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Random Response of Turbine Structures under Seismic Excitation

by

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16. Abstract (Limit: 200 words)

The purpose of this study is to investigate the dynamic behavior of a wind turbine blade under seismic and turbulent wind excitations. Using the Markov process theory and Ito's stochastic differential equation, equations for statistical moments of blade response variables are derived. These equations then can be used to determine certain moment stability conditions for any given set of parameters, and moment responses if the system is stable. Results show that for a constant rpm wind turbine generator the uncoupled flapping, coupled flap-lagging, and coupled flap-lag-torsion of a wind turbine blade are very stable under normal operating conditions and that torsion has little influence on the dynamic behavior of flapping and leadlagging motions. If the system is stable, then the effect of turbulence on moment responses is greater than that of an earthquake; therefore, turbulence is likely the main cause for structural fatigue of wind turbine blades.

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NOMENCLATURE

Symbol		Description
a	:	airfoil lift curve slope
$\overline{A}$	:	constant matrix in the moment equation
b	:	number of blade
B	:	blade-tip loss factor
$\overline{B}$	:	linear matrix in the moment equation
c	:	blade airfoil chord
c_d	:	airfoil profile drag coefficient
C_t	:	thrust coefficient of wind turbine
C_Q	:	power coefficient of wind turbine
C_{ij}	:	damping coefficient
C_{ijo}	:	deterministic term in C_{ij}
C_{ijx}	:	stochastic term in C_{ij}, where x is ξ, η or ν
e_i	:	modulation function
E[ ]	:	ensemble average
F	:	$(3I/16I_\alpha)(c/R)^2$
F_i	:	inhomogeneous forcing function
F_β	:	flapping aerodynamic force
F_ζ	:	leadlagging aerodynamic force
$g, \overline{g}$	:	acceleration of gravity, $\overline{g} = g/R\Omega^2$
g_x, g_y, g_z	:	earthquake accelerations in the X, Y and Z directions
g_{ij}	:	coefficients of stochastic terms in equation of motion
G	:	generalized aerodynamic force for torsion
i_α	:	sectional mass moment of inertia of blade for torsion
I_α	:	feathering mass moment of inertia of blade
I	:	mass moment of inertia of blade for flapping or leadlagging
K_{ij}	:	stiffness coefficient
K_{ijo}	:	deterministic term of K_{ij}
K_{ijx}	:	stochastic term of K_{ij}, where x is ξ, η, ν, g_x, g_y or g_z
K	:	parameter defining vertical gradient of wind velocity

k : RK

m_i : drift coefficients in stochastic equation

M_t : generalized mass for torsion

M_i : first moment, $E[X_i]$

M_{ij} : second moment, $E[X_i X_j]$

M_{ijk} : third moment, $E[X_i X_j X_k]$

$M_\beta, \overline{M}_\beta$: generalized flapping force, $\overline{M}_\beta = M_\beta / I\Omega^2$

$M_\zeta, \overline{M}_\zeta$: generalized lead-lagging force, $\overline{M}_\zeta = M_\zeta / I\Omega^2$

$\overline{N}$: nonlinear matrix in the moment equation

q : aerodynamic torque

r : distance at blade from the hinge

R : blade length

R_{xx} : auto-correlation function of turbulence component

T_i : turbulence component correlation time, $i = 1,2,3$

u : lateral turbulence velocity

U : lateral wind velocity

$U_p, \bar{U}_p$: relative airflow velocity perpendicular to the rotor disc, $\bar{U}_p = U_p/R\Omega$

$U_T, \bar{U}_T$: relative airflow velocity tangent to the rotor disc, $\bar{U}_T = U_T/R\Omega$

v : axial turbulence velocity

v_i : induced velocity

V : axial wind velocity

w : vertical turbulence velocity

W_i : Wiener process

x : r/R

X : Column matrix of the response variables in the equation of motion

Y : state vector of the moment equation

Z_i : white noise process

α : blade-tip torsional angle

β : blade flapping angle

ζ : blade leadlagging angle

$\alpha_e, \beta_e, \zeta_e$: equilibrium position of α, β, ζ

$\delta\alpha, \delta\beta, \delta\zeta$	:	perturbed terms of α, β, ζ
θ	:	pitch angle
θ_e	:	equivalent pitch angle
θ_β	:	pitch-flap coupling parameter
θ_ζ	:	pitch-lag coupling parameter
γ	:	blade lock number, $R^4 \rho ac/I$
ρ	:	air density
$\delta(\ ), \delta_{ij}$	:	Dirac delta function, Kroneker delta
ν	:	$v/R\Omega$
η	:	$u/R\Omega$
ξ	:	$w/R\Omega$
λ	:	$V/R\Omega$
λ_R	:	minimum axial flow rate of designed constant power output
λ_i	:	induced flow rate
μ	:	advanced ratio
μ_i	:	first moment
μ_{ij}	:	second central moment
ψ	:	azimuth angle
Ω	:	rotor angular velocity
σ	:	rotor solidity, $bc/\pi R$
α_i	:	inverse of correlation time of turbulence component
ϕ	:	an arbitrary scalar function
Φ_{ii}	:	spectral density
σ_{ij}	:	diffusion coefficient of stochastic equation
$(\ )^T$	:	matrix transposition
τ	:	a nondimensional time variable
ω_β	:	nondimensional flapping natural frequency
ω_ζ	:	nondimensional lead-lagging natural frequency
ω_α	:	nondimensional torsional natural frequency

Clarifications will be provided in the text whenever the same symbol is used for a different purpose.

CHAPTER 1 INTRODUCTION

1.1 Problem Statement

Wind turbines which convert wind power into mechanical power have been in existence for centuries. In the earlier state, they were used only to pump water and grind grains. It was not until the nineteenth century when wind-powered electricity generation plants began to be built around the world. These earlier electrical projects were discontinued when they became economically less competitive than the fossil-fuel plants [1]. After the energy crisis of 1973, the limited fossil-fuel supply drove the cost of the fuel sky-high, thus generating considerable amount of effort in search of alternative energy sources. Wind power was given a renewed interest as a possible economical energy source.

In general, the cost per kilowatt-hour of electricity generated by a wind turbine decreases as the size of the unit increases. For a large wind turbine, the major cost is that of the rotor blade. It is essential that the blades must be designed for a long service life in order to be economically viable. The dynamic loads acting on a large scale blade include periodically varying deterministic aerodynamic and gravitational forces, as well as random turbulence and seismic loads.

In many respects the analysis of a horizontal-axis wind turbine blade is similar to that of a helicopter blade. For the analysis of wind turbine dynamics, some of the mathematical assumptions and approaches used in the analysis of the airloads and vibrations of helicopter blades are still valid. However, there is a fundamental difference in their functions: a helicopter blade imparts energy into air flow to generate the lifting force whereas a wind turbine blade extracts energy from air flow to generate electricity. Besides, there are several major differences between them. First, the gravity is a steady effect on a helicopter during its forward flight, and it can be ignored when compared with the aerodynamic forces generated by the high speed rotation of the blade. In contrast, the gravitational force is periodic on a wind turbine blade rotating about a horizontal axis, and it is of the same order of magnitude as the aerodynamic forces. Second, a helicopter is normally controlled to operate at large values of pitch angle in order to avoid a negative angle of attack due to inflow. In a wind turbine, however, the angle of attack due to inflow is always positive, and the blade pitch angle is controlled to be nearly zero or even negative to meet the power schedule of the generator. Third, helicopter rotor usually can be trimmed to operate in highly nonuniform flows whereas a wind turbine rotor may not tolerate such nonuniformities. Fourth, the vertical wind velocity gradient will not affect the helicopter dynamic behavior since the rotor rotates almost horizontally. In contrast, the wind turbine rotor operates in a vertical plane, and the velocity gradient causes asymmetry in the airloads of the rotor which may have a significant effect on the stability and response of the rotor system. Fifth, the white noise approximation of turbulence in the analysis of helicopter blade motion [2-5] may not be justifiable for wind turbine blades since the rotating speed is much lower in wind turbine operations. Finally, helicopters will not be exposed to earthquake excitations during their operations. Therefore, a wind turbine is expected to have different dynamic characteristics and operate in different parameter ranges then those of a conventional helicopter.

1.2 A Review of Previous Work

A large number of papers have been published on helicopter rotor dynamics, some of which are useful for the present research and they will be given a brief discussion.

Among the analyses of helicopter rotor dynamics, the simplest case began with a deterministic, single-degree of freedom model. Shulter and Jones [6] analyzed the uncoupled blade flapping motion, in which the reversed flow effect was ignored, using Floquet's theory for periodic systems. Later, Sissingh [7] considered the reversed flow effect which becomes important at high advanced ratio. This analysis was extended by Sissingh and Kuczynski [8] who derived the equations for coupled flap-torsional motion. In their derivation, the blade was assumed to be centrally hinged with an elastic restraint at the center of the hub. In the case of flap-lagging motion, Ormiston and Hodges [9] proposed a simple mathematic model consisting of a rigid blade and root spring system. The elastic constants of the spring system can be adjusted to account for elastic coupling and effects of nonzero feathering angle. This analysis was extended to the nonlinear case by Peters [10].

Next, we will review previous works on wind turbine dynamics. Ormiston [11] has investigated uncoupled flapping blade motion in the presence of axial wind, cross wind, and induced flow, as well as the effects of linear vertical gradients in the axial wind and cross wind, rotor shaft yaw precession and the gravity forces. In the same paper he also discussed the uncoupled lead-lagging response to gravitational and aerodynamic loads. His results showed that for typical parameter values of a wind turbine, both the lead-lag and flap frequencies were high compared to those of typical helicopter rotors. His results also showed that gravity forces dominated the lead-lag response.

Smith, Thresher, Wilson and MacDuff [12] investigated the coupling of rotor flapping and tower translation in an attempt to identify the relevant parameters. In their model, the rotor hub was constrained to rotate about a horizontal axis of fixed direction, and the bearing support structure was allowed to move only in the axial direction. The two blades were hinged at some distance from the center of the hub. A torsional spring at each hinge tended to restore the blade axes to the radial direction. In their investigation, two special cases were considered, both neglecting aerodynamic forces. In the first case, the rotating speed was assumed to be constant and the resulting equations indicated the existence of a parametric exictation due to the gravitational field. In the second case, the rotor speed was assumed to be high enough that gravity could be neglected, and the resulting equations yielded natural modes of vibration. However, they have provided few numerical results based on their equations.

Friedmann [13] has derived a set of general, nonlinear, partial differential equations for coupled flap-lag-torsional motion of a single wind turbine blade and discussed methods for their solutions together with some possible simplifications of the equations. In his derivation, the Davenport's model for the variation of the mean wind velocity in the earth's boundary layer was used. He also recognized the importance of turbulence loading and suggested using an equivalent sinusoidal load to evaluate its effect. However, he has published few numerical results that were based on his equations. Moreover, the deterministic sinusoidal model for turbulence is not satisfactory since atmospheric turbulence is a random phenomenon.

Kaza and Hammond [14] investigated the flap-lag stability of wind turbine rotors in the presence of velocity gradients and of helicopter rotors in a forward flight using an approximate numerical solution. Their results showed a decrease in blade damping with advance ratio in the case of a helicopter rotor and little effect of the wind gradient on wind turbine stability. They have provided only one figure for windmills.

Spera [15] has investigated the blade vibrations of wind turbine rotors. The blade vibrations were limited to the fundamental flapping modes, assumed to be elastic cantilever bending for hingeless rotor blades and rigid-body rotation for teetering blades. The effects of the wind shear and tower wake were taken into consideration. He used a computer program in which aerodynamic coefficients were obtained from a stored table of values, to calculate the airloads on wind turbines and integrated the equations of motion in time. Such an analysis is valuable as a design tool, but is not preferable for basic research. The time domain numerical approach has some disadvantages: 1) the interpretation of time histories is usually difficult, and 2) the complicated nature of a large computer program and the high computing costs prohibit any extensive parameter changes in the mathematical model.

Wei and Peters [16] have studied the flap-lag instability of both helicopter and windmill blades as a function of design parameters and operating conditions for various trimmed and untrimmed conditions. The mathematical techniques used in their study are the perturbation method, multiple time scales and the Floquet theory. Their results indicated that the trimmed and untrimmed autorotation flight conditions were considerably less stable than the power flight condition, and that gravity forces have little effect on stability in the case of axial flow at very high thrust coefficient. Their results also showed that the effects of the axial flow, velocity gradient on blade damping were small. However, their emphasis was focused on helicopter rotors rather than windmill rotors because the operating conditions and control settings used were those of a helicopter.

1.3 Scopes of Present Research

The present research is directed at the dynamics of a wind turbine rotor system under random seismic and turbulence excitations. Three types of blade motion are investigated: 1) uncoupled flapping motion, 2) coupled flap-lagging motion, and 3) coupled flap-lag-torsional motion. Account is taken of the aerodynamic, gravitational and vertical wind gradient effects.

In Chapter 2, a brief review of stochastic processes and stochastic differential equations is presented. Special attention is paid to the generalization of the spectral representation of a stationary random process to that of a non-stationary process. The stochastic differential equation in the sense of Ito is then introduced, followed by the Ito differential rule, which is a useful tool to obtain equations for statistical moments of system response variables. Finally, the stochastic averaging procedure for converting physical equations to Ito's equations and its implication in terms of convergence of sequence of physical processes to a Markov process are briefly discussed.

The equations of uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsional motions are derived in Chapter 3. The effects of axial wind, vertical wind gradients, cross wind and elastic coupling are included. Ground motions and turbulence velocities are treated as random external excitations. The equilibrium solutions of coupled flap-lagging and coupled flap-lag-torsional motions are obtained in this chapter using harmonic balancing method. The control settings of a typical wind turbine are also discussed.

Chapter 4 deals with the mathematical models of the random excitations. Earthquake accelerations are modeled as nonstationary random processes. Turbulence velocities are modeled as stationary processes.

In Chapter 5, the equations of motion obtained in Chapter 3 are converted to the Itô-type stochastic differential equations, from which the moment equations for system response variables are derived. An outline is given of the approaches to obtain the moment stability conditions and response moments.

In Chapter 6, the moment equations derived in Chapter 5 are solved numerically. The effects of turbulence level and earthquake level are investigated and presented graphically. The effects of some other parameters such as turbulence correlation times and elastic coupling are also indicated.

Chapter 7 summarizes the principal conclusions and indicates some topics for further research.

CHAPTER 2

FUNDAMENTALS OF STOCHASTIC DIFFERENTIAL EQUATIONS

2.1 Elements of Stochastic Process

Differential equations governing a physical system subjected to excitations can be written as

$$\dot{x}_i = f_i(\vec{X}, t) + g_{ik}(\vec{X}, t)\, \xi_k(t) \tag{2-1}$$

$$i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, m$$

where x_i are components of a state vector $\vec{X}$, $\xi_k(t)$ are excitations, and a repeated index implies summation over all values of the index. We shall assume that $\xi_k(t)$ are stochastic or random processes. To characterize $\xi_k(t)$, we review the following important concept of stochastic processes.

A. Random process with orthogonal increments [17]

Let $Z(\omega)$ be a complex-valued random process defined on $a < \omega < b$. $Z(\omega)$ is said to have uncorrelated increments if

$$E[\{Z(\omega_2)-Z(\omega_1)\}\{Z^*(\omega_4)-Z^*(\omega_3)\}] = E[Z(\omega_2)-Z(\omega_1)]E[Z^*(\omega_4)-Z^*(\omega_3)] \tag{2-2}$$

for any $a < \omega_1 < \omega_2 < \omega_3 < \omega_4 < b$, where the asterisk denotes the complex conjugate. If the right hand side of Eq. (2-2) is zero, then $Z(\omega)$ is said to be a random process with orthogonal increment.

We define a deterministic function Ψ, such that

$$\Psi(\omega) = \begin{cases} E[|Z(\omega) - Z(\omega_0)|^2], & \text{if } \omega > \omega_0 \\ -E[|Z(\omega_0) - Z(\omega)|^2], & \text{if } \omega < \omega_0 \end{cases} \tag{2-3}$$

where ω_0 is an arbitrarily chosen reference point in the ω-domain. Eq. (2-3) implies that $\Psi(\omega_0) = 0$. It can be shown that if $\omega_2 > \omega_1$

$$E[|Z(\omega_2) - Z(\omega_1)|^2] = \Psi(\omega_2) - \Psi(\omega_1) \tag{2-4}$$

Let $\omega_1 = \omega$, $\omega_2 = \omega + d\omega$ in Eq. (2-4). We obtain

$$E[|dZ(\omega)|^2] = d\Psi(\omega) \tag{2-5}$$

Note that if $\Psi(\omega)$ is differentiable (i.e. $d\Psi(\omega)$ is of the order of $d\omega$), $dZ(\omega)$ is of the order of $(d\omega)^{1/2}$. If $\Psi(\omega)$ is not differentiable, $dZ(\omega)$ is also not differentiable. Therefore, in either case an orthogonal increment process is not differentiable. It also can be shown that an alternative definition for an orthogonal increment process is

$$E[dZ(\omega)dZ^*(\omega')] = \begin{cases} 0, & \text{if } \omega \neq \omega' \\ d\Psi(\omega), & \text{if } \omega = \omega' \end{cases} \tag{2-6}$$

B. Stationary random process

A random process $x(t)$ is said to be stationary (in a strict sense) if its statistical characteristics are invariant under time shifts, i.e., if they remain the same when t is replaced by $t+\tau$ where τ is arbitrary [18].

For a stationary process the mean value must be a constant, say μ, and the correlation function must be a function of time difference τ, i.e.,

$$E[x(t)] = \mu$$

and
$$E[x(t)x(t+\tau)] = R_{xx}(\tau) \tag{2-7}$$

However, these may be true when $x(t)$ is not strictly stationary, in which case, $x(t)$ is said to be weakly stationary.

If $\mu = 0$, then a weakly stationary random process can be expressed as a Fourier-Stieltjes integral representation.

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega) \tag{2-8}$$

where $Z(\omega)$ is an orthogonal increment process. Eq. (2-8) is known as the spectral representation of a stationary process. To compute the correlation function of $x(t)$, we take the ensemble average of $x(t)$ and $x(t+\tau)$:

$$E[x(t)\ x\ (t+\tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega-\omega')t-i\omega'\tau} E[dZ(\omega)dZ^*(\omega')]$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} d\,\Psi(\omega) = R_{xx}(\tau) \tag{2-9}$$

Eq. (2-9) shows that the correlation function of a weakly stationary process also has a Fourier-Stieltjes representation. If $\Psi(\omega)$ is differentiable, then

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\Psi'(\omega)d\omega = \int_{-\infty}^{\infty} e^{-i\omega\tau}\Phi_{xx}(\omega)d\omega \tag{2-10}$$

where "'" denotes differentiation with respect to ω, and $\Phi_{xx}(\omega) = \Psi'(\omega)$ is called the spectral density of x(t). If we let $\Psi(-\infty) = 0$ then

$$\Psi(\omega) = \int_{-\infty}^{\infty} \Phi_{xx}(u)\, du \tag{2-11}$$

In this case, $\Psi(\omega)$ is called the spectral distribution function of x(t) [17]. The mean-square value of x(t) is obtained from

$$E[x^2(t)] = R_{xx}(0) = \Psi(\infty) = \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega \tag{2-12}$$

Eq. (2-12) shows that $\Phi_{xx}(\omega)$ describes the distribution of the mean-square value in the ω (frequency) domain.

C. Non-stationary Random Process

For stationary processes, the spectral representation is well-known and has been used extensively in physical and engineering applications. On the other hand, there has not been a general agreement on a similar representation for non-stationary processes. Several different attempts have been made [19-22]; one of them is the evolutionary spectrum proposed by Priestly [22,23] which has found important applications in earthquake engineering.

This particular class of nonstationary process considered by Priestly has a Stieltjes integral representation,

$$x(t) = \int_{-\infty}^{\infty} a(t,\omega)\, e^{i\omega t} dZ(\omega) \tag{2-13}$$

where $a(t,\omega)$ is a deterministic function of t and ω, and $Z(\omega)$ is again an orthogonal increment process. If $a(\omega,t) = 1$, the representation reduces to the stationary form, Eq. (2-8).

The correlation function of x(t) is

$$\phi_{xx}(t_1,t_2) = E[x(t_1)x(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1,\omega_1)a^*(t_2,\omega_2)e^{i\omega_1 t_1}e^{-i\omega_2 t_2}E[dZ(\omega_1)dZ^*(\omega_2)]$$

$$= \int_{-\infty}^{\infty} a(t_1,\omega)\; a^*(t_2,\omega)e^{i\omega(t_1-t_2)}d\Psi(\omega) \tag{2-14}$$

where $\Psi(\omega)$ is the spectral distribution function of some stationary process. When $t_1 = t_2$, Eq. (2-13) becomes

$$\phi_{xx}(t,t) = \int_{-\infty}^{\infty} |a(t,\omega)|^2\, d\Psi(\omega) \tag{2-15}$$

If $\Psi(\omega)$ is differentiable, then

$$E[x^2(t)] = \int_{-\infty}^{\infty} |a(t,\omega)|^2\, \Phi(\omega)\, d\omega \tag{2-16}$$

As in the stationary process, $|a(t,\omega)|^2\,\Phi(\omega)$ can be interpreted as an energy density if $E[x^2(t)]$ is a measure of the total energy of the process at time t.

If $a(\omega,t)$ does not depend on ω i.e., $a(\omega,t) = e(t)$, Eq. (2-13) becomes

$$x(t) = \int_{-\infty}^{\infty} e(t)e^{i\omega t}dZ(\omega) \tag{2-17}$$

In this special case, x(t) is called a uniformly modulated random process.

D. Markov Process [17]

A continuous stochastic process x(t) is called a Markov process if the following property is satisfied

$$\text{Prob}\ [x(t_n) < x_n | x(t_{n-1}) = x_{n-1},\ x(t_{n-2}) = x_{n-2},\ \ldots,\ x(t_1) = x_1]$$

$$= \text{Prob}\ [x(t_n) < x_n |\ x(t_{n-1}) = x_{n-1}];$$

$$t_n > t_{n-1} > t_{n-2} > \ldots > t_1 \tag{2-18}$$

that is, the past and future of a Markov process are statistically independent when the present is known. A sufficient condition for being Markovian is that x(t) has independent increments.

The conditional probability, Prob $[x(t) < x|\ x(t_o) = x_o]$, is called the transition probability distribution function; its derivative with respect to x is called the transition probability density, to be denoted by $q(x,t|x_o,t_o)$. The transition probability density of a Markov process satisfies a partial differential equation

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} (Aq) - \frac{1}{2} \frac{\partial^2}{\partial x^2} (Bq) + \frac{1}{3} \frac{\partial^3}{\partial x^3} (Cq) - \dots = 0 \qquad (2\text{-}19)$$

where $A = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[x(t+\Delta t) - x(t) \mid x(t) = x]$

$$B = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\{ x(t+\Delta t) - x(t)\}^2 \mid x(t) = x]$$

$$C = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\{ x(t+\Delta t) - x(t)\}^3 \mid x(t) = x]$$

$\vdots$

are called the derivate moments. When x(t) is also Gaussian, the derivate moments of an order higher than two vanish. In this case, Eq. (2-19) reduces to the parabolic type

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} (Aq) - \frac{1}{2} \frac{\partial^2}{\partial x^2} (Bq) = 0 \qquad (2\text{-}20)$$

Eq. (2-20) is called the Fokker-Planck equation or Kolmogorov forward equation.

For a multi-dimensional Markov process, the Fokker-Planck equation can be written in a similar way

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x_i} (a_i q) - \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (b_{ij} q) = 0 \qquad (2\text{-}21)$$

where x_i are the components of a state vector $\vec{X}$, and a_i and b_{ij} are given by

$$a_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[x_i(t+\tau) - x_i(t) | X(t) = X]$$

$$b_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\,[\{x_i(t+\tau) - x_i(t)\}\{x_j(t+\tau) - x_j(t)\} | X(t) = X] \qquad (2\text{-}22)$$

A Markov process whose transition probability density satisfies the Fokker-Planck equation is also known as a diffusion process. The derivate moments $A = \{a_i\}$ and $B = [b_{ij}]$ are called the drift vector and the diffusion matrix of the diffusion process, respectively.

The significance of the transition probability density of a Markov process is that for a given initial state, the transition probability density completely specifies the process x(t).

E. Brownian motion process (Wiener process) [24,25]

A stochastic process W(t) is called a Wiener process if it satisfies:

(i) W(t) is a Gaussian process

(ii) W(0) = 0

(iii) E[W(t)] = 0

(iv) $E[W(t_1)W(t_2)] = \sigma^2 \min(t_1,t_2)$; i.e.

$$E[W(t_1)W(t_2)] = \begin{cases} \sigma^2 t_1, & \text{if } t_1 < t_2 \\ \sigma^2 t_2, & \text{if } t_1 > t_2 \end{cases} \qquad (2\text{-}23)$$

A unit Wiener process is one for which $\sigma = 1$. It can be shown that W(t) has independent increments, and thus satisfies the sufficient condition for being Markovian. In fact, the Wiener process is the simplest Markov process.

Furthermore, since being independent implies being uncorrelated,

$$E[dW(t_1)dW(t_2)] = \begin{cases} \sigma^2 dt, & \text{if } t_1=t_2=t \\ 0, & \text{if } t_1 \neq t_2 \end{cases} \tag{2-24}$$

Therefore, the Wiener process is nowhere differentiable. This property, Eq. (2-24) has provided the motivation for introducing Ito's differential equation and Ito's integral as we will see later.

F. White noise process

A weakly stationary random process with zero mean and a constant spectral density is called a white noise; denoted by Z(t). That is,

$$\Phi_{ZZ}(\omega) = K$$

or

$$R_{ZZ}(\tau) = 2\pi K \delta(\tau), \tau = t_2 - t_1 \tag{2-25}$$

Eq. (2-25) indicates that Z(t) is also a delta-correlated process. A constant spectral density implies that the energy content of the process is uniformly distributed over the entire frequency range. The total energy, which is equal to the infinite integral of the spectral density, is infinite. Therefore, the white noise process is a theoretical idealization, and physically non-existent. Nevertheless, such processes are sometimes useful as approximations for physical processes and they may be used to obtain meaningful results.

The spectral density of a physically realizable process must be negligible beyond some cut off frequency ω_c. Replacing a physical process by a white noise process means that this cut-off frequency is taken to be infinity. This is permissible if the actual ω_c is considerably higher than all frequencies which are important in a given physical problem.

It also can be shown that the derivative of a Wiener process is a white noise, and a Markov process.

G. Exponentially correlated process

Now we discuss the statistical properties of a stochastic process $\xi(t)$ with an exponential correlation function. Such a process satisfies the first order differential equation.

$$\dot{\xi} = -\alpha\xi + Z(t) \tag{2-26}$$

where Z(t) is a white noise with zero mean and $R_{ZZ}(\tau) = 2\pi\Phi_{ZZ}\delta(\tau)$. If the initial condition is $\xi(t_0) = \xi_0$, then Eq. (2-26) has the solution

$$\xi(t) = \xi_0 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-s)} Z(s)ds \tag{2-27}$$

Using Eq. (2-27), we find the following expressions for the mean value and the correlation function of $\xi(t)$:

$$E[\xi(t)] = \xi_0 e^{-\alpha(t-t_0)}$$

$$E[\xi(t_1), \xi(t_2)] = \frac{\pi\Phi_{ZZ}}{\alpha} [e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_1+t_2-2t_0)}] + \xi_0^2 e^{-\alpha(t_1+t_2-2t_0)} \tag{2-28}$$

As $t - t_0$ increases, the correlation function tends to its stationary form

$$R_{\xi\xi}(\tau) = \sigma^2 e^{-\alpha|\tau|} \qquad (2\text{-}29)$$

where

$\sigma^2 = \frac{\pi\Phi_{zz}}{\alpha}$, and $\tau = t_2 - t_1$.

The process $\xi(t)$ becomes an approximation of white noise as $\alpha \to \infty$, $\Phi_{zz} \to \alpha$, but σ^2 = constant.

When the excitations in , Eq. (2-1) are exponentially correlated, the response state vector $\vec{X}$ is not a vector Markov process. However, the expanded (n+m)-dimensional state vector (x_i, ξ_k) where i = 1,2, ... ,n, k = 1,2, ... ,m is Markovian.

2.2 Itô Stochastic Differential Equation

The governing equation for every diffusive Markov process can be written in the following form:

$$dx = m(x,t)dt + \sigma(x,t)\, dW(t) \qquad (2\text{-}30)$$

where W(t) is a unit Wiener process. Eq. (2-30) is equivalent to

$$x(t) = x(0) + \int_0^t m(x(u),\, u)\, du + \int_0^t \sigma(x(u),u)\, dW(u) \qquad (2\text{-}31)$$

The second integral in Eq. (2-31) cannot be interpreted as a usual Stieltjes integral, since the sample functions of W(t) are of unbounded variation. Itô proposed that it be interpreted as a forward integral [27,28].

$$\int_0^t \sigma(x(u),u)\, dW(u) = \underset{\max \Delta \to 0}{\text{l.i.m.}} \sum_{i=1}^{N-1} \sigma(x(u_i),u_i)[W(u_{i+1}) - W(u_i)] \qquad (2\text{-}32)$$

where $\Delta = u_{i+1} - u_i$ and l.i.m. represents the so-called limit in the mean. It is of interest to note that every pair of $\sigma(x(u_i),u_i)$ and $[W(u_{i+1}) - W(u_i)]$ are independent, since the increment in W occurs after the time u_i. Eq. (2-30) is called Ito's stochastic differential equaiton if the integral is interpreted as Eq. (2-32).

In the case of a Markov vector, Eq. (2-30) is generalized to

$$dx_j = m_j(\vec{X},t)dt + \sigma_{jk}(\vec{X}, t)\, dW_k(t); \tag{2-33}$$

$$j = 1,2, \ldots ,n, \quad k = 1,2, \ldots ,m$$

It can be shown that the solution vector $\vec{X}(t)$ generated by Eq. (2-33) is Markovian and has no derivatives [29]. Furthermore, the Fokker-Planck equation for the transition probability density q of Markov vector $\vec{X}$ is given by

$$\frac{\partial q}{\partial t} = -\frac{\partial (m_j q)}{\partial x_j} + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_k}(\sigma_{jl}\sigma_{lk}q) \tag{2-34}$$

Comparing Eq. (2-34) and Eq. (2-21), the first and second derivate moments are seen to be

$$a_j\,(\vec{X},t) = m_j(\vec{X},t) \tag{2-35}$$

$$b_{jk}\,(\vec{X},t) = \sigma_{jl}(\vec{X},t)\,\sigma_{lk}(\vec{X},t) \tag{2-36}$$

2.3 Ito's Differential Rule

As indicated earlier, the transition probability density completely specifies a Markov vector $\vec{X}(t)$ provided that the initial state is known; however, to obtain a closed form solution for the transition probability density is quite difficult if not impossible. The alternative is to obtain the statistical moments of the system response.

For the computation of the statistical moment, we introduce the Ito's differential rule. The advantage of using this rule lies in the fact that similar Ito equations can be derived quite simply and without ambiguity for arbitrary scalar functions of a Markov vector satisfying rather general conditions. The Itô's differential rule may be stated as follows: Let x_i by the ith component of a Markov vector $\vec{X}$, governed by Ito equation (2-33), and let $\phi(\vec{X},t)$ be a scalar function, then [27,28]

$$d\phi = \left(\frac{\partial \phi}{\partial t} + m_i \frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sigma_{ik}\sigma_{kj}\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) dt + \sigma_{ij}\frac{\partial \phi}{\partial x_i} dW_j \tag{2-37}$$

provided that the derivatives on the right hand side exist. The Itô differential rule differs from the classical chain rule in the additional term $\frac{1}{2}\sigma_{ik}\sigma_{kj}\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ which results from the retention of terms of the order $(dW)^2$.

Equations for the first and second moments are obtained by letting the scalar function ϕ be x_i and $x_r x_s$, respectively, and taking the ensemble average of the results; it leads to

$$\frac{d}{dt} E[x_i] = E[m_i] \tag{2-38}$$

$$E[x_r x_s] = E[m_r x_s + m_s x_r + \sigma_{r1}\sigma_{1s}] \tag{2-39}$$

2.4 Approximation of Physical Process by Markov Process

As mentioned earlier, Markov process is an idealized mathematical process, and no physical process can be exactly Markovian. However, it is sometimes reasonable to use such an approximation to obtain meaningful results. The approximation is usually justified on the basis of how close the increments in non-overlapping time intervals are nearly being independent [17].

For convenience of discussion, we introduce the concepts of the relaxation time for a dynamic system and the correlation time of a stochastic process. The relaxation time τ_r is defined as the time required for the amplitude of a free motion to decrease by a factor e^{-1} or increase by a factor e, where e is the base of natural logarithm. The correlation time of a weakly stationary stochastic process may be defined as

$$\tau_c = \int_0^\infty \tau |R(\tau)| d\tau / \int_0^\infty |R(\tau)| d\tau \tag{2-40}$$

where $R(\tau)$ is the correlation function of the stochastic process. When a dynamic system is subjected to a random excitation, the system response will exhibit a Markov-like behavior if it is observed at time intervals greater than the correlation time of the excitation. However, the observation time for the response must not be too far apart to lose the essential characteristics. A rough guideline is that the observation intervals should not be farther apart than the relaxation time of the dynamic system. If $\tau_c \ll \tau_r$, then the response is expected to show a near Markovian behavior when it is sampled at time intervals of the order of τ_r.

When the relaxation time of a dynamic system is not much greater than the correlation time of excitations, the Markov process theory still can be used. This is done by approximating the excitations as outputs of linear filters driven by Gaussian white noises, and extending the dimensions of the Markov vector. The system response variables constitute only some, not all the components of the resulting Markov vector. The exponentially correlated process mentioned earlier is obtained by passing a white noise through a first order filter.

2.5 Stochastic Averaging Method

We shall assume that the Markov process approximation is justified for the physical system, Eq. (2-1). The question now arises as how Eq. (2-1) can be converted to an equivalent Ito equation, Eq. (2-33). The mechanism is provided in what is now known as the stochastic averaging procedure, proposed by Stratonovich [18] in 1961 on physical grounds and later justified rigorously by Khasmiskii [30]. According to this procedure, the drift and diffusion coefficients in the Ito equation corresponding to Eq. (2-1) are

$$m_j = f_j(\tilde{X},t) + \int_{-\infty}^{0} \left[\frac{\partial}{\partial x_l} g_{jk}(\tilde{X},t) \right] g_{ls}(\tilde{X},t+\tau)\, E[\xi_k(t)\xi_s(t+\tau)]\,d\tau \tag{2-41}$$

$$\sigma_{jl}\sigma_{lk} = 2\int_{-\infty}^{0} g_{jr}(\tilde{X},t)\, g_{ks}(\tilde{X},t+\tau)\, E[\xi_r(t)\,\xi_s(t+\tau)]\,d\tau \tag{2-42}$$

The physical implications of this procedure are clear. The first term of Eq. (2-41), f_j, represents the tendency for future drift of the response variables x_j if random excitations were not present. With the random excitations, the tendency is modified due to the correlation between the past excitations at $t+\tau$ and the present excitations at t, where τ is negative. The integral in Eq. (2-41) sums up all the past correlation effects and lumps the total effects at present. Similarly, the integral in Eq. (2-42) sums up the future diffusion tendency due to random excitations up to the present time. The substitution of equivalent drift and diffusion into the Ito equation for Markov vector is necessary, since the excitations dW_k in Eq. (2-33) are independent of the present state $\vec{X}$; by virtue of Ito's interpretation, Eq. (2-32), they affect only future diffusion, and are dissociated with the drift [17].

Eq. (2-42) gives the elements of $\sigma\sigma^T$, instead of matrix σ itself. However, only the product matrix $\sigma\sigma^T$ is required for the calculation of $E[\phi(\vec{X},t)]$ as shown in Eq. (2-37).

In the special case in which ξ_k of eq. (2-1) are physical white noise processes, i.e.,

$$E\,[\xi_r(t)\xi_s(t+\tau)] = 2\pi\Phi_{rs}\delta(\tau) \tag{2-43}$$

where Φ_{rs} are constants, Eqs. (2-41) and (2-42) reduce to

$$m_j = f_j\,(\vec{X},t) + \pi\Phi_{rs}[\frac{\partial}{\partial x_l}\,g_{jr}(\vec{X},t)]\;g_{ls}(\vec{X},t) \tag{2-44}$$

$$\sigma_{jl}\sigma_{lk} = 2\pi\Phi_{rs}g_{jr}(\vec{X},t)\;g_{ks}(\vec{X},t) \tag{2-45}$$

The second term on the right-hand side of Eq. (2-44) is called the Wong-Zakai correction [32,33].

CHAPTER 3

GOVERNING EQUATIONS OF MOTION OF A ROTOR BLADE

3.1 Physical Model

In this section, differential equations governing the coupled flap-lagging motion of a single blade will be derived first which will serve as a basis for later reduction to the uncoupled flapping motion and extension to coupled flap-lag-torsional motion. In general, the assumptions commonly made in the analyses of helicopter blades [9,10], are equally appropriate for wind turbines. These are briefly described as follows:

A. Structural model

(1) For flap and lead-lag motions , the blade is rigid, centrally hinged, with linear elastic restraint at the hinge.

(2) The mass and the elastic centers coincide with the aerodynamic center; and they lie along a straight line.

(3) The blade has uniformly distributed mass along the span.

B. Aerodynamic model

(1) Flow is incompressible and sectionally two-dimensional.

(2) Linear, quasi-steady strip theory is applicable to calculate the aerodynamic forces.

(3) The reversed flow due to the sidewind is negligible.

(4) Flow separation and stall do not occur.

The assumption of uniform mass distribution is not particularly accurate, but it is made to simplify the formulation. However, the results should be valid qualitatively. When a sidewind is present, part of the wind turbine rotor disc will experience a reversed flow as is the case for a helicopter rotor in forward flight; that is, the fluid flow will approach the trailing edge of blade airfoil. For low sidewind velocities, the reversed flow region is very small and can be neglected.

It is well known that the strip theory is not strictly valid near the blade tip. When the blade chord at the tip is finite, the lifting force based on the strip theory is nonzero throughout the entire length of the blade. In fact the lift must decrease to zero at the blade tip where the air flow must be three-dimensional. To compensate for the inaccuracy due to the two-dimensional flow assumption, the physical rotor radius R is usually replaced by an equivalent radius RB for airload calculations where B is called the tip-loss factor.

3.2 Formulation of Equations of Motion

In the derivation of the equations of motion, reference will be made of the coordinate systems illustrated in Figure 3-1. The fixed coordinate system (X,Y,Z) is defined with a vertical Y direction downward and a Z direction upwind. The rotating coordinate system (X',Y',Z') rotates about Z axis at a constant angular velocity Ω. The blade coordinate system (x,y,z) is attached to the blade and it has a y axis along the span, pointing outward. The wind velocities are decomposed into the steady components, U and V, in the X and -Z directions, and the random turbulence components, u, v and w in the X, -Z and Y directions.

Consider a rigid blade mounted on a spring system at its root and rotating about Z-axis at a constant angular velocity Ω. The kinetic energy and

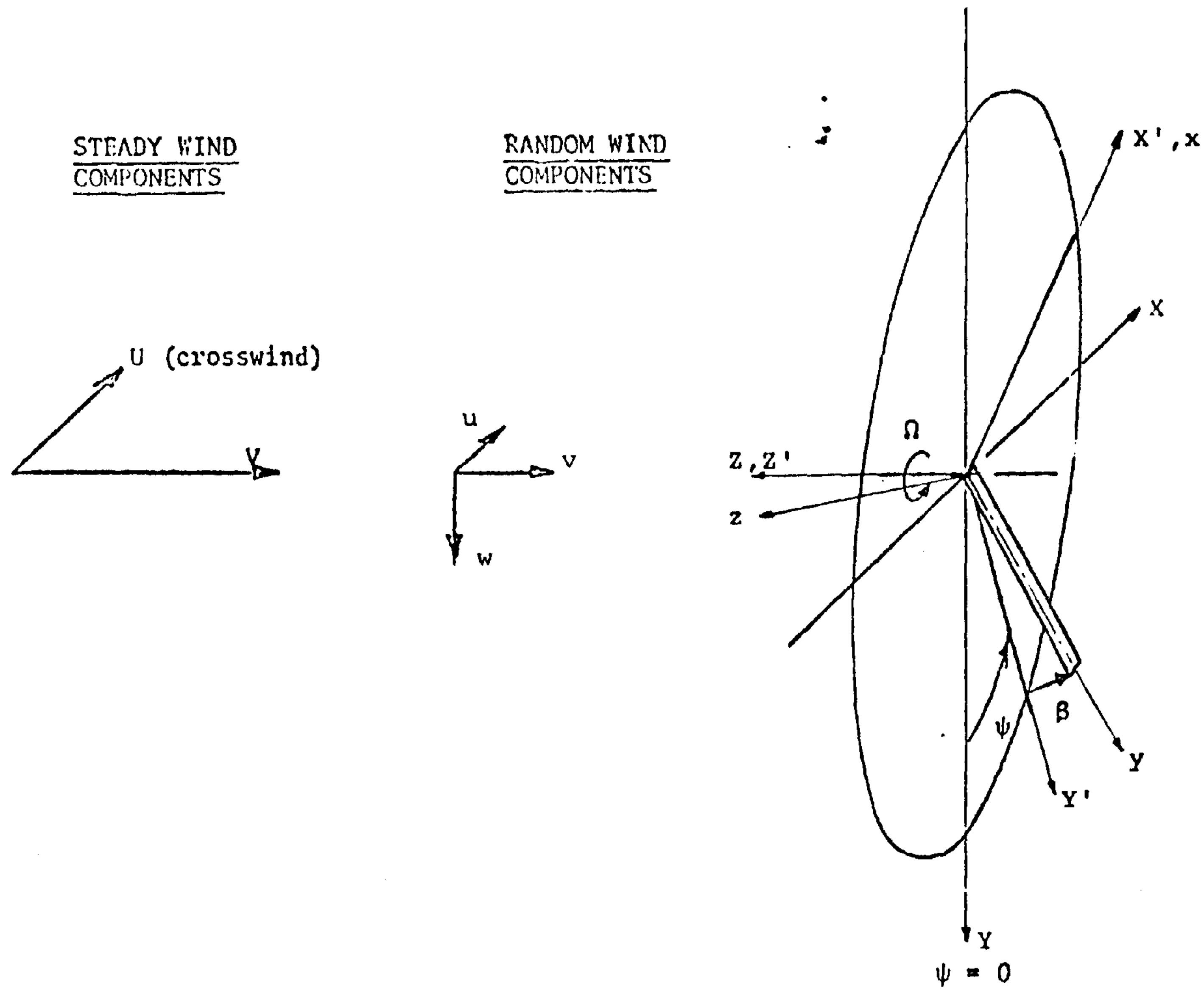

Figure 3-1 Configuration of Wind Turbine Blade.

potential energy for the coupled flap-lagging motion are given by

$$T = \frac{1}{2} I \dot{\beta}^{2} + \frac{1}{2} I (\Omega + \dot{\zeta})^{2} \cos^{2} \beta \tag{3-1}$$

$$V = \frac{1}{2} K^{*}_{\beta\beta} (\beta - \beta_{pc})^{2} + \frac{1}{2} K^{*}_{\zeta\zeta} \zeta^{2} + K^{*}_{\beta\zeta} (\beta - \beta_{pc}) \zeta$$

$$- \frac{1}{2} mgR^{2} \cos \beta \cos (\phi + \zeta) \tag{3-2}$$

where "$\cdot$" denotes one derivative with respect to time t, $I = \frac{mR^3}{3}$, g = the gravity force, β_{pc} = the pre-cone angle, $K^{*}_{\beta\beta}$ = flapping spring constant, $K^{*}_{\zeta\zeta}$ = lead-lagging spring constant, $K^{*}_{\beta\zeta}$ = flap-lagging elastic coupling constant. Let β and ζ be two generalized coordinates. The corresponding generalized forces can be computed as follows.

$$M_{\beta} = \int_{0}^{R} F_{\beta} r \, dr \tag{3-3}$$

$$M_{\zeta} = \int_{0}^{R} F_{\zeta} \, r \cos \beta \, dr \tag{3-4}$$

where F_{β} and F_{ζ} are forces due to aerodynamic and seismic effects.

The equations of motion may be derived by applying the Lagrange equation, and case in the following nondimensional forms:

$$\beta'' + (1+\zeta')^{2} \cos \beta \sin \beta + \overline{K}_{\beta\beta} (\beta - \beta_{pc}) + \overline{K}_{\beta\zeta} \zeta$$

$$+ \frac{3}{2} \overline{g} \sin \beta \cos (\phi + \zeta) = \overline{M}_{\beta} \tag{3-5}$$

$$\zeta'' \cos^2\beta + \overline{K}_{\zeta\zeta}\zeta + \overline{K}_{\beta\zeta}(\beta - \beta_{pc}) - 2\cos\beta\,\sin\beta\,(1+\zeta')\beta'$$

$$+ \frac{3}{2}\,\overline{g}\cos\beta\,\sin\beta\,(\phi + \zeta) = \overline{M}_{\zeta} \qquad (3\text{-}6)$$

where "'" denotes one derivative with respect to the nondimensional time, $\phi = \Omega t$, and

$$\overline{g} = g/R\,\Omega^2$$
$$\overline{K}_{\beta\beta} = K^*_{\beta\beta}/I\,\Omega^2$$
$$\overline{K}_{\zeta\zeta} = K^*_{\zeta\zeta}/I\,\Omega^2$$
$$\overline{K}_{\beta\zeta} = K^*_{\beta\zeta}/I\,\Omega^2$$
$$\overline{M}_{\beta} = M_{\beta}/I\,\Omega^2$$
$$\overline{M}_{\zeta} = M_{\zeta}/I\,\Omega^2$$

Various terms appearing in the equations of motion will now be discussed.

A. Elastic Coupling Model

To estimate the elastic constants $\overline{K}_{\beta\beta}$, $\overline{K}_{\zeta\zeta}$, and $\overline{K}_{\beta\zeta}$ Ormiston and Hodge [9] have proposed a simple model consisting of two sets of orthogonal springs with a collective pitch angle θ between them as shown in Figure 3-2. In our analysis of a wind turbine, the same model will be used to represent the elastic property of the blade-hub assembly.

The blade is assumed to be rectilinear, untwisted, constant chord and without hinge offset. A structural coupling parameter R is introduced, and defined as

$$\mathbb{R} = \frac{1/K_{\beta b} - 1/K_{\zeta b}}{(1/K_{\beta b} + 1/K_{\beta h}) - (1/K_{\zeta b} + 1/K_{\zeta h})} \tag{3-7}$$

The spring constants in eq. (3-7) are those shown in Fig. 3-2; namely $K_{\beta h}(K_{\zeta h})$ is associated with the hub flap (lag) spring that remains aligned with the rotor shaft, and $K_{\beta b}(K_{\zeta b})$ with the blade flap (lag) spring that remains aligned with the blade. The configuration of Figure 3-2 reduces to a simple equivalent single spring system at zero pitch angle which defines the rotor blade nonrotating frequencies, ω_β and ω_ζ. For this reduced case the spring constants are given by, respectively,

$$K_\beta = \frac{K_{\beta b} \; K_{\beta h}}{K_{\beta b} + K_{\beta h}} \tag{3-8}$$

$$K_\zeta = \frac{K_{\zeta b} \; K_{\zeta h}}{K_{\zeta b} + K_{\zeta h}} \tag{3-9}$$

and

$$\omega_\beta^2 = K_\beta / I\Omega^2$$

$$\omega_\zeta^2 = K_\beta / I\Omega^2$$

In terms of these parameters, the equivalent spring constants used in Eq. (3-5) and Eq. (3-6) are

$$\overline{K}_{\beta\beta} = \frac{1}{\Delta}\,[\omega_\beta^2 + \mathbb{R}(\omega_\zeta^2 - \omega_\beta^2)\,\sin^2\theta\,] \tag{3-10}$$

$$\overline{K}_{\zeta\zeta} = \frac{1}{\Delta}\,[\omega_\zeta^2 - \mathbb{R}(\omega_\zeta^2 - \omega_\beta^2)\,\sin^2\theta\,] \tag{3-11}$$

$$\overline{K}_{\beta\zeta} = \frac{\mathbb{R}}{2\Delta}\,[\omega_\zeta^2 - \omega_\beta^2]\,\sin 2\,\theta \tag{3-12}$$

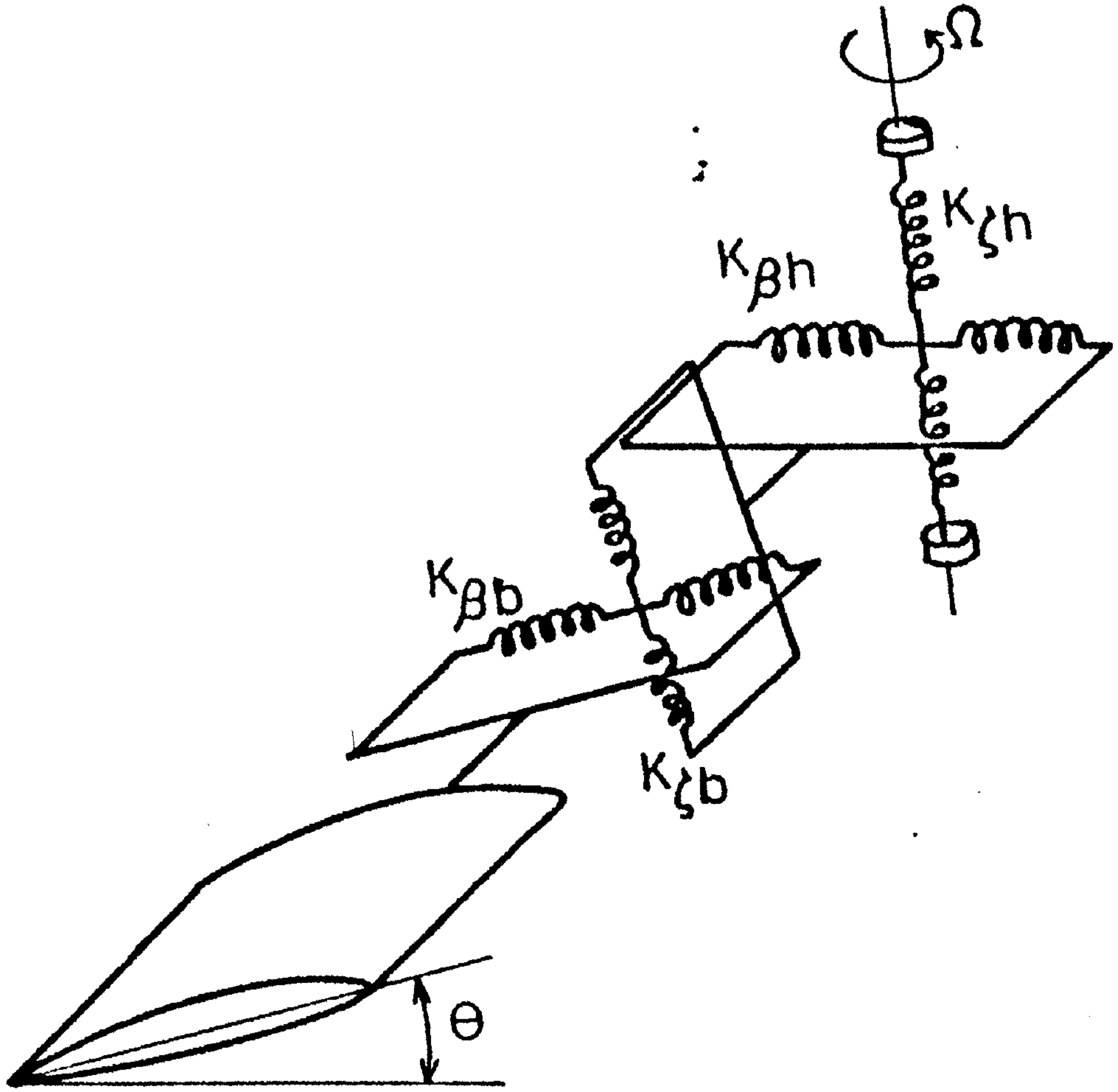

Figure 3-2 Structural Model of Hub-Blade Assembly Simulating the Flap-Leadlag Coupling.

where $\Delta = 1 + R(1-R)\ \sin^2\theta\ (\omega_\zeta^2 - \omega_\beta^2)^2/\omega_\zeta^2\omega_\beta^2$

The case R = 0 corresponds to one where the blade spring system is entirely contained in the hub and the rotor blade does not rotate with pitch angle change. On the contrary, for R = 1, the hub spring system is contained in the blade spring and the rotor blade rotates in accordance with pitch angle change. Variations in elastic coupling are accommodated by intermediate value of R.

B. Seismic Forces

Considering the effect of hub acceleration due to ground motion, let g_x, g_y and g_z be the acceleration components in the X, Y and Z directions, transmitted to the hub due to ground motion. Then, the initial forces corresponding to flapping and leadlagging motions are

$$F_{\beta g} = -m[g_x \sin(\phi+\zeta)\ \sin\beta + g_y \cos(\phi+\zeta)\ \sin\beta + g_z] \tag{3-13}$$

$$F_{\zeta g} = m[g_x \cos(\phi+\zeta) - g_y \sin(\phi+\zeta)] \tag{3-14}$$

The nondimensional generalized forces attributed to ground motion can be obtained by substituting Eq. (3-13) and Eq. (3-14) into Eq. (3-3) and Eq. (3-4) and nondimensionalized,

$$\bar{M}_\beta = -\frac{3}{2}[\bar{g}_x \sin(\phi+\zeta)\sin\beta + \bar{g}_y \cos(\phi+\zeta)\ \sin\beta + \bar{g}_z] \tag{3-15}$$

$$\bar{M}_\zeta = \frac{3}{2}[\bar{g}_x \cos(\phi+\zeta)\ \cos\beta - \bar{g}_y \sin(\phi+\zeta)\ \cos\beta] \tag{3-16}$$

where $\bar{g}_x = g_x/R\,\Omega^2$

$\bar{g}_y = g_y/R\,\Omega^2$

$\bar{g}_z = g_z/R\,\Omega^2$

It is of interest to note that the coefficients of $\bar{g}_x$ and $\bar{g}_y$ are functions of β and ζ, not functions of β' and ζ'. It implies that the ground acceleration components $\bar{g}_x$ and $\bar{g}_y$ appear in the stiffness terms of the equations of motion, but do not affect the system damping. In contrast, the coefficient of $\bar{g}_z$ does not involve β, β', ζ or ζ'; therefore, $\bar{g}_z$ appears only in the inhomogeneous terms of the equations of motion.

C. Aerodynamic Forces

The aerodynamic forces are obtained from a linear, quasi-steady strip theory. The lift and drag components per unit length are

$$\ell = \frac{\rho a c}{2}(U_T^2 + U_p^2)\sin(\theta - \phi) \tag{3-17}$$

$$d = \frac{\rho a c}{2}(U_T^2 + U_p^2)\frac{c_d}{a} \tag{3-18}$$

where $\phi = \tan^{-1}\frac{U_p}{U_T}$, is known as the inflow angle, and U_p and U_T are the relative velocities perpendicular and tangent to the rotor disc, respectively. The symbols ρ, a, c and c_d are defined in the NOTATION list.

The aerodynamic forces $F_{\beta_{aero}}$ and $F_{\zeta_{aero}}$ as shown in Figure 3-3 are

$$F_{\beta_{aero}} = \frac{\rho a c}{2}[U_T\theta - U_T U_p(1 + \frac{c_d}{a})] \tag{3-19}$$

$$F_{\zeta_{aero}} = \frac{\rho a c}{2}[U_p^2 - U_T U_p\theta - U_T^2\frac{c_d}{a}] \tag{3-20}$$

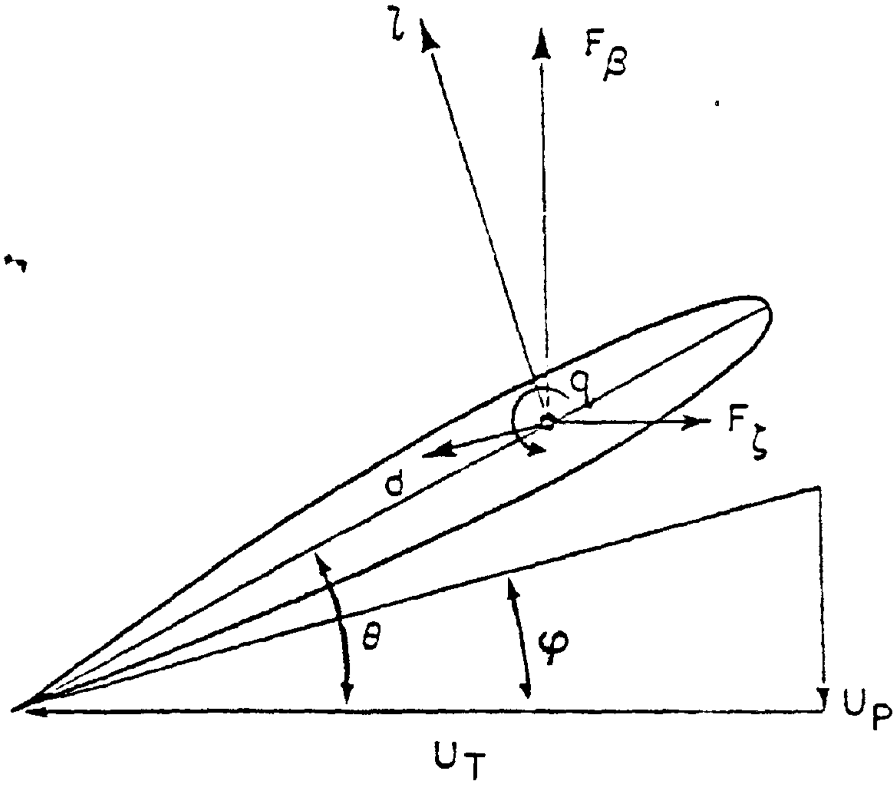

Figure 3-3 Blade Element Geometry.

The velocity components U_p and U_T may be computed as follows:

$$U_p = -(V_o + v - v_i) \cos\beta + (U_o + u) \sin\beta \sin(\psi + \zeta) + r\dot{\beta} \quad (3\text{-}21)$$

$$U_T = -(U_o + u) \cos(\psi + \zeta) + r(\Omega + \dot{\zeta}) \cos\beta + w \sin(\psi + \zeta) \quad (3\text{-}22)$$

where V_o and U_o are the steady axial and crosswind components in the -Z and X direction, v and u are the turbulence components in the directions of V_o and U_o, w is the vertical turbulence, and v_i is the induced inflow velocity.

It is assumed that V_o and U_o vary linearly with the vertical distance Y from the ground. This is an approximation for the actual distribution of velocity in the atmospheric boundary layer, an approximation proposed by Ormiston [11]. Then

$$V_o = V[1 - rK \cos\beta \cos(\psi + \zeta)] \quad (3\text{-}23)$$

$$U_o = U[1 - rK \cos\beta \cos(\psi + \zeta)] \quad (3\text{-}24)$$

where k is a linear velocity gradient of the atmospheric boundary layer, assumed to be the same in x and z directions. Substituting Eqs. (3-23) and (3-24) into Eqs. (3-21) and (3-22), and nondimensionalizing,

$$\overline{U}_p = [1 - xk \cos\beta \cos(\psi + \zeta)] [-\lambda \cos\beta + \mu \sin\beta \sin(\psi + \zeta)]$$

$$-\nu \cos\beta + \lambda_i \cos\beta + \eta \sin\beta \sin(\psi + \zeta) + x\beta' \quad (3\text{-}25)$$

$$\overline{U}_T = -[1-xk\cos\beta\cos(\phi+\zeta)]\mu\cos(\phi+\zeta) - \eta\cos(\phi+\zeta)$$

$$+ x(1+\zeta')\cos\beta + \xi\sin(\phi+\zeta) \qquad (3\text{-}26)$$

where $\overline{U}_p = U_p/R\Omega$

$\overline{U}_T = U_T/R\Omega$

$x = r/R$

$\lambda = V/R\Omega$

$\mu = U/R\Omega$

$\lambda_i = v_i/R\Omega$

$\nu = v/R\Omega$

$\eta = u/R\Omega$

$\xi = w/R\Omega$

Adopting the same terminology as that used in the helicopter analyses, μ will be referred to as the advanced ratio, λ_i the induced flow ratio, and ν, η and ξ non-dimensional turbulent velocity components.

As in the case of helicopter rotor in forward flight, reversed flows can occur where U_T becomes negative near the blade root. For low crosswind velocities, the effect may be neglected.

The following nondimensional generalized forces are obtained by substituting the aerodynamic forces into Eqs. (3-3) and (3-4), integrating along the blade from 0 to RB, and nondimensionalizing:

$$\overline{M}_\beta = \frac{\gamma}{2}\int_0^B [\overline{U}_T^2\theta - \overline{U}_T\overline{U}_p(1 + \frac{c_d}{a})]\, x dx \qquad (3\text{-}27)$$

$$\overline{M}_\zeta = \frac{\gamma}{2}\int_0^B [\overline{U}_p^2 - \overline{U}_T\overline{U}_p\theta - \overline{U}_T^2\frac{c_d}{a}]\cos\beta\; x\, dx \qquad (3\text{-}28)$$

where $\gamma = \frac{\rho a c R^4}{I}$, is known as the Lock number.

3.3 Flap Motion

The equation for uncoupled flapping motion follows from letting $\zeta = 0$, in Eq. (3-5):

$$\beta'' + \cos\beta \sin\beta + \overline{K}_{\beta\beta}(\beta - \beta_{pc}) + \frac{3}{2}\overline{g}\sin\beta\cos\phi = \overline{M}_\beta \qquad (3\text{-}29)$$

where $\overline{M}_\beta$ consists of both the seismic and aerodynamic loads, which have been obtained in Eqs. (3-15) and (3-19). After some algebraic work and neglecting higher order terms of such small quantities as β, β', μ, ξ, η, ν, $\overline{g}_x$, $\overline{g}_y$ and $\overline{g}_z$, the equation of motion may be cast in the following linear form:

$$\beta'' + C\beta' + K\beta = F \qquad (3\text{-}30)$$

where

$$C = C_{110} + C_{11\xi}\xi + C_{11\eta}\eta$$
$$K = K_{110} + K_{11\xi}\xi + K_{11\eta}\eta + K_{11g_x}\overline{g}_x + K_{11g_y}\overline{g}_y$$
$$F = F_{10} + F_{1\xi}\xi + F_{1\eta}\eta + F_{1\nu}\nu + F_{1g_z}\overline{g}_z$$

The coefficients C, K and F are deterministic periodic functions of ϕ, and are listed in Table I.

It should be noted that excitations in the axial direction, ground acceleration $\bar{g}_z$ and turbulence component ν, appear only in the inhomogeneous terms; they do not cause instability of the system. Excitations in the other

Table I

$$C_{11\theta} = -\frac{\gamma}{2}\{(1+\frac{c_d}{a})(\frac{B^3}{3}\mu C - \frac{B^4}{4}k\mu C^2 - \frac{B^4}{4})\}$$

$$C_{11\xi} = -\frac{\gamma}{2}\{-(1+\frac{c_d}{a})\frac{B^3}{3}S\}$$

$$C_{11\eta} = -\frac{\gamma}{2}\{(1+\frac{c_d}{a})\frac{B^3}{3}C\}$$

$$K_{11\theta} = (\overline{K}_{\beta\beta}+1) + \frac{3}{2}\overline{g}C - \frac{\gamma}{2}\{(1+\frac{c_d}{a})(-\frac{B^3}{3}\mu s + \frac{B^4}{4}k\mu CS)\}$$

$$K_{11\xi} = -\frac{\gamma}{2}\{(1+\frac{c_d}{a})(-\frac{B^2}{2}\mu S^2 + \frac{B^3}{3}k\mu CS^2)\}$$

$$K_{11\eta} = -\frac{\gamma}{2}\{(1+\frac{c_d}{a})(\frac{B^2}{2}\mu CS - \frac{B^3}{3}k\mu C^2S + \frac{B^2}{2}\mu CS - \frac{B^3}{3}\mu CS$$
$$- \frac{B^3}{3}S)\}$$

$$K_{11g_x} = \frac{3}{2}S$$

$$K_{11g_y} = \frac{3}{2}C$$

$$F_{10} = \overline{K}_{\beta\beta}\beta_{pc} + \frac{\gamma}{2}\{\theta(\frac{B^4}{4} - \frac{2B^3}{3}\mu C + \frac{B^4}{2}k\mu C^2) + (1+\frac{c_d}{a})(-\frac{B^2}{2}\lambda\mu C$$
$$+ \frac{4B^3}{3}\lambda k\mu C^2 + \frac{B^2}{2}\mu C\lambda_i - \frac{B^4}{4}\lambda k^2\mu C^3 - \frac{2B^3}{3}k\mu\lambda_i C^2 + \frac{B^3}{3}\lambda$$
$$- \frac{B^4}{4}\lambda kC - \frac{B^3}{3}\lambda_i)\}$$

Table I continued

$$F_{1\xi} = \frac{\gamma}{2}\{\theta(-B^2\mu CS + \frac{2B^3}{3}k\mu C^2 S + \frac{2B^3}{3}S) + (1 + \frac{c_d}{a})(\frac{B^2}{2}\lambda S - \frac{B^2}{2}\lambda_i S$$

$$- \frac{B^3}{3}\lambda kCS)\}$$

$$F_{1\eta} = \frac{\gamma}{2}\{\theta(B^2\mu C^2 - \frac{2B^3}{3}k\mu C^3 - \frac{2B^3}{3}C) + (1 + \frac{c_d}{a})(-\frac{B^2}{2}\lambda C + \frac{B^2}{2}\lambda_i C$$

$$+ \frac{B^3}{3}\lambda kC^2)\}$$

$$F_{1\nu} = \frac{\gamma}{2}\{(1 + \frac{c_d}{a})(-\frac{B^2}{2}\mu C + \frac{B^3}{3}k\mu C^2 + \frac{B^3}{3})\}$$

$$F_{1g_z} = -\frac{3}{2}$$

where $C = \cos\phi$, $S = \sin\phi$

directions appear in the coefficients of β and/or β'; They affect the system stability.

3.4 Coupled Flap-lag Motion

The equations of motion for coupled flap-lagging motion are obtained by substituting the generalized forces due to aerodynamic and seismic loads into Eqs. (3-5) and (3-6). They remain to be coupled nonlinear differential equations after judiciously neglecting some small terms. The equations can be linearized by converting them to a corresponding set of equations for small perturbations about a periodic equilibrium solution of the original nonlinear system, ζ_e, β_e and θ_e[16]. Specifically, let

$$\begin{aligned} \zeta &= \zeta_e + \delta\zeta \\ \beta &= \beta_e + \delta\beta \\ \theta &= \theta_e + \theta_\beta \delta\beta + \theta_\zeta \delta\zeta \end{aligned} \tag{3-31}$$

where θ_β and θ_ζ are pitch-flap and pitch-lag coupling parameters which approximate the changes in the blade pitch angle due to changes in flap and leadlag angles. For small angles of β and ζ,

$$\begin{aligned} \sin\beta &= \beta_e + \delta\beta \\ \cos\beta &= 1 - \beta_e \delta\beta \\ \sin\zeta &= \zeta_e + \delta\zeta \\ \cos\zeta &= 1 - \zeta_e \delta\zeta \end{aligned} \tag{3-32}$$

The linearized equations of motion are obtained by substituting Eq. (3-31) into the nonlinear equations, collecting linear terms in $\delta\beta$ and $\delta\zeta$ and their derivatives, and subtracting out the equilibrium solution which will be discussed later, to yield:

$$\begin{Bmatrix}\delta\beta'' \\ \delta\zeta''\end{Bmatrix} + [C]\begin{Bmatrix}\delta\beta' \\ \delta\zeta'\end{Bmatrix} + [K]\begin{Bmatrix}\delta\beta \\ \delta\beta\end{Bmatrix} = \begin{Bmatrix}F\end{Bmatrix} \qquad (3\text{-}33)$$

where [C] and [K] are two by two square matrices and F is a two by one column matrix. The elements of [K], [C] and F are

$$C_{ij} = C_{ijo} + C_{ij\xi}\xi + C_{ij\eta}\eta + C_{ij\nu}\nu$$
$$K_{ij} = K_{ijo} + K_{ijg_x}\overline{g}_x + K_{ijg_y}\overline{g}_y + K_{ijg_z}\overline{g}_z + K_{ij\xi}\xi + K_{ij\eta}\eta + K_{ij\nu}\nu$$
$$F_i = F_{ig_x}\overline{g}_x + F_{ig_y}\overline{g}_y + F_{ig_z}\overline{g}_z + F_{i\xi}\xi + F_{i\eta}\eta + F_{i\nu}\nu$$

where i, j = 1,2 and the coefficients C_{ij}, K_{ij} and F_i are listed in Table II. It can be seen from Table II that, unlike the case of uncoupled flapping motion, the axial turbulence component also appears in the homogeneous terms as the other turbulence components. However, the roles played by various earthquake components remain the same as those in the case of uncoupled flapping motion.

3.5 Coupled Flap-lag-torsion Motion

To investigate the three-way coupling of flap-lag-torsional motion, an assumption is made that the torsional mode is linear [2,8], in addition to those assumptions listed previously in Section 2-1; namely, the torsional angle is proportional to x, $0 < x < 1$. The equation of motion for torsion may be written as follows:

Table II

$$F_{1g_x} = -\frac{3}{2} S \beta_e$$
$$F_{1g_y} = -\frac{3}{2} C \beta_e$$
$$F_{1g_z} = -\frac{3}{2}$$
$$F_{2g_x} = \frac{3}{2} (C - S \zeta_e)$$
$$F_{2g_y} = -\frac{3}{2} (S + C \zeta_e)$$
$$K_{11g_x} = \frac{3}{2} (S + C \zeta_e)$$
$$K_{11g_y} = \frac{3}{2} (C - S \zeta_e)$$
$$K_{21g_x} = \frac{3}{2} C \beta_e$$
$$K_{21g_y} = -\frac{3}{2} S \beta_e$$
$$K_{12g_x} = \frac{3}{2} C \beta_e$$
$$K_{12g_y} = -\frac{3}{2} S \beta_e$$
$$K_{22g_x} = \frac{3}{2} (S + C \zeta_e)$$
$$K_{22g_y} = \frac{3}{2} (C - S \zeta_e)$$

$$F_{1\xi} = \frac{\gamma}{2} [\theta_e(\frac{2B^3}{3} S + \frac{2B^3}{3} \mu k C^2 S - B^2 \mu CS + \frac{2B^3}{3} C \zeta_e + \frac{2B^3}{3} S \zeta_e') - (1 + \frac{c_d}{a})$$
$$(-\frac{B^2}{2} \lambda S + \frac{B^3}{3} \lambda kCS + \frac{B^2}{2} \lambda_i S - \frac{B^2}{2} \lambda C \zeta_e + \frac{B^3}{3} \lambda kC^2 \zeta_e - \frac{B^3}{3} \lambda kS^2 \zeta_e + \frac{B^2}{2} \lambda_i C \zeta_e$$
$$+ \frac{B^3}{3} S \beta_e')]$$

$$F_{1\eta} = \frac{\gamma}{2} [\theta_e(B^2 \mu C^2 - \frac{2B^3}{3} \mu kC^3 - \frac{2B^3}{3} C + \frac{2B^3}{3} S \zeta_e - \frac{2B^3}{3} C \zeta_e') - (1 + \frac{c_d}{a}) (\frac{B^2}{2} \lambda C$$
$$- \frac{B^3}{3} \lambda kC^2 - \frac{B^2}{2} \lambda_i C - \frac{B^2}{2} CS \beta_e - \frac{B^2}{2} \lambda S \zeta_e + \frac{2B^3}{3} \lambda kCS \zeta_e + \frac{B^2}{2} \lambda_i S \zeta_e - \frac{B^3}{3} C \beta_e'$$
$$+ \frac{B^3}{3} S \beta_e)]$$

Table II continued

$$F_{1\nu} = \frac{\gamma}{2}\,[-(1+\frac{c_d}{a})\,(\frac{B^2}{2}\mu C - \frac{B^3}{3}\mu k C^2 - \frac{B^3}{3} - \frac{B^3}{3}\zeta_e')]$$

$$F_{2\xi} = \frac{\gamma}{2}\,[-\theta_e(-\frac{B^2}{2}\lambda S + \frac{B^3}{3}\lambda k CS + \frac{B^2}{2}\lambda_i S - \frac{B^2}{2}\lambda C\zeta_e + \frac{B^3}{3}\lambda k C^2\zeta_e - \frac{B^3}{3}\lambda k S^2\zeta_e$$

$$+ \frac{B^2}{2}\lambda_i C\zeta_e + \frac{B^3}{3} S\beta_e') - \frac{c_d}{a}\frac{2B^3}{3} S]$$

$$F_{2\eta} = \frac{\gamma}{2}\,[-\theta_e(\frac{B^2}{2}\lambda C - \frac{B^3}{3}\lambda k C^2 - \frac{B^2}{2}\lambda_i C - \frac{B^2}{2} CS\beta_e - \frac{B^2}{2}\lambda S\zeta_e + \frac{2B^3}{3}\lambda k CS\zeta_e$$

$$+ \frac{B^2}{2}\lambda_i S\zeta_e - \frac{B^3}{3} C\beta_e' + \frac{B^3}{3} S\beta_e) - B^2\lambda S\beta_e + \beta^2\lambda_i S\beta_e + \frac{2B^3}{3}\lambda k CS\beta_e + \frac{c_d}{a}\frac{2B^3}{3} C]$$

$$F_{2\nu} = \frac{\gamma}{2}\,[-\theta_e(\frac{B^2}{2}\mu C - \frac{B^3}{3}\mu k C^2 - \frac{B^3}{3} - \frac{B^3}{3}\zeta_e') + B^2\lambda - B^2\lambda - \frac{2B^3}{3}\lambda k C + \frac{2B^3}{3}\lambda k S\zeta_e - \frac{2B^3}{3}\beta_e']$$

$$C_{110} = -\frac{\gamma}{2}\,[-(1 + \frac{c_d}{a})\,(-\frac{B^3}{3}\mu C + \frac{B^4}{4}\mu k C^2 + \frac{B^4}{4} + \frac{B^4}{4}\zeta_e')]$$

$$C_{1\xi} = -\frac{\gamma}{2}\,[-(1 + \frac{c_d}{a})\,(\frac{B^3}{3} S + \frac{B^3}{3} C\zeta_e)]$$

$$C_{1\eta} = -\frac{\gamma}{2}\,[-(1 + \frac{c_d}{a})\,(-\frac{B^3}{3} C + \frac{B^3}{3} S\zeta_e)]$$

$$C_{120} = -\frac{\gamma}{2}\,[\theta_e(\frac{B^4}{2} + \frac{B^4}{2}\mu k C^2 - \frac{2B^3}{3}\mu C + \frac{B^4}{2}\zeta_e') - (1 + \frac{c_d}{a})\,(-\frac{B^3}{3}\lambda + \frac{B^4}{4}\lambda k C$$

$$+ \frac{B^3}{3}\lambda_i - \frac{B^4}{4}\lambda k S\zeta_e + \frac{B^4}{4}\beta_e')] + 2\beta_e$$

$$C_{12\xi} = -\frac{\gamma}{2}\,[\theta_e\,(\frac{2B^3}{3} S + \frac{2B^3}{3} C\zeta_e)]$$

Table II continued

$$C_{12\eta} = -\frac{\gamma}{2}[\theta_e\,(-\frac{2B^3}{3}C + \frac{2B^3}{3}S\,\zeta_e) - (1 + \frac{c_d}{a})\,\frac{B^3}{3}S\,\beta_e]$$

$$C_{12\nu} = -\frac{\gamma}{2}[\frac{B^3}{3}(1 + \frac{c_d}{a})]$$

$$C_{210} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^3}{3}\mu C + \frac{B^4}{4}\mu kC^2 + \frac{B^4}{4} + \frac{B^4}{4}\zeta_e') - \frac{2B^3}{3}\lambda + \frac{B^4}{2}\lambda kC + \frac{2B^3}{3}\lambda_i$$

$$+ \frac{B^4}{2}\beta_e' - \frac{B^4}{2}\lambda kS\,\zeta_e] - 2\beta_e$$

$$C_{21\xi} = -\frac{\gamma}{2}[-\theta_e\,(\frac{B^3}{3}S + \frac{B^3}{3}C\,\zeta_e)]$$

$$C_{21\eta} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^3}{3}C + \frac{B^3}{3}S\,\zeta_e) + \frac{2B^3}{3}S\,\beta_e]$$

$$C_{21\nu} = -\frac{\gamma}{2}[-\frac{2B^3}{3}]$$

$$C_{220} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^3}{3}\lambda + \frac{B^4}{4}\lambda kC + \frac{B^3}{3}\lambda_i - \frac{B^4}{4}\lambda kS\,\zeta_e + \frac{B^4}{4}\beta_e') - \frac{c_d}{a}\frac{B^4}{2}]$$

$$C_{22\xi} = -\frac{\gamma}{2}[-\frac{c_d}{a}\frac{2B^3}{3}S]$$

$$C_{22\eta} = -\frac{\gamma}{2}[-\theta_e\frac{B^3}{3}S\,\beta_e + \frac{c_d}{a}\frac{2B^3}{3}C]$$

$$C_{22\nu} = -\frac{\gamma}{2}[\frac{B^3}{3}\theta_e]$$

$$K_{110} = -\frac{\gamma}{2}[\frac{B^4}{4}\theta_\beta - (1+\frac{c_d}{a})\,(\frac{B^3}{3}\mu S - \frac{B^4}{4}k\mu CS + \frac{2B^3}{3}\lambda\beta_e - \frac{3B^4}{4}\lambda kC\,\beta_e - \frac{2B^3}{3}\lambda_i\beta_e)$$

Table II continued

$$- \frac{B^4}{2} \theta_e \beta_e] + \bar{K}_{\beta\beta} + 1 + 2\zeta_e + \frac{3}{2} \bar{g} \, (C - S \zeta_e)$$

$$K_{1\xi} = - \frac{\gamma}{2} [\frac{2B^3}{3} \theta_\beta S - (1 + \frac{c_d}{a}) \, (\frac{B^2}{2} \mu S^2 - \frac{B^3}{3} k\mu CS^2 + \frac{B^2}{2} \lambda S \beta_e - \frac{2B^3}{3} \lambda kCS \beta_e$$

$$- \frac{B^2}{2} \lambda_i S \beta_e) - \frac{2B^3}{3} S \theta_e \beta_e]$$

$$K_{1\eta} = - \frac{\gamma}{2} [- \frac{2B^3}{3} \theta_\beta C - (1 + \frac{c_d}{a})(- \frac{B^2}{2} \mu CS + \frac{2B^3}{3} \theta kC^2 S - \frac{B^2}{2} \mu C + \frac{B^3}{3} S - \frac{B^2}{2} \lambda C \beta_e$$

$$+ \frac{2B^3}{3} \lambda kC^2 \beta_e + \frac{B^2}{2} \lambda_i C \beta_e + \frac{B^3}{3} C \zeta_e + \frac{B^3}{3} S \zeta_e') + \frac{2B^3}{3} C \theta_e \beta_e]$$

$$K_{1\nu} = - \frac{\gamma}{2} [-(1 + \frac{c_d}{a}) \frac{2B^3}{3} \beta_e]$$

$$K_{120} = - \frac{\gamma}{2} [\theta_e (\frac{2B^3}{3} \mu S - B^4 \mu kCS) + \frac{B^4}{4} \theta_\zeta - (1 + \frac{c_d}{a})(- \frac{B^2}{2} \mu\lambda S + \frac{4B^3}{3} \mu\lambda kCS$$

$$- \frac{3B^4}{4} \mu\lambda k^2 C^2 S + \frac{B^2}{2} \mu\lambda_i S - \frac{2B^3}{3} \mu k\lambda_i CS - \frac{B^4}{4} \lambda kS - \frac{B^4}{4} \lambda kC \zeta_e - \frac{B^4}{4} \lambda kS \zeta_e')]$$

$$+ \bar{K}_{\beta\zeta} - \frac{3}{2} \bar{g} S \beta_e$$

$$K_{12\xi} = - \frac{\gamma}{2} [\theta_e (-B^2 \mu C^2 + B^2 \mu S^2 - \frac{4B^3}{3} \mu kCS^2 + \frac{2B^3}{3} \mu kC^3 + \frac{2B^3}{3} C - \frac{2B^3}{3} S \zeta_e + \frac{2B^3}{3} C \zeta_e')$$

$$+ \frac{2B^3}{3} \theta_\zeta S - (1 + \frac{c_d}{a})(- \frac{B^2}{2} \lambda C + \frac{B^3}{3} \lambda kC^2 - \frac{B^3}{3} \lambda kS^2 + \frac{B^2}{2} \lambda_i C + \frac{B^2}{2} \lambda S \zeta_e$$

$$- \frac{B^2}{2} \lambda_i S \zeta_e + \frac{B^3}{3} C \beta_e' - \frac{4B^3}{3} \lambda kCS \zeta_e)]$$

Table II continued

$$K_{12\eta} = -\frac{\gamma}{2}\Big[\theta_e\Big(-2B^2\mu CS + 2B^3\mu kC^2S + \frac{2B^3}{3}S + \frac{2B^3}{3}C\zeta_e + \frac{2B^3}{3}S\zeta_e'\Big) - \frac{2B^3}{3}C\theta_\zeta$$

$$-\Big(1+\frac{c_d}{a}\Big)\Big(-\frac{B^2}{2}\lambda S + \frac{2B^3}{3}\lambda kCS + \frac{B^2}{2}\lambda_iS - \frac{B^2}{2}\lambda C\zeta_e + \frac{B^2}{2}\lambda_iC\zeta_e + \frac{2B^3}{3}\lambda kC^2\zeta_e$$

$$-\frac{2B^3}{3}\lambda kS^2\zeta_e + \frac{B^3}{3}S\beta_e' + \frac{B^3}{3}C\beta_e\Big)\Big]$$

$$K_{12\nu} = -\frac{\gamma}{2}\Big[-\Big(1+\frac{c_d}{a}\Big)\Big(-\frac{B^2}{2}\mu S + \frac{2B^3}{3}\mu kCS\Big)\Big]$$

$$K_{210} = -\frac{\gamma}{2}\Big[\frac{B^3}{3}\theta_\beta\lambda - \frac{B^4}{4}\lambda k\theta_\beta C - \frac{B^3}{3}\lambda_i\theta_\beta - B^3\lambda\theta_e\beta_e + B^4\lambda kC\theta_e\beta_e + \beta^3\lambda_i\theta_e\beta_e$$

$$-B^2\lambda\mu S + \frac{4B^3}{3}k\mu\lambda CS - \frac{B^2}{2}\lambda k^2\mu CS + B^2\mu\lambda_iS - \frac{2B^3}{3}k\mu\lambda_iCS - B^4\lambda^2k^2C^2\beta_e$$

$$-B^2\lambda_i^2\beta_e + 2B^3\lambda^2kC\beta_e + 2B^2\lambda\lambda_i\beta_e - 2B^3k\lambda\lambda_iC\beta_e - B^2\lambda^2\beta_e - \frac{B^2}{2}\lambda^2\beta_e$$

$$-\frac{B^4}{4}\lambda^2k^2C^2\beta_e - \frac{B^2}{2}\lambda_i^2\beta_e + \frac{2B^3}{3}\lambda^2kC\beta_e + B^2\lambda\lambda_i\beta_e - \frac{2B^3}{3}k\lambda_i\lambda C\beta_e - \theta_e$$

$$\Big(-\frac{B^3}{3}\mu S - \frac{B^4}{4}k\mu CS\Big)\Big] + \bar{K}_{\beta\zeta} - 2\beta_e' - \frac{3}{2}\bar{g}S\beta_e$$

$$K_{21\xi} = -\frac{\gamma}{2}\Big[\frac{B^2}{2}\lambda\theta_\beta S - \frac{B^3}{3}\lambda kCS\theta_\beta - \frac{B^2}{2}\lambda\,\theta_iS_\beta + \theta_e\beta_e(-B^2\lambda S + B^3\lambda kCS + B^2\lambda_iS)$$

$$-\theta_e\Big(\frac{B^2}{2}\mu S - \frac{B^3}{3}k\mu CS^2\Big)\Big]$$

$$K_{21\eta} = -\frac{\gamma}{2}\Big[-\theta_\beta\Big(\frac{B^2}{2}\lambda C - \frac{B^3}{3}\lambda kC^2 - \frac{B^2}{2}\lambda_iC\Big) + \theta_e\beta_e(B^2\lambda C - B^3\lambda kC^2 - B^2\lambda_iC) - B^2\lambda S$$

Table II continued

$$+ \frac{2B^3}{3}\lambda kCS + B^2\lambda_i S - B^2\lambda C\zeta_e + \frac{2B^3}{3}\lambda kC^2\zeta_e - \frac{2B^3}{3}\lambda kS^2\zeta_e + B^2\lambda_i C\zeta_e$$

$$+ \frac{2B^3}{3} S\beta_e' - \theta_e(-\frac{B^2}{2}\mu CS + \frac{2B^3}{3}\mu kC^2 S - \frac{B^2}{2}\mu C + \frac{B^3}{3} S + \frac{B^3}{3} C\zeta_e + \frac{B^3}{3} S\zeta_e')]$$

$$K_{21\nu} = -\frac{\gamma}{2}[\frac{B^3}{3}\theta_\beta - B^3\theta_e\beta_e - B^2\mu S + \frac{2B^3}{3} k\mu CS - 3B^2\lambda\beta_e + \frac{8B^3}{3}\lambda kC\beta_e + 3B^2\lambda_i\beta_e]$$

$$K_{220} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^2}{2}\mu\lambda S + \frac{4B^3}{3}\mu\lambda kCS - \frac{3B^4}{4}\mu\lambda k^2C^2S + \frac{B^2}{2}\mu\lambda_i S - \frac{2B^3}{3}\mu k\lambda_i CS$$

$$- \frac{B^4}{4}\lambda ks - \frac{B^4}{4}\lambda kC\zeta_e - \frac{B^4}{4}\lambda kS\zeta_e') + \frac{B^3}{3}\lambda\theta_\zeta - \frac{B^4}{4}\lambda kC\theta_\zeta - \frac{B^3}{3}\lambda_i\theta_\zeta$$

$$- \frac{B^4}{2}\lambda^2k^2CS + \frac{2B^3}{3}\lambda^2kS - \frac{2B^3}{3}k\lambda\lambda_i S - \frac{B^4}{2}\lambda^2k^2C^2\zeta_e + \frac{B^4}{2}k^2\lambda^2S^2\zeta_e$$

$$+ \frac{2B^3}{3}\lambda^2kC\zeta_e - \frac{2B^3}{3}k\lambda\lambda_i C\zeta_e - \frac{B^4}{2}k\lambda S\beta_e'] + \overline{K}_{\zeta\zeta} + \frac{3}{2}\overline{g}\,(C - S\zeta_e)$$

$$K_{22\xi} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^2}{2}\lambda C + \frac{B^3}{3}\lambda kC^2 - \frac{B^3}{3}\lambda kS^2 + \frac{B^2}{2}\lambda_i C + \frac{B^2}{2}\lambda S\zeta_e - \frac{B^2}{2}\lambda_i S\zeta_e$$

$$+ \frac{B^3}{3} C\beta_e' - \frac{4B^3}{3}\lambda kCS\zeta_e) + \frac{B^2}{2}\lambda S\theta_\zeta - \frac{B^3}{3}\lambda kCS\theta_\zeta - \frac{B^2}{2}\lambda_i S\theta_\zeta - \frac{c_d}{a}\frac{2B^3}{3} C]$$

$$K_{22\eta} = -\frac{\gamma}{2}[-\theta_e(-\frac{B^2}{2}\lambda S + \frac{2B^3}{3}\lambda kCS + \frac{B^2}{2}\lambda_i S - \frac{B^2}{2}\lambda C\zeta_e + \frac{2B^3}{3}\lambda kC^2\zeta_e - \frac{2B^3}{3}\lambda kS^2\zeta_e$$

$$+ \frac{B^2}{2}\lambda_i C\zeta_e + \frac{B^3}{3} S\beta_e' + \frac{B^3}{3} C\beta_e) - \frac{B^2}{2}\lambda C\theta_\zeta + \frac{B^3}{3}\lambda kC^2\theta_\zeta + \frac{B^2}{2}\lambda_i C\theta_\zeta$$

$$- B^2\lambda C\beta_e + \frac{2B^3}{3}\lambda kC^2\beta_e - \frac{2B^3}{3}\lambda kS^2\beta_e + \beta^2\lambda_i C\beta_e - \frac{c_d}{a}\frac{2B^3}{3}\zeta]$$

Table II continued

$$K_{22_\nu} = -\frac{\gamma}{2}\left[-\theta_e\left(-\frac{B^2}{2}\mu S + \frac{2B^3}{3}\mu kCS\right) + \frac{B^3}{3}\theta_\zeta + \frac{2B^3}{3}\lambda kS + \frac{2B^3}{3}\lambda kC\zeta_e\right]$$

where C = cos ϕ, S = sin ϕ and the coefficients of C_{ij}, K_{ij} and F_i are zero if they are not listed.

$$M_t\ (\alpha'' + \omega_\alpha^2 \alpha) = G/\Omega^2 \tag{3-34}$$

where ω_α is the torsional natural frequency, and α is the tip torsional angle. For a linear mode, the generalized mass and generalized forces are given by

$$M_t = \int_0^1 x^2 i_\alpha d(Rx) = \frac{1}{3} I_\alpha \tag{3-35}$$

$$G = \int qxd\ (Rx) = R \int_0^B qxdx \tag{3-36}$$

where i_α is the sectional polar mass moment of inertia about the elastic axis, I_α is the feathering mass moment of inertia of the blade, and q is the aerodynamic torque applied at the aerodynamic center of the blade.

Using a quasi-steady aerodynamic theory, the aerodynamic torque may be computed from

$$q = -\frac{\rho a c^3}{16} \Omega^2 R\ \overline{U}_T [x\,\alpha' + (1 + \zeta') \sin \beta]$$

$$\doteq -\frac{\rho a c^3}{16} \Omega^2 R\ \overline{U}_T (x\,\alpha' + \beta) \tag{3-37}$$

The equations for coupled flap-lag-torsional motion are obtained by combining Eqs. (3-5), (3-6) and (3-34), and replacing θ by $\theta + x\alpha$ in the derivation of the generalized forces due to aerodynamic loads in Eqs. (3-5) and (3-6). As before, the equations form a set of nonlinear coupled equations which may be linearized about the equilibrium position, β_e, ζ_e and α_e. In addition to Eq. (3-31), we let $\alpha = \alpha_e + \delta\alpha$ in the nonlinear equations mentioned above. Subtracting out the equilibrium terms which will be discussed later, neglecting the small quantities α_e, β_e^2, ζ_e^2, $\beta_e\zeta_e$ and μ^2 in the aerodynamic loads, and collecting the linear terms of $\delta\beta$, $\delta\zeta$ and $\delta\alpha$ yield:

$$\begin{Bmatrix} \delta\beta'' \\ \delta\zeta'' \\ \delta\alpha'' \end{Bmatrix} + [C] \begin{Bmatrix} \delta\beta' \\ \delta\zeta' \\ \delta\alpha' \end{Bmatrix} + [K] \begin{Bmatrix} \delta\beta \\ \delta\zeta \\ \delta\alpha \end{Bmatrix} = \begin{Bmatrix} F \\ \\ \end{Bmatrix} \tag{3-38}$$

where [C] and [K] are three by three square matrices and F are three by one column matrix. The elements of matrices [C], [K] and F are

$$C_{ij} = C_{ijo} + C_{ij\xi}\xi + C_{ij\eta}\eta + C_{ij\nu}\nu$$

$$K_{ij} = K_{ijo} + K_{ij\xi}\xi + K_{ij\eta}\eta + K_{ij\nu}\nu + K_{ijg_x}\bar{g}_x + k_{ijg_y}\bar{g}_y + k_{ijg_z}\bar{g}_z$$

$$F_i = F_{ig_x}\bar{g}_x + F_{ig_y}\bar{g}_y + F_{ig_z}\bar{g}_z + F_{i\xi}\xi + F_{i\eta}\eta + F_{i\nu}\nu$$

where i, j = 1,2,3 and the coefficients of C_{ij}, K_{ij} and F_i are given in Table II and Table III.

3.6 Periodic Equilibrium Solutions and Control Parameters

To linearize a set of nonlinear equations about the equilibrium solution that equilibrium solution must first be obtained. Here, we shall discuss the equilibrium solution for the flap-lag-torsional coupling which can be reduced to that of flap-lagging coupling by letting torsional angle be equal to zero.

Neglecting the higher order products of μ, k, β_e, ζ_e, β_e', α_e, α_e' and θ_e, the equilibrium equation may be written as the following forms:

Table III

$$F_{3\xi} = \gamma F[-\frac{B^2}{2} S\,\beta_e]$$

$$F_{3\eta} = \gamma F[\frac{B^2}{2} C\,\beta_e]$$

$$C_{320} = -\gamma F[-\frac{2B^3}{3}\beta_e]$$

$$C_{32\xi} = -\gamma F[-\frac{B^2}{2} S\,\beta_e]$$

$$C_{32\eta} = -\gamma F[\frac{B^2}{2} C\,\beta_e]$$

$$C_{330} = -\gamma F[\frac{B^3}{3}\mu C - \frac{B^4}{4}(1 + k\,\mu C^2) - \frac{B^4}{4}\zeta_e']$$

$$C_{33\xi} = -\gamma F[-\frac{B^3}{3} S - \frac{B^3}{3} C\,\zeta_e]$$

$$C_{33\eta} = -\gamma F[\frac{B^3}{3} C - \frac{B^3}{3} S\,\zeta_e]$$

$$K_{130} = -\frac{\gamma}{2}[\frac{B^5}{5} - \frac{B^4}{2}\mu C + \frac{2B^5}{5}\mu k C^2 + \frac{2B^5}{5}\zeta_e']$$

$$K_{13\xi} = -\frac{\gamma}{2}[-\frac{2B^3\mu}{3} SC + \frac{B^4}{2}\mu k C^2 S + \frac{B^4}{2} S + \frac{B^4}{2} C\,\zeta_e + \frac{B^4}{2} S\,\zeta_e']$$

$$K_{13\eta} = -\frac{\gamma}{2}[\frac{2B^3}{3}\mu C^2 - \frac{B^4}{2}\mu k C^3 - \frac{B^4}{2} C - \frac{B^4}{2} C\,\zeta_e' + \frac{B^4}{2} S\,\zeta_e]$$

$$K_{230} = -\frac{\gamma}{2}[-\frac{B^3}{3}\mu C(\lambda-\lambda_i) + \frac{B^4}{4}(\lambda-\lambda_i) + \frac{B^4}{4}\mu k C^2(2\lambda-\lambda_i) - \frac{B^5}{5}\lambda k C(1 + \mu k C^2)$$

Table III continued

$$+ \frac{B^4}{4}(\lambda - \lambda_i)\zeta_e' + \frac{B^5}{5}\lambda k(S\zeta_e - C\zeta_e') - \frac{B^5}{5}\beta_e']$$

$$K_{23\xi} = -\frac{\gamma}{2}[\frac{B^3}{3}S(\lambda-\lambda_1) - \frac{B^4}{4}\lambda kCS + \frac{B^3}{3}C\zeta_e(\lambda-\lambda_i) - \frac{B^4}{4}\lambda k\zeta_e(C^2-S^2) - \frac{B^4}{4}S\beta_e']$$

$$K_{23\eta} = -\frac{\gamma}{2}[-\frac{B^3}{3}C(\lambda-\lambda_i) + \frac{B^4}{4}\lambda kC^2 + \frac{B^3}{3}S\zeta_e(\lambda-\lambda_i) - \frac{B^4}{2}\lambda kCS\zeta_e$$

$$+ \frac{B^4}{4}C\beta_e' - \frac{B^4}{4}S\beta_e]$$

$$K_{23\nu} = -\frac{\gamma}{2}[\frac{B^4}{4} - \frac{B^3}{3}\mu C + \frac{B^4}{4}\mu kC^2 + \frac{B^4}{4}\zeta_e']$$

$$K_{310} = -\gamma F[\frac{B^2}{2}\mu C - \frac{B^3}{3}k\mu C^2 - \frac{B^3}{3} - \frac{2B^3}{3}\zeta_e']$$

$$K_{31\xi} = -\gamma F[-\frac{B^2}{2}S - \frac{B^2}{2}C\zeta_e - \frac{B^2}{2}S\zeta_e']$$

$$K_{31\eta} = -\gamma F[\frac{B^2}{2}C - \frac{B^2}{2}S\zeta_e + \frac{B^2}{2}C\zeta_e']$$

$$K_{32\xi} = -\gamma F[-\frac{B^2}{2}C\beta_e]$$

$$K_{32\eta} = -\gamma F[-\frac{B^2}{2}S\beta_e]$$

$$K_{330} = \omega_\alpha^2$$

where $F = \frac{3Ic^2}{16I_\alpha R^2}$, $C = \cos\phi$ and $S = \sin\phi$. The coefficients of C_{ij}, K_{ij} and F_i are zero if it was not listed above or in Table II.

$$\beta_e'' + (\bar{K}_{\beta\beta}+1)\,\beta_e + \bar{K}_{\beta\zeta}\zeta_e + \frac{3}{2}\,\bar{g}\,C\,\beta_e = \bar{K}_{\beta\beta}\beta_{pc} + \frac{\gamma}{2}\,[\frac{B^4}{4}\,\theta_e - \frac{B^3}{3}\,(\lambda_i - \lambda) + \frac{B^3}{3}\,\mu\,(C\,\beta_e'$$

$$- S\,\beta_e) - \frac{B^4}{4}\,\beta_e' + \frac{B^4}{4}\,\lambda k S\,\zeta_e - \frac{B^4}{4}\,\lambda\;k\,C\,\zeta_e' - \frac{B^3}{3}\,(\lambda_i - \lambda)\,\zeta_e' - \frac{B^2}{2}\,\mu\,(\lambda_i - \lambda)\,S\,\zeta_e$$

$$- \frac{2B^3}{3}\,\theta_e\mu C + \frac{B^2}{2}\,\mu C\,(\lambda_i - \lambda) - \frac{B^4}{4}\,\lambda kC + \frac{B^5}{5}\,\alpha_e - \frac{B^4}{2}\,\mu C\,\alpha_e] \tag{3-39}$$

$$\zeta_e'' + \bar{K}_{\beta\zeta}\beta_e + (\bar{K}_{\zeta\zeta} - \frac{3}{2}\,\bar{g}C)\,\zeta_e = \bar{K}_{\beta\zeta}\beta_{pc} - \frac{3}{2}\,\bar{g}S + \frac{\gamma}{2}\,[\frac{B^2}{2}\,\mu\lambda\theta_e C + \frac{B^2}{2}\,\mu\lambda_i C\,\theta_e$$

$$+ \frac{B^3}{3}\,\lambda\theta_e - \frac{B^4}{4}\,\lambda kC\,\theta_e - \frac{B^3}{3}\,\lambda_i\theta_e + \frac{B^2}{2}\,\lambda^2 + \frac{B^2}{2}\,\lambda_i^2 - B^2\lambda\lambda_i + \frac{2B^3}{3}\,\lambda kC(\lambda_i - \lambda)$$

$$- \frac{2B^3}{3}\,\lambda kS(\lambda_i - \lambda)\zeta_e + B^2\mu S(\lambda_i - \lambda)\beta_e' + \frac{2B^3}{3}\,(\lambda_i - \lambda)\beta_e' + \frac{B^4}{2}\,\lambda k\,C\beta_e'$$

$$- \frac{B^4}{4}\,(\lambda_i - \lambda)\alpha_e - \frac{B^5}{5}\,\lambda kC\,\alpha_e + \frac{B^3}{3}\,\mu C\,\alpha_e(\lambda_i - \lambda)] \tag{3-40}$$

$$\alpha_e'' + \omega_\alpha^2\alpha_e = \gamma F[\frac{B^3\mu}{3}\,C\,\alpha_e' - \frac{B^4}{4}\,\alpha_e' + \frac{B^3\mu}{2}\,C\,\beta_e - \frac{B^3}{3}\,\beta_e] \tag{3-41}$$

Eqs. (3-39), (3-40) and (3-41) are coupled linear differential equations with periodic coefficients. Approximate solutions can be readily obtained for the equilibrium solutions in terms of the input parameters λ, λ_i, θ_e, γ, F, k and β_{pc}. The solutions can be written as Fourier series and will be approximated by truncating the series after the first harmonic terms.

$$\begin{aligned} \beta_e &= \beta_0 + \beta_c \cos\phi + \beta_s \sin\phi \\ \zeta_e &= \zeta_0 + \zeta_c \cos\phi + \zeta_s \sin\phi \\ \alpha_e &= \alpha_0 + \alpha_c \cos\phi + \alpha_s \cos\phi \end{aligned} \tag{3-42}$$

When Eq. (3-42) is substituted into Eqs. (3-39), (3-40) and (3-41), a system of linear algebraic equations is obtained by equating the coefficients of $\cos\alpha\psi$, $\cos\psi$ and $\sin\psi$ of each differential equation to zero. Then, the approximated equilibrium solutions can be written as follows.

$$D\,\vec{X} = H \qquad (3\text{-}43)$$

where $\vec{X}^T = [\beta_0\ \beta_s\ \beta_c\ \zeta_0\ \zeta_s\ \zeta_c\ \alpha_0\ \alpha_s\ \alpha_c]$, D is nine by nine square matrix, and H is nine by one column matrix. The nonzero coefficients of matrices D and H are given in Table IV.

Before the equilibrium solutions of coupled flap-lag and flap-lag-torsion motions can be determined, it is necessary to specify the relationship between the wind velocity ratio λ, induced flow ratio λ_i, and blade pitch angle θ_e. This can be accomplished relatively simply by using momentum theory in conjunction with the aerodynamic force per unit length, Eq. (3-19), in which we assumed the flapping angle is small.

In this theory, the induced inflow are assumed to be uniformly distributed over the rotor disc. By neglecting the sidewind μ, and velocity gradient k which are small compared to others, the dimensionless thrust coefficient and power coefficient are given by

$$C_t = 4\,\frac{\lambda_i}{\lambda}\left(1 - \frac{\lambda_i}{\lambda}\right) \qquad (3\text{-}44)$$

$$C_Q = 4\,\frac{\lambda_i}{\lambda}\left(1 - \frac{\lambda_i}{\lambda}\right)^2 \qquad (3\text{-}45)$$

Table IV

$$d_{11} = \bar{K}_{\beta\beta} + 1$$

$$d_{13} = \frac{3}{4}\,\bar{g}$$

$$d_{14} = \bar{K}_{\beta\zeta}$$

$$d_{15} = \frac{\gamma B^2}{8}\,\mu(\lambda_i - \lambda)$$

$$d_{17} = \frac{-\gamma B^5}{10}$$

$$d_{19} = \frac{\gamma B^4 \mu}{8}$$

$$d_{21} = \frac{\gamma B^3}{6}\,\mu$$

$$d_{22} = \bar{K}_{\beta\beta}$$

$$d_{23} = \frac{-\gamma B^4}{8}$$

$$d_{24} = \frac{\gamma B^2}{4}\,\mu(\lambda_i - \lambda) - \frac{\gamma B^4}{8}\,\lambda k$$

$$d_{25} = \bar{K}_{\beta\zeta}$$

$$d_{26} = \frac{-\gamma B^3}{6}\,(\lambda_i - \lambda)$$

$$d_{28} = \frac{-\gamma B^5}{10}$$

$$d_{31} = \frac{3}{2}\,\bar{g}$$

$$d_{32} = \frac{\gamma B^4}{8}$$

$$d_{33} = \bar{K}_{\beta\beta}$$

$$d_{35} = \frac{\gamma B^3}{6}\,(\lambda_i - \lambda)$$

$$d_{36} = \bar{K}_{\beta\zeta}$$

$$d_{37} = \frac{\gamma B^4}{4}\,\mu$$

$$d_{39} = -\frac{\gamma B^5}{10}$$

$$d_{41} = \bar{K}_{\beta\zeta}$$

$$d_{42} = \frac{-\gamma B^4}{8}\,\lambda k - \frac{\gamma B^2}{4}\,\mu(\lambda_i - \lambda)$$

$$d_{44} = K_{\zeta\zeta}$$

$$d_{45} = \frac{\gamma B^3}{6}\,\lambda k(\lambda_i - \lambda)$$

$$d_{46} = -\frac{3}{4}\,\bar{g}$$

$$d_{47} = \frac{\gamma B^4}{8}\,(\lambda_i - \lambda)$$

$$d_{49} = -\frac{\gamma B^3}{12}\,\mu\,(\lambda_i - \lambda) + \frac{\gamma B^5 \lambda k}{20}$$

$$d_{51} = \frac{-\gamma B^2}{2}\,\mu(\lambda_i - \lambda)$$

$$d_{52} = \bar{K}_{\beta\zeta}$$

$$d_{53} = \frac{\gamma B^3}{3}\,(\lambda_i - \lambda)$$

$$d_{54} = \frac{\gamma B^3}{3}\,\lambda k(\lambda_i - \lambda)$$

$$d_{55} = \bar{K}_{\zeta\zeta} - 1$$

$$d_{58} = \frac{\gamma B^4}{8}\,(\lambda_i - \lambda)$$

$$d_{62} = \frac{-\gamma B^3}{3}\,(\lambda_i - \lambda)$$

$$d_{63} = \bar{K}_{\beta\zeta}$$

$$d_{64} = -\frac{3}{2}\,\bar{g}$$

$$d_{66} = \bar{K}_{\zeta\zeta} - 1$$

$$d_{67} = \frac{\gamma B^5 \lambda k}{10} - \frac{\gamma B^3 \mu}{6}\,(\lambda_i - \lambda)$$

Table IV continued

$$d_{69} = \frac{\gamma B^4}{8} (\lambda_i - \lambda)$$

$$d_{71} = \frac{\gamma F B^3}{3}$$

$$d_{73} = -\frac{\gamma F B^2 \mu}{4}$$

$$d_{77} = \omega_\alpha^2$$

$$d_{78} = -\frac{\gamma F B^3 \mu}{6}$$

$$d_{82} = \frac{\gamma F B^3}{3}$$

$$d_{88} = \omega_\alpha^2 - 1$$

$$d_{89} = -\frac{\gamma F B^4}{4}$$

$$d_{91} = -\frac{\gamma F B^2 \mu}{2}$$

$$d_{93} = \frac{\gamma F B^3}{3}$$

$$d_{98} = \frac{\gamma F B^4}{4}$$

$$d_{99} = \omega_\alpha^2 - 1$$

$$h_1 = \bar{K}_{\beta\beta} \beta_{pc} + \frac{\gamma}{8} [B^4 \theta_e - \frac{4B^3}{3} (\lambda_i - \lambda)]$$

$$h_3 = \frac{\gamma}{8} [2B^2 \mu (\lambda_i - \lambda) - B^4 \lambda k - \frac{8B^3}{3} \theta_e \mu]$$

$$h_4 = K_{\beta\zeta} \beta_{pc} - \frac{\gamma B^3}{6} \theta_e (\lambda_i - \lambda) + \frac{\gamma B^2}{4} (\lambda_i - \lambda)^2$$

$$h_5 = -\frac{3}{2} \bar{g}$$

$$h_6 = \frac{\gamma B^2}{4} \mu \theta_e (\lambda_i - \lambda) - \frac{\gamma B^4}{8} \theta_e \lambda k + \frac{\gamma B^3}{3} \lambda k (\lambda_i - \lambda)$$

The maximum power and maximum thrust which can be determined from Eqs. (3-44) and (3-45) occur when $\lambda_i = \lambda/3$ and $\lambda/2$, respectively. The condition $\lambda_i = \frac{\lambda}{2}$ is not permissible since at the downstream velocity is $\lambda - 2\lambda_i$, and the condition $\lambda_i > \lambda/2$ implies flow reversed in the wake.

A relation between λ, λ_i and θ_e can be obtained by equating the integrated rotor thrust from Eq. (3-19) to the thrust from the elementary momentum theory. The result is

$$\lambda_i = \frac{1}{2}\left(\lambda + \frac{a\sigma B^2}{8}\right) \pm \sqrt{1/4\left(\lambda + \frac{a\sigma B^2}{8}\right)^2 - \frac{a\sigma}{8}\left(\lambda B^2 + \frac{2B^3}{3}\theta_e\right)} \quad (3\text{-}46)$$

where $\sigma = \frac{bc}{\pi R}$, is known as rotor solidity, b is number of blades per rotor. The negative sign on the radical is normally used, since otherwise it will produce unrealistic λ_i, for instance, as $\theta_e = 0$, $\lambda_i = \lambda$ which will cause the flow reversal in the wake as mentioned above. This equation reflects the fact that the induced flow ratio is a dependent variable that can be determined by the independent variable λ and the control parameter θ_e. The pitch angle θ_e can be independently controlled to obtain desired rotor thrust or power. Therefore, Eq. (3-46) can be solved as well as Eq. (3-43) for any given combination of λ and θ_e.

When the blade pitch angle is zero, the solution to Eq. (3-46) is $\lambda_i = \frac{a\sigma B^2}{8}$. The zero power and thrust windmilling condition occurs when $\lambda_i = 0$ and $\theta_e = -3\lambda/2B$. Those two points where θ_e and $\lambda_i = 0$ can be used to define a simple approximation for Eq. (2-46)

$$\lambda_i = \frac{a\sigma B^2}{8} \left(1 + \frac{2B}{3\lambda} \theta_e\right)$$

or

$$\theta_e = \frac{3\lambda}{2B} \left(\frac{\lambda_i}{\frac{a\sigma B^2}{8}} - 1\right) \qquad (3\text{-}47)$$

where the accuracy of the approximation in the practical range will be discussed later.

Eq. (3-46) for the induced inflow is applicable for any arbitrary combination of λ and θ_e. However, for a typical constant rpm wind turbine, the blade pitch angle is controlled to produce the maximum power output as well as to prevent overloading the power generator. A typical operating schedule is the following. At low wind velocities, the pitch angle is controlled to produce the maximum power available. At the rated design condition, the rotor power output equals to the installed generator capacity of the wind power plant and the pitch angle is to be controlled to prevent the further power increase at the higher wind velocities. At the maximum operating condition, the turbine will be shut down to minimize the risk of damaging the rotor.

To operate under this schedule, the induced inflow ratio will be given by [11]

$$\lambda_i = \lambda/3, \ \lambda < \lambda_R$$

$$\lambda_i = \frac{4\lambda/27}{(\lambda/\lambda_R)^3 (1 - \lambda_i/\lambda)^2}, \ \lambda > \lambda_R \qquad (3\text{-}48)$$

where λ_R is the rated design wind velocity of the wind turbine plant.

Figure 3-4 shows that the pitch angle variation with axial wind velocity ratio λ based on Eqs. (3-46) and (3-48), and Eqs. (3-47) and (3-48) for B = .97, $\frac{a\sigma}{8}$ = .053 and λ_R = .1. The result indicates that Eq. (3-47) is a good approximation of Eq. (3-46) in the practical range of the wind turbine operating condition. When λ, λ_i and θ_e are known, the equilibrium solutions of the flap-lagging and flap-lag-torsional motions are ready to be calculated.

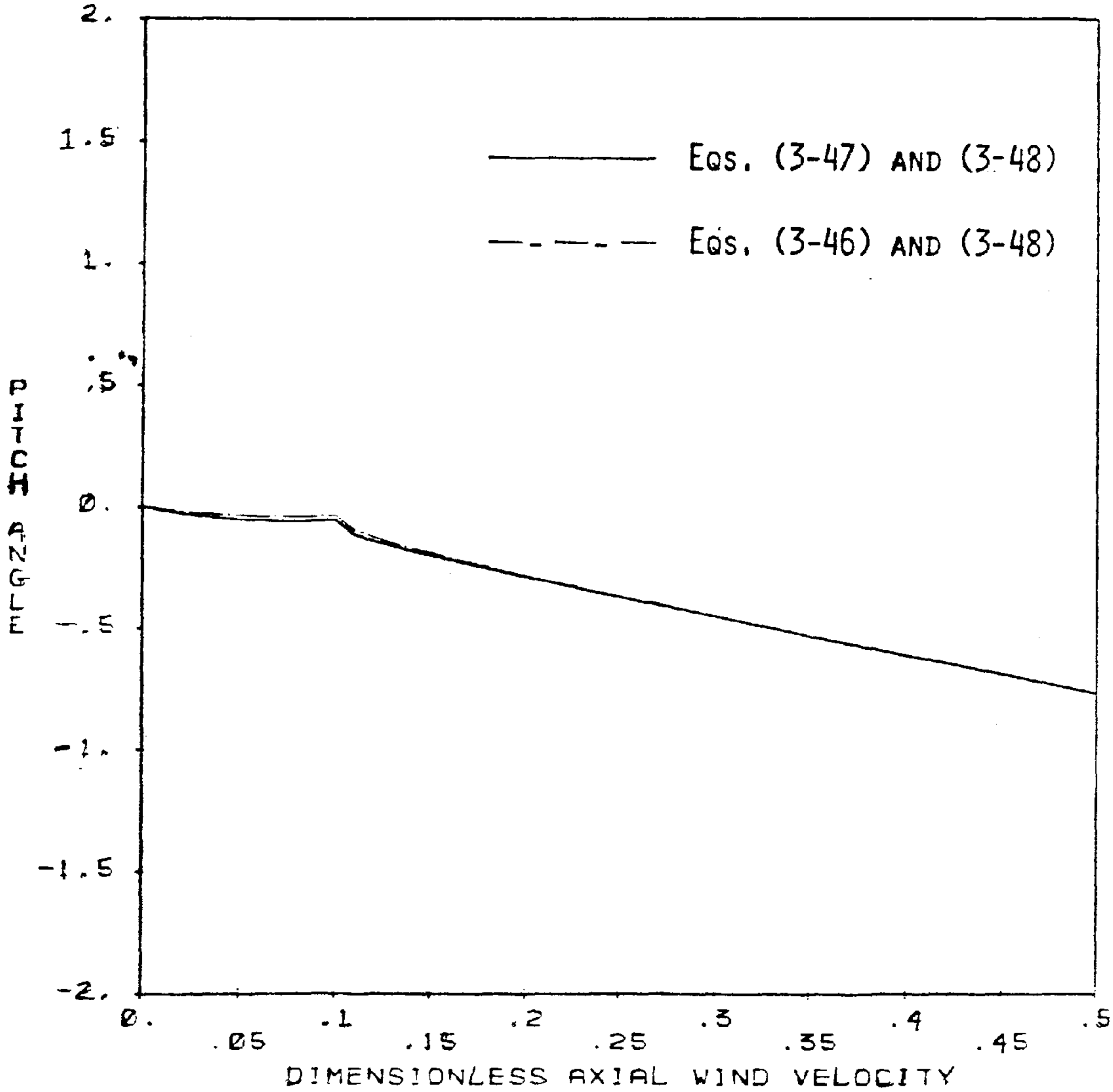

Figure 3-4 Pitch Angle Variation with Axial Wind Velocity.

$B = .97,\ \frac{a\sigma}{8} = .053,\ \lambda_R = .1.$

CHAPTER 4

STOCHASTIC MODELS OF RANDOM EXCITATIONS

4.1 Earthquake Model

Since seismic waves are initiated by irregular slippage along faults followed by numerous random reflections, refractions and attenuations within the complex ground formations through which they pass, stochastic modeling of strong ground motion seems appropriate [34]. If unlimited ground-motion data were available, representative stochastic models could be established directly by statistical analyses. Unfortunately, strong-motion data are limited. Therefore, one is forced to hypothesize forms of models, and to use the available data in checking the appropriateness of these forms.

Accelerograms usually show a phase of nearly constant intensity during the period of most severe oscillation, which suggests that earthquake motions may be modeled as white noise processes of limited duration. The simple stationary white noise process has been used to model earthquake accelerations [35].

However, the entire real accelerogram often shows a short phase of intensity buildup to some maximum level. The intensity then remains fairly constant for some time, after which it decays in an exponential fashion. This appearance suggests that the nonstationary models could be more representative of actual strong ground motions. Several nonstationary models for earthquake accelerations have been proposed [36-38], although their use gives rise to some analytical difficulties, for instance, the statistical linearization technique is no longer simple.

In this study, we assume that ground acceleration can be modeled by the simple expression:

$$g(\tau) = e(\tau)\ Z(\tau) \tag{4-1}$$

in which $Z(\tau)$ is a stationary random process and $e(\tau)$ is an intensity function having an appropriate form based on statistical analyses of real accelerograms. One form which has been suggested [39] is that given in Figure 4-1, or

$$e(\tau) = \begin{cases} \left(\dfrac{\tau-\tau_0}{\tau_1-\tau_0}\right)^2 & , \tau_0 < \tau < \tau_0 + \tau_1 \\ 1 & , \tau_0+\tau_1 < \tau < \tau_0 + \tau_2 \\ e^{-c(\tau-\tau_2-\tau_0)} & , \tau_2 + \tau_0 < \tau \end{cases} \tag{4-2}$$

where τ_0 is the initial time of earthquake and the constants τ_1, τ_2, and c should be assigned only after considering such factors as earthquake magnitude, epicentral distance, etc.

The advantage of using Eq. (4-1) over other earthquake models stems from the fact that the nonstationary character is restricted only to the intensity function $e(\tau)$ and that through the use of the stationary random process $Z(\tau)$, the desirable properties of spectral description and orthogonal decomposition can be preserved.

For the stationary random process Z(t), many forms of power spectral density have been proposed to reflect the influence of the local environments [40]. Among them, the white noise model having a uniform spectral distribution of frequency contents is frequently used for its simplicity and reasonably adequate approximation to real spectra.

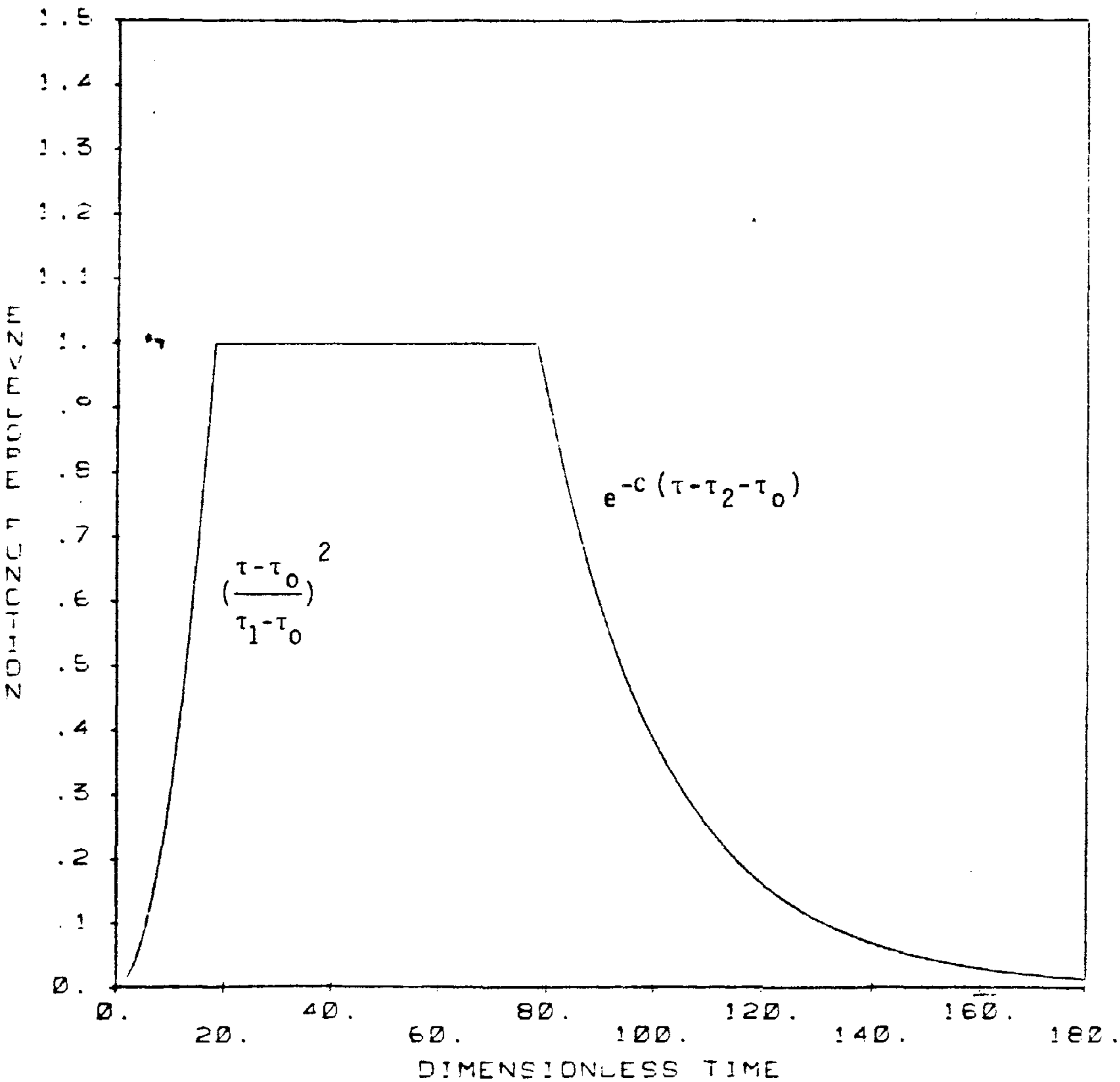

Figure 4-1 Envelop Function of Earthquake.

$\tau_0 = 0$, $\tau_1 = 18$, $\tau_2 = 78$, $c = 0.043$

In our study, the non-dimensional ground acceleration will be decomposed into

$$\begin{aligned} \bar{g}_x(\psi) &= e_1(\psi)Z_1(\psi) \\ \bar{g}_y(\psi) &= e_2(\psi)Z_2(\psi) \\ \bar{g}_z(\psi) &= e_3(\psi)Z_3(\psi) \end{aligned} \tag{4-3}$$

where Z_i are assumed to be uncorrelated Gaussian white noise processes, i.e.

$$E[Z_i(\psi)\ Z_i(\psi+\tau)] = 2\pi\Phi_{ii}\delta(\tau), \qquad i = 1,2,3 \tag{4-4}$$

in which $\delta(\ )$ is the Dirac delta function, Φ_{ii} are spectral constants defined by

$$\Phi_{ii} = \int_{-\infty}^{\infty} E[Z_i(s)\ Z_i(s+\tau)]e^{-i\tau\omega}d\tau \tag{4-5}$$

4.2 Turbulence Model

When a blade rotates in the atmosphere, the relative velocity of the blade to air is comprised of two parts: the velocity of the blade itself and the velocity of the air. The velocity of the air can further be divided into the mean air velocity and the turbulence fluctuating about the mean air velocity. In order to predicts wind turbine response characteristic in the presence of atmospheric turbulence, it is important to identify and characterize the turbulence field which is being convected past the rotor disc when the blades are rotating.

We shall assume that the steady aerodynamics is applicable. Then the aerodynamic forces are determined by the instantaneous air velocity distribution along each of the wind turbine blades. It is thus necessary to characterize the wind turbulence field by a three-dimensional velocity vector which varies randomly not only with time but also with the position in space. The description of this turbulent velocity field requires a complete set of joint probability distributions for different velocity components at different time and different position in space. Clearly, such a description is not possible without considerable simplification.

It is generally agreed that the atmospheric turbulence is approximate Gaussian distributed [41-43]. Being a Gaussian process, the turbulent velocity at each point is completely characterized by the mean and correlation function. Since turbulence is defined as random fluctuation about the mean wind velocity, the mean of the turbulence itself is zero. To model the correlation function, the following assumptions are made. First, the turbulent velocity is assumed to be locally homogenous. Second, the random field is assumed to be isotropic for all separations for which it is homogeneous. Under these assumptions, Holly [44] compared different turbulence models and suggested that the turbulence field may be modeled as stationary random processes with exponential correlation functions for the analysis of horizontal-axis wind turbines. Such processes can be conveniently represented by a first order stochastic differential equation of the form

$$x' = -\alpha x + Z(\phi) \qquad (4\text{-}6)$$

where $Z(\phi)$ is a physical white noise, and α is the reciprocal of correlation time of process $x(\phi)$. As shown earlier, the correlation function of $x(\phi)$ is given by $R_{xx}(\tau) = \sigma^2 e^{-\alpha|\tau|}$, in which τ is the difference of ϕ, and σ^2 is the mean square value of process $x(\phi)$. The normalized correlation function is shown in Fig. 4-2.

In this study, the three components of turbulent velocity can be written as follows.

$$\xi' = -\alpha_1 \xi + Z_4(\phi) \tag{4-7}$$

$$\eta' = -\alpha_2 \eta + Z_5(\phi) \tag{4-8}$$

$$\nu' = -\alpha_3 \nu + Z_6(\phi) \tag{4-9}$$

In which α_1, α_2 and α_3 are the reciprocals of correlation times of processes $\xi(\phi)$, $\eta(\phi)$ and $\nu(\phi)$, respectively, Z_4, Z_5, and Z_6 are assumed to be uncorrelated Gaussian white noise processes with zero means, i.e.,

$$E[Z_i(\phi)\ Z_i(\phi+\tau)] = 2\pi\Phi_{ii}\delta(\tau) \tag{4-10}$$

where i = 4, 5, 6 and Φ_{ii} and $\delta(\ )$ are defined as before.

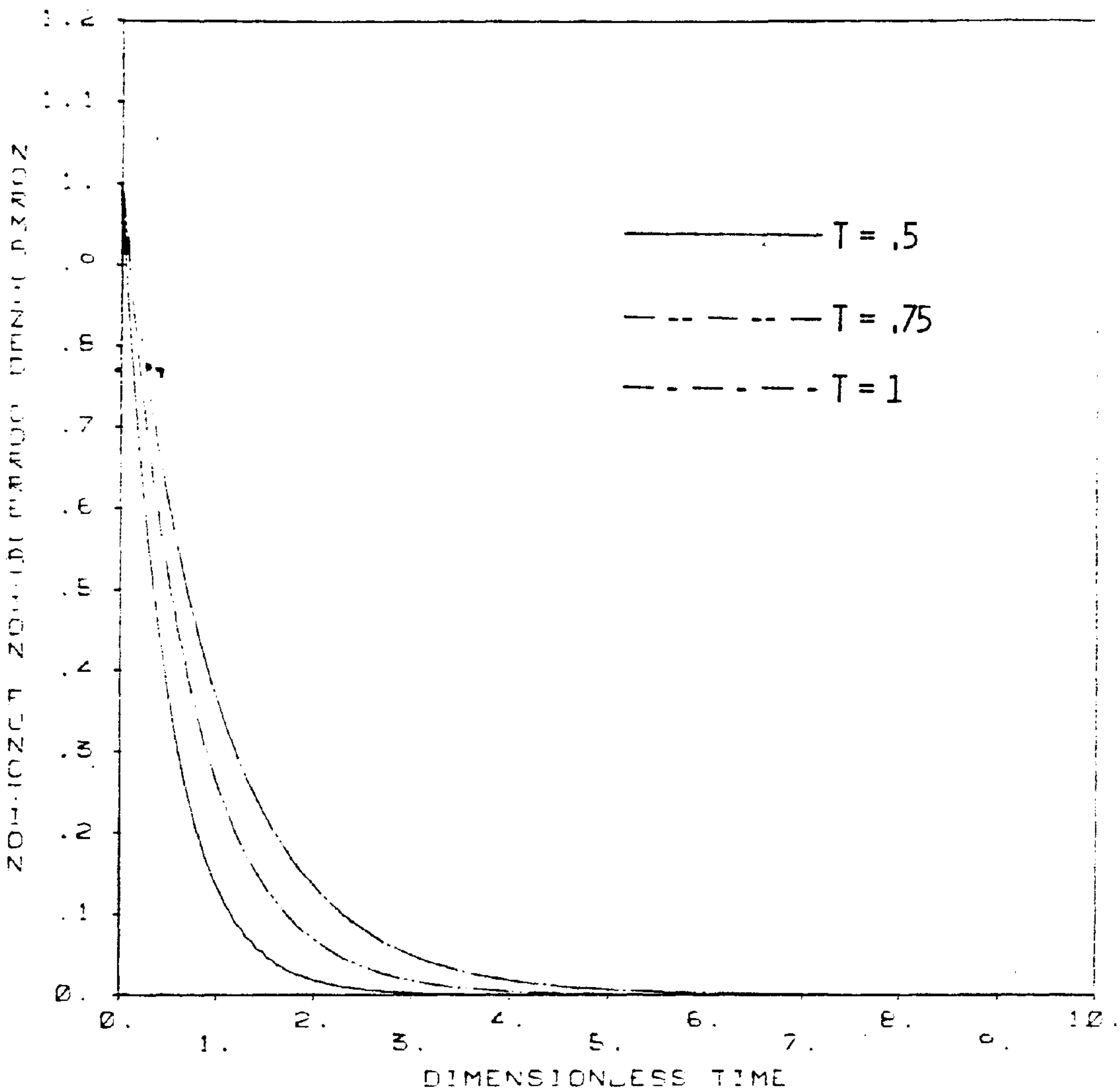

Figure 4-2   Exponential Correlation Functions of Turbulence.

CHAPTER 5

GOVERNING STOCHASTIC DIFFERENTIAL EQUATIONS AND MOMENT RESPONSES

5.1 An Outline of the General Approach

In the linearized equations of motions derived in Chapter 3, random excitations appear both in the coefficients (parametric excitations) and in the inhomogeneous terms (nonparametric excitations). The random excitations include the atmospheric turbulence velocities which are assumed to be stationary processes and the earthquake acceleration components which are assumed to be nonstationary processes. All of the random excitations in the linearized equations of motions are multiplied by periodic modulation functions such as $K_{1\xi}$, $C_{1\xi}$, etc., which are originated from the rotor rotation. Of course, the modulated random processes are not statistically stationary, even though turbulence velocities are assumed to be stationary random processes.

To gain more insight into the properties of the equations of motions, we investigate the corresponding homogeneous equations by dropping the inhomogeneous terms. If the random excitations are also neglected, then these homogeneous equations belong to the class of Hill's equations [45] - a class of linear differential equations with periodic coefficients. In such a case, the system can become unstable under certain combinations of parameters. The addition of random parametric excitations to the system changes the stability conditions. The existence of parametric random excitations can destabilize some systems but stabilize other systems [46,47].

As indicated earlier, each turbulence component is modeled as an exponentially correlated random process, and each earthquake component is modeled as the product of a deterministic modulation function and a white noise process. If only the turbulence excitations are present, then the coefficients and the inhomogeneous terms in the equations of motion are either periodic functions or products of periodic functions and random processes. The state vector of the system is not Markovian. However, if we extend the state vector to include the turbulence components, i.e. adding Eqs. (4-7) through (4-9) to the equations of motion of the dynamic system, then the new state vector becomes Markovian, and a host of well-developed mathematical tools of Markov process theory can be used. The extended system is nonlinear since the parametric turbulence excitations are also treated as unknowns. If the earthquake excitations are also included in the analysis and if each component of the ground acceleration is modeled as the product of an intensity function and a Gaussian white noise process, the Markov process theory can still be used.

For the convenience of applying the Markov process theory, each second order differential equation will be replaced by an equivalent set of two first order differential equations. Then employing the transformation formulas, Eqs. (2-41) and (2-42), the drift and diffusion terms of corresponding Itô's stochastic differential equations can be found. Furthermore, the equations for the moments of the state variables can be obtained by using Itô's differential rule Eq. (2-37) and taking the ensemble average of the resulting differential equations. The moment equations become a sequence of coupled linear periodic differential equations if only turbulence components are present. The periodicity is lost when earthquake excitations are included because the modulating functions of the earthquake models are not periodic. In either case, the moment equations form an infinite hierarchy, in the sense that the higher order moments will appear in the equations for lower order moments. Thus, only approximate solutions for the lower order moments are obtainable by using a suitable closure scheme.

Perhaps the best known scheme is the Gaussian closure in which the higher moments are assumed to be related to the first and second moments in the same way as Gaussian random variables. In particular, the third moments which appear in the equations of first and second moments can be expressed as follows [31]:

$$E[x_i x_j x_k] = E[x_i x_j]\, E[x_k] + E[x_j x_k]\, E[x_i] + E[x_i x_k] E[x_j] - 2E[x_i]\, E[x_j]\, E[x_k] \qquad (5\text{-}1)$$

By substituting the above relationship into the first and second moment equations, these equations become a closed set of coupled nonlinear differential equations.

To examine the stability conditions for the first and second moments we follow a linearization procedure used by Bolotin [50] and Owen [51]. The nonlinear differential equations are re-formulated in terms of the first moments and the second central moments $\mu_i = E[x_i]$ and $\mu_{ij} = E[(x_i - \mu_i)\,(x_j - \mu_j)]$. The products of the type of μ_i and μ_{ij} are neglected, however, the terms involving the mean square values of the turbulence velocity components are retained. For the stability study the modulating functions of the earthquake models are replaced by the constant unity, i.e., the earthquake components are assumed to be physical white noise processes. This is conservative and leads to meaningful results. The equations now become a set of linear periodic differential equations, and the Floquet theory can be used to find the stability condition of the dynamic system.

To calculate the response moments, the original system of nonlinear differential equations obtained from Gaussian closure is used. If all the random excitations are stationary random processes multiplied by deterministic periodic functions, then the statistics of the response will tend to periodic steady-state functions if the response is stable in some sense. In contrast, if the random excitations are not all stationary, which is the case when earthquakes are also present, then the response statistics are transient-like. In our calculations to be presented in Chapter 6, both the steady state response moments of the dynamic system due to turbulence excitation and the transient reponse moments due to earthquake excitation will be included.

5.2 Stochastic Differential Equations and Moment Equations for Response Variables

In this section, the moment equations of uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsional motions are derived. For simplicity, subscripts g_x, g_y, g_z, ξ, η and ν in the matrices [F], [C] and [K] are changed to 1,2, ..., 6, respectively. The first order differential equations corresponding to Eq. (3-30), (3-33) or (3-38), and Eqs. (4-7) through (4-9) are recapitulated as follows:

$$x'_{(2i-1)} = x_{(2i)}$$

$$x'_{(2i)} = F_{io} - \sum_{k=1}^{n}(K_{iko}x_{(2k-1)}+C_{iko}x_{(2k)})$$

$$+ \sum_{j=4}^{6}\{F_{ij} - \sum_{k=1}^{n}(K_{ikj}x_{(2k-1)}+C_{ikj}x_{(2k)})\}\, x_{(j+2n-3)}$$

$$+ \sum_{j=1}^{3}\{F_{ij} - \sum_{k=1}^{n}K_{ikj}x_{(2k-1)}\}\, e_j Z_j(\phi)$$

$$i=1,\ \ldots,\ n$$

$$x'_i = -\alpha_{(i-2n)} x_i + Z_{(i-2n+3)}(\phi)$$
$$i=(2n+1),\ (2n+2),\ (2n+3) \qquad (5\text{-}2)$$

where x_i are the components of state vector $\vec{X}$ and will be defined later, n is the number of the generalized coordinates of the dynamic system, e_j are the earthquake intensity functions, and $Z_j(\phi)$ are uncorrelated Gaussian white noise processes. It must be noted that some of the coefficients are added for the commensurability among the different equations of motions such as K_{116}, C_{116}, K_{113}, F_{11} and F_{12} of flapping motion and F_{io} of coupled flap-lagging and flap-lag-torsional motions. The state vector $\vec{x}$ will now be approximated by a diffusive Markov vector. The governing Itô stochastic differential equations for the Markov vector can be obtained by using the Stratonovich stochastic averaging method. Following Eqs. (2-44) and (2-45) drift coefficients are

$$m_{(2i-1)} = x_{(2i)}$$

$$m_{(2i)} = F_{io} - \sum_{k=1}^{n} (K_{iko} x_{(2k-1)} + C_{iko} x_{(2k)})$$
$$+ \sum_{j=4}^{6} \{F_{ij} - \sum_{k=1}^{n} (K_{ikj} x_{(2k-1)} + C_{ikj} x_{(2k)})\} \ x_{(j+2n-3)} ;$$

$$i = 1,\ 2,\ \ldots,\ n$$

$$m_i = -\alpha_{(i-2n)} x_i ;$$

$$i = (2n+1),\ (2n+2),\ (2n+3) \qquad (5\text{-}3)$$

The nonzero elements of the diffusion matrix are

$$(\sigma\sigma^T)_{(2i)(2j)} = \sum_{k=1}^{3} 2\pi e_k^2 \Phi_{kk}\{F_{ik} - \sum_{\ell=1}^{n} K_{i\ell k} X_{(2\ell-1)}\}\ \{F_{jk} - \sum_{s=1}^{n} K_{jsk} X_{(2s-1)}\}\ ;$$

$i,j=1,\ \ldots,\ n$

$$(\sigma\sigma^T)_{ij} = 2\pi\Phi_{(i+2n-3)(j+2n-3)}\delta_{ij}$$

$i,j=(2n+1),\ (2n+2),(2n+3)$ (5-4)

where δ_{ij} is the Kronecker delta. It is of interest to note that the Wong-Zakai correlation term does not appear in the drift coefficient, Eq. (5-3), as it often does for the random parametric excited systems.

To derive the first and second moment equations, we let the scalar function ϕ be equal to x_i and $x_i x_j$ in Eq. (2-37), and substitute Eqs. (5-3) and (5-4) into Eq. (2-37). After taking the ensemble average of the results, the first and second moment equations can be obtained. The first moment equations are

$$M'_{(2i-1)} = M_{(2i)}$$

$$M'_{(2i)} = F_{io} - \sum_{k=1}^{n}(K_{iko}M_{(2k-1)} + C_{iko}M_{(2k)}) + \sum_{j=4}^{6}\{F_{ij}M_{(j+2n-3)}$$

$$- \sum_{k=1}^{n}(K_{ikj}M_{(2k-1)(j+2n-3)} + C_{ikj}M_{(2k)(j+2n-3)})\}\ ;$$

$i = 1,\ \ldots,\ n$

$$M'_i = -\alpha_{(i-2n)}M_i;$$

$i=(2n+1),\ (2n+2),\ (2n+3)$ (5-5)

The second moment equations are

$$M'_{(2i-1)(2j-1)} = M_{(2i)(2j-1)} + M_{(2j)(2i-1)}$$

$$M'_{(2i)(2j-1)} = M_{(2j)(2i)} + F_{io}M_{(2j-1)} - \sum_{k=1}^{n}(K_{iko}M_{(2k-1)(2j-1)} + C_{iko}M_{(2k)(2j-1)})$$

$$+ \sum_{l=4}^{6}\{F_{il}M_{(l+2n-3)(2j-1)} - \sum_{k=1}^{n}K_{ikl}M_{(2k-1)(l+2n-3)(2j-1)} + C_{ikl}M_{(2k)(l+2n-3)(2j-1)}\}$$

$$M'_{(2i)(2j)} = F_{io}M_{(2j)} + F_{jo}M_{(2i)} - \sum_{k=1}^{n}(K_{iko}M_{(2k-1)(2j)} + C_{iko}M_{(2k)(2j)}$$

$$+ K_{jko}M_{(2k-1)(2i)} + C_{jko}M_{(2k)(2i)}) + \sum_{l=4}^{6}\{F_{il}M_{(l+2n-3)(2j)}$$

$$+ F_{jl}M_{(l+2n-3)(2i)} - \sum_{k=1}^{n}(K_{ikl}M_{(2k-1)(l+2n-3)(2j)}$$

$$+ C_{ikl}M_{(2k)(l+2n-3)(2j)} + K_{jkl}M_{(2k-1)(l+2n-3)(2i)}$$

$$+ C_{jkl}M_{(2k)(l+2n-3)(2i)})\} + \sum_{k=1}^{3} 2\pi e_k^2 \Phi_{kk}\{F_{ik}F_{jk} - F_{ik}\sum_{s=1}^{n}K_{jsk}M_{(2s-1)}$$

$$- F_{jk}\sum_{l=1}^{n}K_{ilk}M_{(2l-1)} + \sum_{l=1}^{n}\sum_{s=1}^{n}K_{ilk}K_{jsk}M_{(2l-1)(2s-1)}\} \;;$$

$i,j = 1, \ldots, n$

$$M'_{(2i-1)j} = M_{(2i)j} - \alpha_{(j-2n)}M_{(2i-1)j}$$

$$M'_{(2i)j} = F_{io}M_j - \sum_{k=1}^{n}(K_{iko}M_{(2k-1)j} + C_{iko}M_{(2k)j}) + \sum_{l=4}^{6}\{F_{il}M_{(l+2n-3)j}$$

$$- \sum_{k=1}^{n}(K_{ikl}M_{(2k-1)(l+2n-3)j} + C_{ikl}M_{(2k)(l+2n-3)j})\} - \alpha_{(j-2n)}M_{(2i)j} \;;$$

$i = 1, \ldots, n,$

$j = (2n+1), (2n+2), (2n+3).$

$$M'_{ij} = -(\alpha_{(i-2n)} + \alpha_{(j-2n)} M_{ij} + 2\pi\Phi_{(i-2n+3)(j-2n+3)} \delta_{ij} ;$$

$$i, j = (2n+1), (2n+2), (2n+3) \qquad (5\text{-}6)$$

where $M_i = E[x_i]$, $M_{ij} = E[x_i x_j]$, and $M_{ijk} = E[x_i x_j x_k]$. It can be seen that the first moment equations contain the first and second moments, and the second moment equations contain the first, second and third moments. To obtain the complete set of the first and second moment equations, we substitute Eq. (5-1) into Eq. (5-6). The second moment equations can be rewritten as the following.

$$M'_{(2i-1)(2j-1)} = M_{(2i)(2j-1)} + M_{(2j)(2i-1)}$$

$$M'_{(2i)(2j-1)} = M_{(2i)(2j)} + F_{io} M_{(2j-1)} - \sum_{k=1}^{n} (K_{iko} M_{(2k-1)(2j-1)} + C_{iko} M_{(2k)(2j-1)})$$

$$+ \sum_{l=4}^{6} \{ F_{il} M_{(l+2n-3)(2j-1)} - \sum_{k=1}^{n} (K_{ikl} [M_{(2k-1)(l+2n-3)} M_{(2j-1)}$$

$$+ M_{(2k-1)(2j-1)} M_{(l+2n-3)} + M_{(l+2n-3)(2j-1)} M_{(2k-1)} - 2M_{(2k-1)} M_{(l+2n-3)} M_{(2j-1)}]$$

$$+ C_{ikl} [M_{(2k)(l+2n-3)} M_{(2j-1)} + M_{(2k)(2j-1)} M_{(l+2n-3)} + M_{(l+2n-3)(2j-1)} M_{(2k)}$$

$$- 2M_{(2k)} M_{(l+2n-3)} M_{(2j-1)}]) \}$$

$$M'_{(2i)(2j)} = F_{io} M_{(2j)} + F_{jo} M_{(2i)} - \sum_{k=1}^{n} (K_{iko} M_{(2k-1)(2j)} + C_{iko} M_{(2k)(2j)}$$

$$+ K_{jko} M_{(2k-1)(2i)} + C_{jko} M_{(2k)(2i)} + \sum_{l=4}^{6} \{ F_{il} M_{(l+2n-3)(2j)}$$

$$+ F_{jl}M_{(l+2n-3)(2i)} - \sum_{k=1}^{n}(K_{ikl}[M_{(2k-1)(l+2n-3)}M_{(2j)} + M_{(2k-1)(2j)}M_{(l+2n-3)}$$

$$+ M_{(l+2n-3)(2j)}M_{(2k-1)} - 2M_{(2k-1)}M_{(l+2n-3)}M_{(2j)}] + C_{ikl}[M_{(2k)(l+2n-3)}M_{(2j)}$$

$$+ M_{(2k)(2j)}M_{(l+2n-3)} + M_{(l+2n-3)(2j)}M_{(2k)} - 2M_{(2k)}M_{(l+2n-3)}M_{(2j)}]$$

$$+ K_{jkl}[M_{(2k-1)(l+2n-3)}M_{(2i)} + M_{(2k-1)(2i)}M_{(l+2n-3)} + M_{(l+2n-3)(2i)}M_{(2k-1)}$$

$$- 2M_{(2k-1)}M_{(l+2n-3)}M_{(2i)}] + C_{jkl}[M_{(2k)(l+2n-3)}M_{(2i)} + M_{(2k)(2i)}M_{(l+2n-3)}$$

$$+ M_{(l+2n-3)(2i)}M_{(2k)} - 2M_{(2k)}M_{(l+2n-3)}M_{(2i)}])\} + \sum_{k=1}^{3} 2\pi e_k^2 \Phi_{kk}$$

$$\{F_{ik}F_{jk} - F_{ik}\sum_{s=1}^{n}K_{jsk}M_{(2s-1)} - F_{jk}\sum_{l=1}^{n}K_{ilk}M_{(2l-1)}$$

$$+ \sum_{l=1}^{n}\sum_{s=1}^{n}K_{ilk}K_{jsk}M_{(2l-1)(2s-1)}\};$$

$i, j = 1, \ldots, n$

$$M'_{(2i-1)j} = M_{(2i)j} - \alpha_{(j-2n)}M_{(2i-1)j}$$

$$M'_{(2i)j} = F_{io}M_j - \sum_{k=1}^{n}(K_{iko}M_{(2k-1)j} + C_{iko}M_{(2k)j}) + \sum_{l=4}^{6}\{F_{il}M_{(l+2n-3)j}$$

$$- \sum_{k=1}^{n}(K_{ikl}[M_{(2k-1)(l+2n-3)}M_j + M_{(2k-1)j}M_{(l+2n-3)} + M_{(l+2n-3)j}M_{(2k-1)}$$

$$- 2M_{(2k-1)}M_{(l+2n-3)}M_j] + C_{ikl}[M_{(2k)(l+2n-3)}M_j + M_{(2k)j}M_{(l+2n-3)}$$

$$+ M_{(1+2n-3)j} M_{(2k)} - 2M_{(2k)}M_{(1+2n-3)}M_j]) \} - \alpha_{(j-2n)}M_{(2i)j};$$

$i = 1, \ldots, n,$

$j = (2n+1), (2n+2), (2n+3)$

$$M'_{ij} = -(\alpha_{(i-2n)} + \alpha_{(j-2n)}) M_{ij} + 2\pi\Phi_{(i-2n+3)(j-2n+3)}\delta_{ij};$$

$$i,j = (2n+1), (2n+2), (2n+3) \tag{5-7}$$

It must be noted that the second moment is symmetric, i.e., $M_{ij} = M_{ji}$.

The system of Eqs. (5-5) and (5-7) forms a complete set of moment equations. They can be cast in the following matrix forms.

$$\vec{Y}' = \bar{A} + \bar{B}\vec{Y} + \bar{N}(\phi, \vec{Y}) \tag{5-8}$$

where $\vec{Y}^T = [M_1 \ldots M_{(2n)} \;\; M_{(2n+1)} \;\; M_{(2n+2)} \;\; M_{(2n+3)} \;\; M_{11} \;\; M_{12} \ldots M_{1(2n+3)}$ $M_{22} \;\; M_{23} \ldots M_{2(2n+3)} \ldots M_{(2n+3)(2n+3)}]$. Vectors $\vec{Y}$, $\bar{A}$ and $\bar{N}$ are of the same order as $(2n+3) + (2n+3) + (2n+2) + \ldots + 1$; and the square matrix $\bar{B}$ is of an order $(2n+3) + (2n+3) + (2n+2) + \ldots + 1$. Vector $\bar{N}$ contains the non-linear terms which are the products of first and second moments.

For moment stability analyses, we replace the moments by the corresponding central moment in Eq. (5-8) and linearized in the sense of Bolotin. The linearized moment equations become

$$\mu'_{(2i-1)} = \mu_{(2i)}$$

$$\mu'_{(2i)} = -\sum_{k=1}^{n}(K_{ik0}\mu_{(2k-1)} + C_{ik0}\mu_{(2k)}) - \sum_{j=4}^{6}\sum_{k=1}^{n}(K_{ijk}\mu_{(2k-1)(j+2n-3)}$$

$$+ C_{ijk}\mu_{(2k)(j+2n-3)});$$

$i=1, \ldots, n$

$$\mu'_{(2i)(2j-1)} = \mu_{(2i)(2j-1)} + \mu_{(2j)(2i-1)}$$

$$\mu'_{(2i)(2j-1)} = \mu_{(2i)(2j)} + F_{i0}\mu_{(2j-1)} + \sum_{l=4}^{6} F_{il}\mu_{(l+2n-3)(2j-1)}$$

$$-\sum_{k=1}^{n}(K_{ik0}\mu_{(2k-1)(2j-1)} + C_{ik0}\mu_{(2k)(2j-1)})$$

$$\mu'_{(2i)(2j)} = F_{i0}\mu_{(2j)} + F_{j0}\mu_{(2i)} - \sum_{k=1}^{n}(K_{ik0}\mu_{(2k-1)(2j)} + C_{ik0}\mu_{(2k)(2j)}$$

$$+K_{jk0}\mu_{(2k-1)(2i)} + C_{jk0}\mu_{(2k)(2i)}) + \sum_{l=4}^{n}(F_{il}\mu_{(l+2n-3)(2j)}$$

$$+ F_{jl}\mu_{(l+2n-3)(2i)}) + \sum_{k=1}^{3} 2\pi e_k^2 \Phi_{kk}\{-F_{ik}\sum_{s=1}^{n} K_{jsk}\mu_{(2s-1)}$$

$$- F_{jk}\sum_{l=1}^{n} K_{ilk}\mu_{(2l-1)} + \sum_{l=1}^{n}\sum_{s=1}^{n} K_{ilk}K_{jsk}\mu_{(2l-1)(2s-1)}\};$$

$i, j = 1, \ldots, n$

$$\mu'_{(2i-1)j} = \mu_{(2i)j} - \alpha_{(j-2n)}\mu_{(2i-1)j}$$

$$\mu'_{(2i)j} = F_{i0}\mu_j - \sum_{k=1}^{n}(K_{ik0}\mu_{(2k-1)j} + C_{ik0}\mu_{(2k)j}) - \alpha_{(j-2n)}\mu_{(2i)j}$$

$$- \sum_{l=4}^{6} \sum_{k=1}^{n} \sigma_{j(l+2n-3)} \delta_{j(l+2n-3)} (K_{ik}\mu_{(2k-1)} + C_{ik}\mu_{(2k)});$$

$i = 1, \ldots, n$

$$j = (2n+1),\ (2n+2),\ (2n+3) \tag{5-9}$$

where σ_{ii} are the mean square of turbulence components, $\mu_i = M_i$, and $\mu_{ij} = M_{ij} - M_i M_j$ are the central moments. The detailed list of the linearized moment equations of uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsion is given in Appendix E. Letting the intensity functions equal to unity and applying the method of Floquet transition matrix, one can determine the stability conditions for any given combinations of parameters based on the operating condition described in Chapter 3.

As mentioned previously, for moment response analyses the steady state response is emphasized if only turbulence is present, and the transient moment responses are required if both turbulence and earthquake are present. In the first case, we let the intensity functions, e_1, e_2 and e_3, be zero in Eqs. (5-5) and (5-7), and integrate the system of differential equations starting from an arbitrary set of initial conditions until the moments converge to periodic functions with period 2π. In the second case, we impose the earthquake excitations upon the system which has reached its stationary state in the presence of some moderate turbulence; that is, we obtain the transient solutions of Eqs. (5-5) and (5-7) with the initial conditions which are the steady state solutions due to some turbulence excitation alone. In our calculations, however, earthquakes are assumed to commence at $\phi = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$, respectively.

In the case of uncoupled flapping motion, n=1 and $\vec{X}^T = [x_1\ x_2\ x_3\ x_4\ x_5] = [\beta\ \beta'\ \xi\ \eta\ \nu]$. The total number of equations for the first and second monents is twenty. The detailed moment equations are given in Appendix A. Since Eq. (3-30) is linearized about the zero flapping angle, the stability of the system of Eq. (5-9) is interpreted as for the zero flapping angle. The solutions of the first and sixth components of vector $\vec{Y}$ represent the mean and mean square flapping angles.

In the case of coupled flap-lagging motion, n = 2 and $\vec{X}^T = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7] = [\delta\beta\ \delta\beta'\ \delta\zeta\ \delta\zeta'\ \xi\ \eta\ \nu]$, giving rise to thirty-five first and second moment eqdations, as detailed in Appendix B. Since Eq. (3-33) describes a perturbed motion from an equilibrium flap-lagging motion, the stability should be interpreted as that of the equilibrium solution, β_e, ζ_e and θ_e. The first and eighth components of vector $\vec{Y}$ are the mean and mean square of the perturbed flapping angle, and the third and twenty-first components are those of the perturbed lead-lagging angle.

Finally, for coupled flap-lag torsion motion, n = 3 and $\vec{X}^T = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9] = [\delta\beta\ \delta\beta'\ \delta\zeta\ \delta\zeta'\ \delta\alpha\ \delta\alpha'\ \xi\ \eta\ \nu]$. Then Eq. (5-8) contains fifty-four equations for the first and second moments. The detailed moment equations are given in Appendix C. Again, the stability is interpreted as that of the equilibrium solutions, β_e, ζ_e, α_e and θ_e. The first and tenth components of vector $\vec{Y}$ are the mean and mean square of the perturbed flapping angle. The third and twenty-seventh components are those for the perturbed leadlagging angle and the fifth and fortieth components for the perturbed torsional angle.

CHAPTER 6

NUMERICAL EXAMPLES

Some numerical results will be presented in this chapter to illustrate the applications of the procedure developed herein. The equilibrium solutions when required are calculated first using Eq. (3-43). Next, Eq. (5-9) is used to determine the moment stability conditions. The statistical moments are then calculated when the motion is stable. The moment responses include two parts: 1) steady state solutions without the presence of earthquakes, 2) transient state solutions with both turbulence and earthquake. To calculate part 2, the intensity functions are assumed to be the same for three earthquake components and are given in Fig. 4-1.

Fig. 6-1 and Fig. 6-2 show the equilibrium solutions of coupled flap-lagging and flap-lag-torsional motions in a typical operating condition. These results are based on the following parametric values: V = 30 ft/sec (9.15 m/sec), U = 6 ft/sec (1.83 m/sec,) **R** = .2, R = 50 ft (15.25 m), Ω = 6 rad/sec, γ = 8 , $\frac{a\sigma B^2}{8}$ = .05, c_d = .01, β_{pc} = 0, θ_β = 0, θ_ζ = 0, λ_R = .1, k = .02, ω_β = 1.414 and ω_ζ = 1.871. For coupled flap-lag-torsional motion, the values ω_α = 33 and F = .72 are used. The periodic equilibrium solutions shown in Fig. 6-1 correspond to β_o = .0118, β_c = -.0007, β_c = -.0012, ζ_o = .0012, ζ_s = -.0108 and ζ_c = .00001. The maximum angles of flapping and leadlagging in the equilibrium solutions are 0.757° and 0.693° which are also the maximum responses for the deterministic case. Fig. 6-2 shows the equilibrium solutions with an additional degree of freedom in torsion. The flapping and leadlagging motions are almost identical to those in Fig. 6-1, and the torsional equilibrium angle is near zero everywhere. This is because of the fact that the last three rows on the right hand side of Eq. (3-43) are zero, and the torsional mode is much stiffer than flapping and leadlagging.

A large number of cases were considered in the analyses of uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsion. All the three modes were found to be very stable, even when the turbulence level is extremely high. Table V gives the largest norm among the eigenvalues of the Floquet transition matrix for different cases. Since computation of eigenvalues of the Floquet transition matrix of a dynamic system involving many equations with periodic coefficients would be very expensive, the stability condition under unusual parameter combinations was not considered. Our main effort in the present study, therefore, was expended to obtain the moment responses for coupled flap-lagging and flap-lag-torsional motions. The parametric values used in the calculation of moment response are the same as those used for determining the equilibrium solutions. Any exceptions will be indicated individually.

The steady state solutions of moment response to turbulence alone are presented first. Fig. 6-3 shows the computed first moments for the uncoupled flapping motion. The results indicate that the turbulence root-mean-square (rms) level has little effect on the mean flapping angle which is indistinguishable from the non-turbulence deterministic solution. However, the rms shown in Fig. 6-4 increases about 18% from the deterministic one for a high turbulence rms level and about 4.7% for a low level. The flapping angle for the deterministic case is the one associated with zero turbulence rms level. The reason for labeling the strength of turbulence in terms of the rms value instead of spectral level is that the spectral level of an exponentially

Table V

The Largest Norm Among Eigenvalues of the Floquet Transition Matrix

Case	Motion	V ft/sec (m/sec)	rms value of u, v and w ft/sec (m/sec)	largest norm among eigenvalues
1	flap	30. (9.14)	3.0 (.914)	.059857
2	flap	30. (9.14)	20.0 (6.10)	.060081
3	flap	50. (15.24)	3.0 (.914)	.059914
4	flap*	50. (15.24)	20.0 (6.10)	.060005
5	flap-lag	30. (9.14)	3.0 (.914)	.948321
6	flap-lag	30. (9.14)	20. (6.10)	.932839
7	flap-lag	50. (15.24)	3.0 (.914)	.944450
8	flap-lag	50. (15.24)	20. (6.10)	.932710
9	flap-lag-torsion	30. (9.14)	3.0 (.914)	.943482
10	flap-lag-torsion	30. (9.14)	20. (6.10)	.932245
11	flap-lag-torsion	50. (15.24)	3.0 (.914)	.944213
12	flap-lag-torsion	50. (15.24)	20. (6.10)	.932616

*$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 1$, $\Phi_{11} = \Phi_{33} = \frac{25}{16}\Phi_{22} = .92698 \times 10^{-7}$, $\tau_0 = 0$; other parameters values are the same as those used for determining the equilibrium solution

correlated process is a function of frequency. The effects of turbulence rms level on the moment responses of the coupled flap-lagging motion are depicted in Figs. 6-5 through 6-8. The mean flapping and leadlagging perturbed angles are shown in Figs. 6-5 and 6-7, whereas the rms perturbed angles are shown in Figs. 6-6 and 6-8. The mean perturbed angles are very small in all cases. The rms perturbed angles are about 58% and 23% of their respective equilibrium angles for flapping and leadlagging motions at a turbulence rms level of 5 ft/sec (1.52 m/sec), and about 29% and 12% at a low level of 2.5 ft/sec (0.76 m/sec). The results indicate that adding the leadlagging degree of freedom tends to lower the rigidity of the flapping mode.

The effects of the turbulence correlation time T_i and of the elastic paremeter R on the responses of coupled flap-lagging motion are also compared. Figs. 6-9 through 6-14 show the variation of moment response with the turbulence correlation time at the same rms turbulence velocity of 5 ft/sec (1.52 m/sec) rms in all three directions. The mean and rms angles of the flapping and leadlagging responses computed for different correlation times are very close to each other. This indicates that within the practical range investigated the excitation correlation time does not have a profound effect on the moment responses. Figs. 6-13 through 6-16 show the moment responses of coupled flap-lagging motion with different values of elastic coupling parameter R at the same turbulence rms level. The mean and rms responses computed for varying value of R are not much different. It must be noted that when we change the value of R, the equilibrium solutions are also changed; however, the changes in the maximum equilibrium angles are small compared to the maximum angles themselves. Therefore, a slight change in the R value still leads to the same general results. Figs. 6-17 through 6-22 illustrate the mean and rms angles of the flapping, leadlagging and torsional moment responses computed for different turbulence rms levels. The mean and rms torsional responses are almost equal to zero; the results for flapping and leadlagging responses are very close to those obtained previously without the torsional degree of freedom. The same conclusion has been reached for equilibrium solutions.

Next, the transient moment responses due to both turbulence and earthquake excitations will be presented. It is assumed that the steady state motion due to turbulence excitation is present when the earthquake excitation occurs at $\phi = 0$. Figs. 6-23 and 6-24 show the transient responses of mean and rms angles of uncoupled flapping motion computed for the same turbulence level but different earthquake levels. The curves indicate that the response statistics remain periodic; that is, an earthquake in the range of level considered does not affect the moment response of uncoupled flapping motion. The mean and rms angles for the coupled flap and leadlag responses are shown in Figs. 6-25 through 6-28. Figs. 6-25 and 6-27 indicate that both the mean flap and leadlag angles are very small, and that there is no influence of earthquake excitation on the mean angles. As shown in Fig. 6-26, the rms flapping angle of the transient solution differs only slightly from the steady state periodic due to turbulence excitation alone. In constrast, the rms angle of the leadlagging response is affected by the earthquake excitation. It is about 8% of the corresponding equilibrium solution for a nondimensional earthquake level of 9.2698×10^{-8} and about 1% for a low earthquake level of 9.2698×10^{-9}. The effect of different starting times of earthquake excitation on the transient solutions of coupled flap-lagging motion is also investigated. Figs. 6-29 through 6-32 illustrate the mean and rms angles of flapping and lead-lagging responses. They show that the effect of different starting times is not very significant. The results for the transient response of coupled flap-lag-torsional motion are expected to be similar to those obtained without the torsional degree of freedom since the torsional mode has been found to be very stiff previously.

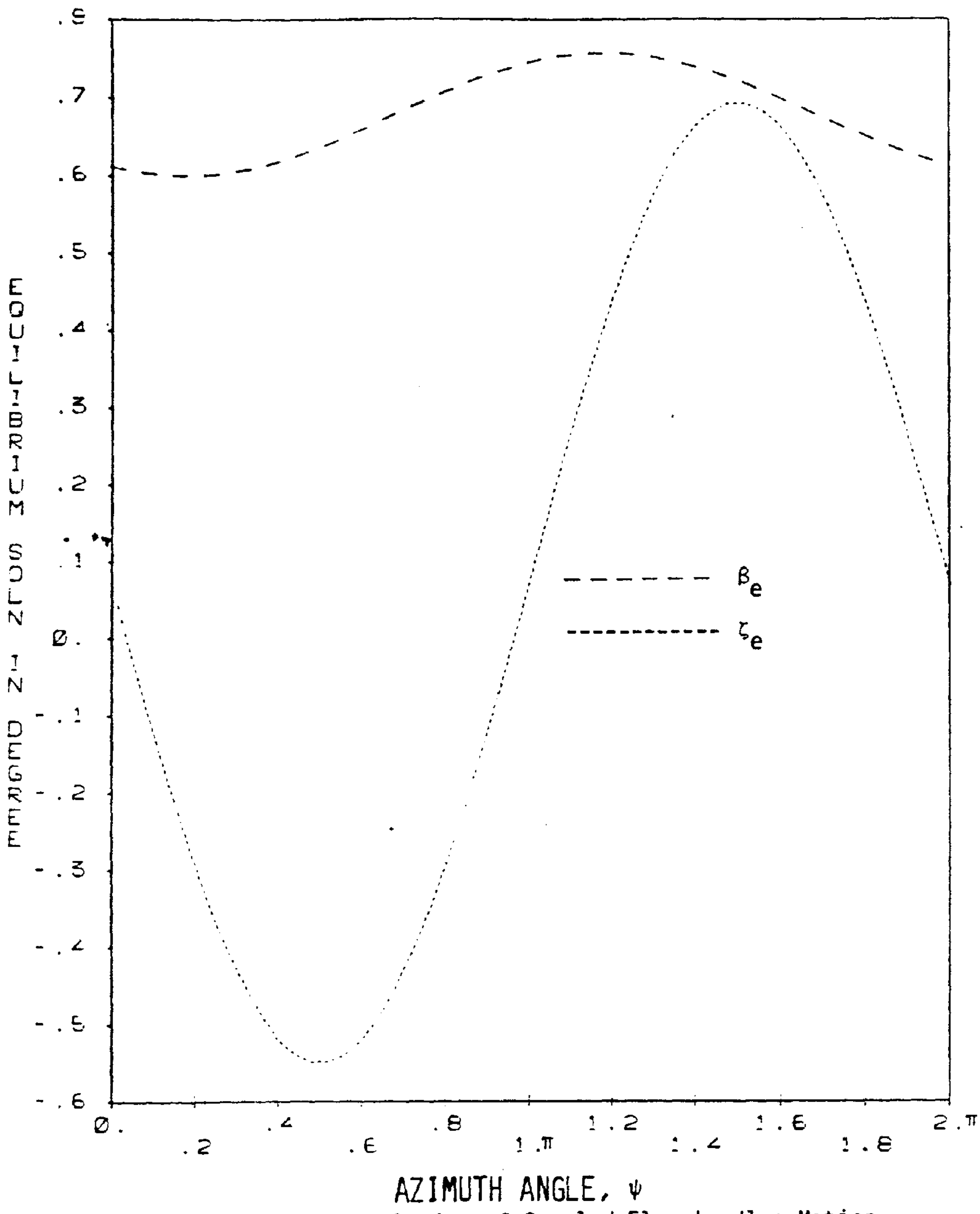

Figure 6-1 Equilibrium Solution of Coupled Flap-Leadlag Motion.

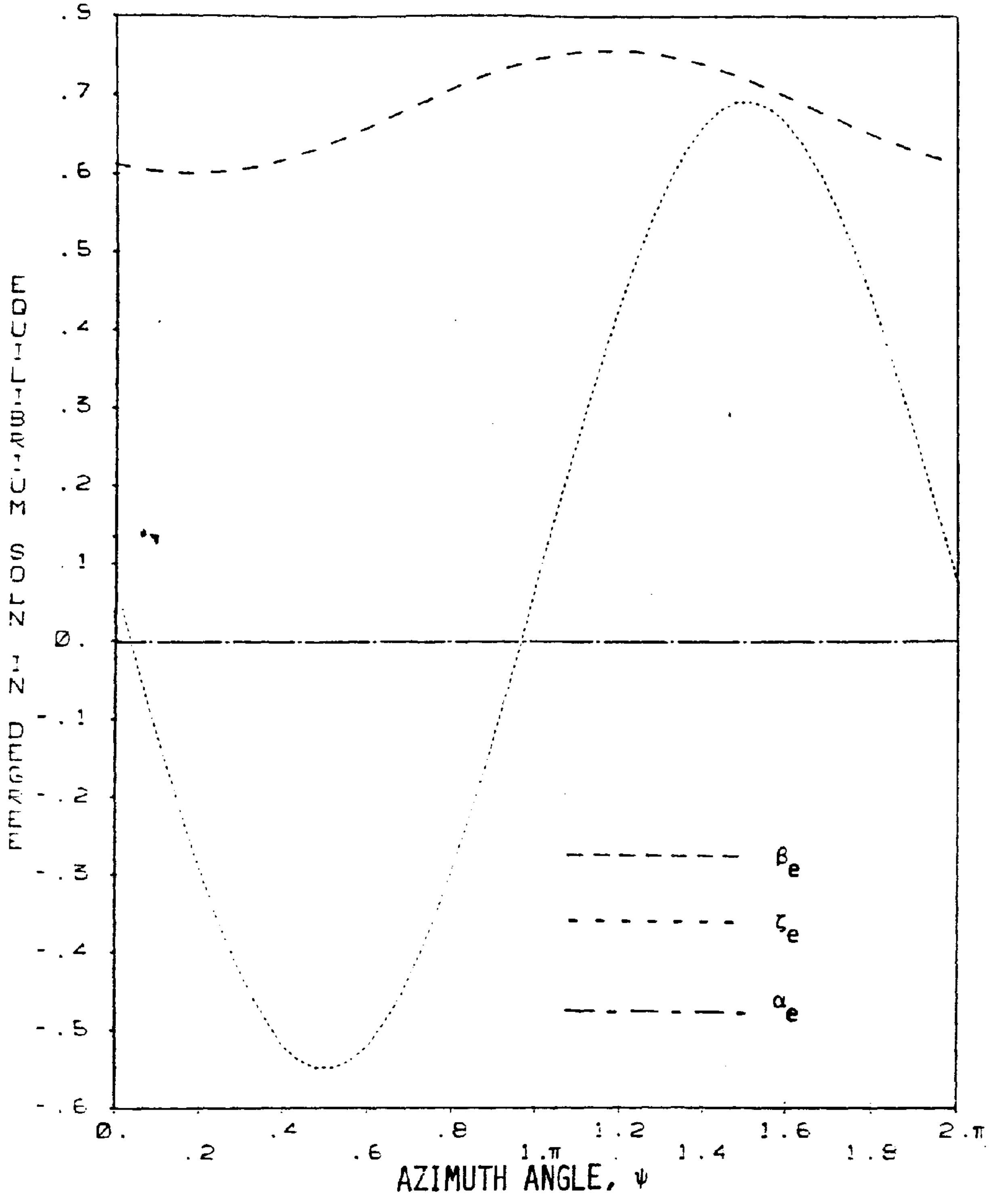

Figure 6-2 Equilibrium Solution of Coupled Flap-Leadlag-Torsion Motion.

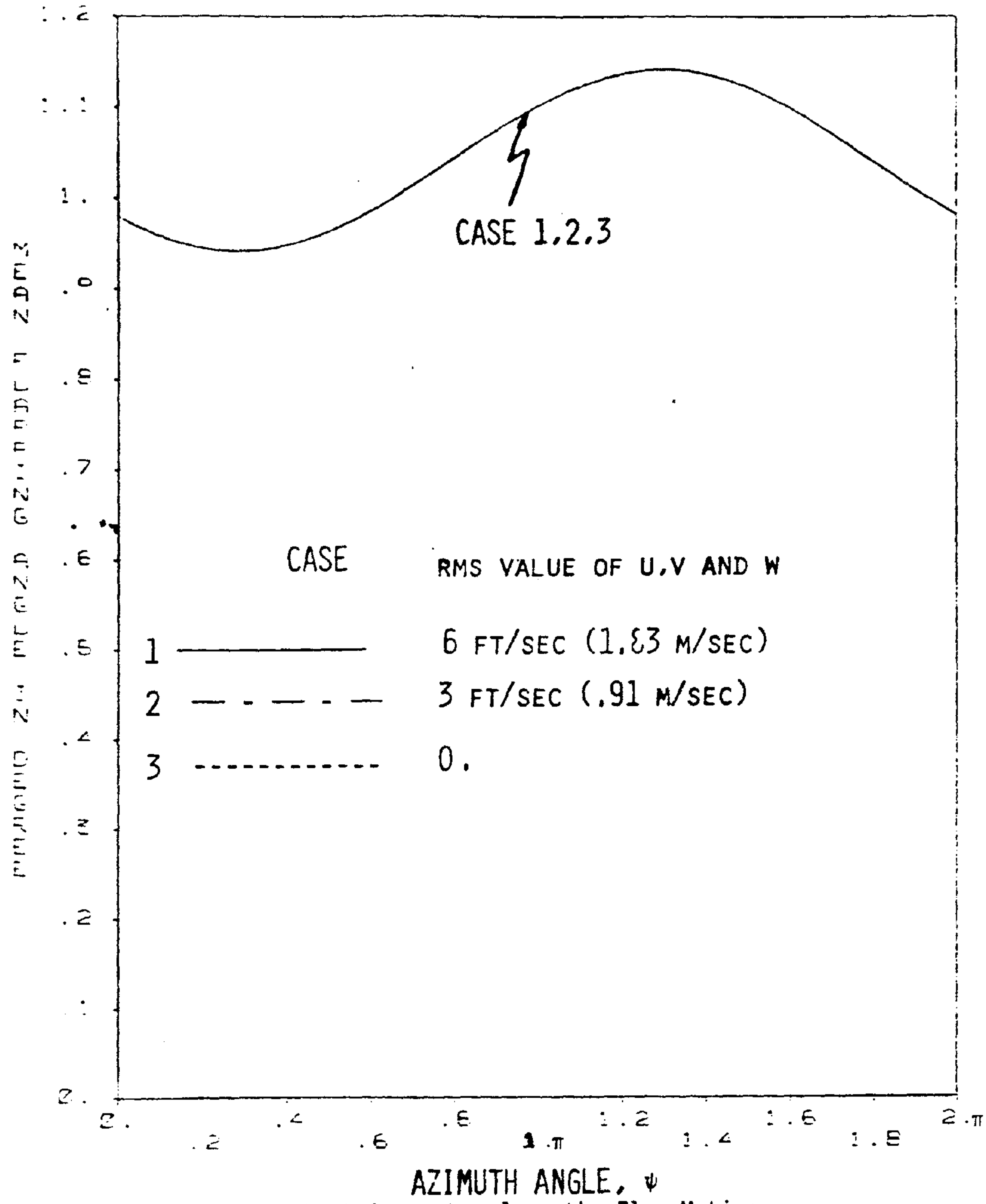

Figure 6-3 Effect of Turbulence Level on the Flap Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0$.

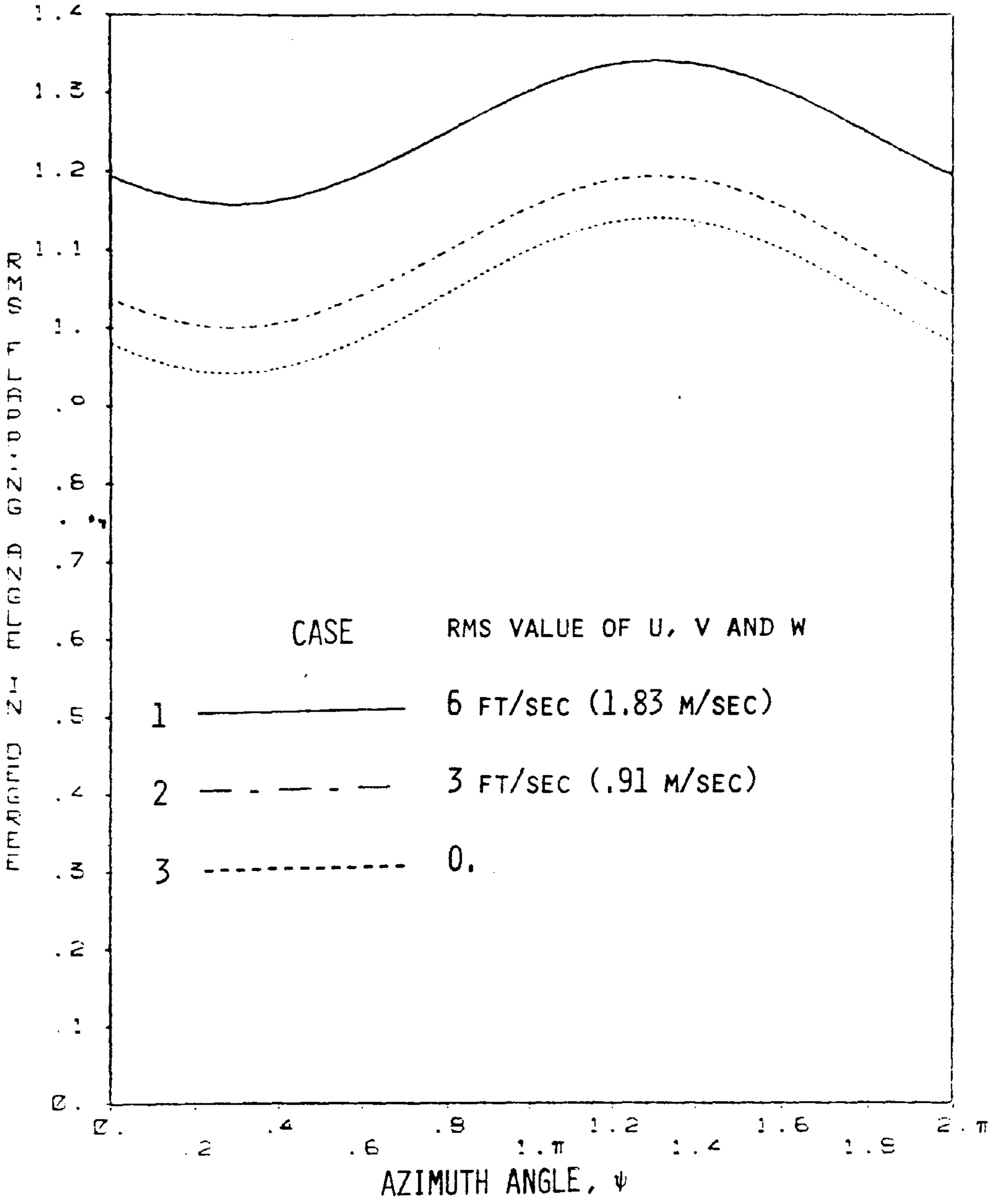

Figure 6-4 Effect of Turbulence Level on the Flap Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0.$

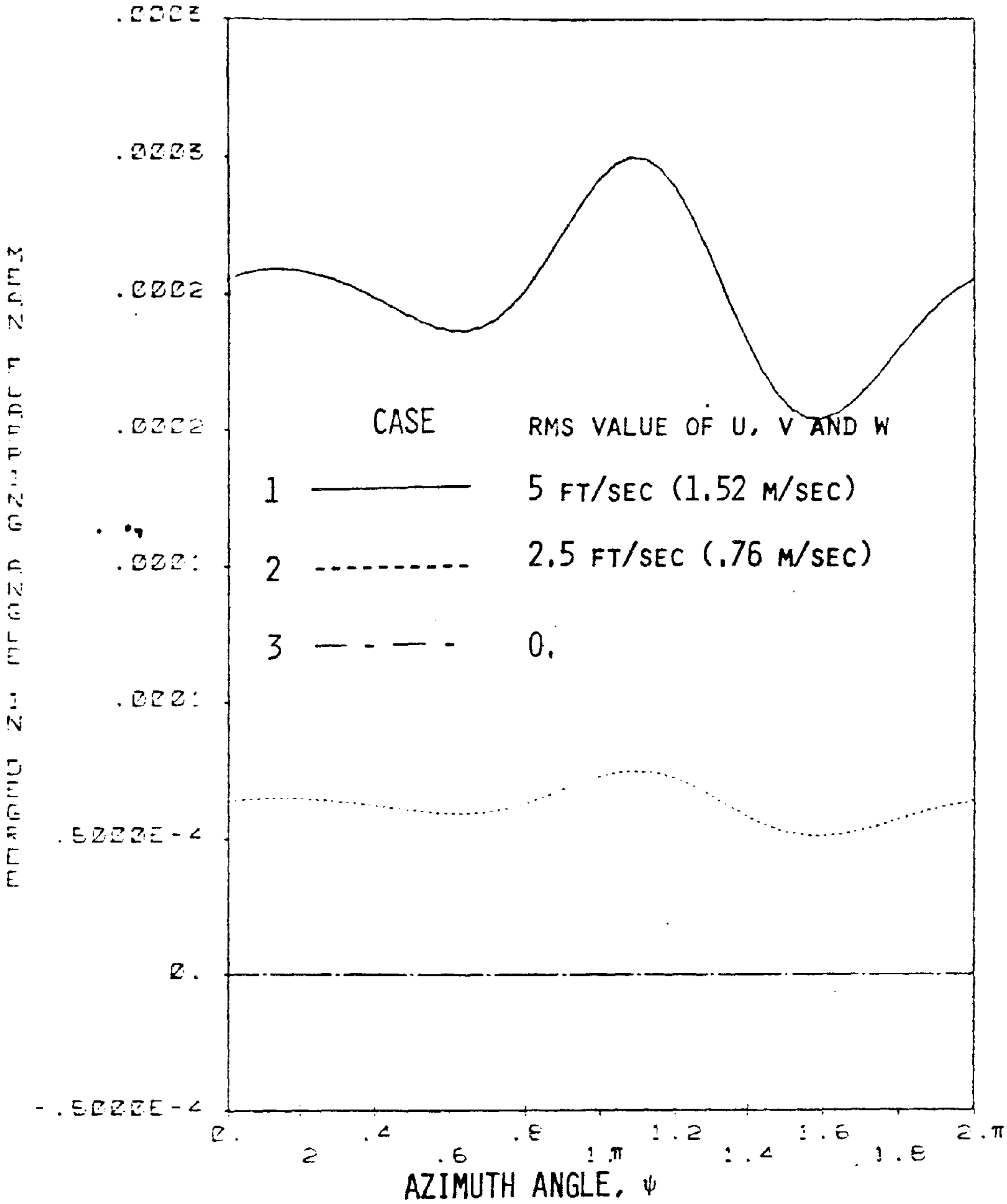

Figure 6-5 Effect of Turbulence Level on the of Coupled Flap-Leadlag Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0$.

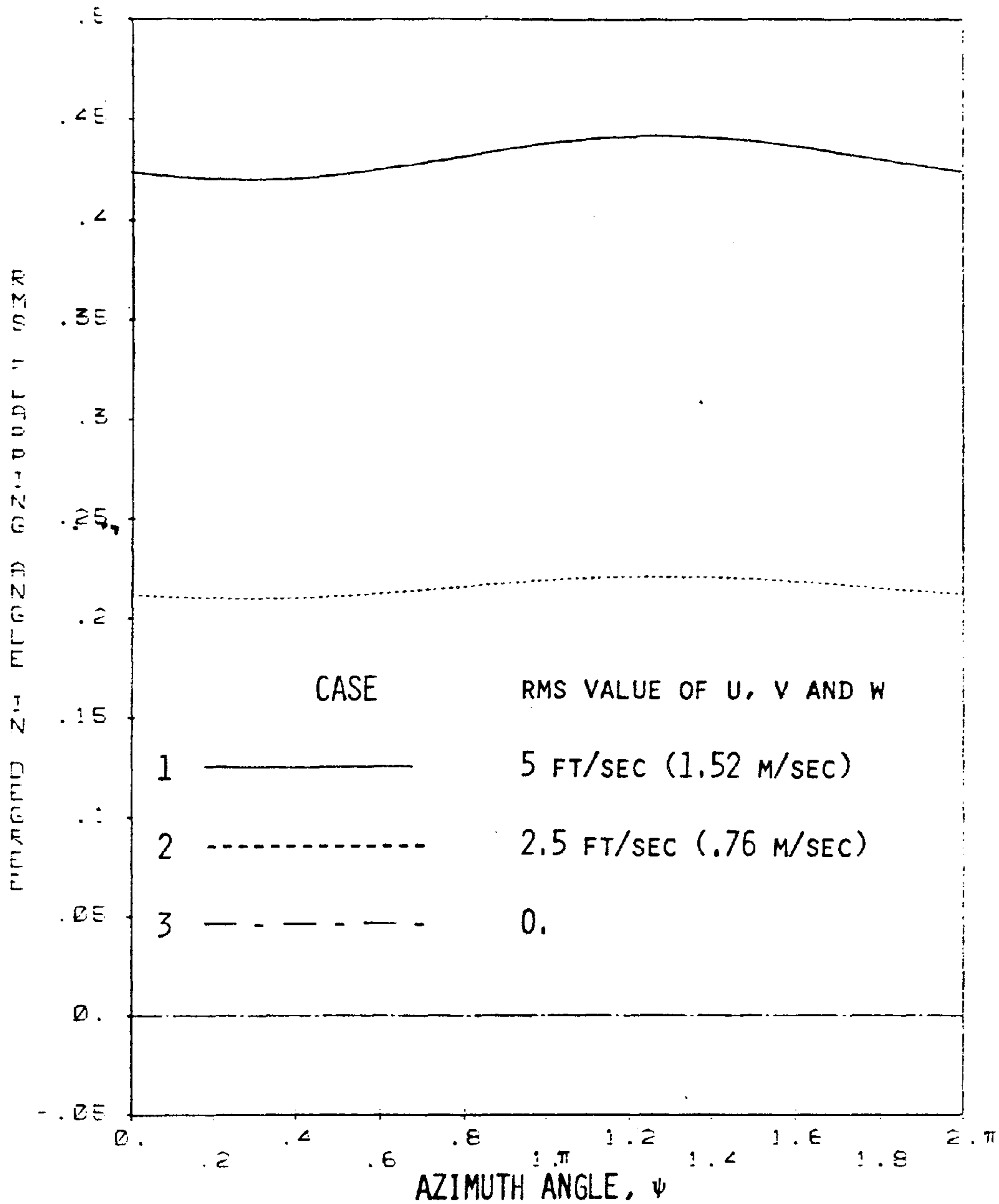

Figure 6-6 Effect of Turbulence Level on the Coupled Flap-Leadlag Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0.$

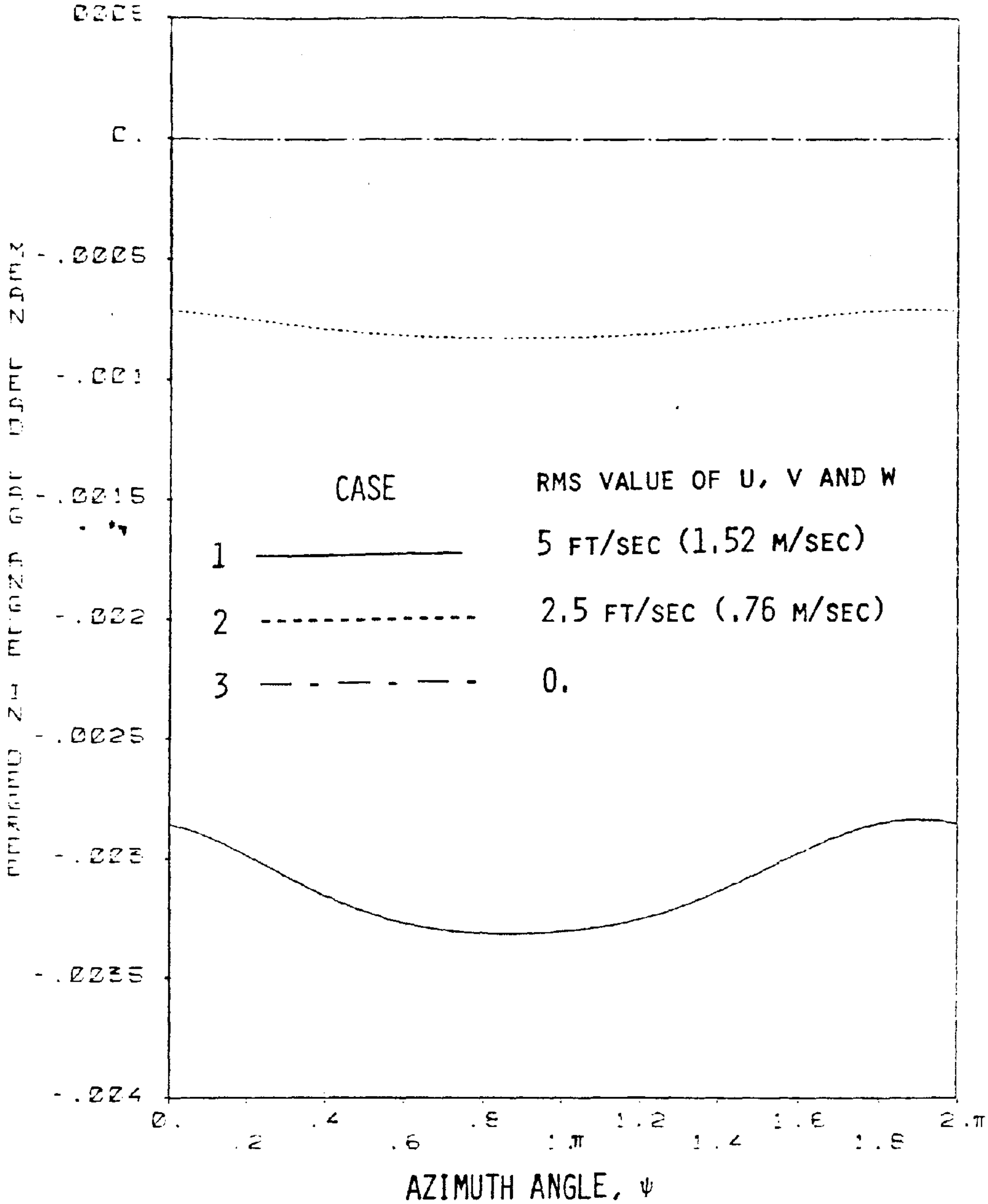

Figure 6-7 Effect of Turbulence Level on the Coupled Flap-Leadlag Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0.$

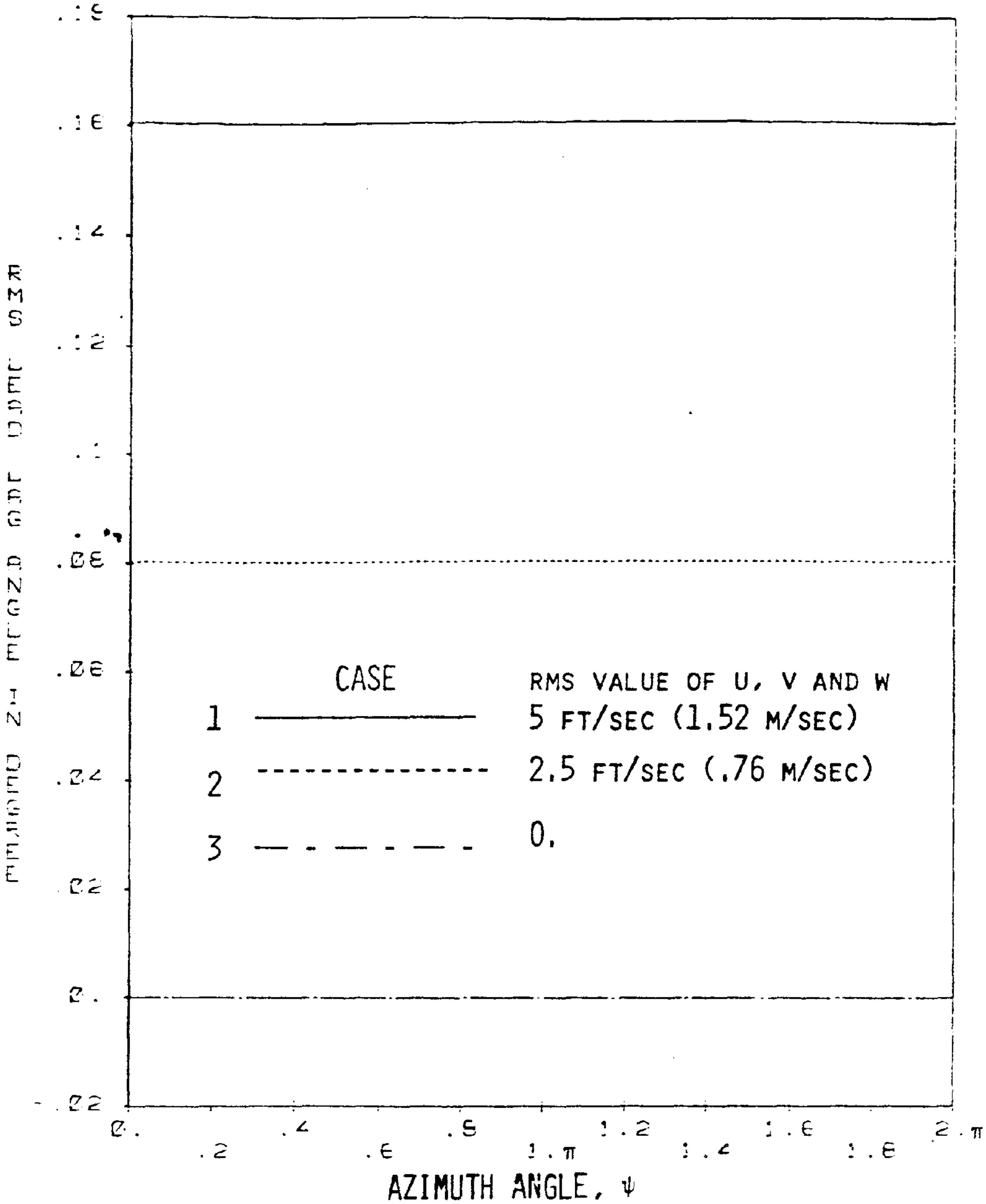

Figure 6-8 Effect of Turbulence Level on the Coupled Flap-Leadlag Motion.

$T_1 = T_2 = T_3 = .6667$, $e_1 = e_2 = e_3 = 0.$

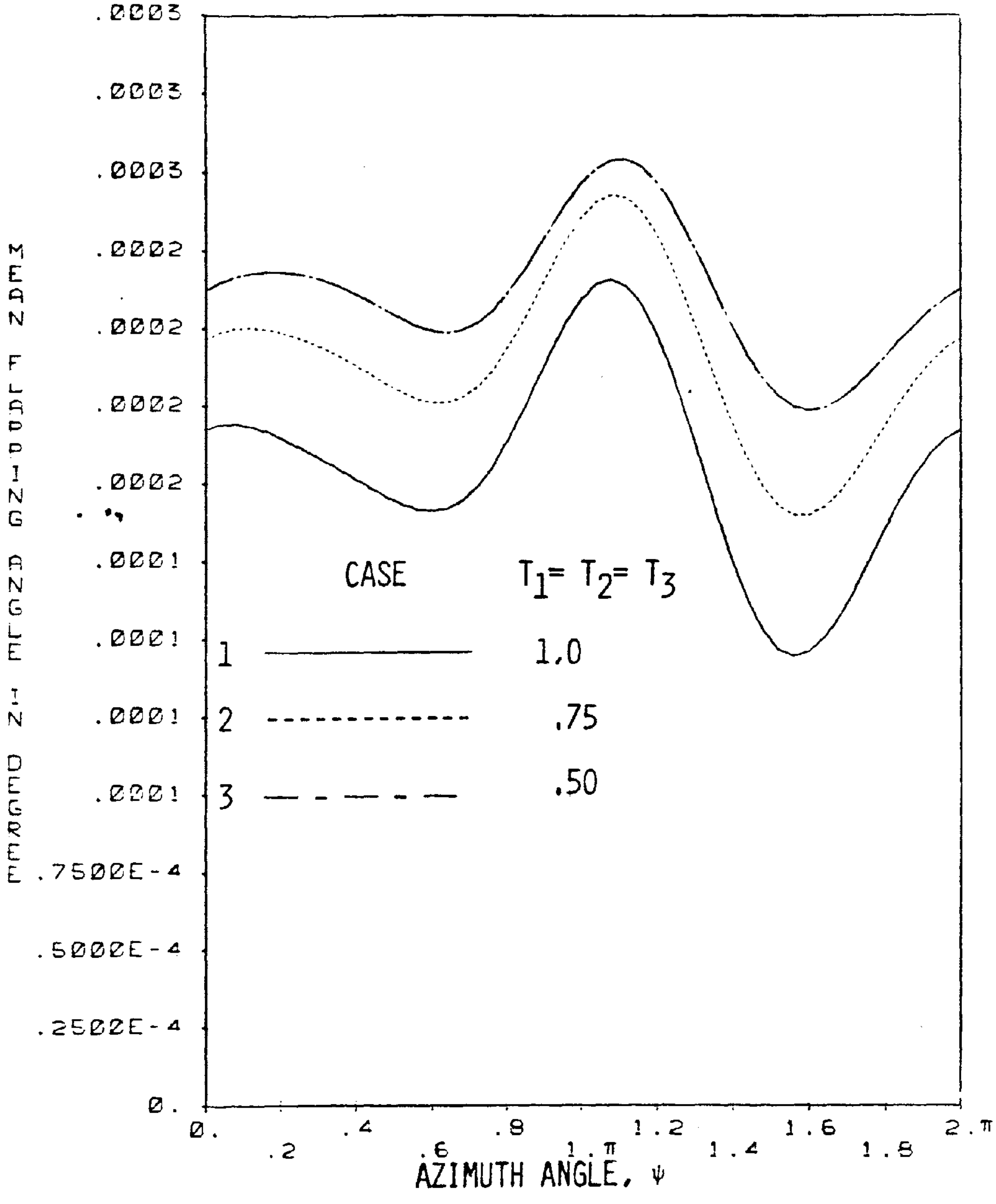

Figure 6-9 Effect of Turbulence Correlation Time on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec,) $e_1 = e_2 = e_3 = 0$.

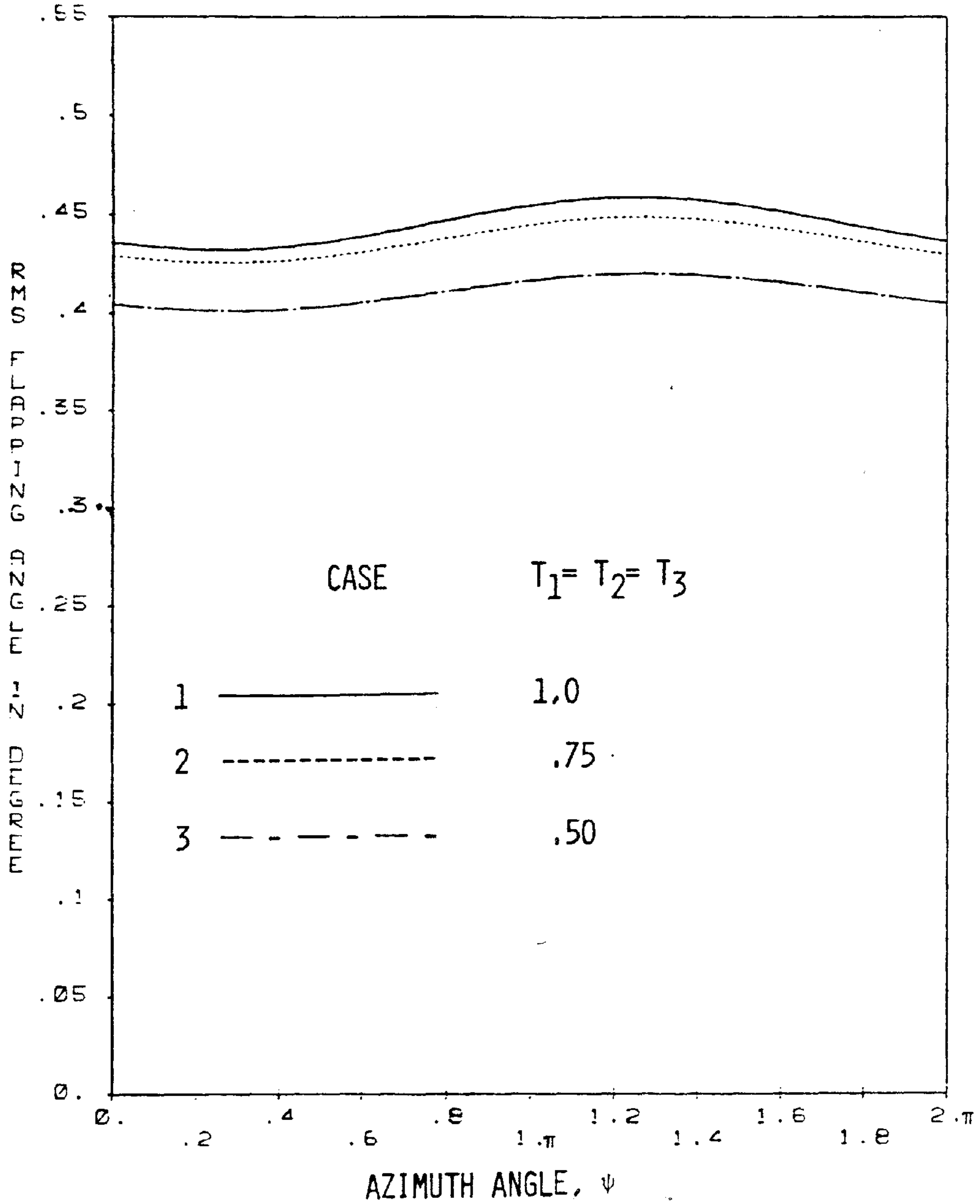

Figure 6-10 Effect of Turbulence Correlation Time on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $e_1=e_2=e_3=0$.

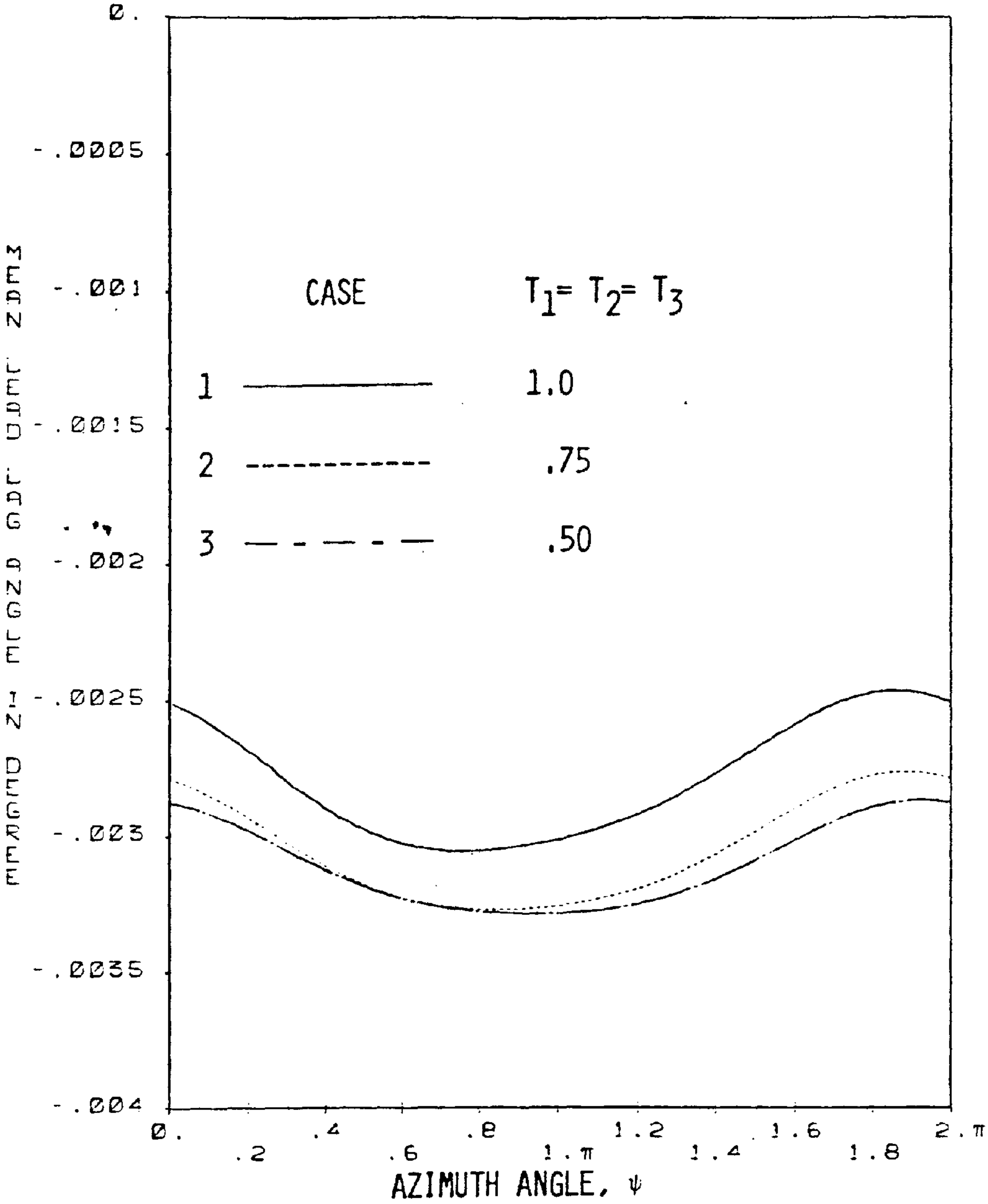

Figure 6-11 Effect of Turbulence Correlation Time on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $e_1=e_2=e_3=0$.

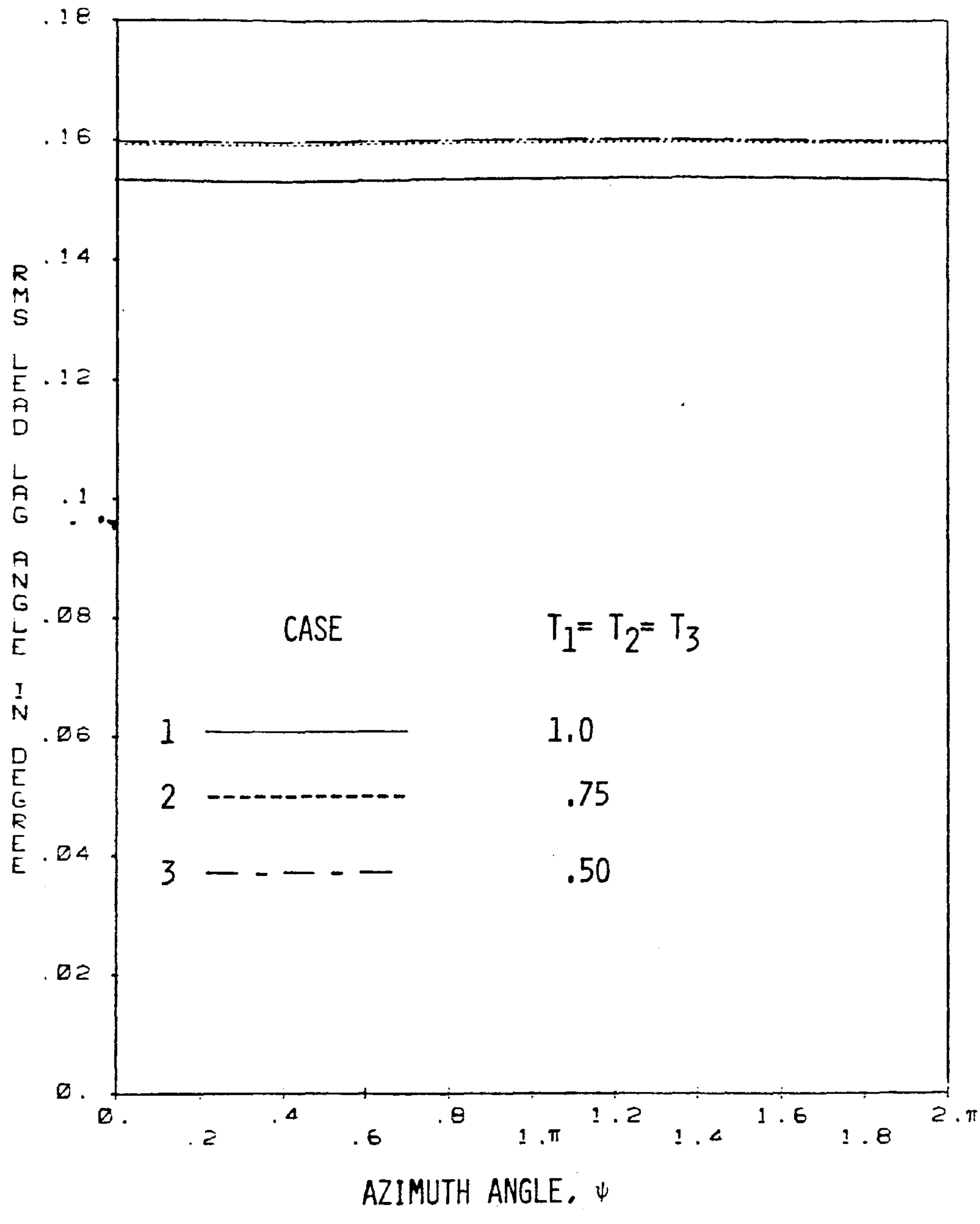

Figure 6-12 Effect of Turbulence Correlation Time on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $e_1=e_2=e_3=0$.

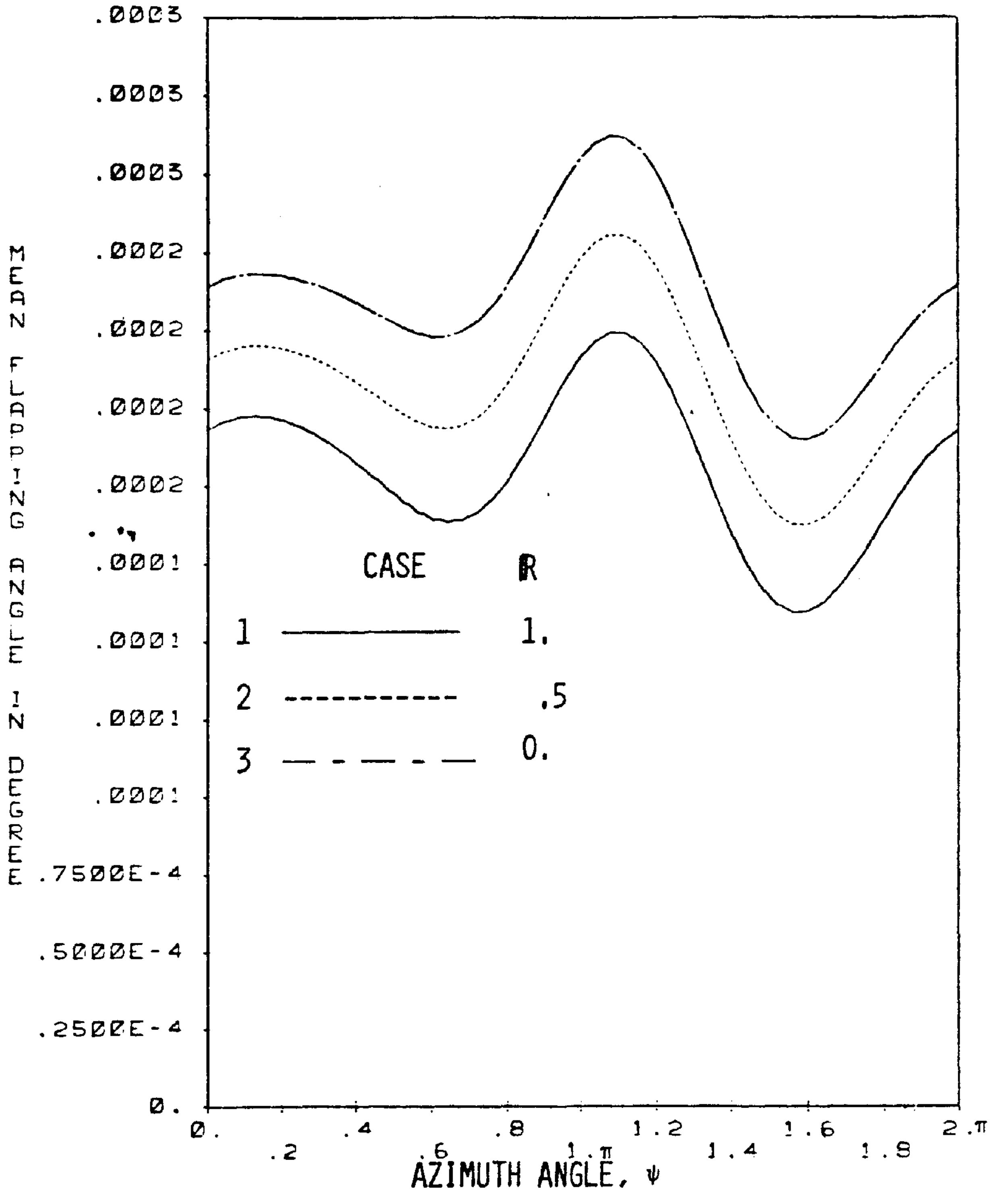

Figure 6-13 Effect of Elastic Coupling Parameter on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $T_1=T_2=T_3=.6667$,

$e_1=e_2=e_3=0$.

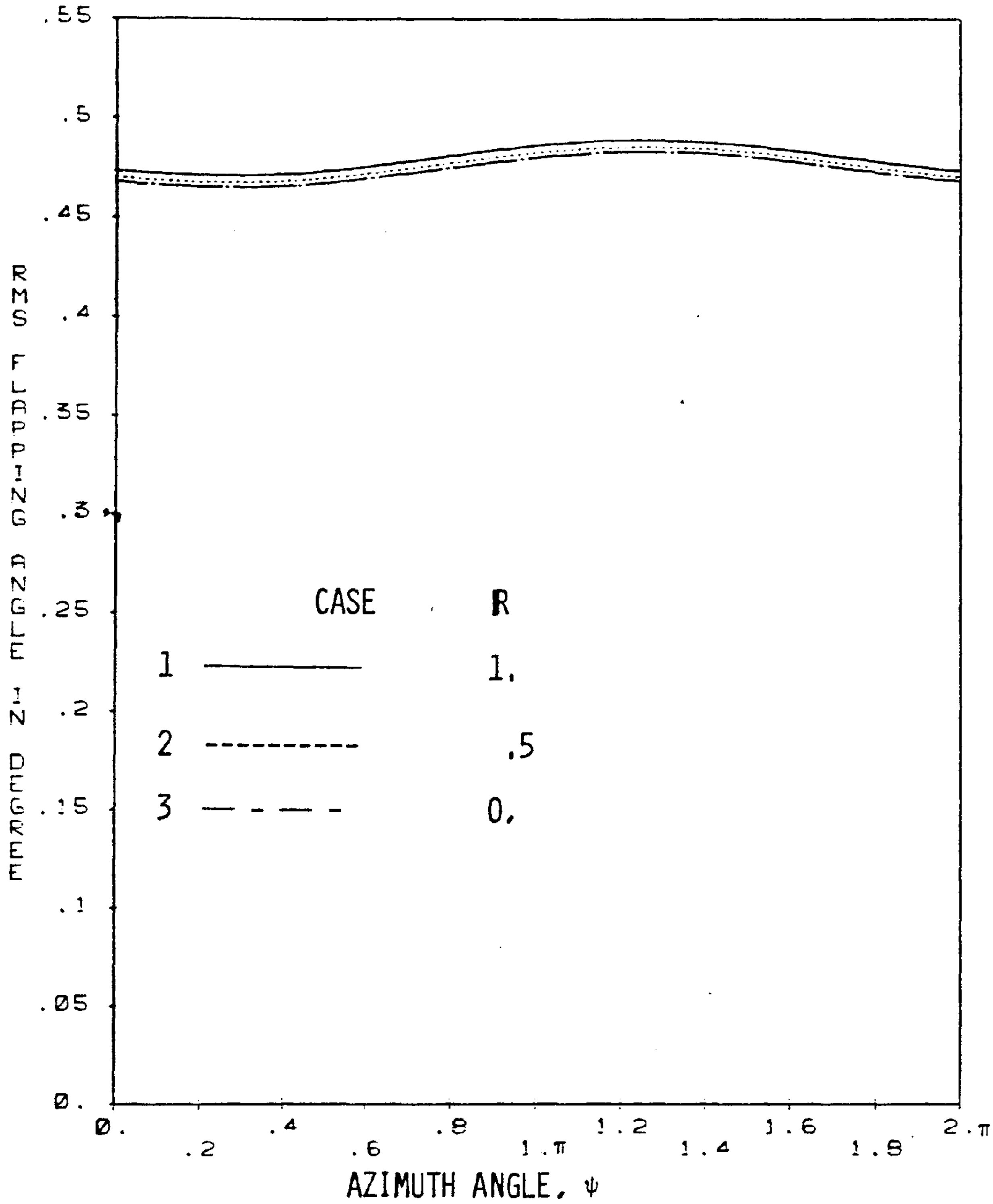

Figure 6-14 Effect of Elastic Coupling Parameter on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

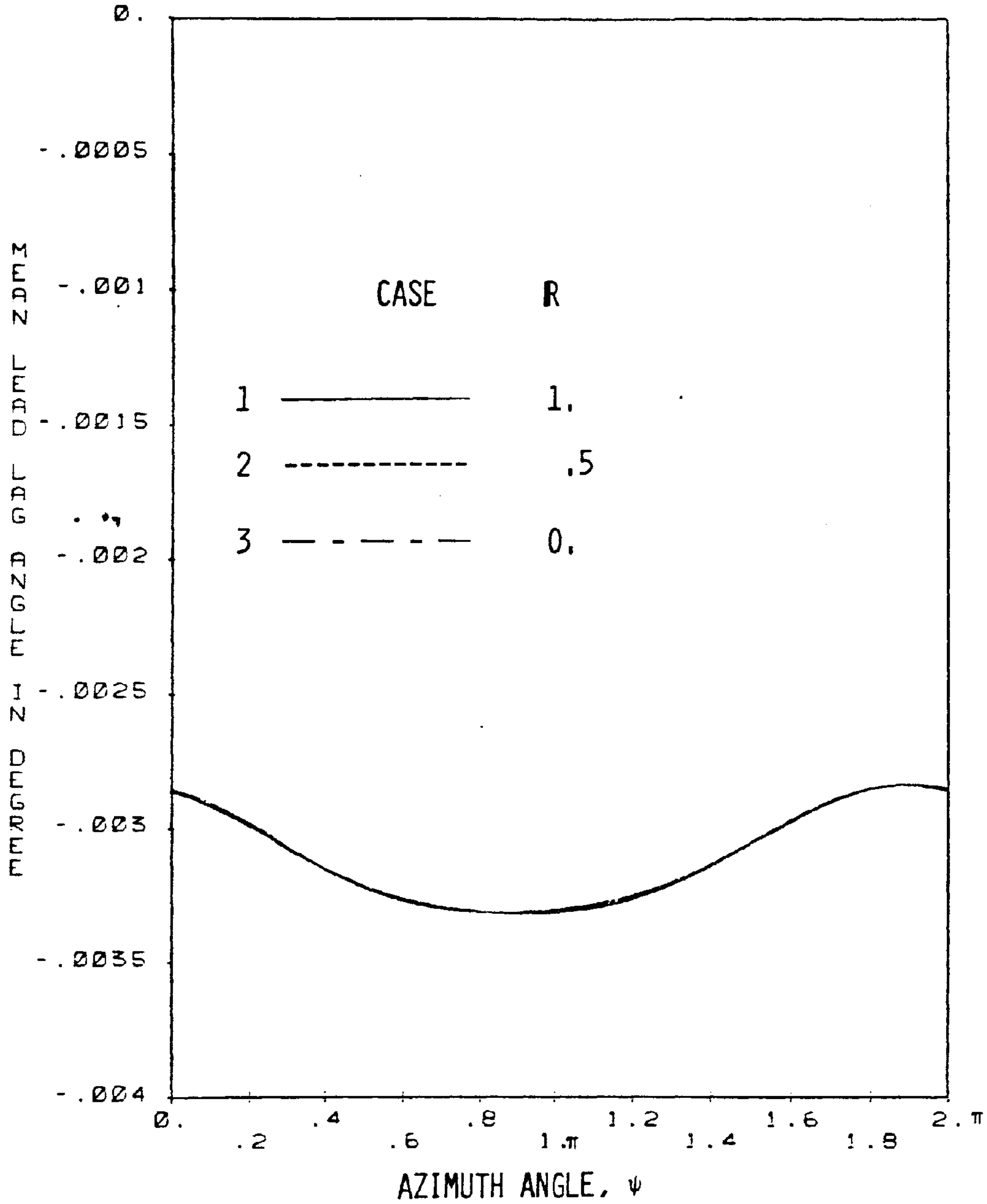

Figure 6-15 Effect of Elastic Coupling Parameter on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

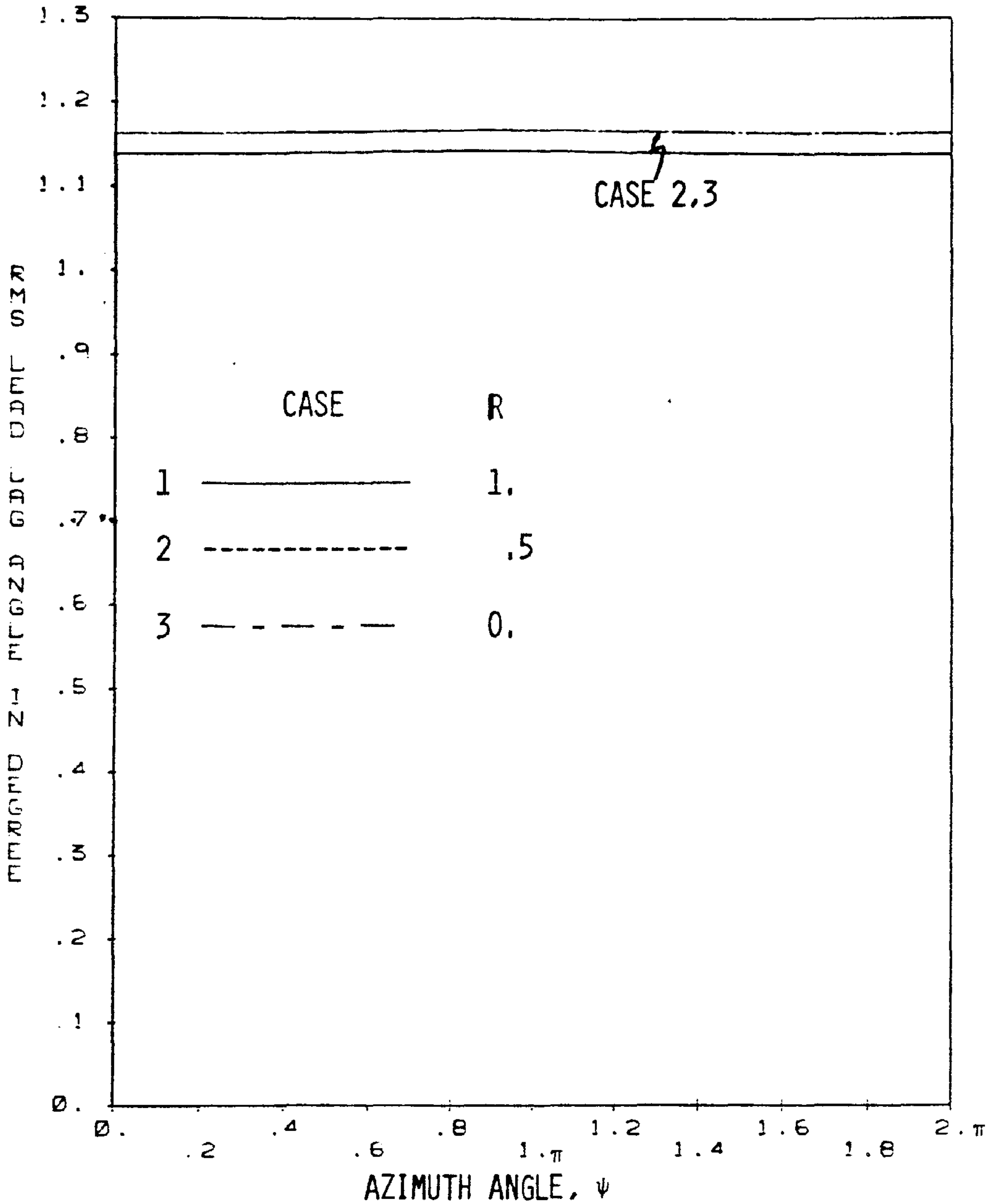

Figure 6-16 Effect of Elastic Coupling Parameter on the Coupled Flap-Leadlag Motion.

rms value of u, v and w = 5 ft/sec (1.52 m/sec), $T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

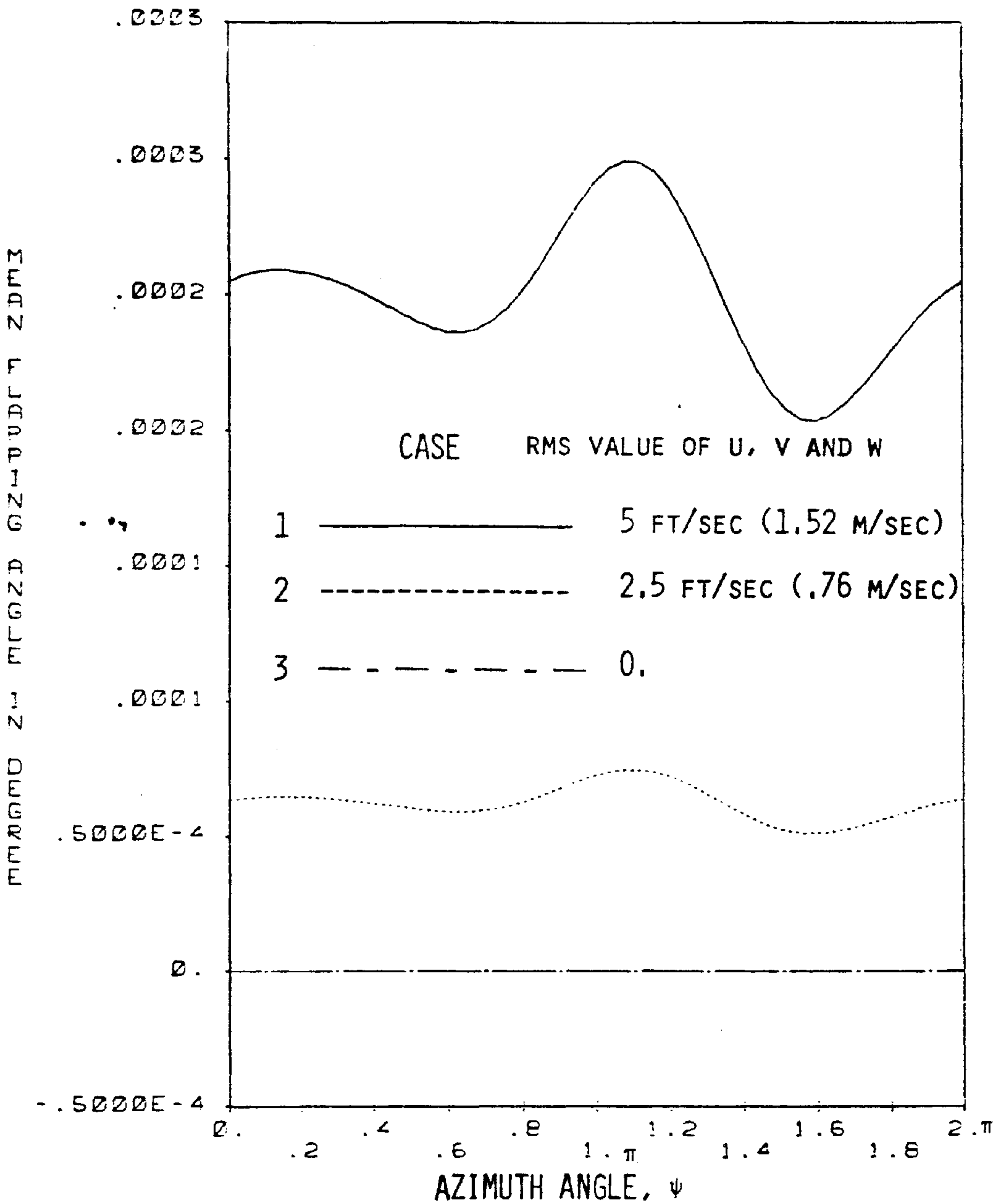

Figure 6-17 Effect of Turbulence Level on the Coupled Flap-Leadlag-Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

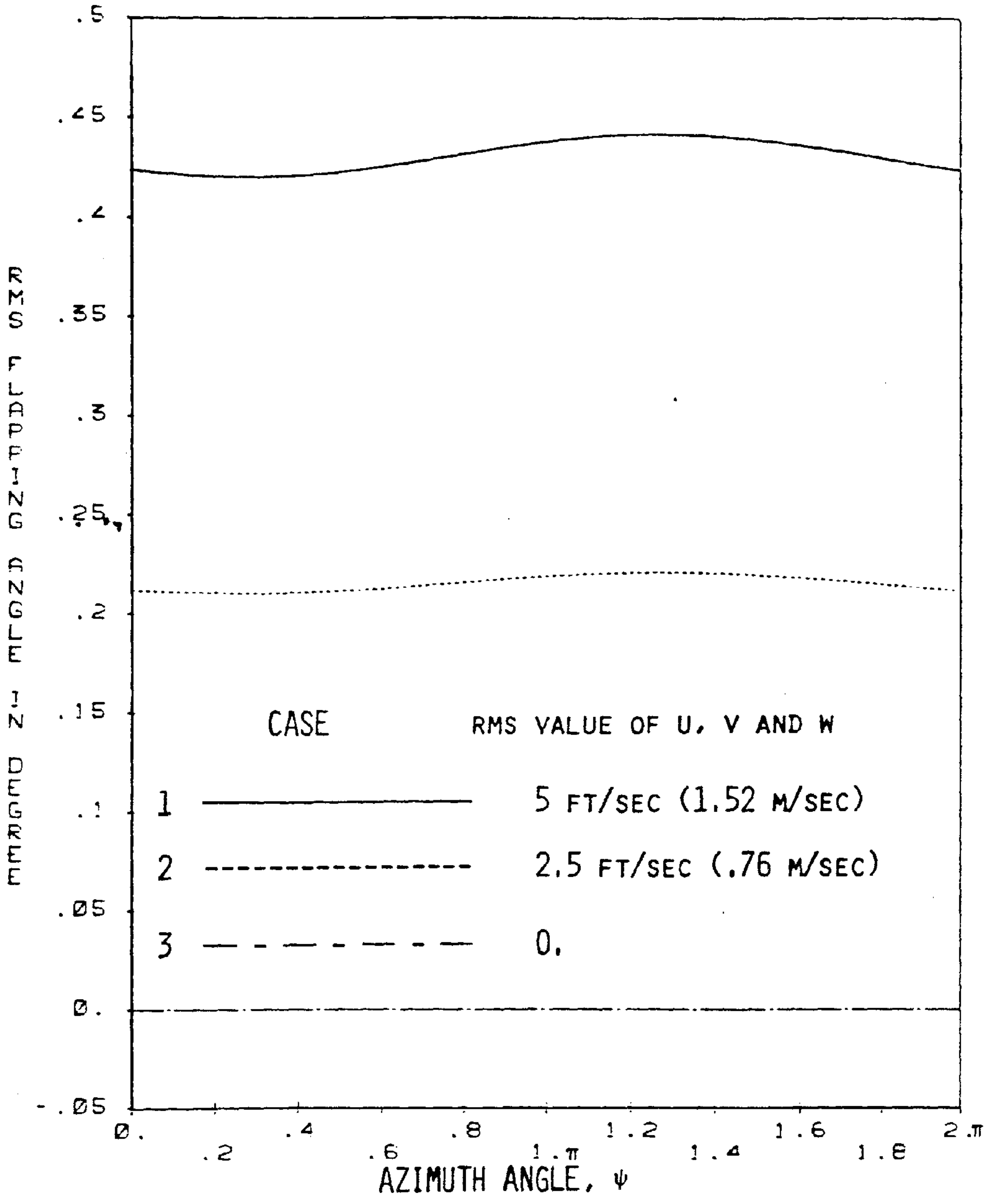

Figure 6-18 Effect of Turbulence Level on the Coupled Flap-Leadlag-Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

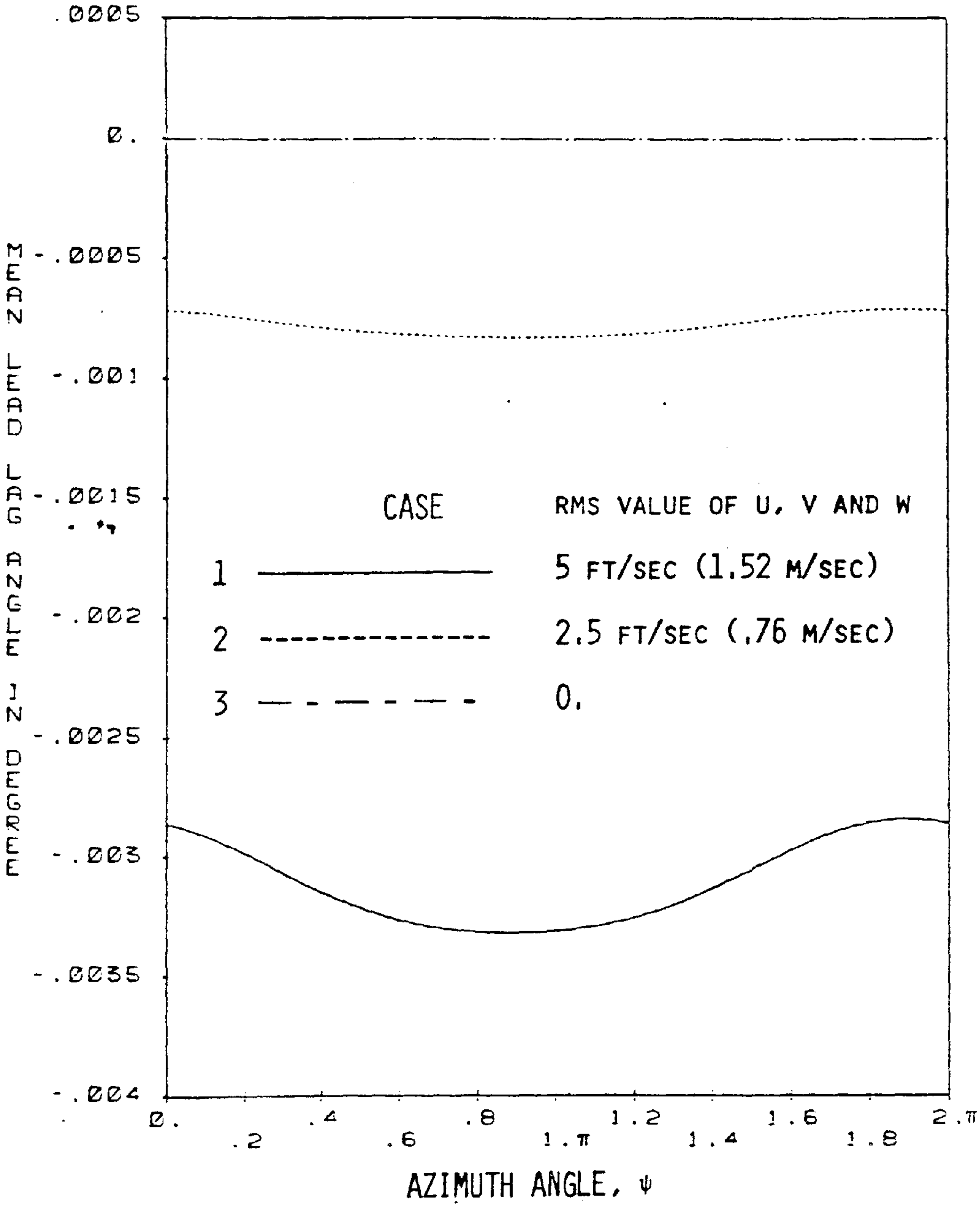

Figure 6-19 Effect of Turbulence Level on the Coupled Flap-Leadlag-Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

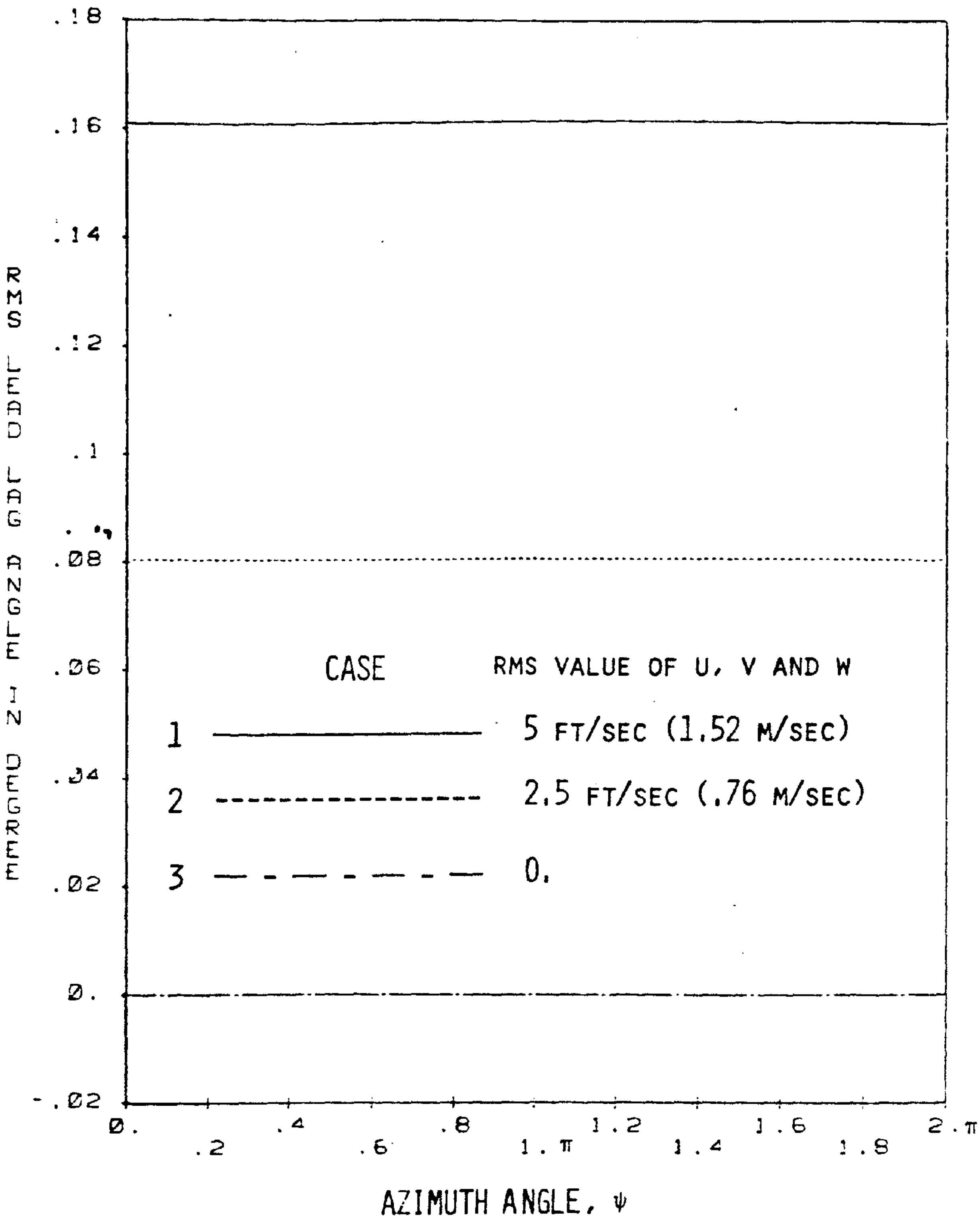

Figure 6-20 Effect of Turbulence Level on the Coupled Flap-Leadlag-Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0.$

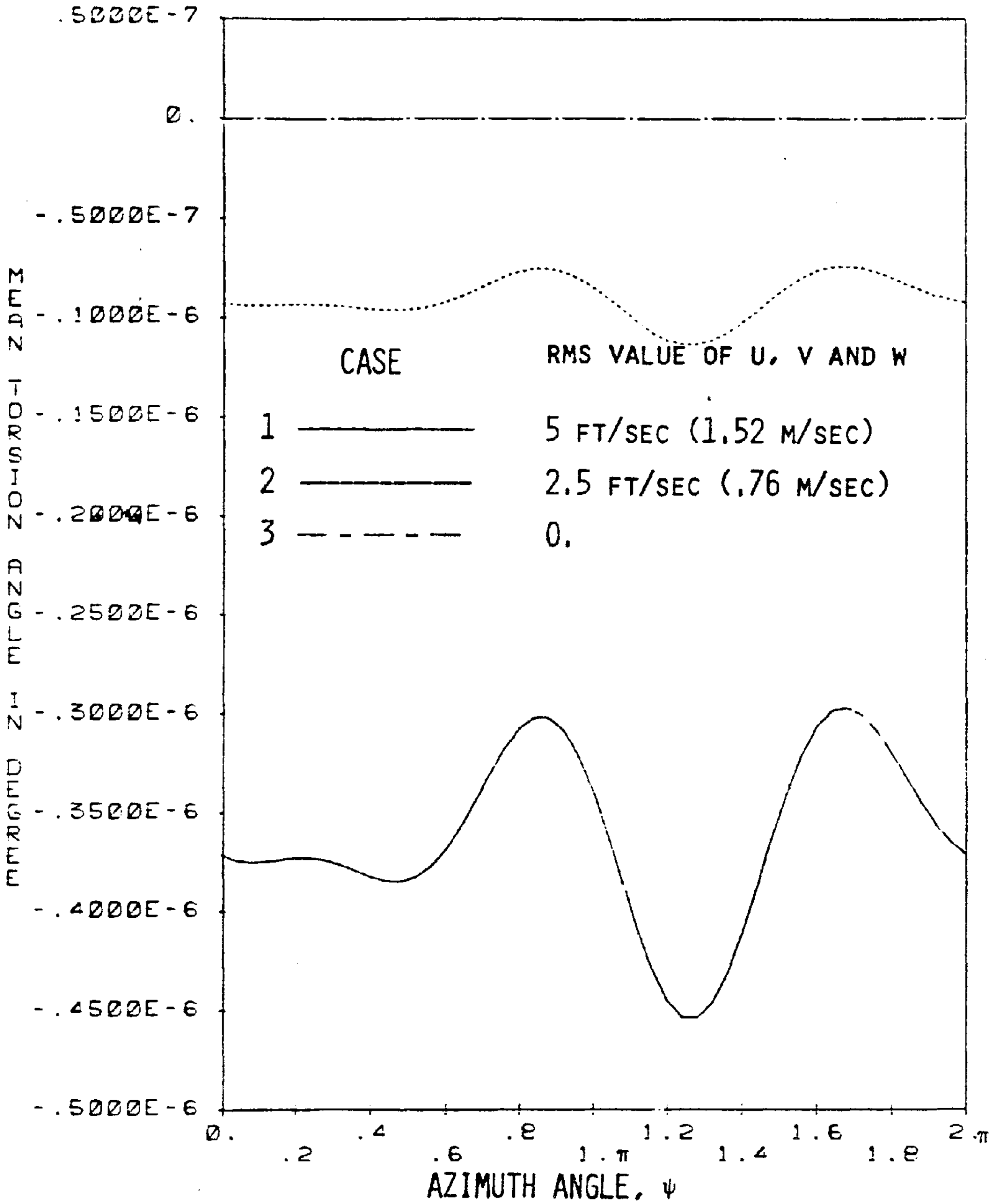

Figure 6-21 Effect of Turbulence Level on the Coupled Flap-Leadlag Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

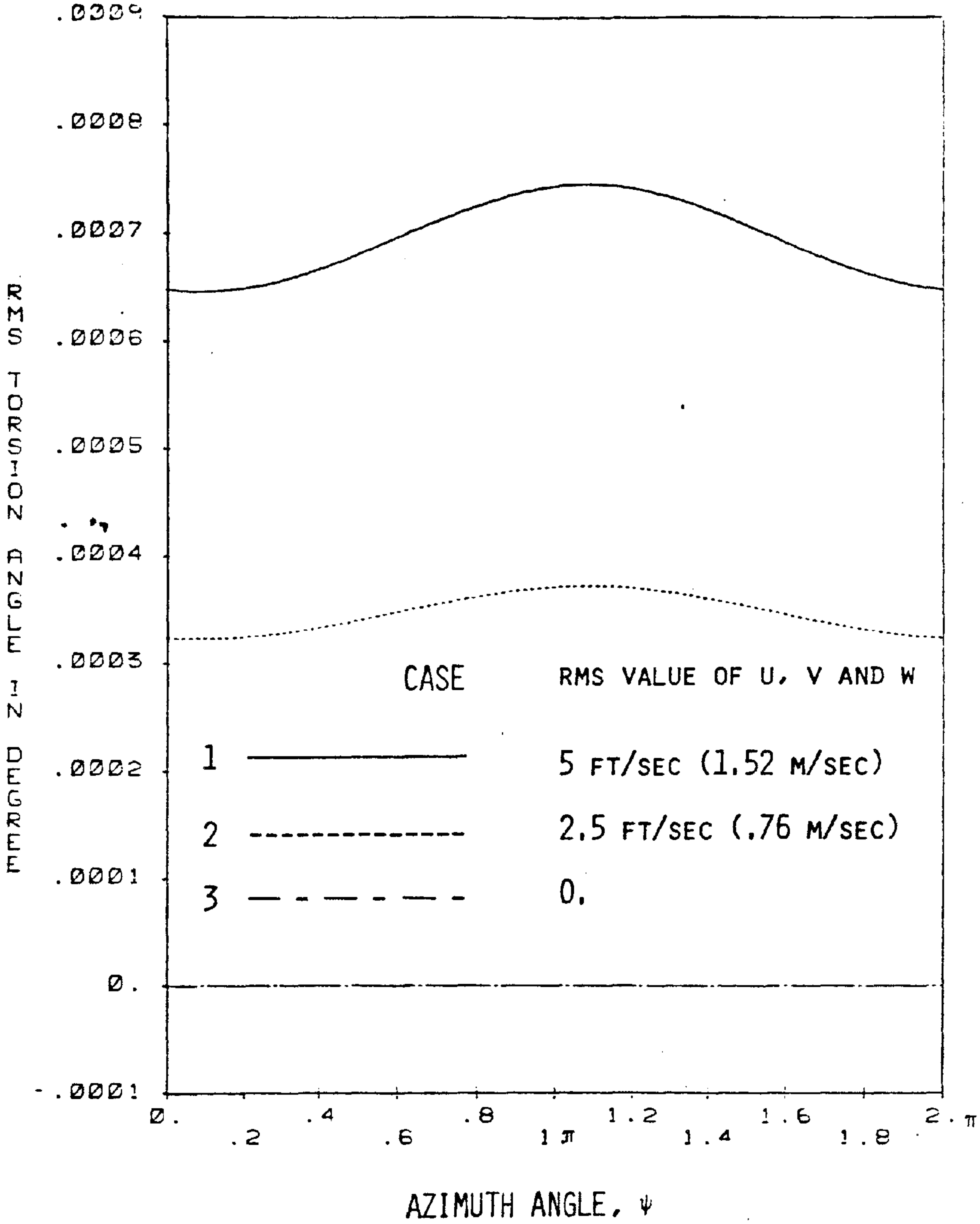

Figure 6-22 Effect of Turbulence Level on the Coupled Flap-Leadlag-Torsion Motion.

$T_1=T_2=T_3=.6667$, $e_1=e_2=e_3=0$.

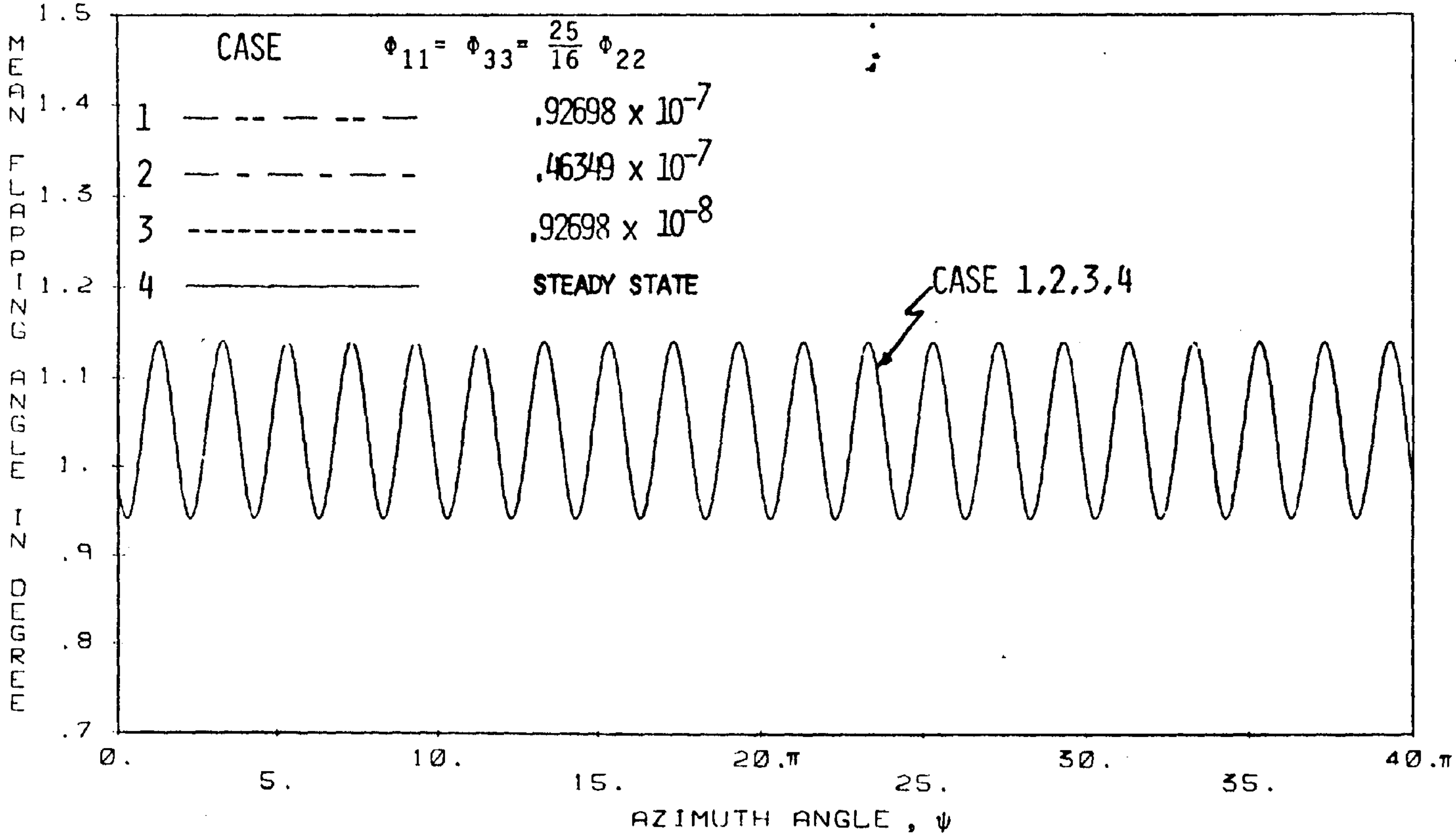

Figure 6-23 Effect of Earthquake Level on the Transient Flap Motion.

$T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$, rms value of u, v and w = 3 ft/sec (.91 m/sec).

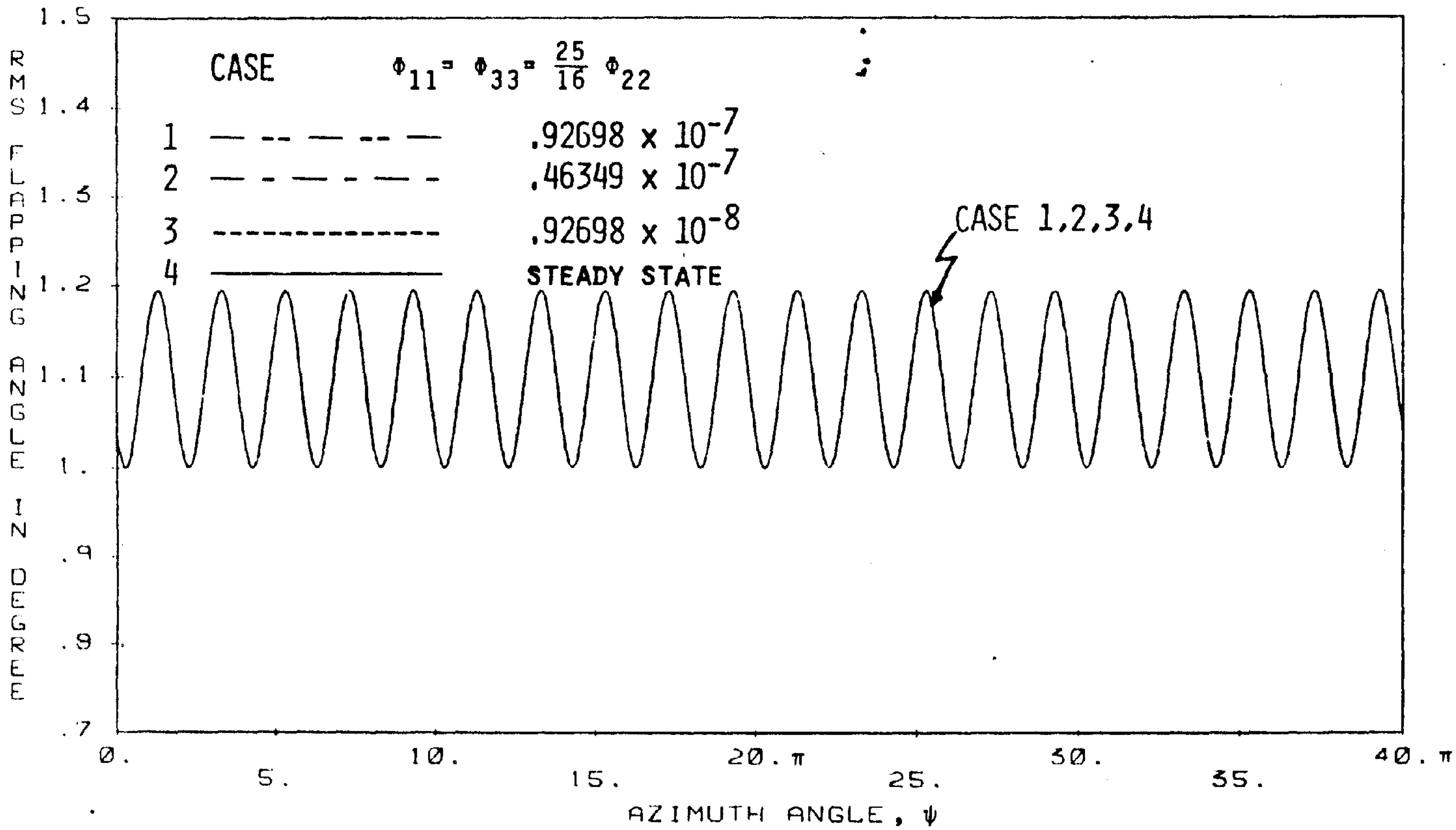

Figure 6-24 Effect of Earthquake Level on the Transient Flap Motion.

$T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$, rms value of u, v and w = 3 ft/sec (.91 m/sec).

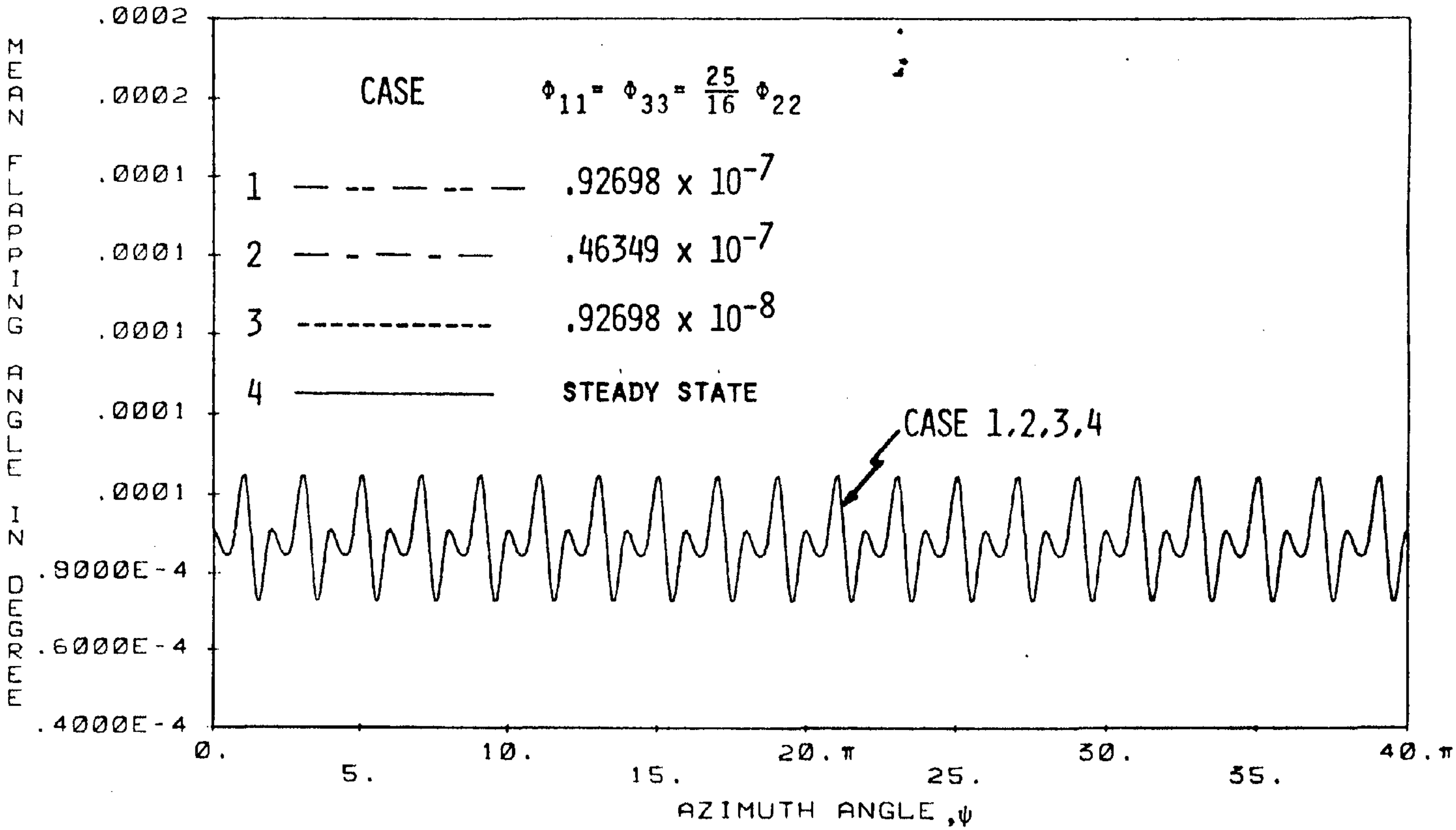

Figure 6-25 Effect of Earthquake Level on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$.

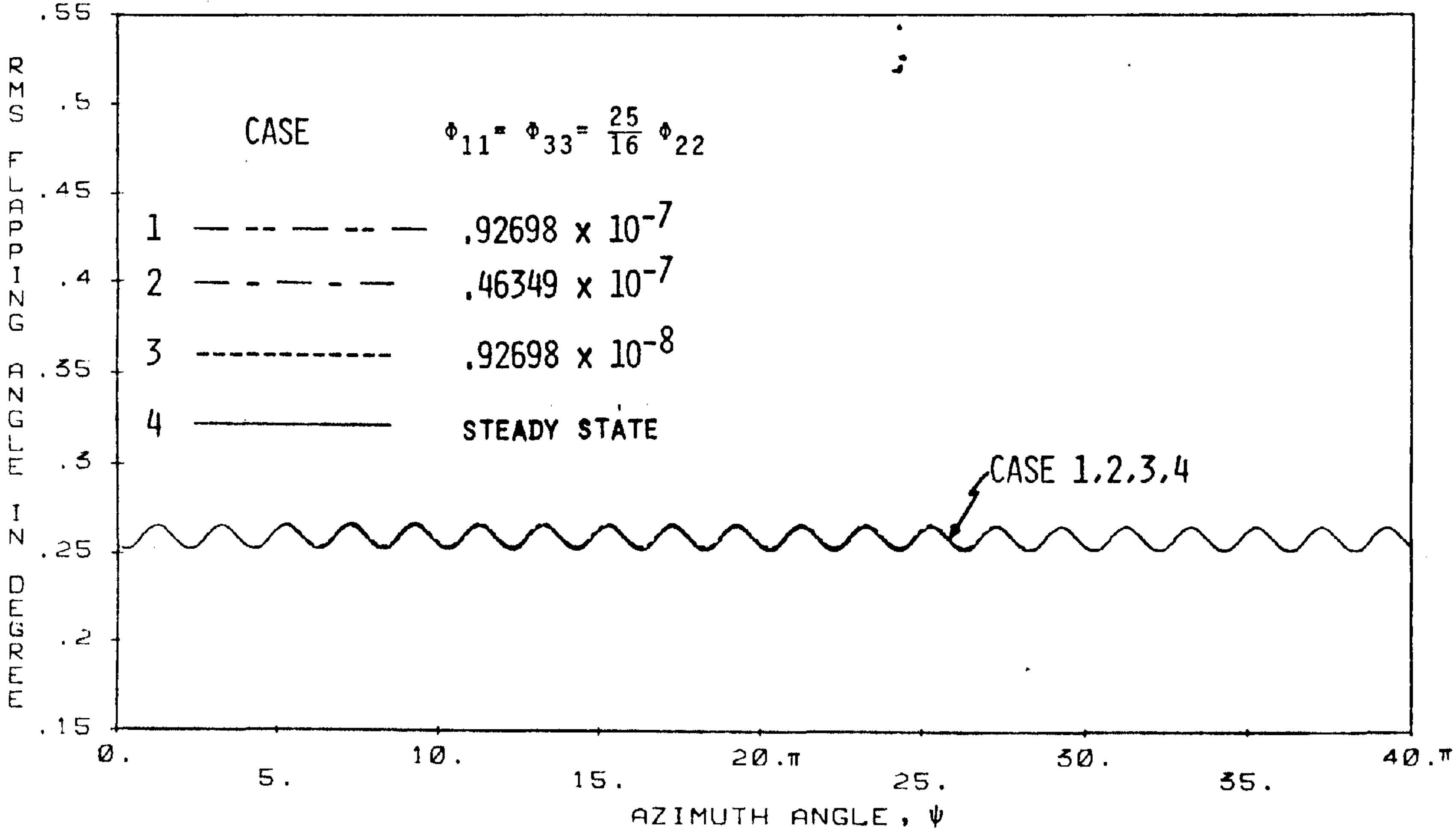

Figure 6-26 Effect of Earthquake Level on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$.

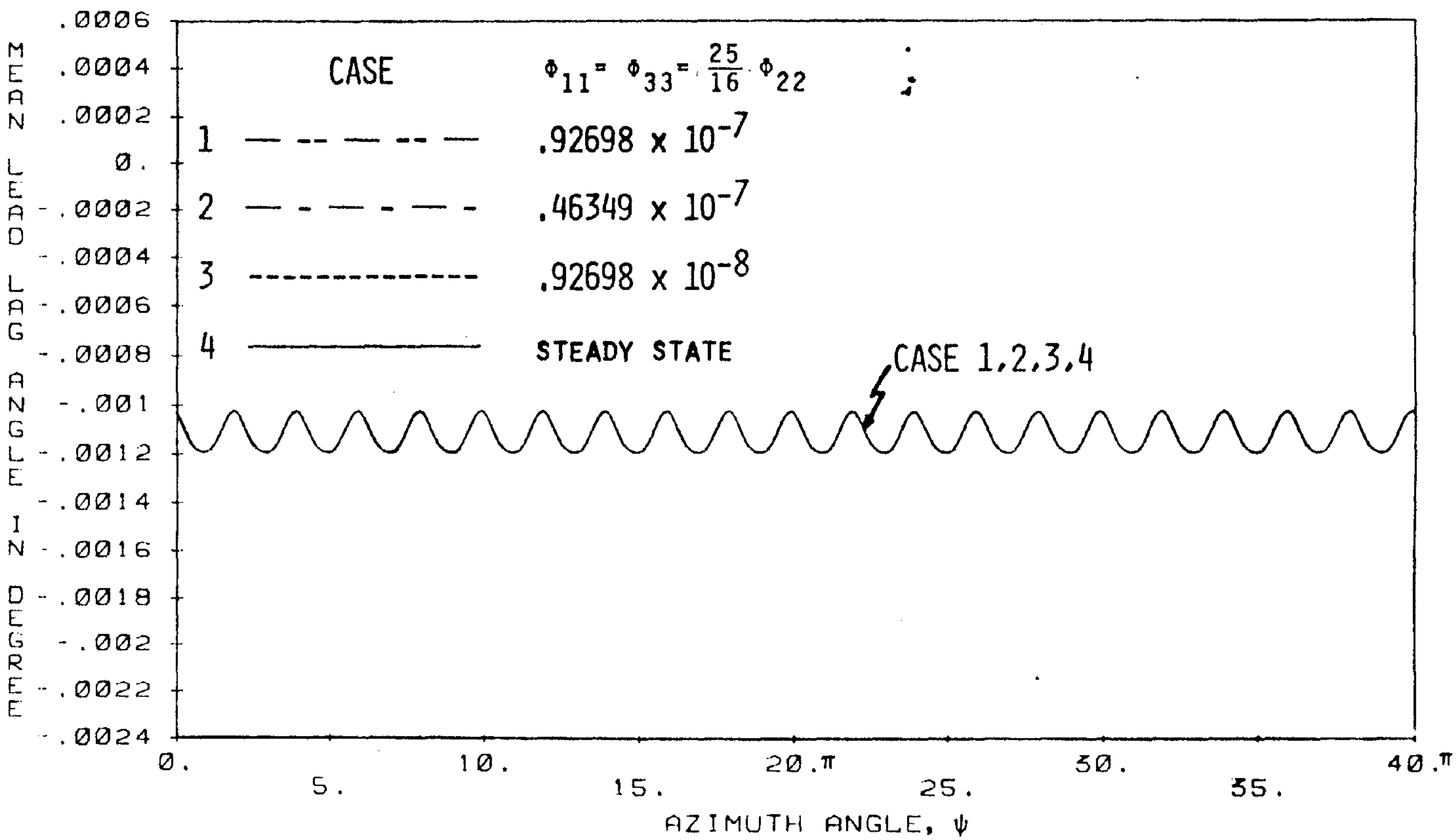

Figure 6-27 Effect of Earthquake Level on the Transient Coupled Flap-Leadlag Motion. rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$.

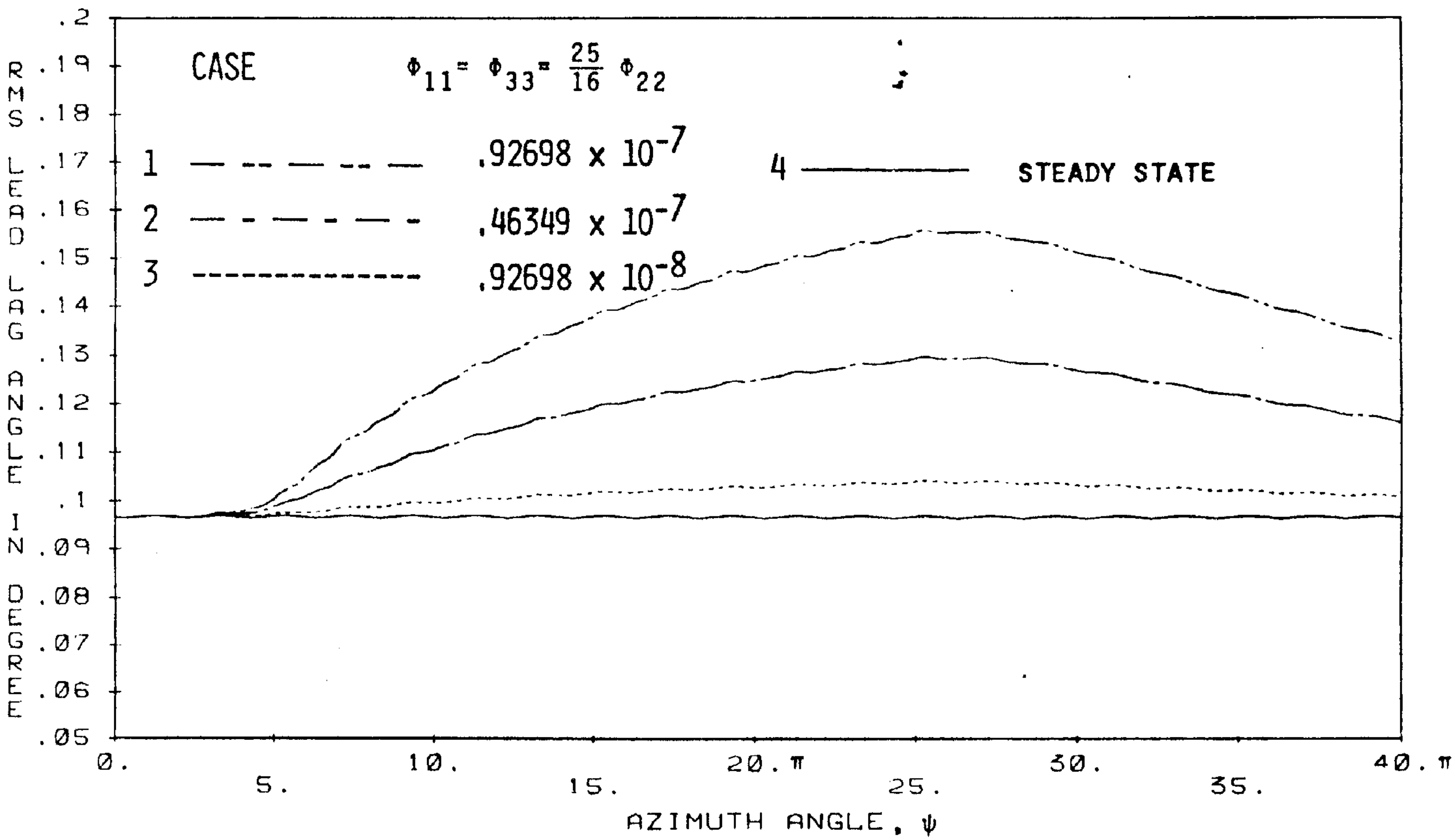

Figure 6-28 Effect of Earthquake Level on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$, $\tau_0 = 0$.

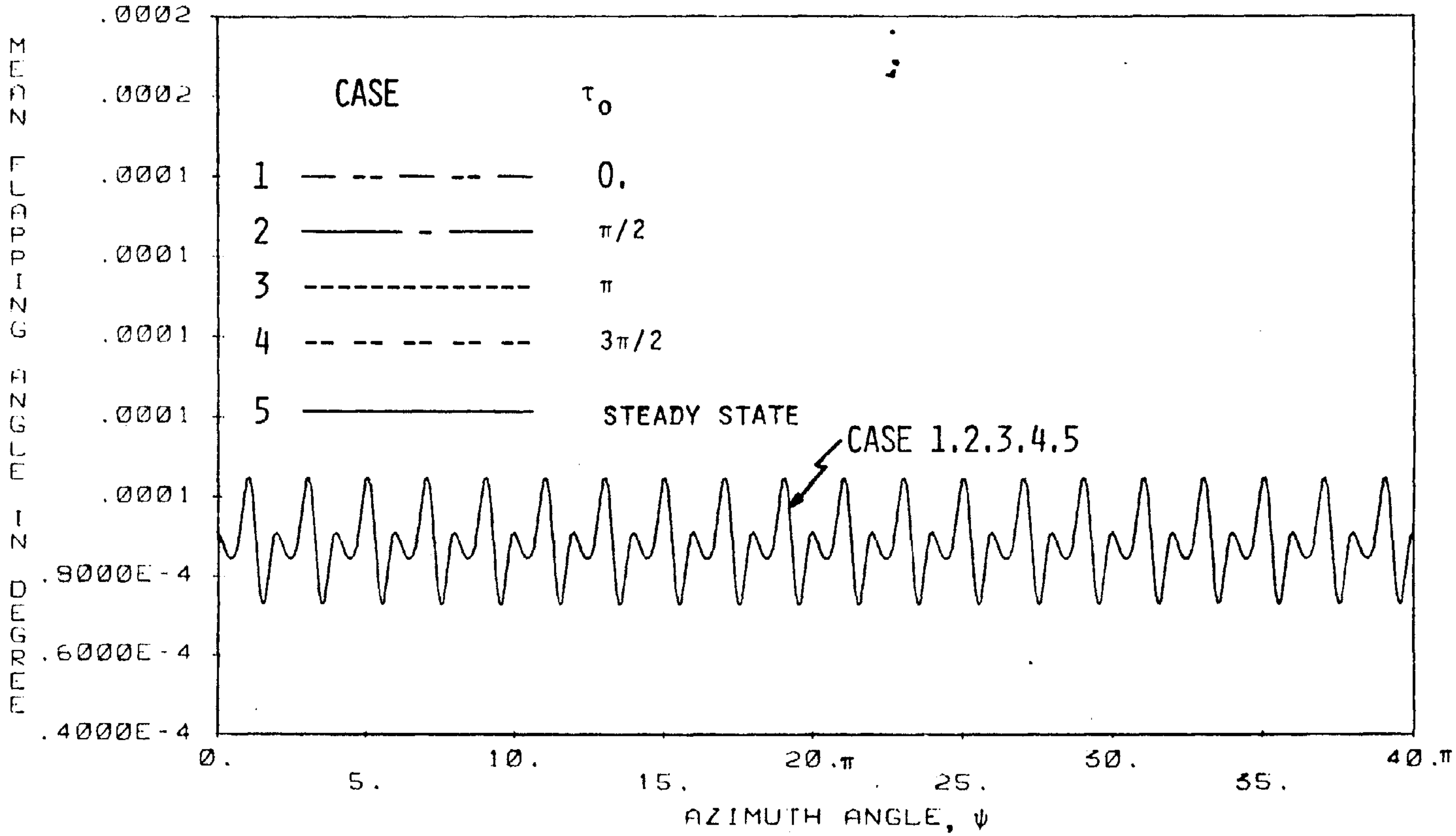

Figure 6-29 Effect of Earthquake Initiation Time on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$,

$\Phi_{11} = \Phi_{33} = \frac{25}{16}\ \Phi_{22} = .92698 \times 10^{-7}$.

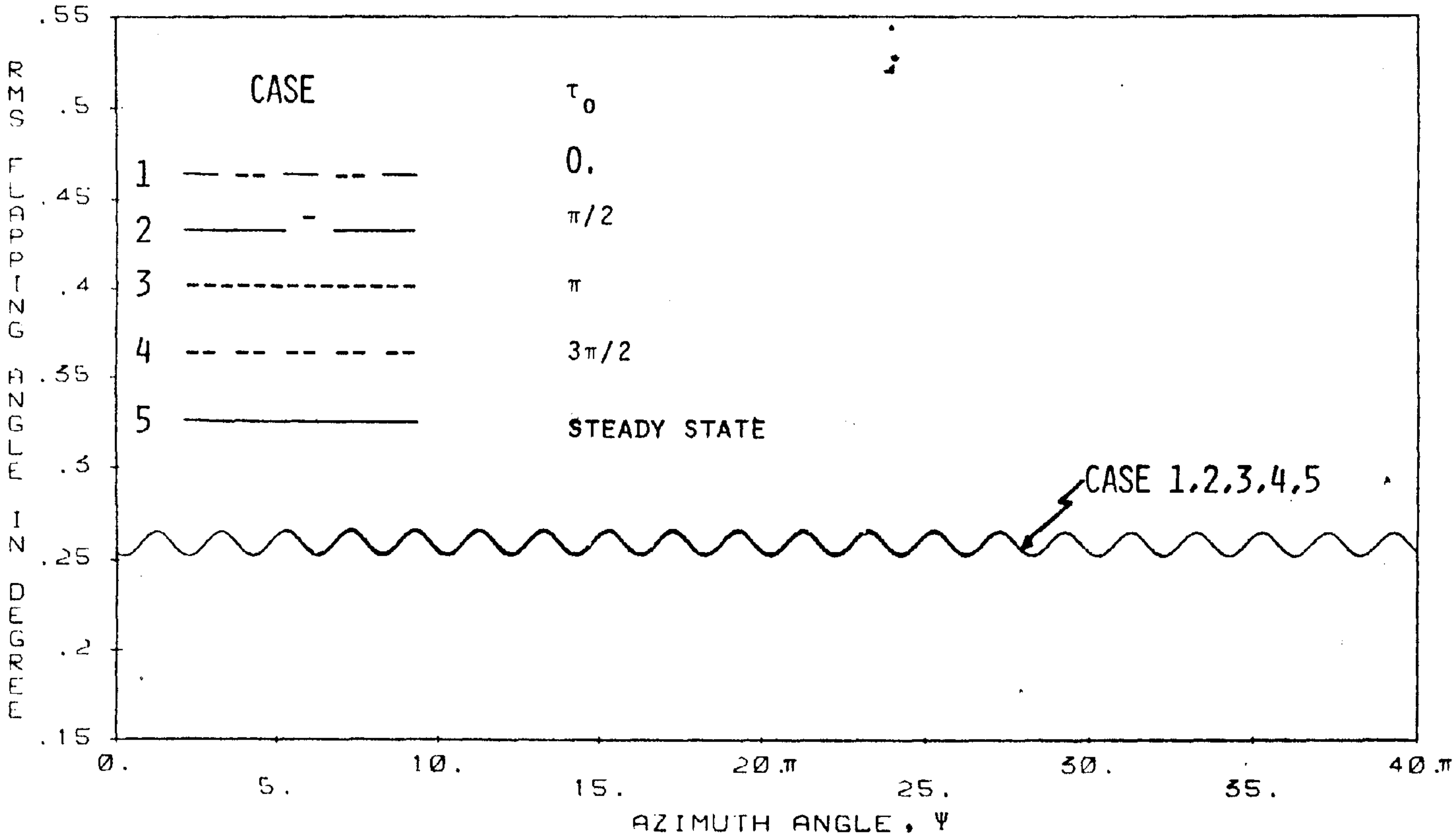

Figure 6-30 Effect of Earthquake Initiation Time on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$,

$\Phi_{11} = \Phi_{33} = \frac{25}{16}\Phi_{22} = .92698 \times 10^{-7}$.

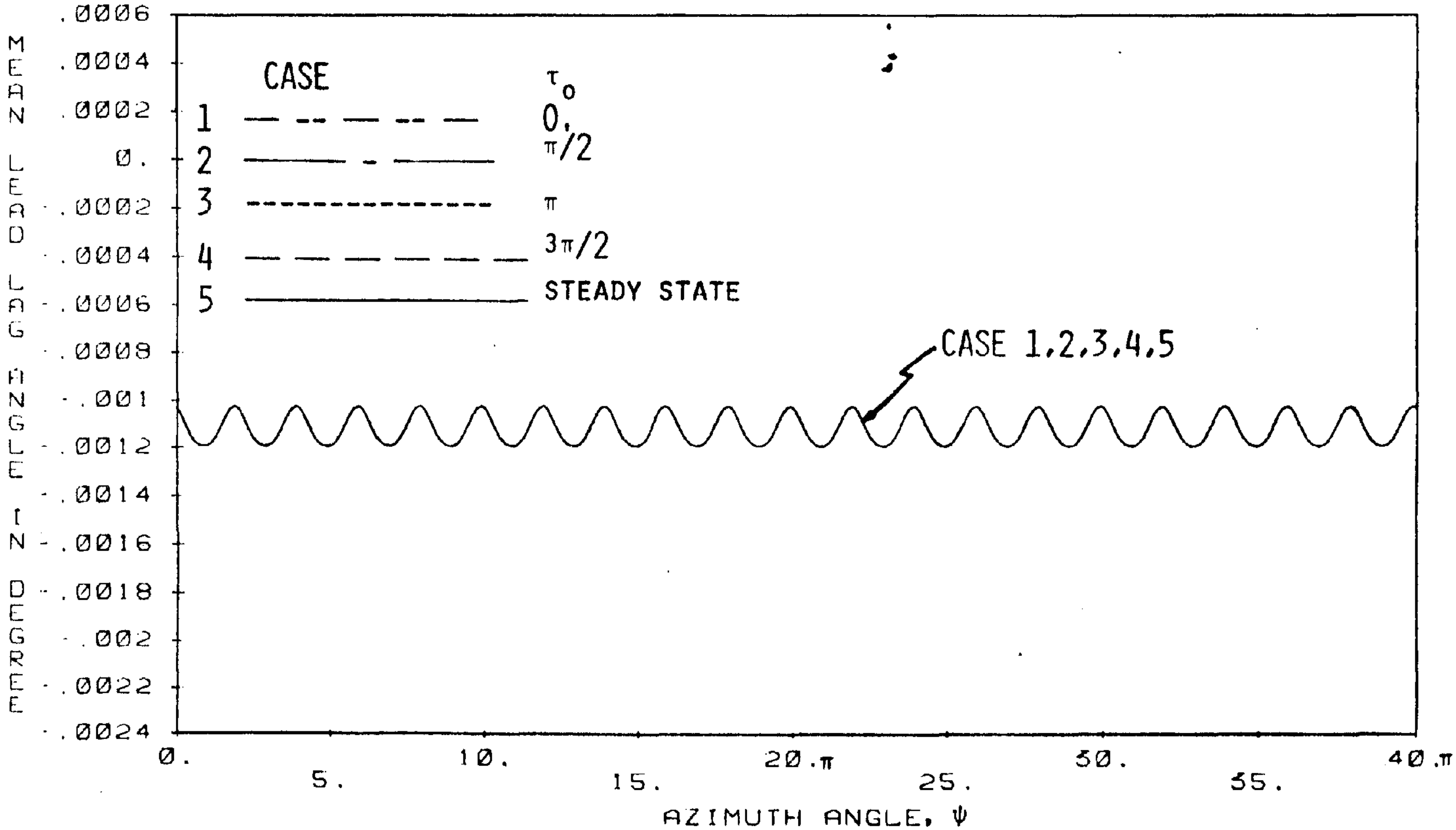

Figure 6-31 Effect of Earthquake Initiation Time on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$,

$\Phi_{11} = \Phi_{33} = \frac{25}{16}\Phi_{22} = .92698 \times 10^{-7}$.

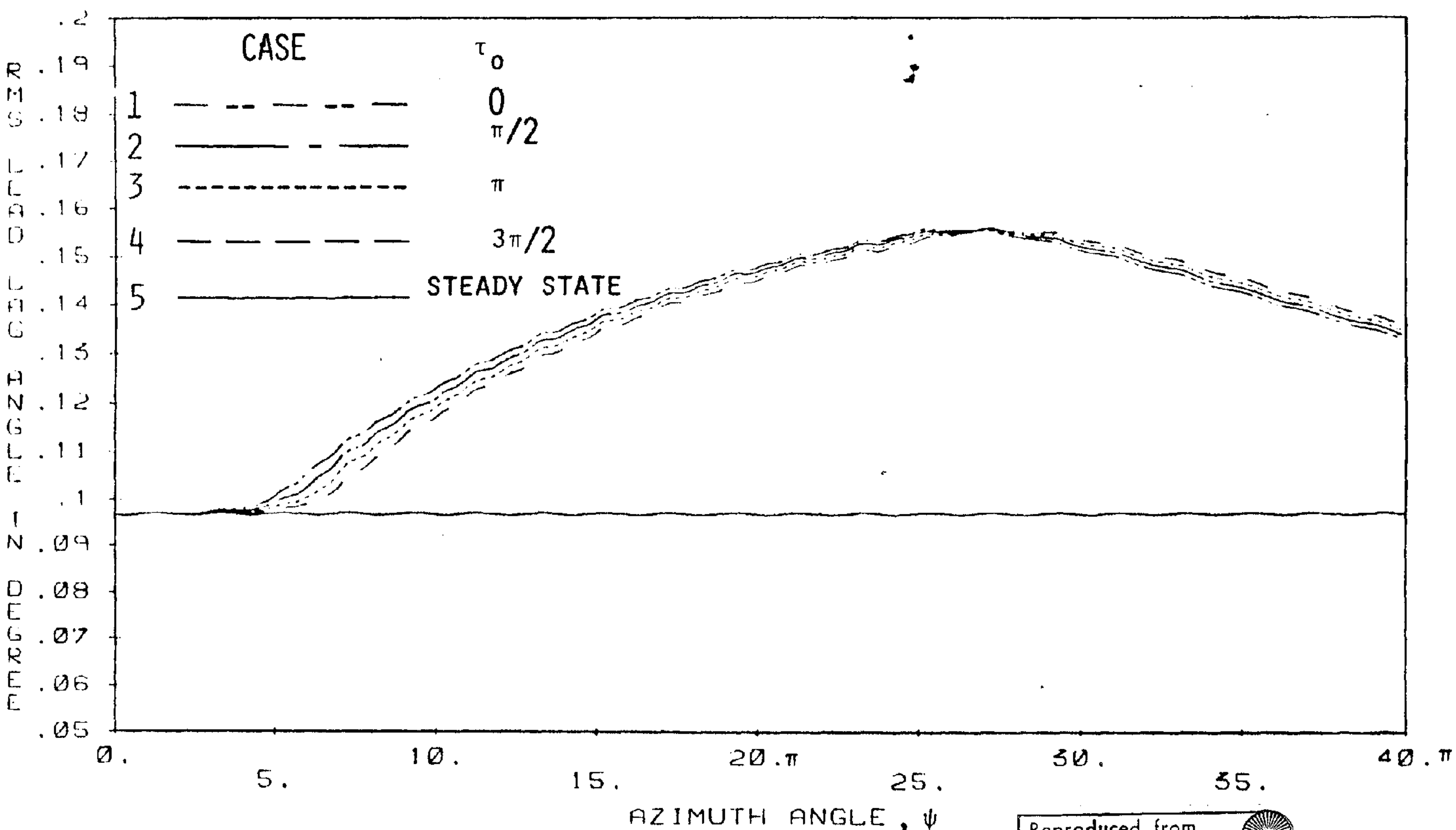

Figure 6-32 Effect of Earthquake Initiation Time on the Transient Coupled Flap-Leadlag Motion.

rms value of u, v and w = 3 ft/sec (.91 m/sec), $T_1 = T_2 = T_3 = .6667$,

$\Phi_{11} = \Phi_{33} = \frac{25}{16}\Phi_{22} = .92698 \times 10^{-7}$.

CHAPTER 7

CONCLUSIONS

7.1 Summary of Present Research

The main results obtained in this study are summarized as follows:

1) For uncoupled flapping motion, the in-plane turbulence velocity components affect the system stability since they appear in the coefficients of the equation of motion, whereas the axial turbulence component does not change the stability condition and it appears only in the inhomogeneous terms. For coupled flap-lagging or flap-lag-torsional motion, all three turbulence components affect the system stability as well as the responses.

2) The in-plane earthquake acceleration components appear in the coefficients of stiffness matrix and the inhomogeneous terms in the equations of motion. The axial earthquake acceleration component appears only in the inhomogeneous terms.

3) The equations for the statistical moments of response variables form an infinite hierarchy for which some closure scheme must be used to obtain approximate solutions. The nonlinearity is originated from modeling the turbulence excitation to be filtered white noise processes.

4) Without the presence of earthquake, the moment equations form a set of differential equations with periodic coefficients and its solutions tend to periodic functions with period 2π. With earthquake, the statistical periodicity no longer exists.

5) The uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsion are found to be very stable under normal operating conditions.

6) The mean and rms responses of torsional motion are very small; therefore the torsional degree of freedom has little influence on the moment responses of the flapping and leadlagging motions. On the other hand, adding the lead-lagging mode softens the flapping mode significantly.

7) The effect of turbulence is higher on the flapping response than on the leadlagging response. However, earthquake has some effect on leadlagging but almost no effect on flapping.

8) The mean response is nearly the same as the deterministic response without random excitations. This is the case for uncoupled flapping, coupled flap-lagging and coupled flap-lag-torsion. The rms responses are strongly dependent on the levels of random excitations. Within the practical range of turbulence level, the rms responses of flapping and leadlagging motions are significant compared to the deterministic responses. However, the rms responses due to an earthquake are small. Therefore, turbulence is likely a main cause for structural fatigue.

7.2 Proposed Areas for Future Research

The following is a list of potential topics for future study:

1. The present single blade analysis can be extended to a multi-blade analysis by using the multiblade coordinate transformation [48].
2. The effect of dynamic inflow may be considered; however, the derivation may be very complicated since inflow becomes position dependent.
3. The dynamic effect of the yawing angular velocity of the rotor axis may be investigated. As the wind direction changes, the wind turbine must be reoriented until the rotor axis is aligned with the wind direction. This rotation will result in large flapwise moments on the blades.

Appendix A

Moment Equations of Flap Motion

In this appendix, the detailed first and second moment equations are given. In these equations, M_i and M_{ij} denote $E[X_i]$ and $E[X_i X_j]$, respectively, and each overdot denotes one differentiation with respect to the non-dimensional time ϕ. The first moment equations are

$$\begin{aligned}
\dot{M}_1 &= M_2 \\
\dot{M}_2 &= F_{10} - K_{110}M_1 - C_{110}M_2 + F_{14}M_3 + F_{15}M_4 + F_{16}M_5 - K_{114}M_{13} \\
&\quad - C_{114}M_{23} - K_{115}M_{14} - C_{115}M_{24} \\
\dot{M}_3 &= -\alpha_1 M_3 \\
\dot{M}_4 &= -\alpha_2 M_4 \\
\dot{M}_5 &= -\alpha_3 M_5
\end{aligned} \qquad \text{(A-1)}$$

The second moment equations are

$$\begin{aligned}
\dot{M}_{11} &= 2M_{12} \\
\dot{M}_{12} &= M_{22} + F_{10}M_1 - K_{110}M_{11} - C_{110}M_{12} + F_{14}M_{13} + F_{15}M_{14} + F_{16}M_{15} \\
&\quad - K_{114}[2M_{13}M_1 + M_{11}M_3 - 2M_1^2M_3] - C_{114}[M_{12}M_3 + M_{13}M_2 + M_1M_{23} \\
&\quad - 2M_1M_2M_3] - K_{115}[2M_{14}M_1 + M_{11}M_4 - 2M_1^2M_4] - C_{115}[M_{12}M_4 + M_1M_{24} + M_{14}M_2 \\
&\quad - 2M_1M_2M_4] \\
\dot{M}_{13} &= M_{23} - \alpha_1 M_{13} \\
\dot{M}_{14} &= M_{24} - \alpha_2 M_{14} \\
\dot{M}_{15} &= M_{25} - \alpha_3 M_{15}
\end{aligned}$$

$$\dot{M}_{22} = 2\pi F_{13}^2 e_3^2 \Phi_{33} + 2F_{10}M_2 - 2K_{110}M_{12} - 2C_{110}M_{22} + 2F_{14}M_{23}$$
$$+ 2F_{15}M_{24} + 2F_{16}M_{25} + 2\pi [K_{111}^2 e_1^2 \Phi_{11} + K_{112}^2 e_2^2 \Phi_{22}] M_{11}$$
$$- 2C_{114}[2M_{23}M_2 + M_{22}M_3 - 2M_2^2 M_3] - 2K_{114}[M_{12}M_3 + M_{13}M_2$$
$$+ M_1M_{23} - 2M_1M_2M_3] - 2C_{115} [2M_{24}M_2 + M_{22}M_4 - 2M_2^2M_4]$$
$$- 2K_{115}[M_{12}M_4 + M_{14}M_2 + M_{24}M_1 - 2M_1M_2M_4]$$
$$\dot{M}_{23} = F_{10}M_3 - K_{110}M_{13} - C_{110}M_{23} + F_{14}M_{33} + F_{15}M_{34} + F_{16}M_{35}$$
$$- \alpha_1 M_{23} - C_{114}[M_2M_{33} + 2M_3M_{23} - 2M_2M_3^2] - K_{114}[M_1M_{33}$$
$$+ 2M_{13}M_3 - 2M_1M_3^2] - C_{115}[M_{23}M_4 + M_{24}M_3 + M_{34}M_2$$
$$- 2M_2M_3M_4] - K_{115}[M_{13}M_4 + M_{14}M_3 + M_{34}M_1 - 2M_1M_3M_4]$$
$$\dot{M}_{24} = F_{10}M_4 - K_{110}M_{14} - C_{110}M_{24} + F_{14}M_{34} + F_{15}M_{44} + F_{16}M_{45}$$
$$- \alpha_2 M_{24} - C_{114}[M_{23}M_4 + M_{24}M_3 + M_{34}M_2 - 2M_2M_3M_4]$$
$$- K_{114} [M_{13}M_4 + M_{14}M_3 + M_{34}M_1 - 2M_1M_3M_4] - C_{115}[2M_{24}M_4$$
$$+ M_2M_{44} - 2M_2M_4^2] - K_{115}[M_1M_{44} + 2M_{14}M_4 - 2M_1M_4^2]$$
$$\dot{M}_{25} = F_{10}M_5 - K_{110}M_{15} - C_{110}M_{25} + F_{14}M_{35} + F_{15}M_{45} + F_{16}M_{55}$$
$$- \alpha_3 M_{25} - C_{114}[M_{23}M_5 + M_{25}M_3 + M_{35}M_2 - 2M_2M_3M_5]$$
$$- K_{114}[M_{13}M_5 + M_{15}M_3 + M_{35}M_1 - 2M_1M_3M_5] - C_{115}[M_{24}M_5$$
$$+ M_{25}M_4 + M_{45}M_2 - 2M_2M_4M_5] - K_{115}[M_{14}M_5 + M_{15}M_4 + M_{45}M_1 - 2M_1M_4M_5]$$
$$\dot{M}_{33} = 2\pi\Phi_{44} - 2\alpha_1 M_{33}$$
$$\dot{M}_{34} = -(\alpha_1 + \alpha_2)\, M_{34}$$
$$\dot{M}_{35} = -(\alpha_1 + \alpha_3)\, M_{35}$$
$$\dot{M}_{44} = 2\pi\Phi_{55} - 2\alpha_2 M_{44}$$
$$\dot{M}_{45} = -(\alpha_2 + \alpha_3)\, M_{45}$$
$$\dot{M}_{55} = 2\pi\Phi_{66} - 2\alpha_3 M_{55} \qquad \text{(A-2)}$$

Appendix B

Moment Equations of Coupled Flag-lag Motions

In this appendix, the detailed first and second moment equations of coupled flap-lag motion are given. The first moment equations are

$$\dot{M}_1 = M_2$$

$$\dot{M}_2 = -K_{110}M_1 - C_{110}M_2 - K_{120}M_3 - C_{120}M_4 + F_{14}M_5 - F_{15}M_6$$
$$+ F_{16}M_7 - K_{114}M_{15} - C_{114}M_{25} - K_{124}M_{35} - C_{124}M_{45}$$
$$- K_{115}M_{16} - C_{115}M_{26} - K_{125}M_{36} - C_{125}M_{46} - K_{116}M_{17}$$
$$- C_{116}M_{27} - K_{126}M_{37} - C_{126}M_{47}$$

$$\dot{M}_3 = M_4$$

$$\dot{M}_4 = -K_{210}M_1 - C_{210}M_2 - K_{220}M_3 - C_{220}M_4 + F_{24}M_5 + F_{25}M_6$$
$$+ F_{26}M_7 - K_{214}M_{15} - C_{214}M_{25} - K_{224}M_{35} - C_{224}M_{45} - K_{215}M_{16}$$
$$- C_{215}M_{26} - K_{225}M_{36} - C_{225}M_{46} - K_{216}M_{17} - C_{216}M_{27}$$
$$- K_{226}M_{37} - C_{226}M_{47}$$

$$\dot{M}_5 = -\alpha_1 M_5$$

$$\dot{M}_6 = -\alpha_2 M_6$$

$$\dot{M}_7 = -\alpha_3 M_7 \qquad \text{(B-1)}$$

The second moment equations are

$$\dot{M}_{11} = 2M_{12}$$

$$\dot{M}_{12} = M_{22} - K_{110}M_{11} - C_{110}M_{12} - K_{120}M_{13} - C_{120}M_{14} + F_{14}M_{15} + F_{15}M_{16}$$
$$+ F_{16}M_{17} - K_{114}[2M_1M_{15} + M_{11}M_5 - 2M_1^2M_5] - C_{114}[M_{12}M_5 + M_{15}M_2$$

$$+ M_{25}M_1 - 2M_1M_2M_5] - K_{124}[M_{13}M_5 + M_{15}M_3 + M_{35}M_1 - 2M_1M_3M_5]$$
$$- C_{124}[M_{14}M_5 + M_{15}M_4 + M_{45}M_1 - 2M_1M_4M_5] - K_{115}[2M_1M_{16} + M_{11}M_6$$
$$- 2M_1^2M_6] - C_{115}[M_{12}M_6 + M_{16}M_2 + M_{26}M_1 - 2M_1M_2M_6] - K_{125}[M_{13}M_6$$
$$+ M_{16}M_3 + M_{36}M_1 - 2M_1M_3M_6] - C_{125}[M_{14}M_6 + M_{16}M_4 + M_{46}M_1 - 2M_1M_4M_6]$$
$$- K_{116}[2M_{17}M_1 + M_{11}M_7 - 2M_1^2M_7] - C_{116}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7]$$
$$- K_{126}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7] - C_{126}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1$$
$$- 2M_1M_4M_7]$$

$$\dot{M}_{13} = M_{23} + M_{14}$$

$$\dot{M}_{14} = M_{24} - K_{210}M_{11} - C_{210}M_{12} - K_{220}M_{13} - C_{220}M_{14} + F_{24}M_{15} + F_{25}M_{16}$$
$$+ F_{26}M_{17} - K_{214}[2M_{15}M_1 + M_{11}M_5 - 2M_1^2M_5] - C_{214}[M_{12}M_5 + M_{25}M_1$$
$$+ M_{15}M_2 - 2M_1M_2M_5] - K_{224}[M_{13}M_5 + M_{15}M_3 + M_{35}M_1 - 2M_1M_3M_5]$$
$$- C_{224}[M_{14}M_5 + M_{15}M_4 + M_{45}M_1 - 2M_1M_4M_5] - K_{215}[2M_{16}M_1 + M_{11}M_6$$
$$- 2M_1^2 M_6] - C_{215}[M_{12}M_6 + M_{16}M_2 + M_{26}M_1 - 2M_1M_2M_6] - K_{225}[M_{13}M_6$$
$$+ M_{16}M_3 + M_{36}M_1 - 2M_1M_3M_6] - C_{225}[M_{14}M_6 + M_{16}M_4 + M_{46}M_1 - 2M_1M_4M_6]$$
$$- K_{216}[2M_{17}M_1 + M_{11}M_7 - 2M_1^2 M_7] - C_{216}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7]$$
$$- K_{226}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7] - C_{226}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1$$
$$- 2M_1M_4M_7]$$

$$\dot{M}_{15} = M_{25} - \alpha_1M_{15}$$

$$\dot{M}_{16} = M_{26} - \alpha_2M_{16}$$

$$\dot{M}_{17} = M_{27} - \alpha_3M_{17}$$

$$\dot{M}_{22} = 2\pi[F_{11}^2\ e_1^2\Phi_{11} + F_{12}^2\ e_2^2\Phi_{22} + F_{13}^2\ e_3^2\Phi_{33}] - 4\pi e_1^2\Phi_{11}[F_{11}K_{111}M_1$$
$$+ F_{11}K_{121}M_3] - 4\pi\ e_2^2\Phi_{22}[F_{12}K_{112}M_1 + F_{12}K_{122}M_3] - 4\pi e_3^2\Phi_{33}$$
$$[F_{13}K_{113}M_1 + F_{13}K_{123}M_3] + 2\pi e_1^2\Phi_{11}[K_{111}^2M_{11} + K_{121}^2 M_{33}$$
$$+ 2K_{111}K_{121}M_{13}] + 2\pi e_2^2\Phi_{22}[K_{112}^2 M_{11} + K_{122}^2M_{33} + 2K_{112} K_{122}M_{13}]$$
$$+ 2\pi e_3^2\Phi_{33}[K_{113}^2 M_{11} + K_{123}^2 M_{33} + 2K_{113}K_{123}M_{13}] - 2K_{110}M_{12}$$
$$- 2C_{110}M_{22} - 2K_{120}M_{23} - 2C_{120}M_{24} + 2F_{14}M_{25} + 2F_{15}M_{26} + 2F_{16}M_{27}$$

$$- 2K_{114}[M_{12}M_5 + M_{15}M_2 + M_{25}M_1 - 2M_1M_2M_5] - 2C_{114}[2M_{25}M_2 + M_{22}M_5$$
$$- 2M_2^2 M_5] - 2K_{124}[M_{23}M_5 + M_{35}M_2 + M_{25}M_3 - 2M_2M_3M_5] - 2C_{124}[M_{24}M_5$$
$$+ M_{25}M_4 + M_{45}M_2 - 2M_2M_4M_5] - 2K_{115}[M_{12}M_6 + M_{16}M_2 + M_{26}M_1 - 2M_1M_2M_6]$$
$$- 2C_{115}[2M_{26}M_2 + M_{22}M_6 - 2M_2^2 M_6] - 2K_{125}[M_{23}M_6 + M_{26}M_3 + M_{36}M_2$$
$$- 2M_2M_3M_6] - 2C_{125}[M_{24}M_6 + M_{46}M_2 + M_{26}M_4 - 2M_2M_4M_6] - 2K_{116}[M_{12}M_7$$
$$+ M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - 2C_{116}[2M_{27}M_2 + M_{22}M_7 - 2M_2^2M_7]$$
$$- 2K_{126}[M_{23}M_7 + M_{37}M_2 + M_{27}M_3 - 2M_2M_3M_7] - 2C_{126}[M_{24}M_7 + M_{27}M_4$$
$$+ M_{47}M_2 - 2M_2M_4M_7]$$

$$\dot{M}_{23} = M_{24} - K_{110}M_{13} - C_{110}M_{23} - K_{120}M_{33} - C_{120}M_{34} + F_{14}M_{35} + F_{15}M_{36}$$
$$+ F_{16}M_{37} - K_{114}[M_{13}M_5 + M_{15}M_3 + M_{35}M_1 - 2M_1M_3M_5] - C_{114}[M_{23}M_5$$
$$+ M_{25}M_3 + M_{35}M_2 - 2M_2M_3M_5] - K_{124}[2M_{35}M_3 + M_{33}M_5 - 2M_3^2M_5]$$
$$- C_{124}[M_{34}M_5 + M_{35}M_4 + M_{45}M_3 - 2M_3M_4M_5] - K_{115}[M_{13}M_6 + M_{16}M_3$$
$$+ M_{36}M_1 - 2M_1M_3M_6] - C_{115}[M_{23}M_6 + M_{26}M_3 + M_{36}M_2 - 2M_2M_3M_6]$$
$$- K_{125}[2M_{36}M_3 + M_{33}M_6 - 2M_3^2M_6] - C_{125}[M_{34}M_6 + M_{36}M_4 + M_{46}M_3$$
$$- 2M_3M_4M_6] - K_{116}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7] - C_{116}[M_{23}M_7$$
$$+ M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - K_{126}[2M_{37}M_3 + M_{33}M_7 - 2M_3^2M_7]$$
$$- C_{126}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7]$$

$$\dot{M}_{24} = 2\pi\ [F_{11}F_{21}\ e_1^2\,\Phi_{11} + F_{12}F_{22}\ e_2^2\,\Phi_{22} + F_{13}F_{23}\ e_3^2\,\Phi_{33}] - 2\pi e_1^2\,\Phi_{11}$$
$$[F_{11}K_{211}M_1 + F_{21}K_{111}M_1 + F_{11}K_{221}M_3 + F_{21}K_{121}M_3] - 2\pi e_2^2\,\Phi_{22}$$
$$[F_{12}K_{212}M_1 + F_{22}K_{112}M_1 + F_{12}K_{222}M_3 + F_{22}K_{122}M_3] - 2\pi e_3^2\,\Phi_{33}$$
$$[F_{13}K_{213}M_1 + F_{23}K_{113}M_1 + F_{13}K_{223}M_3 + F_{23}K_{123}M_3] + 2\pi e_1^2\,\Phi_{11}$$
$$[K_{111}K_{211}M_{11} + K_{121}K_{221}M_{33} + K_{111}K_{221}M_{13} + K_{121}K_{211}M_{13}] + 2\pi e_2^2\,\Phi_{22}$$
$$[K_{112}K_{212}M_{11} + K_{122}K_{222}M_{33} + K_{112}K_{222}M_{13} + K_{122}K_{212}M_{13}] + 2\pi e_2^2\,\Phi_{33}$$
$$[K_{113}K_{213}M_{11} + K_{123}K_{223}M_{33} + K_{113}K_{223}M_{13} + K_{123}K_{213}M_{13}] - K_{110}M_{14}$$
$$- C_{110}M_{24} - K_{120}M_{34} - C_{120}M_{44} + F_{14}M_{45} + F_{15}M_{46} + F_{16}M_{47} - K_{210}M_{12}$$
$$- C_{120}M_{22} - K_{220}M_{23} - C_{220}M_{24} + F_{24}M_{25} + F_{25}M_{26} + F_{26}M_{27}$$

$$- K_{114}[M_{14}M_5 + M_{15}M_4 + M_{45}M_1 - 2M_1M_4M_5] - C_{114}[M_{24}M_5 + M_{25}M_4$$
$$+ M_{45}M_2 - 2M_2M_4M_5] - K_{124}[M_{34}M_5 + M_{35}M_4 + M_{45}M_3 - 2M_3M_4M_5]$$
$$- C_{124}[2M_{45}M_4 + M_{44}M_5 - 2M_4^2M_5] - K_{115}[M_{14}M_6 + M_{16}M_4 + M_{46}M_1$$
$$- 2M_1M_4M_6] - C_{115}[M_{24}M_6 + M_{26}M_4 + M_{46}M_2 - 2M_2M_4M_6] - K_{125}[M_{34}M_6 + M_{46}M_3$$
$$+ M_{36}M_4 - 2M_3M_4M_6] - C_{125}[2M_{46}M_4 + M_{44}M_6 - 2M_4^2M_6] - K_{116}[M_{14}M_7 + M_{17}M_4$$
$$+ M_{47}M_1 - 2M_1M_4M_7] - C_{116}[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - K_{126}[M_{34}M_7$$
$$+ M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - C_{126}[2M_{47}M_4 + M_{44}M_7 - 2M_4^2M_7] - K_{214}[M_{12}M_5$$
$$+ M_{15}M_2 + M_{25}M_1 - 2M_1M_2M_5] - C_{214}[2M_{25}M_2 + M_{22}M_5 - 2M_2^2M_5]$$
$$- K_{224}[M_{23}M_5 + M_{25}M_3 + M_{35}M_2 - 2M_2M_3M_5] - C_{224}[M_{24}M_5 + M_{25}M_4$$
$$+ M_{45}M_2 - 2M_2M_4M_5] - K_{215}[M_{12}M_6 + M_{16}M_2 + M_{26}M_1 - 2M_1M_2M_6]$$
$$- C_{215}[2M_{26}M_2 + M_{22}M_6 - 2M_2^2M_6] - K_{225}[M_{23}M_6 + M_{26}M_3 + M_{36}M_2$$
$$- 2M_2M_3M_6] - C_{225}[M_{24}M_6 + M_{26}M_4 + M_{46}M_2 - 2M_2M_4M_6] - K_{216}[M_{12}M_7$$
$$+ M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - C_{216}[2M_{27}M_2 + M_{22}M_7 - 2M_2^2M_7]$$
$$- K_{226}[M_{23}M_7 + M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - C_{226}[M_{24}M_7 + M_{27}M_4$$
$$+ M_{47}M_2 - 2M_2M_4M_7]$$

$$\dot{M}_{25} = -\alpha_1 M_{25} - K_{110}M_{15} - C_{110}M_{25} - K_{120}M_{35} - C_{120}M_{45} + F_{14}M_{55} + F_{15}M_{56}$$
$$+ F_{16}M_{57} - K_{114}[2M_{15}M_5 + M_{55}M_1 - 2M_1M_5^2] - C_{114}[2M_{25}M_5 + M_2M_{55}$$
$$- 2M_2M_5^2] - K_{124}[2M_{35}M_5 + M_{55}M_3 - 2M_3M_5^2] - C_{124}[2M_{45}M_5 + M_{55}M_4$$
$$- 2M_4M_5^2] - K_{115}[M_{15}M_6 + M_{16}M_5 + M_{56}M_1 - 2M_1M_5M_6] - C_{115}[M_{25}M_6 + M_{26}M_5$$
$$+ M_{56}M_2 - 2M_2M_5M_6] - K_{125}[M_{35}M_6 + M_{36}M_5 + M_{56}M_3 - 2M_3M_5M_6] - C_{125}[M_{45}M_6$$
$$+ M_{46}M_5 + M_{56}M_4 - 2M_4M_5M_6] - K_{116}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7]$$
$$- C_{116}[M_{25}M_7 + M_{27}M_5 + M_{57}M_2 - 2M_2M_5M_7] - K_{126}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3$$
$$- 2M_3M_5M_7] - C_{126}[M_{45}M_7 + M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7]$$

$$\dot{M}_{26} = -\alpha_2 M_{26} - K_{110}M_{16} - C_{110}M_{26} - K_{120}M_{36} - C_{120}M_{46} + F_{14}M_{56} + F_{15}M_{66} + F_{16}M_{67}$$
$$- K_{114}[M_{15}M_6 + M_{16}M_5 + M_{56}M_1 - 2M_1M_5M_6] - C_{114}[M_{25}M_6 + M_{26}M_5 + M_{56}M_2$$
$$- 2M_2M_5M_6] - K_{124}[M_{35}M_6 + M_{36}M_5 + M_{56}M_3 - 2M_3M_5M_6] - C_{124}[M_{45}M_6$$

$$+ M_{46}M_5 + M_{56}M_4 - 2M_4M_5M_6] - K_{115}[2M_{16}M_6 + M_{66}M_1 - 2M_1M_6^2] - C_{115}$$

$$[2M_{26}M_6 + M_{66}M_2 - 2M_2M_6^2] - K_{125}[2M_{36}M_6 + M_{66}M_3 - 2M_3M_6^2] - C_{125}$$

$$[2M_{46}M_6 + M_{66}M_4 - 2M_4M_6^2] - K_{116}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7]$$

$$- C_{116}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{126}[M_{36}M_7 + M_{37}M_6$$

$$+ M_{67}M_3 - 2M_3M_6M_7] - C_{126}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7]$$

$$\dot{M}_{27} = -\alpha_3 M_{27} - K_{110}M_{17} - C_{110}M_{27} - K_{120}M_{37} - C_{120}M_{47} + F_{14}M_{57} + F_{15}M_{67} + F_{16}M_{77}$$

$$- K_{114}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{114}[M_{25}M_7 + M_{57}M_2 + M_{27}M_5$$

$$- 2M_2M_5M_7] - K_{124}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3 - 2M_3M_5M_7] - C_{124}[M_{45}M_7 + M_{47}M_5$$

$$+ M_{57}M_4 - 2M_4M_5M_7] - K_{115}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7] - C_{115}[M_{26}M_7$$

$$+ M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{125}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7]$$

$$- C_{125}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7] - K_{116}[2M_{17}M_7 + M_{77}M_1$$

$$- 2M_1M_7^2] - C_{116}[2M_{27}M_7 + M_{77}M_2 - 2M_2M_7^2] - K_{126}[2M_{37}M_7 + M_3M_{77} - 2M_3M_7^2]$$

$$- C_{126}[2M_{47}M_7 + M_{77}M_4 - 2M_4M_7^2]$$

$$\dot{M}_{33} = 2M_{34}$$

$$\dot{M}_{34} = M_{44} - K_{210}M_{13} - C_{210}M_{23} - K_{220}M_{33} - C_{220}M_{34} + F_{24}M_{35} + F_{25}M_{36} + F_{26}M_{37}$$

$$- K_{214}[M_{13}M_5 + M_{15}M_3 + M_{35}M_1 - 2M_1M_3M_5] - C_{214}[M_{23}M_5 + M_{25}M_3 + M_{35}M_2$$

$$- 2M_2M_3M_5] - K_{224}[2M_{35}M_3 + M_{33}M_5 - 2M_3^2M_5] - C_{224}[M_{34}M_5 + M_{35}M_4 + M_{45}M_3$$

$$- 2M_3M_4M_5] - K_{215}[M_{13}M_6 + M_{16}M_3 + M_{36}M_1 - 2M_1M_3M_6] - C_{215}[M_{23}M_6 + M_3M_{26}$$

$$+ M_{36}M_2 - 2M_2M_3M_6] - K_{225}[2M_{36}M_3 + M_{33}M_6 - 2M_3^2M_6] - C_{225}[M_{34}M_6 + M_{36}M_4$$

$$+ M_{46}M_3 - 2M_3M_4M_6] - K_{216}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7]$$

$$- C_{216}[M_{23}M_7 + M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - K_{226}[2M_{37}M_3 + M_{33}M_7$$

$$- 2M_3^2M_7] - C_{226}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7]$$

$$\dot{M}_{35} = M_{45} - \alpha_1 M_{35}$$

$$\dot{M}_{36} = M_{46} - \alpha_2 M_{36}$$

$$\dot{M}_{37} = M_{47} - \alpha_3 M_{37}$$

$$\dot{M}_{44} = 2\pi[F_{21}^2 e_1^2 \Phi_{11} + F_{22}^2\ e_2^2 \Phi_{22} + F_{23}^2\ e_3^2 \Phi_{33}] - 4\pi\ e_1^2 \Phi_{11}[F_{21}K_{211}M_1$$
$$+ F_{21}K_{221}M_3] - 4\pi e_2^2 \Phi_{22}\ [F_{22}K_{212}M_1 + F_{22}K_{222}M_3] - 4\pi e_3^2 \Phi_{33}$$
$$[F_{23}K_{213}M_1 + F_{23}K_{223}M_3] + 2\pi\ e_1^2 \Phi_{11}[K_{211}^2 M_{11} + K_{221}^2\ M_{33}$$
$$+ 2K_{211}K_{221}M_{13}] + 2\pi\ e_2^2 \Phi_{22}[K_{212}^2\ M_{11} + K_{222}^2 M_{33} + 2K_{212}K_{222}\ M_{13}]$$
$$+ 2\pi\ e_3^2 \Phi_{33}[K_{213}^2\ M_{11} + K_{223}^2\ M_{33} + 2K_{213}K_{223}M_{13}] - 2K_{210}M_{14} - 2C_{210}M_{24}$$
$$- 2K_{220}M_{34} - 2C_{220}M_{44} + 2F_{24}M_{45} + 2F_{25}M_{46} + 2F_{26}M_{47} - 2K_{214}[M_{14}M_5$$
$$+ M_{15}M_4 + M_{45}M_1 - 2M_1M_4M_5] - 2C_{214}[M_{24}M_5 + M_{25}M_4 + M_{45}M_2 - 2M_2M_4M_5]$$
$$- 2K_{224}[M_{34}M_5 + M_{35}M_4 + M_{45}M_3 - 2M_3M_4M_5] - 2C_{224}[2M_{45}M_4 + M_{44}M_5 - 2M_4^2\ M_5]$$
$$- 2K_{215}[M_{14}M_6 + M_{16}M_4 + M_{46}M_1 - 2M_1M_4M_6] - 2C_{215}[M_{24}M_6 + M_{26}M_4 + M_{46}M_2$$
$$- 2M_2M_4M_6] - 2K_{225}[M_{34}M_6 + M_{36}M_4 + M_{46}M_3 - 2M_3M_4M_6] - 2C_{225}[2M_{46}M_4$$
$$+ M_{44}M_6 - 2M_4^2M_6] - 2K_{216}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1 - 2M_1M_4M_7] - 2C_{216}$$
$$[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - 2K_{226}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3$$
$$- 2M_3M_4M_7] - 2C_{226}[2M_{47}M_4 + M_{44}M_7 - 2M_4^2M_7]$$
$$\dot{M}_{45} = -\alpha_1 M_{45} - K_{210}M_{15} - C_{210}M_{25} - K_{220}M_{35} - C_{220}M_{45} + F_{24}M_{55} + F_{25}M_{56}$$
$$+ F_{26}M_{57} - K_{214}[2M_{15}M_5 + M_{55}M_1 - 2M_1M_5^2] - C_{214}[2M_{25}M_5 + M_{55}M_2 - 2M_2M_5^2]$$
$$- K_{224}[2M_{35}M_5 + M_{55}M_3 - 2M_3M_5^2] - C_{224}[2M_{45}M_5 + M_{55}M_4 - 2M_4M_5^2] - K_{215}$$
$$[M_{15}M_6 + M_{16}M_5 + M_{56}M_1 - 2M_1M_5M_6] - C_{215}[M_{25}M_6 + M_{26}M_5 + M_{56}M_2 - 2M_2M_5M_6]$$
$$- K_{225}[M_{35}M_6 + M_{36}M_5 + M_{56}M_3 - 2M_3M_5M_6] - C_{225}[M_{45}M_6 + M_{46}M_5 + M_{56}M_4$$
$$- 2M_4M_5M_6] - K_{216}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{216}[M_{25}M_7 + M_{27}M_5$$
$$+ M_{57}M_2 - 2M_2M_5M_7] - K_{226}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3 - 2M_3M_5M_7] - C_{226}$$
$$[M_{45}M_7 + M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7]$$
$$\dot{M}_{46} = -\alpha_2 M_{46} - K_{210}M_{16} - C_{210}M_{26} - K_{220}M_{36} - C_{220}M_{46} + F_{24}M_{56} + F_{25}M_{66} + F_{26}M_{67}$$
$$- K_{214}[M_{15}M_6 + M_{16}M_5 + M_{56}M_1 - 2M_1M_5M_6] - C_{214}[M_{25}M_6 + M_{26}M_5 + M_{56}M_2$$
$$- 2M_2M_5M_6] - K_{224}[M_{35}M_6 + M_{36}M_5 + M_{56}M_3 - 2M_3M_5M_6] - C_{224}\ [M_{45}M_6$$
$$+ M_{46}M_5 + M_{56}M_4 - 2M_4M_5M_6] - K_{215}[2M_{16}M_6 + M_{66}M_1 - 2M_1M_6^2] - C_{215}$$
$$[2M_{26}M_6 + M_{66}M_2 - 2M_2M_6^2] - K_{225}[2M_{36}M_6 + M_{66}M_3 - 2M_3M_6^2] - C_{225}[2M_{46}M_6$$

$$+ M_{66}M_4 - 2M_4M_6^2] - K_{216}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7] - C_{216}[M_{26}M_7$$

$$+ M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{226}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7]$$

$$- C_{226}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7]$$

$$\dot{M}_{47} = -\alpha_3 M_{47} - K_{210}M_{17} - C_{210}M_{27} - K_{220}M_{37} - C_{220}M_{47} + F_{24}M_{57} + F_{25}M_{67} + F_{26}M_{77}$$

$$- K_{214}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{214}[M_{25}M_7 + M_{27}M_5$$

$$+ M_{57}M_2 - 2M_2M_5M_7] - K_{224}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3 - 2M_3M_5M_7] - C_{224}$$

$$[M_{45}M_7 + M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7] - K_{215}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7]$$

$$- C_{215}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{225}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3$$

$$- 2M_3M_6M_7] - C_{225}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7] - K_{216}[2M_{17}M_7 + M_{77}M_1$$

$$- 2M_1M_7^2] - C_{216}[2M_{27}M_7 + M_{77}M_2 - 2M_2M_7^2] - K_{226}[2M_{37}M_7 + M_{77}M_3$$

$$- 2M_3M_7^2] - C_{226}[2M_{47}M_7 + M_4M_{77} - 2M_4 M_7^2]$$

$$\dot{M}_{55} = 2\pi\, \Phi_{44} - 2\alpha_1 M_{55}$$

$$\dot{M}_{56} = -(\alpha_1 + \alpha_2)\, M_{56}$$

$$\dot{M}_{57} = -\, (\alpha_1 + \alpha_3)\, M_{57}$$

$$\dot{M}_{66} = 2\pi\, \Phi_{55} - 2\alpha_2 M_{66}$$

$$\dot{M}_{67} = -(\alpha_2 + \alpha_3)\, M_{67}$$

$$\dot{M}_{77} = 2\pi\, \Phi_{66} - 2\alpha_3 M_{77} \qquad \text{(B-2)}$$

Appendix C

Moment Equations of Coupled Flap-lag-torsion Motion

In this appendix, the detailed first and second moment equations are given. The first moment equations are

$$\dot{M}_1 = M_2$$

$$\dot{M}_2 = -K_{110}M_1 - C_{110}M_2 - K_{120}M_3 - C_{120}M_4 - K_{130}M_5 - C_{130}M_5$$
$$+ F_{14}M_7 + F_{15}M_8 + F_{16}M_9 - K_{114}M_{17} - C_{114}M_{27} - K_{124}M_{37}$$
$$- C_{124}M_{47} - K_{134}M_{57} - C_{134}M_{67} - K_{115}M_{18} - C_{115}M_{28} - K_{125}M_{38}$$
$$- C_{125}M_{48} - K_{135}M_{58} - C_{135}M_{68} - K_{116}M_{19} - C_{116}M_{29} - K_{126}M_{39}$$
$$- C_{126}M_{49} - K_{136}M_{59} - C_{136}M_{69}$$

$$\dot{M}_3 = M_4$$

$$\dot{M}_4 = -K_{210}M_1 - C_{210}M_2 - K_{220}M_3 - C_{220}M_4 - K_{230}M_5 - C_{230}M_6$$
$$+ F_{24}M_7 + F_{25}M_8 + F_{26}M_9 - K_{214}M_{17} - C_{214}M_{27} - K_{224}M_{37}$$
$$- C_{224}M_{47} - K_{234}M_{57} - C_{234}M_{67} - K_{215}M_{18} - C_{215}M_{28} - K_{225}M_{38}$$
$$- C_{225}M_{48} - K_{235}M_{58} - C_{235}M_{68} - K_{216}M_{19} - C_{216}M_{29} - K_{226}M_{39}$$
$$- C_{226}M_{49} - K_{236}M_{59} - C_{236}M_{69}$$

$$\dot{M}_5 = M_6$$

$$\dot{M}_6 = -K_{310}M_1 - C_{310}M_2 - K_{320}M_3 - C_{320}M_4 - K_{330}M_5 - C_{330}M_6$$
$$+ F_{34}M_7 + F_{35}M_8 + F_{36}M_9 - K_{314}M_{17} - C_{314}M_{27} - K_{234}M_{37}$$
$$- C_{324}M_{47} - K_{334}M_{57} - C_{334}M_{67} - K_{315}M_{18} - C_{315}M_{28} - K_{325}M_{38}$$
$$- C_{325}M_{48} - K_{335}M_{58} - C_{335}M_{68} - K_{316}M_{19} - C_{316}M_{29} - K_{326}M_{39}$$
$$- C_{326}M_{49} - K_{336}M_{59} - C_{336}M_{69}$$

$$\dot{M}_7 = -\alpha_1 M_7$$

$$\dot{M}_8 = -\alpha_2 M_8$$

$$\dot{M}_9 = -\alpha_3 M_9 \qquad \text{(C-1)}$$

The second moment equations are

$$\dot{M}_{11} = 2M_{12}$$

$$\begin{aligned}
\dot{M}_{12} = {} & M_{22} - K_{110}M_{11} - C_{110}M_{12} - K_{120}M_{13} - C_{120}M_{14} - K_{130}M_{15} - C_{130}M_{16} \\
& + F_{14}M_{17} + F_{15}M_{18} + F_{16}M_{19} - K_{114}[2M_{17}M_1 + M_{11}M_7 - 2M_1^2M_7] \\
& - C_{114}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - K_{124}[M_{13}M_7 + M_{17}M_3 \\
& + M_{37}M_1 - 2M_1M_3M_7] - C_{124}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1 - 2M_1M_4M_7] \\
& - K_{134}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{134}[M_{16}M_7 + M_{17}M_6 \\
& + M_{67}M_1 - 2M_1M_6M_7] - K_{115}[2M_{18}M_1 + M_{11}M_8 - 2M_1^2M_8] - C_{115} \\
& [M_{12}M_8 + M_{18}M_2 + M_{28}M_1 - 2M_1M_2M_8] - K_{125}[M_{13}M_8 + M_{18}M_3 + M_{38}M_1 \\
& - 2M_1M_3M_8] - C_{125}[M_{14}M_8 + M_{18}M_4 + M_{48}M_1 - 2M_1M_4M_8] - K_{135} \\
& [M_{15}M_8 + M_{18}M_5 + M_{58}M_1 - 2M_1M_5M_8] - C_{135}[M_{16}M_8 + M_{18}M_6 + M_{68}M_1 \\
& - 2M_1M_6M_8] - K_{116}[2M_{19}M_1 + M_{11}M_9 - 2\,M_1^2M_9] - C_{116}[M_{12}M_9 + M_{19}M_2 \\
& + M_{29}M_1 - 2M_1M_2M_9] - K_{126}[M_{13}M_9 + M_{19}M_3 + M_{39}M_1 - 2M_1M_3M_9] \\
& - C_{126}[M_{14}M_9 + M_{19}M_4 + M_{49}M_1 - 2M_1M_4M_9] - K_{136}[M_{15}M_9 + M_{19}M_5 \\
& + M_{59}M_1 - 2M_1M_5M_9] - C_{136}[M_{16}M_9 + M_{19}M_6 + M_{69}M_1 - 2M_1M_6M_9]
\end{aligned}$$

$$\dot{M}_{13} = M_{23} + M_{14}$$

$$\begin{aligned}
\dot{M}_{14} = {} & M_{24} - K_{210}M_{11} - C_{210}M_{12} - K_{220}M_{13} - C_{220}M_{14} - K_{230}M_{15} - C_{230}M_{16} \\
& + F_{24}M_{17} + F_{25}M_{18} + F_{26}M_{19} - K_{214}[2M_{17}M_1 + M_{11}M_7 - 2M_1^2M_7] \\
& - C_{214}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - K_{224}[M_{13}M_7 + M_{17}M_3 \\
& + M_{37}M_1 - 2M_1M_3M_7] - C_{224}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1 - 2M_1M_4M_7] \\
& - K_{234}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{234}[M_{16}M_7 + M_{17}M_6 \\
& + M_{67}M_1 - 2M_1M_6M_7] - K_{215}[2M_{18}M_1 + M_{11}M_8 - 2M_1^2M_8] - C_{215}[M_{12}M_8
\end{aligned}$$

$$+ M_{18}M_2 + M_{28}M_1 - 2M_1M_2M_8] - K_{225}[M_{13}M_8 + M_{18}M_3 + M_{38}M_1 - 2M_1M_3M_8]$$
$$- C_{225}[M_{14}M_8 + M_{18}M_4 + M_{48}M_1 - 2M_1M_4M_8] - K_{235}[M_{15}M_8 + M_{18}M_5$$
$$+ M_{58}M_1 - 2M_1M_5M_8] - C_{235}[M_{16}M_8 + M_{18}M_6 + M_{68}M_1 - 2M_1M_6M_8]$$
$$- K_{216}[2M_{19}M_1 + M_{11}M_9 - 2M_1^2M_9] - C_{216}[M_{12}M_9 + M_{19}M_2 + M_{29}M_1 - 2M_1M_2M_9]$$
$$- K_{226}[M_{13}M_9 + M_{19}M_3 + M_{39}M_1 - 2M_1M_3M_9] - C_{226}[M_{14}M_9 + M_{19}M_4$$
$$+ M_{49}M_1 - 2M_1M_4M_9] - K_{236}[M_{15}M_9 + M_{19}M_5 + M_{59}M_1 - 2M_1M_5M_9]$$
$$- C_{236}[M_{16}M_9 + M_{19}M_6 + M_{69}M_1 - 2M_1M_6M_9]$$

$$\dot{M}_{15} = M_{25} + M_{16}$$

$$\dot{M}_{16} = M_{26} - K_{310}M_{11} - C_{310}M_{12} - K_{320}M_{13} - C_{320}M_{14} - K_{330}M_{15} - C_{330}M_{16} + F_{34}M_{17}$$
$$+ F_{35}M_{18} + F_{36}M_{19} - K_{314}[2M_{17}M_1 + M_{11}M_7 - 2M_1^2M_7] - C_{314}[M_{12}M_7$$
$$+ M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - K_{324}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7]$$
$$- C_{324}[M_{14}M_7 + M_{17}M_4 + M_{47}M_1 - 2M_1M_4M_7] - K_{334}[M_{15}M_7 + M_{17}M_5$$
$$+ M_{57}M_1 - 2M_1M_5M_7] - C_{334}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7]$$
$$- K_{315}[2M_{18}M_1 + M_{11}M_8 - 2M_1^2M_8] - C_{315}[M_{12}M_8 + M_{18}M_2 + M_{28}M_1$$
$$- 2M_1M_2M_8] - K_{325}[M_{13}M_8 + M_{18}M_3 + M_{38}M_1 - 2M_1M_3M_8] - C_{325}[M_{14}M_8$$
$$+ M_{18}M_4 + M_{48}M_1 - 2M_1M_4M_8] - K_{335}[M_{15}M_8 + M_{18}M_5 + M_{58}M_1$$
$$- 2M_1M_5M_8] - C_{335}[M_{16}M_8 + M_{18}M_6 + M_{68}M_1 - 2M_1M_6M_8] - K_{316}$$
$$[2M_{19}M_1 + M_{11}M_9 - 2M_1^2M_9] - C_{316}[M_{12}M_9 + M_{19}M_2 + M_{29}M_1 - 2M_1M_2M_9]$$
$$- K_{326}[M_{13}M_9 + M_{19}M_3 + M_{39}M_1 - 2M_1M_3M_9] - C_{326}[M_{14}M_9 + M_{19}M_4 + M_{49}M_1$$
$$- 2M_1M_4M_9] - K_{336}[M_{15}M_9 + M_{19}M_5 + M_{59}M_1 - 2M_1M_5M_9] - C_{336}[M_{16}M_9$$
$$+ M_{19}M_6 + M_{69}M_1 - 2M_1M_6M_9]$$

$$\dot{M}_{17} = M_{27} - \alpha_1 M_{17}$$

$$\dot{M}_{18} = M_{28} - \alpha_2 M_{18}$$

$$\dot{M}_{19} = M_{29} - \alpha_3 M_{19}$$

$$\dot{M}_{22} = 2\pi e_1^2 \Phi_{11}[F_{11}^2 - 2F_{11}K_{111}M_1 - 2F_{11}K_{121}M_3 - 2F_{11}K_{131}M_5 + K_{111}^2M_{11}$$
$$+ K_{121}^2M_{33} + K_{131}^2M_{55} + 2K_{111}K_{121}M_{13} + 2K_{111}K_{131}M_{15} + 2K_{121}K_{131}M_{35}]$$

$$+ 2\pi e_2^2 \Phi_{22} [F_{12}^2 - 2F_{12}K_{112}M_1 - 2F_{12}K_{122}M_3 - 2F_{12}K_{132}M_5 + K_{112}^2M_{11} + K_{122}^2M_{33}$$
$$+ K_{132}^2M_{55} + 2K_{112}K_{122}M_{13} + 2K_{112}K_{132}M_{15} + 2K_{122}K_{132}M_{35}]$$
$$+ 2\pi e_3^2 \Phi_{33}[F_{13}^2 - 2F_{13}K_{113}M_1 - 2F_{13}K_{123}M_3 - 2F_{13}K_{133}M_5 + K_{113}^2M_{11}$$
$$+ K_{123}^2M_{33} + K_{133}^2M_{55} + 2K_{113}K_{123}M_{13} + 2K_{113}K_{133}M_{15} + 2K_{123}K_{133}M_{35}]$$
$$+ 2\{-K_{110}M_{12} - C_{110}M_{22} - K_{120}M_{23} - C_{120}M_{24} - K_{130}M_{25} - C_{130}M_{26} + F_{14}M_{27}$$
$$+ F_{15}M_{28} + F_{16}M_{29} - K_{114}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1 - 2M_1M_2M_7] - C_{114}$$
$$[2M_{27}M_2 + M_{22}M_7 - 2M_2^2 M_7] - K_{124}[M_{23}M_7 + M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7]$$
$$- C_{124}[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - K_{134}[M_{25}M_7 + M_{27}M_5 + M_{57}M_2$$
$$- 2M_2M_5M_7] - C_{134}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{115}[M_{12}M_8$$
$$+ M_{18}M_2 + M_{28}M_1 - 2M_1M_2M_8] - C_{115}[2M_{28}M_2 + M_{22}M_8 - 2M_2^2M_8] - K_{125}$$
$$[M_{23}M_8 + M_{28}M_3 + M_{38}M_2 - 2M_2M_3M_8] - C_{125}[M_{24}M_8 + M_{28}M_4 + M_{48}M_2$$
$$- 2M_2M_4M_8] - K_{135}[M_{25}M_8 + M_{28}M_5 + M_{58}M_2 - 2M_2M_5M_8] - C_{135}[M_{26}M_8$$
$$+ M_{28}M_6 + M_{68}M_2 - 2M_2M_6M_8] - K_{116}[M_{12}M_9 + M_{19}M_2 + M_{29}M_1 - 2M_1M_2M_9]$$
$$- C_{116}[2M_{29}M_2 + M_{22}M_9 - 2M_2^2M_9] - K_{126}[M_{23}M_9 + M_{29}M_3 + M_{39}M_2 - 2M_2M_3M_9]$$
$$- C_{126}[M_{24}M_9 + M_{29}M_4 + M_{49}M_2 - 2M_2M_4M_9] - K_{136}[M_{25}M_9 + M_{29}M_5 + M_{59}M_2$$
$$- 2M_2M_5M_9] - C_{136}[M_{26}M_9 + M_{29}M_6 + M_{69}M_2 - 2M_2M_6M_9]\}$$

$$\dot{M}_{23} = M_{24} - K_{110}M_{13} - C_{110}M_{23} - K_{120}M_{33} - C_{120}M_{34} - K_{130}M_{35} - C_{130}M_{36} + F_{14}M_{37}$$
$$+ F_{15}M_{38} + F_{16}M_{39} - K_{114}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7] - C_{114}[M_{23}M_7$$
$$+ M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - K_{124}[2M_{37}M_3 + M_{33}M_7 - 2M_3^2M_7] - C_{124}[M_{34}M_7$$
$$+ M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - K_{134}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3 - 2M_3M_5M_7]$$
$$- C_{134}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7] - K_{115}[M_{13}M_8 + M_{18}M_3 + M_{38}M_1$$
$$- 2M_1M_3M_8] - C_{115}[M_{23}M_8 + M_{28}M_3 + M_{38}M_2 - 2M_2M_3M_8] - K_{125}[2M_{38}M_3$$
$$+ M_{33}M_8 - 2M_3^2M_8] - C_{125}[M_{34}M_8 + M_{38}M_4 + M_{48}M_3 - 2M_3M_4M_8] - K_{135}$$
$$[M_{35}M_8 + M_{38}M_5 + M_{58}M_3 - 2M_3M_5M_8] - C_{135}[M_{36}M_8 + M_{38}M_6 + M_{68}M_3$$
$$- 2M_3M_6M_8] - K_{116}[M_{13}M_9 + M_{19}M_3 + M_{39}M_1 - 2M_1M_3M_9] - C_{116}[M_{23}M_9 + M_{29}M_3$$
$$+ M_{39}M_2 - 2M_2M_3M_9] - K_{126}[2M_{39}M_3 + M_{33}M_9 - 2M_3^2M_9] - C_{126}[M_{34}M_9$$

$+ M_{39}M_4 + M_{49}M_3 - 2M_3M_4M_9] - K_{136}[M_{35}M_9 + M_{39}M_5 + M_{59}M_3 - 2M_3M_5M_9]$

$- C_{136}[M_{36}M_9 + M_{39}M_6 + M_{69}M_3 - 2M_3M_6M_9]$

$M_{24} = 2\pi e_1^2 \Phi_{11}[F_{11}F_{21} - (F_{11}K_{211} + F_{21}K_{111})M_1 - (F_{11}K_{221} + F_{21}K_{121})M_3$

$- (F_{11}K_{231} + F_{21}K_{131}) M_5 + K_{111}K_{211}M_{11} + K_{121}K_{221}M_{33} + K_{131}K_{231}M_{55}$

$+ (K_{111}K_{221} + K_{211}K_{121})M_{13} + (K_{111}K_{231} + K_{131}K_{211})M_{15} + (K_{121}K_{231}$

$+ K_{131}K_{221})M_{35}] + 2\pi e_2^2 \Phi_{22} [F_{12}F_{22} - (F_{12}K_{212} + F_{22}K_{112})M_1 - (F_{12}K_{222}$

$+ F_{22}K_{122})M_3 - (F_{12}K_{232} + F_{22}K_{132})M_5 + K_{112}K_{212}M_{11} + K_{122}K_{222}M_{33}$

$+ K_{132}K_{232}M_{55} + (K_{112}K_{222} + K_{212}K_{122})M_{13} + (K_{112}K_{232} + K_{132}K_{212})M_{15}$

$+ (K_{122}K_{232} + K_{132}K_{222})M_{35}] + 2\pi e_3^2 \Phi_{33}[F_{13}F_{23} - (F_{13}K_{213} + F_{23}K_{113})M_1$

$- (F_{13}K_{223} + F_{23}K_{123})M_3 - (F_{13}K_{233} + F_{23}K_{133})M_5 + K_{113}K_{213}M_{11} + K_{123}K_{223}M_{33}$

$+ K_{133}K_{233}M_{55} + (K_{113}K_{223} + K_{213}K_{123})M_{13} + (K_{113}K_{233} + K_{133}K_{213})M_{15}$

$+ (K_{123}K_{233} + K_{133}K_{223})M_{35}] - K_{110}M_{14} - C_{110}M_{24} - K_{120}M_{34} - C_{120}M_{44}$

$- K_{130}M_{45} - C_{130}M_{46} + F_{14}M_{47} + F_{15}M_{48} + F_{16}M_{49} - K_{210}M_{12} - C_{210}M_{22} - K_{220}M_{23}$

$- C_{220}M_{24} - K_{230}M_{25} - C_{230}M_{26} + F_{24}M_{27} + F_{25}M_{28} + F_{26}M_{29} - K_{114}[M_{14}M_7 + M_{17}M_4$

$+ M_{47}M_1 - 2M_1M_4M_7] - C_{114}[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - K_{124}[M_{34}M_7$

$+ M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - C_{124}[2M_{47}M_4 + M_{44}M_7 - 2M_4^2M_7] - K_{134}[M_{45}M_7$

$+ M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7] - C_{134}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7]$

$- K_{115}[M_{14}M_8 + M_{18}M_4 + M_{48}M_1 - 2M_1M_4M_8] - C_{115}[M_{24}M_8 + M_{28}M_4 + M_{48}M_2$

$- 2M_2M_4M_8] - K_{125}[M_{34}M_8 + M_{38}M_4 + M_{48}M_3 - 2M_3M_4M_8] - C_{125}[2M_{48}M_4 + M_{44}M_8$

$- 2M_4^2M_8] - K_{135}[M_{45}M_8 + M_{48}M_5 + M_{58}M_4 - 2M_4M_5M_8] - C_{135}[M_{46}M_8 + M_{48}M_6$

$+ M_{68}M_4 - 2M_4M_6M_8] - K_{116}[M_{14}M_9 + M_{19}M_4 + M_{49}M_1 - 2M_1M_4M_9] - C_{116}[M_{24}M_9$

$+ M_{29}M_4 + M_{49}M_2 - 2M_2M_4M_9] - K_{126}[M_{34}M_9 + M_{39}M_4 + M_{49}M_3 - 2M_3M_4M_9]$

$- C_{126}[2M_{49}M_4 + M_{44}M_9 - 2M_4^2M_9] - K_{136}[M_{45}M_9 + M_{49}M_5 + M_{59}M_4 - 2M_4M_5M_9]$

$- C_{136}[M_{46}M_9 + M_{49}M_6 + M_{69}M_4 - 2M_4M_6M_9] - K_{214}[M_{12}M_7 + M_{17}M_2 + M_{27}M_1$

$- 2M_1M_2M_7] - C_{214}[2M_{27}M_2 + M_{22}M_7 - 2M_2^2M_7] - K_{224}[M_{23}M_7 + M_{27}M_3 + M_{37}M_2$

$- 2M_2M_3M_7] - C_{224}[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - K_{234}[M_{25}M_7 + M_{27}M_5$

$$+ M_{57}M_2 - 2M_2M_5M_7] - C_{234}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{215}[M_{12}M_8$$

$$+ M_{18}M_2 + M_{28}M_1 - 2M_1M_2M_8] - C_{215}[2M_{28}M_2 + M_{22}M_8 - 2M_2^2M_8] - K_{225}[M_{23}M_8$$

$$+ M_{28}M_3 + M_{38}M_2 - 2M_2M_3M_8] \quad C_{225}[M_{24}M_8 + M_{28}M_4 + M_{48}M_2 - 2M_2M_4M_8]$$

$$- K_{235}[M_{25}M_8 + M_{28}M_5 + M_{58}M_2 - 2M_2M_5M_8] - C_{235}[M_{26}M_8 + M_{28}M_6 + M_{68}M_2$$

$$- 2M_2M_6M_8] - K_{216}[M_{12}M_9 + M_{19}M_2 + M_{29}M_1 - 2M_1M_2M_9] - C_{216}[2M_{29}M_2 + M_{22}M_9$$

$$- 2M_2^2M_9] - K_{226}[M_{23}M_9 + M_{29}M_3 + M_{39}M_2 - 2M_2M_3M_9] - C_{226}[M_{24}M_9 + M_{29}M_4 + M_{49}M_2$$

$$- 2M_2M_4M_9] - K_{236}[M_{25}M_9 + M_{29}M_5 + M_{59}M_2 - 2M_2M_5M_9] - C_{236}[M_{26}M_9 + M_{29}M_6$$

$$+ M_{69}M_2 - 2M_2M_6M_9]$$

$$\dot{M}_{25} = M_{26} - K_{110}M_{15} - C_{110}M_{25} - K_{120}M_{35} - C_{120}M_{45} - K_{130}M_{55} - C_{130}M_{56} + F_{14}M_{57}$$

$$+ F_{15}M_{58} + F_{16}M_{59} - K_{114}[M_{15}M_7 + M_{17}M_5 + M_{57}M_1 - 2M_1M_5M_7] - C_{114}[M_{25}M_7$$

$$+ M_{27}M_5 + M_{57}M_2 - 2M_2M_5M_7] - K_{124}[M_{35}M_7 + M_{37}M_5 + M_{57}M_3 - 2M_3M_5M_7]$$

$$- C_{124}[M_{45}M_7 + M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7] - K_{134}[2M_{57}M_5 + M_{55}M_7 - 2M_5^2M_7]$$

$$- C_{134}[M_{56}M_7 + M_{57}M_6 + M_{67}M_5 - 2M_5M_6M_7] - K_{115}[M_{15}M_8 + M_{18}M_5 + M_{58}M_1$$

$$- 2M_1M_5M_8] - C_{115}[M_{25}M_8 + M_{28}M_5 + M_{58}M_2 - 2M_2M_5M_8] - K_{125}[M_{35}M_8$$

$$+ M_{38}M_5 + M_{58}M_5 - 2M_3M_5M_8] - C_{125}[M_{45}M_8 + M_{48}M_5 + M_{58}M_4 - 2M_4M_5M_8]$$

$$- K_{135}[2M_{58}M_5 + M_{55}M_8 - 2M_5^2M_8] - C_{135}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5 - 2M_5M_6M_8]$$

$$- K_{116}[M_{15}M_9 + M_{19}M_5 + M_{59}M_1 - 2M_1M_5M_9] - C_{116}[M_{25}M_9 + M_{29}M_5 + M_{59}M_2$$

$$- 2M_2M_5M_9] - K_{126}[M_{35}M_9 + M_{39}M_5 + M_{59}M_3 - 2M_3M_5M_9] - C_{126}[M_{45}M_9 + M_{49}M_5$$

$$+ M_{59}M_4 - 2M_4M_5M_9] - K_{136}[2M_{59}M_5 + M_{55}M_9 - 2M_5^2M_9] - C_{136}[M_{56}M_9$$

$$+ M_{59}M_6 + M_{69}M_5 - 2M_5M_6M_9]$$

$$\dot{M}_{26} = 2\pi e_1^2 \Phi_{11} [F_{11}F_{31} - (F_{11}K_{311} + F_{31}K_{111})M_1 - (F_{11}K_{321} + F_{31}K_{121})M_3 - (F_{11}K_{331}$$

$$+ F_{31}K_{131})M_5 + K_{111}K_{311}M_{11} + K_{121}K_{321}M_{33} + K_{131}K_{331}M_{55} + (K_{111}K_{321} + K_{121}K_{311})$$

$$M_{13} + (K_{111}K_{331} + K_{131}K_{311})M_{15} + (K_{121}K_{331} + K_{131}K_{321})M_{35}] + 2\pi e_2^2 \Phi_{22}$$

$$[F_{12}F_{32} - (F_{12}K_{312} + F_{32}K_{112})M_1 - (F_{12}K_{322} + F_{32}K_{122})M_3 - (F_{12}K_{332} + F_{32}K_{132})M_5$$

$$+ K_{112}K_{312}M_{11} + K_{122}K_{322}M_{33} + K_{132}K_{332}M_{55} + (K_{112}K_{322} + K_{122}K_{312})M_{13}$$

$$+ (K_{112}K_{332} + K_{132}K_{312})M_{15} + (K_{122}K_{332} + K_{132}K_{322})M_{35}] + 2\pi e_3^2 \Phi_{33}$$

$$[F_{13}F_{33}-(F_{13}K_{313}+F_{33}K_{113})M_1-(F_{13}K_{323}+F_{33}K_{123})M_3-(F_{13}K_{333}+F_{33}K_{133})M_5$$
$$+ K_{113}K_{313}M_{11} + K_{123}K_{323}M_{33} + K_{133}K_{333}M_{55} + (K_{113}K_{323} + K_{123}K_{313})M_{13}$$
$$+ (K_{113}K_{333} + K_{133}K_{313})M_{15} + (K_{123}K_{333} + K_{133}K_{323})M_{35}] - K_{110}M_{16}$$
$$- C_{110}M_{26} - K_{120}M_{36} - C_{120}M_{46} - K_{130}M_{56} - C_{130}M_{66} + F_{14}M_{67} + F_{15}M_{68} + F_{16}M_{69}$$
$$- K_{310}M_{12} - C_{310}M_{22} - K_{320}M_{23} - C_{320}M_{24} - K_{330}M_{25} - C_{330}M_{26} + F_{34}M_{27} + F_{35}M_{28}$$
$$+ F_{36}M_{29} - K_{114}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7] - C_{114}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2$$
$$- 2M_2M_6M_7] - K_{124}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7] - C_{124}[M_{46}M_7 + M_{47}M_6$$
$$+ M_{67}M_4 - 2M_4M_6M_7] - K_{134}[M_{56}M_7 + M_{57}M_6 + M_{67}M_5 - 2M_5M_6M_7] - C_{134}[2M_{67}M_6$$
$$+ M_{66}M_7 - 2M_6^2M_7] - K_{115}[M_{16}M_8 + M_{18}M_6 + M_{68}M_1 - 2M_1M_6M_8] - C_{115}[M_{26}M_8 + M_{28}M_6$$
$$+ M_{68}M_2 - 2M_2M_6M_8] - K_{125}[M_{36}M_8 + M_{38}M_6 + M_{68}M_3 - 2M_3M_6M_8] - C_{125}[M_{46}M_8$$
$$+ M_{48}M_6 + M_{68}M_4 - 2M_4M_6M_8] - K_{135}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5 - 2M_5M_6M_8] - C_{135}[$$
$$+ 2M_{68}M_6 + M_{66}M_8 - 2M_6^2M_8] - K_{116}[M_{16}M_9 + M_{19}M_6 - M_{69}M_1 - 2M_1M_6M_9] - C_{116}$$
$$[M_{26}M_9 + M_{29}M_6 + M_{96}M_2 - 2M_2M_6M_9] - K_{126}[M_{36}M_9 + M_{39}M_6 + M_{69}M_3 - 2M_3M_6M_9]$$
$$- C_{126}[M_{46}M_9 + M_{49}M_6 + M_{69}M_4 - 2M_4M_6M_9] - K_{136}[M_{56}M_9 + M_{59}M_6 + M_{69}M_5$$
$$- 2M_5M_6M_9] - C_{136}[2M_{69}M_6 + M_{66}M_9 - 2M_6^2M_9] - K_{314}[M_{12}M_7 + M_{17}M_2$$
$$+ M_{27}M_1 - 2M_1M_2M_7] - C_{314}[2M_{27}M_2 + M_{22}M_7 - 2M_2^2M_7] - K_{324}[M_{23}M_7 + M_{27}M_3$$
$$+ M_{37}M_2 - 2M_2M_3M_7] - C_{324}[M_{24}M_7 + M_{27}M_4 + M_{47}M_2 - 2M_2M_4M_7] - K_{334}[M_{25}M_7$$
$$+ M_{27}M_5 + M_{57}M_2 - 2M_2M_5M_7] - C_{334}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7]$$
$$- K_{315}[M_{12}M_8 + M_{18}M_2 + M_{28}M_1 - 2M_1M_2M_8] - C_{315}[2M_{28}M_2 + M_{22}M_8 - 2M_2^2M_8]$$
$$- K_{325}[M_{23}M_8 + M_{28}M_3 + M_{38}M_2 - 2M_2M_3M_8] - C_{325}[M_{24}M_8 + M_{28}M_4 + M_{48}M_2$$
$$- 2M_2M_4M_8] - K_{335}[M_{25}M_8 + M_{28}M_5 + M_{58}M_2 - 2M_2M_5M_8] - C_{335}[M_{26}M_8 + M_{28}M_6$$
$$+ M_{68}M_2 - 2M_2M_6M_8] - K_{316}[M_{12}M_9 + M_{19}M_2 + M_{29}M_1 - 2M_1M_2M_9] - C_{316}[2M_{29}M_2$$
$$+ M_{22}M_9 - 2M_2^2M_9] - K_{326}[M_{23}M_9 + M_{29}M_3 + M_{39}M_2 - 2M_2M_3M_9] - C_{326}[M_{24}M_9$$
$$+ M_{29}M_4 + M_{49}M_2 - 2M_2M_4M_9] - K_{336}[M_{25}M_9 + M_{29}M_5 + M_{59}M_2 - 2M_2M_5M_9]$$
$$- C_{336}[M_{26}M_9 + M_{29}M_6 + M_{69}M_2 - 2M_2M_6M_9]$$

$$\dot{M}_{27} = -\alpha_1 M_{27} - K_{110}M_{17} - C_{110}M_{27} - K_{120}M_{37} - C_{120}M_{47} - K_{130}M_{57} - C_{130}M_{67} + F_{14}M_{77}$$

$$+ F_{15}M_{78} + F_{16}M_{79} - K_{114}[2M_{17}M_7 + M_{77}M_1 - 2M_7^2M_1] - C_{114}[2M_{27}M_7 + M_{77}M_2$$
$$- 2M_7^2M_2] - K_{124}[2M_{37}M_7 + M_{77}M_3 - 2M_7^2M_3] - C_{124}[2M_{47}M_7 + M_{77}M_4 - 2M_7^2M_4]$$
$$- K_{134}[2M_{57}M_7 + M_{77}M_5 - 2M_7^2M_5] - C_{134}[2M_{67}M_7 + M_{77}M_6 - 2M_7^2M_6] - K_{115}$$
$$[M_{17}M_8 + M_{18}M_7 + M_{78}M_1 - 2M_1M_7M_8] - C_{115}[M_{27}M_8 + M_{28}M_7 + M_{78}M_2 - 2M_2M_7M_8]$$
$$- K_{125}[M_{37}M_8 + M_{38}M_7 + M_{78}M_3 - 2M_3M_7M_8] - C_{125}[M_{47}M_8 + M_{48}M_7 + M_{78}M_4$$
$$- 2M_4M_7M_8] - K_{135}[M_{57}M_8 + M_{58}M_7 + M_{78}M_5 - 2M_5M_7M_8] - C_{135}[M_{67}M_8 + M_{68}M_7$$
$$+ M_{78}M_6 - 2M_6M_7M_8] - K_{116}[M_{17}M_9 + M_{19}M_7 + M_{79}M_1 - 2M_1M_7M_9] - C_{116}[M_{27}M_9$$
$$+ M_{29}M_7 + M_{79}M_2 - 2M_2M_7M_9] - K_{126}[M_{37}M_9 + M_{39}M_7 + M_{79}M_3 - 2M_3M_7M_9] - C_{126}[M_{47}M_9$$
$$+ M_{49}M_7 + M_{79}M_4 - 2M_4M_7M_9] - K_{136}[M_{57}M_9 + M_{59}M_7 + M_{79}M_5 - 2M_5M_7M_9] - C_{136}$$
$$[M_{67}M_9 + M_{69}M_7 + M_{79}M_6 - 2M_6M_7M_9]$$

$$\dot{M}_{28} = -\alpha_2 M_{28} - K_{110}M_{18} - C_{110}M_{28} - K_{120}M_{38} - C_{120}M_{48} - K_{130}M_{58} - C_{130}M_{68} + F_{14}M_{78}$$
$$+ F_{15}M_{88} + F_{16}M_{89} - K_{114}[M_{17}M_8 + M_{18}M_7 + M_{78}M_1 - 2M_1M_7M_8] - C_{114}[M_{27}M_8$$
$$+ M_{28}M_7 + M_{78}M_2 - 2M_2M_7M_8] - K_{124}[M_{37}M_8 + M_{38}M_7 + M_{78}M_3 - 2M_3M_7M_8]$$
$$- C_{124}[M_{47}M_8 + M_{48}M_7 + M_{78}M_4 - 2M_4M_7M_8] - K_{134}[M_{57}M_8 + M_{58}M_7 + M_{78}M_5$$
$$- 2M_5M_7M_8] - C_{134}[M_{67}M_8 - M_{68}M_7 + M_{78}M_6 - 2M_6M_7M_8] - K_{115}[2M_{18}M_8 + M_{88}M_1$$
$$- 2M_8^2M_1] - C_{115}[2M_{28}M_2 + M_{88}M_2 - 2M_8^2M_2] - K_{125}[2M_{38}M_8 + M_{88}M_5 - 2M_8^2M_3]$$
$$- C_{125}[2M_{48}M_8 + M_{88}M_4 - 2M_8^2M_4] - K_{135}[2M_{58}M_8 + M_{88}M_5 - 2M_8^2M_5] - C_{135}[2M_{68}M_8$$
$$+ M_{88}M_6 - 2M_8^2M_6] - K_{116}[M_{18}M_9 + M_{19}M_8 + M_{89}M_1 - 2M_1M_8M_9] - C_{116}[M_{28}M_9$$
$$+ M_{29}M_8 + M_{89}M_2 - 2M_2M_8M_9] - K_{126}[M_{38}M_9 + M_{39}M_8 + M_{89}M_3 - 2M_3M_8M_9] - C_{126}$$
$$[M_{48}M_9 + M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9] - K_{136}[M_{58}M_9 + M_{59}M_8 + M_{89}M_5 - 2M_5M_8M_9]$$
$$- C_{136}[M_{68}M_9 + M_{69}M_8 + M_{89}M_6 - 2M_6M_8M_9]$$

$$\dot{M}_{29} = -\alpha_3 M_{29} - K_{110}M_{19} - C_{110}M_{29} - K_{120}M_{39} - C_{120}M_{49} - K_{130}M_{59} - C_{130}M_{69} + F_{14}M_{79}$$
$$+ F_{15}M_{89} + F_{16}M_{99} - K_{114}[M_{17}M_9 + M_{19}M_7 + M_{79}M_1 - 2M_1M_7M_9] - C_{114}[M_{27}M_9$$
$$+ M_{29}M_7 + M_{79}M_2 - 2M_2M_7M_9] - K_{124}[M_{37}M_9 + M_{39}M_7 + M_{79}M_3 - 2M_3M_7M_9]$$
$$- C_{124}[M_{47}M_9 + M_{49}M_7 + M_{79}M_4 - 2M_4M_7M_9] - K_{134}[M_{57}M_9 + M_{59}M_7 + M_{79}M_5$$
$$- 2M_5M_7M_9] - C_{134}[M_{67}M_9 + M_{69}M_7 + M_{79}M_6 - 2M_6M_7M_9] - K_{115}[M_{18}M_9 + M_{19}M_8$$

$$+ M_{89}M_1 - 2M_1M_8M_9] - C_{115}[M_{28}M_9 + M_{29}M_8 + M_{89}M_2 - 2M_2M_8M_9] - K_{125}[M_{38}M_9$$

$$+ M_{39}M_8 + M_{89}M_3 - 2M_3M_8M_9] - C_{125}[M_{48}M_9 + M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9]$$

$$- K_{135}[M_{58}M_9 + M_{59}M_8 + M_{89}M_5 - 2M_5M_8M_9] - C_{135}[M_{68}M_9 + M_{69}M_8 + M_{89}M_6$$

$$- 2M_6M_8M_9] - K_{116}[2M_{19}M_9 + M_{99}M_1 - 2M_9^2M_1] - C_{116}[2M_{29}M_9 + M_{99}M_2 - 2M_9^2M_2]$$

$$- K_{126}[2M_{39}M_9 + M_{99}M_3 - 2M_9^2M_3] - C_{126}[2M_{49}M_9 + M_{99}M_4 - 2M_9^2M_4]$$

$$- K_{136}[2M_{59}M_9 + M_{99}M_5 - 2M_9^2M_5] - C_{136}[2M_{69}M_9 + M_{99}M_6 - 2M_9^2M_6]$$

$$\dot{M}_{33} = 2M_{34}$$

$$\dot{M}_{34} = M_{44} - K_{210}M_{13} - C_{210}M_{23} - K_{220}M_{33} - C_{220}M_{34} - K_{230}M_{35} - C_{230}M_{36} + F_{24}M_{37}$$

$$+ F_{25}M_{38} + F_{26}M_{39} - K_{214}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7] - C_{214}$$

$$[M_{23}M_7 + M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - K_{224}[2M_{37}M_3 + M_{33}M_7 - 2M_3^2M_7]$$

$$- C_{224}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - K_{234}[M_{35}M_7 + M_{37}M_5$$

$$+ M_{57}M_3 - 2M_3M_5M_7] - C_{234}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7]$$

$$- K_{215}[M_{13}M_8 + M_{18}M_3 + M_{38}M_1 - 2M_1M_3M_8] - C_{215}[M_{23}M_8 + M_{28}M_3$$

$$+ M_{38}M_2 - 2M_2M_3M_8] - K_{225}[2M_{38}M_3 + M_{33}M_8 - 2M_3^2M_8] - C_{225}[M_{34}M_8$$

$$+ M_{38}M_4 + M_{48}M_3 - 2M_3M_4M_8] - K_{235}[M_{35}M_8 + M_{38}M_5 + M_{58}M_3 - 2M_3M_5M_8]$$

$$- C_{235}[M_{36}M_8 + M_{38}M_6 + M_{68}M_3 - 2M_3M_6M_8] - K_{216}[M_{13}M_9 + M_{19}M_3$$

$$+ M_{39}M_1 - 2M_1M_3M_9] - C_{216}[M_{23}M_9 + M_{29}M_3 + M_{39}M_2 - 2M_2M_3M_9]$$

$$- K_{226}[2M_{39}M_3 + M_{33}M_9 - 2M_3^2M_9] - C_{226}[M_{34}M_9 + M_{39}M_4 + M_{49}M_3$$

$$- 2M_3M_4M_9] - K_{236}[M_{35}M_9 + M_{39}M_5 + M_{59}M_3 - 2M_3M_5M_9] - C_{236}[M_{36}M_9$$

$$+ M_{39}M_6 + M_{69}M_3 - 2M_3M_6M_9]$$

$$\dot{M}_{35} = M_{45} + M_{36}$$

$$\dot{M}_{36} = M_{46} - K_{310}M_{13} - C_{310}M_{23} - K_{320}M_{33} - C_{320}M_{34} - K_{330}M_{35} - C_{330}M_{36}$$

$$+ F_{34}M_{37} + F_{35}M_{38} + F_{36}M_{39} - K_{314}[M_{13}M_7 + M_{17}M_3 + M_{37}M_1 - 2M_1M_3M_7]$$

$$- C_{314}[M_{23}M_7 + M_{27}M_3 + M_{37}M_2 - 2M_2M_3M_7] - K_{324}[2M_{37}M_3 + M_{33}M_7 - 2M_3^2M_7]$$

$$- C_{324}[M_{34}M_7+M_{37}M_4+M_{47}M_3-2M_3M_4M_7]-K_{334}[M_{35}M_7+M_{37}M_5+M_{57}M_3-2M_3M_5M_7]$$

$$- C_{334}[M_{36}M_7+M_{37}M_6+M_{67}M_3-2M_3M_6M_7]-K_{315}[M_{13}M_8+M_{18}M_3+M_{38}M_1-2M_1M_3M_8]$$

$$- C_{315}[M_{23}M_8 + M_{28}M_3 + M_{38}M_2 - 2M_2M_3M_8] - K_{325}[2M_{38}M_3 + M_{33}M_8 - 2M_3^2M_8]$$

$$- C_{325}[M_{34}M_8+M_{38}M_4+M_{48}M_3-2M_3M_4M_8]-K_{335}[M_{35}M_8+M_{38}M_5+M_{58}M_3-2M_3M_5M_8]$$

$$- C_{335}[M_{36}M_8+M_{38}M_6+M_{68}M_3-2M_3M_6M_8]-K_{316}[M_{13}M_9+M_{19}M_3+M_{39}M_1-2M_1M_3M_9]$$

$$- C_{316}[M_{23}M_9+M_{29}M_3+M_{39}M_2-2M_2M_3M_9]-K_{326}[2M_{39}M_3+M_{33}M_9-2M_3^2M_9]$$

$$- C_{326}[M_{34}M_9+M_{39}M_4+M_{49}M_3-2M_3M_4M_9]-K_{336}[M_{35}M_9+M_{39}M_5+M_{59}M_3-2M_3M_5M_9]$$

$$- C_{336}[M_{36}M_9 + M_{39}M_6 + M_{69}M_3 - 2M_3M_6M_9]$$

$$\dot{M}_{37} = M_{47} - \alpha_1 M_{37}$$

$$\dot{M}_{38} = M_{48} - \alpha_2 M_{38}$$

$$\dot{M}_{39} = M_{49} - \alpha_3 M_{39}$$

$$\dot{M}_{44} = 2\pi e_1^2 \Phi_{11}[F_{21}^2 - 2F_{21}K_{211}M_1-2F_{21}K_{221}M_3-2F_{21}K_{231}M_5 + K_{211}^2M_{11} + K_{221}^2M_{33}$$

$$+ K_{231}^2M_{55} + 2K_{211}K_{221}M_{13} + 2K_{211}K_{231}M_{15} + 2K_{221}K_{231}M_{35}] + 2\pi e_2^2 \Phi_{22}$$

$$[F_{22}^2 -2F_{22}K_{212}M_1-2F_{22}K_{222}M_3-2F_{22}K_{232}M_5+K_{212}^2M_{11}+K_{222}^2M_{33}+K_{232}^2 M_{55}$$

$$+ 2K_{212}K_{232}M_{13} + 2K_{212}K_{232}M_{15} + 2K_{222}K_{232}M_{35}] + 2\pi e_3^2 \Phi_{33}[F_{23}^2$$

$$- 2F_{23}K_{213}M_1-2F_{23}K_{223}M_3-2F_{23}K_{233}M_5 + K_{213}^2 M_{11}^2 + K_{223}^2 M_{33}^2 + K_{233}^2 M_{55}^2$$

$$+ 2K_{213}K_{223}M_{13} + 2K_{213}K_{233}M_{15} + 2K_{223}K_{233}M_{35}] + 2\{-K_{210}M_{14} - C_{210}M_{24}$$

$$- K_{220}M_{34} - C_{220}M_{44} - K_{230}M_{45} - C_{230}M_{46} + F_{24}M_{47} + F_{25}M_{48} + F_{26}M_{49}$$

$$- K_{214}[M_{14}M_7+M_{17}M_4+M_{47}M_1-2M_1M_4M_7]-C_{214}[M_{24}M_7+M_{27}M_4+M_{47}M_2-2M_2M_4M_7]$$

$$- K_{224}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - C_{224}[2M_{47}M_4 + M_{44}M_7 - 2M_4^2M_7]$$

$$- K_{234}[M_{45}M_7+M_{47}M_5+M_{57}M_4-2M_4M_5M_7]-C_{234}[M_{46}M_7+M_{47}M_6+M_{67}M_4-2M_4M_6M_7]$$

$$- K_{215}[M_{14}M_8+M_{18}M_4+M_{48}M_1-2M_1M_4M_8]-C_{215}[M_{24}M_8+M_{28}M_4+M_{48}M_2-2M_2M_4M_8]$$

$$- K_{225}[M_{34}M_8 + M_{38}M_4 + M_{48}M_3 - 2M_3M_4M_8] - C_{225}[2M_{48}M_4 + M_{44}M_8 - 2M_4^2M_8]$$

$$- K_{235}[M_{45}M_8+M_{48}M_5+M_{58}M_4-2M_4M_5M_8]-C_{235}[M_{46}M_8+M_{48}M_6+M_{68}M_4-2M_4M_6M_8]$$

$$- K_{216}[M_{14}M_9+M_{19}M_4+M_{49}M_1-2M_1M_4M_9]-C_{216}[M_{24}M_9+M_{29}M_4+M_{49}M_2- 2M_2M_4M_9]$$

$$- K_{226}[M_{34}M_9 + M_{39}M_4 + M_{49}M_3 - 2M_3M_4M_9] - C_{226}[2M_{49}M_4 + M_{44}M_9 - 2M_4^2M_9]$$

$$- K_{236}[M_{45}M_9+M_{49}M_5+M_{59}M_4-2M_4M_5M_9]-C_{236}[M_{46}M_9+M_{49}M_6+M_{69}M_4-2M_4M_6M_9]\}$$

$$\dot{M}_{45} = M_{46}-K_{210}M_{15}-C_{210}M_{25}-K_{220}M_{35}-C_{220}M_{45}-K_{230}M_{55}-C_{230}M_{56}+F_{24}M_{57}$$

$+ F_{25}M_{58}+F_{26}M_{59}-K_{214}[M_{15}M_7+M_{17}M_5+M_{57}M_1-2M_1M_5M_7]-C_{214}[M_{25}M_7+M_{27}M_5$

$+ M_{57}M_2-2M_2M_5M_7]-K_{224}[M_{35}M_7+M_{37}M_5+M_{57}M_3-2M_3M_5M_7]-C_{224}[M_{45}M_7+M_{47}M_5$

$+ M_{57}M_4-2M_4M_5M_7]-K_{234}[2M_{57}M_5+M_{53}M_7-2M_5^2M_7]-C_{234}[M_{56}M_7+M_{57}M_6+M_{67}M_5$

$- 2M_5M_6M_7]-K_{215}[M_{15}M_8+M_{18}M_5+M_{58}M_1-2M_1M_5M_8]-C_{215}[M_{25}M_8+M_{28}M_5+M_{58}M_2$

$- 2M_2M_5M_8]-K_{225}[M_{35}M_8+M_{38}M_5+M_{58}M_3-2M_3M_5M_8]-C_{225}[M_{45}M_8+M_{48}M_5+M_{58}M_4$

$- 2M_4M_5M_8] - K_{235}[2M_{58}M_5 + M_{55}M_8 - 2M_5^2M_8] - C_{235}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5$

$- 2M_5M_6M_8]-K_{216}[M_{15}M_9+M_{19}M_5+M_{59}M_1-2M_1M_5M_9]-C_{216}[M_{25}M_9+M_{29}M_5+M_{59}M_2$

$- 2M_2M_5M_9]-K_{226}[M_{35}M_9+M_{39}M_5+M_{59}M_3-2M_3M_5M_9]-C_{226}[M_{45}M_9+M_{49}M_5+M_{59}M_4$

$- 2M_4M_5M_9]-K_{236}[2M_{59}M_5+M_{55}M_9-2M_5^2M_9]-C_{236}[M_{56}M_9+M_{59}M_6+M_{69}M_5 - 2M_5M_6M_9]$

$\dot{M}_{46} = 2\pi e_1^2 \Phi_{11}[F_{21}F_{31}-(F_{21}K_{311}+F_{31}K_{211})M_1-(F_{21}K_{321}+F_{31}K_{221})M_3-(F_{21}K_{331}$

$+ F_{31}K_{231})M_5+K_{211}K_{311}M_{11}+K_{221}K_{321}M_{33}+K_{231}K_{331}M_{55}+(K_{211}K_{321}+K_{221}K_{311})$

$M_{13}+(K_{211}K_{331}+K_{311}K_{231})M_{15}+(K_{221}K_{331}+K_{231}K_{321})M_{35}]+ 2\pi e_2^2 \Phi_{22}[F_{22}F_{32}$

$- (F_{22}K_{312} + K_{32}K_{212})M_1 - (F_{22}K_{322} + F_{32}K_{222})M_3 - (F_{22}K_{332} + F_{32}K_{232})M_5$

$+ K_{212}K_{312}M_{11} + K_{222}K_{322}M_{33} + K_{232}K_{332}M_{55} + (K_{212}K_{322} + K_{222}K_{312})M_{13}$

$+ (K_{212}K_{332} + K_{312}K_{232})M_{15} + (K_{222}K_{332} + K_{232}K_{322})M_{35}] + 2\pi e_3^2 \Phi_{33}[F_{23}F_{33}$

$- (F_{23}K_{313} + F_{33}K_{213})M_1 - (F_{23}K_{323} + F_{33}K_{223})M_3 - (F_{23}K_{333} + F_{33}K_{233})M_5$

$+ K_{213}K_{313}M_{11} + K_{223}K_{323}M_{33} + K_{233}K_{333}M_{55} + (K_{213}K_{323} + K_{223}K_{313})M_{13}$

$+ (K_{213}K_{333} + K_{313}K_{233})M_{15} + (k_{223}K_{333} + K_{233}K_{323})M_{35}] - K_{210}M_{16} - C_{210}M_{26}$

$- K_{220}M_{36} - C_{220}M_{46} - K_{230}M_{56} - C_{230}M_{66} + F_{24}M_{67} + F_{25}M_{68} + F_{26}M_{69} - K_{214}$

$[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7]-C_{214}[M_{26}M_7+M_{27}M_6+M_{67}M_2-2M_2M_6M_7]$

$- K_{224}[M_{36}M_7+M_{37}M_6+M_{67}M_3-2M_3M_6M_7] \quad C_{224}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7]$

$- K_{234}[M_{56}M_7 + M_{57}M_6 + M_{67}M_5 - 2M_5M_6M_7] - C_{234}[2M_{67}M_6 + M_{66}M_7 - 2M_6^2M_7]$

$- K_{215}[M_{16}M_8+M_{18}M_6+M_{68}M_1-2M_1M_6M_8]-C_{215}[M_{26}M_8+M_{28}M_6+M_{68}M_2-2M_2M_6M_8]$

$- K_{225}[M_{36}M_8+M_{38}M_6+M_{68}M_3-2M_3M_6M_8]-C_{225}[M_{46}M_8+M_{48}M_6+M_{68}M_4-2M_4M_6M_8]$

$- K_{235}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5 - 2M_5M_6M_8] - C_{235}[2M_{68}M_6 + M_{66}M_8 - 2M_6^2M_8]$

$- K_{216}[M_{16}M_9 + M_{19}M_6 - 2M_1M_6M_9] - C_{216}[M_{26}M_9 + M_{29}M_6 + M_{69}M_2 - 2M_2M_6M_9]$

$$- K_{226}[M_{36}M_9+M_{39}M_6+M_{69}M_3-2M_3M_6M_9]-C_{226}[M_{46}M_9+M_{49}M_6+M_{69}M_4-2M_4M_6M_9]$$

$$- K_{236}[M_{56}M_9+M_{59}M_6+M_{69}M_5-2M_5M_6M_9]-C_{236}[2M_{69}M_6+M_{66}M_9-2M_6^2M_9]-K_{310}M_{14}$$

$$- C_{310}M_{24}-K_{320}M_{34}-C_{320}M_{44}-K_{330}M_{45}-C_{330}M_{46}+F_{34}M_{47}+F_{35}M_{48}+F_{36}M_{49}$$

$$- K_{314}[M_{14}M_7+M_{17}M_4+M_{47}M_1-2M_1M_4M_7]-C_{314}[M_{24}M_7+M_{27}M_4+M_{47}M_2-2M_2M_4M_7]$$

$$- K_{324}[M_{34}M_7 + M_{37}M_4 + M_{47}M_3 - 2M_3M_4M_7] - C_{324}[2M_{47}M_4 + M_{44}M_7 - 2M_4^2M_7]$$

$$- K_{334}[M_{45}M_7+M_{47}M_5+M_{57}M_4-2M_4M_5M_7]-C_{334}[M_{46}M_7+M_{47}M_6+M_{67}M_4-2M_4M_6M_7]$$

$$- K_{315}[M_{14}M_8+M_{18}M_4+M_{48}M_1-2M_1M_4M_8]-C_{315}[M_{24}M_8+M_{28}M_4+M_{48}M_2-2M_2M_4M_8]$$

$$- K_{325}[M_{34}M_8 + M_{38}M_4 + M_{48}M_3 - 2M_3M_4M_8] - C_{325}[2M_{48}M_4 + M_{44}M_8 - 2M_4^2M_8]$$

$$- K_{335}[M_{45}M_8+M_{48}M_5+M_{58}M_4-2M_4M_5M_8]-C_{325}[M_{46}M_8+M_{48}M_6+M_{68}M_4-2M_4M_6M_8]$$

$$- K_{316}[M_{14}M_9+M_{19}M_4+M_{49}M_1-2M_1M_4M_9]-C_{316}[M_{24}M_9+M_{29}M_4+M_{49}M_2-2M_2M_4M_9]$$

$$- K_{326}[M_{34}M_9 + M_{39}M_4 + M_{49}M_3 - 2M_3M_4M_9] - C_{326}[2M_{49}M_4 + M_{44}M_9 - 2M_4^2M_9]$$

$$- K_{336}[M_{45}M_9+M_{49}M_5+M_{59}M_4-2M_4M_5M_9]-C_{336}[M_{46}M_9+M_{49}M_6+M_{69}M_4-2M_4M_6M_9]$$

$$\dot{M}_{47} = -\alpha_1 M_{47}-K_{210}M_{17}-C_{210}M_{27}-K_{220}M_{37}-C_{220}M_{47}-K_{230}M_{57}-C_{230}M_{67}+F_{24}M_{77}+F_{25}M_{78}$$

$$+ F_{26}M_{79} - K_{214}[2M_{17}M_7 + M_{77}M_1 - 2M_1M_7^2] - C_{214}[2M_{27}M_7 + M_{77}M_2 - 2M_7^2M_2]$$

$$- K_{224}[2M_{37}M_7 + M_{77}M_3 - 2M_7^2M_3]- C_{224}[2M_{47}M_7 + M_{77}M_4 -2M_7^2M_4] - K_{234}[2M_{57}M_7$$

$$+ M_{77}M_5-2M_7^2M_5]-C_{234}[2M_{67}M_7+M_{77}M_6-2M_7^2M_6]-K_{215}[M_{17}M_8+M_{18}M_7+M_{78}M_1-2M_1M_7M_8]$$

$$- C_{215}[M_{27}M_8+M_{28}M_7+M_{78}M_2-2M_2M_7M_8]-K_{225}[M_{37}M_8+M_{38}M_7+M_{78}M_3-2M_3M_7M_8]$$

$$- C_{225}[M_{47}M_8+M_{48}M_7+M_{78}M_4-2M_4M_7M_8]-K_{235}[M_{57}M_8+M_{58}M_7+M_{78}M_5-2M_5M_7M_8]$$

$$- C_{235}[M_{67}M_8+M_{68}M_7+M_{78}M_6-2M_6M_7M_8]-K_{216}[M_{17}M_9+M_{19}M_7+M_{79}M_1-2M_1M_7M_9]$$

$$- C_{216}[M_{27}M_9+M_{29}M_7+M_{79}M_2-2M_2M_7M_9]-K_{226}[M_{37}M_9+M_{39}M_7+M_{79}M_3-2M_3M_7M_9]$$

$$- C_{226}[M_{47}M_9+M_{49}M_7+M_{79}M_4-2M_4M_7M_9]-K_{236}[M_{57}M_9+M_{59}M_7+M_{79}M_5-2M_5M_7M_9]$$

$$- C_{236}[M_{67}M_9+M_{69}M_7+M_{79}M_6-2M_6M_7M_9]$$

$$\dot{M}_{48} = -\alpha_2 M_{48}-K_{210}M_{18}-C_{210}M_{28}-K_{220}M_{38}-C_{220}M_{48}-K_{230}M_{58}-C_{230}M_{68}+F_{24}M_{78}+F_{25}M_{88}+F_{26}M_{89}$$

$$- K_{214}[M_{17}M_8+M_{18}M_7+M_{78}M_1-2M_1M_7M_8]-C_{214}[M_{27}M_8+M_{28}M_7+M_{78}M_2-2M_2M_7M_8]$$

$$- K_{224}[M_{37}M_8+M_{38}M_7+M_{78}M_3-2M_3M_7M_8]-C_{224}[M_{47}M_8+M_{48}M_7+M_{78}M_4-2M_4M_7M_8]$$

$$- K_{234}[M_{57}M_8+M_{58}M_7+M_{78}M_5-2M_5M_7M_8]-C_{234}[M_{67}M_8+M_{68}M_7+M_{78}M_6-2M_6M_7M_8]$$

$$- K_{215}[2M_{18}M_8+M_{88}M_1-2M_8^2M_1] - C_{215}[2M_{28}M_8+M_{88}M_2-2M_8^2M_2]-K_{225}[2M_{38}M_8 + M_{88}M_3$$

$$- 2M_8^2M_3]-C_{225}[2M_{48}M_8+M_{88}M_4-2M_8^2M_4]-K_{235}[2M_{58}M_8+M_{88}M_5-2M_8^2 M_5]-C_{235}$$

$$[2M_{68}M_8+M_{88}M_6-2M_8^2M_6]-K_{216}[M_{18}M_9+M_{19}M_8+M_{89}M_1-2M_1M_8M_9]-C_{216}[M_{28}M_9$$

$$+ M_{29}M_8+M_{89}M_2-2M_2M_8M_9]-K_{226}[M_{38}M_9+M_{39}M_8+M_{89}M_3-2M_3M_8M_9]-C_{226}[M_{48}M_9$$

$$+ M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9] - K_{236}[M_{58}M_9 + M_{59}M_8 + M_{89}M_5 - 2M_5M_8M_9] - C_{236}$$

$$[M_{68}M_9 + M_{69}M_8 + M_{89}M_6 - 2M_6M_8M_9]$$

$$\dot{M}_{49}= -\alpha_3M_{49}-K_{210}M_{19}-C_{210}M_{29}-K_{220}M_{39}-C_{220}M_{49}-K_{320}M_{59}-C_{230}M_{69}+F_{24}M_{79}+F_{25}M_{89}+F_{26}M_{99}$$

$$- K_{214}[M_{17}M_9+M_{19}M_7+M_{79}M_1-2M_1M_7M_9]-C_{214}[M_{27}M_9+M_{29}M_7+M_{79}M_2-2M_2M_7M_9]$$

$$- K_{224}[M_{37}M_9+M_{39}M_7+M_{79}M_3-2M_3M_7M_9]-C_{224}[M_{47}M_9+M_{49}M_7+M_{79}M_4- 2M_4M_7M_9]$$

$$- K_{234}[M_{57}M_9+M_{59}M_7+M_{79}M_5 - 2M_5M_7M_9] - C_{234}[M_{67}M_9 + M_{69}M_7 + M_{79}M_6 - 2M_6M_7M_9]$$

$$- K_{215}[M_{18}M_9+M_{19}M_8+M_{89}M_1 - 2M_1M_8M_9] - C_{215}[M_{28}M_9 + M_{29}M_8 + M_{89}M_2 - 2M_2M_8M_9]$$

$$- K_{225}[M_{38}M_9+M_{39}M_8+M_{89}M_3 - 2M_3M_8M_9] - C_{225}[M_{48}M_9 + M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9]$$

$$- K_{235}[M_{58}M_9+M_{59}M_8+M_{89}M_5 - 2M_5M_8M_9] - C_{235}[M_{68}M_9 + M_{69}M_8 + M_{89}M_6 - 2M_6M_8M_9]$$

$$- K_{216}[2M_{19}M_9+M_{99}M_1-2M_9^2M_1]-C_{216}[2M_{29}M_9+M_{99}M_2-2M_9^2M_2]-K_{226}[2M_{39}M_9+M_{99}M_3-2M_9^2M_3]$$

$$- C_{226}[2M_{49}M_9+M_{99}M_4-2M_9^2M_4]-K_{236}[2M_{59}M_9+M_{99}M_5-2M_9^2M_5]-C_{236}[2M_{69}M_9+M_{99}M_6- 2M_9^2M_6]$$

$$\dot{M}_{55}= 2M_{56}$$

$$\dot{M}_{56}= M_{66}-K_{310}M_{15}-C_{310}M_{25}-K_{320}M_{35}-C_{320}M_{45}-K_{330}M_{55}-C_{330}M_{56}+F_{34}M_{57}+F_{35}M_{58}+F_{36}M_{59}$$

$$- K_{314}[M_{15}M_7+M_{17}M_5+M_{57}M_1 - 2M_1M_5M_7] - C_{314}[M_{25}M_7 + M_{27}M_5 + M_{57}M_2 - 2M_2M_5M_7]$$

$$- K_{324}[M_{35}M_7+M_{37}M_5+M_{57}M_3 - 2M_3M_5M_7] - C_{324}[M_{45}M_7 + M_{47}M_5 + M_{57}M_4 - 2M_4M_5M_7]$$

$$- K_{334}[2M_{57}M_5 + M_{55}M_7 - 2M_5^2M_7] - C_{334}[M_{56}M_7 + M_{57}M_6 + M_{67}M_5 - 2M_5M_6M_7]$$

$$- K_{315}[M_{15}M_8+M_{18}M_5+M_{58}M_1 - 2M_1M_5M_8] - C_{315}[M_{25}M_8 + M_{28}M_5 + M_{58}M_2 - 2M_2M_5M_8]$$

$$- K_{325}[M_{35}M_8+M_{38}M_5+M_{58}M_3 - 2M_3M_5M_8] - C_{325}[M_{45}M_8 + M_{48}M_5 + M_{58}M_4 - 2M_4M_5M_8]$$

$$- K_{335}[2M_{58}M_5+M_{55}M_8-2M_5^2M_8] - C_{335}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5 - 2M_5M_6M_8]$$

$$- K_{316}[M_{15}M_9+M_{19}M_5+M_{59}M_1-2M_1M_5M_9]-C_{316}[M_{25}M_9+M_{29}M_5+M_{59}M_2-2M_2M_5M_9]$$

$$- K_{326}[M_{35}M_9+M_{39}M_5+M_{39}M_3 - 2M_3M_5M_9] - C_{326}[M_{45}M_9 + M_{49}M_5 + M_{59}M_4 - 2M_4M_5M_9]$$

$$- K_{336}[2M_{59}M_5 + M_{55}M_9 - 2M_5^2M_9] - C_{336}[M_{56}M_9 + M_{59}M_6 + M_{69}M_5 - 2M_5M_6M_9]$$

$$\dot{M}_{57}= M_{67} - \alpha_1M_{57}$$

$$\dot{M}_{58} = M_{68} - \alpha_2 M_{58}$$

$$\dot{M}_{59} = M_{69} - \alpha_3 M_{59}$$

$$\dot{M}_{66} = 2\pi e_1^2 \Phi_{11}[F_{31}^2 - 2F_{31}K_{311}M_1 - 2F_{31}K_{321}M_3 - 2F_{31}K_{331}M_5 + K_{311}^2 M_{11} + K_{321}^2 M_{33} + K_{331}^2 M_{55}$$
$$+ 2K_{311}K_{321}M_{13} + 2K_{311}K_{331}M_{15} + 2K_{321}K_{331}M_{15}] + 2\pi e_2^2 \Phi_{22}[F_{32}^2 - 2F_{32}K_{312}M_1$$
$$- 2F_{32}K_{322}M_3 - 2F_{32}K_{332}M_5 + K_{312}^2 M_{11} + K_{322}^2 M_{33} + K_{332}^2 M_{55} + 2K_{312}K_{322}M_{13}$$
$$+ 2K_{312}K_{332}M_{15} + 2K_{322}K_{332}M_{35}] + 2\pi e_3^2 \Phi_{33}[F_{33}^2 - 2F_{33}K_{313}M_1 - 2F_{33}K_{323}M_3$$
$$- 2F_{33}K_{332}M_5 + K_{313}^2 M_{11} + K_{323}^2 M_{33} + K_{333}^2 M_{55} + 2K_{313}K_{323}M_{13} + 2K_{313}K_{333}M_{15}$$
$$+ 2K_{323}K_{333}M_{35}] + 2\{-K_{310}M_{16} - C_{310}M_{26} - K_{320}M_{36} - C_{320}M_{46} - K_{330}M_{56}$$
$$- C_{330}M_{66} + F_{34}M_{67} + F_{35}M_{68} + F_{36}M_{69} - K_{314}[M_{16}M_7 + M_{17}M_6 + M_{67}M_1 - 2M_1M_6M_7]$$
$$- C_{314}[M_{26}M_7 + M_{27}M_6 + M_{67}M_2 - 2M_2M_6M_7] - K_{324}[M_{36}M_7 + M_{37}M_6 + M_{67}M_3 - 2M_3M_6M_7]$$
$$- C_{324}[M_{46}M_7 + M_{47}M_6 + M_{67}M_4 - 2M_4M_6M_7] - K_{334}[M_{56}M_7 + M_{57}M_6 + M_{67}M_5 - 2M_5M_6M_7]$$
$$- C_{334}[2M_{67}M_6 + M_{66}M_7 - 2M_6^2M_7] - K_{315}[M_{16}M_8 + M_{18}M_6 + M_{68}M_1 - 2M_1M_6M_8]$$
$$- C_{315}[M_{26}M_8 + M_{28}M_6 + M_{68}M_2 - 2M_2M_6M_8] - K_{325}[M_{36}M_8 + M_{38}M_6 + M_{68}M_3 - 2M_3M_6M_8]$$
$$- C_{325}[M_{46}M_8 + M_{48}M_6 + M_{68}M_4 - 2M_4M_6M_8] - K_{335}[M_{56}M_8 + M_{58}M_6 + M_{68}M_5 - 2M_5M_6M_8]$$
$$- C_{335}[2M_{68}M_6 + M_{66}M_8 - 2M_6^2M_8] - K_{316}[M_{16}M_9 + M_{19}M_6 + M_{69}M_1 - 2M_1M_6M_9]$$
$$- C_{316}[M_{26}M_9 + M_{29}M_6 + M_{69}M_2 - 2M_2M_6M_9] - K_{326}[M_{36}M_9 + M_{39}M_6 + M_{69}M_3 - 2M_3M_6M_9]$$
$$- C_{326}[M_{46}M_9 + M_{49}M_6 + M_{69}M_4 - 2M_4M_6M_9] - K_{336}[M_{56}M_9 + M_{59}M_6 + M_{69}M_5 - 2M_5M_6M_9]$$
$$- C_{336}[2M_{69}M_6 + M_{66}M_9 - 2M_6^2M_9]\}$$

$$\dot{M}_{67} = -\alpha_1 M_{67} - K_{310}M_{17} - C_{310}M_{27} - K_{320}M_{37} - C_{320}M_{47} - K_{330}M_{57} - C_{330}M_{67} + F_{34}M_{77} + F_{35}M_{78}$$
$$+ F_{36}M_{79} - K_{314}[2M_{17}M_7 + M_{77}M_1 - 2M_7^2M_1] - C_{314}[2M_{27}M_7 + M_{77}M_2 - 2M_7^2M_2] - K_{324}[2M_{37}M_7$$
$$+ M_{77}M_3 - 2M_7^2M_3] - C_{324}[2M_{47}M_7 + M_{77}M_4 - 2M_7^2M_4] - K_{334}[2M_{57}M_7 + M_{77}M_5 - 2M_7^2M_5]$$
$$- C_{334}[2M_{67}M_7 + M_{77}M_6 - 2M_7^2M_6] - K_{315}[M_{17}M_8 + M_{18}M_7 + M_{78}M_1 - 2M_1M_7M_8]$$
$$- C_{315}[M_{27}M_8 + M_{28}M_7 + M_{78}M_2 - 2M_2M_7M_8] - K_{325}[M_{37}M_8 + M_{38}M_7 + M_{78}M_3 - 2M_3M_7M_8]$$
$$- C_{325}[M_{47}M_8 + M_{48}M_7 + M_{78}M_4 - 2M_4M_7M_8] - K_{335}[M_{57}M_8 + M_{58}M_7 + M_{78}M_5 - 2M_5M_7M_8]$$
$$- C_{335}[M_{67}M_8 + M_{68}M_7 + M_{78}M_6 - 2M_6M_7M_8] - K_{316}[M_{17}M_9 + M_{19}M_7 + M_{79}M_1 - 2M_1M_7M_9]$$
$$- C_{316}[M_{27}M_9 + M_{29}M_7 + M_{79}M_2 - 2M_2M_7M_9] - K_{326}[M_{37}M_9 + M_{39}M_7 + M_{79}M_3 - 2M_3M_7M_9]$$
$$- C_{326}[M_{47}M_9 + M_{49}M_7 + M_{79}M_4 - 2M_4M_7M_9] - K_{336}[M_{57}M_9 + M_{59}M_7 + M_{79}M_5 - 2M_5M_7M_9]$$

$$- C_{336}[M_{67}M_9 + M_{69}M_7 + M_{79}M_6 - 2M_6M_7M_9]$$

$$\dot{M}_{68} = -\alpha_2 M_{68} - K_{310}M_{18} - C_{310}M_{28} - K_{320}M_{38} - C_{320}M_{48} - K_{330}M_{58} - C_{330}M_{68} + F_{34}M_{78} + F_{35}M_{88}$$

$$+ F_{36}M_{89} - K_{314}[M_{17}M_8 + M_{18}M_7 + M_{78}M_1 - 2M_1M_7M_8] - C_{314}[M_{27}M_8 + M_{28}M_7 + M_{78}M_2$$

$$- 2M_2M_7M_8] - K_{324}[M_{37}M_8 + M_{38}M_7 + M_{78}M_3 - 2M_3M_7M_8] - C_{324}[M_{47}M_8 + M_{48}M_7 + M_{78}M_4$$

$$- 2M_4M_7M_8] - K_{334}[M_{57}M_8 + M_{58}M_7 + M_{78}M_5 - 2M_5M_7M_8] - C_{334}[M_{67}M_8 + M_{68}M_7 + M_{78}M_6$$

$$- 2M_6M_7M_8] - K_{315}[2M_{18}M_8 + M_{88}M_1 - 2M_8^2M_1] - C_{315}[2M_{28}M_8 + M_{88}M_2 - 2M_8^2M_2] - K_{325}[2M_{38}M_8$$

$$+ M_{88}M_3 - 2M_8^2M_3] - C_{325}[2M_{48}M_8 + M_{88}M_4 - 2M_8^2M_4] - K_{335}[2M_{58}M_8 + M_{88}M_5 - 2M_8^2M_5]$$

$$- C_{335}[2M_{68}M_8 + M_{88}M_6 - 2M_8^2M_6] - K_{316}[M_{18}M_9 + M_{19}M_8 + M_{89}M_1 - 2M_1M_8M_9]$$

$$- C_{316}[M_{28}M_9 + M_{29}M_8 + M_{89}M_2 - 2M_2M_8M_9] - K_{326}[M_{38}M_9 + M_{39}M_8 + M_{89}M_3 - 2M_3M_8M_9]$$

$$- C_{326}[M_{48}M_9 + M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9] - K_{336}[M_{58}M_9 + M_{59}M_8 + M_{89}M_5 - 2M_5M_8M_9]$$

$$- C_{336}[M_{68}M_9 + M_{69}M_8 + M_{89}M_6 - 2M_6M_8M_9]$$

$$\dot{M}_{69} = -\alpha_3 M_{69} - K_{310}M_{19} - C_{310}M_{29} - K_{320}M_{39} - C_{320}M_{49} - K_{330}M_{59} - C_{330}M_{69} + F_{34}M_{79} + F_{35}M_{89} + F_{36}M_{99}$$

$$- K_{314}[M_{17}M_9 + M_{19}M_7 + M_{79}M_1 - 2M_1M_7M_9] - C_{314}[M_{27}M_9 + M_{29}M_7 + M_{79}M_2 - 2M_2M_7M_9]$$

$$- K_{324}[M_{37}M_9 + M_{39}M_7 + M_{79}M_3 - 2M_3M_7M_9] - C_{324}[M_{47}M_9 + M_{49}M_7 + M_{79}M_4 - 2M_4M_7M_9]$$

$$- K_{334}[M_{57}M_9 + M_{59}M_7 + M_{79}M_5 - 2M_5M_7M_9] - C_{334}[M_{67}M_9 + M_{69}M_7 + M_{79}M_6 - 2M_6M_7M_9]$$

$$- K_{315}[M_{18}M_9 + M_{19}M_8 + M_{89}M_1 - 2M_1M_8M_9] - C_{315}[M_{28}M_9 + M_{29}M_8 + M_{89}M_2 - 2M_2M_8M_9]$$

$$- K_{325}[M_{38}M_9 + M_{39}M_8 + M_{89}M_3 - 2M_3M_8M_9] - C_{325}[M_{48}M_9 + M_{49}M_8 + M_{89}M_4 - 2M_4M_8M_9]$$

$$- K_{335}[M_{58}M_9 + M_{59}M_8 + M_{89}M_5 - 2M_5M_8M_9] - C_{335}[M_{68}M_9 + M_{69}M_8 + M_{89}M_6 - 2M_{65}M_8M_9]$$

$$- K_{316}[2M_{19}M_9 + M_{99}M_1 + 2M_9^2M_1] - C_{316}[2M_{29}M_9 + M_{99}M_2 - 2M_9^2M_2] - K_{326}[2M_{39}M_9 + M_{99}M_3 - 2M_9^2M_3]$$

$$- C_{326}[2M_{49}M_9 + M_{99}M_4 - 2M_9^2M_4] - K_{336}[2M_{59}M_9 + M_{99}M_5 - 2M_9^2M_5] - C_{336}[2M_{69}M_9 + M_{99}M_6 - 2M_9^2M_6]$$

$$\dot{M}_{77} = -2\alpha_1 M_{77} + 2\pi\, \Phi_{44}$$

$$\dot{M}_{78} = -(\alpha_1 + \alpha_2)M_{78}$$

$$\dot{M}_{79} = -(\alpha_1 + \alpha_3)M_{79}$$

$$\dot{M}_{88} = -2\alpha_2 M_{88} + 2\pi\, \Phi_{55}$$

$$\dot{M}_{89} = -(\alpha_2 + \alpha_3)M_{89}$$

$$\dot{M}_{99} = -2\alpha_3 M_{99} + 2\pi\, \Phi_{66} \qquad \text{(C-2)}$$

Appendix D

Comparison of Equilibrium Solutions

In this appendix, the equilibrium solutions given in Chapter 3 are compared with those obtained by other more precise methods. Restituting the nonlinear terms into the equations for the flap-lag-torsional motion, Eqs. (3-39), (3-40) and (3-41), we obtain

$$\beta_e'' + 2\beta_e \zeta_e' - \overline{K}_{\beta\beta}\beta_{pc} + (\overline{K}_{\beta\beta} + 1)\beta_e + \overline{K}_{\beta\zeta}\zeta_e + \frac{3}{2}\,\overline{g}\,C\beta_e - \frac{3}{2}\,\overline{g}\,S\beta_e\zeta_e$$

$$+ \frac{\gamma B^2}{4}\mu(\lambda_i - \lambda)\, S\zeta_e + \frac{\gamma B^3}{6}\mu S\beta_e - \frac{\gamma B^3}{6}\mu C\beta_e' + \frac{\gamma B^4}{8}\beta_e' - \frac{\gamma B^4}{8}\lambda k S\zeta_e$$

$$+ \frac{\gamma B^4}{8}\lambda k C\zeta_e' + \frac{\gamma B^3}{6}(\lambda_i - \lambda)\zeta_e' - \frac{\gamma B^5}{10}\alpha_e + \frac{\gamma B^4}{4}\mu C\alpha_e$$

$$= \frac{\gamma B^4}{8}\theta_e + \frac{\gamma B^2}{4}\mu C(\lambda_i - \lambda) - \frac{\gamma B^3}{6}(\lambda_i - \lambda) - \frac{\gamma B^4}{8}\lambda k C - \frac{\gamma B^3}{3}\mu C\theta_e \qquad \text{(D-1)}$$

$$\zeta_e'' - 2\beta_e(1 + \zeta_e')\beta_e' + \overline{K}_{\zeta\zeta}\zeta_e + \overline{K}_{\beta\zeta}(\beta_e - \beta_{pc}) + \frac{3}{2}\,\overline{g}S - \frac{3}{2}\,\overline{g}\,C\zeta_e$$

$$+ \frac{\gamma B^3}{3}\lambda k S(\lambda_i - \lambda)\zeta_e - \frac{\gamma B^2}{2}\mu S(\lambda_i - \lambda)\beta_e - \frac{\gamma B^3}{3}(\lambda_i - \lambda)\beta_e' - \frac{\gamma B^4}{4}\lambda k\, C\beta_e'$$

$$- \frac{\gamma B^3}{6}\mu C(\lambda_i - \lambda)\alpha_e + \frac{\gamma B^4}{8}(\lambda_i - \lambda)\alpha_e + \frac{\gamma B^5}{10}\lambda k C\alpha_e$$

$$= \frac{\gamma}{2}\Big[\frac{B^2}{2}\mu C(\lambda_i - \lambda)\theta_e - \frac{B^3}{3}(\lambda_i - \lambda)\theta_e - \frac{B^4}{4}\lambda k\, C\theta_e + \frac{B^2}{2}(\lambda_i - \lambda)^2$$

$$+ \frac{2B^3}{3}\lambda k C(\lambda_i - \lambda)\Big] \qquad \text{(D-2)}$$

$$\alpha_e'' + \omega_\alpha^2 \alpha_e - \frac{\gamma F B^3}{3} \mu C \alpha_e' + \frac{\gamma F B^4}{4} \alpha_e' - \frac{\gamma F B^2}{2} \mu C \beta_e + \frac{\gamma F B^3}{3} \beta_e = 0 \qquad \text{(D-3)}$$

Those nonlinear terms are $2\beta_e \zeta_e' - \frac{3}{2} \bar{g} S \beta_e \zeta_e$ in Eq. (D-1) and $-2\beta_e(1 + \zeta_e')\beta_e'$ in Eq. (D-2).

The system of Eqs. (D-1), (D-2) and (D-3), can be replaced by six first order differential equations by letting $x^T = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6] = [\beta_e\ \beta_e'\ \zeta_e\ \zeta_e'\ \alpha_e\ \alpha_e']$. They are

$$x_1' = x_2$$

$$x_2' = \frac{\gamma B^4}{8} \theta_e - \frac{\gamma B^3}{3} \mu C \theta_e + \frac{\gamma B^2}{4} \mu C (\lambda_i - \lambda) - \frac{\gamma B^3}{6} (\lambda_i - \lambda) - \frac{\gamma B^4}{8} \lambda k C$$

$$- 2x_1 x_4 + \bar{K}_{\beta\beta} \beta_{pc} - (\bar{K}_{\beta\beta} + 1) x_1 - \bar{K}_{\beta\zeta} x_3 - \frac{3}{2} \bar{g} C\, x_1 + \frac{3}{2} \bar{g} S x_1 x_3$$

$$- \frac{\gamma B^2}{4} \mu (\lambda_i - \lambda)\ S x_3 - \frac{\gamma B^3}{6} \mu S x_1 + \frac{\gamma B^3}{6} \mu C x_2 - \frac{\gamma B^4}{8} x_2 + \frac{\gamma B^4}{8} \lambda k S x_3$$

$$- \frac{\gamma B^4}{8} \lambda k C x_4 - \frac{\gamma B^3}{6} (\lambda_i - \lambda)\ x_4 + \frac{\gamma B^5}{10} x_5 - \frac{\gamma B^4}{4} \mu C x_5$$

$$x_3' = x_4$$

$$x_4' = \frac{\gamma}{2} [\frac{B^2}{2} \mu C (\lambda_i - \lambda)\, \theta_e - \frac{B^3}{3} (\lambda_i - \lambda)\, \theta_e - \frac{B^4}{4} \lambda k C \theta_e + \frac{B^2}{2} (\lambda_i - \lambda)^2$$

$$+ \frac{2B^3}{3} \lambda k C (\lambda_i - \lambda)] + 2x_1 (1 + x_4) x_2 - \bar{K}_{\zeta\zeta} x_3 - \bar{K}_{\beta\zeta} (x_1 - \beta_{pc})$$

$$- \frac{3}{2}\bar{g}S + \frac{3}{2}\bar{g}Cx_3 - \frac{\gamma B^3}{3}\lambda kS(\lambda_i-\lambda)x_3 + \frac{\gamma B^2}{2}\mu S(\lambda_i-\lambda)x_1$$

$$+ \frac{\gamma B^3}{3}(\lambda_i-\lambda)x_2 + \frac{\gamma B^4}{4}\lambda kCx_2 + \frac{\gamma B^3}{6}\mu C(\lambda_i-\lambda)x_5 - \frac{\gamma B^4}{8}(\lambda_i-\lambda)x_5$$

$$- \frac{\gamma B^5}{10}\lambda kCx_5$$

$$x_5' = x_6$$

$$x_6' = -\omega_\alpha^2 x_5 + \frac{\gamma FB^3}{3}\mu Cx_6 - \frac{\gamma FB^4}{4}x_6 + \frac{\gamma FB^2}{2}\mu Cx_1 - \frac{\gamma FB^3}{3}x_1 \qquad \text{(D-4)}$$

Eq. (D-4) can be solved numerically for a given parameter combination, using a suitable computer routine; e.g. DGEAR routine of IMSL computer software.

Another alternative to finding the equilibrium solutions is similar to that used in Chapter 3, except that the nonlinear terms are retained in Eqs. (D-1) and (D-2). By substituting Eq. (3-42) into Eqs. (D-1), (D-2) and (D-3), and equating the coefficients of cos $\alpha\psi$, cos ψ and sin ψ, we obtain a system of nine algebraic equations which can be expressed as the summation of Eq. (3-43) and some additional terms due to the nonlinearity; namely,

$$D\vec{X} + N = H \qquad \text{(D-5)}$$

where $\vec{X}$, D and H are the same as Eq. (3-43). N is a nine by one column matrix of nonlinear terms. The nonzero components of N are

$$n_1 = -\beta_S\zeta_C + \beta_C\zeta_S - \frac{3}{4}\bar{g}\beta_0\zeta_S - \frac{3}{4}\bar{g}\beta_S\zeta_0$$

$$n_2 = -2\beta_0\zeta_C - \frac{3}{2}\bar{g}\beta_0\zeta_0 - \frac{9}{8}\bar{g}\beta_S\zeta_S - \frac{3}{8}\bar{g}\beta_C\zeta_C$$

$$n_3 = 2\beta_0\zeta_S - \frac{3}{8}\bar{g}\beta_S\zeta_C - \frac{3}{8}\bar{g}\beta_C\zeta_S$$

$$n_4 = -\beta_0\beta_C\zeta_C - \beta_0\beta_S\zeta_S$$

$$n_5 = 2\beta_0\beta_C - \beta_S\beta_C\zeta_C + \frac{1}{2}\beta_C^2\zeta_S - \frac{1}{2}\beta_S^2\zeta_S$$

$$n_6 = -\frac{1}{2}\beta_C^2\zeta_C - \beta_S\beta_C\zeta_S - 2\beta_0\beta_S + \frac{1}{2}\beta_S^2\zeta_C \qquad \text{(D-6)}$$

Eq. (D-5) has the same form as that studied by Klotter [49]. He utilized the orthogonality of functions $\cos 0\psi$, $\cos\psi$ and $\sin\psi$ and suggested that for the first approximation the nine algebraic equations can be decomposited into three sets of three algebraic equations multiplying each of the orthogonal functions and integrating over the period 2π.

The system of Eq. (D-5) also can be solved numerically without being simplified to three separate sets.

For the comparison of the equilibrium solutions, the plot of periodic solutions obtained by the two methods mentioned above and the one used in Chapter 3, are presented. Because of a large number of parameters involved, the comparison is limited to those parameter values given in Chapter 6. Fig. D-1 shows that the equilibrium solutions obtained from Eq. (3-43) and Eq. (D-5) respectively are almost identical, and are only slightly different from those obtained from the original Eqs. (D-1), (D-2) and (D-3). These results indicate that the nonlinear terms have little effect on the equilibrium solutions and that the higher harmonic components in the exact solutions are negligible. Therefore, the linearized harmonic balancing method is an adequate approach in the present case.

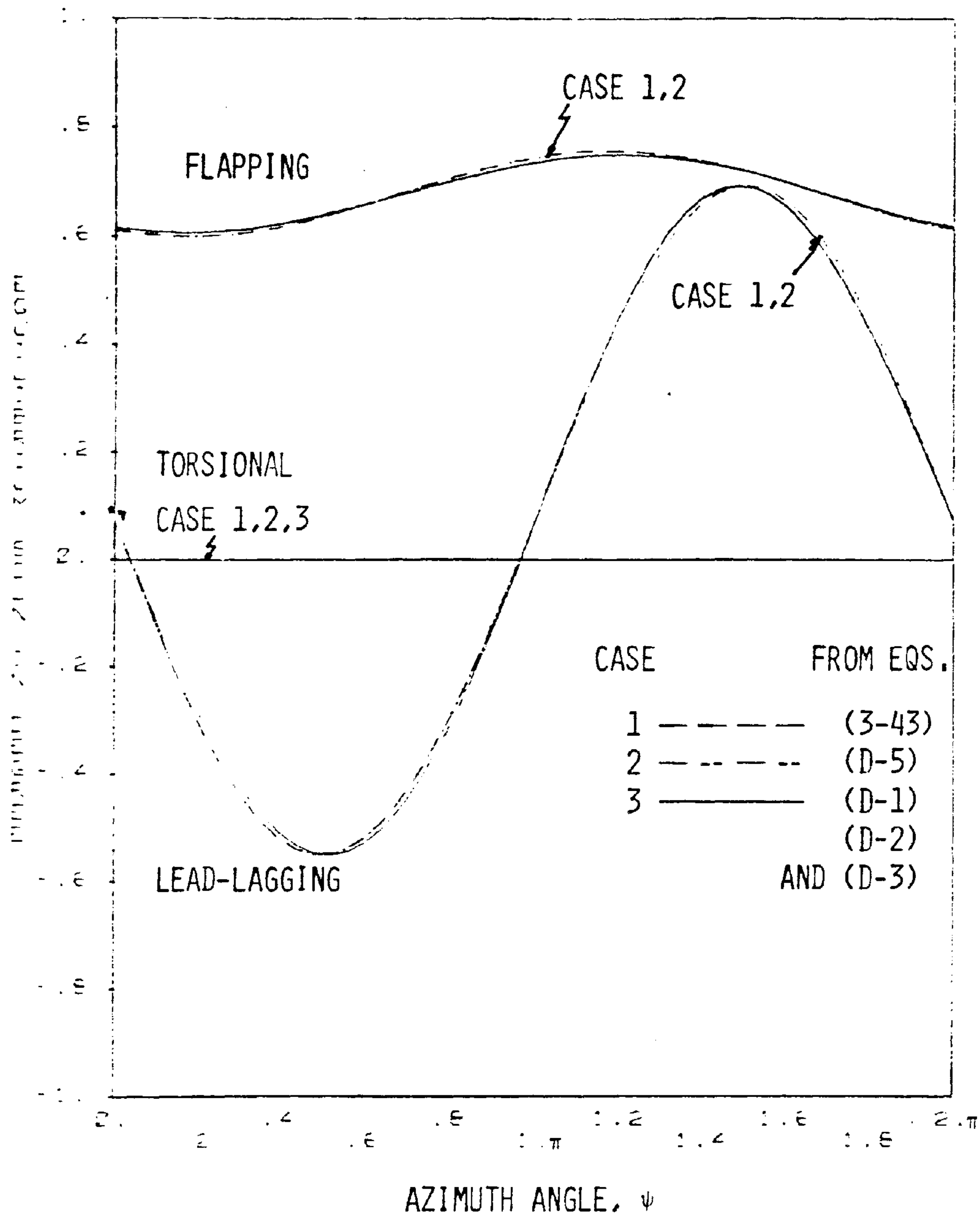

Figure D-1 Comparison of Equilibrium Solutions Obtained from Different Methods.

Appendix E

Linearized Moment Equations for Stability Analyses in the Sense of Bolotin

In this appendix, the detailed moment equations linearized according to the Bolotin-Owen procedure are given. In these equations, μ_i and μ_{ij} denote $E[X_i]$ and $E[(X_i-\mu_i)(X_j-\mu_j)]$ respectively; the overdot denotes one differentiation with respect to the non-dimensional time ψ. The moment equations for the stability analysis of flapping motion are

$$\dot{\mu}_1 = \mu_2$$

$$\dot{\mu}_2 = -K_{110}\mu_1 - C_{110}\mu_2 - K_{114}\mu_{13} - C_{114}\mu_{23} - K_{115}\mu_{14} - C_{115}\mu_{24}$$

$$\dot{\mu}_{11} = 2\mu_{12}$$

$$\dot{\mu}_{12} = \mu_{22} + F_{10}\mu_1 - K_{110}\mu_{11} - C_{110}\mu_{12} + F_{14}\mu_{13} + F_{15}\mu_{14} + F_{16}\mu_{15}$$

$$\dot{\mu}_{13} = \mu_{23} - \alpha_1\mu_{13}$$

$$\dot{\mu}_{14} = \mu_{24} - \alpha_2\mu_{14}$$

$$\dot{\mu}_{15} = \mu_{25} - \alpha_3\mu_{15}$$

$$\dot{\mu}_{22} = 2F_{10}\mu_2 - 2K_{110}\mu_{12} - 2C_{110}\mu_{22} + 2F_{14}\mu_{23} + 2F_{15}\mu_{24} + 2F_{16}\mu_{25} + 2\pi[K_{111}^2 e_1^2 \Phi_{11} + K_{112}^2 e_2^2 \Phi_{22}]\,\mu_{11}$$

$$\dot{\mu}_{23} = -K_{110}\mu_{13} - C_{110}\mu_{23} - \alpha_1\mu_{23} - \sigma_{33}(K_{114}\mu_1 + C_{114}\mu_2)$$

$$\dot{\mu}_{24} = -K_{110}\mu_{14} - C_{110}\mu_{24} - \sigma_{44}(K_{115}\mu_1 + C_{115}\mu_2) - \alpha_2\mu_{24}$$

$$\dot{\mu}_{25} = -K_{110}\mu_{15} - C_{110}\mu_{25} - \alpha_3\mu_{25} \qquad \text{(E-1)}$$

The moment equations for the stability analysis of coupled flap-lag are

$$
\begin{aligned}
\dot{\mu}_1 &= \mu_2 \\
\dot{\mu}_2 &= -K_{110}\mu_1 - C_{110}\mu_2 - K_{120}\mu_3 - C_{120}\mu_4 \\
&\quad - K_{114}\mu_{15} - C_{114}\mu_{25} - K_{124}\mu_{35} - C_{124}\mu_{45} \\
&\quad - K_{115}\mu_{16} - C_{115}\mu_{26} - K_{125}\mu_{36} - C_{125}\mu_{46} - K_{116}\mu_{17} \\
&\quad - C_{116}\mu_{27} - K_{126}\mu_{37} - C_{126}\mu_{47} \\
\dot{\mu}_3 &= \mu_4 \\
\dot{\mu}_4 &= -K_{210}\mu_1 - C_{210}\mu_2 - K_{220}\mu_3 - C_{220}\mu_4 \\
&\quad -K_{214}\mu_{15} - C_{214}\mu_{25} - K_{224}\mu_{35} - C_{224}\mu_{45} - K_{215}\mu_{16} \\
&\quad - C_{215}\mu_{26} - K_{225}\mu_{36} - C_{225}\mu_{46} - K_{216}\mu_{17} - C_{216}\mu_{27} \\
&\quad - K_{226}\mu_{37} - C_{226}\mu_{47} \\
\dot{\mu}_{11} &= 2\mu_{12} \\
\dot{\mu}_{12} &= \mu_{22} - K_{110}\mu_{11} - C_{110}\mu_{12} - K_{120}\mu_{13} - C_{120}\mu_{14} + F_{14}\mu_{15} + F_{15}\mu_{16} \\
&\quad + F_{16}\mu_{17} \\
\dot{\mu}_{13} &= \mu_{23} + \mu_{14} \\
\dot{\mu}_{14} &= \mu_{24} - K_{210}\mu_{11} - C_{210}\mu_{12} - K_{220}\mu_{13} - C_{220}\mu_{14} + F_{24}\mu_{15} + F_{25}\mu_{16} \\
&\quad + F_{26}\mu_{17} \\
\dot{\mu}_{15} &= \mu_{25} - \alpha_1\mu_{15} \\
\dot{\mu}_{16} &= \mu_{26} - \alpha_2\mu_{16} \\
\dot{\mu}_{17} &= \mu_{27} - \alpha_3\mu_{17} \\
\dot{\mu}_{22} &= -4\pi e_1^2 \Phi_{11}[F_{11}K_{111}\mu_1 + F_{11}K_{121}\mu_3] \\
&\quad - 4\pi e_2^2 \Phi_{22}[F_{12}K_{112}\mu_1 + F_{12}K_{122}\mu_3] - 4\pi e_3^2 \Phi_{33} \\
&\quad [F_{13}K_{113}\mu_1 + F_{13}K_{123}\mu_3] + 2\pi e_1^2 \Phi_{11}[K_{111}^2\mu_{11} + K_{121}^2\mu_{33} \\
&\quad + 2K_{111}K_{121}\mu_{13}] + 2\pi e_2^2 \Phi_{22}[K_{112}^2\mu_{11} + K_{122}^2\mu_{33} + 2K_{112}K_{122}\mu_{13}] \\
&\quad + 2\pi e_3^2 \Phi_{33}[K_{113}^2\mu_{11} + K_{123}^2\mu_{33} + 2K_{113}K_{123}\mu_{13}] - 2K_{110}\mu_{13}
\end{aligned}
$$

$$- 2C_{110}\mu_{22} - 2K_{120}\mu_{23} - 2C_{120}\mu_{24} + 2F_{14}\mu_{25} + 2F_{15}\mu_{26} + 2F_{16}\mu_{27}$$

$$\dot{\mu}_{23} = \mu_{24} - K_{110}\mu_{13} - C_{110}\mu_{23} - K_{120}\mu_{33} - C_{120}\mu_{34} + F_{14}\mu_{35} + F_{15}\mu_{36} + F_{16}\mu_{37}$$

$$\dot{\mu}_{24} = -2\pi e_1^2 \Phi_{11} [F_{11}K_{211}\mu_1 + F_{21}K_{111}\mu_1 + F_{11}K_{221}\mu_3 + F_{21}K_{121}\mu_3]$$
$$- 2\pi e_2^2 \Phi_{22}[F_{12}K_{212}\mu_1 + F_{22}K_{112}\mu_1 + F_{12}K_{222}\mu_3 + F_{22}K_{122}\mu_3] - 2\pi e_3^2 \Phi_{33} \cdot$$
$$[F_{13}K_{213}\mu_1 + F_{23}K_{113}\mu_1 + F_{13}K_{223}\mu_3 + F_{23}K_{123}\mu_3] + 2\pi e_1^2 \Phi_{11}$$
$$[K_{111}K_{211}\mu_{11} + K_{121}K_{221}\mu_{33} + K_{111}K_{221}\mu_{13} + K_{121}K_{211}\mu_{13}] + 2\pi e_2^2 \Phi_{22}$$
$$[K_{112}K_{212}\mu_{11} + K_{122}K_{222}\mu_{33} + K_{112}K_{222}\mu_{13} + K_{122}K_{212}\mu_{13}] + 2\pi e_2^2 \Phi_{33}$$
$$[K_{113}K_{213}\mu_{11} + K_{123}K_{223}\mu_{33} + K_{113}K_{223}\mu_{13} + K_{123}K_{213}\mu_{13}] - K_{110}\mu_{14}$$
$$- C_{110}\mu_{24} - K_{120}\mu_{34} - C_{120}\mu_{44} + F_{14}\mu_{45} + F_{15}\mu_{46} + F_{16}\mu_{47} - K_{210}\mu_{12}$$
$$- C_{120}\mu_{22} - K_{220}\mu_{23} - C_{220}\mu_{24} + F_{24}\mu_{25} + F_{25}\mu_{26} + F_{26}\mu_{27}$$

$$\dot{\mu}_{25} = -\alpha_1\mu_{25} - K_{110}\mu_{15} - C_{110}\mu_{25} - K_{120}\mu_{35} - C_{120}\mu_{45} - \sigma_{55}(K_{114}\mu_1 + C_{114}\mu_2 + K_{124}\mu_3 + C_{124}\mu_4)$$

$$\dot{\mu}_{26} = -\alpha_2\mu_{26} - K_{110}\mu_{16} - C_{110}\mu_{26} - K_{120}\mu_{36} - C_{120}\mu_{46} - \sigma_{66}(K_{115}M_1 + C_{115}M_2 + K_{125}\mu_3 + C_{125}\mu_4)$$

$$\dot{\mu}_{27} = -\alpha_3\mu_{27} - K_{110}\mu_{17} - C_{110}\mu_{27} - K_{120}\mu_{37} - C_{120}\mu_{47} - \sigma_{77}(K_{116}\mu_1 + C_{116}\mu_2 + K_{126}\mu_3 + C_{126}\mu_4)$$

$$\dot{\mu}_{33} = 2\mu_{34}$$

$$\dot{\mu}_{34} = \mu_{44} - K_{210}\mu_{13} - C_{210}\mu_{23} - K_{220}\mu_{33} - C_{220}\mu_{34} + F_{24}\mu_{35} + F_{25}\mu_{36} + F_{26}\mu_{37}$$

$$\dot{\mu}_{35} = \mu_{45} - \alpha_1\mu_{35}$$

$$\dot{\mu}_{36} = \mu_{46} - \alpha_2\mu_{36}$$

$$\dot{\mu}_{37} = \mu_{47} - \alpha_3\mu_{37}$$

$$\dot{\mu}_{44} = -4\pi e_1^2 \Phi_{11}[F_{21}K_{211}\mu_1 + F_{21}K_{221}\mu_3]$$
$$- 4\pi e_2^2 \Phi_{22}[F_{22}K_{212}\mu_1 + F_{22}K_{222}\mu_3] - 4\pi e_3^2 \Phi_{33}$$
$$[F_{23}K_{213}\mu_1 + F_{23}K_{223}\mu_3] + 2\pi e_1^2 \Phi_{11}[K_{211}^2\mu_{11} + K_{221}^2\mu_{33}$$
$$+ 2K_{211}K_{221}\mu_{13}] + 2\pi e_2^2 \Phi_{22}[K_{212}^2\mu_{11} + K_{222}^2\mu_{33} + 2K_{212}K_{222}\mu_{13}]$$
$$+ 2\pi e_3^2 \Phi_{33}[K_{213}^2\mu_{11} + K_{223}^2\mu_{33} + 2K_{213}K_{223}\mu_{13}] - 2K_{210}\mu_{14} - 2C_{210}\mu_{24}$$

$$- 2K_{220}\mu_{34} - 2C_{220}\mu_{44} + 2F_{24}\mu_{45} + 2F_{25}\mu_{46} + 2F_{26}\mu_{47}$$

$$\dot{\mu}_{45} = -\alpha_1\mu_{45} - K_{210}\mu_{15} - C_{210}\mu_{25} - K_{220}\mu_{35} - C_{220}\mu_{45} - \sigma_{55}(K_{214}\mu_1 + C_{214}\mu_2 + K_{224}\mu_3 + C_{224}\mu_4)$$

$$\dot{\mu}_{46} = -\alpha_2\mu_{46} - K_{210}\mu_{16} - C_{210}\mu_{26} - K_{220}\mu_{36} - C_{220}\mu_{46} - \sigma_{66}(K_{215}\mu_1 + C_{215}\mu_2 + K_{225}\mu_3 + C_{225}\mu_4)$$

$$\dot{\mu}_{47} = -\alpha_3\mu_{47} - K_{210}\mu_{17} - C_{210}\mu_{27} - K_{220}\mu_{37} - C_{220}\mu_{47} - \sigma_{77}(K_{216}\mu_1 + C_{216}\mu_2 + K_{226}\mu_3 + C_{226}\mu_4) \qquad \text{(E-2)}$$

Finally, the moment equations for the stability analysis of the coupled flap-lag-torsion motion are

$$\dot{\mu}_1 = \mu_2$$

$$\begin{aligned}\dot{\mu}_2 = & -K_{110}\mu_1 - C_{110}\mu_2 - K_{120}\mu_3 - K_{130}\mu_5 - C_{130}\mu_5 \\ & - K_{114}\mu_{17} - C_{114}\mu_{27} - K_{124}\mu_{37} \\ & - C_{124}\mu_{47} - K_{134}\mu_{57} - C_{134}\mu_{67} - K_{115}\mu_{18} - C_{115}\mu_{28} - K_{125}\mu_{38} \\ & - C_{125}\mu_{48} - K_{135}\mu_{58} - C_{135}\mu_{68} - K_{116}\mu_{19} - C_{116}\mu_{29} - K_{126}\mu_{39} \\ & - C_{126}\mu_{49} - K_{136}\mu_{59} - C_{136}\mu_{69}\end{aligned}$$

$$\dot{\mu}_3 = \mu_4$$

$$\begin{aligned}\dot{\mu}_4 = & -K_{210}\mu_1 - C_{210}\mu_2 - K_{220}\mu_3 - C_{220}\mu_4 - K_{230}\mu_5 - C_{230}\mu_6 \\ & - K_{214}\mu_{17} - C_{214}\mu_{27} - K_{224}\mu_{37} \\ & - C_{224}\mu_{47} - K_{234}\mu_{57} - C_{234}\mu_{67} - K_{215}\mu_{18} - C_{215}\mu_{28} - K_{225}\mu_{38} \\ & - C_{225}\mu_{48} - K_{235}\mu_{58} - C_{235}\mu_{68} - K_{216}\mu_{19} - C_{216}\mu_{29} - K_{226}\mu_{39} \\ & - C_{226}\mu_{49} - K_{236}\mu_{59} - C_{236}\mu_{69}\end{aligned}$$

$$\dot{\mu}_5 = \mu_6$$

$$\dot{\mu}_6 = -K_{310}\mu_1 - C_{310}\mu_2 - K_{320}\mu_3 - C_{320}\mu_4 - K_{330}\mu_5 - C_{330}\mu_6$$
$$- K_{314}\mu_{17} - C_{314}\mu_{27} - K_{234}\mu_{37}$$
$$- C_{324}\mu_{47} - K_{334}\mu_{57} - C_{334}\mu_{67} - K_{315}\mu_{18} - C_{315}\mu_{28} - K_{325}\mu_{38}$$
$$- C_{325}\mu_{48} - K_{335}\mu_{58} - C_{335}\mu_{68} - K_{316}\mu_{19} - C_{316}\mu_{29} - K_{326}\mu_{39}$$
$$- C_{326}\mu_{49} - K_{336}\mu_{59} - C_{336}\mu_{69}$$

$$\dot{\mu}_{11} = 2\mu_{12}$$

$$\dot{\mu}_{12} = \mu_{22} - K_{110}\mu_{11} - C_{110}\mu_{12} - K_{120}\mu_{13} - C_{120}\mu_{14} - K_{130}\mu_{15} - C_{130}\mu_{16}$$
$$+ F_{14}\mu_{17} + F_{15}\mu_{18} + F_{16}\mu_{19}$$

$$\dot{\mu}_{13} = \mu_{23} + \mu_{14}$$

$$\dot{\mu}_{14} = \mu_{24} - K_{210}\mu_{11} - C_{210}\mu_{12} - K_{220}\mu_{13} - C_{220}\mu_{14} - K_{230}\mu_{15} - C_{230}\mu_{16}$$
$$+ F_{24}\mu_{17} + F_{25}\mu_{18} + F_{26}\mu_{19}$$

$$\dot{\mu}_{15} = \mu_{25} + \mu_{16}$$

$$\dot{\mu}_{16} = \mu_{26} - K_{310}\mu_{11} - C_{310}\mu_{12} - K_{320}\mu_{13} - C_{320}\mu_{14} - K_{330}\mu_{15} - C_{330}\mu_{16} + F_{34}\mu_{17}$$
$$+ F_{35}\mu_{18} + F_{36}\mu_{19}$$

$$\dot{\mu}_{17} = \mu_{27} - \alpha_1\mu_{17}$$

$$\dot{\mu}_{18} = \mu_{28} - \alpha_2\mu_{18}$$

$$\dot{\mu}_{19} = \mu_{29} - \alpha_3\mu_{19}$$

$$\dot{\mu}_{22} = 2\pi e_1^2 \Phi_{11}[- 2F_{11}K_{111}\mu_1 - 2F_{11}K_{121}\mu_3 - 2F_{11}K_{131}\mu_5 + K_{111}^2 \mu_{11}$$
$$+ K_{121}^2\mu_{33} + K_{131}^2\mu_{55} + 2K_{111}K_{121}\mu_{13} + 2K_{111}K_{131}\mu_{15} + 2K_{121}K_{131}\mu_{35}]$$
$$+ 2\pi e_2^2 \Phi_{22}[-2F_{12}K_{112}\mu_1 - 2F_{12}K_{122}\mu_3 - 2F_{12}K_{132}\mu_5 + K_{112}^2\mu_{11} + K_{122}^2\mu_{33}$$
$$+K_{132}^2\mu_{55} + 2K_{112}K_{122}\mu_{13} + 2K_{112}K_{132}\mu_{15} + 2K_{122}K_{132}\mu_{35}]$$
$$+ 2\pi e_3^2 \Phi_{33}[- 2F_{13}K_{113}\mu_1 - 2F_{13}K_{123}\mu_3 - 2F_{13}K_{133}\mu_5 + K_{113}^2\mu_{11}$$
$$+K_{123}^2\mu_{33} + K_{133}^2\mu_{55} + 2K_{113}K_{123}\mu_{13} + 2K_{113}K_{133}\mu_{15} + 2K_{123}K_{133}\mu_{35}]$$

$$+ 2\{ -K_{110}\mu_{12} - C_{110}\mu_{22} - K_{120}\mu_{23} - C_{120}\mu_{24} - K_{130}\mu_{25} - C_{130}\mu_{26} + F_{14}\mu_{27}$$
$$+ F_{15}\mu_{28} + F_{16}\mu_{29}\}$$
$$\dot{\mu}_{23} = \mu_{24} - K_{110}\mu_{13} - C_{110}\mu_{23} - K_{120}\mu_{33} - C_{120}\mu_{34} - K_{130}\mu_{35} - C_{130}\mu_{36} + F_{14}\mu_{37}$$
$$+ F_{15}\mu_{38} + F_{16}\mu_{39}$$
$$\dot{\mu}_{24} = 2\pi e_1^2 \Phi_{11} [- (F_{11}K_{211} + F_{21}K_{111})\mu_1 - (F_{11}K_{221} + F_{21}K_{121})\mu_3$$
$$- (F_{11}K_{231} + F_{21}K_{131})\mu_5 + K_{111}K_{211}\mu_{11} + K_{121}K_{221}\mu_{33} + K_{131}K_{231}\mu_{55}$$
$$+ (K_{111}K_{221} + K_{211}K_{121})\mu_{13} + (K_{111}K_{231} + K_{131}K_{211})\mu_{15} + (K_{121}K_{231}$$
$$+ K_{131}K_{221})\mu_{35}] + 2\pi e_2^2 \Phi_{22} [- (F_{12}K_{212} + F_{22}K_{112})\mu_1 - (F_{12}K_{222}$$
$$+ F_{22}K_{122})\mu_3 - (F_{12}K_{232} + F_{22}K_{132})\mu_5 + K_{112}K_{212}\mu_{11} + K_{122}K_{222}\mu_{33}$$
$$+ K_{132}K_{232}\mu_{55} + (K_{112}K_{222} + K_{212}K_{122})\mu_{13} + (K_{112}K_{232} + K_{132}K_{212})\mu_{15}$$
$$+ (K_{122}K_{232} + K_{132}K_{222})\mu_{35}] + 2\pi e_3^2 \Phi_{33}[- (F_{13}K_{213} + F_{23}K_{113})\mu_1$$
$$- (F_{13}K_{223} + F_{23}K_{123})\mu_3 - (F_{13}K_{233} + F_{23}K_{133})\mu_5 + K_{113}K_{213}\mu_{11} + K_{123}K_{223}\mu_{33}$$
$$+ K_{133}K_{233}\mu_{55} + (K_{113}K_{223} + K_{213}K_{123})\mu_{13} + (K_{113}K_{233} + K_{133}K_{213})\mu_{15}$$
$$+ (K_{123}K_{233} + K_{133}K_{223})\mu_{35}] - K_{110}\mu_{14} - C_{110}\mu_{24} - K_{120}\mu_{34} - C_{120}\mu_{44}$$
$$- K_{130}\mu_{45} - C_{130}M_{46} + F_{14}\mu_{47} + F_{15}\mu_{48} + F_{16}\mu_{49} - K_{210}M_{12} - C_{210}\mu_{22} - K_{220}\mu_{23}$$
$$- C_{220}\mu_{24} - K_{230}\mu_{25} - C_{230}\mu_{26} + F_{24}\mu_{27} + F_{25}\mu_{28} + F_{26}\mu_{29}$$
$$\dot{\mu}_{25} = \mu_{26} - K_{110}\mu_{15} - C_{110}\mu_{25} - K_{120}\mu_{35} - C_{120}\mu_{45} - K_{130}\mu_{55} - C_{130}\mu_{56} + F_{14}\mu_{57}$$
$$+ F_{15}\mu_{58} + F_{16}\mu_{59}$$
$$\dot{\mu}_{26} = 2\pi e_1^2 \Phi_{11} [-(F_{11}K_{311}+F_{31}K_{111})\mu_1 -(F_{11}K_{321}+F_{31}K_{121})\mu_3 -(F_{11}K_{331}$$
$$+ F_{31}K_{131})\mu_5 + K_{111}K_{311}\mu_{11} + K_{121}K_{321}\mu_{33} + K_{131}K_{331}\mu_{55} + (K_{111}K_{321}+K_{121}K_{311})$$
$$\mu_{13} + (K_{111}K_{331} + K_{131}K_{311})\mu_{15} + (K_{121}K_{331} + K_{131}K_{321})\mu_{35}] + 2\pi e_2^2 \Phi_{22}$$
$$[-(F_{12}K_{312}+F_{32}K_{112})\mu_1 -(F_{12}K_{322}+F_{32}K_{122})\mu_3 -(F_{12}K_{332}+F_{32}K_{132})\mu_5$$
$$+ K_{112}K_{312}\mu_{11} + K_{122}K_{322}\mu_{33} + K_{132}K_{332}\mu_{55} + (K_{112}K_{322} + K_{122}K_{312})\mu_{13}$$
$$+ (K_{112}K_{332} + K_{132}K_{312})\mu_{15} + (K_{122}K_{332} + K_{132}K_{322})\mu_{35}] + 2\pi e_3^2 \Phi_{33}$$
$$[-(F_{13}K_{313}+F_{33}K_{113})\mu_1 -(F_{13}K_{323}+F_{33}K_{123})\mu_3 -(F_{13}K_{333}+F_{33}K_{133})\mu_5$$
$$+ K_{113}K_{313}\mu_{11} + K_{123}K_{323}\mu_{33} + K_{133}K_{333}\mu_{55} + (K_{113}K_{323} + K_{123}K_{313})\mu_{13}$$

$$+ (K_{113}K_{333} + K_{133}K_{313})\mu_{15} + (K_{123}K_{333} + K_{133}K_{323})\mu_{35}] - K_{110}\mu_{16}$$
$$- C_{110}\mu_{26} - K_{120}\mu_{36} - C_{120}\mu_{46} - K_{130}M_{56} - C_{130}\mu_{66} + F_{14}\mu_{67} + F_{15}\mu_{68} + F_{16}\mu_{69}$$
$$- K_{310}\mu_{12} - C_{310}\mu_{22} - K_{320}\mu_{23} - C_{320}\mu_{24} - K_{330}\mu_{25} - C_{330}\mu_{26} + F_{34}\mu_{27} + F_{35}\mu_{28}$$
$$+ F_{36}\mu_{29}$$

$$\dot{\mu}_{27} = -\alpha_1\mu_{27} - K_{110}\mu_{17} - C_{110}\mu_{27} - K_{120}\mu_{37} - C_{120}\mu_{47} - K_{130}\mu_{57} - C_{130}\mu_{67}$$
$$- \sigma_{77}(K_{114}\mu_1 + C_{114}\mu_2 + K_{124}\mu_3 + C_{124}\mu_4 + K_{134}\mu_5 + C_{134}\mu_6)$$

$$\dot{\mu}_{28} = -\alpha_2\mu_{28} - K_{110}\mu_{18} - C_{110}\mu_{28} - K_{120}\mu_{38} - C_{120}\mu_{48} - K_{130}\mu_{58} - C_{130}\mu_{68}$$
$$- \sigma_{88}K_{115}\mu_1 + C_{115}\mu_2 + K_{125}\mu_3 + C_{125}\mu_4 + K_{135}\mu_5 + C_{135}\mu_6)$$

$$\dot{\mu}_{29} = -\alpha_3\mu_{29} - K_{110}\mu_{19} - C_{110}\mu_{29} - K_{120}\mu_{39} - C_{120}\mu_{49} - K_{130}\mu_{59} - C_{130}\mu_{69}$$
$$- \sigma_{99}(K_{116}\mu_1 + C_{116}\mu_2 + K_{126}\mu_3 + C_{126}\mu_4 + K_{136}\mu_5 + C_{136}\mu_6)$$

$$\dot{\mu}_{33} = 2\mu_{34}$$

$$\dot{\mu}_{34} = \mu_{44} - K_{210}\mu_{13} - C_{210}\mu_{23} - K_{220}\mu_{33} - C_{220}\mu_{34} - K_{230}\mu_{35} - C_{230}\mu_{36} + F_{24}\mu_{37}$$
$$+ F_{25}\mu_{38} + F_{26}\mu_{39}$$

$$\dot{\mu}_{35} = \mu_{46} + \mu_{36}$$

$$\dot{\mu}_{36} = \mu_{46} - K_{310}\mu_{13} - C_{310}\mu_{23} - K_{320}\mu_{33} - C_{320}\mu_{34} - K_{330}\mu_{35} - C_{330}\mu_{36}$$
$$+ F_{34}\mu_{37} + F_{35}\mu_{38} + F_{36}\mu_{39}$$

$$\dot{\mu}_{37} = \mu_{47} - \alpha_1\mu_{37}$$

$$\dot{\mu}_{38} = \mu_{48} - \alpha_2\mu_{38}$$

$$\dot{\mu}_{39} = \mu_{49} - \alpha_3\mu_{39}$$

$$\dot{\mu}_{44} = 2\pi e_1^2 \Phi_{11}[- 2F_{21}K_{211}M_1 - 2F_{21}K_{221}\mu_3 - 2F_{21}K_{231}\mu_5 + K_{211}^2\mu_{11} + K_{221}^2\mu_{33}$$
$$+ K_{231}^2\mu_{55} + 2K_{211}K_{221}\mu_{13} + 2K_{211}K_{231}\mu_{15} + 2K_{221}K_{231}\mu_{35}] + 2\pi e_2^2 \Phi_{22}$$
$$[-2F_{22}K_{212}\mu_1 - 2F_{22}K_{222}\mu_3 - 2F_{22}K_{232}\mu_5 + K_{212}^2\mu_{11} + K_{222}^2\mu_{33} + K_{232}^2\mu_{55}$$
$$+ 2K_{212}K_{232}\mu_{13} + 2K_{212}K_{232}\mu_{15} + 2K_{222}K_{232}\mu_{35}] + 2\pi e_3^2 \Phi_{33}$$
$$- 2F_{23}K_{213}\mu_1 - 2F_{23}K_{223}\mu_3 - 2F_{23}K_{233}\mu_5 + K_{213}^2\mu_{11}^2 + K_{223}^2\mu_{33}^2 + K_{233}^2\mu_{55}^2$$
$$+ 2K_{213}K_{223}\mu_{13} + 2K_{213}K_{233}\mu_{15} + 2K_{223}K_{233}\mu_{35}] + 2\{-K_{210}\mu_{14} - C_{210}\mu_{24}$$
$$- K_{220}\mu_{34} - C_{220}\mu_{44} - K_{230}\mu_{45} - C_{230}\mu_{46} + F_{24}\mu_{47} + F_{25}\mu_{48} \quad F_{26}\mu_{49}$$

$$\dot{\mu}_{45} = \mu_{46} - K_{210}\mu_{15} - C_{210}\mu_{25} - K_{220}\mu_{35} - C_{220}\mu_{45} - K_{230}\mu_{55} - C_{230}\mu_{56} + F_{24}\mu_{57} + F_{25}\mu_{58} + F_{26}\mu_{59}$$

$$\dot{\mu}_{46} = 2\pi e_1^2 \Phi_{11} [-(F_{21}K_{311}+F_{31}K_{211})\mu_1 - (F_{21}K_{321}+F_{31}K_{221})\mu_3 - (F_{21}K_{331} + F_{31}K_{231})\mu_5 + K_{211}K_{311}\mu_{11} + K_{221}K_{321}\mu_{33} + K_{231}K_{331}\mu_{55} + (K_{211}K_{321}+K_{221}K_{311})\mu_{13} + (K_{211}K_{331}+K_{311}K_{231})\mu_{15} + (K_{221}K_{331}+K_{231}K_{321})\mu_{35}] + 2\pi e_2^2 \Phi_{22} [-(F_{22}K_{312} + K_{32}K_{212})\mu_1 - (F_{22}K_{322} + F_{32}K_{222})\mu_3 - (F_{22}K_{332} + F_{32}K_{232})\mu_5 + K_{212}K_{312}\mu_{11} + K_{222}K_{322}\mu_{33} + K_{232}K_{332}\mu_{55} + (K_{212}K_{322} + K_{222}K_{312})\mu_{13} + (K_{212}K_{332} + K_{312}K_{232})\mu_{15} + (K_{222}K_{332} + K_{232}K_{322})\mu_{35}] + 2\pi e_3^2 \Phi_{33} [-(F_{23}K_{313} + F_{33}K_{213})\mu_1 - (F_{23}K_{323} + F_{33}K_{223})\mu_3 - (F_{23}K_{333} + F_{33}K_{233})\mu_5 + K_{213}K_{313}\mu_{11} + K_{223}K_{323}\mu_{33} + K_{233}K_{333}\mu_{55} + (K_{213}K_{323} + K_{223}K_{313})\mu_{13} + (K_{213}K_{333} + K_{313}K_{233})\mu_{15} + (K_{223}K_{333} + K_{233}K_{323})\mu_{35}] - K_{210}\mu_{16} - C_{210}\mu_{26} - K_{220}\mu_{36} - C_{220}\mu_{46} - K_{230}\mu_{56} - C_{230}\mu_{66} + F_{24}\mu_{67} + F_{25}\mu_{68} + F_{26}\mu_{69} - K_{310}\mu_{14} - C_{310}\mu_{24} - K_{320}\mu_{34} - C_{320}\mu_{44} - K_{330}\mu_{45} - C_{330}\mu_{46} + F_{34}\mu_{47} + F_{35}\mu_{48} + F_{36}\mu_{49}$$

$$\dot{\mu}_{47} = -\alpha_1\mu_{47} - K_{210}\mu_{17} - C_{210}\mu_{27} - K_{220}\mu_{37} - C_{220}\mu_{47} - K_{230}\mu_{57} - C_{230}\mu_{67} - \sigma_{77}(K_{214}\mu_1 + C_{214}\mu_2 + K_{224}\mu_3 + C_{224}\mu_4 + K_{234}\mu_5 + C_{234}\mu_6)$$

$$\dot{\mu}_{48} = -\alpha_2\mu_{48} - K_{210}\mu_{18} - C_{210}\mu_{28} - K_{220}\mu_{38} - C_{220}\mu_{48} - K_{230}\mu_{58} - C_{230}\mu_{68} - \sigma_{88}(K_{215}\mu_1 + C_{215}\mu_2 + K_{225}\mu_3 + C_{225}\mu_4 + K_{235}\mu_5 + C_{235}\mu_6)$$

$$\dot{\mu}_{49} = -\alpha_3\mu_{49} - K_{210}\mu_{19} - C_{210}\mu_{29} - K_{220}\mu_{39} - C_{220}\mu_{49} - K_{320}\mu_{59} - C_{230}\mu_{69} - \sigma_{99}(K_{216}\mu_1 + C_{216}\mu_2 + K_{226}\mu_3 + C_{226}\mu_4 + K_{236}\mu_5 + C_{236}\mu_6)$$

$$\dot{\mu}_{55} = 2\mu_{56}$$

$$\dot{\mu}_{56} = \mu_{66} - K_{310}\mu_{15} - C_{310}\mu_{25} - K_{320}\mu_{35} - C_{320}\mu_{45} - K_{330}\mu_{55} - C_{330}\mu_{56} + F_{34}\mu_{57} + F_{35}\mu_{58} + F_{36}\mu_{59}$$

$$\dot{\mu}_{57} = \mu_{67} - \alpha_1\mu_{57}$$

$$\dot{\mu}_{58} = \mu_{68} - \alpha_2\mu_{58}$$

$$\dot{\mu}_{59} = \mu_{69} - \alpha_3\mu_{59}$$

$$\dot{\mu}_{66} = 2\pi e_1^2 \Phi_{11}[-2F_{31}K_{311}\mu_1 - 2F_{31}K_{321}\mu_3 - 2F_{31}K_{331}\mu_5 + K_{311}^2\mu_{11} + K_{321}^2\mu_{33} + K_{331}^2\mu_{55} + 2K_{311}K_{321}\mu_{13} + 2K_{311}K_{331}\mu_{15} + 2K_{321}K_{331}\mu_{15}] + 2\pi e_2^2 \Phi_{22} [-2F_{32}K_{312}\mu_1$$

$$- 2F_{32}K_{322}\mu_3 - 2F_{32}K_{332}\mu_5 + K_{312}^2\mu_{11} + K_{322}^2\mu_{33} + K_{332}^2\mu_{55} + 2K_{312}K_{322}\mu_{13}$$

$$+ 2K_{312}K_{332}\mu_{15} + 2K_{322}K_{332}\mu_{35}] + 2\pi e_3^2 \Phi_{33} [- 2F_{33}K_{313}\mu_1 - 2F_{33}K_{323}\mu_3$$

$$- 2F_{33}K_{333}\mu_5 + K_{313}^2\mu_{11} + K_{323}^2\mu_{33} + K_{333}^2\mu_{55} + 2K_{313}K_{323}\mu_{13} + 2K_{313}K_{333}\mu_{15}$$

$$+ 2K_{323}K_{333}\mu_{35}] + 2\{-K_{310}\mu_{16} - C_{310}\mu_{26} - K_{320}\mu_{36} - C_{320}\mu_{46} - K_{330}\mu_{56}$$

$$- C_{330}\mu_{66} + F_{34}\mu_{67} + F_{35}\mu_{68} + F_{36}\mu_{69}\}$$

$$\dot{\mu}_{67} = -\alpha_1\mu_{67} - K_{310}\mu_{17} - C_{310}\mu_{27} - K_{320}\mu_{37} - C_{320}\mu_{47} - K_{330}\mu_{57} - C_{330}\mu_{57}$$

$$- \sigma_{77}(K_{314}\mu_1 + C_{314}\mu_2 + K_{324}\mu_3 + C_{324}\mu_4 + K_{334}\mu_5 + C_{334}\mu_6)$$

$$\dot{\mu}_{68} = -\alpha_2 M_{68} - K_{310}\mu_{18} - C_{310}\mu_{28} - K_{320}\mu_{38} - C_{320}\mu_{48} - K_{330}\mu_{58} - C_{330}\mu_{68}$$

$$- \sigma_{88}(K_{315}\mu_1 + C_{315}\mu_2 + K_{325}\mu_3 + C_{325}\mu_4 + K_{325}\mu_5 + C_{335}\mu_6)$$

$$\dot{\mu}_{69} = -\alpha_3\mu_{69} - K_{310}\mu_{19} - C_{310}\mu_{29} - K_{320}\mu_{39} - C_{320}\mu_{49} - K_{330}\mu_{59} - C_{330}\mu_{69}$$

$$- \sigma_{99}(K_{316}\mu_1 + C_{316}\mu_2 + K_{326}\mu_3 + C_{326}\mu_4 + K_{336}\mu_5 + C_{336}\mu_6) \qquad \text{(E-3)}$$

Appendix F

Stability Analyses of Nonlinear Moment Equations

In this Appendix, the stability of the system governed by Eq. (5-8) is re-examined using the variational equations of Poincaré in the neighborhoods of its singular points. The singular points can be determined by letting the right hand side of Eq. (5-8) equal to zero. If $\vec{Y}^i$ is a singular point of the system, then it satisfies

$$\bar{A} + \bar{B}\vec{Y}^i + \bar{N}(\psi, \vec{Y}^i) = 0 \tag{F-1}$$

The variational equation of the system can be obtained by substituting $\vec{Y} = \vec{Y}^i + \varepsilon\vec{X}$ into Eq. (5-8), and expanding the nonlinear matrix $\bar{N}$ in a Taylor series,

$$\dot{X} = [\bar{B} + \frac{\bar{N}}{\vec{Y}} \Big|_{\vec{Y}=\vec{Y}^i}] \vec{X} + 0 (\ ^2) \tag{F-2}$$

where

$$\frac{\bar{N}}{Y} \Big|_{Y=Y^i} = \begin{bmatrix} \frac{N_1}{Y_1} & \frac{N_1}{Y_2} & \cdots\cdots & \frac{N_1}{Y_n} \\ \frac{N_2}{Y_1} & \cdots & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{N_n}{Y_1} & \cdots\cdots\cdots\cdots & & \frac{N_n}{Y_n} \end{bmatrix}_{Y=Y^i}$$

and an overdot denotes one derivative with respect to the nondimensional time ψ. The first order variational equation of the system is given by

$$\dot{\vec{X}} = (\bar{B} + \frac{\partial \vec{N}}{\partial \vec{Y}} \Big|_{\vec{Y}=\vec{Y}^i}) \vec{X} \quad \text{(F-3)}$$

Since $\vec{Y}^i$ is a known singular point Eq. (F-3) is a linear periodic equation if the intensity functions of the earthquake are assumed to be equal to unity. Then, the Floquet theory can be used to determine the stability condition of the dynamic system at $\vec{Y}^i$ for any given combination of parameters (associated with a given operating condition as described in Chapter 3).

It must be noted that since Eq. (F-1) is a set of nonlinear algebraic equations multiple singular points are possible, each of which must be investigated separately. However, in the present study only one singular point is found to be meaningful physically, and it can be obtained numerically by iteration. A first trial may be obtained by neglecting the nonlinear terms in Eq. (F-1) because without the turbulence the system is linear and with turbulence the solution is expected to deviate only slightly from the linear one. An alternative first trial may be the trivial zero solution. In fact, the solutions obtained from these two approaches converged in our numerical calculations.

Our calculations were carried out, however, only for the coupled flap-lag motion because of the following reasons: 1) the uncoupled flapping motion is very stable, 2) the torsional degree of freedom has little influence on the dynamic behavior of flap and leadlag motions and 3) the cost is very high for computing the eigenvalue of the Floquet transition matrix of a dynamic system involving a large number of equations with periodic coefficients. The values of parameters used in the calculations were the same as those of cases 5, 6, 7 and 8 in Table V. The largest norms of the state vector defined by Eq. (5-9) at the singular point over one period are .0000769 for case 5, .0027677 for case 6, .0000779 for case 7 and .0028446 for case 8. These results indicate that under the excitation of a high level turbulence the singular point is shifted farther away from the zero position; and for a moderate turbulence level the singular point is quite close to the zero position. The largest norms of the eigenvalues of the Floquet transition matrices are .943535 for case 5, .932597 for case 6, .944804 for case 7 and .932468 for case 8. These results differ from those obtained from the Bolotin method less than 1%. Therefore, the Bolotin method is sufficiently accurate if the turbulence level is moderate.

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