset{\sim}{u}_{3i}$ given by Eq. (2.53) and $\underset{\sim}{u}_{1i}$ by Eq. (2.58), give

$$\overset{\star}{\theta}_{ij} = -\,\overset{\star}{\theta}_{ji} \tag{2.62}$$

and therefore

$$\theta^{*}_{\ell\ell} = \theta^{*}_{mm} = \theta^{*}_{m\ell,m\ell} = \theta^{*}_{2m,2m} = 0 \qquad (2.63)$$

We note here that the coefficients $\theta^{*}_{\ell m}$ and $\theta^{*}_{m\ell,2m}$ still remain unknown and cannot be obtained with the current five-terms expansion used in Eqs. (2.23) and (2.24). To obtain them, we will have to extend our expansions up to six terms, tha is, we will have to consider terms of $O(\varepsilon^{5/2})$.

Finally we need to examine the correction terms of $O(\varepsilon^2)$. We multiply Eq. (2.29) on the left by $\underset{\sim}{u}^T_{oj}$, replace the following expansion for $\underset{\sim}{u}_{4i}$:

$$u_{4i} = \sum_{k=1}^{2m} \tilde{\theta}_{ik}\underset{\sim}{u}_{ok} \qquad ; \quad i=\ell,m,m\ell,2m \qquad (2.64)$$

and apply the orthogonality conditions of the zero order eigenvalue problem, to obtain:

$$p_{4i}\delta_{ij} = (p_{oj}-p_{oi})\tilde{\theta}_{ij} - p_{2i}\hat{\theta}_{ij} + \sum_{k=1}^{2m} \hat{\theta}_{ik}\underset{\sim}{u}^T_{oj}A_1\underset{\sim}{u}_{ok} + \underset{\sim}{u}^T_{oj}[A_2 - p_{oi}B_2]\underset{\sim}{u}_{oi} \qquad (2.65)$$

considering the case i=j and with the help of Eqs. (2.46), (2.47) and (2.59) we obtain for the correction terms p_{4i}:

$$p_{4i} = \underset{\sim}{u}^T_{oi}[A_2 - p_{oi}B_2]\underset{\sim}{u}_{oi} + \sum_{\substack{k=1 \\ k\neq\ell,m,m\ell}}^{2m-1} (p_{oi} - p_{ok})(\hat{\theta}_{ik})^2 \qquad ; \; i = \ell,m,m\ell,2m \qquad (2.66)$$

Letting $i\neq j$ in Eq. (2.65) it follows that:

$$\tilde{\theta}_{ij} = \frac{1}{p_{oi} - p_{oj}} \{\underset{\sim}{u}^T_{oj}[A_2 - p_{oi}B_2]\underset{\sim}{u}_{oi} - p_{2i}\hat{\theta}_{ij} \sum_{k=1}^{2m} (p_{ok} - p_{oj})\hat{\theta}_{ik}\hat{\theta}_{kj}\} \tag{2.67}$$

The above expression is valid for $i = \ell, m$; $j = 1,\ldots,2m$; $j \neq \ell, m$ and for $i = m\ell, 2m$; $j = 1,\ldots,2m-1$; $j \neq m\ell$. For the cases $i = \ell$, $j = m$ and $i = m\ell$, $j = 2m$, from Eq. (2.65) we obtain instead the coefficients:

$$\hat{\theta}_{\ell m} = \frac{1}{p_{2\ell}-p_{2m}} \{\underset{\sim}{u}^T_{om}[A_2-p_{o\ell}B_2]\underset{\sim}{u}_{o\ell} + \sum_{\substack{k=1 \\ k\neq \ell,m,m\ell}}^{2m-1} (p_{om}-p_{ok})\hat{\theta}_{\ell k}\hat{\theta}_{mk}\} \tag{2.68}$$

$$\hat{\theta}_{m\ell,2m} = \frac{1}{p_{2m\ell}-p_{2,2m}} \{\underset{\sim}{u}^T_{o2m} [A_2 - p_{om\ell}B_2]\underset{\sim}{u}_{om\ell} + \sum_{\substack{k=1 \\ k\neq \ell,m,m\ell}}^{2m-1} (p_{o2m}-p_{ok})\hat{\theta}_{m\ell,k},\hat{\theta}_{2m,k}\} \tag{2.69}$$

From the orthogonality conditions, Eq. (2.34), after substituting Eqs. (2.43), (2.58) and (2.64), we obtain:

$$\tilde{\theta}_{ij} + \tilde{\theta}_{ji} = - \sum_{k=1}^{2m} \hat{\theta}_{jk}\hat{\theta}_{ik} - \underset{\sim}{u}^T_{oj}B_2\underset{\sim}{u}_{oi} \tag{2.70}$$

and hence:

$$\tilde{\theta}_{ii} = - \frac{1}{2} \{\sum_{k=1}^{2m} (\hat{\theta}_{ik})^2 + \underset{\sim}{u}^T_{oi}B_2\underset{\sim}{u}_{oi}\} \qquad ; \quad i=\ell,m,m\ell,2m \tag{2.71}$$

We conclude here with the analysis of the terms of $O(\varepsilon^2)$. However, the coefficients $\tilde{\theta}_{\ell m}$ and $\tilde{\theta}_{m\ell,2m}$ remain undefined. In order to obtain them we have to include up to terms of $O(\varepsilon^3)$ in the expansions (2.23) and (2.24).

It is interesting to note that although we used the half power expansions in Eqs. (2.23) and (2.24), almost all the terms associated with the half powers have been found to be zero. For example, the correction terms $\underset{\sim}{u}_{1i}$ are identically zero and at least the elements of $\underset{\sim}{u}_{3i}$ which could be obtained from the five-term expansion are all zero. A casual observations, thus, seems to suggest that since the half power terms in Eqs. (2.23) and (2.24) are inconsequential, the expansions assumed in the detuned case is all we need. A more careful review, however, reveals subtle differences. For instance, the expressions obtained for $\hat{\theta}_{\ell m}$ and $\hat{\theta}_{m\ell,2m}$ obtained here, Eqs. (2.68) and (2.69) are quite different from the corresponding expressions obtained for the detuned case, Eq. (2.20).

CHAPTER 3

EIGENPROPERTIES OF A CLASSICALLY DAMPED COMBINED STRUCTURE-EQUIPMENT SYSTEM

3.1 INTRODUCTION

In this chapter, a combined structure and equipment system is analyzed. It is assumed that the supporting primary structure as well as the combined system is classically damped. In some situations a combined system can be nonclassically damped. Such a case can be handled as in Chapter 4, or the effect of the nonclassicality can be indirectly included in the calculation of reponse as described in References 22 and 23.

Since the primary structure and the combined system are assumed to be classically damped, the eigenproperties will be real valued. Thus, the numerical results are obtained for the real eigenproperties of the combined system in terms of real eigenproperties of the primary structure and the equipment parameters. The results are obtained both for light and heavy equipment to test the applicability of the method.

3.2 EIGENVALUE ANALYSIS

The equations of motion for a system composed of a damped single degree of freedom oscillator and a classically damped supporting structure subjected to a base motion $\ddot{X}_g(t)$ are

$$[M]\ddot{\underset{\sim}{x}} + [C]\dot{\underset{\sim}{x}} + [K]\underset{\sim}{x} = -[M]\underset{\sim}{r}\ddot{X}_g(t) \tag{3.1}$$

where $\underset{\sim}{x}$ is the relative (with respect to ground) displacement vector of the combined system and

$$[M] = \begin{bmatrix} M_p & \underset{\sim}{0} \\ \underset{\sim}{0} & m_e \end{bmatrix} \tag{3.2}$$

$$[C] = \begin{bmatrix} C_p & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [C_c] \qquad (3.3)$$

$$[K] = \begin{bmatrix} K_p & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [K_c] \qquad (3.4)$$

$$\underset{\sim}{r}^T = [\underset{\sim}{r}_p^T, r_e] \qquad (3.5)$$

in which $[M_p]$, $[C_p]$ and $[K_p]$ are the mass, damping and stiffness matrices, respectively, of the primary system; $[C_c]$ and $[K_c]$ are the damping and stiffness coupling matrices, respectively, containing the damping coefficient and stiffness of the oscillator in their non-zero elements. The vector $\{r\}$ is the displacement influence vector of the combined system, composed of $\underset{\sim}{r}_p$, vector of influence coefficients of the primary system and r_e, the influence coefficient of the equipment. The displacement influence coefficient r_e is set equal to 1 if the equipment is constrained to move in the direction of the excitation and 0 otherwise.

If the oscillator is assumed to be attached to the K^{th} degree of freedom of the primary system, the coupling matrices $[K_c]$ and $[C_c]$ can be written as follows

$$[K_c] = m_e \omega_e^2 [\underset{\sim}{v}\underset{\sim}{v}^T] \qquad (3.6)$$

$$[C_c] = 2\beta_e \omega_e m_e [\underset{\sim}{v}\underset{\sim}{v}^T] \qquad (3.7)$$

where $\underset{\sim}{v}$ is a (n+1)-dimensional vector with only two non-zero entries at the K^{th} and $(n+1)^{th}$ positions:

$$\underset{\sim}{v}^T = [0,\ldots,1,\ldots,-1] \qquad (3.8)$$

and ω_e, β_e and m_e, respectively, are the natural frequency, damping ratio and mass of the equipment.

Since we are interested in the eigenproperties of the undamped combined system we have to solve the eigenvalue problem associated with the system of Eq. (3.1):

$$[K]\hat{\underset{\sim}{\phi}}_j = \lambda_j [M]\hat{\underset{\sim}{\phi}}_j \quad ; \quad j = 1,\ldots,m \tag{3.9}$$

where we call m = n + 1 the number of dof of the combined system.

We introduce the following transformation in Eq. (4.9):

$$\hat{\underset{\sim}{\phi}}_j = \begin{bmatrix} \Phi_p & \underset{\sim}{0} \\ \underset{\sim}{0} & \psi_e \end{bmatrix} \underset{\sim}{\phi}_j = [T]\underset{\sim}{\phi}_j \tag{3.10}$$

where:

$$\psi_e = \frac{1}{\sqrt{m_e}} \tag{3.11}$$

and $[\Phi_p]$ is the matrix of eigenvectors of the primary system. It is assumed that the primary system eigenvectors are normalized such that:

$$[\Phi_p]^T[M_p][\Phi_p] = [I] \tag{3.12}$$

Introducing Eq. (3.10) in (3.9) and premultiplying by the transpose of [T] we obtain the following transformed eigenvalue problem:

$$\left[\begin{bmatrix} \omega_{p1}^2 & & & \\ & \ddots & & \\ & & \omega_{pn}^2 & \\ & & & 0 \end{bmatrix} + m_e\omega_e^2[T]^T\underset{\sim}{v}\underset{\sim}{v}^T[T]\right]\underset{\sim}{\phi}_j = \lambda_j\underset{\sim}{\phi}_j \quad ; \quad j = 1,\ldots,m \tag{3.13}$$

where ω_{pi}, i = 1,...,n, are the natural frequencies of the primary system. Introducing the vector ν, defined as:

$$\underset{\sim}{\nu}^T = [\phi_{p1}(K),\ldots,\phi_{pi}(K),\ldots,\phi_{pn}(K)] \tag{3.14}$$

in which $\phi_{pi}(K)$ is the K^{th} element of the i^{th} modal vector $\underset{\sim}{\phi}_{pi}$ of the primary system, the transformed eigenvalue problem can be written as follows:

$$\left[[\Lambda] + m_e\omega_e^2 \left[\begin{array}{c|c} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & -\psi_e\underset{\sim}{\nu} \\ \hline -\psi_e\underset{\sim}{\nu}^T & 0 \end{array} \right] \right] \underset{\sim}{\phi}_j = \lambda_j \underset{\sim}{\phi}_j \quad ; \quad j = 1,\ldots,m \qquad (3.15)$$

where:

$$[\Lambda] = \begin{bmatrix} \omega_{p1}^2 & & & 0 \\ & \ddots & & \\ & & \omega_{pn}^2 & \\ 0 & & & \omega_e^2 \end{bmatrix} \qquad (3.16)$$

We are interested in this transformed eigenproblem instead of the original problem because the closed form expressions for the eigenvalues and eigenvectors can be obtained for this case. Futhermore, the resulting expressions will be independent of the analytical model of the primary structure; only the frequencies and mode shapes of the supporting structure will be required for the solution of the combined eigenvalue problem. We observe that due to the different orders of magnitude of the elements in the second matrix in the left hand side of Eq. (3.15) we can write

$$\left[[\Lambda] + m_e\omega_e^2 \left[\begin{array}{c|c} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & -\psi_e\underset{\sim}{\nu} \\ \hline -\psi_e\underset{\sim}{\nu}^T & 0 \end{array} \right] \right] = [A_o] + [A_1] + [A_2] \qquad (3.17)$$

where:

$$[A_o] = [\Lambda] \qquad (3.18)$$

$$[A_1] = -\sqrt{m_e}\ \omega_e^2 \left[\begin{array}{c|c} 0 & \underset{\sim}{\nu} \\ \hline \underset{\sim}{\nu}^T & 0 \end{array} \right] \qquad (3.19)$$

$$[A_2] = m_e\omega_e^2 \left[\begin{array}{c|c} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & 0 \end{array} \right] \qquad (3.20)$$

The elements of matrix $[A_2]$ are proportional to the ratio of the equipment mass to the floor mass, while the elements of $[A_1]$ are proportional to the square root of this ratio. Therefore, for light equipment, if we consider the equipment mass-to-the-floor mass ratios, or equivalently, the elements of $[A_2]$ to be small quantities of order ε^2, the elements of $[A_o]$ and $[A_1]$ will be of order ε^0 and ε respectively. Introducing a small parameter ε to help to trace the order of magnitude of the different quantities involved, the can write the eigenvalue problem of Eq. (3.15) as follows:

$$[[A_o] + \varepsilon[A_1] + \varepsilon^2[A_2]]\underset{\sim}{\phi}_j = \lambda_j \underset{\sim}{\phi}_j \quad ; \quad j = 1,\ldots,m \qquad (3.21)$$

3.2.1 Closed Form Expressions for the Eigenproperties of a Detuned Case.

Equation (3.21) has essentially the same form as Eq. (2.1), except that $[B_o]$ is equal to the identity matrix and $[B_2]$ is equal to zero. Therefore, we can apply the expressions obtained in Chapter 2 to obtain the eigenproperties of the combined system directly. However, because of the special nature of the matrices $[A_o]$, $[A_1]$ and $[A_2]$ and the eigenvectors $\underset{\sim}{u}_{oj}$, we can also obtain the closed form expressions for the desired eigenvalues and eigenvectors.

With $[A_o]$ given by Eq. (3.18), from the zero order eigenvalue problem, Eq. (2.4), we obtain:

$$p_{oj} = \lambda_{pj} \quad ; \quad j = 1,\ldots,n \qquad (3.22)$$

$$p_{om} = \omega_e^2 \qquad (3.23)$$

We also obtain the eigenvectors as:

$$\underset{\sim}{u}_{oj}^T = [0,\ldots,1,\ldots,0] \qquad (3.24)$$

where 1 is at the j^{th} location. By substituting $\underset{\sim}{u}_{oj}$ defined as above and $[A_1]$ from Eq. (3.19) in Eqs. (2.13) and (2.14) we obtain:

$$p_{1j} = 0 \quad ; \quad j = 1,\ldots,n \qquad (3.25)$$

$$\theta_{ji} = 0 \quad ; \quad i,j = 1,\ldots,n \tag{3.26}$$

$$\theta_{mi} = -\theta_{im} = \frac{\sqrt{m_e}\,\omega_e^2 \nu_i}{\lambda_{pi} - \omega_e^2} \quad ; \quad i = 1,\ldots,n \tag{3.27}$$

Substituting Eqs. (3.24) through (3.27) in Eq. (2.19) and considering the definition of matrix $[A_2]$, Eq. (3.20), we obtain the correction terms p_{2j} as:

$$p_{2j} = m_e \omega_e^2 \nu_j^2 \frac{\lambda_{pj}}{\lambda_{pj} - \omega_e^2} \quad ; \quad i = 1,\ldots,n \tag{3.28}$$

$$p_{2m} = -m_e \omega_e^4 \sum_{k=1}^{n} \frac{\nu_k^2}{\lambda_{pk} - \omega_e^2} \tag{3.29}$$

For calculating the coefficient $\hat{\theta}_{ji}$, two different cases need to be considered: (1) for $j = m$ and $i = 1,\ldots,n$ and (2) for $i,j = 1,\ldots,n$ with $i \neq j$. A substitution of $[A_2]$, $\underset{\sim}{u}_{oj}$, p_{1j} and θ_{ji} in Eq. (2.20) gives:

$$\hat{\theta}_{mi} = \hat{\theta}_{im} = 0 \quad ; \quad i = 1,\ldots,n \tag{3.30}$$

$$\hat{\theta}_{ji} = \frac{m_e \omega_e^2 \nu_i \nu_j}{(\lambda_{pi} - \lambda_{pj})(\omega_e^2 - \lambda_{pj})} \quad ; \quad i,j = 1,\ldots,n;\ i \neq j \tag{3.31}$$

And from Eq. (2.22), considering Eqs. (3.26) and (3.27), we obtain:

$$\hat{\theta}_{jj} = -\frac{m_e \omega_e^4 \nu_i^2}{2(\lambda_{pi} - \omega_e^2)^2} \quad ; \quad i = 1,\ldots,n \tag{3.32}$$

For $i = m$, a different expression is obtained:

$$\hat{\theta}_{mm} = -\frac{1}{2} m_e \omega_e^4 \sum_{k=1}^{n} \left(\frac{\nu_k}{\lambda_{pk} - \omega_e^2}\right)^2 \tag{3.33}$$

The modal vectors of the combined system can now be obtained from Eq. (2.3), with the substitution of $\underset{\sim}{u}_{1j}$ and $\underset{\sim}{u}_{2j}$ from Eqs. (2.11) and (2.17):

$$\underset{\sim}{\phi}_j = \underset{\sim}{u}_{oj} + \sum_{k=1}^{m} (\theta_{jk} + \hat{\theta}_{jk})\underset{\sim}{u}_{ok} \qquad (3.34)$$

With $\underset{\sim}{u}_{ok}$ given by Eq. (3.24) and considering Eqs. (3.26) and (3.30) the j^{th} eigenvector $\underset{\sim}{\phi}_j$, with $j \neq m$, can be written as:

$$\underset{\sim}{\phi}_j^T = [\hat{\theta}_{j1}, \hat{\theta}_{j2}, \ldots, 1 + \hat{\theta}_{jj}, \ldots, \theta_{jm}] \quad ; \quad j = 1,\ldots,n \qquad (3.25)$$

and, similarly, the last eigenvector $\underset{\sim}{\phi}_m$ becomes:

$$\underset{\sim}{\phi}_m^T = [\theta_{m1}, \theta_{m2}, \ldots, \theta_{mn}, 1 + \hat{\theta}_{mm}] \qquad (3.36)$$

Knowing the expressions for $\hat{\theta}_{ji}$, etc., we can readily obtain the elements of the transformed system eigenvectors as follows. The first n eigenvectors are defined by:

$$\phi_{i,j} = \frac{m_e \omega_e^2 \lambda_{pj} \nu_i \nu_j}{(\lambda_{pi} - \lambda_{pj})(\omega_e^2 - \lambda_{pj})} \quad ; \quad i = 1,\ldots,n; \; i \neq j \qquad (3.37)$$

$$\phi_{j,j} = 1 - \frac{1}{2} \frac{m_e \omega_e^4 \nu_j^2}{(\lambda_{pj} - \omega_e^2)^2} \qquad (3.38)$$

$$\phi_{m,j} = - \frac{\sqrt{m_e}\, \omega_e^2 \nu_j}{\lambda_{pj} - \omega_e^2} \qquad (3.39)$$

And the elements of the m^{th} eigenvector are:

$$\phi_{i,m} = - \phi_{m,i} \quad ; \quad i = 1,\ldots,n \qquad (3.40)$$

$$\phi_{m,m} = 1 - \frac{m_e \omega_e^4}{2} \sum_{k=1}^{n} \left(\frac{\nu_k}{\lambda_{pk} - \omega_e^2}\right)^2 \qquad (3.41)$$

Similarly, substituting for p_{oj}, p_{1j} and p_{2j} from Eqs. (3.22), (3.23), (3.25), (3.28) and (3.29) into Eq. (2.2) and setting the book-keeping parameter ε equal to 1, the following explicit expressions are obtained for the eigenvalues of the combined system:

$$\lambda_j = \lambda_{pj}\left(1 + \frac{m_e \omega_e^2 \lambda_i^2}{\lambda_{pi} - \omega_e^2}\right) \quad ; \quad j = 1,\ldots,n \qquad (3.42)$$

$$\lambda_m = \omega_e^2(1 - m_e\omega_e^2 \sum_{k=1}^{n} \frac{\nu_k^2}{\lambda_{pk} - \omega_e^2}) \tag{3.43}$$

3.2.2 Closed Form Expressions for the Eigenproperties of a Tuned Case.

The expansions proposed in the previous section break down when the equipment frequency is equal or nearly equal to one of the structural frequencies. For example, consider the expression for the corrected eigenvalue, Eq. (3.42), rewritten here as follows:

$$\lambda_j = \lambda_{pj}\left[1 + \frac{m_e\nu_j^2}{\frac{\lambda_{pj}}{\omega_e^2} - 1}\right] \quad ; \quad j = 1,\ldots,n \tag{3.44}$$

It is seen that for:

$$\left|\frac{\lambda_{pj}}{\omega_e^2} - 1\right| \le m_e\nu_j^2 \tag{3.45}$$

The second term in the parenthesis of Eq. (3.44), which is also the correction term, will be of the same order as the first term, thus invalidating the assumed form of the expansion in Eq. (2.2). Another more stringent condition is obtained if we consider the corrected eigenvector element:

$$\phi_{j,j} = 1 - \frac{1}{2}\,\frac{m_e\nu_j^2}{\left(\frac{\lambda_{pj} - \omega_e^2}{\omega_e^2}\right)^2} \quad ; \quad j = 1,\ldots,n \tag{3.46}$$

we observe that the correction terms will be of the same order as the first term when:

$$\left|\frac{\lambda_{pj}}{\omega_e^2} - 1\right| \le \frac{1}{\sqrt{2}}\sqrt{m_e}\;|\nu_j| \tag{3.47}$$

Therefore, since the term in the right hand side of Eq. (3.47) is one order of magnitude larger than the corresponding term in Eq. (3.45), it governs the validity of the perturbation expansions. It is easy to see that in the case of an oscillator tuned or nearly tuned to one of the frequencies of the primary structure, Eq. (3.47) can be easily satisfied. For such cases, the expansion used in the previous section cannot be used and an alternative expansion is required. For such a case, the expansion proposed in Section 2.4 must be used.

The eigenvalue problem analyzed in Section 2.4 is more general in scope than the case we are considering here. However, considering the zero order matrix $[B_0]$ equal to the identity matrix, the second order perturbation matrix $[B_2]$ equal to zero and letting the subscript i take the values ℓ or m only, we can readily use the expressions obtained for the corrected eigenproperties in Section 2.4.

In order to include in our analysis not only the tuned case, but also the nearly tuned situation, we introduce two "detuning parameters" δ_1 and δ_2 such that if the value $(1 - \lambda_{p\ell}/\omega_e^2)$ is of order ε we write:

$$1 - \frac{\lambda_{p\ell}}{\omega_e^2} = \delta_1 \tag{3.48}$$

and δ_2 equal to zero. On the other hand, if $(1 - \lambda_{p\ell}/\omega_e^2)$ is $O(\varepsilon^2)$ we equate the left hand side of Eq. (3.48) to δ_2 and take δ_1 to be equal to zero. In terms of these parameters, the equipment frequency can be written in terms of the nearly tuned structural frequency as follows:

$$\omega_e^2 = \lambda_{p\ell} + \varepsilon\delta_1\omega_e^2 + \varepsilon^2\delta_2\omega_e^2 \tag{3.49}$$

As per Eq. (3.49), we now slightly change the matrices $[A_1]$ and $[A_2]$ in Eqs. (3.19) and (3.20) such that their $(m,m)^{th}$ elements are not zero but are now equal to $\delta_1\omega_e^2$ and $\delta_2\omega_e^2$ respectively. Correspondingly, the $(m,m)^{th}$ element of $[A_0]$ in Eq. (3.18) is now changed to $\lambda_{p\ell}$.

The zero order eigenvalue problem in Eq. (2.25) for $i = \ell$ and $i = m$ is;

$$\begin{bmatrix} \lambda_{p1} & & & & 0 \\ & \ddots & & & \\ & & \lambda_{p\ell} & & \\ & & & \ddots & \\ 0 & & & & \lambda_{p\ell} \end{bmatrix} \underset{\sim}{u}_{oi} = p_{oi} \underset{\sim}{u}_{oi} \quad ; \quad i = \ell, m \tag{3.50}$$

This directly gives:

$$p_{o\ell} = p_{om} = \lambda_{p\ell} \tag{3.51}$$

To obtain the eigenvectors $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$ we write them as follows:

$$\underset{\sim}{u}_{o\ell}^T = [0,\ldots,\alpha,\ldots,\beta] \tag{3.52}$$

$$\underset{\sim}{u}_{om}^T = [0,\ldots,\gamma,\ldots,\Delta] \tag{3.53}$$

where the undetermined coefficients α, γ and β, Δ are in the ℓ^{th} and m^{th} position, respectively. These coefficients can be related to each other by employing the orthogonality conditions (2.30) as follows:

$$\alpha = \frac{\Delta}{\sqrt{1 + \Delta^2}} \tag{3.54}$$

$$\gamma = -\beta = \frac{1}{\sqrt{1 + \Delta^2}} \tag{3.55}$$

To find Δ we substitute the eigenvectors $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$ from Eqs. (3.52) and (3.53) in Eq. (2.47) and obtain the quadratic equation:

$$\sqrt{m_e}\, \nu_\ell \Delta^2 + \delta_1 \Delta + \sqrt{m_e}\, \nu_\ell = 0 \tag{3.56}$$

which when solved for Δ gives:

$$\Delta = -\frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell} + \sqrt{1 + \left(\frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell}\right)^2} \tag{3.57}$$

The minus sign before the radical term in Eq. (3.57) is also admissible. It can be shown that this other choice of the sign will simply exchange the values of the two tuned eigenvalues and corresponding eigenvectors.

Using Eqs. (3.52) and (3.53) in Eq. (2.45) we obtain:

$$p_{2\ell} = \frac{\omega_e^2}{1 + \Delta^2} (\delta_1 + 2\sqrt{m_e}\ \nu_\ell \Delta) \tag{3.58}$$

$$p_{2m} = \frac{\omega_e^2}{1 + \Delta^2} (\delta_1 \Delta^2 - 2\sqrt{m_e}\ \nu_1 \Delta) \tag{3.59}$$

Furthermore, with substitution of Eqs. (3.24), (3.52) and (3.53) in Eq. (2.46) we obtain:

$$\hat{\theta}_{\ell j} = \frac{\sqrt{m_e}\ \omega_e^2\ \nu_j}{\lambda_{p\ell} - \lambda_{pj}} \frac{1}{\sqrt{1 + \Delta^2}} \tag{3.60}$$

$$; \quad j = 1,\ldots,n,\ j \neq \ell$$

$$\hat{\theta}_{mj} = \frac{\sqrt{m_e}\ \omega_e^2 \nu_j}{\lambda_{p\ell} - \lambda_{pj}} \frac{\Delta}{\sqrt{1 + \Delta^2}} \tag{3.61}$$

To find the correction terms $p_{4\ell}$ and p_{4m} we first need to obtain:

$$\underset{\sim}{u}_{oi}^T A_2 \underset{\sim}{u}_{oi} = \begin{cases} \dfrac{\omega_e^2}{1 + \Delta^2} (\delta_2 + m_e \nu_\ell^2 \Delta^2) \quad ; \quad i = \ell & (3.62) \\[2ex] \dfrac{\omega_e^2}{1 + \Delta^2} (\delta_2 \Delta^2 + m_e \nu_\ell^2) \quad ; \quad i = m & (3.63) \end{cases}$$

Substituting Eqs. (3.60)-(3.63) into Eq.(2.66) it follows that:

$$p_{4\ell} = \frac{\omega_e^2}{1 + \Delta^2} [\delta_2 + m_e \nu_\ell^2 \Delta^2 + m_e \omega_e^2 \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{\nu_k^2}{\lambda_{p\ell} - \lambda_{pk}}] \tag{3.64}$$

$$p_{4m} = \frac{\omega_e^2}{1 + \Delta^2} [\delta_2 \Delta + m_e \nu_\ell^2 + m_e \omega_e^2 \Delta^2 \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{\nu_k^2}{\lambda_{p\ell} - \lambda_{pk}}] \tag{3.65}$$

In order to define the coefficients $\hat{\theta}_{\ell m}$ we use Eqs. (3.20), (3.52), (3.53), (3.60) and (3.61) to obtain the following partial results

$$u_{\sim om}^{T} A_2 u_{\sim o\ell} = \frac{\omega_e^2 \Delta}{1 + \Delta^2} (m_e \nu_\ell^2 - \delta_2) \tag{3.66}$$

$$\sum_{\substack{k=1 \\ k \neq \ell}}^{n} (p_{om} - p_{ok}) \hat{\theta}_{\ell k} \hat{\theta}_{mk} = - \frac{\Delta \omega_e^2 \sigma}{1 + \Delta^2} \tag{3.67}$$

where:

$$\sigma = m_e \omega_e^2 \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{\nu_k^2}{\lambda_{p\ell} - \lambda_{pk}} \tag{3.68}$$

Introducing Eqs. (3.58), (3.59), (3.66) and (3.67) into Eq. (2.68) yields:

$$\hat{\theta}_{\ell m} = \frac{m_e \nu_\ell^2 - \delta_2 - \sigma}{\delta_1 (1 - \Delta^2)/\Delta + 4\sqrt{m_e}\, \nu_\ell} \tag{3.69}$$

With the help of Eqs. (3.24), (3.52), (3.53), (3.58)-(3.61) and (3.69), from Eq. (2.67) we obtain:

$$\tilde{\theta}_{\ell j} = \frac{m_e \omega_e^2 \nu_j}{\lambda_{p\ell} - \lambda_{pj}} \frac{\Delta}{\sqrt{1 + \Delta^2}} [\nu_j - \frac{2\nu_\ell}{1+\Delta^2} \frac{\omega_e^2}{\lambda_{p\ell} - \lambda_{pj}} (1 + \frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell \Delta}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}]$$

$$j = 1, \ldots, n; \; j \neq \ell \tag{3.70}$$

$$\tilde{\theta}_{mj} = \frac{m_e \omega_e^2 \nu_j}{\lambda_{p\ell} - \lambda_{pj}} \frac{1}{\sqrt{1 + \Delta^2}} [\nu_j - \frac{2\nu_\ell \Delta^2}{1 + \Delta^2} \frac{\omega_e^2}{\lambda_{p\ell} - \lambda_{pj}} (1 - \frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}] \tag{3.71}$$

The tuned eigenvectors can now be obtained from the expansion (2.24). Setting the book-keeping parameter ε equal to one and considering Eqs. (2.35), (2.43), (2.53) and (2.64), we obtain:

$$\underset{\sim}{\phi}_i = u_{\sim oi} + \sum_{k=1}^{m} (\theta_{ik} + \hat{\theta}_{ik} + \overset{\star}{\theta}_{ik} + \tilde{\theta}_{ik}) u_{\sim ok} \tag{3.72}$$

Note that the terms $\overset{\star}{\theta}_{\ell m}$ and $\tilde{\theta}_{\ell m}$ could not be obtained with the five-terms expansion considered here. As mentioned earlier, to obtain these, more terms in the perturbation expansion would be required. Expanding Eq. (3.72) and ignoring the coefficients in the summation that are zero or undefined, we obtain the eigenvectors $\underset{\sim}{\phi}_\ell$ and $\underset{\sim}{\phi}_m$ as follows:

$$\underset{\sim}{\phi}_\ell^T = [\hat{\theta}_{\ell 1} + \tilde{\theta}_{\ell 1}, \ldots, (\alpha + \hat{\theta}_{\ell m}\gamma), \ldots, (-\gamma + \hat{\theta}_{\ell m}\alpha)] \qquad (3.73)$$

$$\underset{\sim}{\phi}_m^T = [\hat{\theta}_{m1} + \tilde{\theta}_{m1}, \ldots, (\gamma - \hat{\theta}_{\ell m}\alpha), \ldots, (\alpha + \hat{\theta}_{\ell m}\gamma)] \qquad (3.74)$$

By substituting the coefficients $\hat{\theta}_{\ell i}$, etc., we obtain the following closed form expressions for the elements of these tuned eigenvectors:

$$\phi_{i,\ell} = \frac{\omega_e^2 \nu_i}{\lambda_{pi} - \lambda_{p\ell}} \frac{1}{\sqrt{1 + \Delta^2}} \{\sqrt{m_e} + m_e\Delta[\nu_i + \frac{2\nu_\ell}{1 + \Delta^2} \frac{\omega_e^2}{\lambda_{pi} - \lambda_{p\ell}}$$

$$(1 + \frac{\delta_1}{2\sqrt{m_e}\,\nu_\ell\Delta}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}]\} \quad ; \quad i = 1,\ldots,n;\ i \neq \ell \qquad (3.75)$$

$$\phi_{\ell,\ell} = \frac{1}{\sqrt{1 + \Delta^2}} (\Delta + \hat{\theta}_{\ell m}) \qquad (3.76)$$

$$\phi_{m,\ell} = \frac{1}{\sqrt{1 + \Delta^2}} (-1 + \Delta\hat{\theta}_{\ell m}) \qquad (3.77)$$

$$\phi_{i,m} = \frac{\omega_e^2 \nu_i}{\lambda_{pi} - \lambda_{p\ell}} \frac{\Delta}{\sqrt{1 + \Delta^2}} \{\sqrt{m_e} - m_e[\nu_i + \frac{2\nu_\ell\Delta^2}{1 + \Delta^2} \frac{\omega_e^2}{\lambda_{pi} - \lambda_{p\ell}}$$

$$(1 - \frac{\delta_1}{2\sqrt{m_e}\,\nu_\ell}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}]\} \quad ; \quad i = 1,\ldots,n;\ i \neq \ell \qquad (3.78)$$

$$\phi_{\ell,m} = -\phi_{m,\ell} \qquad (3.79)$$

$$\phi_{m,m} = \phi_{\ell,\ell} \qquad (3.80)$$

Combining the expressions found for the correction terms for the eigenvalues, Eqs. (3.51), (3.58), (3.59), (3.64) and (3.65), the two tuned eigenvalues can be written as follows:

$$\lambda_\ell = \lambda_{p\ell} + \frac{\omega_e^2}{1 + \Delta^2} [\delta_1 + \delta_2 + 2\sqrt{m_e}\ \nu_\ell \Delta + m_e \nu_\ell^2 \Delta^2 + \sigma] \quad (3.81)$$

$$\lambda_m = \lambda_{p\ell} + \frac{\omega_e^2}{1 + \Delta^2} [(\delta_1 + \delta_2)\Delta^2 - 2\sqrt{m_e}\ \nu_\ell \Delta + m_e \nu_\ell^2 + \Delta^2 \sigma] \quad (3.82)$$

with σ defined in Eq. (3.68). The non-tuned eigenvalues and eigenvectors are still obtained from Eqs. (3.42)-(3.43) and (3.37)-(3.41), respectively.

Having obtained the eigenproperties of the transformed system of Eq. (3.15), the eigenvectors of the original system of Eq. (3.9) can now be obtained from:

$$\hat{\underset{\sim}{\phi}}_j = [T]\underset{\sim}{\phi}_j \quad ; \quad j = 1,\ldots,m \quad (3.83)$$

The eigenvectors $\hat{\underset{\sim}{\phi}}_j$ obtained as indicated above are approximately orthonormal with respect to the mass matrix of the combined system:

$$\hat{\underset{\sim}{\phi}}_i^T [M] \hat{\underset{\sim}{\phi}}_j = \delta_{ij} + \text{terms of } O(\varepsilon^3) \quad ; \quad i,j = 1,\ldots,m \quad (3.84)$$

3.3 PARTICIPATION FACTORS OF THE COMBINED SYSTEM

To obtain the system response from Eq. (3.1) for a given ground motion input by modal analysis, we also need the modal participation factors. These participation factors can be obtained in terms of the eigenvectors of the original system or the transformed system. In terms of the eigenvectors of the original system, by definition, the participation factors are:

$$\gamma_j = \hat{\underset{\sim}{\phi}}_j^T [M] \underset{\sim}{r} \quad ; \quad j = 1,\ldots,m \quad (3.85)$$

In terms of the eigenvectors of the transformed system, the participation factors are obtained as follows:

$$\gamma_j = \phi_j^T \begin{Bmatrix} \gamma_p \\ \\ \sqrt{m_e}\, r_e \end{Bmatrix}; \quad j = 1,\ldots,m \tag{3.86}$$

where γ_p is the vector of participation factors of the primary system. Substituting $\phi_{i,j}$ and $\phi_{m,j}$ from Eqs. (3.37)-(3.39), we obtain for the detuned case the following:

$$\gamma_j = \gamma_{pj} - \frac{m_e\omega_e^2\nu_j}{\lambda_{pj} - \omega_e^2} [r_e + \frac{1}{2} \frac{\omega_e^2\nu_j^2\gamma_{pj}}{\lambda_{pj} - \omega_e^2} + \lambda_{pj} \sum_{i \neq j}^{n} \frac{\nu_i\gamma_{pi}}{\lambda_{pi} - \lambda_{pj}}] \; ; \; j = 1,\ldots,n \tag{3.87}$$

$$\gamma_m = \sqrt{m_e}\, r_e + \sqrt{m_e}\, \omega_e^2 \sum_{i=1}^{n} [\frac{\nu_i\gamma_{pi}}{\lambda_{pi} - w_e^2} - \frac{1}{2} \frac{m_e r_e \omega_e^2 \nu_i^2}{(\lambda_{pi} - \omega_e^2)^2}] \tag{3.88}$$

Similarly, substituting for $\phi_{i,\ell}$, $\phi_{\ell,\ell}$ and $\phi_{m,\ell}$ from Eqs. (3.75) to (3.78), and $\phi_{i,m}$, $\phi_{\ell,m}$ and $\phi_{m,m}$ from Eqs. (3.78) to (3.80), we obtain the participation factors corresponding to the two tuned frequencies as follows:

$$\gamma_\ell = \frac{1}{\sqrt{1 + \Delta^2}} \{\Delta\gamma_{p\ell} - \sqrt{m_e}\, r_e + (\gamma_{p\ell} + \Delta\sqrt{m_e}\, r_e)\hat{\theta}_{\ell m} - \omega_e^2 \sum_{\substack{i=1 \\ i \neq \ell}}^{n} \frac{\nu_i\gamma_{pi}}{\lambda_{pi} - \lambda_{p\ell}}$$

$$\{\sqrt{m_e} + m_e\Delta[\nu_i + \frac{2\nu_\ell}{1 + \Delta^2} \frac{\omega_e^2}{\lambda_{pi} - \lambda_{p\ell}} (1 + \frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell\Delta}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}]\}\} \tag{3.89}$$

$$\gamma_m = \frac{1}{\sqrt{1 + \Delta^2}} \{\gamma_{p\ell} + \Delta\sqrt{m_e}\, r_e + (-\Delta\gamma_{p\ell} + \sqrt{m_e}\, r_e)\hat{\theta}_{\ell m} + \omega_e^2 \sum_{\substack{i=1 \\ i \neq \ell}}^{n} \frac{\nu_i\gamma_{pi}}{\lambda_{pi} - \lambda_{p\ell}}$$

$$\{\sqrt{m_e} - m_e[\nu_i + \frac{2\nu_\ell\Delta^2}{1 + \Delta^2} \frac{\omega_e^2}{\lambda_{pi} - \lambda_{p\ell}} (1 - \frac{\delta_1}{2\sqrt{m_e}\, \nu_\ell}) - \frac{\hat{\theta}_{\ell m}}{\sqrt{m_e}}]\}\} \tag{3.90}$$

Knowing the complete eigenproperties of the combined system any response quantity of interest can now be obtained as described in References 22 and 23. In response analysis for tuned cases, it is sometimes important to consider the nonclassical nature of the combined system damping. This can also be included by adopting the approach in References 22 and 23.

3.4 NUMERICAL RESULTS

The numerical results are presented for a four degrees of freedom structure shown in Fig. 3.1. The mass of each floor, m, is = 5×10^5 Kg. The interstory stiffness, k, is = 2×10^9 N/m. The natural frequencies and mode shapes of the primary structure are given in Table 3.1.

The frequencies calculated by the perturbation approach developed here are shown in Tables 3.2, 3.3 and 3.4. The values are shown for three different equipment-to-floor mass ratios of 1/1000, 1/10 and 1/2. These values are compared with the values obtained by an extended precision direct eigenvalue analyses of the complete equipment-structure system; the magnitude of the relative errors between the exact and approximate eigenvalues are also shown in these tables. The results in Table 3.2 are for an equipment not tuned to any structural frequency. The results in Tables 3.3 and 3.4, on the other hand, are for the cases when the equipment is tuned to the lowest and the highest structural frequencies, respectively. The magnitude of the errors in these tables clearly shows that the perturbation approach gives rather very accurate estimate of the frequencies even for the mass ratio as high as 1/2. The error becomes a little higher for the equipment tuned to the highest frequency, although still within the acceptable range. As will be noted later, this is, however, not the case with the error in the eigenvector values.

The eigenvectors calculated by the perturbation approach are shown in Tables 3.5 and 3.6 for the detuned and the tuned cases. In the tuned case, the equipment frequency is very near the first structural frequency. The results in both tables are for the equipment-to-floor mass ratio of 1/10. The error in the calculated values when they are compared with the corresponding exact values are also shown parenthetically in the tables. In particular, the maximum error in the elements of the modal matrix corresponding to the equipment degree of freedom, which is shown in the last rows of Tables 3.5 and 3.6, is only 5.3 percent.

In Table 3.7 are shown the values of the participation factors and the elements of the modal matrix associated the equipment degree of freedom, calculated for four different equipment frequencies. Except for the second case with ω_e = 42 rad/sec., all other values are for the tuned cases. The first set of values are for the equipment tuned to the first frequency, the third set for the equipment tuned to third frequency and the last set is for the equipment tuned to the highest frequency. The error between these values and the exact values calculated by direct eigenvalue analysis of the combined system are also given in parentheses. It is noted that the errors in the higher mode participation factors, and also to some extent in the eigenvectors, are rather large when the equipment is tuned to the higher mode frequencies. It is primarily due to the fact that in this case, because of ω_e being large, the elements of matrix $[A_0]$ are not necessarily much larger than the corresponding elements of matrices $[A_1]$ and $[A_2]$, and thus the conditions for application of the perturbation expansion are weakened. However, the accuracy can be improved somewhat if a seven-term expansion is assumed thus enabling us to obtain the remaining terms, $\tilde{\theta}_{ik}$, in expansion (2.64) which we have been unable to obtain now. However, the tuning with the higher modes is rarely of any importance as the contribution of such modes to the response is usually small. If more accurate values of the modal properties are desired, especially for heavy equipment, the approach developed by the authors in Reference 24 can be used. In that case, the values obtained by the perturbation approach proposed here, then provide excellent initial guesses to be used in the iterative solution of a nonlinear equation developed in Reference 24.

3.5 SUMMARY AND CONCLUSIONS

A systematic matrix perturbation approach is applied for calculating the eigenproperties of a combined equipment-structure system in terms of the eigenproperties of the individual systems. Both the detuned and tuned equipment are considered. The perturbation analysis of the tuned case requires a special expansion scheme to obtain meaningful results. Closed form expressions are provided to calculate the frequencies, mode shapes and participation factors, both for a detuned and a tuned case.

The numerical results are obtained and compared with the results obtained by a direct eigenvalue analysis of the combined system. It is observed that this analysis can provide quite accurate estimates of the frequencies for equipment as heavy as 1/2 the mass of the supporting floor. The accuracy in the calculated eigenvectors is, however, acceptable only for equipment not heavier than 1/10 the floor mass. The error in the higher modal properties becomes rather large when the equipment is tuned to some very high frequency modes relative to the first mode frequency and if the equipment is heavy. These errors can be reduced somewhat by considering a higher order expansion scheme. However, as the contribution of the higher modes to the response is usually much less when compared with the contribution of the lower modes, the errors in the calculated higher modal properties are usually inconsequential.

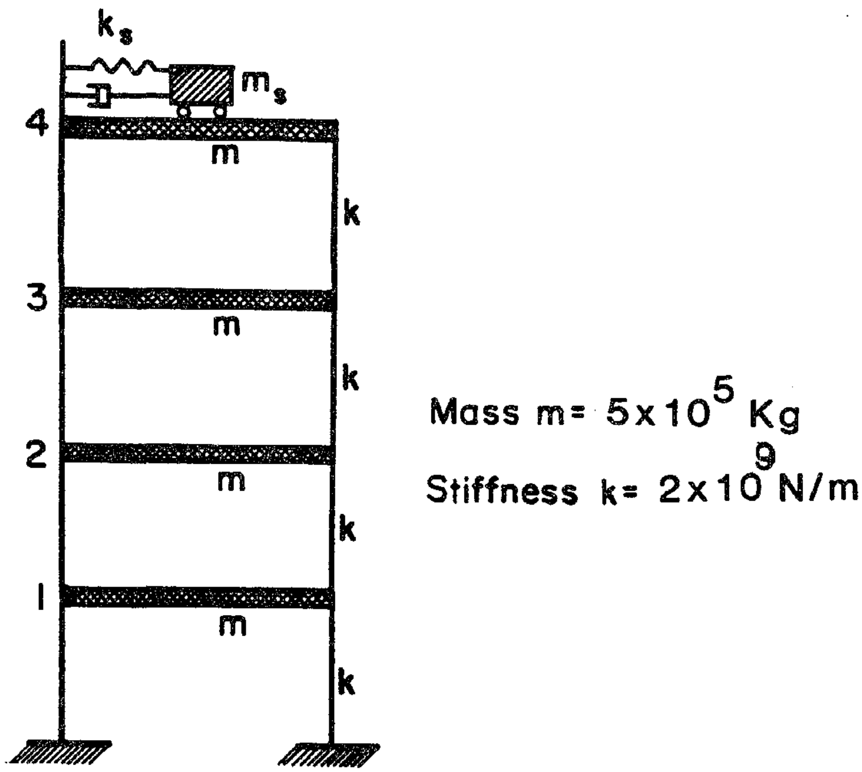

Figure 3.1 A Four Degrees of Freedom Primary Structure and Equipment System

Table 3.1 Eigenproperties of the example primary structure.

	Natural Frequencies (rad/sec)			
	21.9649	63.2455	96.8978	118.8628
Degree of Freedom	Mode Shapes, $x10^{-3}$			
1	.32246	-.81650	.92848	.60602
2	.60602	-.81650	-.32246	-.92848
3	.81650	.00000	-.81650	.81650
4	.92848	.81650	.60602	-.32246

Table 3.2 Natural frequencies of the structure-equipment system for a detuned case

Equipment Frequency = 42 rad/sec.

Freq. No.	Mass Ratio 1/1000 Freq. rad/sec.	Error %	1/10 Freq. rad/sec.	Error %	1/2 Freq. rad/sec.	Error %
1	21.96	0.00	21.30	.21	18.42	5.05
2	42.00	0.00	42.58	.10	41.17	6.59
3	63.25	0.00	64.07	.01	67.03	.41
4	96.90	0.00	97.10	.00	97.92	.11
5	118.86	0.00	118.91	.00	119.08	.02

Table 3.3 Natural frequencies of the structure-equipment system for a tuned case.

Equipment Frequency = 22 rad/sec.

Freq. No.	Mass Ratio 1/1000 Freq. rad/sec.	Error %	1/10 Freq. rad/sec.	Error %	1/2 Freq. rad/sec.	Error %
1	21.75	0.0	19.79	.01	17.43	.34
2	22.21	0.0	24.35	.01	27.44	.08
3	63.25	0.0	63.39	.00	63.97	.01
4	96.90	0.0	96.95	.00	97.14	.01
5	118.86	0.0	118.87	.00	118.92	.00

Table 3.4 Natural frequencies of the structure-equipment system for a tuned case: Equipment tuned to the highest mode.

Equipment Frequency = 118 rad/sec.

Freq. No.	Mass Ratio 1/1000 Freq. rad/sec.	Error %	1/10 Freq. rad/sec.	Error %	1/2 Freq. rad/sec.	Error %
1	21.96	0.0	21.47	.09	19.36	2.37
2	63.23	0.0	61.75	.25	55.36	5.08
3	96.87	0.0	94.13	.89	82.11	11.01
4	117.90	0.0	118.57	1.30	130.26	11.42
5	119.05	0.0	126.76	.73	146.48	3.23

Table 3.5 Eigenvectors of combined system obtained by perturbation approach for a detuned case with mass ratio = 1/10

Equipment Frequency = 42 rad/sec.

Degree of Freedom	Mode No. 1	2	3	4	5
1	.30496 E-3 (0.50)*	.18688 E-3 (2.88)	.80803 E-3 (0.06)	.92481 E-3 (0.01)	.60356 E-3 (0.01)
2	.57544 E-3 (0.45)	.29135 E-3 (3.65)	.78656 E-3 (0.08)	-.33044 E-3 (0.04)	-.92630 E-3 (0.01)
3	.78085 E-3 (0.37)	.26733 E-3 (5.52)	-.42943 E-4 (1.86)	-.80684 E-3 (0.03)	.81804 E-3 (0.00)
4	.89792 E-3 (0.25)	.12542 E-3 (13.03)	-.82950 E-3 (0.16)	.61880 E-3 (0.01)	-.32915 E-3 (0.04)
5	.12780 E-3 (5.29)	-.42407 E-2 (0.36)	.64414 E-3 (3.24)	-.14020 E-3 (1.55)	.46005 E-4 (1.99)

*Error in percent.

Table 3.6 Eigenvectors of combined system obtained by perturbation approach for a tuned case with mass ratio = 1/10: Equipment tuned to the first mode

Equipment Frequency = 22 rad/sec.

Degree of Freedom	Mode No. 1	2	3	4	5
1	.20639 E-3 (0.04)*	.25301 E-3 (0.02)	.81624 E-3 (0.00)	.92771 E-3 (0.00)	.60542 E-3 (0.00)
2	.39306 E-3 (0.17)	.46922 E-3 (0.13)	.81249 E-3 (0.00)	-.32436 E-3 (0.00)	-.92795 E-3 (0.00)
3	.54007 E-3 (0.00)	.61506 E-3 (0.05)	-.74931 E-3 (0.47)	-.81430 E-3 (0.00)	.81689 E-3 (0.00)
4	.63618 E-3 (0.21)	.67193 E-3 (0.32)	-.81999 E-3 (0.00)	.60908 E-3 (0.00)	-.32413 E-3 (0.00)
5	.33363 E-2 (0.20)	-.29882 E-2 (0.17)	.11240 E-3 (0.11)	-.32938 E-4 (0.40)	.11438 E-4 (0.18)

*Error in percent.

Table 3.7 Participation factors and eigenvectors of the combined system for mass ratio = 1/10.

Equipment Frequency	Participation Factor γ_i	$\hat{\phi}_{m,i}$
22 rad/sec	1054.7 (0.13)* 855.2 (0.15) 406.2 (0.00) 197.4 (0.00) 85.7 (0.00)	.33363 E-2 (0.20) -.29882 E-2 (0.17) .11240 E-3 (0.11) -.32938 E-4 (0.40) .11438 E-4 (0.50)
42 rad/sec	1343.5 (0.11) 223.5 (11.33) 393.3 (0.15) 196.2 (0.01) 85.4 (0.01)	.12780 E-2 (5.29) -.42407 E-2 (0.36) .64414 E-3 (3.24) -.14020 E-3 (1.55) .46005 E-4 (1.99)
97 rad/sec	1351.7 (0.0) 411.3 (0.39) 209.1 (9.94) 101.0 (0.3) 79.2 (3.13)	.97867 E-3 (2.04) -.14203 E-2 (0.0) .27121 E-2 (7.60) -.36124 E-2 (11.5) .64289 E-3 (22.85)
118 rad/sec	1352.0 (0.00) 413.9 (0.10) 200.9 (1.19) 156.4 (66.4) 53.6 (235.5)	.96181 E-3 (1.88) -.11456 E-2 (9.79) .18608 E-2 (44.88) -.18040 E-2 (3.73) .45206 E-2 (23.74)

*Error in percent.

Chapter 4

EIGENPROPERTIES OF CLASSICALLY DAMPED PRIMARY STRUCTURE AND EQUIPMENT SYSTEMS WITH NONCLASSICAL DAMPING EFFECTS

4.1. INTRODUCTION

In the preceding chapter the combined system was assumed to be proportionally or classically damped and thus the undamped eigenvalues and eigenvectors of the primary system were used to obtain the real-valued eigenproperties of the composite structure. However, in the studies related to the calculation of equipment response [9,22], it has been reported that in some cases the combined equipment-structure system becomes nonclassically damped even though the supporting structure is classically damped. This is especially the case when the equipment is in resonance with one of the supporting structure's frequencies and the damping ratios of the two systems are significantly different. In such a case it may be necessary to include this nonclassicality in the calculation of accurate equipment response.

Some approaches [10,23] have been proposed to include the effect of this nonclassicality in the calculation of equipment response from the combined undamped eigenproperties. However, a mathematically consistent and more effective approach would be to obtain the combined damped eigenproperties. Therefore in this chapter, a systematic perturbation approach is applied to obtain the damped eigenproperties of a combined equipment- structure system from the undamped eigenproperties of the individual systems. Both the cases of the detuned and tuned equipment are considered. For the detuned case the formulas obtained with the conventional perturbation scheme breaks down and different expressions are needed. This alternative formulation was presented in Chapter 2 in a general form and is used here to obtain the closed form expressions for the eigenproperties of the combined system which are tuned. These combined properties can be used to calculate the response which will incorporate the effect of the dynamic interaction between the equipment and the supporting structure as well as the effect of the nonclassicality of the combined system damping. The numerical results demonstrating the effectiveness of the proposed perturbation approach are presented.

4.2. EIGENVALUE ANALYSIS

The equations of motion for a system composed of a damped single dof oscillator and a classically damped supporting structure excited by a ground motion $\ddot{X}_g(t)$ are

$$[M]\ddot{\underset{\sim}{x}} + [C]\dot{\underset{\sim}{x}} + [K]\underset{\sim}{x} = -[M]\underset{\sim}{r}\ \ddot{X}_g(t) \tag{4.1}$$

The mass, damping and stiffness matrices [M], [C] and [K] of the combined system are defined by Eqs. (3.2)-(3.4) and (3.6)-(3.8) for an oscillator attached to the K^{th} dof of the primary system. The vector of influence coefficients of the combined system $\underset{\sim}{r}$ is given by Eq. (3.5). We will assume here that the properties of the supporting structure and equipment are such that the damping matrix of the joint system cannot be diagonalized or nearly diagonalized by a pre- and postmultiplication by the eigenvector matrix obtained from an eigenvalue analysis of the undamped case. The primary system, however, is regarded as proportionally or classically damped, that is to say, its damping matrix $[C_p]$ is such that

$$\underset{\sim}{\phi}_{pj}^T[C_p]\underset{\sim}{\phi}_{pj} = 2\beta_{pj}\omega_{pj} \quad ; \quad j = 1,\ldots,n \tag{4.2}$$

In order to obtain the system response by modal analysis we have to resort to the state vector approach [13] where the equations of motion are cast in a 2m-dimensional form. We introduce a 2m-dimensional vector, as follows:

$$\underset{\sim}{z} = \begin{Bmatrix} \underset{\sim}{x} \\ \dot{\underset{\sim}{x}} \end{Bmatrix} \tag{4.3}$$

and put Eq. (4.1) in the state vector form to obtain:

$$\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \dot{\underset{\sim}{z}} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \underset{\sim}{z} = -\begin{Bmatrix} 0 \\ M\underset{\sim}{r} \end{Bmatrix} \ddot{X}_g(t) \tag{4.4}$$

This system of equations can be decoupled with the eigenvectors provided by the following eigenvalue problem:

$$\begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} \hat{\underset{\sim}{\psi}}_j = p_j \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \hat{\underset{\sim}{\psi}}_j \quad ; \quad j = 1,\ldots,2m \tag{4.5}$$

Before we attempt to obtain the complex-valued eigenproperties of the above system, we introduce the following transformation in Eq. (4.5)

$$\hat{\underset{\sim}{\psi}}_j = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \underset{\sim}{\psi}_j = [T]^T \underset{\sim}{\psi}_j \quad ; \quad j = 1,\ldots,2m \tag{4.6}$$

where:

$$[U] = \begin{bmatrix} \Phi_p & 0 \\ 0 & 1/\sqrt{m_e} \end{bmatrix} \tag{4.7}$$

in which $[\Phi_p]$ is the real-valued eigenvector matrix of the primary system normalized with respect to the mass matrix. Multiplying on the left

Eq. (4.5) by $[T]^T$ and considering the orthonormality properties of the primary system eigenvectors we can write:

$$\begin{bmatrix} I & 0 \\ 0 & -K_t \end{bmatrix} \underset{\sim}{\psi}_j = p_j \begin{bmatrix} 0 & I \\ I & C_t \end{bmatrix} \underset{\sim}{\psi}_j \qquad ; \quad j = 1,\ldots,2m \qquad (4.8)$$

The matrices $[K_t]$ and $[C_t]$ are defined as:

$$[K_t] = [A_o'] - \omega_e^2 m_e [B_1'] + \omega_e^2 m_e [B_2'] \qquad (4.9)$$

$$[C_t] = [B_o'] - 2\beta_e \omega_e \sqrt{m_e}\, [B_1'] + 2\beta_e \omega_e m_e [B_2'] \qquad (4.10)$$

om which $[A_o']$ and $[B_o']$ are diagonal matrices, defined in terms of the unperturbed frequencies and damping ratios of the two systems as:

$$[A_o'] = [\diagdown \omega_i^2 \diagdown] \qquad (4.11)$$

$$[B_o'] = [\diagdown 2\beta_i \omega_i \diagdown] \qquad (4.12)$$

It is noted that the mth elements of these two matrices pertain to the equipment:

$$\omega_m^2 = \omega_e^2 \qquad (4.13)$$

$$2\beta_m \omega_m = 2\beta_e \omega_e \qquad (4.14)$$

whereas the remaining elements are associated with the modal frequencies and damping ratios of the primary structure:

$$\omega_i^2 = \omega_{pi}^2 \qquad (4.15)$$

$$; \quad i = 1,\ldots,n$$

$$2\beta_i \omega_i = 2\beta_{pi} \omega_{pi} \qquad (4.16)$$

The remaining matrices $[B_1']$ and $[B_2']$ in Eqs. (4.9) and (4.10) are defined as:

$$[B_1'] = \begin{bmatrix} 0 & \underset{\sim}{\nu} \\ \underset{\sim}{\nu}^T & 0 \end{bmatrix} \tag{4.17}$$

$$[B_2'] = \begin{bmatrix} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} \tag{4.18}$$

The vector $\underset{\sim}{\nu}$ is defined in terms of the K^{th} elements of the n eigenvectors of the primary system as in Eq. (3.14).

From Eqs. (4.9) and (4.10) we note that the matrices $[K_t]$ and $[C_t]$ are composed of matrices with elements of different orders of magnitude. For an equipment of small mass, we assume that the ratio of the equipment mass to a floor mass is of order ε^2. The second term on the right hand side of Eq. (4.9) will then be $O(\varepsilon)$ and the third term $O(\varepsilon^2)$. Furthermore, if we assume that the damping ratios of the two systems are of order ε then the second and third terms in Eq. (4.10) will be $O(\varepsilon^2)$ and $O(\varepsilon^3)$, respectively. Discarding the terms of order ε^3 and separating the matrices of different orders in Eq. (4.8), we obtain:

$$[A_o + \varepsilon A_1 + \varepsilon^2 A_2]\underset{\sim}{\psi}_j = p_j[B_o + \varepsilon^2 B_2]\underset{\sim}{\psi}_j \;; \quad j = 1,\ldots,2m \tag{4.19}$$

where a bookkeeping parameter ε has been introduced to keep track of the order of magnitude of different quantities. The matrices $[A_o]$, $[A_1]$, etc., are defined as follows:

$$[A_o] = \begin{bmatrix} I & 0 \\ 0 & -A_o' \end{bmatrix} \quad ; \quad [B_o] = \begin{bmatrix} 0 & I \\ I & B_o' \end{bmatrix} \tag{4.20}$$

$$[A_1] = \omega_e^2\sqrt{m_e} \begin{bmatrix} 0 & 0 \\ 0 & B_1' \end{bmatrix} \tag{4.21}$$

$$[A_2] = -\omega_e^2 m_e \begin{bmatrix} 0 & 0 \\ 0 & B_2' \end{bmatrix} \quad ; \quad [B_2] = -2\beta_e\omega_e\sqrt{m_e} \begin{bmatrix} 0 & 0 \\ 0 & B_1' \end{bmatrix} \tag{4.22}$$

It is noted that the eigenvalue problem of Eq. (4.19) has the same form as Eq. (2.1) in Chapter 2 which was analyzed by the matrix perturbation methods. Therefore, we can readily apply the expressions obtained there to obtain the perturbed eigenvalues and eigenvectors for the present problem. To obtain the eigenproperties of the combined system we must consider the case of a tuned and detuned equipment separately. First we examine the case of a detuned equipment. The case of a tuned oscillator will be presented later.

4.2.1. Closed Form Expressions for the Eigenproperties of a Detuned Case.

Considering the unperturbed eigenvalue problem in Eq. (4.19), we observe that the upper and the lower parts of these eigenvectors are related as:

$$\underset{\sim}{u}_{oj}^{u} = p_{oj}\underset{\sim}{u}_{oj}^{\ell} \quad ; \; j = 1,\ldots,2m \tag{4.23}$$

where the superscripts u and ℓ refer to the upper and lower m elements of $\underset{\sim}{u}_{oj}$. From the last m equations of the eigenproblem of order 0 we obtain:

$$\underset{\sim}{u}_{oj}^{\ell} = [0,\ldots,1,\ldots,0] \tag{4.24}$$

where 1 is at the j^{th} location, and:

$$p_{oj} = - \beta_j\omega_j + i\omega_j\sqrt{1 - \beta_j^2} \quad ; \; j = 1,2,\ldots,m \tag{4.25}$$

where the remaining m eigenvalues are the complex conjugate of these. Here i denotes the complex number. The eigenvectors $\underset{\sim}{u}_{oj}$ when normalized according to Eq. (2.8) become:

$$\underset{\sim}{u}_{oj}^T = \alpha_j[0,\ldots,p_{oj},\ldots,1,\ldots,0] \quad ; \; i,j = 1,2,\ldots,2m \tag{4.26}$$

where:

$$\alpha_j = \frac{(1 - i)}{\sqrt{4\omega_j(1 - \beta_j^2)^{1/2}}} \quad ; \; j = 1,2,\ldots,m \tag{4.27}$$

$$\alpha_{j+m} = \bar{\alpha}_j \tag{4.28}$$

Hereafter, a bar over a quantity will denote its complex conjugate.

Substituting for $[A_1]$ from Eq. (4.21) and $\underset{\sim}{u}_{oj}$ from Eq. (4.26) in Eqs. (2.13) and (2.14) we find:

$$p_{1j} = 0 \quad ; \; j = 1,2,\ldots,m \tag{4.29}$$

$$\theta_{jk} = 0 \quad ; \; j,k = 1,2,\ldots,2m-1, \; j,k \neq m \tag{4.30}$$

$$\theta_{jm} = \frac{\sqrt{m_e}\,\omega_e^2\nu_j\alpha_j\alpha_m}{p_{oj} - p_{om}} = -\theta_{mj} \quad ; \; j = 1,2,\ldots,2m-1, \; j \neq m \tag{4.31}$$

$$\theta_{j,2m} = \bar{\theta}_{j,m} \quad ; \; j = 1,2,\ldots,2m-1, \; j \neq m \tag{4.32}$$

where the following notation is used to define ν_j for $j \geq m$:

$$\nu_{j+m} = \nu_j \quad ; \quad j = 1,2,\ldots,m \tag{4.33}$$

In order to find the second order correction terms we substitute the expressions for $\underset{\sim}{u}_{oj}$, $[A_2]$ and $[B_2]$ in Eq. (2.19) and obtain:

$$p_{2j} = m_e\omega_e^2\nu_j^2\alpha_j^2[\omega_e^2(\frac{\alpha_m^2}{p_{oj} - p_{om}} + \frac{\bar{\alpha}_m^2}{p_{oj} - \bar{p}_{om}}) - 1] \quad ; \; j = 1,\ldots,2m-1, \; j \neq m \tag{4.34}$$

which after some simplification can be written as:

$$p_{2j} = - m_e\omega_e^2\nu_j^2\alpha_j^2 \frac{p_{oj}^2 + 2\beta_e\omega_e p_{oj}}{p_{oj}^2 + 2\beta_e\omega_e p_{oj} + \omega_e^2} = \bar{p}_{2,j+m} \quad ; \quad j = 1,2,..,m-1 \quad (4.35)$$

And for the case j = m the same set of equations leads to:

$$p_{2m} = m_e\omega_e^4\alpha_m^2 \sum_{k=1}^{m-1} \nu_k^2\left(\frac{\alpha_k^2}{p_{om} - p_{ok}} + \frac{\bar{\alpha}_k^2}{p_{om} - \bar{p}_{ok}}\right) = \bar{p}_{2,2m} \quad (4.36)$$

which can also be simplified to:

$$p_{2m} = m_e\omega_e^4\alpha_m^2 \sum_{k=1}^{m-1} \frac{\nu_k^2}{p_{om}^2 + 2\beta_k\omega_k p_{om} + \omega_k^2} = \bar{p}_{2,2m} \quad (4.37)$$

With a similar substitution for $[A_2]$, $[B_2]$ etc. in Eq. (2.20) and after some simplification we obtain:

$$\hat{\theta}_{ji} = \frac{m_e\omega_e^2\nu_j\nu_i\alpha_j\alpha_i}{p_{ok} - p_{oj}} \frac{p_{oj}^2 + 2\beta_e\omega_e p_{oj}}{p_{oj}^2 + 2\beta_e\omega_e p_{oj} + \omega_e^2} \quad ; \quad \begin{array}{l} j,i = 1,...,2m-1 \\ j \neq i \\ j,i \neq m \end{array} \quad (4.38)$$

and if i = m or j = m they become:

$$\hat{\theta}_{jm} = \frac{p_{oj}}{p_{oj} - p_{om}} 2\beta_e\omega_e\sqrt{m_e}\, \nu_j\alpha_j\alpha_m \quad (4.39)$$

$$; \quad \begin{array}{l} j = 1,...,2m-1 \\ j \neq m \end{array}$$

$$\hat{\theta}_{mj} = \frac{p_{oj}}{p_{om} - p_{oj}} 2\beta_e\omega_e\sqrt{m_e}\, \nu_j\alpha_j\alpha_m \quad (4.40)$$

For the case j = m and i = 2m we can similarly show that:

$$\hat{\theta}_{m,2m} = \frac{m_e\omega_e^4|\alpha_m|^2}{p_{om} - \bar{p}_{om}} \sum_{k=1}^{n} \frac{\nu_k^2}{p_{om}^2 + 2\beta_k\omega_k p_{om} + \omega_k^2} = \bar{\hat{\theta}}_{2m,m} \quad (4.41)$$

When both subscripts are equal we obtain:

$$\hat{\theta}_{ii} = - \frac{1}{2} m_e\omega_e^4\nu_i^2\alpha_i^2\left[\left(\frac{\alpha_m}{p_{ok} - p_{om}}\right)^2 + \left(\frac{\bar{\alpha}_m}{p_{ok} - \bar{p}_{om}}\right)^2\right] \quad ; \quad \begin{array}{l} i = 1,...,2m-1 \\ i \neq m \end{array} \quad (4.42)$$

which after some simplification can be written as:

$$\hat{\theta}_{ii} = -m_e\omega_e^4\nu_i^2\alpha_i^2 \frac{p_{oi} + \beta_e\omega_e}{(p_{oi}^2 + 2\beta_e\omega_e p_{oi} + \omega_e^2)^2} \quad ; \quad \begin{array}{l} i = 1,...,2m-1 \\ i \neq m \end{array} \quad (4.43)$$

For the case i = m a different expression is obtained:

$$\hat{\theta}_{mm} = -m_e\omega_e^4\alpha_m^2 \sum_{k=1}^{n} \nu_k^2 \frac{(p_{om} + \beta_k\omega_k)}{(p_{om} + 2\beta_k\omega_k p_{om} + \omega_k^2)^2} = \bar{\hat{\theta}}_{2m,2m} \qquad (4.44)$$

Once all the coefficients θ_{ji} and $\hat{\theta}_{ji}$ of Eqs. (2.11) and (2.17) are found, the perturbed eigenvectors $\underset{\sim}{\psi}_j$ can be obtained from the expansion (2.3). Setting the coefficient ε equal to 1, we obtain:

$$\underset{\sim}{\psi}_j = \underset{\sim}{u}_{oj} + \sum_{k=1}^{2m} (\theta_{jk} + \hat{\theta}_{jk})\underset{\sim}{u}_{ok} \quad ; \; j = 1,\ldots,2m \qquad (4.45)$$

It is noted that we only need to know the lower half of these vectors, as their upper halves are related to the lower halves as follows:

$$\underset{\sim}{\psi}_j^u = p_j\underset{\sim}{\psi}_j^\ell \quad ; \; j = 1,\ldots,2m \qquad (4.46)$$

Substituting for $\underset{\sim}{u}_{oK}$ in Eq. (4.45) and taking advantage of the fact that several elements of these vectors are zero, we obtain the simplified expressions for the elements of the lower half of $\underset{\sim}{\psi}_j$ as follows:

$$\psi_{i+m,j} = \hat{\theta}_{ji}\alpha_i + \hat{\theta}_{j,i+m}\bar{\alpha}_i \quad ; \quad \begin{matrix} i = 1,\ldots,m-1 \\ i \neq j \end{matrix} \qquad (4.47)$$

$$\psi_{j+m,j} = \alpha_j + \hat{\theta}_{jj}\alpha_j + \hat{\theta}_{j,j+m}\bar{\alpha}_j \qquad (4.48)$$

$$\psi_{2m,j} = (\theta_{jm} + \hat{\theta}_{jm})\alpha_m + (\theta_{j,2m} + \hat{\theta}_{j,2m})\bar{\alpha}_m \qquad (4.49)$$

Substituting the coefficients $\hat{\theta}_{ji}$, $\hat{\theta}_{jj}$, etc. already obtained, and after some algebra, we finally arrive at:

$$\psi_{i+m,j} = -m_e\omega_e^2\nu_i\nu_j\alpha_j \frac{p_{oj}^2 + 2\beta_e\omega_e p_{oj}}{p_{oj}^2 + 2\beta_i\omega_i p_{oj} + \omega_i^2} X \quad ; \quad \begin{matrix} i,j = 1,\ldots,n \\ i \neq j \end{matrix} \qquad (4.50)$$

$$\psi_{j+m,j} = \alpha_j + m_e\omega_e^2\nu_j^2\alpha_j^5[p_{oj}^4 + 4\beta_e\omega_e p_{oj}^3 + 4\beta_e^2\omega_e^2 p_{oj}^2 + \beta_e\omega_e^3 p_{oj} +$$

$$+ \beta_e\omega_e^3\bar{p}_{oj} + \omega_e^2\omega_j^2]X^2 \quad ; \quad j = 1,\ldots,n \qquad (4.51)$$

$$\psi_{2m,j} = \sqrt{m_e}\ \nu_j\alpha_j[2\beta_e\omega_e p_{oj} + \omega_e^2]X \quad ; \quad j = 1,\ldots,n \tag{4.52}$$

where:

$$X = \frac{1}{p_{oj}^2 + 2\beta_e\omega_e p_{oj} + \omega_e^2} \tag{4.53}$$

Equations (4.50)-(4.53) only define the first m - 1 vectors. The expressions for the elements of the m^{th} vector, $\underset{\sim}{\psi}_m$, are:

$$\psi_{j+m,m} = (\theta_{jm} + \hat{\theta}_{jm})\alpha_m + (\theta_{j,2m} + \hat{\theta}_{j,2m})\bar{\alpha}_m \quad ; \quad j = 1,\ldots,m-1 \tag{4.54}$$

$$\psi_{2m,m} = \alpha_m + \hat{\theta}_{mm}\alpha_m + \hat{\theta}_{m,2m}\bar{\alpha}_m \tag{4.55}$$

Substituting the values of $\theta_{jm}, \hat{\theta}_{jm}$ etc., and after some simplifications we obtain:

$$\psi_{jm,m} = \sqrt{m_e}\ \nu_j\alpha_m \frac{2\beta_e\omega_e p_{om} + \omega_e^2}{p_{om}^2 + 2\beta_j\omega_j p_{om} + \omega_j^2} \quad ; \quad j = 1,2,\ldots,m-1 \tag{4.56}$$

$$\psi_{2m,m} = \alpha_m - m_e\omega_e^4\alpha_m^3 \sum_{k=1}^{n} \nu_k^2 \frac{\alpha_m^2 p_{om}^2 + (1 + 2\beta_k\omega_k\alpha_m^2)p_{om} + (\beta_k + \omega_k\alpha_m^2)\omega_k}{(p_{om}^2 + 2\beta_k\omega_k p_{om} + \omega_k^2)^2} \tag{4.57}$$

The perturbed eigenvalues are obtained by substitution of Eqs. (4.29), (4.35) and (4.37) into Eq. (2.2) with ε equal to 1, as follows:

$$p_j = p_{oj}[1 - m_e\omega_e^2\nu_j\alpha_j^2 \frac{p_{oj} + 2\beta_e\omega_e}{p_{oj}^2 + 2\beta_e\omega_e p_{oj} + \omega_e^2}] \quad ; \quad j = 1,2,\ldots,n \tag{4.58}$$

$$p_m = p_{om} + m_e\omega_e^4\alpha_m^2 \sum_{k=1}^{n} \frac{\nu_k^2}{p_{om}^2 + 2\beta_k\omega_k p_{om} + \omega_k^2} \tag{4.59}$$

4.2.2. Closed Form Expressions for the Eigenproperties of a Tuned Case.

When the equipment natural frequency and modal damping ratio are equal or nearly equal to one of the modal frequencies and the corresponding damping ratio of the primary system, the expansions obtained in the previous section are no longer valid. From Eq. (4.34) we observe that the expansion (2.2) does not hold whenever:

$$|p_{oi} - p_{om}| \leq m_e \omega_e^2 \nu_i^2 |\alpha_i^2| \tag{4.60}$$

The condition for the validity of the expansion (2.3) is, however, different. According to Eq. (4.42), whenever the following inequality is true

$$|p_{oi} - p_{om}| \leq \frac{1}{\sqrt{2}} \sqrt{m_e} \; |\nu_i||\alpha_i| \; |\alpha_m|\omega_e^2 \tag{4.61}$$

the assumed expansion for the eigenvectors breaks down because the correction terms become of the same order as (or larger than) the zero order terms. For this case, therefore, we need a different set of expansions that will avoid the "non-uniformity" [15] of the foregoing expansions. These alternative expansions were already obtained in Chapter 2. We will assume in the sequel that the values of the ℓ^{th} structural frequency and corresponding modal damping ratio are such that condition (4.61) is satisfied. We will introduce some "detuning parameters" defined as follows:

$$1 - \frac{\omega_\ell^2}{\omega_e^2} = \begin{cases} \delta_1 \\ \text{or} \\ \delta_2 \end{cases} \tag{4.62}$$

If the values of ω_e and ω_ℓ are such that $(1 - \omega_e^2/\omega_\ell^2)$ is of order ε, that is, of the order of the square root of the ratio of the equipment mass to the floor mass, we define δ_1 from Eq. (4.62) and set δ_2 equal to zero. On the other hand, if $(1 - \omega_e^2/\omega_\ell^2)$ is of order ε^2, or of the order of the ratio of the equipment mass to the floor mass, we use Eq. (4.62) to define δ_2 and take δ_1 equal to zero. In terms of these detuning parameters, the equipment eigenvalue can be written as follows:

$$\omega_e^2 = \omega_\ell^2 + \varepsilon\delta_1\omega_e^2 + \varepsilon^2\delta_2\omega_e^2 \tag{4.63}$$

In a similar way we introduce an additional detuning parameter:

$$1 - \frac{\beta_\ell\omega_\ell}{\beta_e\omega_e} = \sigma_1 \tag{4.64}$$

so that the quantity $\beta_e\omega_e$ can be expressed in the form:

$$\beta_e\omega_e = \beta_\ell\omega_\ell + \varepsilon^2\sigma_1\beta_e\omega_e \tag{4.65}$$

We now redefine the matrices $[A_0]$, $[B_0]$, $[A_1]$, $[A_2]$ and $[B_2]$ slightly differently by utilizing Eq. (4.63) and (4.64). The last diagonal elements, that is the $(2m,2m)^{th}$ elements of matrices $[A_0]$ and $[B_0]$ of Eq. (4.20) are set equal to $-\omega_\ell^2$ and $2\beta_\ell\omega_\ell$, respectively, whereas the corresponding elements of $[A_1]$, $[A_2]$ and $[B_2]$ which were zero in Eqs. (4.21) and (4.22) are now changed to $-\delta_1\omega_e^2$, $-\delta_2\omega_e^2$ and $\sigma_1\beta_e\omega_e$, respectively.

Since several elements of matrices $[A_0]$, $[B_0]$, etc. are zero, we can take advantage of this again to obtain the closed form expressions for various correction terms defined before. We start by examining the unperturbed eigenvalue problem (2.25). From the definition of $[A_0]$ it follows that:

$$p_{o\ell} = p_{om} = \overline{p}_{om\ell} = \overline{p}_{o2m} = -\beta_\ell\omega_\ell + i\omega_\ell\sqrt{1 - \beta_\ell^2} \tag{4.66}$$

Considering the orthonormality conditions (2.30), the unperturbed tuned eigenvectors can be written as follows:

$$\underset{\sim}{u}_{o\ell}^T = \overline{\underset{\sim}{u}}_{om\ell}^T = [0,\ldots,p_{o\ell},\ldots,p_{o\ell}\Delta,\ldots,1,\ldots,\Delta]\,\frac{\alpha_\ell}{\sqrt{1 + \Delta^2}} \tag{4.67}$$

$$\underset{\sim}{u}_{om}^{T} = \bar{\underset{\sim}{u}}_{o2m}^{T} = [0,\ldots,p_{o\ell}\Delta,\ldots,-p_{o\ell},\ldots,\Delta,\ldots,-1] \frac{\alpha_\ell}{\sqrt{1+\Delta^2}} \quad (4.68)$$

where the non-zero entries in the above arrays are at the ℓ^{th}, m^{th}, $m\ell^{th}$ and $2m^{th}$ locations respectively and α_ℓ is defined by Eq. (4.27).

The value of the constant Δ can be defined with the condition given by Eq. (2.47). It can be shown that:

$$\Delta = -\frac{\delta_1}{2\sqrt{\bar{m}_e}\,\nu_\ell} + \sqrt{1 + \left(\frac{\delta_1}{2\sqrt{\bar{m}_e}\,\nu_\ell}\right)^2} \quad (4.69)$$

The choice of the sign of the square root is irrelevant. Indeed, it can be shown that if the minus sign is chosen, this will only interchange the respective values of p_ℓ and p_m and corresponding eigenvectors $\underset{\sim}{\psi}_\ell$ and $\underset{\sim}{\psi}_m$.

Utilizing $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$ defined by Eqs. (4.67) and (4.68) in Eq. (2.45) we obtain:

$$p_{2\ell} = \frac{\alpha_\ell^2\omega_e^2\Delta}{1+\Delta^2}\left(-\delta_1\Delta + 2\sqrt{\bar{m}_e}\,\nu_\ell\right) \quad (4.70)$$

$$p_{2m} = -\frac{\alpha_\ell^2\omega_e^2}{1+\Delta^2}\left(\delta_1 + 2\sqrt{\bar{m}_e}\,\nu_\ell\Delta\right) \quad (4.71)$$

With $\underset{\sim}{u}_{oj}$ defined by Eq. (4.26) and with the previous definitions of $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$, the coefficients $\hat{\theta}_{\ell j}$ and $\hat{\theta}_{mj}$ of Eq. (2.46) become:

$$\hat{\theta}_{\ell j} = \frac{\alpha_\ell\Delta}{\sqrt{1+\Delta^2}}\,\frac{\sqrt{\bar{m}_e}\,\omega_e^2\nu_j\alpha_j}{p_{o\ell}-p_{oj}} \quad (4.72)$$

$$; \; j = 1,2,\ldots,2m-1 \qquad j \neq \ell, m, m\ell$$

$$\hat{\theta}_{mj} = -\frac{\alpha_\ell}{\sqrt{1+\Delta^2}}\,\frac{\sqrt{\bar{m}_e}\,\omega_e^2\nu_j\alpha_j}{p_{o\ell}-p_{oj}} \quad (4.73)$$

To obtain the coefficient $\hat{\theta}_{\ell m}$ we need to obtain first the expressions for the following terms in terms of the quantities defined above as follows:

$$p_{2\ell} - p_{2m} = \frac{\alpha_\ell^2 \omega_e^2}{1 + \Delta^2} [\delta_1(1 - \Delta^2) + 4\sqrt{m_e}\ \omega_\ell \Delta] \qquad (4.74)$$

$$\sum_{\substack{k=1 \\ k\neq \ell,m,m\ell}}^{2m-1} (p_{om} - p_{ok})\hat{\theta}_{\ell k}\hat{\theta}_{mk} = - m_e \omega_e^4 \frac{\alpha_\ell^2 \Delta}{1+\Delta^2} \sum_{\substack{k=1 \\ k\neq \ell}}^{n} \frac{\nu_k^2}{p_{o\ell}^2 + 2\beta_k \omega_k p_{o\ell} + \omega_k^2} \qquad (4.75)$$

$$\underset{\sim}{u}_{om}^T [A_2 - p_{o\ell} B_2] \underset{\sim}{u}_{o\ell} = \frac{\alpha_\ell^2 \omega_e^2}{1 + \Delta^2} \{\Delta(\delta_2 - m_e \nu_\ell^2) + \frac{2\beta_e p_{o\ell}}{\omega_e} [\sigma_1 \Delta - \sqrt{m_e}\ \nu_\ell (1 - \Delta^2)]\} \qquad (4.76)$$

Substituting Eqs. (4.74)-(4.76) into Eq. (2.68) we obtain for $\hat{\theta}_{\ell m}$:

$$\hat{\theta}_{\ell m} = - \hat{\theta}_{m\ell} = \frac{1}{\delta_1(1 - \Delta^2) + 4\sqrt{m_e}\ \nu_\ell \Delta} \{\Delta(\delta_2 - m_e \nu_\ell^2) + \frac{2\beta_e p_{o\ell}}{\omega_e} [\sigma_1 \Delta - \sqrt{m_e}\ \nu_\ell (1 - \Delta^2)] - m_e \omega_e^2 \Delta \sum_{\substack{k=1 \\ k\neq \ell}}^{n} \frac{\nu_k^2}{p_{o\ell}^2 + 2\beta_k \omega_k p_{o\ell} + \omega_k^2}\} \qquad (4.77)$$

We need also to obtain the coefficients $\hat{\theta}_{\ell,m\ell}$, $\hat{\theta}_{\ell,2m}$, $\hat{\theta}_{m,m\ell}$ and $\hat{\theta}_{m,2m}$ which cannot be defined with Eqs. (4.72) and (4.73). From Eq. (2.46) and with the help of Eqs. (4.67) and (4.68), it can be shown that these coefficients can be written as follows:

$$\hat{\theta}_{\ell,m\ell} = - \frac{\alpha_\ell^2 |\alpha_\ell|^2 \omega_e^2 \Delta}{1 + \Delta^2} [\delta_1 \Delta - 2\sqrt{m_e}\ \nu_\ell] \qquad (4.78)$$

$$\hat{\theta}_{\ell,2m} = \hat{\theta}_{m,m\ell} = \frac{\alpha_\ell^2 |\alpha_\ell|^2 \omega_e^2}{1 + \Delta^2} [\delta_1 \Delta - \sqrt{m_e}\ \nu_\ell (1 - \Delta^2)] \qquad (4.79)$$

$$\hat{\theta}_{m,2m} = - \frac{\alpha_\ell^2 |\alpha_\ell|^2 \omega_e^2}{1 + \Delta^2} [\delta_1 + 2\sqrt{m_e}\ \nu_\ell \Delta] \qquad (4.80)$$

Introducing Eqs. (4.67) and (4.72) into Eq. (2.66) for $i = \ell$, considering the previous definitions of matrices $[A_2]$ and $[B_2]$ and after some algebraic manipulations we can obtain the correction term $p_{4\ell}$ as follows:

$$p_{4\ell} = \bar{p}_{4m\ell} = - \frac{\alpha_\ell^2 \omega_e^2}{1 + \Delta^2} [\delta_2 \Delta^2 + m_e \nu_\ell^2 + \frac{2\beta_e p_{o\ell} \Delta}{\omega_e} (\sigma_1 \Delta - 2\sqrt{m_e}\, \nu_\ell)$$

$$- m_e \omega_e^2 \Delta^2 \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{\nu_k^2}{p_{o\ell}^2 + 2\beta_k \omega_k p_{o\ell} + \omega_k^2}] \tag{4.81}$$

Proceeding in a similar way but now substituting Eqs. (4.68) and (4.73) into Eq. (2.66) for $i = m$, we obtain for the correction term p_{4m}:

$$p_{4m} = \bar{p}_{4,2m} = - \frac{\alpha_\ell^2 \omega_e^2}{1 + \Delta^2} [\delta_2 + m_e \nu_\ell^2 \Delta^2 + \frac{2\beta_e p_{o\ell}}{\omega_e} (\sigma_1 + 2\sqrt{m_e}\, \nu_\ell \Delta)$$

$$- m_e \omega_e^2 \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \frac{\nu_k^2}{p_{o\ell}^2 + 2\beta_k \omega_k p_{o\ell} + \omega_k^2}] \tag{4.82}$$

The tuned eigenvectors are retrieved from Eq. (2.24) as follows:

$$\underset{\sim}{\psi}_i = \underset{\sim}{u}_{oi} + \sum_{k=1}^{2m} \hat{\theta}_{ik} \underset{\sim}{u}_{ok} \quad ; \quad i = \ell, m \tag{4.83}$$

where Eq. (2.58) was considered and the correction terms $\underset{\sim}{u}_{3i}$ and $\underset{\sim}{u}_{4i}$ were disregarded since they could not be completely defined with the assumed five-term expansion. We will be only concerned with the lower m elements of the vectors $\underset{\sim}{\psi}_\ell$ and $\underset{\sim}{\psi}_m$ since the upper m elements can be obtained with

$$\psi_{k,i} = p_i \psi_{k+m,i} \quad ; \quad \begin{matrix} i = \ell, m \\ k = 1, \ldots, m \end{matrix} \tag{4.84}$$

and also:

$$\underset{\sim}{\psi}_{m\ell} = \bar{\underset{\sim}{\psi}}_{\ell} \tag{4.85}$$

$$\underset{\sim}{\psi}_{2m} = \bar{\underset{\sim}{\psi}}_{m} \tag{4.86}$$

From Eq. (4.83) and with the proper substitution for $\underset{\sim}{u}_{ok}$ from Eqs. (4.26), (4.67) or (4.68), the lower m elements of $\underset{\sim}{\psi}_{\ell}$ become:

$$\psi_{i+m,\ell} = \hat{\theta}_{\ell i}\alpha_i + \hat{\theta}_{\ell,i+m}\bar{\alpha}_i \quad ; \quad i = 1, \ldots, n, \; i \neq \ell \tag{4.87}$$

$$\psi_{m+\ell,\ell} = \frac{1}{\sqrt{1 + \Delta^2}} \left[(1 + \Delta\hat{\theta}_{\ell m})\alpha_\ell + (\hat{\theta}_{\ell,m\ell} + \Delta\hat{\theta}_{\ell,2m})\bar{\alpha}_\ell \right] \tag{4.88}$$

$$\psi_{2m,\ell} = \frac{1}{\sqrt{1 + \Delta^2}} \left[(\Delta - \hat{\theta}_{\ell m})\alpha_\ell + (\Delta\hat{\theta}_{\ell,m\ell} - \hat{\theta}_{\ell,2m})\bar{\alpha}_\ell \right] \tag{4.89}$$

and similarly, the lower m elements of $\underset{\sim}{\psi}_m$ are given by the following expressions:

$$\psi_{i+m,m} = \hat{\theta}_{mi}\alpha_i + \hat{\theta}_{m,i+m}\bar{\alpha}_i \quad ; \quad i = 1, \ldots, n, \; i \neq \ell \tag{4.90}$$

$$\psi_{m+\ell,m} = \frac{1}{\sqrt{1 + \Delta^2}} \left[(\Delta - \hat{\theta}_{\ell m})\alpha_\ell + (\hat{\theta}_{m,m\ell} + \Delta\hat{\theta}_{m,2m})\bar{\alpha}_\ell \right] \tag{4.91}$$

$$\psi_{2m,m} = \frac{1}{\sqrt{1 + \Delta^2}} \left[-(1 + \Delta\hat{\theta}_{\ell m})\alpha_\ell + (\Delta\hat{\theta}_{m,m\ell} - \hat{\theta}_{m,2m})\bar{\alpha}_\ell \right] \tag{4.92}$$

Substituting Eq. (4.72) into (4.87) we obtain:

$$\psi_{i+m,\ell} = \frac{\Delta}{\sqrt{1 + \Delta^2}} \frac{\sqrt{m_e}\,\omega_e^2\alpha_\ell\nu_i}{p_{o\ell}^2 + 2\beta_i\omega_i p_{o\ell} + \omega_i^2} \quad ; \quad \begin{matrix} i = 1, \ldots, n \\ i \neq \ell \end{matrix} \tag{4.93}$$

Introducing Eqs. (4.77), (4.78) and (4.79) into Eq. (4.88), it follows that:

$$\psi_{m+\ell,\ell} = \frac{\alpha_\ell}{\sqrt{1+\Delta^2}} [1 + |\alpha_\ell|^4 \sqrt{m_e}\, \omega_e^2 \nu_\ell \Delta - \Delta Y]. \qquad (4.94)$$

where:

$$Y = \frac{1}{\delta_1(1-\Delta^2)/\Delta + 4\sqrt{m_e}\nu_\ell \Delta} \{\delta_2 - m_e\nu_\ell^2 + \frac{2\beta_e p_{o\ell}}{\omega_e} [\sigma_1 - \sqrt{m_e}\, \nu_\ell \frac{(1-\Delta^2)}{\Delta}] - Z\} \qquad (4.95)$$

$$Z = m_e\omega_e^2 \sum_{\substack{k=1 \\ k\neq\ell}}^{n} \frac{\nu_k^2}{p_{o\ell}^2 + 2\beta_k\omega_k p_{o\ell} + \omega_k^2} \qquad (4.96)$$

Similarly, substitution of Eqs. (4.77)-(4.79) into Eq. (4.89) leads to:

$$\psi_{2m,\ell} = \frac{\alpha_\ell}{\sqrt{1+\Delta^2}} [\Delta - |\alpha_\ell|^4 \omega_e^2(\delta_1\Delta - \sqrt{m_e}\, \nu_\ell) - Y] \qquad (4.97)$$

From Eqs. (4.73) and (4.90) we obtain:

$$\psi_{i+m,m} = -\frac{1}{\sqrt{1+\Delta^2}} \frac{\sqrt{m_e}\, \omega_e^2 \alpha_\ell \nu_i}{p_{o\ell}^2 + 2\beta_i\omega_i p_{o\ell} + \omega_i^2} \quad ; \quad \begin{matrix} i = 1,2,\ldots,n \\ i \neq \ell \end{matrix} \qquad (4.98)$$

When Eqs. (4.77), (4.79) and (4.80) are substituted into Eq. (4.91), it yields:

$$\psi_{m+\ell,m} = \frac{\alpha_\ell}{\sqrt{1+\Delta^2}} [\Delta - |\alpha_\ell|^4 \sqrt{m_e}\, \omega_e^2 \nu_\ell - Y] \qquad (4.99)$$

Finally, introducing Eqs. (4.77), (4.79) and (4.80) into Eq. (4.92), we obtain:

$$\psi_{2m,m} = \frac{\alpha_\ell}{\sqrt{1+\Delta^2}} [-1 + |\alpha_\ell|^4 \omega_e^2(\delta_1 + \sqrt{m_e}\, \nu_\ell\Delta) - \Delta Y] \qquad (4.100)$$

Combining the correction terms for the eigenvalues, Eqs. (4.70), (4.71), (4.81) and (4.82), the tuned complex eigenvalues can now be written as follows:

$$p_\ell = p_{o\ell} + \frac{\alpha_\ell^2\omega_e^2}{1 + \Delta^2} [-(\delta_1 + \delta_2)\Delta^2 + 2\sqrt{m_e}\ \nu_\ell\Delta - m_e\nu_\ell^2$$

$$- \frac{2\beta_e p_{o\ell}\Delta}{\omega_e} (\sigma_1\Delta - 2\sqrt{m_e}\ \nu_\ell) + \Delta^2 Z] \qquad (4.101)$$

$$p_m = p_{o\ell} + \frac{\alpha_\ell^2\omega_e^2}{1 + \Delta^2} [-(\delta_1 + \delta_2) - 2\sqrt{m_e}\ \nu_\ell\Delta - m_e\nu_\ell^2\Delta^2$$

$$- \frac{2\beta_e p_{o\ell}}{\omega_e} (\sigma_1 + 2\sqrt{m_e}\ \nu_\ell\Delta) + Z] \qquad (4.102)$$

Equations (4.84)-(4.86) and (4.93)-(4.102) define completely the tuned modal shapes and eigenvalues of the combined system. The non-tuned eigenvalues and eigenvectors are still given by Eqs. (4.46), (4.50)-(4.52), (4.56)-(4.59) of the previous section. The matrix of eigenvectors of the original system of Eq. (4.4) are obtained with the transformation:

$$\hat{\underset{\sim}{\psi}}_j = \begin{Bmatrix} p_j[\underset{\sim}{U}]\psi_j^\ell \\ [U]\underset{\sim}{\psi}_j^\ell \end{Bmatrix} ; \quad j = 1,\dots,2m \qquad (4.103)$$

The eigenvectors $\hat{\underset{\sim}{\psi}}_j$ obtained from Eq. (4.103) are approximately (up to second order terms) orthonormal in the following sense:

$$\underset{\sim}{\psi}_i^T \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \underset{\sim}{\psi}_j = \delta_{ij} + O(\varepsilon^3) \qquad (4.104)$$

4.3. NUMERICAL RESULTS

The numerical results are obtained for the complex-valued frequencies, eigenvectors and participation factors for the equipment-structure shown in Fig. 3.1 by the perturbation approach proposed here. To ascertain the accuracy of these results, they are then compared with the exact values obtained by a direct analysis of the combined structure-equipment system. The floor mass and interstory stiffness of the example structure are identical to those of the primary structure used in Chapter 3.

The primary structure is assumed to be classically damped. The natural frequencies and undamped mode shapes of the primary structure are given in Table 3.1. The modal shape matrix is normalized according to Eq. (3.12). The damping ratio in each mode is assumed to be 0.03. These values have been utilized in calculating the following results.

Tables 4.1, 4.2 and 4.3 show the eigenvalues of the combined system obtained by the perturbation approach for the equipment-to-floor mass ratios of 1/1000, 1/100 and 1/10. Table 4.1 is for a detuned equipment, whereas Tables 4.2 and 4.3 are for the equipment tuned to the first and the last structural frequencies, respectively. Both the modulus and argument of the complex quantities are shown. These values are compared with the exact values obtained by a direct analysis of the combined system. The error, in percentage, obtained between these two values is shown in parenthesis directly beneath the calculated quantity. It is seen that even for heavy equipment tuned to the highest frequency the largest error between the exact value and the value calculated by the proposed approach is less than 2 percent.

Table 4.4 shows similar results obtained for the eigenvector elements $\hat{\psi}_{2m,j}$ and the modal participation factors. The modal participation factors are defined according to Reference 27 as follows:

$$F_j = \{\psi_j^{\ell}\}^T \left\{ \begin{array}{c} \gamma_p \\ \sqrt{m_e}\; r_e \end{array} \right\} \; ; \quad j = 1,2,\ldots,m \qquad (4.105)$$

where ψ_j^{ℓ} is the vector formed by the m-lower elements of ψ_j and γ_p is the vector of participation factors of the primary system. It is seen that even for the heavy equipment, the error is rather quite small. The error, however, increases when the equipment is tuned to the higher modes. As the higher modes usually do not contribute much to the response, such errors are often inconsequential. The response results substantiating this are given in Table 4.8 and are discussed later.

As the response, in particular the absolute acceleration response [23], is primarily determined by the product of $q_j = p_j^2 F_j \hat{\psi}_{2m,j}$, it is of interest to compare the error obtained in this product. Table 4.5 shows the numerical values of this product, as well as the error when these values are compared with the exact values. Here also the error is not more that 3 percent.

A proper consideration of the nonclassicality of the combined system is essential for accurate calculation of the response, especially when the damping characteristics of the primary system and the equipment are quite different and the equipment is light and tuned to a structural frequency (see References 9 and 22). This can be clearly seen from the results given in Table 4.6. The values in Column 3 of this table are the effective modal damping ratios for the two tuned frequencies shown in Column 2. These are calculated from the complex-valued eigenvalues obtained by the proposed approach. The expressions for calculating these frequencies and modal damping ratios are:

$$\Omega_j = |p_j| \; ; \; \xi_j = -\text{Real}(p_j)/\Omega_j \quad ; \quad j = 1,2,\ldots,m \qquad (4.106)$$

Columns 4 and 5 show the exact frequencies and modal damping ratios calculated with no regard for the nonclassicality of the combined system. The differences in the two quantities due to this disregard of nonclassicality are shown in Columns 6 and 7. It is noted that although the difference in frequency is small, it is quite large in the corresponding damping ratio. This latter difference can cause quite large errors in the calculated response quantities as is shown by the following results.

Table 4.7 shows the absolute acceleration response values calculated for equipment tuned to each of the structural frequencies. All the results in this table were obtained for seismic input defined by a set of ground response spectrum curves. The mass ratio of the equipment is 1/1000. The results in Column 2 are the exact values obtained by a direct analysis of the non-classically damped combined system made with an extended numerical precision algorithm to include the effect of very light equipment. They are exact in the sense that no approximations have been made in calculating the modal properties. The values in Column 3 are obtained with the eigenproperties calculated by the proposed approach. For calculating the response from these eigenproperties, the method developed by Singh [27] was used. The error in these values when compared with the exact values in Column 2 are shown in Column 4. These are quite small for the whole range of frequencies. Column 5 shows the exact response values calculated with no regard for the nonclassicality of damping of the combined system. It is seen that the error between these values and the exact values of Column 1 obtained from the damped case are quite large. Thus in such cases, the use of the approach proposed here is advocated for calculating the combined eigenproperties of equipment-structure systems to incorporate the non-classical damping effects.

4.4. SUMMARY AND CONCLUSIONS

A second order perturbation analysis is developed to obtain the complex-valued eigenvalues and eigenvectors of a combined equipment-structure system. Both cases of detuned and tuned equipment, requiring different perturbation expansion schemes, are considered. Closed form expressions are obtained for the eigenvalues and eigenvectors of the combined system in terms of the undamped eigenproperties of the two systems. Numerical results show that this approach can be effectively used to obtain the eigenproperties for light equipment without much error when compared with the exact values. The result showing the need of such an analysis to incorporate the nonclassical damping effects in the response are also presented.

Table 4.1 Eigenvalues of the combined structure-equipment system for a detuned case.

Equipment frequency = 42.0 rad/sec - Equipment damping = 0.03

Eigenvalue No.	Mass Ratio					
	1/1000		1/100		1/10	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	21.9584 (0.00)	-88.2791 (0.00)	21.8990 (0.01)	-88.2629 (0.03)	21.3145 (0.16)	-88.0963 (0.28)
2	42.0059 (0.00)	-88.2815 (0.00)	42.0587 (0.00)	-88.2879 (0.01)	42.5874 (0.10)	-88.3504 (0.10)
3	63.2539 (0.00)	-88.2812 (0.00)	63.3287 (0.00)	-88.2843 (0.01)	64.0767 (0.00)	-88.3150 (0.11)
4	96.8999 (0.00)	-88.2809 (0.00)	96.9184 (0.00)	-88.2813 (0.00)	97.1036 (0.00)	-88.2855 (0.02)
5	118.8632 (0.00)	-88.2809 (0.00)	118.8672 (0.00)	-88.2809 (0.00)	118.9068 (0.00)	-88.2816 (0.00)

Table 4.2 Eigenvalues of the combined structure-equipment system for a tuned case.

Equipment tuned to the lowest structural frequency - Equipment damping = 0.03

Eigenvalue No.	Mass Ratio					
	1/1000		1/100		1/10	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	21.7390 (0.01)	-88.2990 (0.00)	21.2645 (0.06)	-88.3407 (0.00)	19.8901 (0.57)	-88.4944 (0.04)
2	22.1903 (0.01)	-88.2630 (0.00)	22.6592 (0.16)	-88.2240 (0.00)	23.9770 (1.43)	-88.0970 (0.00)
3	63.2470 (0.00)	-88.2809 (0.00)	63.2600 (0.00)	-88.2813 (0.00)	63.3901 (0.00)	-88.2855 (0.03)
4	96.8983 (0.00)	-88.2809 (0.00)	96.9026 (0.00)	-88.2809 (0.00)	96.9460 (0.00)	-88.2818 (0.01)
5	118.8629 (0.00)	-88.2809 (0.00)	118.8638 (0.00)	-88.2809 (0.00)	118.8737 (0.00	-88.2810 (0.00)

Table 4.3 Eigenvalues of the combined structure-equipment system for a tuned case.

Equipment tuned to the highest structural frequency - Equipment damping = 0.03.

Eigenvalue No.	Mass Ratio					
	1/1000		1/100		1/10	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	21.9601 (0.00)	-88.2770 (0.00)	21.9161 (0.00)	-88.2419 (0.05)	21.4766 (0.05)	-87.8830 (0.50)
2	63.2309 (0.00)	-88.2795 (0.00)	63.0987 (0.00)	-88.2670 (0.02)	71.7776 (0.21)	-88.1395 (0.21)
3	96.8713 (0.00)	-88.2798 (0.00)	96.6329 (0.01)	-88.2698 (0.02)	94.2486 (0.80)	-88.1671 (0.18)
4	118.4761 (0.00)	-88.2885 (0.00)	117.9251 (0.03)	-88.3152 (0.02)	118.7513 (1.35)	-88.4879 (0.22)
5	119.3362 (0.00)	-88.2762 (0.00)	120.6662 (0.04)	-88.2762 (0.03)	127.6301 (0.62)	-88.3622 (0.60)

Table 4.4 Complex participation factors and eigenvector elements $\hat{\psi}_{2m,j}$ for a tuned case.

Equipment tuned to the lowest structural frequency - Equipment damping = 0.03.

Mass Ratio	Frequency No.	Complex Participation Factor		Eigenvector $\hat{\psi}_{2m,j}$	
		Amplitude	Phase	Amplitude	Phase
1/1000	1	140.4685 (0.00)	-45.0013 (0.01)	.47196 E-2 (0.00)	-44.9993 (0.01)
	2	144.8682 (0.00)	-44.9987 (0.01)	.48248 E-2 (0.00)	-45.0006 (0.01)
	3	36.3050 (0.00)	-44.9984 (0.00)	.10059 E-4 (0.00)	-37.7046 (0.00)
	4	14.2010 (0.00)	-45.0012 (0.00)	.24223 E-5 (0.00)	-32.8706 (0.00)
	5	5.5652 (0.00)	-44.9987 (0.00)	.77091 E-6 (0.00)	-29.7564 (0.00)
1/100	1	149.8405 (0.02)	-44.9965 (0.01)	.15619 E-2 (0.03)	-45.9981 (0.02)
	2	135.9720 (0.02)	-45.0041 (0.04)	.14563 E-2 (0.03)	-44.9981 (0.02)
	3	36.2854 (0.00)	-44.9844 (0.05)	.10060 E-4 (0.02)	-37.7046 (0.00)
	4	14.2073 (0.00)	-45.0124 (0.02)	.24223 E-5 (0.04)	-32.8706 (0.02)
	5	5.5646 (0.00)	-44.9872 (0.03)	.77091 E-6 (0.05)	-29.7565 (0.03)
1/10	1	167.2062 (0.13)	-44.9906 (0.07)	.53008 E-3 (0.23)	-45.0052 (0.04)
	2	123.3643 (0.27)	-45.0149 (0.14)	.42435 E-3 (0.39)	-44.9935 (0.07)
	3	36.0810 (0.02)	-44.8431 (0.47)	.10059 E-4 (0.21)	-37.7046 (0.01)
	4	14.1801 (0.00)	-45.1247 (0.23)	.24223 E-5 (0.37)	-32.8706 (0.18)
	5	5.5587 (0.00)	-44.8724 (0.34)	.77090 E-6 (0.49)	-29.7564 (0.29)

Table 4.5 Response quantity q_j for the combined damped system for a tuned case.

Equipment tuned to the highest structural frequency - Equipment damping = 0.03.

Frequency No.	Mass Ratio					
	1/1000		1/100		1/10	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	330.3174 (0.01)	-86.5975 (0.00)	105.8283 (0.10)	-86.6797 (0.00)	35.0647 (1.03)	-86.9847 (0.07)
2	326.4446 (0.03)	-86.5266 (0.00)	101.6664 (0.33)	-86.4502 (0.01)	30.0959 (2.95)	-86.2024 (0.03)
3	1.4609 (0.00)	-79.2649 (0.00)	1.4607 (0.02)	-79.2517 (0.02)	1.4588 (0.21)	-79.1187 (0.20)
4	0.3232 (0.00)	-74.4336 (0.00)	0.3232 (0.03)	-74.4450 (0.02)	0.3228 (0.37)	-74.5590 (0.18)
5	0.0606 (0.00)	-71.3169 (0.00)	0.0606 (0.00)	-71.3055 (0.01)	0.0606 (0.00)	-71.1909 (0.17)

Table 4.6 Comparison between frequencies and modal damping coefficients obtained from the classical and non-classical damping analysis of the combined system.

Mass ratio = 1/1000 - Equipment damping = 0.005 - Structure modal damping = 0.09

Frequency No.	Non-Classical Damping Analysis		Classical Damping Analysis		Error in %	
	Frequency	Modal Damping	Frequency	Modal Damping	Frequency	Modal Damping
Equipment tuned to the 1st frequency						
1	21.9647	0.08873	21.7378	0.04739	1.03	46.59
2	21.9646	0.00627	22.1938	0.04761	1.04	659.33
Equipment tuned to the 2nd frequency						
2	63.2450	0.08902	62.6762	0.04785	0.90	46.25
3	63.2564	0.00601	63.8307	0.04717	0.91	684.86
Equipment tuned to the 3rd frequency						
3	96.8973	0.08946	96.2659	0.04897	0.65	45.26
4	96.9433	0.00561	97.5793	0.04612	0.66	722.10
Equipment tuned to the 4th frequency						
4	118.8626	0.08985	118.4764	0.05183	0.32	42.31
5	118.9446	0.00528	119.3372	0.04333	0.33	720.64

Table 4.7 Comparison between absolute acceleration response values obtained from non-classical and classical damping analysis.

Mass ratio = 1/1000 - Equipment damping = 0.005 - Structure modal damping = 0.09.

	Absolute acceleration in G units.				
	Non-classically damped system			Classically damped system	
Tuned Frequency	Exact Values	Perturbation Approach	Error in %	Exact Values	Error in %
1	0.2620	0.2566	2.06	0.2028	22.60
2	0.0628	0.0619	1.43	0.0449	28.50
3	0.0298	0.0298	0.00	0.0271	9.06
4	0.0246	0.0246	0.00	0.0243	1.22

CHAPTER 5

EIGENPROPERTIES OF NONCLASSICALLY DAMPED PRIMARY STRUCTURE AND EQUIPMENT SYSTEMS

5.1. INTRODUCTION

In the preceding chapter, the combined system was considered to be nonclassically damped, though the primary structure was assumed to be classically damped. Thus, the new modal properties of the composite system were obtained in terms of the undamped eigenproperties of the supporting structure and the equipment. There are cases, however, in which not only the combined system but also the primary structure itself cannot be regarded as proportionally damped. Examples of these are the structures composed of parts with large differences in their energy dissipation rates, like massive structures on soft soil. In some cases it is satisfactory to neglect the off-diagonal terms in the damping matrix corresponding to the normal coordinates [28], but this approximation may lead to unacceptable errors.

The response of such nonclassically damped combined systems can be obtained via modal decomposition by employing the state vector approach [13,18]. This requires a knowledge of the complex-valued eigenvalues and eigenvectors of the composite systems. These, in principle can be obtained by solving an eigenvalue problem of double size with respect to the undamped case. However, as mentioned earlier, due to the gross differences in the mass and stiffness properties of the two subsystems, some numerical problems may appear. And even though these problems may be overcome by using extended precision algebra, when different characteristics or locations of the equipment need to be considered, like in the generation of floor response spectra, the process becomes costly and cumbersome. For the nonclassically damped case these difficulties are even more manifest because of the increased size of each eigenproblem. It is desirable, therefore, to have available the closed form expressions to define the eigenvalues and eigenvectors of the combined system in terms of the equipment characteristics and the modal properties of the primary structure. To that end, in this chapter these eigenproperties are obtained through the general second order perturbation expansions developed in Chapter 2. The case when the equipment damped frequency is well-separated from all of the structure's eigenvalues as well as the case when it is equal or nearly equal to some of them are examined. Numerical results showing the accuracy of the proposed approach both in the complex eigenproperties and floor response spectra are presented. This approach provides reasonably accurate results for equipment as heavy as 1/5 the mass of the supporting floor.

5.2. EIGENVALUE ANALYSIS

Consider a non-classically damped structure with n degrees of freedom. The primary structure properties are described by the stiffness matrix $[K_p]$, damping matrix $[C_p]$ and mass matrix $[M_p]$. If an equipment modelled as a single degree of freedom oscillator of frequency ω_e, damping ratio β_e and mass m_e is attached to the K^{th} dof of the structure and the system is subjected to a base excitation $\ddot{X}_g(t)$, the equations of motion in state vector form [13] for the combined structure are

$$\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{Bmatrix} \ddot{\underset{\sim}{x}} \\ \dot{\underset{\sim}{x}} \end{Bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \dot{\underset{\sim}{x}} \\ \underset{\sim}{x} \end{Bmatrix} = - \begin{Bmatrix} \underset{\sim}{0} \\ [M]\underset{\sim}{r} \end{Bmatrix} \ddot{X}_g(t) \qquad (5.1)$$

where the mass matrix [M], damping matrix [C] and stiffness matrix [K] of the combined system are defined by Eqs. (3.2)-(3.4) and (3.6)-(3.8).

The vector of influence coefficients of the combined system $\tilde{r}$ comprises the vector of influence coefficients of the primary system $\tilde{r}_p$ and the equipment influence coefficient r_e and is given by Eq. (3.5). The combined system posseses $m = n+1$ degress of freedom.

To solve Eq. (5.1), we need to obtain the 2m eigenvalues p_j and eigenvectors $\hat{\tilde{\psi}}_j$ of its associated eigenvalue problem. We would like, however, to obtain the eigenvalues and eigenvectors in terms of the eigenproperties of the primary structure and equipment. Let λ_{pj} and $\tilde{\psi}_{pj}$ be the j^{th} eigenvalue and eigenvector of the damped primary structure, obtained from the solution of the following eigenvalue problem of order 2n

$$\begin{bmatrix} M_p & 0 \\ 0 & -K_p \end{bmatrix} \tilde{\psi}_{pj} = \lambda_{pj} \begin{bmatrix} 0 & M_p \\ M_p & C_p \end{bmatrix} \tilde{\psi}_{pj} \quad ; \; j = 1,\ldots,2n \qquad (5.2)$$

We also assume that these eigenvectors are orthonormalized with respect to the right hand side matrix of Eq. (5.2). Similarly we also define the complex eigenproperties of the oscillator which are obtained as a solution of the following equation

$$\begin{bmatrix} m_e & 0 \\ 0 & -m_e\omega_e^2 \end{bmatrix} \tilde{\psi}_{sj} = \lambda_{sj} \begin{bmatrix} 0 & m_e \\ m_e & 2\beta_e\omega_e m_e \end{bmatrix} \tilde{\psi}_{sj} \quad ; \; j = 1,2 \qquad (5.3)$$

These eigenvalues and eigenvectors are

$$\lambda_{s1} = \bar{\lambda}_{s2} = \lambda_e = -\beta_e\omega_e + i\omega_e\sqrt{1-\beta_e^2} \qquad (5.4)$$

$$\tilde{\psi}_{s1}^T = \bar{\tilde{\psi}}_{s2}^T = [\phi_e\lambda_e; \; \phi_e] \qquad (5.5)$$

$$\phi_e = \frac{1-i}{2\sqrt{m_e\omega_e}(1-\beta_e^2)^{1/2}} \qquad (5.6)$$

where a bar over a complex quantity denote its complex conjugate. These complex eigenvectors are also normalized with respect to the matrix in the right hand side of Eq. (5.3).

Before we solve the eigenvalue problem associated with Eq. (5.1), it will be advantageous to consider a transformed eigenvalue problem, with the transformation defined by:

$$\hat{\psi}_j = \left[\begin{array}{cc|cc} \Psi_p^u & 0 & \bar{\Psi}_p^u & 0 \\ 0 & \phi_e \lambda_e & 0 & \bar{\phi}_e \bar{\lambda}_e \\ \hline \Psi_p^\ell & 0 & \bar{\Psi}_p^\ell & 0 \\ 0 & \phi_e & 0 & \bar{\phi}_e \end{array}\right] \psi_j = [T] \psi_j \tag{5.7}$$

where $[\Psi_p^u]$ and $[\Psi_p^\ell]$ are (nxn) submatrices of $[\Psi_p]$, the eigenvector matrix of the primary system. The submatrix $[\Psi_p^u]$ is composed of the upper n rows and first n eigenvectors, while $[\Psi_p^\ell]$ contains the lower n elements of the first n eigenvectors.

By introducing the transformation of Eq. (5.7) in the eigenvalue problem associated with the system of Eq. (5.1), premultiplying by the transpose of [T] and using the orthonormality properties of the eigenvectors ψ_{pj}, it can be shown [27] that we obtain:

$$\left[\begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix} - m_e \omega_e^2 \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & \bar{D}_{11} \end{bmatrix}\right] \psi_j = p_j \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + 2\beta_e \omega_e m_e \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & \bar{D}_{11} \end{bmatrix}\right] \psi_j$$

$$; \quad j = 1,\ldots,2m \tag{5.8}$$

where:

$$[\Lambda] = \begin{bmatrix} \lambda_{p1} & & & 0 \\ & \ddots & & \\ & & \lambda_{pn} & \\ 0 & & & \lambda_e \end{bmatrix} \tag{5.9}$$

$$[D_{11}] = \begin{bmatrix} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & -\phi_e \underset{\sim}{\nu}^T \\ -\phi_e \underset{\sim}{\nu}^T & 0 \end{bmatrix} \tag{5.10}$$

$$[D_{12}] = \begin{bmatrix} \underset{\sim}{\nu}\bar{\underset{\sim}{\nu}}^T & -\bar{\phi}_e \underset{\sim}{\nu} \\ -\phi_e \bar{\underset{\sim}{\nu}}^T & 0 \end{bmatrix} \tag{5.11}$$

$$\underset{\sim}{\nu}^T = [\psi_1^{(p)}(K+n),\ldots,\psi_i^{(p)}(K+n),\ldots,\psi_n^{(p)}(K+n)] \tag{5.12}$$

and $\psi_i^{(p)}(K+n)$ is the $(K+n)^{th}$ element of the i^{th} eigenvector of the primary system.

An examination of the second matrix in the left hand side of Eq. (5.8) shows that except for the m^{th} and $(2m)^{th}$ rows and columns, all its elements are of the order of the ratio of the equipment mass to primary system mass elements. We will assume here that these ratios are small quantities of order ε^2 and therefore these elements are $O(\varepsilon^2)$, while the remaining elements in the mth and $(2m)^{th}$ rows and columns are $O(\varepsilon)$. Similarly, if we assume that β_e is $O(\varepsilon)$, then the second matrix in the right hand side of Eq. (5.8) is composed of the elements of order ε and ε^2. As we intend to obtain the eigenproperties up to terms $O(\varepsilon^2)$, we will discard the third order elements in the eigenvalue problem (5.8) to obtain the following:

$$[A_o + \varepsilon A_1 + \varepsilon^2 A_2]\underset{\sim}{\psi}_j = p_j[I + \varepsilon^2 B_2]\underset{\sim}{\psi}_j \; ; \; j = 1,\ldots,2m \tag{5.13}$$

where we have discarded the elements of order ε^3 and a bookkeeping parameter ε is introduced to trace the order of the different quantities involved. The matrices $[A_o]$, $[A_1]$ and $[A_2]$ in Eq. (5.13) are

$$[A_o] = \begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix} \tag{5.14}$$

$$[A_1] = m_e \omega_e^2 \phi_e \left[\begin{array}{cc|cc} 0 & \underset{\sim}{\nu} & 0 & \underset{\sim}{\nu} \\ \underset{\sim}{\nu}^T & 0 & \underset{\sim}{\bar{\nu}}^T & 0 \\ \hline 0 & \underset{\sim}{\bar{\nu}} & 0 & \underset{\sim}{\bar{\nu}} \\ \underset{\sim}{\nu}^T & 0 & \underset{\sim}{\bar{\nu}}^T & 0 \end{array}\right] \tag{5.15}$$

$$[A_2] = -m_e \omega_e^2 \left[\begin{array}{cc|cc} \underset{\sim}{\nu}\underset{\sim}{\nu}^T & \underset{\sim}{0} & \underset{\sim}{\nu}\underset{\sim}{\bar{\nu}}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 & \underset{\sim}{0} & 0 \\ \hline \underset{\sim}{\bar{\nu}}\underset{\sim}{\nu}^T & \underset{\sim}{0} & \underset{\sim}{\bar{\nu}}\underset{\sim}{\bar{\nu}}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 & \underset{\sim}{0} & 0 \end{array}\right] \tag{5.16}$$

$$[B_2] = -2\beta_e \omega_e m_e \phi_e \left[\begin{array}{cc|cc} 0 & \underset{\sim}{\nu} & 0 & \underset{\sim}{\nu} \\ \underset{\sim}{\nu}^T & 0 & \underset{\sim}{\bar{\nu}}^T & 0 \\ \hline 0 & \underset{\sim}{\bar{\nu}} & 0 & \underset{\sim}{\bar{\nu}} \\ \underset{\sim}{\nu}^T & 0 & \underset{\sim}{\bar{\nu}}^T & 0 \end{array}\right] \tag{5.17}$$

As it will be evident later, we need to consider two separate cases in the solution of the eigenproblem (5.8). First we assume that all of the primary system eigenvalues do not have numerical values close to the equipment eigenvalue λ_e (detuned case). If any eigenvalue of the structure is equal or nearly equal to λ_e, a different analysis is required for the two closely spaced eigenvalues and eigenvectors (tuned case).

5.2.1 Closed Form Expressions for the Eigenproperties of a Detuned Case.

In Chapter 2 we obtained general expressions for the perturbed eigenvalues and eigenvectors for any eigenvalue problem that can be cast in the form of Eq. (2.1). The eigenproblem that we are considering in this chapter has the same form as Eq. (2.1) if we set $[B_0] = [I]$ and let n be equal to 2m. Because of the simple form of the matrices $[A_1]$, $[A_2]$

and $[B_2]$, we can obtain compact expressions for the eigenvalues and eigenvectors of the transformed system (5.8). We begin examining the zero order eigenvalue problem (2.4). In our case it is:

$$\begin{bmatrix} \Lambda & 0 \\ 0 & \bar{\Lambda} \end{bmatrix} \underset{\sim}{u}_{oj} = p_{oj} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \underset{\sim}{u}_{oj} \qquad ; \; j = 1,\ldots,2m \qquad (5.18)$$

from which we conclude that the unperturbed eigenvalues are:

$$p_{oj} = \lambda_{pj} \quad ; \quad p_{o,j+m} = \bar{\lambda}_{pj} \qquad ; \; j = 1,\ldots,n \qquad (5.19)$$

$$p_{om} = \lambda_e \quad ; \quad p_{o,2m} = \bar{\lambda}_e \qquad (5.20)$$

and the unperturbed eigenvectors are:

$$\underset{\sim}{u}^T_{oj} = [0,\ldots,1,\ldots,0] \quad ; \quad j = 1,\ldots,2m \qquad (5.21)$$

where 1 is at the j^{th} row.

The first order correction terms to the eigenvalues will be obtained from Eq. (2.13). With Eqs. (5.15) and (5.21) it follows that:

$$\underset{\sim}{u}^T_{oi} A_1 \underset{\sim}{u}_{oj} = \begin{cases} 0 & ; \; i,j = 1,\ldots,2m-1 \; ; \; i,j \neq m \qquad (5.22) \\ m_e \omega_e^2 \nu_i \phi_e & ; \; i = 1,\ldots,2m-1 \; ; \; i \neq m \; ; \; j = m \qquad (5.23) \\ m_e \omega_e^2 \nu_i \bar{\phi}_e & ; \; i = 1,\ldots,2m-1 \; ; \; i \neq m \; ; \; j = 2m \qquad (5.24) \end{cases}$$

where the following notation is used to define ν_i for $i \geq m$:

$$\nu_{i+m} = \bar{\nu}_i \qquad ; \; i = 1,\ldots,n \qquad (5.25)$$

Direct substitution of Eqs. (5.19)-(5.24) in Eqs. (2.13)-(2.16) leads to:

$$p_{1j} = 0 \qquad ; \; j = 1,\ldots,2m \qquad (5.26)$$

$$\theta_{ji} = 0 \qquad ; \; i,j = 1,\ldots,2m-1 \; ; \; i,j \neq m \qquad (5.27)$$

$$\theta_{m,2m} = \theta_{2m,m} = \theta_{m,m} = \theta_{2m,2m} = 0$$

$$\theta_{mi} = \frac{m_e \omega_e^2 \phi_e \nu_i}{p_{om}-p_{oi}} \qquad (5.28)$$

$$; \; i = 1,\ldots,2m-1 \; ; \; i \neq m$$

$$\theta_{2m,i} = \frac{m_e \omega_e^2 \bar{\phi}_e \nu_i}{\bar{p}_{om}-p_{oi}} \qquad (5.29)$$

To obtain the second order correction terms we examine first the products $\underset{\sim}{u}_{oi}^T [A_2] \underset{\sim}{u}_{oj}$ and $\underset{\sim}{u}_{oi}^T [B_2] \underset{\sim}{u}_{oj}$. From Eqs. (5.16) and (5.21) we obtain first:

$$\underset{\sim}{u}_{oi}^T A_2 \underset{\sim}{u}_{oj} = \begin{cases} -m_e \omega_e^2 \nu_i \nu_j & ; \; i,j = 1,\ldots,2m-1 \; ; \; i,j \neq m \qquad (5.30) \\ 0 & ; \; i = 1,\ldots,2m \; ; \; j = m,2m \qquad (5.31) \end{cases}$$

From the definitions of $[B_2]$, Eq. (5.17), and $\underset{\sim}{u}_{oj}$, Eq. (5.21), it follows that:

$$\underset{\sim}{u}_{oi}^T B_2 \underset{\sim}{u}_{oj} = \begin{cases} 0 & ; \; i,j = 1,\ldots,2m-1 \; ; \; i,j \neq m \\ & ; \; i = j = m,2m \\ & ; \; i = m \; ; \; j = 2m \qquad (5.32) \\ & ; \; i = 2m \; ; \; j = m \\ -2\beta_e \omega_e m_e \phi_e \nu_i & ; \; i = 1,\ldots,2m-1 \; ; \; i \neq m \; ; \; j = m \qquad (5.33) \\ -2\beta_e \omega_e m_e \bar{\phi}_e \nu_i & ; \; i = 1,\ldots,2m-1 \; ; \; i \neq m \; ; \; j = 2m \qquad (5.34) \end{cases}$$

The correction terms p_{2j} can be obtained from Eqs. (2.19), (5.30) and (5.32), as follows:

$$p_{2j} = \sum_{k=1}^{2m} (\theta_{jk})^2 (p_{oj}-p_{ok}) - m_e \omega_e^2 \nu_j^2 \quad ; \; j = 1,\ldots,2m-1 \; ; \; j \neq m \qquad (5.35)$$

But since θ_{jk} is different from zero only for $k = m,2m$, substituting Eqs. (5.28) and (5.29), we obtain:

$$p_{2j} = m_e^2 \omega_e^4 \nu_j^2 \left(\frac{\phi_e^2}{p_{oj}-p_{om}} + \frac{\bar{\phi}_e^2}{p_{oj}-\bar{p}_{om}}\right) - m_e \omega_e^2 \nu_j^2 \qquad (5.36)$$

From the definitions of p_{oj}, ϕ_e and p_{om} we get:

$$\frac{\phi_e^2}{p_{oj}-p_{om}} + \frac{\bar{\phi}_e^2}{p_{oj}-\bar{p}_{om}} = \frac{1/m_e}{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}+\omega_e^2} \tag{5.37}$$

And substituting the above expression in Eq. (5.36), p_{2j} becomes:

$$p_{2j} = -m_e\omega_e^2\nu_j^2 \frac{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}}{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}+\omega_e^2} \quad ; \quad j = 1,\ldots,2m-1 \quad j \neq m \tag{5.38}$$

The correction term p_{2m} is obtained by replacing Eqs. (5.27), (5.31) and (5.32) in Eq. (2.19) for j = m :

$$p_{2m} = m_e^2\omega_e^4\phi_e^2 \sum_{k=1}^{n} \left(\frac{\nu_k^2}{p_{om}-p_{ok}} + \frac{\bar{\nu}_k^2}{p_{om}-\bar{p}_{ok}}\right) \tag{5.39}$$

Introducing the following constants:

$$\begin{aligned} a_k &= 2m_e\text{Real}(\nu_k) \quad ; \quad b_k = -2m_e\text{Real}(\nu_k^2\bar{\lambda}_{pk}) \\ c_k &= -2\ \text{Real}(\lambda_{pk}) \quad ; \quad d_k = |\lambda_{pk}|^2 \end{aligned} \tag{5.40}$$

we can write for p_{2m}:

$$p_{2m} = m_e\omega_e^4\phi_e^2 \sum_{k=1}^{n} \frac{a_k\lambda_e+b_k}{\lambda_e^2+c_k\lambda_e+d_k} \tag{5.41}$$

In order to examine the coefficients $\hat{\theta}_{ij}$ we need to consider several cases separately. First, for the case $i,j \neq m,2m$ and $i \neq j$, substituting Eqs. (5.26), (5.28)-(5.29) and (5.32) in Eq. (2.20) we obtain:

$$\hat{\theta}_{ji} = \frac{m_e\omega_e^2}{p_{oj}-p_{oi}} \left[m_e\omega_e^2\nu_i\nu_j \left(\frac{\phi_e^2}{p_{oj}-p_{om}} + \frac{\bar{\phi}_e^2}{p_{oj}-\bar{p}_{om}}\right) - \nu_i\nu_j\right] \tag{5.42}$$

and with Eq. (5.37) the coefficients $\hat{\theta}_{ji}$ becomes:

$$\hat{\theta}_{ji} = \frac{m_e\omega_e^2\nu_i\nu_j}{\lambda_{pi}-\lambda_{pj}} \frac{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}}{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}+\omega_e^2} \quad ; \quad \begin{matrix} i,j = 1,\ldots,2m-1 \\ i,j \neq m \quad ; \quad i \neq j \end{matrix} \tag{5.43}$$

For the case i = m, Eq. (2.20) reduces to:

$$\hat{\theta}_{jm} = \frac{\underset{\sim}{u}_{om}^T[A_2-p_{oj}B_2]\underset{\sim}{u}_{oj}}{p_{oj}-p_{om}} \quad ; \quad \begin{matrix} j = 1,\ldots,2m-1 \\ j \neq m \end{matrix} \tag{5.44}$$

and from Eqs. (5.31) and (5.33) we obtain:

$$\hat{\theta}_{jm} = 2\beta_e\omega_e m_e\phi_e\nu_j \frac{\lambda_{pj}}{\lambda_{pj}-\lambda_e} \quad ; \quad \begin{matrix} j = 1,\ldots,2m-1 \\ j \neq m \end{matrix} \tag{5.45}$$

In a similar fashion we can find that:

$$\hat{\theta}_{j,2m} = 2\beta_e\omega_e m_e\bar{\phi}_e\nu_j \frac{\lambda_{pj}}{\lambda_{pj}-\bar{\lambda}_e} \quad ; \quad \begin{matrix} j = 1,\ldots,2m-1 \\ j \neq m \end{matrix} \tag{5.46}$$

The coefficients $\hat{\theta}_{m,2m}$ require a different expression. Starting from Eq. (2.20) for j = m, i = 2m and with the help of Eqs. (5.28)-(5.32) we conclude that:

$$\hat{\theta}_{m,2m} = \frac{m_e^2\omega_e^4|\phi_e|^2}{\lambda_e-\bar{\lambda}_e} \sum_{k=1}^{n} \left(\frac{\nu_k^2}{p_{om}-p_{ok}} + \frac{\bar{\nu}_k^2}{p_{om}-\bar{p}_{ok}}\right) \tag{5.47}$$

and with the definitions of λ_e, ϕ_e and Eq. (5.40), it follows that:

$$\hat{\theta}_{m,2m} = m_e^2\omega_e^4\phi_e^2|\phi_e|^2 \sum_{k=1}^{n} \frac{a_k\lambda_e+b_k}{\lambda_e^2+c_k\lambda_e+d_k} \tag{5.48}$$

We will investigate next the coefficients $\hat{\theta}_{ii}$ given by Eq. (2.22). For $i \neq m,2m$ and using the results already obtained, Eq. (2.22) reduces to:

$$\hat{\theta}_{ii} = -\frac{1}{2} m_e^2 \omega_e^4 \nu_i^2 \left[\frac{\phi_e^2}{(p_{oi}-p_{om})^2} + \frac{\bar{\phi}_e^2}{(p_{oi}-\bar{p}_{om})^2}\right] \quad ; \quad \begin{matrix} i = 1,\ldots,2m-1 \\ i \neq m \end{matrix} \qquad (5.49)$$

From the definitions of p_{oi}, ϕ_e, and p_{om}, it can be shown that the term in square brackets can be expressed as:

$$\frac{\phi_e^2}{(p_{oi}-p_{om})^2} + \frac{\bar{\phi}_e^2}{(p_{oi}-\bar{p}_{om})^2} = \frac{2}{m_e} \frac{\lambda_{pi}+\beta_e\omega_e}{(\lambda_{pi}+2\beta_e\omega_e\lambda_{pi}+\omega_e^2)^2} \qquad (5.50)$$

and therefore:

$$\hat{\theta}_{ii} = -m_e\omega_e^4\nu_i^2 \frac{\lambda_{pi}+\beta_e\omega_e}{(\lambda_{pi}+2\beta_e\omega_e\lambda_{pi}+\omega_e^2)^2} \quad ; \quad \begin{matrix} i = 1,\ldots,2m-1 \\ i \neq m \end{matrix} \qquad (5.51)$$

Finally we study the terms $\hat{\theta}_{mm}$ and $\hat{\theta}_{2m,2m}$. Introducing Eqs. (5.28) and (5.33) into Eq. (2.22) it follows that $\hat{\theta}_{mm}$ is:

$$\hat{\theta}_{mm} = -\frac{1}{2} m_e^2\omega_e^4\phi_e^2 \sum_{k=1}^{n} \left[\frac{\nu_k^2}{(\lambda_e-\lambda_{pk})^2} + \frac{\bar{\nu}_k^2}{(\lambda_e-\bar{\lambda}_{pk})^2}\right] \qquad (5.52)$$

This can also be written as:

$$\hat{\theta}_{mm} = -\frac{1}{2} m_e^2\omega_e^4\phi_e^2 \sum_{k=1}^{n} \left[\frac{a_k\lambda_e^2+2b_k\lambda_e+e_k}{(\lambda_e^2+c_k\lambda_e+d_k)^2}\right] \qquad (5.53)$$

where:

$$e_k = 2m_e \text{ Real } (\nu_k^2\bar{\lambda}_{pk}^2) \qquad (5.54)$$

Proceeding in a similar way we find that:

$$\hat{\theta}_{2m,2m} = \bar{\hat{\theta}}_{mm} \tag{5.55}$$

The expressions for the eigenvectors of the transformed system can be obtained from Eq. (2.3) setting the bookkeeping parameter ε equal to 1:

$$\underset{\sim}{\psi}_j = \underset{\sim}{u}_{oj} + \sum_{k=1}^{2m} (\theta_{jk}+\hat{\theta}_{jk})\underset{\sim}{u}_{ok} \quad ; \quad j = 1,\ldots,2m \tag{5.56}$$

With $\underset{\sim}{u}_{ok}$ given by Eq. (5.21) and discarding the terms θ_{jk} and $\hat{\theta}_{jk}$ that were found to be zero, the elements of $\underset{\sim}{\psi}_j$, $j \neq m,2m$, become:

$$\psi_{i,j} = \hat{\theta}_{ji} \quad ; \quad \begin{matrix} i,j = 1,\ldots,2m-1 \\ i \neq j \; ; \; i,j \neq m,2m \end{matrix} \tag{5.57}$$

$$\psi_{j,j} = 1 + \hat{\theta}_{jj} \tag{5.58}$$

$$\psi_{m,j} = \theta_{jm} + \hat{\theta}_{jm} \quad ; \; j = 1,\ldots,2m-1 \; ; \; j \neq m \tag{5.59}$$

$$\psi_{2m,j} = \theta_{j,2m} + \hat{\theta}_{j,2m} \tag{5.60}$$

The elements of vector $\underset{\sim}{\psi}_m$ are:

$$\psi_{i,m} = \theta_{mi} + \hat{\theta}_{mi} \quad ; \; i = 1,\ldots,2m-1 \; ; \; i \neq m \tag{5.61}$$

$$\psi_{m,m} = 1 + \hat{\theta}_{mm} \tag{5.62}$$

$$\psi_{2m,m} = \hat{\theta}_{m,2m} \tag{5.63}$$

With the expressions for the coefficients $\hat{\theta}_{ji}$, etc., substituted in Eqs. (5.57)-(5.60), we obtain the final expressions for the elements of the eigenvectors $\underset{\sim}{\psi}_j$ for $j = 1,\ldots,n$:

$$\psi_{i,j} = \frac{m_e\omega_e^2\nu_i\nu_j}{\lambda_{pi}-\lambda_{pj}} \frac{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}}{\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}+\omega_e^2} \quad ; \quad \begin{matrix} i = 1,\ldots,2m-1 \\ i \neq m, \neq j \end{matrix} \tag{5.64}$$

$$\psi_{j,j} = 1 - m_e\omega_e^4\nu_j^2 \frac{\lambda_{pj}+\beta_e\omega_e}{(\lambda_{pj}^2+2\beta_e\omega_e\lambda_{pj}+\omega_e^2)^2} \tag{5.65}$$

$$\psi_{m,j} = m_e \phi_e \nu_j \frac{2\beta_e \omega_e \lambda_{pj} + \omega_e^2}{\lambda_{pj} - \lambda_e} \tag{5.66}$$

$$\psi_{2m,j} = m_e \bar{\phi}_e \nu_j \frac{2\beta_e \omega_e \lambda_{pj} + \omega_e^2}{\lambda_{pj} - \bar{\lambda}_e} \tag{5.67}$$

Substituting θ_{mi}, etc., in Eqs. (5.61)-(5.63), the elements of the m^{th} eigenvector are given by the following expressions:

$$\psi_{i,m} = m_e \phi_e \nu_i \frac{2\beta_e \omega_e \lambda_e + \omega_e^2}{\lambda_e - \lambda_{pi}} \quad ; \quad \begin{array}{l} i = 1,\ldots,2m-1 \\ i \neq m \end{array} \tag{5.68}$$

$$\psi_{m,m} = 1 - \frac{1}{2} m_e \omega_e^4 \phi_e^2 \sum_{k=1}^{n} \frac{a_k \lambda_e^2 + 2b_k \lambda_e + e_k}{(\lambda_e^2 + c_k \lambda_e + d_k)^2} \tag{5.69}$$

$$\psi_{2m,m} = m_e^2 \omega_e^4 |\phi_e|^2 \phi_e^2 \sum_{k=1}^{n} \frac{a_k \lambda_e + b_k}{\lambda_e + c_k \lambda_e + d_k} \tag{5.70}$$

where the constants $a_k,\ldots,d_k$ are defined by Eq. (5.40) and e_k is given by Eq. (5.54).

The eigenvalues of the combined system are obtained from Eq. (2.2) by setting ε equal to 1 and considering Eqs. (5.19), (5.20), (5.26), (5.38) and (5.41) as:

$$p_j = \bar{p}_{j+m} = \lambda_{pj}\left(1 - m_e \omega_e^2 \nu_j^2 \frac{\lambda_{pj} + 2\beta_e \omega_e}{p_{oj}^2 + 2\beta_e \omega_e p_{oj} + \omega_e^2}\right) \quad ; \quad j = 1,\ldots,n \tag{5.71}$$

$$p_m = \bar{p}_{2m} = \lambda_e + m_e \omega_e^4 \phi_e^2 \sum_{k=1}^{n} \frac{a_k \lambda_e + b_k}{\lambda_e^2 + c_k \lambda_e + d_k} \tag{5.72}$$

5.2.2 Closed Form Expressions for the Eigenproperties of a Tuned Case.

Examining the expressions found in the previous section for the eigenvectors we can see that the assumed expansion for $\underset{\sim}{\psi}_j$ breaks down whenever

$$\frac{|\lambda_{pi}-\lambda_e|}{\omega_e^2} \le \frac{1}{\sqrt{2}} m_e |\nu_i||\phi_e| \tag{5.73}$$

since for this case the correction terms are equal to, or larger than the unperturbed terms. If the ℓ^{th} eigenvalue $\lambda_{p\ell}$ satisfies the condition (5.73), we need different expansions for the ℓ^{th}, m^{th}, $(m+\ell)^{th}$ and $(2m)^{th}$ eigenvalues and eigenvectors. These expansions are available from Chapter 2 for an eigenvalue problem in the general form of Eq. (2.1).

For the tuned case also, we can use these generalized expressions to obtain the closed form expressions for the tuned eigenvalues and eigenvectors for the specialized form of matrices $[A_o]$, $[A_1]$ and $[A_2]$.

We introduce here two "detuning parameters" defined as follows

$$1 - \frac{\lambda_{p\ell}}{\lambda_e} = \begin{cases} \delta_1 \\ \text{or} \\ \delta_2 \end{cases} \tag{5.74}$$

with the understanding that if $|(1-\lambda_{p\ell}/\lambda_e)|$ is of order ε, we will set δ_2 equal to zero and define δ_1 from the above expression. On the contrary, if $|(1-\lambda_{p\ell}/\lambda_e)|$ is $O(\varepsilon^2)$, then δ_1 is taken equal to zero and Eq. (5.74) defines δ_2. From Eq. (5.74) thus, the equipment eigenvalue can be expressed in the form

$$\lambda_e = \lambda_{p\ell} + \varepsilon\delta_1\lambda_e + \varepsilon^2\delta_2\lambda_e \tag{5.75}$$

This form of Eq. (5.75), in turn, requires some modifications in the matrices $[A_0]$, $[A_1]$ and $[A_2]$ of Eqs. (5.14)-(5.16). The mth and (2m)th diagonal elements of $[A_0]$ now change to $\lambda_{p\ell}$ and $\bar{\lambda}_{p\ell}$, respectively. The $(m,m)^{th}$ elements of $[A_1]$ and $[A_2]$ are now equal to $\delta_1\lambda_e$ and $\delta_2\lambda_e$, respectively. The $(2m,2m)^{th}$ elements are the complex conjugate values of the $(m,m)^{th}$ elements. The remaining elements of these matrices remain unchanged.

From the zero order eigenvalue problem (2.25) we obtain that the tuned eigenvalues and eigenvectors are:

$$p_{o\ell} = p_{om} = \lambda_{p\ell} \tag{5.76}$$

$$\underset{\sim}{u}_{o\ell}^T = [0,\ldots,\Delta,\ldots,-1,\ldots,0]\ 1/\sqrt{1+\Delta^2} \tag{5.77}$$

$$\underset{\sim}{u}_{om}^T = [0,\ldots,1,\ldots,\Delta,\ldots,0]\ 1/\sqrt{1+\Delta^2} \tag{5.78}$$

where the non-zero entries are at the ℓ^{th} and m^{th} locations, and:

$$p_{om\ell} = p_{o2m} = \lambda_{p\ell} \tag{5.79}$$

$$\underset{\sim}{u}_{om\ell}^T = [0,\ldots,\tau,\ldots,-1]\ 1/\sqrt{1+\tau^2} \tag{5.80}$$

$$\underset{\sim}{u}_{o2m}^T = [0,\ldots,1,\ldots,\tau]\ 1/\sqrt{1+\tau^2} \tag{5.81}$$

where the only two nonzero terms are at the $m\ell^{th}$ and $(2m)^{th}$ rows.

The values of the constants Δ and τ can be obtained using the conditions given by Eq. (2.47), with the above eigenvectors and the modified matrix $[A_1]$. It can be shown that they are:

$$\Delta = \bar{\tau} = \frac{\delta_1\lambda_e}{2m_e\omega_e^2\nu_\ell\phi_e} + \sqrt{1 + \left(\frac{\delta_1\lambda_e}{2m_e\omega_e^2\nu_\ell\phi_e}\right)^2} \tag{5.82}$$

The correction terms p_{2i} are obtained by substituting $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$ from Eqs. (5.77) and (5.78) in Eq. (2.45).

$$p_{2\ell} = \frac{\delta_1 \lambda_e - 2m_e \omega_e^2 \phi_e \nu_\ell \Delta}{1 + \Delta^2} \tag{5.83}$$

$$p_{2m} = \frac{\delta_1 \lambda_e \Delta^2 + 2m_e \omega_e^2 \phi_e \nu_\ell \Delta}{1 + \Delta^2} \tag{5.84}$$

By direct evaluation of Eq. (2.46) with Eqs. (5.21) and (5.79)-(5.81) we obtain the coefficients $\hat{\theta}_{ik}$ in the expansion for $\underset{\sim}{u}_{2i}$:

$$\hat{\theta}_{\ell j} = \frac{m_e \omega_e^2 \phi_e \nu_j}{\lambda_{pj} - \lambda_{p\ell}} \frac{1}{\sqrt{1+\Delta^2}} \tag{5.85}$$

$$\hat{\theta}_{mj} = -\Delta \hat{\theta}_{\ell j} \tag{5.86}$$

$$\hat{\theta}_{m\ell,j} = \frac{m_e \omega_e^2 \bar{\phi}_e \nu_j}{\lambda_{p_j} - \bar{\lambda}_{p\ell}} \frac{1}{\sqrt{1+\bar{\Delta}^2}} \quad ; \quad j = 1,\ldots,2m-1, \quad j \neq \ell, m, m\ell \tag{5.87}$$

$$\hat{\theta}_{2m,j} = -\bar{\Delta}\, \hat{\theta}_{m\ell,j} \tag{5.88}$$

Similarly, evaluating Eq. (2.46) for $j = m\ell$ and $j = 2m$ we obtain the remaining coefficients:

$$\hat{\theta}_{\ell,m\ell} = -2\alpha \text{ Real } (\bar{\phi}_e \nu_\ell \Delta) \tag{5.89}$$

$$\hat{\theta}_{\ell,2m} = \alpha(\bar{\phi}_e \nu_\ell |\Delta|^2 - \phi_e \bar{\nu}_\ell) \tag{5.90}$$

$$\hat{\theta}_{m,m\ell} = -\hat{\theta}_{\ell,2m} \tag{5.91}$$

$$\hat{\theta}_{m,2m} = 2\alpha \text{ Real } (\bar{\phi}_e \nu_\ell \bar{\Delta}) \tag{5.92}$$

where:

$$\alpha = \frac{m_e \omega_e^2}{\lambda_{p\ell} - \bar{\lambda}_{p\ell}} \frac{1}{\sqrt{(1+\Delta^2)(1+\bar{\Delta}^2)}} \tag{5.93}$$

To obtain the correction terms p_{4i}, we consider first the expressions for $\underset{\sim}{u}_{o\ell}$ and $\underset{\sim}{u}_{om}$, Eqs. (5.77) and (5.78), and the definitions of $[A_2]$ and $[B_2]$ to obtain:

$$\underset{\sim}{u}_{o\ell}^T A_2 \underset{\sim}{u}_{o\ell} = (\delta_2\lambda_e - m_e\omega_e^2\nu_\ell^2\Delta^2)/(1+\Delta^2) \tag{5.94}$$

$$\underset{\sim}{u}_{om}^T A_2 \underset{\sim}{u}_{om} = (\delta_2\lambda_e\Delta^2 - m_e\omega_e^2\nu_\ell^2)/(1+\Delta^2) \tag{5.95}$$

$$\underset{\sim}{u}_{o\ell}^T B_2 \underset{\sim}{u}_{o\ell} = 2\beta_e\omega_e m_e\phi_e\nu_\ell \frac{\Delta}{1+\Delta^2} \tag{5.96}$$

$$\underset{\sim}{u}_{om}^T B_2 \underset{\sim}{u}_{om} = -2\beta_e\omega_e m_e\phi_e\nu_\ell \frac{\Delta}{1+\Delta^2} \tag{5.97}$$

We also need to obtain the second terms in Eq. (2.66) for i = ℓ and i = m. With Eqs. (5.85) and (5.86) we get:

$$\sum_{\substack{k=1 \\ k\neq\ell,m\ell,m}}^{2m-1} (p_{o\ell}-p_{ok})(\hat{\theta}_{\ell k})^2 = \frac{m_e^2\omega_e^4\phi_e^2}{1+\Delta^2} \sum_{\substack{k=1 \\ k\neq\ell}}^{n} \left(\frac{\nu_k^2}{p_{o\ell}-p_{ok}} + \frac{\bar{\nu}_k^2}{p_{o\ell}-\bar{p}_{ok}}\right) \tag{5.98}$$

$$\sum_{\substack{k=1 \\ k\neq\ell,m\ell,m}}^{2m-1} (p_{om}-p_{ok})(\hat{\theta}_{mk})^2 = \frac{m_e^2\omega_e^4\phi_e^2\Delta^2}{1+\Delta^2} \sum_{\substack{k=1 \\ k\neq\ell}}^{n} \left(\frac{\nu_k^2}{p_{o\ell}-p_{ok}} + \frac{\bar{\nu}_k^2}{p_{o\ell}-\bar{p}_{ok}}\right) \tag{5.99}$$

With Eqs. (5.94)-(5.99) we conclude that the correction terms $p_{4\ell}$ and p_{4m} are:

$$p_{4\ell} = \frac{1}{1+\Delta^2} (\delta_2\lambda_e - m_e\omega_e^2\nu_\ell^2\Delta^2 - 2\beta_e\omega_e m_e\phi_e\nu_\ell\Delta\lambda_{p\ell} + m_e\omega_e^4\phi_e^2\sigma) \tag{5.100}$$

$$p_{4m} = \frac{1}{1+\Delta^2} (\delta_2\lambda_e\Delta^2 - m_e\omega_e^2\nu_\ell^2 + 2\beta_e\omega_e m_e\phi_e\nu_\ell\Delta\lambda_{p\ell} + m_e\omega_e^4\phi_e^2\Delta^2\sigma) \tag{5.101}$$

in which:

$$\sigma = \sum_{\substack{k=1 \\ k\neq\ell}}^{n} \frac{a_k\lambda_{p\ell}+b_k}{\lambda_{p\ell}^2+c_k\lambda_{p\ell}+d_k} \tag{5.102}$$

Next we study the coefficients $\hat{\theta}_{\ell m}$ and $\hat{\theta}_{m\ell,2m}$. According to Eq. (2.68), we need first to obtain the following expressions:

$$\underset{\sim}{u}_{om}^T[A_2-p_{o\ell}B_2]\underset{\sim}{u}_{o\ell} = -\frac{1}{1+\Delta^2}[\delta_2\lambda_e\Delta+m_e\omega_e^2\nu_\ell^2\Delta+2\beta_e\omega_e m_e\phi_e\nu_\ell(1-\Delta^2)\lambda_{p\ell}] \tag{5.103}$$

$$p_{2\ell}-p_{2m} = \frac{1}{1+\Delta^2}[\delta_1\lambda_e(1-\Delta^2)-4m_e\omega_e^2\phi_e\nu_\ell\Delta] \tag{5.104}$$

$$\sum_{\substack{k=1 \\ k\neq\ell,m,m\ell}}^{2m-1} (p_{om}-p_{ok})\hat{\theta}_{\ell k}\hat{\theta}_{mk} = -m_e\omega_e^4\phi_e^2\frac{\Delta}{1+\Delta^2}\sigma \tag{5.105}$$

Substituting Eqs. (5.103)-(5.105) in Eq. (2.68), we obtain:

$$\hat{\theta}_{\ell m} = \frac{1}{\delta_1\lambda_e(\Delta^2-1)+4m_e\omega_e^2\phi_e\nu_\ell\Delta}[\delta_2\lambda_e\Delta+m_e\omega_e^2\nu_\ell^2\Delta+2\beta_e\omega_e m_e\phi_e\nu_\ell(1-\Delta^2)\lambda_{p\ell}+m_e\omega_e^4\phi_e^2\Delta\sigma] \tag{5.106}$$

Similar manipulations with Eq. (2.69), show that:

$$\hat{\theta}_{m\ell,2m} = \bar{\hat{\theta}}_{\ell,m} \tag{5.107}$$

Now we can obtain the closed form expressions for the tuned eigenvectors by setting the bookkeeping parameter equal to one in Eq. (2.24) and ignoring the correction terms which are zero and those not completely defined:

$$\underset{\sim}{\psi}_i = \underset{\sim}{u}_{oi} + \sum_{k=1}^{2m}\hat{\theta}_{ik}\underset{\sim}{u}_{ok} \quad ; \quad i = \ell,m,m\ell,2m \tag{5.108}$$

According to Eqs. (5.21), (5.77) (5.78), (5.80) and (5.81), it follows that the elements are defined by five different expressions:

$$\psi_{i,\ell} = \frac{m_e \omega_e^2 \phi_e \nu_i}{\lambda_{pi} - \lambda_{p\ell}} \frac{1}{\sqrt{1+\Delta^2}} \quad ; \quad \begin{matrix} i = 1,\ldots,2m-1 \\ i \neq \ell, m, m\ell \end{matrix} \tag{5.109}$$

$$\psi_{\ell,\ell} = \frac{\Delta + \hat{\theta}_{\ell m}}{\sqrt{1+\Delta^2}} \tag{5.110}$$

$$\psi_{m,\ell} = \frac{-1 + \hat{\theta}_{\ell m} \Delta}{\sqrt{1+\Delta^2}} \tag{5.11}$$

$$\psi_{m\ell,\ell} = \frac{\hat{\theta}_{\ell,m\ell} \bar{\Delta} + \hat{\theta}_{\ell,2m}}{\sqrt{1+\bar{\Delta}^2}} \tag{5.112}$$

$$\psi_{2m,\ell} = \frac{-\hat{\theta}_{\ell,m\ell} + \hat{\theta}_{\ell,2m} \Delta}{\sqrt{1+\bar{\Delta}^2}} \tag{5.113}$$

and substituting $\hat{\theta}_{\ell,m\ell}$ and $\hat{\theta}_{\ell,2m}$ from Eqs. (5.89) and (5.90), it follows that the last two terms can be expressed as:

$$\psi_{m\ell,\ell} = -m_e^2 \omega_e^2 \phi_e^3 \bar{\nu}_\ell \frac{1}{\sqrt{1+\Delta^2}} \tag{5.114}$$

$$\psi_{2m,\ell} = m_e^2 \omega_e^2 \bar{\phi}_e^2 \phi_e \nu_\ell \frac{\Delta}{\sqrt{1+\Delta^2}} \tag{5.115}$$

Similarly, expanding Eq. (5.108) and considering Eqs. (5.21), (5.77), (5.78), (5.80) and (5.81), the elements of the other tuned eigenvector $\underset{\sim}{\psi}_m$ are described by the following set of expressions:

$$\psi_{i,m} = -\Delta \psi_{i,\ell} \quad ; \quad \begin{matrix} i = 1,\ldots,2m-1 \\ i \neq \ell, m, m\ell \end{matrix} \tag{5.116}$$

$$\psi_{\ell,m} = -\psi_{m,\ell} \quad (5.117)$$

$$\psi_{m,m} = \psi_{\ell\ell} \quad (5.118)$$

$$\psi_{m\ell,m} = \frac{\hat{\theta}_{m,m\ell}\bar{\Delta}+\hat{\theta}_{m,2m}}{\sqrt{1+\bar{\Delta}^2}} \quad (5.119)$$

$$\psi_{2m,m} = \frac{\hat{\theta}_{m,2m}\bar{\Delta}-\hat{\theta}_{m,m\ell}}{\sqrt{1+\bar{\Delta}^2}} \quad (5.120)$$

Substituting $\hat{\theta}_{m,m\ell}$ and $\hat{\theta}_{m,2m}$ from Eqs. (5.91) and (5.92) in Eqs. (5.119) and (5.120) and comparing the expressions obtained with Eqs. (5.114) and (5.115), we conclude that the last two elements can also be written as follows:

$$\psi_{m\ell,m} = -\Delta\psi_{m\ell,\ell} \quad (5.121)$$

$$\psi_{2m,m} = \Delta\psi_{2m,\ell} \quad (5.122)$$

Finally, combining Eq. (5.83) with (5.100) and Eq. (5.84) with (5.101), the tuned eigenvalues are defined by the following equations:

$$p_\ell = \bar{p}_{m\ell} = p_{o\ell} + \frac{1}{1+\Delta^2}\,[(\delta_1+\delta_2)\lambda_e - m_e\omega_e^2\nu_\ell\Delta(2\phi_e+\nu_\ell\Delta) - 2\beta_e\omega_e m_e\phi_e\Delta\nu_\ell\lambda_{p\ell} + m_e\omega_e^4\phi_e^2\sigma] \quad (5.123)$$

$$p_m = \bar{p}_{2m} = p_{o\ell} + \frac{1}{1+\Delta^2}\,[(\delta_1+\delta_2)\lambda_e\Delta^2 + m_e\omega_e^2\nu_\ell(2\phi_e\Delta-\nu_\ell) + 2\beta_e\omega_e\omega_e\phi_e\Delta\nu_\ell\lambda_{p\ell} + m_e\omega_e^4\phi_e^2\sigma\Delta^2] \quad (5.124)$$

The detuned eigenvalues and eigenvectors are still defined as in the previous section by Eqs. (5.64) through (5.72).

The remaining m eigenvectors are related to the first m eigenvectors according to the following expressions:

$$\left.\begin{array}{l}\psi_{i,j+m} = \overline{\psi}_{i+m,j} \\ \psi_{i+m,j+m} = \overline{\psi}_{i,j}\end{array}\right\} \quad ; \quad i,j = 1,\ldots,m \qquad (5.125)$$

The eigenvectors $\hat{\underset{\sim}{\psi}}_j$ of the original system are then recovered with the transformation of Eq. (5.7). Since the eigenvectors $\underset{\sim}{\psi}_j$ of the transformed system satisfy approximately the orthonormality condition given by Eq. (2.7), the eigenvectors $\hat{\underset{\sim}{\psi}}_j$ are nearly orthonormal (up to second order terms) in the following sense:

$$\hat{\underset{\sim}{\psi}}_i \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \hat{\underset{\sim}{\psi}}_j = \delta_{ij} + \text{terms } O(\varepsilon^3) \quad ; \quad i,j = 1,\ldots,2m \qquad (5.126)$$

5.3 PARTICIPATION FACTORS OF THE COMBINED SYSTEM

If we are seeking the response of the structure-equipment system to a base excitation by modal superposition, we need to obtain the complex participation factors F_j defined as follows:

$$F_j = \hat{\underset{\sim}{\psi}}_j^T \begin{Bmatrix} \underset{\sim}{0} \\ \begin{bmatrix} M_p & \underset{\sim}{0} \\ \underset{\sim}{0} & m_e \end{bmatrix} \underset{\sim}{r} \end{Bmatrix} \quad ; \quad j = 1,\ldots,2m \qquad (5.127)$$

where the eigenvectors of the combined system are normalized as in Eq. (5.126). The participation factors F_j can be more conveniently obtained in terms of the complex participation factors of the primary system F_{pj}:

$$F_{pj} = \underset{\sim}{\psi}_{pj}^T \begin{Bmatrix} \underset{\sim}{0} \\ M_p \underset{\sim}{r}_p \end{Bmatrix} ; \quad j = 1,\ldots,n \qquad (5.128)$$

Indeed, it is not difficult to show that the combined system participation factors can be obtained as:

$$F_j = \overline{F}_{j+m} = \sum_{i=1}^{m} \psi_{i,j} F_{pi} + \psi_{i+m,j} \overline{F}_{pi} \quad ; \quad j = 1,\ldots,m \qquad (5.129)$$

where we defined:

$$F_{pm} = r_e \phi_e m_e \qquad (5.130)$$

5.4 NUMERICAL RESULTS

A simple six degree of freedom model representing a shear building showed in Figure 5.1 is chosen as the primary system to examine the accuracy of the proposed method. The primary system is regarded as nonclassically damped with the damping matrix given in Table 5.1. The floor masses are: $m_1 = m_2 = 7\text{x}10^7$ Kg, $m_3 = m_4 = 5/7\ m_1$, $m_5 = m_6 = 4/7\ m_1$. The interstory stiffnesses are: $k_1 = k_2 = 5\text{x}10^{11}$ N/m, $k_3 = k_4 = 0.8\ k_1$, $k_5 = k_6 = 0.7\ k_1$. The complex eigenvalues of the structure are listed in Table 5.2.

Tables 5.3, 5.4 and 5.5 shows the complex eigenvalues of the combined structure-equipment system obtained with the present approach. The errors in percent in the amplitudes and phases of the eigenvalues when they are compared with the exact values obtained by a combined analysis of the structure-equipment system are given in the parentheses. Three different oscillator-to-the-floor mass ratio values are examined: 1/100, 1/10 and 1/5. The oscillator is located on the fifth floor. In Table 5.3, the equipment eigenvalue is not tuned to any of the primary systems eigenvalues while in Tables 5.4 and 5.5, respectively, it is tuned to the lowest and fourth structural eigenvalues. From Table 5.5, it is observed that the errors in the values obtained by the perturbation method do increase somewhat when the equipment is heavy and tuned to a higher eigenvalue.

In Table 5.6, results of the eigenvectors obtained with the proposed method are presented. Only the absolute value (or amplitude) of the half lower part of the eigenvectors of the combined system are shown. The equipment-to-floor mass ratio is 1/10 and the equipment eigenvalue is tuned to the lowest structural eigenvalue. It is noted again that the largest error in the eigenvector elements is 0.37%. This error increases when the equipment is tuned to a higher mode. However, the effect of this error on the calculated response is usually insignificant because the higher modes usually do not count much to the response. This error can be further removed if the mode acceleration formulation is utilized [19].

Since the complex participation factors defined by Eq. (5.129) and the elements of the eigenvector corresponding to the equipment degree of freedom are used to obtain the equipment response it is also relevant to examine the accuracy obtained for these quantities. Moreover, the errors in the participation factors give a measure of the overall accuracy of the eigenvectors obtained with the present approach. It is seen that even for the cases of a rather heavy equipment, the errors in these quantities remain small (less than 2%), as indicated by the results in Table 5.7. These errors will, however, increase if a heavier equipment is considered and also if it also is tuned a high frequency mode.

The eigenproperties obtained with the closed form expressions are next used to calculate the floor response spectra values. These values, obtained for the three mass ratios 1/100, 1/10 and 1/5, are given in

Columns (2), (4) and (6) of Table 5.8. The procedure for the calculation of these response quantities is described in Reference 27. The seismic input for these results is defined in terms of average pseudo and relative velocity response spectrum curves obtained for an ensemble of 75 synthetically generated accelerograms. The errors in these spectrum values, when they are compared with the exact spectrum values (obtained with the exact eigenproperties), are shown in Columns (3), (5) and (7). The errors are mostly less than 4%, but at the frequencies where the transition from the tuned to detuned case is rather abrupt; this error is about 10%.

5.5 SUMMARY AND CONCLUSIONS

The eigenproperties of a nonproportionally damped structure-equipment system are obtained via a systematic second order matrix perturbation analysis. Two different cases are analyzed. For the case when the equipment eigenvalue is well separated from all of the primary system eigenvalues, the standard perturbation expansions for the combined system eigenvalues and eigenvectors are used to obtain the closed form expressions. These eigenproperties are expressed in terms of the complex modal properties of the main structure and equipment characteristics. When the equipment natural frequency and damping ratio are such that its eigenvalue is equal or nearly equal to an eigenvalue of the structure, the conventional expansion for the eigenvalues and eigenvectors breaks down and alternative expansions are required. These special expansions are used in this chapter to obtain the closed form expression for the eigenproperties. The accuracy of the proposed method is tested through several numerical examples by comparing the exact and approximate eigenproperties. The expressions provided for the eigenvalues for both the tuned and detuned cases are quite accurate for equipment with mass as large as 1/5 of the supporting floor mass. Because of the nature of the perturbation method the results tend to deteriorate for heavy equipment which are tuned to the higher structural frequencies. Although this error is more severe in the calculation of the eigenvectors, it does not affect significantly the equipment response. Examples of floor response spectra obtained with the modal properties of the composite system obtained with the proposed method are also presented and compared with those obtained by using the exact eigenproperties.

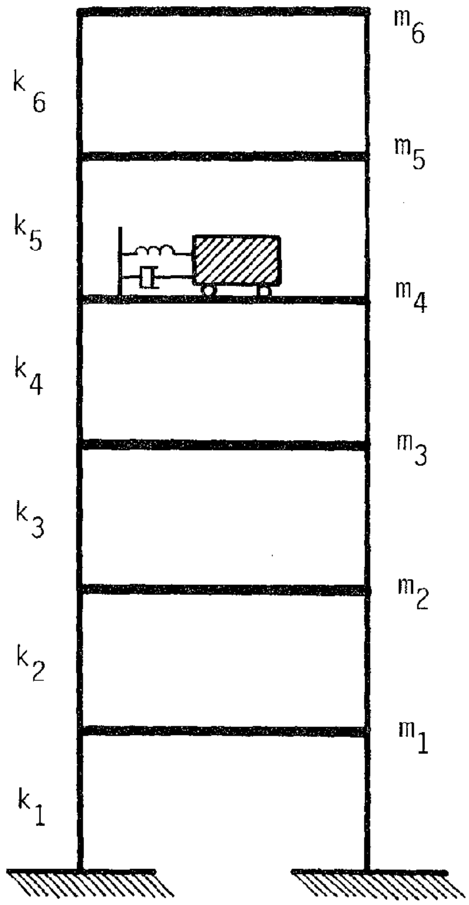

Figure 5.1 A Six Degrees of Freedom Primary Structure Supporting an Equipment

Table 5.1 - Damping matrix of the nonclassically damped primary structure of the example problem.

$$[C_p] = \begin{bmatrix} 20.0 & -4.0 & -0.4 & -0.1 & -0.08 & -0.06 \\ & 9.0 & -4.0 & -0.3 & -0.2 & -0.15 \\ & & 8.0 & -4.0 & -0.3 & -0.2 \\ & & & 7.0 & -2.0 & -0.6 \\ & \text{SYMM} & & & 5.0 & -3.0 \\ & & & & & 4.0 \end{bmatrix} \times 10^8 \text{ [Kg/sec]}$$

Table 5.2 - Complex-eigenvalues of the nonclassically damped primary structure of the example problem.

Eigenvalues of the primary system		
No.	Real	Imaginary
1	-0.2923	23.8724
2	-3.7472	61.5101
3	-7.2993	97.1178
4	-11.7546	132.5499
5	-11.7583	153.0566
6	-12.1126	170.3976

Table 5.3 - Eigenvalues of the combined structure-equipment system for a detuned case.

Equipment undamped frequency = 40.0 rad/sec - Equipment damping ratio = 0.03

Eigenvalue No.	Mass Ratio					
	1/100		1/10		1/5	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	23.8295 (0.00)	-89.2850 (0.02)	23.4272 (0.07)	-89.1609 (0.17)	22.9804 (0.29)	-89.0180 (0.34)
2	61.6426 (0.00)	-86.5159 (0.00)	61.8087 (0.00)	-86.5341 (0.03)	61.9933 (0.01)	-86.5543 (0.05)
3	97.3919 (0.00)	-85.7017 (0.00)	97.3932 (0.00)	-85.7013 (0.00)	97.3947 (0.00)	-85.7009 (0.00)
4	133.0770 (0.00)	-84.9320 (0.00)	133.1390 (0.00)	-84.9294 (0.01)	133.2079 (0.01)	-84.9266 (0.01)
5	153.5219 (0.00)	-85.6070 (0.00)	153.6512 (0.00)	-85.6069 (0.015)	153.7949 (0.01)	-85.6067 (0.03)
6	170.8429 (0.00)	-85.9354 (0.00)	170.9810 (0.05)	-85.9485 (0.02)	171.1345 (0.01)	-85.9630 (0.03)
7	40.0535 (0.00)	-88.2878 (0.00)	40.5345 (0.05)	-88.3497 (0.10)	41.0691 (0.19)	-88.4167 (0.20)

Table 5.4 - Eigenvalues of the combined damped structure-equipment system with the equipment frequency tuned to the lowest structure eigenvalue.

Equipment undamped frequency = 24.0 rad/sec - Equipment damping ratio = 0.01

Eigenvalue No.	Mass Ratio					
	1/100		1/10		1/5	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	24.5419 (0.03)	-89.3611 (0.01)	25.9367 (0.31)	-89.3575 (0.06)	26.8396 (0.61)	-89.3556 (0.09)
2	61.6287 (0.00)	-86.5142 (0.00)	61.6698 (0.00)	-86.5171 (0.00)	61.7153 (0.00)	-86.5205 (0.01)
3	97.3917 (0.00)	-85.7017 (0.00)	97.3922 (0.00)	-85.7016 (0.00)	97.3927 (0.00)	-85.7015 (0.00)
4	133.0725 (0.00)	-84.9321 (0.00)	133.0934 (0.00)	-84.9312 (0.00)	133.1168 (0.00)	-84.9301 (0.00)
5	153.5125 (0.00)	-85.6070 (0.00)	153.5570 (0.00)	-85.6068 (0.00)	153.6064 (0.00)	-85.6067 (0.01)
6	170.8329 (0.00)	-85.9345 (0.00)	170.8809 (0.00)	-85.9390 (0.00)	170.9343 (0.00)	-85.9439 (0.01)
7	23.3578 (0.03)	-89.3649 (0.01)	22.1923 (0.30)	-89.3694 (0.05)	21.5442 (0.61)	-89.3724 (0.07)

Table 5.5 - Eigenvalues of the combined damped structure-equipment system with the equipment frequency tuned to the 4th structure eigenvalue.

Equipment undamped frequency = 133.0 rad/sec - Equipment damping ratio = 0.08

Eigenvalue No.	Mass Ratio					
	1/100		1/10		1/5	
	Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
1	23.8450 (0.00)	-89.2337 (0.07)	23.5834 (0.01)	-88.6434 (0.75)	23.2958 (0.00)	-87.9721 (1.51)
2	61.5922 (0.00)	-86.5034 (0.01)	61.3042 (0.01)	-86.4094 (0.15)	60.9844 (0.05)	-86.3039 (0.29)
3	97.3900 (0.00)	-85.7021 (0.00)	97.3747 (0.00)	-85.7051 (0.00)	97.3576 (0.01)	-85.7085 (0.01)
4	134.8448 (0.03)	-85.0321 (0.05)	137.1382 (0.10)	-84.8228 (0.03)	137.2825 (0.21)	-84.6323 (0.23)
5	154.1024 (0.03)	-85.6140 (0.08)	159.4565 (1.29)	-85.6744 (0.50)	165.4057 (3.87)	-85.7369 (0.66)
6	171.2331 (0.02)	-85.9751 (0.06)	174.8868 (0.48)	-86.3364 (0.68)	178.9543 (1.27)	-86.7205 (1.50)
7	130.6917 (0.03)	-85.2992 (0.12)	123.6131 (0.83)	-85.3828 (0.54)	118.1527 (2.37)	-85.4301 (0.74)

Table 5.6 - Amplitude of the lower-half part of the combined system eigenvectors for a tuned case.

Equipment undamped frequency = 24.0 rad/sec - Equipment damping ratio = 0.01

Degree of Freedom	Amplitude of eigenvectors, x 10^{-5}						
	1	2	3	4	5	6	7
1	.19449 (0.33)	.43626 (0.00)	.42087 (0.00)	.46499 (0.00)	.28627 (0.00)	.07620 (0.00)	.18429 (0.28)
2	.37138 (0.24)	.64163 (0.00)	.28979 (0.00)	.23316 (0.00)	.37761 (0.00)	.16058 (0.02)	.35641 (0.21)
3	.54991 (0.08)	.47387 (0.00)	.35497 (0.00)	.38164 (0.00)	.36136 (0.02)	.36470 (0.01)	.54179 (0.05)
4	.68311 (0.11)	.08370 (0.05)	.57037 (0.00)	.30582 (0.01)	.06714 (0.33)	.44168 (0.01)	.69463 (0.12)
5	.77092 (0.37)	.41151 (0.00)	.04662 (0.00)	.31287 (0.00)	.45685 (0.01)	.48717 (0.01)	.82154 (0.36)
6	.83392 (0.28)	.72821 (0.00)	.52851 (0.00)	.30453 (0.00)	.26985 (0.01)	.20882 (0.01)	.86947 (0.27)
7	4.76588 (0.32)	.07321 (0.06)	.03005 (0.04)	.01051 (0.35)	.01145 (0.24)	.00977 (0.30)	5.44083 (0.29)

Table 5.7 - Complex participation factors and $(2m)^{th}$ eigenvector elements of the combined system with the equipment frequency tuned to the lowest structure eigenvalue.

Equipment undamped frequency = 24.0 rad/sec - Equipment damping ratio = 0.01

Mass Ratio	Frequency No.	Complex Participation Factor		Eigenvector $\hat{\psi}_{2m,j}$	
		Amplitude	Phase	Amplitude	Phase
1/100	1	1515.13 (0.06)	-43.689 (0.31)	.16579 E-3 (0.00)	-46.2394 (0.04)
	2	579.13 (0.00)	-45.165 (0.01)	.73213 E-6 (0.01)	-49.6609 (0.00)
	3	227.68 (0.00)	-37.305 (0.00	.30054 E-7 (0.00)	-33.5202 (0.00)
	4	137.30 (0.00)	-30.347 (0.01)	.10512 E-6 (0.04)	-43.2076 (0.01)
	5	64.17 (0.00)	-19.022 (0.03)	.11446 E-6 (0.03)	-43.9303 (0.01)
	6	12.08 (0.00)	-18.068 (0.08)	.97735 E-7 (0.03)	-50.6005 (0.01)
	7	1809.64 (0.13)	-46.239 (0.03)	.15705 E-3 (0.09)	-43.6917 (0.21)
1/5	1	1412.19 (0.28)	-44.800 (0.06)	.32281 E-4 (0.55)	-45.3190 (0.10)
	2	575.95 (0.01)	-45.085 (0.12)	.73213 E-6 (0.11)	-49.6609 (0.06)
	3	227.73 (0.00)	-37.299 (0.01)	.30054 E-7 (0.07)	-33.5202 (0.07)
	4	137.35 (0.01)	-30.355 (0.14)	.10512 E-6 (0.70)	-43.2076 (0.13)
	5	63.95 (0.08)	-18.936 (0.66)	.11446 E-6 (0.47)	-43.9303 (0.15)
	6	11.82 (0.59)	-18.440 (1.65)	.97735 E-7 (0.60)	-50.6005 (0.21)
	7	1976.17 (0.08)	-45.272 (0.06)	.39891 E-4 (0.42)	-44.7418 (0.04)

Table 5.8 - Comparison of floor response spectrum values obtained with the exact and approximate modal properties of the combined system.

Equipment damping ratio = 0.01.

Equipment Natural Frequency, in rad/sec	Mass Ratio					
	1/100		1/10		1/5	
	Acceleration G-units	Error in %	Acceleration G-units	Error in %	Acceleration G-units	Error in %
10.0	0.3176	0.00	0.3166	0.07	0.3154	0.13
15.0	0.6376	0.16	0.6280	1.45	0.6172	2.66
22.0	3.6915	10.22	1.8756*	3.37	1.4340*	3.03
24.0	6.0681*	0.11	2.1641*	0.52	1.4947*	1.23
28.0	2.2574	4.04	1.5326	4.19	1.2480	4.35
40.0	0.8932	0.46	0.8797	4.16	0.8637	7.48
50.0	0.7209	0.19	0.7056	1.69	0.6882	2.81
60.0	0.7103	1.96	0.6513*	1.78	0.6239*	1.82
62.0	0.9138	3.39	0.6573*	4.20	0.6219*	2.73
70.0	0.6095	0.34	0.5812	1.44	0.5634	1.66
97.0	0.5061	0.13	0.4975*	1.43	0.4879*	2.91
130.0	0.4908	0.18	0.4809*	1.89	2.63*	2.63

(*): These cases were considered as tuned.

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NOMENCLATURE

$[A_o]$	=	left hand side symmetric matrix of the unperturbed eigenvalue problem.
$[A_1]$	=	left hand side symmetric perturbation matrix or order ε.
$[A_2]$	=	left hand side symmetric perturbation matrix of order ε^2.
$[A'_o]$	=	auxiliary diagonal matrix.
$[B_o]$	=	right hand side symmetric matrix of the unperturbed eigenvalue problem.
$[B_2]$	=	right hand side symmetric perturbation matrix of order ε^2.
$[B'_o]$	=	auxiliary diagonal matrix.
$[B'_1]$	=	auxiliary $(m \times m)$ matrix.
$[B'_2]$	=	auxiliary $(m \times m)$ matrix.
$[C]$	=	$(m \times m)$ damping matrix of the combined system.
$[C_c]$	=	$(m \times m)$ damping coupling matrix.
$[C_p]$	=	$(n \times n)$ damping matrix of primary system.
$[C_t]$	=	transformed damping matrix.
c_e	=	damping coefficient of the equipment
$[D_{11}]$	=	auxiliary matrix composed of $\underset{\sim}{\nu}$ and ϕ_e.
$[D_{12}]$	=	auxiliary matrix composed of $\underset{\sim}{\nu}$, $\overline{\underset{\sim}{\nu}}$ and ϕ_e.
F_j	=	jth complex-valued participation factor of the nonclassically damped combined system.
F_{pj}	=	jth complex-valued participation factor of the nonclassically damped primary system.
$[I]$	=	identity matrix
$[K]$	=	$(m \times m)$ stiffness matrix of the combined system.
$[K_c]$	=	$(m \times m)$ stiffness coupling matrix.
$[K_p]$	=	$(n \times n)$ stiffness matrix of primary system.
$[K_t]$	=	transformed stiffness matrix.
k_e	=	stiffness of equipment.

$[M]$ = $(m \times m)$ mass matrix of combined system.

$[M_p]$ = $(n \times n)$ mass matrix of primary system.

m = number of dof of combined structure - oscillator system.

m_e = mass of equipment.

n = number of dof of primary system.

p_j = jth complex-valued eigenvalue of nonclassically damped combined system.

p_{oj} = jth unperturbed eigenvalue.

p_{1j} = first correction term for the jth eigenvalue.

p_{2j} = second correction term for the jth eigenvalue.

p_{3j} = third correction term for the jth eigenvalue.

p_{4j} = fourth correction term for the jth eigenvalue.

$\tilde{r}$ = displacement influence vector of combined system.

r_e = displacement influence coefficient of equipment.

r_m = ratio of equipment mass-to-supporting floor mass

$\tilde{r}_p$ = displacement influence vector of primary system.

$[T]$ = auxiliary transformation matrix.

$[U]$ = auxiliary transformation matrix.

$\tilde{u}_{oj}$ = jth eigenvector of the general unperturbed eigenproblem.

$\tilde{u}_{1j}$ = first vector of correction terms for the jth eigenvector.

$\tilde{u}_{2j}$ = second vector of correction terms for the jth eigenvector.

$\tilde{u}_{3j}$ = third vector of correction terms for the jth eigenvector.

$\tilde{u}_{4j}$ = fourth vector of correction terms for the jth eigenvector.

$\tilde{u}_{oj}^{u}$ = vector formed by the upper m elements of $\tilde{u}_{oj}$.

$\tilde{u}_{oj}^{\ell}$ = vector formed by the lower m elements of $\tilde{u}_{oj}$.

$\tilde{v}$ = m-dimensional vector with 2 non-zero entries at the Kth and mth positions.

X = auxiliary constant.

$\ddot{X}_g(t)$ = ground excitation.

$\underset{\sim}{x}$ = relative displacement vector of the combined system.

$\dot{\underset{\sim}{x}}$ = relative velocity vector of the combined system.

$\ddot{\underset{\sim}{x}}$ = relative acceleration vector of the combined system.

Y = auxiliary constant.

Z = auxiliary constant.

$\underset{\sim}{z}$ = 2m-dimensional state vector.

α_j = jth complex constant.

β_j = structure modal damping ratio for j = 1,..,n and equipment damping ratio for j = m.

β_e = damping ratio of equipment.

β_{pj} = jth modal damping ratio of the primary structure.

γ_j = jth real-valued participation factor of the combined system.

γ_{pj} = jth real-valued participation factor of the primary system.

Δ = auxiliary constant for the definition of the tuned eigen vectors.

δ_1 = detuning parameter of order ε.

δ_2 = detuning parameter of order ε^2.

δ_{ij} = Kronecker delta.

ε = bookkeeping parameter indicating the order of the accompanying quantity.

θ_{ij}, $\tilde{\theta}_{ij}$, $\hat{\theta}_{ij}$, θ^{*}_{ij} = coefficients used in the expansions of $\underset{\sim}{u}_{1j}$, $\underset{\sim}{u}_{2j}$, etc.

$[\Lambda]$ = (m×m) diagonal matrix comprising the natural frequencies of the primary system and equipment.

$[\Lambda_p]$ = (n×n) diagonal matrix with the primary system eigenvalues.

λ_e = complex eigenvalue of single dof equipment.

λ_j = jth real-valued eigenvalue of the combined system.

λ_{pj} = jth complex-valued eigenvalue of the nonclassically damped primary system.

λ_{sj} = jth eigenvalue of a (2×2) eigenproblem associated with the equipment.

$\underset{\sim}{\nu}$ = vector composed of the elements of the eigenvector matrix of the primary system associated with the attachment point of equipment.

σ = auxiliary constant.

ϕ_e = complex eigenvector element of the equipment.

$\underset{\sim}{\phi}_j$ = jth real-valued eigenvector of transformed combined system.

$\underset{\sim}{\phi}_{pj}$ = jth real-valued eigenvector of primary system.

$\hat{\underset{\sim}{\phi}}_j$ = jth real-valued eigenvector of combined system.

$[\Psi_p]$ = complex (2n×2n) eigenvector matrix of the nonclassically damped primary system.

$[\Psi_p^{\ell}]$ = (n×n) submatrix with the first n rows and columns of $[\Psi_p]$.

$[\Psi_p^{\ell}]$ = (n×n) submatrix with the lower n rows and first n columns of $[\Psi_p]$.

ψ_e = equivalent eigenvector element of the equipment = $1/\sqrt{m_e}$.

$\underset{\sim}{\psi}_j$ = jth complex eigenvector of a transformed nonclassically damped combined system.

$\underset{\sim}{\psi}_{pj}$ = jth complex eigenvector of the nonclassically damped combined system

$\underset{\sim}{\psi}_{sj}$ = jth eigenvector of a (2×2) eigenproblem associated with the equipment.

$\hat{\underset{\sim}{\psi}}_j$ = jth complex eigenvector of the nonclassically damped combined system.

$\underset{\sim}{\psi}_j^{\ell}$ = lower half of the eigenvector $\underset{\sim}{\psi}_j$.

$\underset{\sim}{\psi}_j^{u}$ = upper half of the eigenvector $\underset{\sim}{\psi}_j$.

ω_e = natural frequency of equipment in rad/sec.

ω_j = structure natural frequency for j = 1,..,n and equipment frequency for j = m.

ω_{pj} = jth natural frequency of primary system in rad/sec.