MODE SYNTHESIS APPROACH FOR THE ANALYSIS OF SECONDARY SYSTEMS

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16. Abstracts

A mode synthesis-based direct approach is presented to calculate seismic response of equipment supported on structures. The approach incorporates the effect of the dynamic interaction between the equipment and the supporting structure. The modal properties of the combined structure-equipment system are obtained by synthesizing the modal properties of the individual structures. The seismic input defined in terms of smoothed ground response spectra can be directly utilized in this approach. Both the heavy and light equipment can be considered by the approach equally effectively. Numerical examples demonstrating the effectiveness of the proposed approach are presented.

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1. INTRODUCTION

For seismic design of secondary systems, floor response spectra are commonly used as the design input. Like the ground response spectra, the floor spectra define the maximum response characteristics of a floor motion. It has been a common practice to ignore the dynamic interaction or the feed-back effect between the secondary and primary systems in the analyses used for generation of floor response spectra [1-3]. That is, in these analyses, the equipment or the secondary system is considered decoupled from, and in cascade with, the primary systems. The primary structure is analyzed for the specified ground motion to obtain the motion of the floor. The floor motion is then used as the input to the equipment to calculate its maximum response. This decoupled analysis is acceptable in most cases, especially when the equipment or the secondary system is very light. There are, however, situations where it is important to consider the feed-back effect to obtain more accurate response. This especially happens when the equipment is in resonance with one of the dominant structural frequencies.

To incorporate this dynamic interaction in the seismic analysis of an equipment-structure system, a novel approach was originally proposed by Sackman and Kelly [4]. An analytical procedure was developed to obtain the small perturbations caused in the frequencies and modes of the primary system by an addition of a tuned and detuned oscillator. This approach was formulated for deterministic ground inputs originally. It was then extended to stochastic inputs by Der Kiureghian and Sackman and their colleagues [5-7]. A somewhat similar approach has also been proposed by Gupta [8-9] to obtain the combined frequencies of the primary and light secondary systems and the modified mode shapes of the primary system. This enables one to study the effect on the primary system response of the interaction between the two systems. A recent paper by Hernried and Sackman [10] in this area is also of direct relevance.

This paper presents a mode synthesis-based approach for obtaining the combined modal properties and a response spectrum approach for calculating the response of the two systems. The approach is not restricted to light equipment; that is, an equipment as heavy as (or heavier than) its support can be considered and the combined eigenproperties as well as the system response can be calculated as accurately as desired. The basic inputs required for this approach are the modal properties of the supporting primary system and the equipment characteristics. The seismic input defined in terms of the smoothed ground response spectra can be directly used. The proposed response spectrum approach is based on the random vibration analysis. The applicability of the approach is demonstrated by numerical examples.

2. DYNAMIC ANALYSIS OF COUPLED SYSTEMS

It is desirable that the methods to be used in the analysis of a combined structure-equipment system employ only the modal properties of the two systems. This is because, sometimes, it is more convenient to define and store the dynamic characteristics of a structure in terms of the natural frequencies and mode shapes rather than the complete physical characteristics (stiffness and mass matrices). The stiffness coupling method is one of such methods in which the dynamic characteristics of a complete system are obtained using only the vibrational characteristics of each component substructure. Here, for the analysis of a main structure supporting an equipment, represented as a single degree-of-freedom (dof) oscillator, the two subsystems are considered connected by a flexible link of finite stiffness equal to the stiffness of the oscillator. The equations of motion for this n+1 dof coupled system subjected to a base motion of $\ddot{X}_g(t)$ are:

$$\begin{bmatrix} M_p & \underset{\sim}{0} \\ \underset{\sim}{0} & m_e \end{bmatrix} \begin{Bmatrix} \ddot{\underset{\sim}{x}}_p \\ \ddot{x}_e \end{Bmatrix} + \left[\begin{bmatrix} C_p & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [C_c] \right] \begin{Bmatrix} \dot{\underset{\sim}{x}}_p \\ \dot{x}_e \end{Bmatrix} + \left[\begin{bmatrix} K_p & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [K_c] \right] \begin{Bmatrix} \underset{\sim}{x}_p \\ x_e \end{Bmatrix} = - \begin{Bmatrix} M_p \underset{\sim}{r} \\ m_e r_e \end{Bmatrix} \ddot{X}_g(t) \quad (1)$$

where $\underset{\sim}{x}_p$ = relative displacement response vector of the primary system; x_e = relative displacement of the oscillator with respect to ground; $[M_p]$, $[C_p]$ and $[K_p]$ are the mass, damping and stiffness matrices, respectively, of the primary system; m_e = equipment mass; $[C_c]$ and $[K_c]$ are the coupling matrices associated with the damping and stiffness forces and contain the damping coefficient and stiffness of the oscillator in their non-zero elements; and $\underset{\sim}{r}$ = displacement influence vector of the primary system. The displacement influence coefficient, r_e, of the equipment is equal to 1 if the equipment vibrates in the direction of excitation and zero if it is constrained to move in a perpendicular direction. Here, in general, the subscripts p, e and c refer to the primary elements, equipment and coupling elements, respectively. We introduce the following transformation in equation (1)

$$\begin{Bmatrix} \underset{\sim}{x}_p \\ x_e \end{Bmatrix} = \begin{bmatrix} \Phi_p & \underset{\sim}{0} \\ \underset{\sim}{0} & \phi_e \end{bmatrix} \begin{Bmatrix} \underset{\sim}{q}_p \\ q_e \end{Bmatrix} = [U]\{q\} \quad (2)$$

where: $[\Phi_p]$ = eigenvector matrix of the primary system, normalized such that:

$$[\Phi_p]^T [M_p][\Phi_p] = [I] \quad (3)$$

where the superscript T denotes transpose. Also,

$$\phi_e = \frac{1}{\sqrt{m_e}} \tag{4}$$

Premultiplying by $[U]^T$, we obtain:

$$[M^*]\{\ddot{q}\} + [C^*]\{\dot{q}\} + [K^*]\{q\} = -\{\gamma^*\}\ddot{x}_g(t) \tag{5}$$

where:

$$[M^*] = [I] \tag{6}$$

$$[C^*] = \begin{bmatrix} 2\beta_{pi}\omega_{pi} \searrow & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [U]^T[C_c][U] \tag{7}$$

$$[K^*] = \begin{bmatrix} \omega_{pi}^2 \searrow & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + [U]^T[K_c][U] \tag{8}$$

$$\{\gamma^*\} = \begin{Bmatrix} \underset{\sim}{\gamma}_p \\ \hline \sqrt{m_e}\ r_e \end{Bmatrix} \tag{9}$$

Here ω_{pi}, β_{pi} and $\underset{\sim}{\gamma}_p$, respectively, are the ith natural frequency, ith modal damping ratio and the vector of participation factors of the primary structure.

If the oscillator is attached to the kth point of the primary structure, the coupling terms in Eqs. (7) and (8) can be expressed as follows:

$$[U]^T[K_c][U] = m_e\omega_e^2\{\nu\}\{\nu\}^T \tag{10}$$

$$[U]^T[C_c][U] = 2m_e\omega_e\beta_e\{\nu\}\{\nu\}^T \tag{11}$$

where ω_e and β_e, respectively, are the equipment frequency and damping ratio, and

$$\{\nu\}^T = [\phi_{k1}^{(p)}, \phi_{k2}^{(p)}, \ldots, \phi_{kn}^{(p)}, -\phi_e] \tag{12}$$

in which $\phi_{ki}^{(p)}$ is the $(k,i)^{th}$ element of $[\phi_p]$ or the kth element of the eigenvector $\{\phi_p\}_i$ of the primary structure.

To obtain any response quantity of interest for the combined structure-equipment system, one only needs to solve Eq. (5) in conjunction with transformation of Eq. (2). The numerical inaccuracy which could possibly occur in the solution of Eq. (1) due to ill-conditioning of the matrices caused by the lightness of the equipment, is now avoided in Eq. (5), as all diagonal elements of the matrices in this equation are of the same order. The system of equations (5) in general may be nonclassical and thus may require the state vector approach [11,12] for their solution. Here, however, a classical normal mode approach is proposed. This approach requires a second eigenvalue analysis. To save the computation cost, the size of this second eigenvalue problem can, however, be reduced as described in the following section.

2.1 Combined Modal Properties: Dynamic Transformation

To obtain the eigenproperties of the combined primary-secondary system, the following eigenvalue problem must be solved:

$$[\ K^* - \omega_j^2 M^*\ ]\{\phi_j^*\} = \{0\}; \quad j = 1,2,\ldots n+1 \qquad (13)$$

where $\{\phi_j^*\}$ = jth eigenvector and ω_j = jth frequency. On substitution of $[K^*]$ and $[M^*]$, we obtain

$$\left[\begin{bmatrix} \diagdown\omega_{pi}^2\diagdown & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + m_e\omega_e^2\ [\{\nu\}\{\nu\}^T]\right]\{\phi_j^*\} = \omega_j^2\{\phi_j^*\}; \quad j = 1,2,\ldots n+1 \qquad (14)$$

In this section, the methods to reduce the size of the above eigenvalue problem are described. In the classical mode synthesis approach the order of this eigenvalue problem is reduced by truncating or omitting (generally) the higher modes at the substructure level. Here, we utilize a somewhat different size reduction technique. We partition the component eigenvectors, modal coordinates and other associated vectors and matrices in two n_r (reduced) and n_k (kept) sets as:

$$\{\phi_j^*\} = \begin{Bmatrix} \underset{\sim}{\phi}_j^k \\ \underset{\sim}{\phi}_j^r \end{Bmatrix}; \quad \{\nu\} = \begin{Bmatrix} \underset{\sim}{\nu}^k \\ \underset{\sim}{\nu}^r \end{Bmatrix} \tag{15}$$

where it is understood that the kept modes include the modal value ϕ_e corresponding to the oscillator's degree of freedom. With this division, the eigenvalue problem in Eq. (13) can be expressed as follows:

$$\left[\begin{bmatrix} K^{kk} & K^{kr} \\ K^{rk} & K^{rr} \end{bmatrix} - \omega_j^2 \begin{bmatrix} I^k & 0 \\ 0 & I^r \end{bmatrix}\right] \begin{Bmatrix} \underset{\sim}{\phi}_j^k \\ \underset{\sim}{\phi}_j^r \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{0} \end{Bmatrix} \tag{16}$$

where:

$$[K^{kk}] = \begin{bmatrix} \diagdown (\omega_p^k)^2 \diagdown & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + m_e\omega_e^2[\{\nu^k\}\{\nu^k\}^T] \tag{17}$$

$$[K^{rr}] = [\diagdown (\omega_p^r)^2 \diagdown] + m_e\omega_e^2[\{\nu^r\}\{\nu^r\}^T] \tag{18}$$

$$[K^{kr}] = [K^{rk}]^T = m_e\omega_e^2[\{\nu^k\}\{\nu^r\}^T] \tag{19}$$

In Eqs. (15) through (19) and also in the following formulation, the superscripts k and r on the vectors, matrices and scalar quantities refer to the kept and reduced modal coordinates, respectively.

If the contribution of the reduced modes is neglected, as done in the classical mode synthesis methods, only the following reduced eigenvalue problem need to be solved:

$$[K^{kk} - \omega_j^2 I^k]\{\phi_j^k\} = \{0\}; \quad j = 1,2,\ldots n_k \tag{20}$$

However, by using the dynamic transformation proposed by Kuhar and Stahle [13], the effect of the eliminated modes can be included approximately by relating the set of n_r reduced eigenvectors to the n_k kept eigenvectors, by considering the lower set of Eq. (16), as:

$$\{\phi_j^r\} = [R]\{\phi_j^k\} \; ; \; j = 1,2,\ldots n_k \tag{21}$$

where the transformation matrix [R] is defined as:

$$[R] = -[K^{rr} - \omega_j^2 I^r]^{-1}[K^{kr}]^T \tag{22}$$

Substituting $\{\phi_j^r\}$ in Eq. (16), the resulting eigenvalue problem becomes now:

$$[K - \omega_j^2 I^k]\{\phi_j^k\} = \{0\}; \quad j = 1,2,\ldots n_k \tag{23}$$

where:

$$[K] = [K^{kk}] + [K^{kr}][R] \tag{24}$$

Once this eigenvalue problem is solved, the elements of the reduced eigenvectors associated with the reduced degrees-of-freedom can be obtained from Eq. (21).

It is noted that the solution obtained with the dynamic transformation will be exact provided that the jth eigenvalue is used in the definition of the transformation matrix [R] in Eq. (22). However, ω_j is not known a priori. Often, ω_j is taken to be zero in Eq. (22). In this case, the dynamic transformation is then the same as the well-known Guyan reduction technique [14]. However, the accuracy can be improved if we use the equipment frequency ω_e for ω_j in Eq. (22), since the changes in the structural frequencies are expected to be significant only when the equipment is tuned to a natural frequency or to a cluster of them. The remaining frequencies originally higher than the oscillator frequency will be increased and those originally lower will be decreased by an amount depending on the lightness of the equipment [15].

It is pointed out that the order n_k of the reduced eigenvalue problem in Eq. (23) is considerably smaller than the dimension (n+1) of the original problem. For example, if the oscillator is tuned to, say, the mth structural frequency, then the vector $\underset{\sim}{\nu}^k$ in Eq. (15) may be defined as:

$$\{\nu^k\}^T = (\phi^{(p)}_{k,m-1}, \phi^{(p)}_{k,m}, \phi^{(p)}_{k,m+1}, -\phi_e) \tag{25}$$

which leads to a 4x4 eigenvalue problem. The solution of such a reduced eigenvalue problem usually provides very accurate values for the n_k eigenvalues and eigenvectors of the combined system (when compared with the values obtained from the solution of the complete (n+1)x(n+1) eigenvalue problem). In Eq. (25) we are assuming that only three adjacent eigenproperties are affected by the equipment. Thus in the selection of the kept modes, one mode on either side of the tuned mode plus the tuned and equipment modes only are retained. However, if the equipment is tuned to a cluster of primary modes, then in Eq. (25) one should include the terms corresponding to the cluster plus a mode on either side along with the equipment mode. If desired, the accuracy can be improved further by including more modes around the cluster with, of course, an increase in the size of the eigenvalue problem.

In this approach, the eigenproperties of the reduced modes, far removed from the equipment frequency, are assumed to be unaffected and thus an eigenvector can be written as

$$\{\phi^*_j\}^T \approx (0, \ldots 1 \ldots 0., \phi^*_{n+1,j}); \quad j = (n_r+1)\ldots(n+1) \tag{26}$$

where 1 is at the jth location. This is explained in Appendix I, where it is shown that the changes in the eigenvector elements associated with the degrees-of-freedom of the primary system are of the second order compared to the change in the last element which is of the first order. Thus, if we neglect the second order correction terms, the first n rows of Eq. (14) will give the first n terms of vector (26) directly.

We must also obtain the last element of this vector which is required for calculating the equipment response. This is especially required if the modes which were kept in the reduced eigenvalue analysis, did not include the first few modes which usually contribute most to the response. These elements of all the reduced eigenvectors can be obtained as follows. Substituting Eq. (26) into Eq. (14), and examining only the last row, we obtain

$$m_e \omega_e^2 \nu_{n+1} (\nu_j + \nu_{n+1} \phi^*_{n+1,j}) = \omega_j^2 \phi^*_{n+1,j} \qquad (27)$$

From Eq. (12), we note that $\nu_j = \phi_{kj}^{(p)}$ and $\nu_{n+1} = -\phi_e = -1/\sqrt{m_e}$. By substituting these, we can solve for $\phi^*_{n+1,j}$ as

$$\phi^*_{n+1,j} = \sqrt{m_e}\; \phi_{kj}^{(p)} / \{1 - (\frac{\omega_{pj}}{\omega_e})^2\} \qquad (28)$$

In Eq. (28), ω_{pj} is the same as the unaffected primary structure frequency. It is noted that Eq. (28) is to be used only for those frequencies which are not in the vicinity of the equipment frequency. The tuned and nearly tuned modes were already included in Eq. (23).

Equations (23) with Eq. (21) and Eq. (26) with Eq. (28) now define the complete set of eigenvectors of Eq. (5). To obtain the modal matrix of the original system represented by Eq. (1), we can use the transformation in Eq. (2) as

$$[\Phi] = [U][\Phi^*] \tag{29}$$

where $[\Phi]$ is the modal matrix of the original system.

It is straightforward to show that if the modal matrix $[\Phi^*]$ is normalized such that $[\Phi^*]^T[\Phi^*] = [I]$, then the modal matrix $[\Phi]$ is orthonormal with respect to the original mass matrix. That is,

$$[\Phi]^T \begin{bmatrix} M_p & \underset{\sim}{0} \\ \underset{\sim}{0} & m_e \end{bmatrix} [\Phi] = [I] \tag{30}$$

2.2 An Alternative Dynamic Condensation Technique

In the aforementioned dynamic condensation technique, we require a prior knowledge of ω_j. In addition, an inversion of a matrix of size $n_r \times n_r$ was also required to define [R] as in Eq. (22). These two problems can be eliminated if the equipment is tuned to one of the lower structural frequencies.

In this case, only a first few of the lower modes are retained (or kept) in the condensation. For such kept modes,

$$(\omega_i^k)^2 < (\omega_j^r)^2; \qquad i = 1,2\ldots n_k;\ j = 1,2\ldots n_r \tag{31}$$

where ω_i^k = ith frequency of the kept modes and ω_j^r = jth frequency of the reduced modes. This condition enables us to use a numerically more efficient scheme whereby the inversion in Eq. (22) is completely avoided.

We examine the matrix to be inverted in Eq. (22), which with the use of Eq. (18) can also be written as:

$$[\ K^{rr} - \lambda_j\ I^r\ ] = [\diagdown(\omega_p^r)^2\diagdown] - [\lambda_j\ I^r - m_e\omega_e^2\underset{\sim}{\nu}^r\underset{\sim}{\nu}^{r^T}] \tag{32}$$

where $\lambda_j = \omega_j^2$. Its inverse can be written as:

$$[\,K^{rr} - \lambda_j\, I^r\,]^{-1} = \Big[I^r - [\lceil(\omega_p^r)^{-2}\rfloor][\lambda_j\, I_r - m_e\omega_e^2 \underset{\sim}{\nu}^r \underset{\sim}{\nu}^{r^T}]\Big]^{-1} [\lceil(\omega_p^r)^{-2}\rfloor] \tag{33}$$

Expanding the inverse on the right hand side of Eq. (33) in a power series, we obtain

$$\begin{aligned}[\,I^r - [\lceil(\omega_p^r)^{-2}\rfloor]\,[\lambda_j\, I^r - m_e\omega_e^2 \underset{\sim}{\nu}^r \underset{\sim}{\nu}^{r^T}]]^{-1} &= [I^r] - [\lceil(\omega_p^r)^{-2}\rfloor][\lambda_j\, I^r - m_e\omega_e^2 \underset{\sim}{\nu}^r \underset{\sim}{\nu}^{r^T}] \\ &\quad + ([\lceil(\omega_p^r)^{-2}\rfloor][\lambda_j\, I^r - m_e\omega_e^2 \underset{\sim}{\nu}^r \underset{\sim}{\nu}^{r^T}])^2 + \ldots\end{aligned} \tag{34}$$

Equation (31) is a necessary condition for the convergence of the above matrix series expansion. However, it is not a sufficient condition. The condition that guarantees the convergence of the series in Eq. (34) is that the spectral radius of the second matrix on the left hand side of Eq. (34) be less than 1. As shown in Appendix II, in our case this requirement is satisfied if

$$(\omega_{n_k}^k)^2 + n_r m_e \omega_e^2 \max|\nu_i^r \nu_\ell^r| \le (\omega_1^r)^2 \qquad 1 \le i,\ell \le n_r \tag{35}$$

In most cases, however, condition in Eq. (31) will be sufficient. Discarding the terms of order higher than two and substituting Eq. (34) into Eq. (33) we obtain:

$$[\,K^{rr} - \lambda_j\, I^r\,]^{-1} \simeq [\lceil(\omega_p^r)^{-2}\rfloor] + \lambda_j\,[\lceil(\omega_p^r)^{-4}\rfloor] - m_e\omega_e^2\,[\lceil(\omega_p^r)^{-2}\rfloor]\,[\underset{\sim}{\nu}^r \underset{\sim}{\nu}^{r^T}]\,[\lceil(\omega_p^r)^{-2}\rfloor] \tag{36}$$

Introducing the vector:

$$\{\alpha\} = [\lceil(\omega_p^r)^{-2}\rfloor]\ \{\nu^r\} \tag{37}$$

with one of its element

$$\alpha_i = \frac{\nu_i^r}{(\omega_{pi}^r)^2} \tag{38}$$

the transformation matrix [R] in Eq. (22) can be expressed now as:

$$[R] = -\Big[[\overline{\ \ }(\omega_p^r)^{-2}\underline{\ \ }] + \lambda_j\,[\overline{\ \ }(\omega_p^r)^{-4}\underline{\ \ }] - m_e\omega_e^2[\underset{\sim}{\alpha}\ \underset{\sim}{\alpha}^T]\Big][K^{kr}]^T \tag{39}$$

With the above transformation matrix, the reduced eigenvalue problem becomes:

$$[K]\{\phi_j^k\} = \lambda_j[M]\{\phi_j^k\}; \qquad j = 1,2,\ldots n_k \tag{40}$$

where the new matrices [K] and [M] are defined now as:

$$[K] = [K^{kk}] - [K^{kr}]\,[\overline{\ \ }(\omega_p^r)^{-2}\underline{\ \ }]\,[K^{kr}]^T + m_e\omega_e^2[K^{kr}][\underset{\sim}{\alpha}\ \underset{\sim}{\alpha}^T][K^{kr}]^T \tag{41}$$

$$[M] = [I^k] + [K^{kr}]\,[\overline{\ \ }(\omega_p^r)^{-4}\underline{\ \ }]\,[K^{kr}]^T \tag{42}$$

It is possible to simplify further the definitions of [K] and [M] with the help of Eq. (19). The first triple product in Eq. (41) becomes:

$$[K^{kr}]\,[\overline{\ \ }(\omega_p^r)^{-2}\underline{\ \ }]\,[K^{kr}]^T = m_e^2\omega_e^4[\underset{\sim}{\nu}^k\underset{\sim}{\nu}^{r^T}][\overline{\ \ }(\omega_p^r)^{-2}\underline{\ \ }][\underset{\sim}{\nu}^r\underset{\sim}{\nu}^{k^T}] \tag{43}$$

Introducing:

$$X = \sum_{i=1}^{n_r} \left(\frac{\nu_i^r}{\omega_{pi}^r}\right)^2 \tag{44}$$

the above expression becomes:

$$[K^{kr}]\,[\overline{\ \ }(\omega_p^r)^{-2}\underline{\ \ }]\,[K^{kr}]^T = m_e^2\omega_e^4\,X[\underset{\sim}{\nu}^k\underset{\sim}{\nu}^{k^T}] \tag{45}$$

Similarly, by letting

$$Y = \sum_{i=1}^{n_r} \frac{(\nu_i^r)^2}{(\omega_{pi}^r)^4} \tag{46}$$

the triple product in Eq. (42) can be written as:

$$[K^{kr}] \left[\diagdown(\omega_p^r)^{-4}\diagdown\right] [K^{kr}]^T = m_e^2\omega_e^4 \, Y[\underset{\sim}{\nu}^k \underset{\sim}{\nu}^{kT}] \tag{47}$$

Also, considering Eqs. (38) and (44), the second triple product in Eq. (41) becomes:

$$[K^{kr}][\underset{\sim}{\alpha} \, \underset{\sim}{\alpha}^T][K^{kr}]^T = m_e^2\omega_e^4 \, X^2[\underset{\sim}{\nu}^k \underset{\sim}{\nu}^{kT}] \tag{48}$$

Finally, combining Eqs. (17), (41), (42), (45), (47) and (48), the matrices [K] and [M] can be written as follows:

$$[K] = \begin{bmatrix} \diagdown (\omega_p^k)^2 \diagdown & \underset{\sim}{0} \\ \underset{\sim}{0} & 0 \end{bmatrix} + m_e\omega_e^2(1-m_e\omega_e^2X + m_e^2\omega_e^4X^2)[\underset{\sim}{\nu}^k \, \underset{\sim}{\nu}^{k^T}] \tag{49}$$

$$[M] = [I^k] + m_e^2\omega_e^4 \, Y[\underset{\sim}{\nu}^k \, \underset{\sim}{\nu}^{k^T}] \tag{50}$$

After solving the eigenproblem of Eq. (40), the procedure of the previous section is followed.

3. DAMPING COUPLING

When a dynamic analysis of a structure is performed, it is generally assumed that the standard normal coordinate transformation uncouples not only the inertia and elastic forces but also the damping forces. When the damping matrix is proportional to the mass or stiffness matrix or can be derived from a linear combination of these matrices, it can always be diagonalized by the transformation to normal coordinates. As shown by Caughey [16], some damping matrices that are not restricted to one of these forms can also be diagonalized in this way. A general damping matrix, however, cannot be diagonalized by the normal coordinate transformation and therefore it introduces coupling between the undamped modal coordinates. The systems where this modal coupling occurs are often referred to as nonproportionally damped, or more properly, nonclassically damped. Well-known examples of nonclassically damped structures are systems composed of massive structures, like nuclear reactor facilities, founded on soft soil. In these cases the substantial differences in the energy loss mechanisms of the components give rise to the nonclassical damping effects. The nonclassical damping effects also become important in the calculation of an equipment response when the equipment is tuned to its supporting structure and the damping characteristics of the equipment and structure differ significantly. Based on the study of a 2 dof structure-equipment system, Igusa et al [17] have opined that the nonclassical damping effect is important when: (1) The ratio r_m between the equipment mass and the structure mass is small; (2) the equipment frequency is tuned or nearly tuned to the structure frequency; and (3) the damping ratios of the equipment β_e and the structure β_{st} satisfy the following inequality

$$(\beta_e - \beta_{st})^2 > r_m \tag{51}$$

In order to examine whether the above conditions hold also for a multi-dof structure-equipment system, the matrix $[\hat{C}]$ resulting from the product

$$[\hat{C}] = [\Phi]^T[C][\Phi] = [\Phi^*]^T[C^*][\Phi^*] \tag{52}$$

was obtained for several values of the mass, frequency and damping ratio of the equipment and for the primary structure described in Section 5. The absolute value of the ratio $\hat{c}_{ij}/\hat{c}_{ii}$ between the elements of $[\hat{C}]$ was taking as an indicator of the degree of coupling obtained for each case.

The ratio r_m is defined here as the quotient between the equipment mass and the floor mass where the equipment is supported. The damping ratio β_{st} is taken equal to the structure modal damping ratio, assumed constant for all modes.

Table 1 shows the ratios $\hat{c}_{ij}/\hat{c}_{ii}$ for the equipment tuned to the lowest structural frequency and different values of r_m, with modal damping ratio $\beta_{st} = 0.05$. In Table 2, the relation $(\beta_e-\beta_{st})^2/r_m = 16$ is kept constant, while the equipment frequency is tuned to different structural frequencies. The mass ratio r_m is 1/10000, with $\beta_{st} = 0.05$ and $\beta_e = 0.01$. Table 3 presents the results obtained with the same value of $(\beta_e-\beta_{st})^2/r_m$ as in Table 2 but now r_m is set equal to 1/2000, and $\beta_{st} = 0.095$, $\beta_e = 0.005$. From the results of Table 1, it is observed that some coupling effect is present even if the condition (51) is not satisfied, although from Tables 2 and 3 we conclude that the coupling is more significant when the condition (51) is met. Also from Tables 2 and 3, it is seen that if the difference between β_e and β_{st} is large, the effect of lowering the ratio r_m does not affect significantly the degree of coupling. For example, for the same value of $(\beta_e-\beta_{st})^2/r_m$ in Tables 2 and 3, widening the difference between β_e and β_{st} increases the coupling more than decreasing the ratio r_m. Finally, it is noted that only the off-diagonal terms of $[\hat{C}]$ associated with the tuned frequencies are significant compared to the diagonal terms.

Based on these results, we confirm that the damping coupling can affect some normal coordinates in the modal response equations of the combined system. Therefore, the floor response spectrum formulation developed in the next section also considers this damping coupling effect.

4. EQUIPMENT RESPONSE: FLOOR RESPONSE SPECTRA

Once the (n+1) modes of the combined system are obtained, the equations of motion (1) can be decoupled, if the combined system can be assumed to be classically damped, with the help of the following standard transformation

$$\{x\} = [\Phi]\{\eta\} \tag{53}$$

where $\{\eta\}$ is the vector of principal coordinates. This transformation gives:

$$[I]\{\ddot{\eta}\} + [\ulcorner 2\beta_i\omega_i \lrcorner]\{\dot{\eta}\} + [\ulcorner \omega_i^2 \lrcorner]\{\eta\} = -\{\gamma\}\ddot{X}_g(t) \tag{54}$$

The participation factor vector of the combined system, $\{\gamma\}$ is obtained from:

$$\{\gamma\} = [\Phi]^T \begin{Bmatrix} M_p \underset{\sim}{r} \\ m_e r_e \end{Bmatrix} = [\Phi^*]^T [U]^T \begin{Bmatrix} M_p \underset{\sim}{r} \\ m_e r_e \end{Bmatrix} \tag{55}$$

Or:

$$\{\gamma\} = [\Phi^*]^T \begin{Bmatrix} \underset{\sim}{\gamma}_p \\ \sqrt{m_e}\, r_e \end{Bmatrix} \tag{56}$$

In Eq. (54), it is assumed that the triple product $[\Phi^*]^T[C^*][\Phi^*]$ leads to a diagonal matrix, i.e., the combined system exhibits proportional or classical damping. But, as mentioned earlier, this is not always the case in structure-equipment interaction even if the primary structure is classically damped. Nevertheless, this general case of non-classical damping can always be handled by employing the state vector approach. The size of the problem to be solved by the state vector approach can be reduced by realizing that, as mentioned earlier, these damping coupling terms will be predominant only near the equipment frequency. Thus, only the equations corresponding to these frequencies need to be analyzed by the state vector approach. Here, the Eqs. (52) are rearranged such that the strongly coupled equations are the first ones in the set. Assuming that there are n_c strongly coupled equations, we write for them as:

$$\ddot{\eta}_i + \hat{c}_{ii}\dot{\eta}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n_c} \hat{c}_{ij}\dot{\eta}_j + \omega_i^2 \eta_i = -\gamma_i \ddot{X}_g(t); \qquad i = 1,2,\ldots n_c \quad (57)$$

where:

$$\hat{c}_{ij} = \{\phi^*\}_i^T [C^*] \{\phi^*\}_j; \qquad i,j = 1,2,\ldots n_c \quad (58)$$

In most cases, the number of equations that need to be regarded as coupled need not be more than the number of tuned frequencies, including the equipment frequency. The accuracy can be further improved by including a few more adjacent modes but, of course, with an increased computational effort. The remaining $(n+1 - n_c)$ equations are essentially uncoupled and can be written as:

$$\ddot{\eta}_i + \hat{c}_{ii}\dot{\eta}_i + \omega_i^2 \eta_i = -\gamma_i \ddot{X}_g(t); \qquad i = n_c+1,\ldots,n+1 \quad (59)$$

Equation (57) can be decoupled with the state vector approach [11,12] in which they are cast in the following form:

$$[A]\{y\} + [B]\{\dot{y}\} = -\begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{\gamma} \end{Bmatrix} \ddot{X}_g(t) \quad (60)$$

where:

$$\{y\}^T = [\dot{\eta}_1, \dot{\eta}_2, \ldots \dot{\eta}_{n_c}, \eta_1, \eta_2, \ldots \eta_{n_c}] \quad (61)$$

and:

$$[A] = \begin{bmatrix} -I & 0 \\ 0 & \omega_i^2 \end{bmatrix} \qquad [B] = \begin{bmatrix} 0 & I \\ I & c_{ij} \end{bmatrix} \quad (62)$$

To decouple Eqs. (60), we employ the eigenvectors of the following associated eigenvalue problem

$$-[A]\{\psi_i\} = p_i[B]\{\psi_i\} \tag{63}$$

in the transformation

$$\{y\} = [\Psi]\{z\} \tag{64}$$

where $[\psi]$ is modal matrix of Eq. (60). Substituting, Eq. (64) into Eq. (60), we obtain n_c decoupled complex and conjugate equations

$$\dot{z}_i - p_i z_i = F_i \ddot{X}_g(t); \qquad i = 1,2,\ldots n_c \tag{65}$$

where the complex participation factors F_i are defined as:

$$F_i = -\frac{1}{a_i^*}\{\psi_i\}^T \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{\Upsilon} \end{Bmatrix}; \qquad i = 1,2,\ldots n_c \tag{66}$$

and

$$a_i^* = \{\psi_i\}^T[B]\{\psi_i\}; \qquad i = 1,2,\ldots n_c \tag{67}$$

To obtain the absolute acceleration floor spectrum value, we need to obtain the maximum of the oscillator acceleration response. The absolute acceleration vector is:

$$\{\ddot{x}_a\} = \{\ddot{x}\} + \begin{Bmatrix} \underset{\sim}{r} \\ r_e \end{Bmatrix} \ddot{X}_g(t) \tag{68}$$

in which the relative acceleration vector $\{\ddot{x}\}$ is:

$$\{\ddot{x}\} = \sum_{j=1}^{n_c} \{\phi\}_j \ddot{\eta}_j + \sum_{j=n_c+1}^{n+1} \{\phi\}_j \ddot{\eta}_j \tag{69}$$

The acceleration associated with the principle coordinates of the coupled equations can be written as:

$$\ddot{\eta}_j = \sum_{\ell=1}^{2n_c} \psi_\ell(j)\dot{z}_\ell = \sum_{\ell=1}^{2n_c} \psi_\ell(j)(F_\ell \ddot{X}_g(t) + p_\ell z_\ell); \qquad j = 1,2,\ldots,n_c \tag{70}$$

where $\psi_\ell(j)$ is the jth element of the upper part of the eigenvector $\{\psi_\ell\}$, and the acceleration associated with principal coordinates of the decoupled equations as,

$$\ddot{\eta}_j = -\gamma_j \ddot{X}_g(t) - 2\beta_j\omega_j\dot{\eta}_j - \omega_j^2\eta_j; \qquad j = n_c+1,\ldots,n+1 \qquad (71)$$

Substituting Eqs. (70) and (71) into Eq. (69), and after some manipulations, it can be shown that:

$$\{\ddot{x}_a\} = \sum_{j=1}^{n_c} \underset{\sim}{\phi}_j \sum_{\ell=1}^{2n_c} \psi_\ell(j)p_\ell z_\ell - \sum_{j=n_c+1}^{n+1} \underset{\sim}{\phi}_j(2\beta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j) \qquad (72)$$

Therefore, the absolute acceleration of the oscillator mass $\ddot{x}_e(t)$ which is associated with the uth degree of freedom is:

$$\ddot{x}_e(t) = \sum_{j=1}^{n_c} \phi_j(u) \sum_{\ell=1}^{2n_c} \psi_\ell(j)p_\ell z_\ell - \sum_{j=n_c+1}^{n+1} \phi_j(u)(2\beta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j)$$

$$= \ddot{x}_c(t) + \ddot{x}_u(t) \qquad (73)$$

where $\ddot{x}_c(t)$ and $\ddot{x}_u(t)$ are the contributions to the equipment acceleration from the coupled and uncoupled modes of Eq. (54). These are defined as:

$$\ddot{x}_c(t) = \sum_{j=1}^{n_c} \phi_j(u) \sum_{\ell=1}^{2n_c} \psi_\ell(j)p_\ell z_\ell \qquad (74)$$

$$\ddot{x}_u(t) = -\sum_{j=n_c+1}^{n+1} \phi_j(u)\ (2\beta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j) \qquad (75)$$

In terms of the earlier notations, u = n+1 and thus $\phi_j(u) = \phi_{n+1,j}$

To obtain the floor response spectrum, we would like to obtain the maximum value of $\ddot{x}_e(t)$ for all possible ground motion that can occur at the site. To consider all possible motions at the site, herein the design ground motion is modeled as a random process. For such random ground motions, the maximum response can be conveniently expressed as the amplified root mean square response. That is

$$R_a^2 = p_f^2 \, E[\ddot{x}_e^2(t)] \tag{76}$$

defines the floor response spectrum, R_a, in terms of the mean square response $E[\ddot{x}_e^2(t)]$ and the peak factor p_f. Here $E(\cdot)$ means the ensemble average of $(\cdot)$. To obtain the mean square response, we obtain the auto-correlation function of $\ddot{x}_e(t)$ defined by equation (71) as follows:

$$\begin{aligned} E[\ddot{x}_e(t_1)\ddot{x}_e(t_2)] &= E[\ddot{x}_c(t_1)\ddot{x}_c(t_2)] + E[\ddot{x}_u(t_1)\ddot{x}_u(t_2)] \\ &\quad + E[\ddot{x}_c(t_1)\ddot{x}_u(t_2) + \ddot{x}_c(t_2)\ddot{x}_u(t_1)] \end{aligned} \tag{77}$$

The auto and cross correlation terms in Eq. (77) can be obtained in terms of the stochastic characteristics of ground motion and structural properties by employing Eqs. (74) and (75). To simplify the algebra here, it is assumed that the ground motion is stochastically stationary with spectral density function $\Phi_g(\omega)$. It is realized that earthquake motions are inherently nonstationary. Yet, however, results of practical importance have been obtained in earlier studies with this assumption. Verification of the results obtained here through a simulation study with realistic nonstationary ensemble of ground motion is reported later.

Each of the correlation terms in Eq. (77) can be expressed in terms of the ground motion spectral density function. The analytical development of the correlation term $E[\ddot{x}_c(t_1)\ddot{x}_c(t_2)]$ is given in Appendix III. The development of $E[\ddot{x}_u(t_1)\ddot{x}_u(t_2)]$ is explained in Appendix IV. The third term is examined in Appendix V. As the design ground motions are usually prescribed in terms of the ground response spectra, the mean square values associated with these three correlation terms can also be expressed in terms of ground response spectra (see Appendices III-VI). In terms of these three mean square values, the floor response spectrum value, as defined by Eq. (76), can now be written as follows:

$$R_a^2 = p_f^2(R_1 + R_2 + R_{12}) \tag{78}$$

where R_1 and R_2 are the contributions to the mean square values by the coupled and uncoupled modes, respectively, and R_{12} is the contribution of the cross modes. These contributions can be expressed in terms of the ground response spectrum values as follows:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u)[4 \sum_{\ell=1}^{n_c} (a_{j\ell}a_{k\ell}I_{2\ell} + Z_{\ell jk}I_{1\ell})$$

$$+ 2 \sum_{\ell=1}^{n_c-1} \sum_{m=\ell+1}^{n_c} (\frac{E_{\ell m}I_{1\ell}}{r_2^2} + F_{\ell m}I_{2\ell} + G_{\ell m}I_{1m} + H_{\ell m}I_{2m})] \tag{79}$$

$$R_2 = \sum_{j=n_c+1}^{n+1} \gamma_j^2\phi_j^2(u)\omega_j^2[I_{1j} + 4\beta_j^2I_{2j}]$$

$$+ 2 \sum_{j=n_c+1}^{n} \sum_{k=j+1}^{n+1} \gamma_j\gamma_k\phi_j(u)\phi_k(u)\omega_k^2[\frac{A_{jk}I_{1j}}{r_1^2} + B_{jk}I_{2j} + C_{jk}I_{1k} + D_{jk}I_{2k}] \tag{80}$$

$$R_{12} = 2 \sum_{k=1}^{n_c} \sum_{j=n_c+1}^{n+1} \gamma_j\phi_j(u)\phi_k(u) \sum_{m=1}^{n_c} \omega_m[\frac{J_{jm}I_{1j}}{r_3^2} + K_{jm}I_{2j} + L_{jm}I_{1m} + M_{jm}I_{2m}] \tag{81}$$

where r_1, r_2 and r_3 and the coefficients A_{jk}, B_{jk}, ... M_{jm} are defined in Appendix VI. The terms $a_{j\ell}$, $a_{k\ell}$ and $Z_{\ell jk}$ are defined in Appendix VI in terms of the complex-valued eigen properties of the Eq. (60). I_{1j} and I_{2j}, respectively, are the mean square values of the pseudo velocity and relative velocity responses of an oscillator with parameter ω_j and β_j, excited by the design ground motion. These mean square values can be defined in terms of the pseudo velocity and relative velocity ground response spectrum values as:

$$I_{1j} = \int_{-\infty}^{\infty} \Phi_g(\omega)\omega_j^2|H_j(\omega)|^2 d\omega = R_{pj}^2/C_{pj}^2 \tag{82}$$

$$I_{2j} = \int_{-\infty}^{\infty} \Phi_g(\omega)\omega^2|H_j(\omega)|^2 d\omega = R_{rj}^2/C_{rj}^2 \tag{83}$$

In Eqs. (82) and (83), R_{pj} and R_{rj}, respectively, are the pseudo velocity and relative velocity ground response spectrum values at frequency ω_j and damping ratio β_j; and C_{pj} and C_{rj} are the peak factors associated with the pseudo velocity and relative velocity responses of the oscillator. $H_j(\omega)$ is the frequency response function, as defined in Appendix III.

The peak factors can be approximately obtained by several of the available methods. The calculation of these factors requires knowledge of the spectral density function of the process. However, it has been observed in seismic response studies made earlier [18] that all the peak factors involved in Eqs. (78), (82) and (83) can be assumed equal, without affecting the results much. This assumption makes Eq. (78) independent of the peak factor values. The numerical results demonstrating the acceptability of this assumption are presented in the following section.

5. NUMERICAL RESULTS

To illustrate the application of the proposed approach, the structure shown in Figure 1 is analyzed to obtain the floor response spectra for various cases. The floor mass is taken equal to 1.2 kips-sec^2/in for the first floor and equal to 1.0 kips/sec^2/in for the remaining floors. The flexural stiffness is 2000 kips/in for the first story and 1800 kips/in for the remaining stories. The modal damping ratio for the structure is assumed to be .05 for all modes. The natural frequencies of the structure are given in Table 4.

In Figure 2 are shown the acceleration floor response spectra curves for floor 10, obtained by the proposed approach, for the equipment to floor mass ratio, r, being equal to 1/2, 1/20 and 1/200. The abscissa in this and in all other spectrum curves is in terms of the oscillator or equipment period. To evaluate the applicability of the proposed approach, a comparison of a floor response spectrum obtained by the proposed approach with the corresponding floor spectrum obtained by a time history ensemble analysis is shown in Figure 4. For development of the spectrum obtained by the proposed approach, the seismic input was defined by the averaged pseudo acceleration and relative velocity ground response spectra obtained for an ensemble of 75 synthetically generated acceleration time histories. The synthetic time histories with frequency characteristics defined by a broad-band Kanai-Tajimi type of spectral density function were generated by a standard randomly phased harmonic summation process. The time histories were also modulated by a deterministic time function with a build-up phase of 2 seconds, strong motion phase of 4 seconds and a decaying phase of 9 seconds. The averaged spectra of these time histories, which were used as seismic input in the proposed approach, are shown in Figure 3. Whereas the time history floor spectrum in Figure 4 is the average of the 75 floor spectra obtained for the very same ensemble of 75 time histories. Thus, the inputs used for development of the two spectra in Figure 4 are consistent. The floor spectrum for acceleration time histories was obtained by integrating Eqs. (59) and (65) for each mode by a Duhamel integral approach assuming linear variation between data points. The development of this approach for the nonclassically damped case is given in Reference 19. The comparison of the two spectra is seen to be very good. This verifies the applicability of the proposed approach in spite of the assumption of the stationarity of earthquake and equality of the peak factors made in the development.

To show the importance of the effect of coupling through damping on the equipment response, an oscillator mass of 1/200 of the floor mass with a damping ratio of .005 is considered. The modal damping ratio of the structure is taken to be .095 for all modes; this has been chosen to be rather on the high side to accentuate the effect of damping coupling. The continuous curve in Figure 5 shows the spectrum in which the damping coupling effect has been neglected; that is, it is assumed that the combined damping matrix can be diagonalized by the combined undamped modes of the structure-equipment system. The spectrum values shown by circles in the same figure are obtained with a proper consideration of the nonclassicality of the damping matrix. The discrepancy between the two values can be noted. It may thus be necessary to include this effect in a floor spectrum generation process, especially at the oscillator frequencies which are in tune with one of the structural frequencies. Similar observations have been made earlier in Reference 7.

The effect of the dynamic interaction between the equipment and structure on the floor spectra is shown in Figure 6. The results reported here are for an equipment with the mass ratio of 1/10. The discontinuous curve in this figure is obtained by the conventional methods [18] where the two systems are assumed decoupled and the effect of the interaction is ignored. The continuous curve on the other hand has been obtained by the proposed approach with a proper consideration of the interaction effect. It is noted that the effect of disregarding the interaction leads to an over-estimation of floor response spectrum. Again this effect is most pronounced at the frequencies where the equipment is tuned to one of the dominant modes of the structure.

6. SUMMARY AND CONCLUSIONS

The paper presents a method for generation of floor response spectra which incorporate the effect of the dynamic interaction between the primary structure and supported equipment. First, the dynamic properties of the combined system are obtained via a mode synthesis approach wherein the modal properties of the two components are used. The approach requires a second eigenvalue analysis. To reduce the computational cost associated with this second eigenvalue analysis, techniques are presented to reduce the size of the eigenvalue problem in which only a few selected modes of the primary system are used. The effect of the truncated modes is taken into account through a dynamic transformation. Next, a direct method for generation of floor spectra with the seismic input defined in terms of pseudo and relative velocity spectra, is presented. This step employs the combined eigen properties of the system. The nonclassicality of the damping matrix of the combined system is also considered in the formulation.

The method can deal with light as well as with heavy equipment, since no assumption concerning small variations in the original structural frequencies and modes of the primary structure is made. If the equipment is tuned to a cluster of structural frequencies, all of them will contribute to affect the new mode shapes and frequencies of the combined system. In the proposed approach, the effect of all such modes in a cluster can be simultaneously considered.

From the numerical examples studied, it is observed that the dynamic interaction plays an important role when the equipment frequency is tuned or nearly tuned to some structural frequencies, or when its mass is not negligible compared to the floor mass. The commonly used classical floor response spectra may be too conservative for such tuned and heavy equipment.

The effect of the modal coupling due to the non-proportionality of damping of the combined system is also found to be quite important especially when the difference between the damping ratio of the primary and secondary systems is large and the oscillator is tuned to some structural frequency.

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Table 1. Relative values of the elements of the modal damping matrix of the combined system: equipment tuned to the lowest frequency.

(a) Equipment damping: 0.02

Mass ratio r_m	$\hat{c}_{11}/\hat{c}_{12}$	$\hat{c}_{13}/\hat{c}_{11}$	$\hat{c}_{14}/\hat{c}_{11}$	$\hat{c}_{12}/\hat{c}_{22}$	$\hat{c}_{23}/\hat{c}_{22}$
1/2	0.552	0.011	0.056	0.333	0.028
1/20	0.465	0.006	0.018	0.395	0.008
1/200	0.434	0.002	0.006	0.418	0.002
1/2000	0.431	0.001	0.002	0.427	0.001
1/20000	0.431	0.000	0.001	0.426	0.000

(b) Equipment damping: 0.05

Mass ratio r_m	$\hat{c}_{11}/\hat{c}_{12}$	$\hat{c}_{13}/\hat{c}_{11}$	$\hat{c}_{14}/\hat{c}_{11}$	$\hat{c}_{12}/\hat{c}_{22}$	$\hat{c}_{23}/\hat{c}_{22}$
1/2	0.010	0.166	0.180	0.005	0.150
1/20	0.001	0.051	0.054	0.001	0.049
1/200	0.000	0.016	0.007	0.000	0.016
1/2000	0.000	0.005	0.005	0.000	0.005
1/20000	0.000	0.002	0.002	0.000	0.002

Table 2 - Relative values of the elements of the modal damping matrix of the combined system: mass ratio = 1/10000 - Equipment damping = 0.01.

Equipment Frequency	$\hat{c}_{12}/\hat{c}_{11}$	$\hat{c}_{21}/\hat{c}_{22}$	$\hat{c}_{23}/\hat{c}_{22}$	$\hat{c}_{32}/\hat{c}_{33}$	$\hat{c}_{34}/\hat{c}_{33}$	$\hat{c}_{43}/\hat{c}_{44}$
ω_1	0.676	0.657	0.001	0.000	0.000	0.000
ω_2	0.003	0.002	0.669	0.664	0.003	0.001
ω_3	0.000	0.000	0.004	0.004	0.668	0.668

Table 3 - Relative values of the elements of the modal damping matrix of the combined system: mass ratio = 1/2000 - Equipment damping = 0.005.

Equipment Frequency	$\hat{c}_{12}/\hat{c}_{11}$	$\hat{c}_{21}/\hat{c}_{22}$	$\hat{c}_{23}/\hat{c}_{22}$	$\hat{c}_{32}/\hat{c}_{33}$	$\hat{c}_{34}/\hat{c}_{33}$	$\hat{c}_{43}/\hat{c}_{44}$
ω_1	0.912	0.888	0.004	0.001	0.000	0.000
ω_2	0.008	0.005	0.904	0.896	0.011	0.004
ω_3	0.000	0.000	0.010	0.012	0.889	0.889

Table 4. Natural Frequencies of Primary Structure.

Frequency No.	Natural Frequency (rad/sec)
1	6.3990
2	18.9961
3	31.0110
4	42.1542
5	52.2907
6	61.3871
7	69.3352
8	75.8963
9	80.7992
10	83.8283

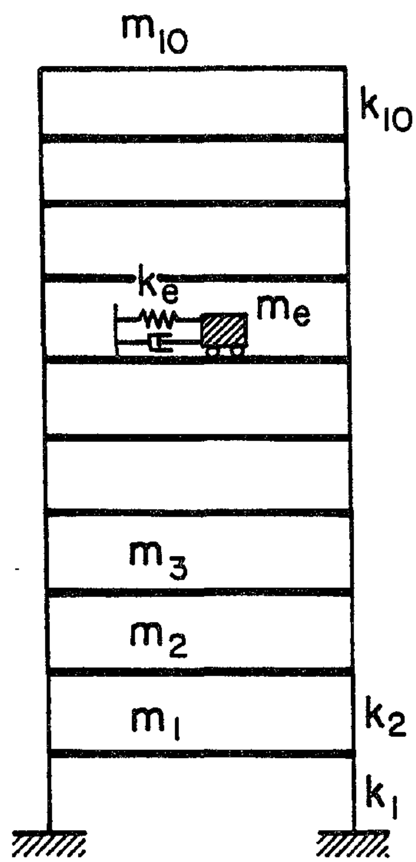

Fig. 1. COMBINED STRUCTURE AND SINGLE DEGREE-OF-FREEDOM OSCILLATOR SYSTEM

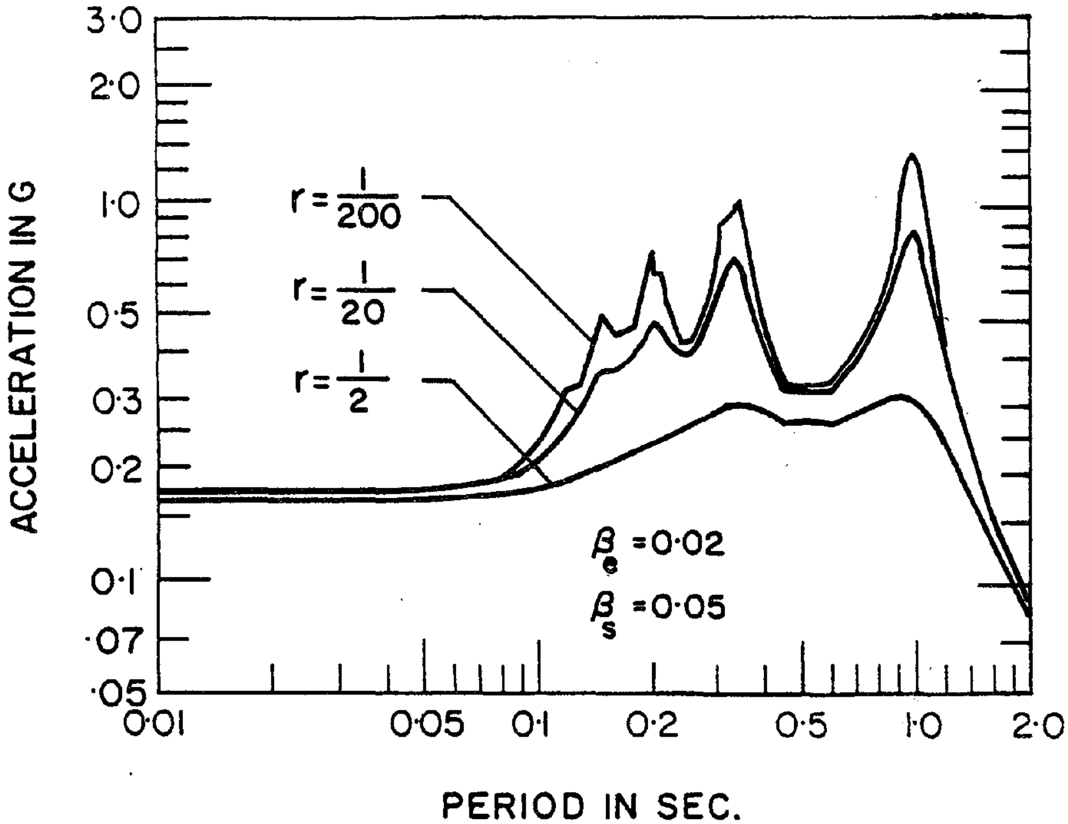

Fig. 2. FLOOR RESPONSE SPECTRA FOR FLOOR 10 FOR DIFFERENT MASS RATIOS

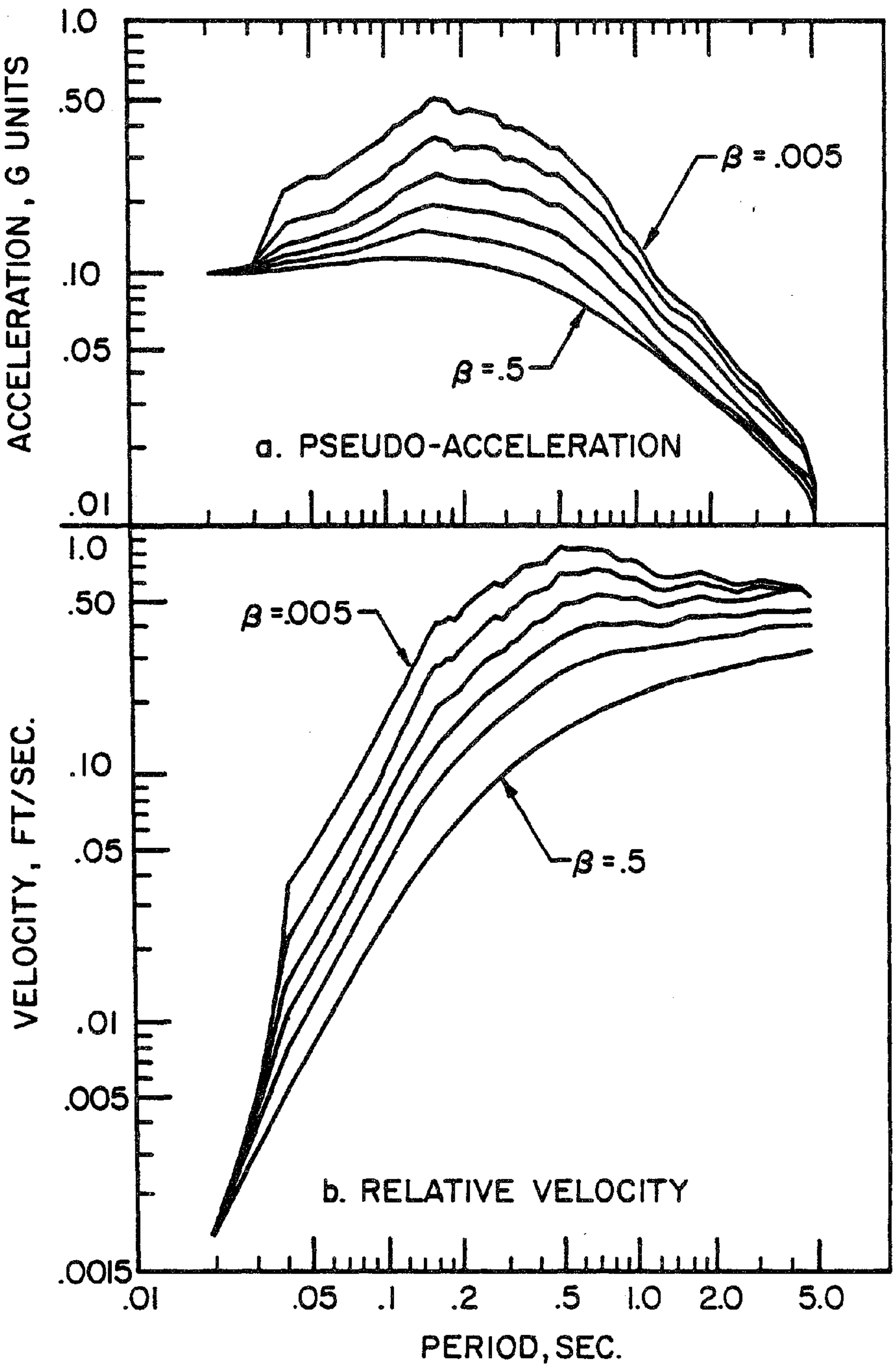

Fig. 3. INPUT GROUND RESPONSE SPECTRA FOR .005, .02, .05, .10, .2 AND .5 PERCENT DAMPING RATIOS

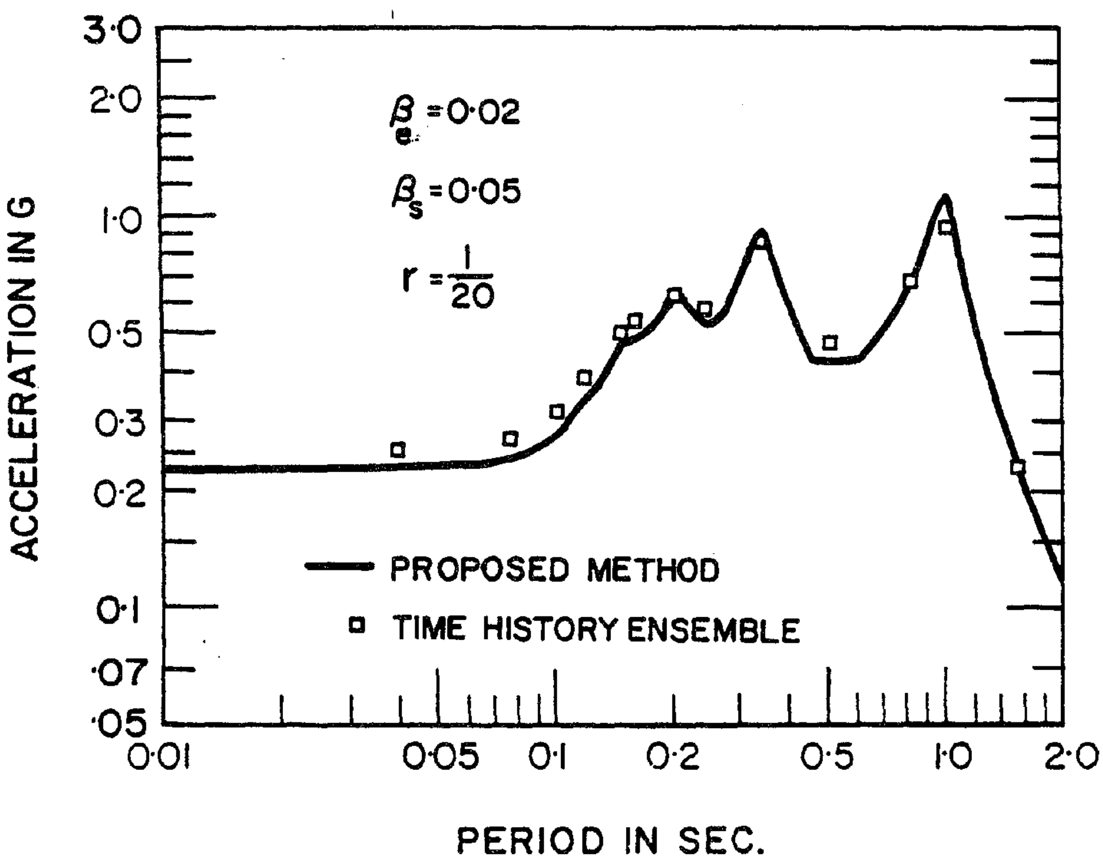

Fig. 4. COMPARISON OF THE FLOOR RESPONSE SPECTRUM FOR FLOOR 10 OBTAINED BY THE PROPOSED APPROACH WITH THE SPECTRUM OBTAINED BY TIME HISTORY ENSEMBLE ANALYSIS

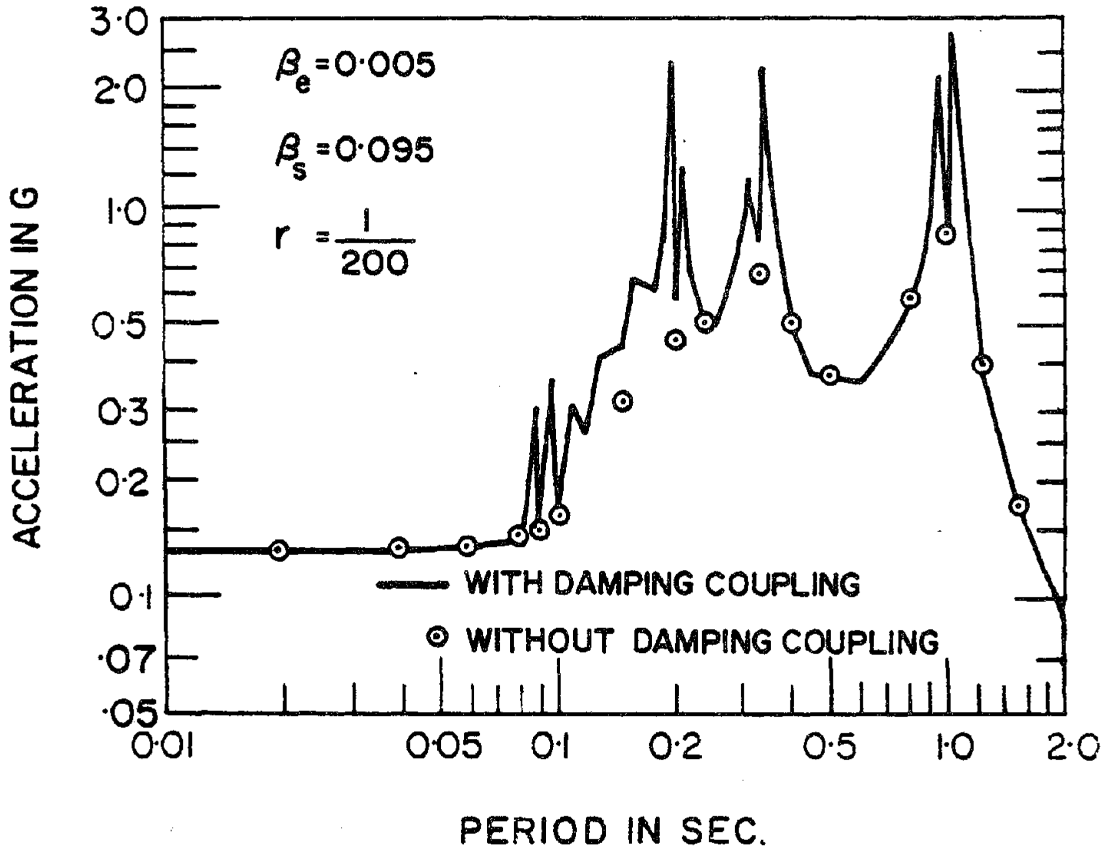

Fig. 5. FLOOR RESPONSE SPECTRA OBTAINED WITH AND WITHOUT THE EFFECT OF THE DAMPING COUPLING IN MODES

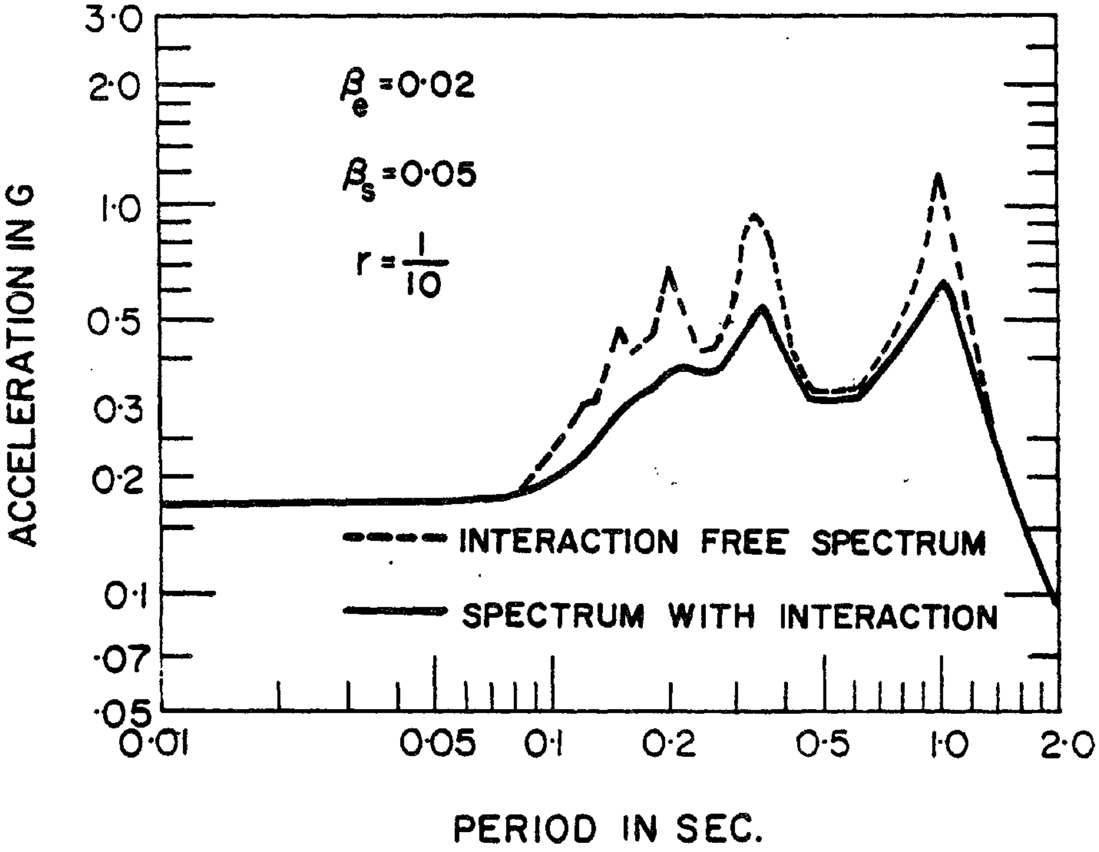

Fig. 6. FLOOR RESPONSE SPECTRA OBTAINED WITH AND WITHOUT THE EFFECT OF DYNAMIC INTERACTION BETWEEN STRUCTURE AND EQUIPMENT

APPENDIX I

APPROXIMATE EXPRESSIONS FOR THE DETUNED EIGENVECTORS

From the last row of the eigenvalue problem of Eq. (14) we obtain:

$$m_e \omega_e^2 \nu_{n+1} \underset{\sim}{\nu}^T \underset{\sim}{\phi}_j^* = \omega_j^2 \phi_{n+1,j}^* \qquad (I.1)$$

Considering the definition of vector $\underset{\sim}{\nu}$, Eq. (12), we can write:

$$\omega_e^2 \phi_{n+1,j}^* - \sqrt{m_e}\ \omega_e^2 \sum_{\ell=1}^{n} \phi_{k,\ell}^{(p)} \phi_{\ell,j}^* = \omega_j^2 \phi^*_{n+1,j} \qquad (I.2)$$

Introducing the definition

$$\Delta_j = \sum_{\ell=1}^{n} \phi_{k,\ell}^{(p)} \phi_{\ell,j}^* \qquad (I.3)$$

and solving for $\phi_{n+1,j}^*$ we obtain:

$$\phi_{n+1,j}^* = \frac{\sqrt{m_e}\ \omega_e^2\ \Delta_j}{\omega_e^2 - \omega_j^2} \qquad (I.4)$$

For light equipment, the term $\sqrt{m_e}\ \Delta_j$ and, consequently, $\phi_{n+1,j}^*$ are small quantities of order equal to the square root of the equipment-to-floor mass ratio. It is noted that the tuned case (i.e. when $\omega_j \simeq \omega_e$) is not being considered here as the modes corresponding to the tuned eigenvalues were already included in the reduced eigenproblem (20).

The ith row of the eigenvalue problem of Eq. (14) is

$$\omega_{pi}^2 \phi_{i,j}^* + m_e \omega_e^2 \nu_i \underset{\sim}{\nu}^T \underset{\sim}{\phi}_j^* = \omega_j^2 \phi_{i,j}^* \quad ; \quad i = 1,\ldots,n \qquad (I.5)$$

and with the definitions of vector $\underset{\sim}{\nu}$ and Δ_j we obtain

$$\omega_{pi}^2 \phi_{i,j}^* + \omega_e^2 \phi_{k,i}^{(p)} (m_e \Delta_j - \sqrt{m_e}\ \phi_{n+1,j}^*) = \omega_j^2 \phi_{i,j}^* \quad ; \quad i = 1,\ldots,n \quad (I.6)$$

Substituting for $\phi^{\star}_{n+1,j}$ from Eq. (I.4) it follows that

$$\omega_{pi}\phi^{\star}_{i,j} - m_e\omega_e^2\Delta_j\phi^{(p)}_{k,i}\frac{\omega_j^2}{\omega_e^2-\omega_j^2} = \omega_j^2\phi^{\star}_{i,j} \quad ; \quad i = 1,..,n \qquad (I.7)$$

In the second term in the left hand side of the above expression, the term $m_e\Delta_j\phi^{(p)}_{k,i}$ is of order equal to the ratio equipment-to-floor mass. Since we are seeking approximate expressions for the detuned eigenvectors that are correct up to terms of the order equal to the square root of the mass ratio, we neglect the second term in Eq. (I.7) and obtain:

$$\omega_{pi}^2\phi^{\star}_{i,j} = \omega_j^2\phi^{\star}_{i,j} \; ; \quad i = 1,...,n \qquad (I.8)$$

If we assume that the frequencies of those modes far from the equipment frequency remain unaffected by the addition of the equipment we have that:

$$\omega_{pj}^2 \simeq \omega_j^2 \quad ; \quad j = n_r+1,...,n+1 \qquad (I.9)$$

and therefore, for $i \neq j$

$$\phi^{\star}_{i,j} \simeq 0 \qquad (I.10)$$

and for $i = j$

$$\phi^{\star}_{j,j} \simeq 1 \qquad (I.11)$$

The complete eigenvector $\underset{\sim}{\phi}^{\star}_j$ can now be written as

$$\underset{\sim}{\phi}^{\star}_j = \left\{ \begin{matrix} 0 \\ \vdots \\ 1 \\ \vdots \\ \phi^{\star}_{n+1,j} \end{matrix} \right\} \; ; \quad j = n_r+1,...,n+1 \qquad (I.12)$$

Note that the element $\phi^{\star}_{j,j}$ was set equal to 1 to render the eigenvector $\underset{\sim}{\phi}^{\star}_j$ approximately orthonormal in the sense

$$\underset{\sim}{\phi}^{\star T}_j \underset{\sim}{\phi}^{\star}_j = 1 + O(\varepsilon^2) \quad ; \quad j = n_r+1,...,n+1 \qquad (I.13)$$

where $O(\varepsilon^2)$ represents the terms of order equal to (or smaller than) the ratio equipment-to-floor mass.

APPENDIX II

CONVERGENCE OF THE MATRIX SERIES EXPANSION

The matrix series expansion of Eq. (34) has the form

$$[I-D]^{-1} = I + D + D^2 + \dots \qquad (II.1)$$

where, in our case, matrix [D] is:

$$[D] = [\diagdown(\omega_p^r)^{-2}\diagdown] \; [\lambda_j [I^r] - m_e \omega_e \underset{\sim}{\nu}^r \underset{\sim}{\nu}^{rT}] \qquad (II.2)$$

For any arbitrary matrix [D] the series expansion (II.1) is convergent provided that [20]

$$\rho(D) < 1 \qquad (II.3)$$

where $\rho(D)$ is the spectral radius of matrix [D], defined as follows [20]:

$$\rho(D) = \max_{1 \le i \le n} |\hat{\lambda}_i| \qquad (II.4)$$

in which $\hat{\lambda}_i$ are the eigenvectors of matrix [D].

We can rewrite matrix [D] of Eq. (II.2) in the following way:

$$[D] = \begin{bmatrix} \diagdown & & \\ & \dfrac{\lambda_j}{(\omega_p^r)^2} & \\ & & \diagdown \end{bmatrix} - \underset{\sim}{a} \, \underset{\sim}{\nu}^{rT} \qquad (II.5)$$

where the elements of vector $\underset{\sim}{a}$ are:

$$a_i = m_e \left(\frac{\omega_e}{\omega_{pi}^r}\right)^2 \nu_i^r \quad ; \quad i = 1, \dots, n_r \qquad (II.6)$$

In order to examine the convergence of the series (II.1) we need to estimate in some way the spectral radius $\rho(D)$ since, obviously, the eigenvalues of [D] are unknown. According to Reference 20, a simple upper bound for the spectral radius of [D] can be obtained as follows

$$\rho(D) \le \max_{1 \le i \le n_r} \sum_{\ell=1}^{n_r} |d_{i\ell}| \qquad (II.7)$$

where $d_{i\ell}$ are the elements of matrix [D]:

$$d_{ii} = \frac{\lambda_j}{(\omega_{pi}^r)^2} - m_e \left(\frac{\omega_e}{\omega_{pi}^r}\right)^2 (\nu_i^r)^2 \quad ; \quad i = 1,\ldots,n_r \qquad (II.8)$$

$$d_{i\ell} = -m_e \left(\frac{\omega_e}{\omega_{pi}^r}\right)^2 \nu_i^r \nu_\ell^r \quad ; \quad i \ne \ell \qquad (II.9)$$

Therefore, the condition for convergence can be written as

$$\rho(D) \le \max_{1 \le i \le n_r} \left[\frac{\lambda_j}{(\omega_{pi}^r)^2} + m_e \left(\frac{\omega_e}{\omega_{pi}^r}\right)^2 |\nu_i^r| \sum_{\ell=1}^{n_r} |\nu_\ell^r|\right] \le 1 \quad (II.10)$$

From the condition of Eq. (31), we can write

$$\lambda_j < (\omega_{pi}^r)^2 \quad ; \quad j = 1,\ldots,n_k \; ; \; i = 1,\ldots,n_r \qquad (II.11)$$

In particular, we can take $\lambda_j \simeq (\omega_{pnk}^k)^2$ as the largest value of the set of eigenvalues λ_j and ω_{pi}^r equal to ω_{p1}^r, the smallest value in the set ω_{pi}^r to consider the most stringent case in condition (II.10). We can also simplify the condition (II.10) writing

$$\max_{1 \le i \le n_r} |\nu_i^r| \sum_{\ell=1}^{nr} |\nu_\ell^r| < n_r \max_{1 \le i,\ell \le n_r} |\nu_i^r \nu_\ell^r| \qquad (II.12)$$

With these considerations, Eq. (II.10) becomes

$$\rho(D) \le \left(\frac{\omega_{pnk}^k}{\omega_{p1}^r}\right)^2 + m_e \left(\frac{\omega_e}{\omega_{p1}^r}\right)^2 n_r \max_{1 \le i,\ell \le n_r} |\nu_i^r \nu_\ell^r| \le 1 \qquad (II.13)$$

Or:

$$(\omega_{nk}^{k})^2 + n_r m_e \omega_e^2 \max |\nu_i^r \nu_\ell^r| \leq (\omega_1^r)^2 \qquad \text{(II.14)}$$

where the subscript p was dropped to simplify the notation. It is emphasized that Eq. (II.14) is only a sufficient condition for the convergence of the series expansion (34). In many cases the series will converge even if condition (II.14) is not satisfied.

APPENDIX III

CONTRIBUTION OF THE COUPLED MODES TO THE FLOOR RESPONSE SPECTRUM

In this appendix the contribution of the first term in Eq. (77) is analyzed separately.

The first term in Eq. (77) can be written as:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \sum_{\ell=1}^{2n_c} \sum_{m=1}^{2n_c} \psi_\ell(j)\psi_m(k)p_\ell p_m E[z_\ell(t_1)z_m(t_2)] \quad \text{(III.1)}$$

Substituting the steady state solution of Eq. (65) in Eq. (III.1)

$$z_m(t) = F_m \int_0^t \ddot{X}_g(\tau)e^{p_m(t-\tau)} d\tau \quad \text{(III.2)}$$

and assuming that the ground motion can be represented as a stationary random process with a spectral density function $\Phi_g(\omega)$, Eq. (III.1) becomes

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \sum_{\ell=1}^{2n_c} \sum_{m=1}^{2n_c} q_{j\ell}q_{km} \int_{-\infty}^{\infty} \Phi_g(\omega) \int_0^{t_1} \int_0^{t_2}$$

$$e^{p_\ell(t_1-\tau_1)} e^{p_m(t_2-\tau_2)} e^{i\omega(\tau_1-\tau_2)} d\tau_1 d\tau_2 d\omega \quad \text{(III.3)}$$

where we introduced the notation:

$$q_{j\ell} = F_\ell \psi_j(\ell) p_\ell \quad \text{(III.4)}$$

Extending the inner summations to n_c only by combining the complex and conjugate terms, it can be shown that, for a stationary response, Eq. (III.3) becomes:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \sum_{\ell=1}^{n_c} \sum_{m=1}^{n_c} \int_{-\infty}^{\infty} [\frac{q_{j\ell}q_{km}}{(-p_\ell+i\omega)(-p_m-i\omega)} + \frac{\bar{q}_{j\ell}\bar{q}_{km}}{(-\bar{p}_\ell+i\omega)(-\bar{p}_m-i\omega)}$$

$$+ \frac{q_{j\ell}\bar{q}_{km}}{(-p_\ell+i\omega)(-\bar{p}_m+i\omega)} + \frac{\bar{q}_{j\ell}q_{km}}{(-\bar{p}_\ell+i\omega)(-p_m-i\omega)}] \; \Phi_g(\omega)e^{i\omega\tau}d\omega \quad \text{(III.5)}$$

where $\tau = t_1-t_2$ and a bar over a quantity denote its complex conjugate value. In order to put this equation in terms of modal response, the undamped natural frequencies and modal damping ratios of Eq. (60) are defined in terms of its complex eigenvalues p_ℓ as:

$$\omega_\ell = |p_\ell|; \; \beta_\ell = -\text{Real} \frac{p_\ell}{|p_\ell|} \; ; \quad \ell = 1,2,\ldots,n_c \qquad \text{(III.6)}$$

Next, combining the four terms in Eq. (III.5) and collecting real and imaginary parts, we obtain:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \sum_{\ell=1}^{n_c} \sum_{m=1}^{n_c} \int_{-\infty}^{\infty} [X_{\ell m}+iY_{\ell m}]H_\ell(\omega)H_m(\omega)\Phi_g(\omega)e^{i\omega\tau}d\omega \qquad \text{(III.7)}$$

where $H_\ell(\omega)$ is the frequency response function of a single degree-of-freedom oscillator defined as:

$$H_\ell(\omega) = \frac{1}{\omega_\ell^2 - \omega^2 + 2i\beta_\ell\omega_\ell\omega} \qquad \text{(III.8)}$$

and:

$$X_{\ell m} = \omega^2 D_1 + \omega_m^2 D_2 \; , \; Y_{\ell m} = 4\omega\omega_m E_3 \qquad \text{(III.9)}$$

with D_1, D_2 and E_3 defined as in Appendix VI.

Separating terms with $\ell = m$ from those with $\ell \neq m$, and discarding terms with odd powers of ω that vanish when the integration is carried out, Eq. (III.7) yields:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \; \{\sum_{\ell=1}^{n_c} \int_{-\infty}^{\infty} [D_1\omega^2 + 4Z_{\ell jk}\omega_\ell^2]|H_\ell(\omega)|^2\Phi_g(\omega)e^{i\omega\tau}d\omega$$

$$+ 2 \sum_{\ell=1}^{n_c-1} \sum_{m=\ell+1}^{n_c} \int_{-\infty}^{\infty} [D_1\omega^6 + (C_1D_1+D_2+E_1)\omega_m^2\omega^4 + (C_1D_2+r_2^2D_1+E_2)\omega_m^4\omega^2$$

$$+ r_2^2D_2\omega_m^6]|H_\ell(\omega)|^2|H_m(\omega)|^2\Phi_g(\omega)e^{i\omega\tau}d\omega\} \qquad \text{(III.10)}$$

where the coefficient $Z_{\ell jk}$, C_1, E_1, E_2 and r_2 are defined in Appendix VI. Expanding in partial fractions the integrand in the second term of Eq. (III.10) and setting $\tau = 0$ to obtain the contribution to the mean square response we obtain:

$$R_1 = \sum_{j=1}^{n_c} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u) \; \{\sum_{\ell=1}^{n_c} [D_1 \int_{-\infty}^{\infty} \Phi_g(\omega)\omega^2|H_\ell(\omega)|^2d\omega + 4Z_{\ell jk} \int_{-\infty}^{\infty} \Phi_g(\omega)\omega_\ell^2|H_\ell(\omega)|^2d\omega]$$

$$+ 2 \sum_{\ell=1}^{n_c-1} \sum_{m=\ell+1}^{n_c} \int_{-\infty}^{\infty} [(E_{\ell m}\omega_m^2+F_{\ell m}\omega^2)|H_\ell(\omega)|^2 + (G_{\ell m}\omega_m^2+H_{\ell m}\omega^2)|H_m(\omega)|^2]\Phi_g(\omega)d\omega\} \qquad \text{(III.11)}$$

where the coefficients $E_{\ell m}$, etc., are defined in Appendix VI.

Finally, considering the definitions of pseudo and relative velocity response spectra given by Eqs. (82) and (83), the contribution R_1 from the coupled modes to R_a can be expressed as in Eq. (79).

APPENDIX IV

CONTRIBUTION OF THE UNCOUPLED MODES TO THE FLOOR RESPONSE SPECTRUM

We examine here the contribution to the mean square response from the second term in Eq. (77). Using Eq. (75) we can write:

$$R_2 = \sum_{j=n_c+1}^{n+1} \sum_{k=nc+1}^{n+1} \phi_j(u)\phi_k(u)\{\omega_j^2\omega_k^2 E[\eta_j(t_1)\eta_k(t_2)] + 4\beta_j\beta_k\omega_j\omega_k E[\dot{\eta}_j(t_1)\dot{\eta}_k(t_2)]$$

$$+ 2\beta_j\omega_j\omega_k^2 E[\dot{\eta}_j(t_1)\eta_k(t_2)] + 2\beta_k\omega_k\omega_j^2 E[\eta_j(t_1)\dot{\eta}_k(t_2)]\} \qquad \text{(IV.1)}$$

Considering the solution of Eq. (59):

$$\eta_j(t) = -\gamma_j \int_0^t \ddot{X}_g(t)h_j(t-\tau)d\tau \qquad \text{(IV.2)}$$

where $h_j(t-\tau)$ is the impulse response function, the first expected value in Eq. (IV.1) can be written as

$$E[\eta_j(t_1)\eta_k(t_2)] = \int_0^{t_1}\int_0^{t_2} E[\ddot{X}_g(\tau_1)\ddot{X}_g(\tau_2)]h_j(t_1-\tau_1)h_k(t_2-\tau_2)d\tau_1 d\tau_2 \qquad \text{(IV.3)}$$

Expressing the autocorrelation function of the ground motion in terms of its power spectral density function and considering the stationary response, it can be shown that Eq. (IV.3) becomes:

$$E[\eta_j(t_1)\eta_k(t_2)] = \int_{-\infty}^{\infty} \Phi_g(\omega)H_j(\omega)\overline{H}_k(\omega)e^{i\omega\tau}d\omega \qquad \text{(IV.4)}$$

where $\tau = t_2-t_1$. From Eq. (IV.4) it follows that:

$$E[\dot{\eta}_j(t_1)\dot{\eta}_k(t_2)] = \int_{-\infty}^{\infty} \omega^2\Phi_g(\omega)H_j(\omega)\overline{H}_k(\omega)e^{i\omega\tau}d\omega \qquad \text{(IV.5)}$$

$$E[\dot{\eta}_j(t_1)\eta_k(t_2)] = -E[\eta_j(t_1)\dot{\eta}_k(t_2)] = i \int_{-\infty}^{\infty} \omega\Phi_g(\omega)H_j(\omega)\overline{H}_k(\omega)e^{i\omega\tau}d\omega \quad \text{(IV.6)}$$

Substituting Eqs. (IV.4)-(IV.6) in Eq. (IV.1) and evaluating separating the terms with $j = k$ and $j \neq k$ with $\tau = 0$, we obtain:

$$R_1 = \sum_{j=n_c+1}^{n+1} \gamma_j^2\phi_j^2(u) \int_{-\infty}^{\infty} (4\beta_j^2\omega_j^2\omega^2+\omega_j^4)\Phi_g(\omega)|H_j(\omega)|^2 d\omega$$

$$+ 2 \sum_{j=nc+1}^{n} \sum_{k=j+1}^{n+1} \gamma_j\gamma_k\phi_j(u)\phi_k(u)\omega_k^2 \int_{-\infty}^{\infty} (W_4\omega^6+W_3\omega_k^2\omega^4+W_2\omega_k^4\omega^2$$

$$+ W_1\omega_k^6)|H_j(\omega)|^2|H_k(\omega)|^2\Phi_g(\omega)d\omega \quad \text{(IV.7)}$$

where the coefficients W_1, W_2, W_3 and W_4 are defined by Eq. (VI.2) in Appendix VI. Solving the integrand in the second term of the above equation in partial fractions we can write

$$R_1 = \sum_{j=n_c+1}^{n+1} \gamma_j^2\phi_j^2(u)\omega_j^2 \int_{-\infty}^{\infty} (\omega_j^2+4\beta_j^2\omega^2)\Phi_g(\omega)|H_j(\omega)|^2 d\omega$$

$$+ 2 \sum_{j=n_c+1}^{n} \sum_{k=j+1}^{n+1} \gamma_j\gamma_k\phi_j(u)\phi_k(u)\omega_k^2 \int_{-\infty}^{-\infty} [(A_{jk}\omega_k^2+B_{jk}\omega^2)|H_j(\omega)|^2$$

$$+ (C_{jk}\omega_k^2+D_{jk}\omega^2)|H_k(\omega)|^2]\Phi_g(\omega)d\omega \quad \text{(IV.8)}$$

where the coefficients A_{jk}, etc., are obtained as explained in Appendix VI. From Eqs. (82) and (83), it follows that that contribution of the uncoupled modes to the floor response spectrum can be written as in Eq. (80).

APPENDIX V

CONTRIBUTION OF THE CROSS TERMS TO THE FLOOR RESPONSE SPECTRUM

In this appendix the contribution of the last term in Eq. (77) will be expressed in terms of modal response spectra values.

The first part of the third term in Eq. (77) can be written as:

$$R_{12}' = - \sum_{j=n_c+1}^{n+1} \sum_{k=1}^{n_c} \phi_j(u)\phi_k(u)\{\omega_j^2 \sum_{m=1}^{2n_c} \psi_m(k)p_m E[\eta_j(t_1)z_m(t_2)]$$

$$+ 2\beta_j\omega_j \sum_{m=1}^{2n_c} \psi_m(k)p_m E[\dot{\eta}_j(t_1)z_m(t_2)]\} \qquad (V.1)$$

With $\eta_j(t_1)$ and $Z_m(t_2)$ given by Eqs. (IV.2) and (III.2) respectively, the first expected value in Eq. (V.1) becomes:

$$E[\eta_j(t_1)z_m(t_2)] = -\gamma_j F_m \int_0^{t_1}\int_0^{t_2} E[\ddot{X}_g(\tau_1)\ddot{X}_g(\tau_2)]e^{p_m(t_2-\tau_2)} h_j(t_1-\tau_1)d\tau_1 d\tau_2 \qquad (V.2)$$

Expressing the autocorrelation of the ground motion in terms of its spectral density function for the stationary response, Eq. (V.2) can be written as:

$$E[\eta_j(t_1)z_m(t_2)] = -\gamma_j F_m \int_{-\infty}^{\infty} \frac{H_j(\omega)}{(-p_m-i\omega)} \Phi_g(\omega)e^{i\omega\tau}d\omega \qquad (V.3)$$

and similarly, the second expected value term in Eq. (V.1) can be shown to be:

$$E[\dot{\eta}_j(t_1)z_m(t_2)] = -i\gamma_j F_m \int_{-\infty}^{\infty} \frac{\omega H_j(\omega)}{(-p_m-i\omega)} \Phi_g(\omega)e^{i\omega\tau}d\omega \qquad (V.4)$$

With Eqs. (V.3) and (V.4), the terms in the curled brackets in Eq. (V.1) become:

$$\hat{R}_{12} = -\gamma_j \sum_{m=1}^{2n_c} \int_{-\infty}^{\infty} \frac{q_{km}}{(-p_m - i\omega)} (\omega_j^2 + i2\beta_j\omega_j\omega) H_j(\omega)\Phi_g(\omega) e^{i\omega\tau} d\omega \quad (V.5)$$

Combining the complex and conjugate terms in Eq. (V.5), we obtain

$$\hat{R}_{12} = -\gamma_j \sum_{m=1}^{n_c} \int_{-\infty}^{\infty} \left[\frac{q_{km}}{(-p_m - i\omega)} + \frac{\bar{q}_{km}}{(-\bar{p}_m - i\omega)}\right] (\omega_j^2 + i2\beta_j\omega_j\omega) H_j(\omega)\Phi_g(\omega) e^{i\omega\tau} d\omega \quad (V.6)$$

After some manipulations, the term in square brackets can be written as:

$$\frac{q_{km}}{-p_m - i\omega} + \frac{\bar{q}_{km}}{-\bar{p}_m - i\omega} = [(A_2\omega_m\omega^2 + A_0\omega_m^3) + i(2a_{km}\omega^3 + A_1\omega_m^2\omega)]|H_m(\omega)|^2 \quad (V.7)$$

where A_0, A_1 and A_2 are given in Appendix VI. It can also be shown that:

$$(\omega_j^2 + i2\beta_j\omega_j\omega) H_j(\omega) = [(4\beta_j^2 - 1)\omega_j^2\omega^2 + \omega_j^4 + i(-2\beta_j\omega_j\omega^3)]|H_j(\omega)|^2 \quad (V.8)$$

Substituting Eqs. (V.7) and (V.8) in (V.6) we obtain:

$$\hat{R}_{12} = -\gamma_j \sum_{m=1}^{n_c} \omega_m \int_{-\infty}^{\infty} (W_4\omega^6 + W_3\omega_m^2\omega^4 + W_2\omega_m^4\omega^2 + W_1\omega_m^6)|H_j(\omega)|^2|H_m(\omega)|^2\Phi_g(\omega)e^{i\omega\tau}d\omega \quad (V.9)$$

where W_1, W_2, etc., are given by Eqs. (V.4) in Appendix VI. Expanding the above integrand in partial fractions, the contribution of the first part of the third term in Eq. (77) is:

$$R'_{12} = \sum_{j=n_c+1}^{n+1} \sum_{k=1}^{n_c} \gamma_j\phi_j(u)\phi_k(u) \sum_{m=1}^{n_c} \omega_m \int_{-\infty}^{\infty} [(J_{jm}\omega_m^2 + K_{jm}\omega^2)|H_j(\omega)|^2 + (L_{jm}\omega_m^2 + M_{jm}\omega^2)|H_m(\omega)|^2]\Phi_g(\omega)d\omega \quad (V.10)$$

where the constants are obtained as indicated in Appendix VI.

If the second part of the third term in Eq. (77) is analyzed in the same way as above, it leads to an expression identical to Eq. (V.10). Considering Eqs. (82) and (83) and combining the contribution of the first and second part of the last term of Eq. (77), we obtain R_{12} as in Eq. (81).

APPENDIX VI

COEFFICIENTS OF PARTIAL FRACTIONS

The coefficients A_{jk}, B_{jk}, etc. in Eqs. (79-81) are obtained from the solution of the following equations:

$$\begin{bmatrix} 1 & 0 & x & 0 \\ y & 1 & z & x \\ 1 & y & 1 & z \\ 0 & 1 & 0 & 1 \end{bmatrix} \left\{ V_{jk} \right\} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} \qquad \text{(VI.1)}$$

Three different cases must be considered to obtain all the partial coefficients. To obtain $A_{jk}, \ldots D_{jk}$, we solve Eq. (VI.1) for vector V_{jk} with following values:

$$\begin{cases} r_1 = \omega_j/\omega_k \\ x = r_1^4 \\ y = -2(1-2\beta_k^2) \\ z = -2(1-2\beta_j^2)r_1^2 \end{cases} \qquad \begin{cases} W_1 = r_1^4 \\ W_2 = r_1^2[4(\beta_j^2 + r_1^2\beta_k^2)-r_1^2-1] \\ W_3 = r_1^2[16\beta_j^2\beta_k^2-4(\beta_j^2+\beta_k^2)+1] \\ W_4 = 4r_1\beta_j\beta_k \end{cases} \qquad \text{(VI.2)}$$

To obtain $E_{\ell m}, \ldots H_{\ell m}$ we solve Eq. (VI.1) for vector V_{jk} with following values:

$$\begin{cases} r_2 = \omega_\ell/\omega_m \\ x = r_2^4 \\ y = -2(1-2\beta_m^2) \\ z = -2(1-2\beta_\ell^2)r_2^2 \end{cases} \qquad \begin{cases} W_1 = r_2^2 D_2 \\ W_2 = C_1 D_2 + r_2^2 D_1 + E_2 \\ W_3 = C_1 D_1 + D_2 + E_1 \\ W_4 = D_1 \end{cases} \qquad \text{(VI.3)}$$

To obtain $J_{jm}, \ldots, M_{jm}$, we solve Eq. (VI.1) for vector V_{jk} with following values:

$$\begin{cases} r_3 = \omega_j/\omega_m \\ x = r_3^4 \\ y = -2(1-2\beta_m^2) \\ z = -2(1-2\beta_j^2)r_3^2 \end{cases} \qquad \begin{cases} W_1 = r_3^4 A_0 \\ W_2 = r_3^4 A_2 + A_0 A_3 \\ W_3 = A_2 A_3 + 2\beta_j r_3 A_1 \\ W_4 = 4\beta_j a_{km} r_3 \end{cases} \qquad \text{(VI.4)}$$

where:

$$\begin{cases} C_1 = -(1+r_2^2-4\beta_\ell\beta_m r_2) \ ; \ D_1 = 4a_{j\ell}a_{km} \\ D_2 = 4r_2[a_{j\ell}a_{km}\beta_\ell\beta_m+b_{j\ell}b_{km}\sqrt{1-\beta_\ell^2}\ \sqrt{1-\beta_m^2}\ -a_{j\ell}b_{km}\beta_\ell\sqrt{1-\beta_m^2}\ -a_{km}b_{j\ell}\beta_m\sqrt{1-\beta_\ell^2}\ ] \\ E_1 = -8(\beta_\ell r_2-\beta_m)E_3 \ ; \ E_2 = -8r_2(\beta_m r_2-\beta_\ell)E_3 \\ E_3 = a_{j\ell}a_{km}(\beta_m-\beta_\ell r_2)-a_{j\ell}b_{km}\sqrt{1-\beta_m^2}\ +a_{km}b_{j\ell}r_2\sqrt{1-\beta_\ell^2} \end{cases} \qquad \text{(VI.5)}$$

$$\begin{cases} A_0 = 2(a_{km}\beta_m-b_{km}\sqrt{1-\beta_m^2} \ ; \ A_1 = -2[a_{km}(1-2\beta_m^2)+2b_{km}\beta_m\sqrt{1-\beta_m^2}] \\ A_2 = 2(a_{km}\beta_m+b_{km}\sqrt{1-\beta_m^2}) \ ; \ A_3 = (4\beta_j^2-1)r_3^2 \end{cases} \qquad \text{(VI.6)}$$

$$\begin{cases} a_{j\ell} = \text{Real Part of } (q_{j\ell}) \\ b_{j\ell} = \text{Imaginary Part of } (q_{j\ell}) \end{cases} \qquad \text{(VI.7)}$$

$$Z_{\ell jk} = a_{j\ell}a_{k\ell}\beta_\ell(\beta_\ell - \sqrt{1-\beta_\ell^2}) + b_{j\ell}b_{k\ell}(1-\beta_\ell^2-\beta_\ell\sqrt{1-\beta_\ell^2}) \qquad \text{(VI.8)}$$

NOMENCLATURE

$[A]$ = $(2n_c x 2n_c)$ matrix of the eigenvalue problem associated with the coupled coordinates

A_0, A_1, A_2, A_3 = auxiliary constants

$A_{jk}, B_{jk}, C_{jk}, D_{jk}$ = coefficients of partial fractions associated with the uncoupled modes

$\underset{\sim}{a}$ = auxiliary vector used in the definition of [D]

$a_{j\ell}$ = real part of the quantity $q_{j\ell}$

a_i^* = ith complex constant

$[B]$ = $(2n_c x 2n_c)$ matrix of the eigenvalue problem associated with the coupled coordinates.

$b_{j\ell}$ = imaginary part of the quantity $q_{j\ell}$

$[C]$ = damping matrix of the combined system

$[C_c]$ = damping coupling matrix

$[C_p]$ = (nxn) damping matrix of the primary system

C_1, D_1, D_2, E_1, E_2 = auxiliary constants

C_{pj} = peak factor associated with the pseudo velocity response

C_{rj} = peak factor associated with the relative velocity response

$[C^*]$ = transformed damping matrix of the combined system

$[\hat{C}]$ = non-diagonal modal damping matrix of the combined system

$\hat{c}_{ij}$ = a generic element of matrix $[\hat{C}]$

$[D]$ = matrix in the power series expansion

$d_{i\ell}$ = a generic element of matrix [D]

E[...] = expected value of [...]

$E_{\ell m}, F_{\ell m}, G_{\ell m}, H_{\ell m}$ = coefficients of partial fractions associated with the coupled modes

F_j = jth complex-valued participation factor

$H_j(\omega)$ = frequency response function for an oscillator of frequency ω_j and damping ratio β_j

$h_j(\omega)$ = impulse response function for an oscillator of frequency ω_j and damping ratio β_j

$[I]$ = identity matrix

$[I^k]$ = $(n_k x n_k)$ identity matrix

$[I^r]$ = $(n_r x n_r)$ identity matrix

I_{1j} = mean square value of the pseudo velocity

I_{2j} = mean square value of the relative velocity

$J_{jm}, K_{jm}, L_{jm}, M_{jm}$ = coefficients of partial fractions associated with the cross terms

$[K]$ = $(n_k x n_k)$ reduced transformed stiffness matrix

$[K_c]$ = stiffness coupling matrix

$[K_p]$ = (nxn) stiffness matrix of the primary system

$[K^{kk}]$ = $(n_k x n_k)$ submatrix of [K*] associated with the kept coordinates

$[K^{kr}]$ = $(n_k x n_r)$ submatrix of [K*] composed of products of kept and reduced coordinates

$[K^{rr}]$ = $(n_k x n_r)$ submatrix of [K*] associated with the reduced coordinates

$[K^*]$ = transformed stiffness matrix of the combined system

$[M]$ = $(n_k x n_k)$ reduced transformed mass matrix

$[M_p]$ = (nxn) mass matrix of the primary system

$[M^*]$ = transformed mass matrix of the combined system

m_e = mass of the equipment

n = number of degrees of freedom of the primary system

n_c = number of strongly coupled principal coordinates due to the nonclassical damping effect

n_k = number of kept modal coordinates

n_r = number of reduced modal coordinates

p_f = peak factor

p_j = jth complex-valued eigenvalue

q_e = transformed coordinate associated with x_e

q_p = transformed coordinate vector associated with $\underset{\sim}{x}_p$

[R] = transformation matrix relating the reduced to the kept coordinates

R_a = floor response spectrum value

R_{pj} = pseudo velocity ground response spectrum

R_{rj} = relative velocity ground response spectrum

R_1 = part of R_a due to the contribution of the coupled modes

R_2 = part of R_a due to the contribution of the uncoupled modes

R_{12} = part of R_a due to the contribution of the cross terms

$\underset{\sim}{r}$ = displacement influence coefficient of the equipment

r_e = displacement influence vector of the primary system

r_m = ratio equipment mass-to-supporting floor mass

r_1, r_2, r_3 = frequency ratios

t, t_1, t_2 = time

[U] = auxiliary transformation matrix

W_1, W_2, W_3, W_4 = constants used to define the vector of independent coefficients for the system of equations of Appendix VI

X = auxiliary constant for the definition of [K]

$X_{\ell m}$ = auxiliary variable

$\ddot{X}_g(t)$ = ground motion

x_e = relative displacement of the equipment

$\underset{\sim}{x}_p$ = relative displacement vector corresponding to the dof's of the primary system

$\ddot{\underset{\sim}{x}}$ = relative acceleration vector of the combined system

$\ddot{\underset{\sim}{x}}_a$ = absolute acceleration vector of the combined system

$\ddot{x}_c(t)$ = contribution to $\ddot{x}_m(t)$ from the coupled modes

$\ddot{x}_e(t)$ = absolute acceleration of the equipment

$\ddot{x}_u(t)$ = contribution to $\ddot{x}_m(t)$ from the uncoupled modes

Y = auxiliary constant for the definition of [M]

$Y_{\ell m}$ = auxiliary variable

$\underset{\sim}{y}$ = $2n_c$-dimensional state vector

$Z_{\ell jk}$ = auxiliary constant

z_j = jth principal coordinate associated with the coupled modal coordinates

$\underset{\sim}{\alpha}$ = auxiliary n_r-dimensional vector

β_j = jth modal damping ratio of the combined system

β_{pj} = jth modal damping ratio of the primary system

$\underset{\sim}{\gamma}$ = vector of participation factors of the combined system

$\underset{\sim}{\gamma}_p$ = vector of participation factors of the primary system

Δ_j = constant used in the definition of $\phi^*_{n+1,j}$

$\underset{\sim}{\eta}$ = (n+1)-dimensional vector of principal coordinates

λ_j = jth eigenvalue of combined system

$\hat{\lambda}_j$ = jth eigenvalue of matrix [D]

$\underset{\sim}{\nu}$ = vector composed of the kth elements of the primary system eigenvector and the equipment eigenvector

$\underset{\sim}{\nu}^k$ = vector formed by the elements of $\underset{\sim}{\nu}$ associated with the n_k kept coordinates

$\underset{\sim}{\nu}^r$ = vector formed by the elements of $\underset{\sim}{\nu}$ associated with the n_r reduced coordinates

$\rho(D)$ = spectral radius of matrix [D]

τ = time difference

τ_1, τ_2 = dummy time variables

$[\Phi]$ = real-valued eigenvector matrix of the combined system

$\Phi_g(\omega)$ = spectral density function of the ground motion

$[\Phi_p]$ = mass-normalized eigenvector matrix of the primary system

$[\Phi^*]$ = real-valued eigenvector matrix of the transformed combined system

ϕ_e = equivalent eigenvector element associated with the equipment

$\underset{\sim}{\phi}_j^k$ = part of the eigenvector $\underset{\sim}{\phi}_j^*$ associated with the kept coordinates

$\underset{\sim}{\phi}_j^r$ = part of the eigenvector $\underset{\sim}{\phi}_j^*$ associated with the reduced coordinates

$\underset{\sim}{\phi}_j^*$ = jth eigenvector of the primary system

$\underset{\sim}{\psi}_j$ = jth complex eigenvector associated with the coupled modal coordinates

ω = variable frequency in rad/sec.

ω_e = natural frequency of the equipment

ω_j = jth natural frequency of the combined system

ω_{pj} = jth natural frequency of the primary system

ω_{pj}^k = natural frequency of the primary system corresponding to a kept modal coordinate

ω_{pj}^r = natural frequency of the primary system corresponding to a reduced modal coordinate