MODAL ANALYSIS OF NONCLASSICALLY DAMPED STRUCTURAL SYSTEMS USING CANONICAL TRANSFORMATION

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NOTICE

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16. Abstract (Limit: 200 words) An alternate modal decomposition method for dynamic analysis of non-classically damped structural systems is presented. The resulting decoupled equations contain only real parameters. Hence, the solution can be obtained in the real field. Several procedures are outlined to solve these equations for both deterministic and nonstationary random ground excitations. Prior work has shown that the effect of nonclassical damping may be significant for the response of light equipment attached to a structure. Therefore, the proposed solution technique is applied to find the response of a light equipment that is attached to a multi-degree-of-freedom structure.

Numerical results obtained from deterministic and nonstationary random vibration analyses indicate that the effect of nonclassical damping on the response of tuned equipment is significant only when the mass ratio and damping ratio of the equipment are small. Under this circumstance, the approximate classically damped solution, i.e., the solution obtained using the undamped modal matrix and disregarding the off-diagonal terms of the resulting damping matrix, is usually unconservative.

For detuned equipment, neglecting the effect of nonclassical damping generally results in an equipment response that is close to the exact results. However, for certain equipment detuned at high frequency, neglecting nonclassical damping results in conservative equipment responses.

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ABSTRACT

An alternate modal decomposition method for dynamic analysis of nonclassically damped structural systems is presented. The resulting decoupled equations contain only real parameters. Hence, the solution can be obtained in the real field. Several procedures are outlined to solve these equations for both deterministic and nonstationary random ground excitations. Prior work has shown that the effect of nonclassical damping may be significant for the response of light equipment attached to a structure. Therefore, the proposed solution technique is applied to find the response of a light equipment that is attached to a multi-degree-of-freedom structure.

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INTRODUCTION

In seismic analysis of linear multi-degree-of-freedom viscously damped structures, it is quite common to assume that the damping matrix is of the classical (proportional) form (i.e., the form specified by Caughey and O'Kelly [1]). This assumption enables one to decouple the equations of motion by using the undamped eigenvectors of the system. After the equations are decoupled, the response quantities of interest can be obtained by either the response spectrum approach or numerical integration of the decoupled equations [2]. Solution of each decoupled equation represents the contribution of a particular mode of vibration of the structure to the total response. Furthermore, the response in most situations can be approximated by contributions from only a few dominant modes.

In general, however, real structures are not classically damped. Therefore, the damping matrix can not be diagonalized by the eigenvectors of the undamped system. Under this circumstance, one can use a step by step integration procedure to evaluate the response of structures. This, however, may involve numerical difficulties unless the time steps are sufficiently small. In general, the use of this procedure can be justified when the structure is behaving nonlinearly.

One may still decouple the equations of motion of a nonclassically damped structural system using the eigenvectors of the damped system. However, for such structures, the damped eigenvectors are complex valued. This procedure, first proposed by Foss [4] and Traill-Nash [11], decouples the equations of motion of an n degree-of-freedom nonclassically damped system into a set of 2n first order equations. These decoupled equations contain complex parameters. Recently, Singh and Ghafory-Ashtiany [10]

developed a step by step numerical integration algorithm to solve these decoupled complex valued equations. Igusa, Der Kiureghian, and Sackman [6], used random vibration approach to obtain statistical moments of the response of nonclassically damped system subjected to stationary white noise excitations. Veletsos and Ventura [12] made a review of the dynamic properties of nonclassically damped structures, and compared the exact solutions of such systems with those computed using the undamped eigen vectors and disregarding the off-diagonal terms of the resulting damping matrix under deterministic excitations.

In this paper an alternate modal decomposition approach employing canonical transformation is presented. The resulting decoupled equations involve only real parameters, thus avoiding computations to be carried out in the complex field. Several procedures are outlined to solve these equations for both deterministic and nonstationary random ground excitations. Prior work [6] has shown that the effect of nonclassical damping may be significant for the response of light equipment attached to a structure. The proposed canonical modal analysis technique is employed to carry out a parametric study for the effect of nonclassical damping on the response of a secondary system that is attached to a primary structure. Numerical results indicate that for tuned light equipments, neglecting nonclassical damping can result in a significantly unconservative equipment response.

Finally, the effect of nonclassical damping on the response of primary system is studied. An eight story shear beam type structure is considered and the distribution of the damping is varied so that a variety of nonclassically damped systems can be studied.

SECTION 2

BACKGROUND

The response of an n degree-of-freedom structure subjected to a ground excitation, $\ddot{x}_g$, can be obtained by solving the following differential equations of motion

$$\underline{M}\, \ddot{\underline{X}} + \underline{C}\, \dot{\underline{X}} + \underline{K}\, \underline{X} = - \underline{M}\, \underline{r}\, \ddot{x}_g \tag{1}$$

in which $\underline{M}$, $\underline{C}$, and $\underline{K}$ denote the (nxn) mass, damping, and stiffness matrices of the structure, respectively, and $\underline{X}$ is an n vector denoting the displacements of the structure relative to the moving base. The vector $\underline{r}$ is a unit vector, $\underline{r} = [1,1,\ldots,1]'$. A super dot ($\cdot$) represents differentiation with respect to time and an under bar (−) denotes a vector or a matrix. In Eq. (1), the argument of time, t, for $\underline{X}$ and $\ddot{x}_g$ have been omitted for simplicity.

Caughey and O'Kelly [1] showed that if the damping matrix statisfies the identity $\underline{C}\, \underline{M}^{-1}\, \underline{K} = \underline{K}\, \underline{M}^{-1}\, \underline{C}$, the eigenvectors of the undamped system can be used to transform the equations of motion, Eq. (1), into a set of n decoupled equations. The system with damping matrix satisfying this condition is said to be classically damped. However, the eigenvectors of the undamped system will no longer diagonalize the damping matrix that is not of the classical form.

For the nonclassically damped system, the approach proposed by Foss [4] and Traill-Nash [11] can be used. In this approach, the n second-order equations of motion are converted into 2n first-order equations as follows:

$$\underline{A}_1\, \dot{\underline{Y}} + \underline{B}\, \underline{Y} = \underline{P}\, \ddot{x}_g \tag{2}$$

in which $\underline{A}_1$ and $\underline{B}$ are (2nx2n) symmetric matrices and $\underline{P}$ is a 2n vector defined as

$$\underline{A}_1 = \left[\begin{array}{c|c} \underline{0} & \underline{M} \\ \hline \underline{M} & \underline{C} \end{array}\right] \quad , \quad \underline{B} = \left[\begin{array}{c|c} -\underline{M} & \underline{0} \\ \hline \underline{0} & \underline{K} \end{array}\right] \quad , \quad \underline{P} = \left\{\begin{array}{c} \underline{0} \\ \cdots \\ -\underline{M}\, \underline{r} \end{array}\right\} \tag{3}$$

and $\underline{Y}$ is a 2n vector, referred to as the state vector,

$$\underline{Y} = \left\{\begin{array}{c} \dot{\underline{X}} \\ \cdots \\ \underline{X} \end{array}\right\} \tag{4}$$

The eigenvalue problem of Eq. (2), $|\lambda\, \underline{A}_1 + \underline{B}| = 0$, can be expressed as follows

$$\lambda_j\, \underline{A}_1\, \underline{\phi}_j + \underline{B}\, \underline{\phi}_j = 0 \tag{5}$$

in which λ_j and $\underline{\phi}_j$ are the jth eigenvalue and eigenvector, respectively. From the definition of the state vector $\underline{Y}$ given by Eq. (4), the jth eigenvector, $\underline{\phi}_j$, has the following form

$$\underline{\phi}_j = \left\{\begin{array}{c} \lambda_j\, \underline{\psi}_j \\ \cdots \\ \underline{\psi}_j \end{array}\right\} \tag{6}$$

in which $\underline{\psi}_j$ represents the displacement eigenvector. Since matrices $\underline{A}_1$ and $\underline{B}$ do not commute, i.e., $\underline{A}_1\, \underline{B} \neq \underline{B}\, \underline{A}_1$, the resulting eigenvalues and eigenvectors are complex valued [3]. Furthermore, if the system is underdamped, the

eigenvalues and eigenvectors are n pairs of complex conjugate. The jth pair of eigenvalues can be written as

$$\lambda_{2j-1,2j} = - \xi_j \omega_j \pm \underline{i} \, \omega_j \sqrt{1-\xi_j^2} = - \xi_j \omega_j \pm \underline{i} \, \omega_{Dj} \tag{7}$$

in which $\underline{i} = \sqrt{-1}$, $\omega_{Dj} = \omega_j (1-\xi_j^2)^{1/2}$ and ω_j is different from the frequency of the corresponding undamped system. The above notations was first introduced by Singh [9].

The response state vector $\underline{Y}$ can be expressed as a linear combination of the eigenvectors

$$\underline{Y} = \underline{\phi} \, \underline{Z} \tag{8}$$

where $\underline{\phi} = \left[\phi_1, \phi_2, \ldots, \phi_{2n} \right]$ is a (2nx2n) complex modal matrix. Substituting Eq. (8) into Eq. (2) and premultiplying it by the inverse of the $\underline{\phi}$ matrix, $\underline{\phi}^{-1}$, one obtains a set of 2n decoupled differential equations

$$\dot{Z}_j = \lambda_j Z_j + Q_j \, \ddot{x}_g \qquad j = 1, 2, \ldots, 2n \tag{9}$$

in which Q_j is the jth element of the $\underline{Q}$ vector where $\underline{Q} = \underline{\phi}^{-1} \left\{ \begin{array}{c} -\underline{r} \\ \cdots \\ \underline{0} \end{array} \right\}$.

The solutions of Eqs. (9) together with the transformation of Eq. (8) yield the response state vector $\underline{Y}(t)$ of the structural system. Note that Eq. (9) involves complex coefficients and the solutions for Z_j (j = 1, 2, ...) are complex. The solution for the state vector $\underline{Y}$, however, is real. In order to avoid the numerical solution for the complex differential equations, we propose the following alternate approach [13]. In this

approach, the only complex calculations are the determination of the eigenvalues and eigenvectors.

SECTION 3

CANONICAL MODAL ANALYSIS

The equations of motion for the state vector, Eq. (2), are rewritten as follows:

$$\dot{\underline{Y}} = \underline{A}\ \underline{Y} + \underline{W}\ \ddot{x}_g \tag{10}$$

where

$$\underline{A} = \left[\begin{array}{c|c} -\underline{M}^{-1}\underline{C} & -\underline{M}^{-1}\underline{K} \\ \hline \underline{I} & \underline{0} \end{array}\right] \quad ; \quad \underline{W} = \left\{\begin{array}{c} -\underline{r} \\ \cdots \\ \underline{0} \end{array}\right\} \tag{11}$$

The eigenvalues and eigenvectors of matrix $\underline{A}$ are identical to those of Eq. (5), denoted by λ_j and $\underline{\phi}_j$, for $j = 1, 2, \ldots, 2n$. Further the jth pair of eigenvectors can be expressed as

$$\underline{\phi}_{2j-1} = \underline{a}_j + \underline{i}\ \underline{b}_j \tag{12}$$

$$\underline{\phi}_{2j} = \underline{a}_j - \underline{i}\ \underline{b}_j \quad , \qquad j = 1, 2, \ldots, n$$

in which $\underline{a}_j$ and $\underline{b}_j$ are 2n real vectors

The (2nx2n) real matrix $\underline{T}$ constructed in the following

$$\underline{T} = \left[\underline{a}_1, \ \underline{b}_1, \ \underline{a}_2, \ \underline{b}_2, \ldots, \underline{a}_j, \ \underline{b}_j, \ldots, \ \underline{a}_n, \ \underline{b}_n\right] \tag{13}$$

will transform the matrix $\underline{A}$ into a canonical form $\underline{\Lambda}$ [8], i.e.,

$$\underline{\Lambda} = \underline{T}^{-1}\ \underline{A}\ \underline{T} \tag{14}$$

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in which $\underline{T}^{-1}$ is the inverse of the $\underline{T}$ matrix and

$$\underline{\Lambda} = \begin{bmatrix} \underline{\Lambda}_1 & & & 0 \\ & \underline{\Lambda}_2 & & \\ & & \ddots & \\ 0 & & & \underline{\Lambda}_n \end{bmatrix} \tag{15}$$

where

$$\underline{\Lambda}_j = \begin{bmatrix} -\xi_j \omega_j & \omega_{Dj} \\ -\omega_{Dj} & -\xi_j \omega_j \end{bmatrix} \qquad , \quad j = 1, 2, \ldots, n \tag{16}$$

Let the transformation of the state vector be

$$\underline{Y} = \underline{T} \, \underline{\nu} \tag{17}$$

Substituting Eq. (17) into Eq. (10) and premultiplying it by the inverse of the $\underline{T}$ matrix, $\underline{T}^{-1}$, one obtains

$$\dot{\underline{\nu}} = \underline{\Lambda} \, \underline{\nu} + \underline{F} \, \ddot{x}_g \tag{18}$$

in which $\underline{\Lambda}$ is given by Eq. (15) and

$$\underline{F} = \underline{T}^{-1} \left\{ \begin{matrix} -\underline{r} \\ \cdots \\ \underline{0} \end{matrix} \right\} \tag{19}$$

Equation (18) consists of n pairs of decoupled equations. Each pair of equations represents one vibrational mode, and it is uncoupled with other pairs. However, the two equations in each pair are coupled. The

3-2

transformation given in Eq. (17) is referred to as the canonical transformation [13, 14]. All the parameters in Eq. (18) are real.

The jth pair of coupled equations in Eq. (18), corresponding to the jth vibrational mode, is given as follows:

$$\dot{\nu}_{2j-1} = -\xi_j \omega_j \, \nu_{2j-1} + \omega_{Dj} \, \nu_{2j} + F_{2j-1} \, \ddot{x}_g \tag{20a}$$

$$\dot{\nu}_{2j} = -\omega_{Dj} \, \nu_{2j-1} - \xi_j \omega_j \, \nu_{2j} + F_{2j} \, \ddot{x}_g \tag{20b}$$

in which F_{2j-1} and F_{2j} are the 2j-1th and the 2jth elements of the $\underline{F}$ vector, respectively. Solutions of Eqs. (20) together with the transformation of Eq. (17) yield the response state vector $\underline{Y}(t)$ of the structural system.

The advantage of the formulation given above is that the computations for the solutions are all in the real field. The modal decomposition approach described above is referred to as the canonical modal decomposition.

Equation (20) can be solved easily using either impulse response function approach or frequency response function approach to be described later. Likewise, it can further be decoupled, if one wishes to obtain one equation in terms of each unknown, although this procedure is unnecessary. The decoupled differential equations are second order, and the pair corresponding to mode j is

$$\ddot{\nu}_{2j-1} + 2\xi_j \omega_j \, \dot{\nu}_{2j-1} + \omega_j^2 \, \nu_{2j-1} = (\xi_j \omega_j \, F_{2j-1} + \omega_{Dj} \, F_{2j}) \, \ddot{x}_g$$
$$+ F_{2j-1} \, \dddot{x}_g \tag{21a}$$

$$\ddot{\nu}_{2j} + 2\xi_j \omega_j \, \dot{\nu}_{2j} + \omega_j^2 \, \nu_{2j} = (\xi_j \omega_j \, F_{2j} - \omega_{Dj} \, F_{2j-1}) \, \ddot{x}_g + F_{2j} \, \dddot{x}_g \tag{21b}$$

The solutions of the above decoupled equations represent the response of the jth mode. However, from the computational standpoint, Eq. (20) appears to be more advantageous than Eq. (21).

The canonical transformation given by Eq. (17) is applied to Eq. (10) in which the $\underline{A}$ matrix is , in general, nonsymmetric. Since the equation of motion given by Eq. (2) is identical to Eq. (10), the same canonical transformation can be applied to Eq. (2) to arrive at the modal decomposition equations given by Eqs. (18) and (20). As such, some orthogonality conditions can be used advantageously, since $\underline{A}_1$ and $\underline{B}$ matrices are symmetric. Detail derivations for the canonical transformation of Eq. (2) are presented in the Appendix.

By virtue of the canoniclal transformation presented in the Appendix, elements of the $\underline{F}$ vector in Eqs. (19) and (20) can be expressed as follows

$$F_{2j-1} = \frac{\left[(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{a}_j') + (\underline{a}_j' \, \underline{A}_1 \, \underline{b}_j)(\underline{b}_j')\right]\underline{P}}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \tag{22a}$$

$$F_{2j} = \frac{\left[(\underline{a}_j' \, \underline{A}_1 \, \underline{b}_j)(\underline{a}_j') - (\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{b}_j')\right]\underline{P}}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \tag{22b}$$

in which a prime indicates the transpose of a vector or matrix.

3.1 Reduction to Classically Damped Structures

If the structural system is classically damped, then the displacement part of the eigenvector is real, i.e., $\underline{\psi}_j$ in Eq. (6) is real. The velocity part of the eigenvector, $\lambda_j \, \underline{\psi}_j$, is simply the eigenvalue, λ_j, multiplied by the displacement part of the eigenvector $\underline{\psi}_j$ (see Eq. 6). Hence, $\underline{a}_j$ and $\underline{b}_j$ can be expressed as

$$\underline{a}_j = \begin{bmatrix} -\xi_j \ \omega_j \ \underline{\Phi}_j \\ \cdots\cdots\cdots \\ \underline{\Phi}_j \end{bmatrix} \quad , \quad \underline{b}_j = \begin{bmatrix} \omega_{Dj} \ \underline{\Phi}_j \\ \cdots\cdots \\ \underline{0} \end{bmatrix} \tag{23}$$

in which $\underline{\Phi}_j$ is the lower half of $\underline{a}_j$ vector representing the displacement eigenvector. Thus, for classically damped systems, $\underline{\Phi}_j$ is simply the jth eigenvector of the undamped system.

The elements F_{2j-1} and F_{2j} given by Eq. (22) can be simplified as follows [see the Appendix]

$$F_{2j-1} = 0 \tag{24a}$$

$$F_{2j} = - \frac{\underline{\Phi}'_j \ \underline{M} \ \underline{r}}{\underline{\Phi}'_j \ \underline{M} \ \underline{\Phi}_j \ \omega_{Dj}} \tag{24b}$$

Finally, the simplified form of Eqs. (21a-b) become

$$\ddot{\nu}_{2j-1} + 2\xi_j\omega_j \ \dot{\nu}_{2j-1} + \omega_j^2 \ \nu_{2j-1} = - \frac{\underline{\Phi}'_j \ \underline{M} \ \underline{r}}{\underline{\Phi}'_j \ \underline{M} \ \underline{\Phi}_j} \ \ddot{x}_g \tag{25a}$$

$$\ddot{\nu}_{2j} + 2\xi_j\omega_j \ \dot{\nu}_{2j} + \omega_j^2 \ \nu_{2j} = + \frac{\underline{\Phi}'_j \ \underline{M} \ \underline{r}}{\underline{\Phi}'_j \ \underline{M} \ \underline{\Phi}_j \ \omega_{Dj}} \ \dddot{x}_g - \frac{\xi_j\omega_j \ \underline{\Phi}'_j \ \underline{M} \ \underline{r}}{\underline{\Phi}'_j \ \underline{M} \ \underline{\Phi}_j \ \omega_{Dj}} \ \ddot{x}_g \tag{25b}$$

Substituting Eq. (23) into Eq. (13), and then into Eq. (17), one obtains the displacement response vector $\underline{X}$, that is the lower half of $\underline{Y}$ vector as follows

$$\underline{X} = \underline{\Phi} \ \underline{U} \tag{26}$$

in which $\underline{\Phi}$ is a (nxn) modal matrix of the undamped system and U is an n vector with ν_{2j-1} as the jth element, representing the generalized coordinate of the undamped system, i.e.,

$$\underline{\Phi} = \left[\underline{\Phi}_1, \underline{\Phi}_2, \ldots, \underline{\Phi}_n\right] \quad , \quad \underline{U} = \left[\nu_1, \nu_3, \ldots, \nu_{2j-1}, \ldots, \nu_{2n-1}\right]' \quad (27)$$

The solution for the displacement vector $\underline{X}$ given by Eqs. (26) and (27) along with Eq. (25a) is simply the solution one would have obtained from the modal analysis of a classically damped system. It can also be shown that the solution of the velocity response $\underline{\dot{X}}$ also reduces to that of the classically damped system using Eqs. (25a) and (25b).

In the following sections different approaches that can be used to solve Eqs. (20a-b) for both deterministic and non-stationary stochastic ground excitations are described.

SECTION 4

DETERMINISTIC RESPONSE TO SPECIFIC EXCITATION

4.1 Direct Numerical Integration

Given the record of ground excitation, Eqs. (20a-b) can be numerically integrated and the response state vector $\underline{Y}(t)$ can be obtained by using the transformation of Eq. (17). Again, all the parameters in these equations are real.

4.2 Impulse Response Function

One can obtain the impulse response vector $\underline{h}_{\nu_j}(t)$ for mode j, $\underline{\nu}_j(t) = \left[\nu_{2j-1}, \nu_{2j}\right]'$, due to the ground acceleration $\ddot{x}_g(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function. The jth modal impulse response vector is given by

$$\underline{h}_{\nu_j}(t) = \begin{Bmatrix} (e^{-\xi_j\omega_j t}\cos\omega_{Dj}t)\,F_{2j-1} + (e^{-\xi_j\omega_j t}\sin\omega_{Dj}t)\,F_{2j} \\ -(e^{-\xi_j\omega_j t}\sin\omega_{Dj}t)\,F_{2j-1} + (e^{-\xi_j\omega_j t}\cos\omega_{Dj}t)\,F_{2j} \end{Bmatrix} \qquad (28)$$

and the response vector, $\underline{\nu}_j(t)$, corresponding to mode j at time t is

$$\underline{\nu}_j(t) = \int_0^t \underline{h}_{\nu_j}(t-\tau)\,\ddot{x}_g(\tau)d\tau \qquad (29)$$

Again Eq. (29) can be integrated numerically and the response state vector $\underline{Y}(t)$ can be obtained through the transformation of Eq. (17).

4.3 Frequency Response Function

Let $\underline{H}_{\nu_j}(\omega)$ be the complex frequency response vector for mode j due to ground acceleration, $\ddot{x}_g = e^{\underline{i}\omega t}$. Then, $\underline{H}_{\nu_j}(\omega)$ can be obtained easily from Eqs. (20a-b) as follows

$$
\underline{H}_{\nu_j}(\omega) = \left\{
\begin{array}{c}
\dfrac{\xi_j\omega_j\; F_{2j-1} + \omega_{Dj}\; F_{2j} + \underline{i}\omega\; F_{2j-1}}{-\omega^2 + 2\underline{i}\; \xi_j\omega_j\omega + \omega_j^2} \\[4mm]
\dfrac{\xi_j\omega_j\; F_{2j} - \omega_{Dj}\; F_{2j-1} + \underline{i}\omega\; F_{2j}}{-\omega^2 + 2\underline{i}\; \xi_j\omega_j\omega + \omega_j^2}
\end{array}
\right\}
\tag{30}
$$

Note that the vectors $\underline{H}_{\nu_j}(\omega)$ and $\underline{h}_{\nu_j}(t)$ are related through the Fourier transform pair, i.e.,

$$
\underline{h}_{\nu_j}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{H}_{\nu_j}(\omega)\; e^{\underline{i}\omega t}\; d\omega \quad ; \quad \underline{H}_{\nu_j}(\omega) = \int_{-\infty}^{\infty} \underline{h}_{\nu_j}(t)\; e^{-\underline{i}\omega t}\; dt \tag{31}
$$

If $\underline{\nu}_j(\omega)$ denotes the Fourier transform of the response vector of mode j, $\underline{\nu}_j(t)$, then

$$
\underline{\nu}_j(\omega) = \underline{H}_{\nu_j}(\omega)\; \ddot{x}_g(\omega) \tag{32}
$$

in which $\ddot{x}_g(\omega)$ is the Fourier transform of the ground acceleration, $\ddot{x}_g(t)$. Finally, the jth modal response vector in the time domain can be obtained as follows

$$
\underline{\nu}_j(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\nu}_j(\omega)\; e^{\underline{i}\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{H}_{\nu_j}(\omega)\; \ddot{x}_g(\omega)\; e^{\underline{i}\omega t} d\omega \tag{33}
$$

the above calculation can be carried out efficiently using the Fast Fourier transform (FFT) algorithm.

SECTION 5

STOCHASTIC RESPONSE TO RANDOM EXCITATION

The canonical modal decomposition method presented above can be used conveniently to obtain the response of a nonclassically damped system to a nonstationary random ground acceleration [14, 15]. The expressions for mean squares of the response state vector, $\underline{Y}(t)$, will be derived in the following.

Often, the earthquake ground excitation, $\ddot{x}_g(t)$, can be modeled as a uniformly modulated nonstationary random process with zero mean

$$\ddot{x}_g(t) = \alpha(t) \ \ddot{x}_0(t) \tag{34}$$

in which $\alpha(t)$ = a deterministic non-negative modulating or envelope function and $\ddot{x}_0(t)$ is a stationary random process with zero mean and a power spectral density, $\phi_{\ddot{x}\ddot{x}}(\omega)$. A commonly used functional form for the spectral density is

$$\phi_{\ddot{x}\ddot{x}}(\omega) = \frac{1 + 4 \ \xi_g^2 \ (\omega/\omega_g)^2}{\left[1 - (\omega/\omega_g)^2\right]^2 + 4 \ \xi_g^2 \ (\omega/\omega_g)^2} \ S^2 \tag{35}$$

in which ξ_g, ω_g, and S are parameters depending on the intensity and the characteristics of the earthquake at a particular geological location.

Various types of the envelope function $\alpha(t)$ have been suggested in the literature to introduce the nonstationarity of the ground acceleration into Eq. (34). One possible form of $\alpha(t)$ is: $\alpha(t) = (t/t_1)^2$ for $0 \leq t \leq t_1$, $\alpha(t) = 1$ for $t_1 \leq t \leq t_2$, and $\alpha(t) = \exp[-\beta(t-t_2)]$ for $t > t_2$. Note that t_1, t_2 and β can be selected to reflect the shape and duration of the earthquake ground acceleration. When $\alpha(t) = 1$, it follows from Eq. (34) that the

ground acceleration is a stationary random process. The stationary assumption is reasonable when the duration of the strong shaking of the earthquake ground motion is much longer than the natural period of the structure.

Since the ground acceleration $\ddot{x}_g(t)$ is a zero mean random process, the mean value of all the response quantities are zero as well. Let $\underline{h}_\nu(t)$ be the impulse response vector of $\underline{\nu}(t)$, i.e., $\underline{h}_\nu(t) = \left[\underline{h}'_{\nu 1}(t), \underline{h}'_{\nu 2}(t), \ldots, \underline{h}'_{\nu n}(t)\right]'$. Then, the response state vector $\underline{Y}(t)$ is given by

$$\underline{Y}(t) = \int_0^t \underline{T} \, \underline{h}_\nu(\tau) \, \ddot{x}_g(t-\tau) \, d\tau \qquad (36)$$

The covariance matrix $\underline{K}_Y(t,t)$ of the response state vector, $\underline{Y}(t)$, is obtained from Eq. (36) as

$$\underline{K}_Y(t,t) = E\left[\int_0^t \underline{T} \, \underline{h}_\nu(\tau_1) \, \ddot{x}_g(t-\tau_1) d\tau_1 \int_0^t \underline{h}'_\nu(\tau_2) \, \underline{T}' \, \ddot{x}_g(t-\tau_2) d\tau_2 \right]$$

$$= \int_0^t \int_0^t \underline{T} \, \underline{h}_\nu(\tau_1) \, \alpha(t-\tau_1) \, R_{\ddot{x}\ddot{x}}(\tau_1-\tau_2) \, \underline{h}'_\nu(\tau_2) \, \underline{T}' \, \alpha(t-\tau_2) \, d\tau_1 d\tau_2 \qquad (37)$$

in which Eq. (34) has been used, and $R_{\ddot{x}\ddot{x}}(t)$ is the autocorrelation function of the stationary random process $\ddot{x}_0(t)$, which is related to the spectral density $\phi_{\ddot{x}\ddot{x}}(\omega)$ through the Wiener-Khinchin's relation

$$R_{\ddot{x}\ddot{x}}(\tau_1-\tau_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{\ddot{x}\ddot{x}}(\omega) \, e^{-i\omega(\tau_1-\tau_2)} \, d\omega \qquad (38)$$

Substituting Eq. (38) into Eq. (37), one obtains

$$\underset{\sim}{K}_Y(t,t) = \int_{-\infty}^{\infty} \underset{\sim}{M}_Y(t,\omega) \; \underset{\sim}{M}_Y^{'*}(t,\omega) \; \phi_{\ddot{x}\ddot{x}}(\omega) d\omega \qquad (39)$$

in which a star denotes the complex conjugate and

$$\underset{\sim}{M}_Y(t,\omega) = \int_0^t \underset{\sim}{T} \; \underset{\sim}{h}_\nu(\tau) \; \alpha(t-\tau) \; e^{-i\omega\tau} \; d\tau \qquad (40)$$

The variance vector $\underset{\sim}{\sigma}_Y^2(t)$ of $\underset{\sim}{Y}(t)$, which is equal to the mean square response vector of $\underset{\sim}{Y}(t)$, consists of the diagonal elements of $\underset{\sim}{K}_Y(t,t)$ and it can be expressed as follows

$$\underset{\sim}{\sigma}_Y^2(t) = \int_{-\infty}^{\infty} \mid \underset{\sim}{M}_Y(t,\omega) \mid^2 \phi_{\ddot{x}\ddot{x}}(\omega) \; d\omega \qquad (41)$$

in which $\mid \underset{\sim}{M}_Y(t,\omega) \mid^2 \phi_{\ddot{x}\ddot{x}}(\omega)$ is the evolutionary power spectral density cf the state vector, $\underset{\sim}{Y}(t)$, and $\mid \underset{\sim}{M}_Y(t,\omega) \mid^2$ is a vector in which its elements are the squares of the absolute value of the corresponding elements of $\underset{\sim}{M}_Y(t,\omega)$ given by Eq. (40).

The nonstationary mean square response given by Eq. (41) can be computed easily as follows. Firstly, the impulse response vector $\underset{\sim}{h}_\nu(t)$ can either be computed directly from Eq. (28) or indirectly from the frequence response vector $\underset{\sim}{H}_\nu(\omega)$ given by Eq. (30) using the FFT technique. Secondly, the vector $\underset{\sim}{M}_Y(t,\omega)$ is evaluated from Eq. (40) using the FFT technique again. Finally, the time dependent root mean square response, $\underset{\sim}{\sigma}_Y(t)$, is obtained by numerically integrating Eq. (41) and taking the square root.

SECTION 6

NUMERICAL EXAMPLES

The canonical modal analysis presented in this paper will be used to study the response behavior of nonclassically damped structural systems subjected to earthquake-type excitations. Emphasis will be placed on the response behavior of primary-secondary structural system. Parametric studies will be conducted for the response of secondary system, such as a light equipment attached to a structure. In particular, under what condition the nonclassical damping may be significant. Frequently, it may be assumed that the structure and equipment are classically damped individually. However, because of different damping characteristics of the equipment and the structure, the combined equipment-structure system generally is not classically damped.

Igusa, Der Kiureghian, and Sackman [6] considered a single-degree-of-freedom equipment attached to a single-degree-of-freedom structure and subjected to a stationary white noise ground excitation. They showed that at tuning (when frequencies of the equipment and structure coincide) the effect of nonclassical damping on the response of light equipment becomes significant. Here we investigate the effect of nonclassical damping on somewhat complex equipment-structure systems excited by an earthquake ground acceleration. Young and Lin [16] and HoLung, Cai and Lin [5] conducted extensive parametric studies for the frequency response function of the primary-secondary structural system. Here we examine the response of the equipment-structure system to both deterministic and nonstationary stochastic ground excitations.

Of particular interest is the case in which the primary structure itself is nonclassically damped. Hence, the second class of problems studied herein deals with the effect of nonclassical damping on the

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structure itself. An eight story shear beam type building is considered and the distribution of the damping within the structure is slightly varied to study the effect of nonclassical damping on the response.

All the example problems studied are subjected to either a simulated sample ground motion or the ground motion that is modeled as a nonstationary random process described in Eqs. (31-32). The parameters that define the envelope function, $\alpha(t)$, and the spectral density, $\phi_{xx}(\omega)$, of the earthquake model are: $t_1 = 3\text{sec.}$, $t_2 = 13 \text{ sec.}$, $\beta = 0.26$, $\omega_g = 3.0 \text{ Hz}$, $\xi_g = 0.65$ and $S^2 = 74.7 \times 10^{-4} \text{ m}^2/\text{sec.}^3/\text{rad}$. A sample function of the earthquake ground acceleration is simulated and shown in Fig. 1.

The first equipment-structure example consists of a two-degree-of-freedom shear beam type structure with a single-degree-of-freedom light equipment attached to it as shown in Fig. 2(a). This primary structure is classically damped if $c_1/k_1 = c_2/k_2$ and the combined equipment-structure system is classically damped if $c_1/k_1 = c_2/k_2 = c_e/k_e$ in which the subscript e refers to the equipment. The mass and stiffness of each story unit of the primary structure are: $m_1 = m_2 = m = 30$ tons; $k_1 = k_2 = k = 19,379$ kN/m. The natural frequencies of the primary structure are 2.5 and 6.5 Hz, respectively. Parametric studies will be carried out by varying the distribution of the damping of the primary structure and the equipment damping.

Let the values of c_1 and c_2 be equal to 123.4 kN/m/sec., so that the primary structure is classically damped with the first modal damping ratio of approximately 5%. Given a mass ratio, the damping ratio of the equipment is varied and the response of the equipment, i.e., displacement relative to the attachment point, is evaluated by the exact method presented in this paper. The results are compared against those obtained using the approximate classically damped approach. The approximate classically damped approach is to decouple the second order equations of motion using

eigenvectors of the undamped system and disregard the off diagonal terms of the $\underline{\Phi}' \underline{C} \underline{\Phi}$ matrix, where $\underline{\Phi}$ is the (nxn) modal matrix of the undamped primary -secondary system, Eq. (27).

Let the equipment frequency, ω_e, be tuned to the fundamental frequency of the primary system, i.e., $\omega_e - 2.5$ Hz. The response of the equipment to the simulated deterministic ground acceleration shown in Fig. 1 is presented in Fig. 3 for different values of equipment damping ratio, ξ_e, and mass ratio, γ. The mass ratio, γ, of the equipment is the ratio of the equipment mass to first modal mass of the primary structure that is equal to 30 tons. In Fig. 3, the ordinate is the maximum response of the equipment relative to the attachment point in 30 seconds of earthquake episode, and the abscissa, η, is the ratio of the equipment damping ratio, ξ_e, to the unique damping ratio of the equipment, ξ_{ec}, that would make the combined equipment-structure system classically damped. For the classically damped shear beam type primary system of Fig. 2(a), the damping ratio of the equipment, ξ_{ec}, that results in a classically damped primary-secondary system can be obtained using the Caughey-O'Kelly identity, $\xi_{ec} - (\omega_e/2)(c_i/k_i)$, in which i refers to any of the degrees of freedom of the primary system. Likewise, when the primary system is classically damped, ξ_{ec} is equal to the jth modal damping ratio of the primary structure if the equipment is tuned to the jth mode of the primary structure. Hence in the present example, ξ_{ec} is equal to the first modal damping ratio of the primary structure which is 5%.

Fig. 3 presents the results for three different values of the equipment mass ratio, γ. From this figure it is clear that the equipment response increases as its damping ratio ξ_e decreases. The results based on the approximate classically damped approach start to deviate from the exact

solution only when $\xi_e < \xi_{ec}$. The deviation increases as the equipment damping ratio or the mass ratio decreases. Further, the solutions obtained using the approximate classically damped approach in the region $\xi_e < \xi_{ec}$ are nonconservative. In other words, the effect of nonclassical damping becomes significant only when the equipment damping ratio, ξ_e, is smaller than ξ_{ec} that results in a classical damping for the combined equipment-structure system. It is also evident that the smaller the mass ratio or the quipment damping, the more pronounced is the effect of neglecting the off-diagonal terms of the $\underline{\Phi}' \underline{C} \underline{\Phi}$ matrix.

Figures 4(a), (b) and (c) show the effect of nonclassical damping on the response of the equipment that is not tuned to any of the frequencies of the primary structure. The equipment frequency ω_e is chosen to be the average of the first two frequencies of the primary structure, i.e., $\omega_e = (\omega_1 + \omega_2)/2 = 4.5$ Hz. From these figures, it is clear that for detuned equipment attached to the two-story primary structure of Fig. 2(a), the effect of nonclassical damping may be ignored without causing any problem. Likewise, the maximum equipment response is not sensitive to the mass ratio.

The observations made from Figs. 3 and 4 above hold for a SDOF equipment attached to a two-story shear beam type primary structure. The second example consists of an eight-degree-of-freedom shear beam type primary structure with a single-degree-of-freedom equipment attached to the seventh story unit as shown in Fig. 2(b). The mass and stiffness of each story unit of the primary structure are: $m_1 = m_2 = \ldots = m_8 = m = 345.6$ tons and $k_1 = k_2 = \ldots = k_8 = k = 3.4 \times 10^5$ kN/m. The undamped natural frequencies of the primary structure are $\omega_1 = 0.92$, $\omega_2 = 2.73$, $\omega_3 = 4.45$, $\omega_4 = 6.02$, $\omega_5 = 7.38$, $\omega_6 = 8.49$, $\omega_7 = 9.32$, $\omega_8 = 9.82$ Hz and the first modal mass is 345.6 tons. Let the values of c_1 through c_8 be equal to 2,937 kN/m/sec, such that the primary structure is classically damped with the first modal damping ratio

of approximately 2.5 %. In this example the equipment frequency, ω_e, is tuned to the <u>third natural frequency</u> of the primary structure, i.e., ω_e = 4.45 Hz. This results in the value of ξ_{ec} of approximately 12%. Again, the simulated earthquake ground acceleration shown in Fig. 1 is used as the input excitation.

Figure 5 shows the response of the equipment for various values of equipment damping ratio, ξ_e, and mass ratio, γ. Similar to the results for the two-degree-of-freedom primary system, the effect of nonclassical damping is generally significant only when the equipment mass ratio is small and its damping ratio is smaller than ξ_{ec}. However, the results seem to indicate that for certain tuned equipment structure systems, ignoring the nonclassical damping may indeed result in conservative equipment response, see Fig. 5(c)

Figure 6(a) shows the effect of nonclassical damping on the response of a detuned equipment attached to the eight story primary structure of Fig. 2(b). The frequency of the equipment is set to be equal to the average of the first and second frequency of the primary structure, i.e., ω_e =$(\omega_1+\omega_2)/2$ = 1.83 Hz, which results in a value of ξ_{ec} of approximately 5%. Fig. 6(b) shows the effect of nonclassical damping for the same equipment-structure system except that the frequency of the equipment is now set to be equal to the average of the second and third frequency of the primary system. This results in an equipment frequency of 3.6 Hz and a ξ_{ec} value of approximately 9.7%. From these figures it is clear that the effect of nonclassical damping is insignificant at all.

However, as the frequency of the detuned equipment increases, the situation may be different. Figure 6(c) presents the equipment response when the equipment frequency is equal to the average of the fourth and fifth frequencies of the primary structure, i.e., ω_e = $(\omega_4+\omega_5)/2$ = 6.70 Hz. Note

that for this equipment-structure system, the value of ξ_{ec} is approximately 18%. When the equipment frequency is set to be equal to the average of the seventh and eighth frequencies of the primary structure, i.e., $\omega_e = (\omega_7 + \omega_8)/2$ = 9.57 Hz, the effect of nonclassical damping is shown in Fig. 6(d). Figure 6(e) presents the same results as those of Fig. 6(c) except that the mass ratio of the equipment is now increased by a factor of hundred.

Figures 6(c) through 6(e) indicate that the effect of nonclassical damping becomes significant for detuned equipments when (i) the frequency of the detuned equipment is high, and (ii) the equipment damping ratio is small. Under this circumstance the approximate classically damped procedure results in a higher equipment response than the exact solution. This phenomenon can be explained in the following. The spacing between natural frequencies of the primary system reduces in the high frequency range. In other words, higher natrual frequencies tend to be closely spaced as evidenced by the primary structure considered herein. When the equipment has a high frequency, although detuned, it will interact with its neighboring structural frequencies due to its clcseness to them. Because of such modal coupling and interaction, the equipment response tends to decrease as observed also in Ref. 5. Such a trend becomes stronger as the damping of the primary system increases. Hence, the observation made above holds for high frequency equipment. However, at such a high frequency, the damping of the primary structure is also pretty high, and it is questionable whether the viscous damping assumption is still valid for the primary structure.

Suppose the damping of each story unit of the primary structure is reduced by a factor of five. This results in a value of ξ_{ec} five times smaller. When the equipment is detuned at a frequency of $\omega_e = (\omega_4 + \omega_5)/2$, the response of the equipment is displayed in Fig. 6(f). This figure

presents the same results as those of Fig. 6(c) except that the damping of the primary structure is now five times smaller. From this figure, the effect of nonclassical damping for this primary-secondary system is not nearly as significant as that of Fig. 6(c). As a result, damping of the primary structure also plays an important role for the response of high frequency equipment. In other words, the modal coupling and interaction increases not only as the equipment frequency increases, but also as the structural damping increases. Therefore, it is reasonable to infer that for detuned equipment at high frequency, the approximate classically damped procedure results in significantly higher equipment response than the exact procedure, if the modal damping of the primary structure adjacent to the equipment node is high and the damping of the equipment is smaller than ζ_{ec}.

When the primary system is nonclassically damped, the response of the secondary system to the simulated ground motion in Fig. 1 will be investigated. Again, consider the equiment-structure system of Fig. 2(a), in which values of c_1 and c_2 are selected such that the primary structure itself is nonclassically damped. Two different damping distributions for the primary strucure are considered. In the first case, all damping of the primary structure of Fig. 2(a) is placed in the first story unit; with the results $c_1 = 246.8$ kN/m/sec and $c_2 = 0.0$. In the second case, all damping of the structure is placed in the second story unit, leading to the results $c_1 = 0.0$ and $c_2 = 246.8$ kN/m/sec. The damping ratio of the equipment is varied and the response behavior of the equipment will be examined. It has been demonstrated above that ignoring the effect of nonclassical damping results in unconservative responses for tuned equipment with small mass ratio γ and small damping ratio, ζ_e. Equipment with such characteristics will be considered in the following.

Figures 7 and 8 show the maximum response in 30 seconds of the time history for an equipment that is attached to a nonclassically damped primary structure described above. The equipment frequency, ω_e, is tuned to the undamped fundamental frequency of the primary structure. In these figures, the ordinate is the maximum displacement of the equipment relative to the attachment point and the abscissa, $\eta' = \xi_e/\xi'_{ec}$, is the ratio of the damping ratio of the equipment to the approximate ξ_{ec}, denoted by ξ'_{ec}, as described in the following.

For tuned equipment attached to classically damped primary structures, the damping ratio ξ_{ec} of the equipment which results in a classically damped primary-secondary system is the same as the modal damping ratio of the primary structure in which the equipment is tuned to. However, when the primary structure is nonclassically damped, one can obtain an approximate value for ξ_{ec}, denoted by ξ'_{ec}, by treating the primary system as being an equivalent classically damped system. An equivalent classically damped primary system is obtained from the original primary system by neglecting the off-diagonal terms of the $\bar{\Phi}\ \bar{C}\ \bar{\Phi}$ matrix, where $\bar{C}$ is the damping matrix of the primary structure and $\bar{\Phi}$ is the modal matrix of the undamped primary structure. Thus, the jth equivalent classical modal damping ratios of the primary structure, denoted by ξ'_j , is obtained by dividing the diagnonal terms of the $\bar{\Phi}\ \bar{C}\ \bar{\Phi}$ matrix by twice the corresponding jth undamped modal frequency. Finally, ξ'_{ec} can be obtained from the equivalent classical modal damping ratios of the primary structure (see Ref. [1]). For the two nonclassically damped primary-secondary structures considered above, the approximate first modal damping ratios are $\xi'_1 = \xi'_{ec} = 7\%$ ($c_1=2c$, $c_2=0$) and $\xi'_1 = \xi'_{ec} = 2.8\%$ ($c_1=0$, $c_2=2c$), respectively. Recall that for this primary structure, distributing the damping equally between the two-story units

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results in a classically damped structure with the first modal damping ratio of approximately 5%, i.e., $\xi_1' = \xi_{ec} = 5\%$.

As expected, the results shown in Figs. 7 and 8 indicate that the distribution of damping in the primary structure has a significant effect on the response of the tuned equipment. Likewise, the effect of nonclassical damping is significant only when the equipment damping is small. When the damping of the equipment ξ_e is equal to ξ_{ec}', i.e., $\eta' = \xi_e/\xi_{ec}'-1$, it is observed from Figs. 7 and 8 that the exact equipment response and the approximate classically damped results are almost identical. Such a solution at ξ_e/ξ_{ec}' is denoted by $(U_{ec})\max$. To examine whether ξ_{ec}' is a useful parameter for measuring the effect of nonclassical damping, results in Figs. 7 and 8 are replotted in Fig. 9 in a different form. In this figure the ordinate is the exact maximum equipment response, denoted by $(U_e)\max$, normalized by the value $(U_{ec})\max$. Also plotted in Fig. 9 are the corresponding results when the primary stucture is classically damped $(c_1=c_2=c)$. From this figure it is clear that the approximate modal damping ratios obtained from the diagonal terms of the $\underline{\Phi}'$ $\underline{C}$ $\underline{\Phi}$ matrix of the primary structure can be used as a useful measure in determining the effect of nonclassical damping on the response of tuned equipment even if the primary structure is not classically damped.

A nonstationary stochastic ground acceleration with a power-spectral density, $\phi_{\ddot{x}\ddot{x}}$ (ω), and an envelope function, $\alpha(t)$, described previously is considered as the input excitation. The primary structure is a two-degree-of-freedom classically damped structure shown in Fig. 2(a). Since the mean value of earthquake ground acceleration is zero, the mean value of response quantities are all zero. Therefore, the root mean square (rms) of the response is equal to the standard deviation. The time dependent root mean square (rms) response of an equipment tuned to the first mode, i.e., $\omega_e =$

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ω_1, is shown in Fig. 10 for several values of equipment damping ratio. The corresponsing results for a detuned equipment with $\omega_e = (\omega_1+\omega_2)/2$ are displayed in Fig. 11. As expected, the rms response increases as the equipment damping reduces and the response for a tuned equipment is at least one order of magnitude larger than that of a detuned equipment. Further, a comparison between Figs. 10 and 11 indicates that the root mean square response of a tuned equipment is quite sensitive to the equipment damping ξ_e, whereas this is not the case for a detuned equipment.

Extensive numerical results show that the effect of nonclassical damping on the rms response of a detuned equipment with $\omega_e = (\omega_1+\omega_2)/2$ is insignificant. For tuned equipment with $\omega_e = \omega_1$, the effect of nonclassical damping on the maximum rms reponse in 30 seconds is shown in Fig. 12 for three different values of mass ratio, γ. In Fig. 12 the ordinate is the maximum rms of the equipment response in 30 seconds of earthquake episode, and the abscissa is $\eta = \xi_e/\xi_{ec} = \xi_e/0.05$ as described in the first example. The results based on exact solution and approximate classically damped approach are presented in the figure. Similar to the response of equipment subjected to a simulated deterministic excitation presented previously, the effect of nonclassical damping for tuned equipment attached to the two story primary structure is pronounced only when the equipment is light and its damping ratio ξ_e is smaller than ξ_{ec}, i.e., $\eta = \xi_e/\xi_{ec} < 1$. Further, the approximate classically damped solutions are unconservative.

Next, the effect of nonclassical damping on the maximum rms response of the equipment that is detuned at the higher frequency is investigated. Again, consider the eight-story primary-secondary structure of Fig. 2(b) in which the equipment frequency is equal to the average of the fourth and fifth frequencies of the primary structure, i.e., $\omega_e = (\omega_4+\omega_5)/2 = 6.7$ Hz. Figs. 13(a)-(b) show the maximum rms of the equipment response, and Fig.

13(c) shows the same results as those of Fig. 13(a) except that the damping of the primary structure is reduced by a factor of 5. From these figures, it is observed that the effect of nonclassical damping is significant for the detuned equipment, when it satisfies the following conditions simultaneously: (i) the equipment is detuned at high frequency with a high ξ_{ec} value, (ii) the equipment damping is small, i.e., $\xi_e/\xi_{ec} < 1$, and (iii) the mass ratio is small. Under this circumstance, the rms response obtained using approximate classically damped procedures is always higher than the exact solution. Again, this is due to the modal coupling and interaction of the equipment and the primary system. These conclusions are identical to those obtained previously, when the excitation is a simulated deterministic sample earthquake ground motion.

The intensity of earthquake ground acceleration usually consists of three segments as shown by the envelope function $\alpha(t)$, Eq. (34). The intensity builds up in the first segment $(0, t_1)$ and reaches a stationary magnitude in the second segment (t_1, t_2), representing the most intense portion of earthquake. The response history of structures also consists of three segments and its stationary rms value in the second segment may be obtained using the stationary random vibration analysis. The transient response in the first segment resulting from zero initial conditions as well as transient earthquake excitations is usually smaller than the stationary response in the second segment. However, under suitable conditions, such as flight vehicles subjected to atmospheric turbulences, the transient response at some point in time in the first segment may exceed the stationary response in the second segment [7]. Such an overshooting phenomenon is important in the design of structures. The overshooting phenomenon may occur if the excitation is applied (or builds up) suddenly. Under ordinary conditions where the initial conditions for the structure are zero and where

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the earthquake excitation also builds up from zero at time zero, such an overshooting phenomenon is unlikely. For tuned equipment, however, the displacement or acceleration response may overshoot. Parametric studies will be made to examine the possibility of overshooting, in particular, the rate of increase of the envelope function will be varied.

The two-degree-of-freedom primary structure with a single-degree-of freedom equipment illustrated in the first example is considered. The equipment is tuned to the first natural frequency of the primary structure, i.e., $\omega_e - \omega_1 - 2.5$ Hz and the damping ratio of the equipment, is set at 5%. The first and second modal damping ratios of the primary structure are 2% and 5.2%, respectively. The power spectral density, $\phi_{\ddot{x}\ddot{x}}(\omega)$, of the stationary ground acceleration is identical to that described previously except the value of ω_g is now at 2.5 Hz. The envelope function $\alpha(t)$, is given as follows: $\alpha(t) - (t/t_1)^4$ for $0 \leq t \leq t_1$. $\alpha(t) - 1$ for $t_1 < t < t_2$ and $\alpha(t) - \exp[-\beta(t-t_2)]$ for $t < t_2$. This envelope function results in a ground acceleration that builds up toward a stationary value $\alpha(t) - 1$ at a faster rate than the envelope function used in the previous example. The time dependent rms responses of the primary and secondary structure, including the rms of the relative displacement and acceleration, are shown in Figs. 14(a) through 14(f). From these figures and extensive results obtained from parametric study, including variation of envelope function, variation of primary-secondary system, tuned or detuned equipments, etc., it is observed that the overshooting phenomenon for the displacement or acceleration response does not occur for either the primary or secondary system.

Finally, the effect of nonclassical damping on the response of a structure without any equipment is examined. An eight-story primary structure shown in Fig. 2(b) is considered and the effect of the distribution of damping on the response will be examined. Six different

distributions of damping for this structure are considered. In all cases, the distribution of damping among the story units are varied but the total damping in the structure is kept the same as the classically damped case (i.e., $c_1 = c_2 = \ldots = c_8 = 2,937$ KN/m/sec.) In case 1, the damping in each story unit is proportional to the height of the story, i.e., $c_i = ic_0$, in which i is the story number and $i = 1$ is the ground level. The second case is the opposite of the first case, i.e., the damping in each story unit is inversely proportional to its height, i.e., $c_i = (9-i)c_0$. In the third case, the largest damping is placed in the fourth story unit and the damping in other story units decreases linearly with respect to their distances from the fourth story unit. The fourth case is similar to the third case except that the largest damping is placed in the fifth story unit. In the fifth case, total damping is distributed equally among the first, third, fifth and seventh story units. The last case is similar to the fifth case, except that total damping is equally distributed among the second, fourth, sixth, and eighth story units.

The six nonclassically damped structures are subjected to the simulated ground acceleration of Fig. 1. Tables 1 and 2 present the results of floor displacements and story deformations for these structures. Also presented in these tables are the results obtained by neglecting the off diagonal terms of $\Phi' C \Phi$ matrix, where Φ is the modal matrix of the undamped structure. The results for the classically damped structure, i.e., (i.e., $c_1 = c_2 = \ldots = c_8 = 2,937$ KN/sec.) are presented in Table 3. An examination of these tables indicates that the effect of neglecting the off-diagonal terms of $\Phi' C \Phi$ matrix is insignificant for all the cases considered, where the distribution of damping along the building height varies slowly. The maximum error using the approximate solution was less than 4% for story displacement and less than 2% for story deformation.

Therefore, it may be concluded that for primary structures, the effect of nonclassical damping may be ignored if the damping distribution within the structure does not change drastically. Such a conclusion does not hold if extra high dampings are added to one or two story units, such as active or passive control devices.

TABLE 1

MAXIMUM FLOOR DISPLACEMENT (cm.)

		X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
	EXACT	3.33	6.55	9.58	12.71	14.51	16.23	17.40	17.99
CASE 1	APPROX. CLASS. DAMPED	3.31	6.53	9.55	12.24	14.49	16.22	17.38	17.96
	EXACT	2.66	5.23	7.65	9.80	11.61	13.00	13.92	14.38
CASE 2	APPROX. CLASS. DAMPED	2.66	5.24	7.66	9.81	11.60	12.97	13.90	14.37
	EXACT	2.90	5.72	8.34	10.65	12.59	14.09	15.10	15.61
CASE 3	APPROX. CLASS. DAMPED	2.89	5.70	8.32	10.64	12.58	14.07	15.08	15.58
	EXACT	3.00	5.90	8.62	11.03	13.03	14.58	15.63	16.16
CASE 4	APPROX. CLASS. DAMPED	2.99	5.88	8.59	11.00	13.02	14.56	15.60	16.13
	EXACT	2.84	5.59	8.16	10.50	12.42	13.90	14.90	15.40
CASE 5	APPROX. CLASS. DAMPED	2.83	5.57	8.14	10.43	12.34	13.81	14.82	15.32
	EXACT	3.08	6.06	8.90	11.38	13.48	15.08	16.15	16.69
CASE 6	APPROX. CLASS. DAMPED	3.08	6.06	8.84	11.32	13.39	14.98	16.04	16.57

TABLE 2

MAXIMUM FLOOR DEFORMATION (cm.)

		U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
CASE 1	EXACT	3.33	3.22	3.03	2.70	2.25	1.72	1.18	0.61
	APPROX. CLASS. DAMPED	3.31	3.22	3.02	2.70	2.25	1.73	1.18	0.61
C	EXACT	2.66	2.58	2.42	2.16	1.80	1.39	0.95	0.49
	APPROX. CLASS. DAMPED	2.66	2.58	2.42	2.15	1.79	1.38	0.95	0.49
CASE 3	EXACT	2.90	2.81	2.63	2.33	1.95	1.50	1.02	0.52
	APPROX. CLASS. DAMPED	2.89	2.80	2.62	2.34	1.94	1.49	1.01	0.52
CASE 4	EXACT	3.00	2.91	2.72	2.42	2.01	1.55	1.06	0.54
	APPROX. CLASS. DAMPED	2.99	2.90	2.71	2.42	2.02	1.55	1.06	0.54
CASE 5	EXACT	2.84	2.76	2.58	2.33	1.93	1.48	1.01	0.52
	APPROX. CLASS. DAMPED	2.83	2.74	2.57	2.30	1.91	1.47	1.01	0.51
CASE 6	EXACT	3.08	2.99	2.84	2.51	2.10	1.60	1.10	0.56
	APPROX. CLASS. DAMPED	3.08	2.98	2.79	2.49	2.07	1.59	1.08	0.55

TABLE 3

8-STORY CLASSICALLY DAMPED

STORY NUMBER (1)	1	2	3	4	5	6	7	8
MAXIMUM FLOOR DISPLACEMENT (X_i) (cm.)	2.94	5.80	8.47	10.85	12.83	14.35	15.39	15.91
MAXIMUM FLOOR DISPLACEMENT (U_i) (cm).	2.94	2.85	2.67	2.39	1.99	1.53	1.04	0.53
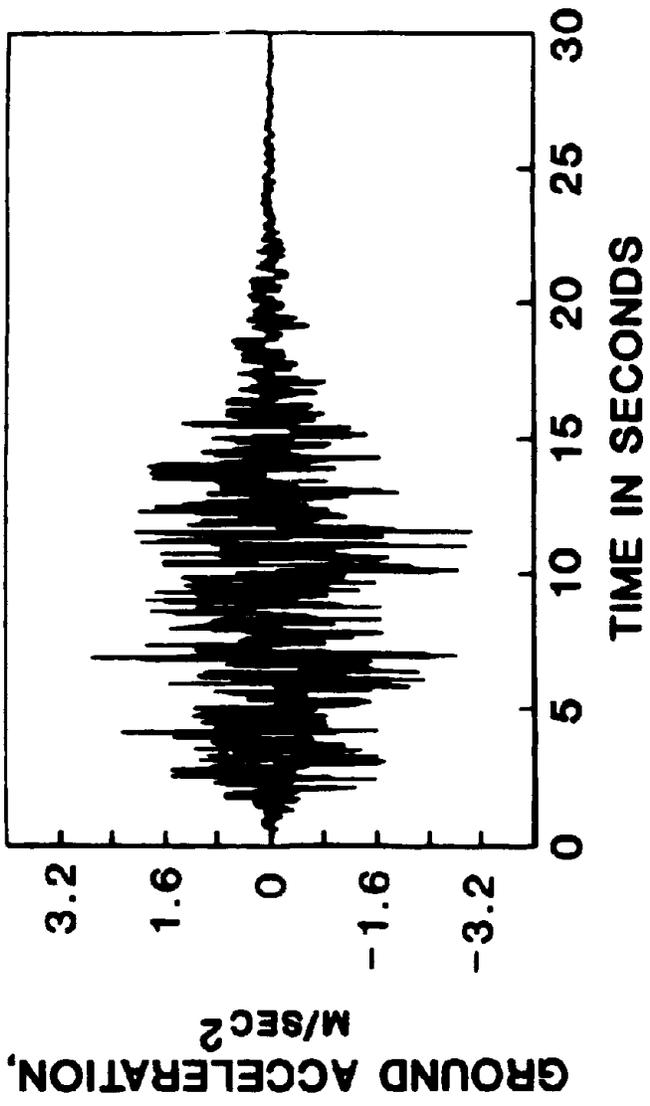
FIGURE 1   Simulated Ground Acceleration.

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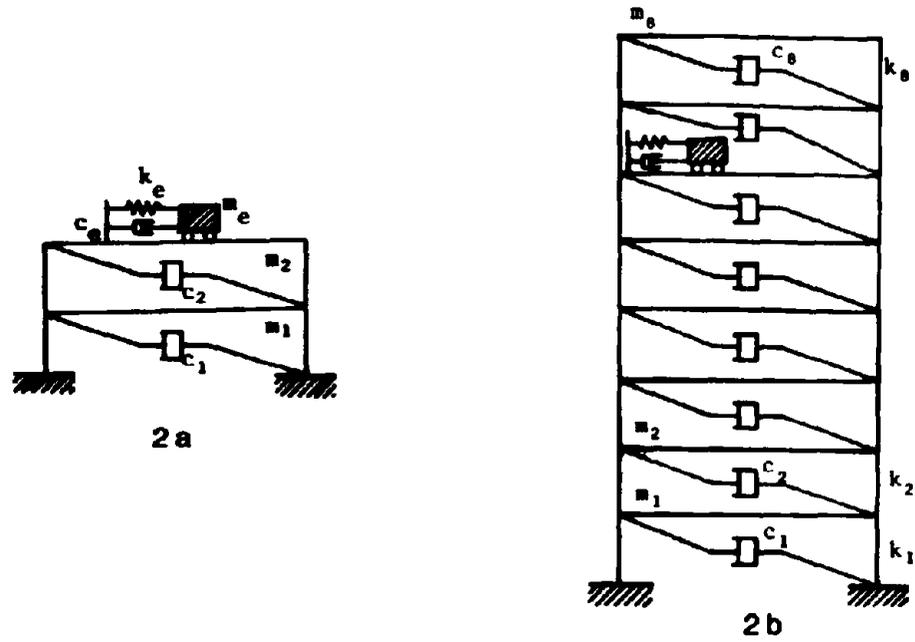
FIGURE 2   Primary Structure and Single Degree-of-Freedom Equipment.

(a) Two Story Primary Structure, (b) Eight Story Primary Structure.
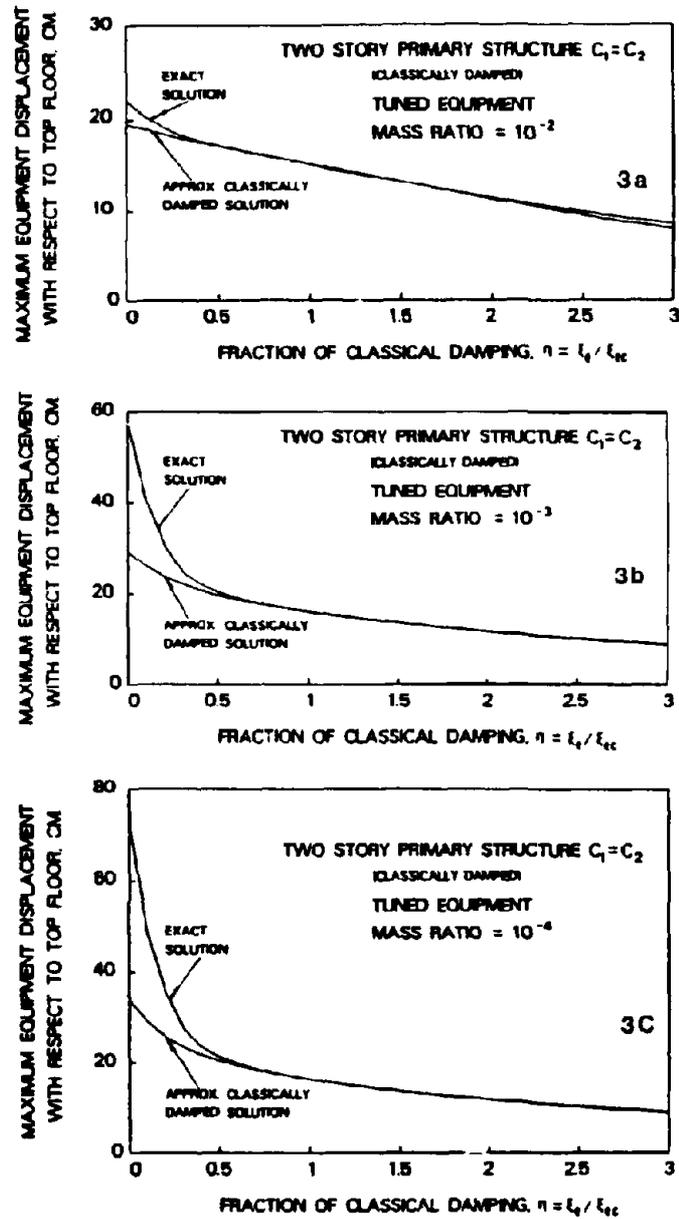
FIGURE 3   Maximum Relative Displacement of Tuned Equipment Attached To Classically Damped Two Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e/\xi_{ec}$.

(a) Mass Ratio $\gamma = 10^{-2}$, (b) Mass Ratio $\gamma = 10^{-3}$, (c) Mass Ratio $\gamma = 10^{-4}$.
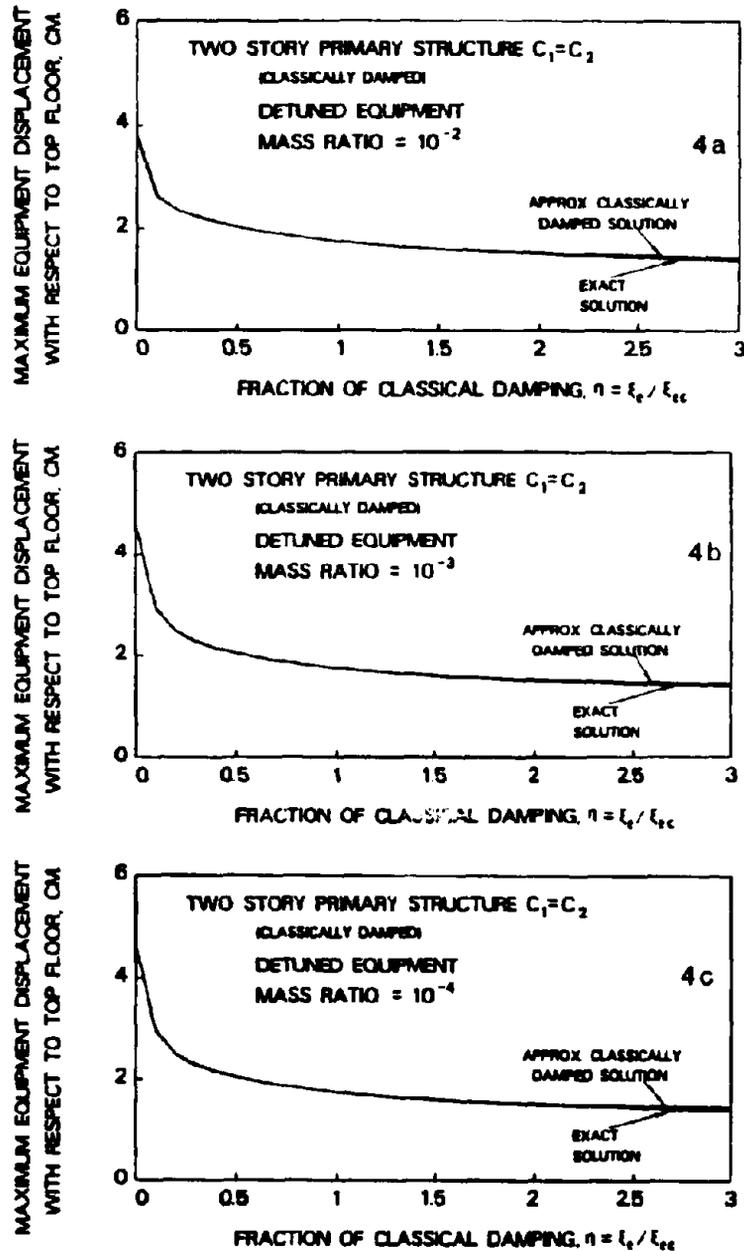
FIGURE 4   Maximum Relative Displacement of Detuned Equipment Attached to Classically Damped Two Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e/\xi_{ec}$.

(a) Mass Ratio $\gamma = 10^{-2}$, (b) Mass Ratio $\gamma = 10^{-3}$, (c) Mass Ratio $\gamma = 10^{-4}$.
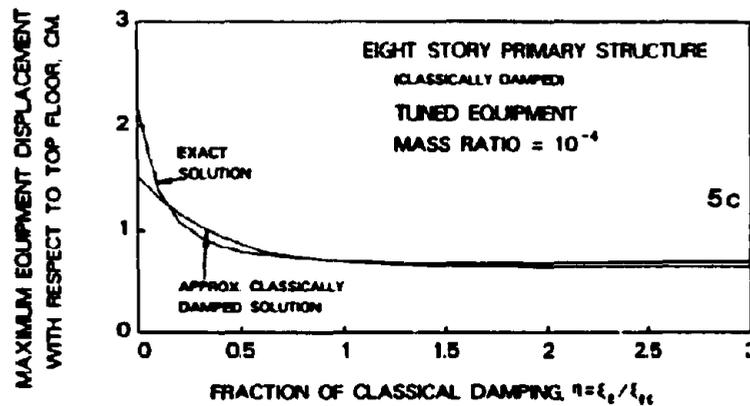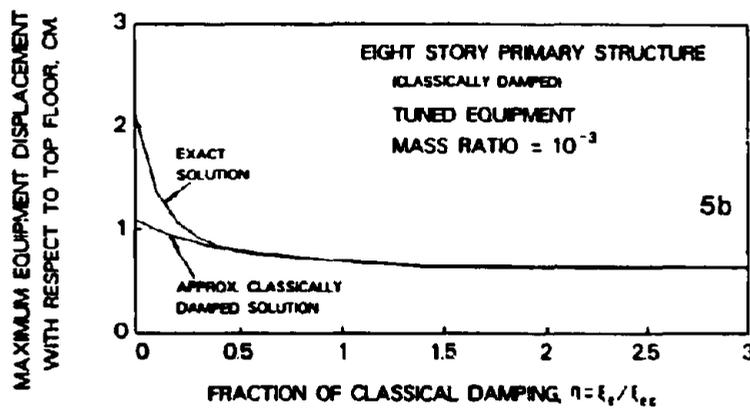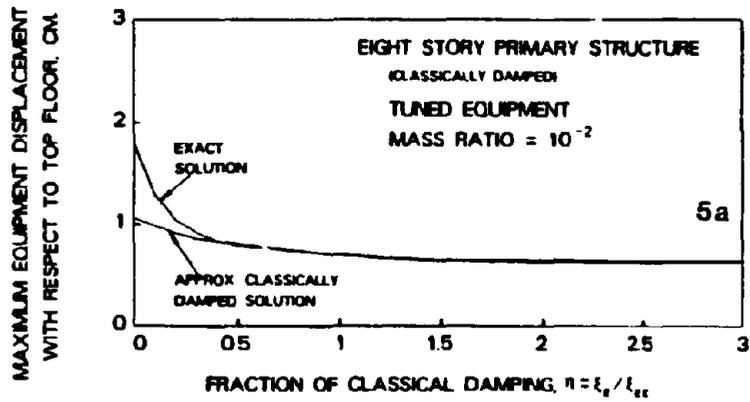
FIGURE 5   Maximum Relative Displacement of Tuned Equipment Attached to Classically Damped Eight Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e/\xi_{ec}$.

(a) Mass Ratio $\gamma = 10^{-2}$, (b) Mass Ratio $\gamma = 10^{-3}$, (c) Mass Ratio $\gamma = 10^{-4}$.
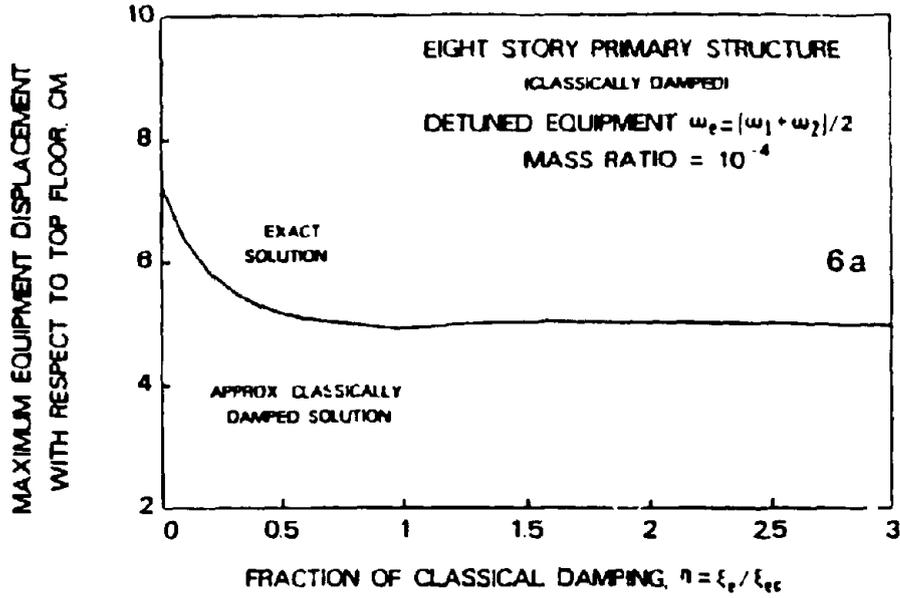
(a) EQUIPMENT FREQUENCY $\omega_e = (\omega_1 + \omega_2)/2$, MASS RATIO $\gamma = 10^{-4}$, $\xi_{ec} = 5\%$.
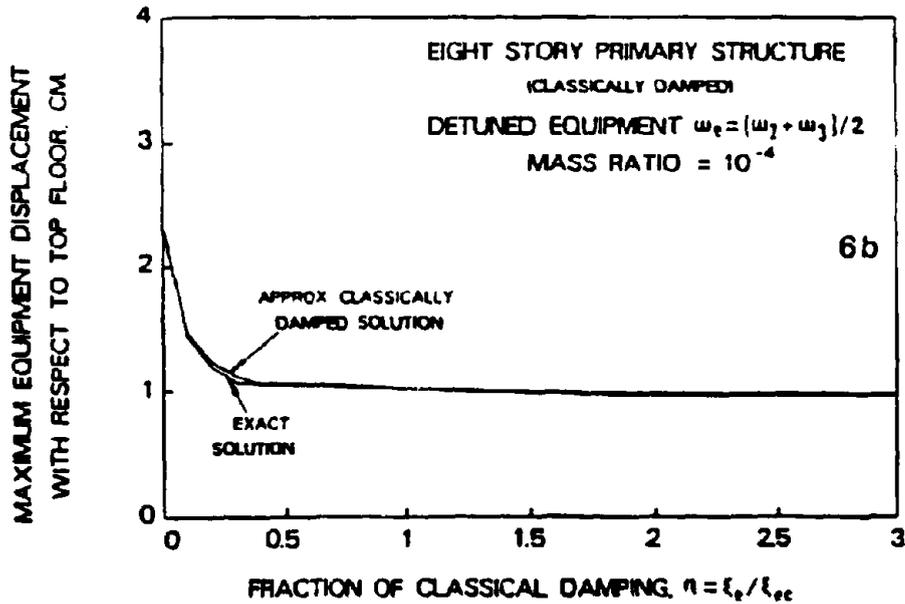
(b) EQUIPMENT FREQUENCY $\omega_e = (\omega_2 + \omega_3)/2$, MASS RATIO $\gamma = 10^{-4}$, $\xi_{ec} = 9.7\%$.

FIGURE 6   Maximum Relative Displacement of Detuned Equipment Attached to Eight Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e / \xi_{ec}$.

MAXIMUM EQUIPMENT DISPLACEMENT WITH RESPECT TO TOP FLOOR, MM

EIGHT STORY PRIMARY STRUCTURE
(CLASSICALLY DAMPED)

DETUNED EQUIPMENT $\omega_e = [\omega_4 + \omega_5]/2$
MASS RATIO = 10^{-4}

6c

APPROX CLASSICALLY DAMPED SOLUTION

EXACT SOLUTION

FRACTION OF CLASSICAL DAMPING, $\eta = \xi_e / \xi_{ec}$

(c) EQUIPMENT FREQUENCY $\omega_e - (\omega_4 + \omega_5)/2$, MASS RATIO $\gamma - 10^{-4}$, $\xi_{ec} - 18\%$.

MAXIMUM EQUIPMENT DISPLACEMENT WITH RESPECT TO TOP FLOOR, MM

EIGHT STORY PRIMARY STRUCTURE
(CLASSICALLY DAMPED)

DETUNED EQUIPMENT $\omega_e = [\omega_7 + \omega_8]/2$
MASS RATIO = 10^{-4}

6d

APPROX CLASSICALLY DAMPED SOLUTION

EXACT SOLUTION

FRACTION OF CLASSICAL DAMPING, $\eta = \xi_e / \xi_{ec}$
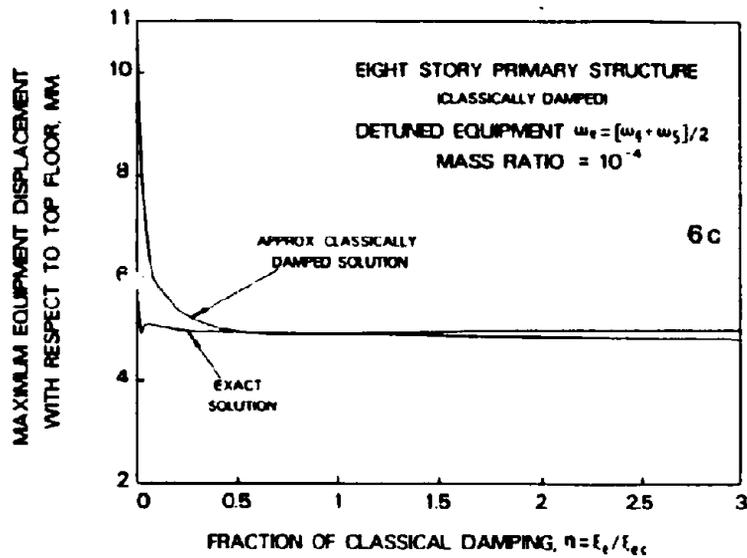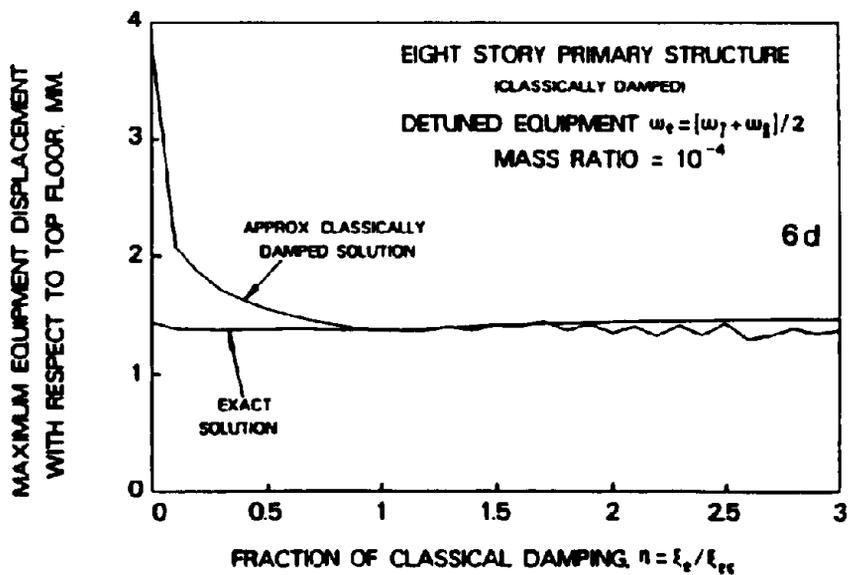
(d) EQUIPMENT FREQUENCY $\omega_e - (\omega_7 + \omega_8)/2$, MASS RATIO $\gamma - 10^{-4}$, $\xi_{ec} - 26\%$.

FIGURE 6   Maximum Relative Displacement of Detuned Equipment Attached to Eight Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e / \xi_{ec}$ (Cont'd).
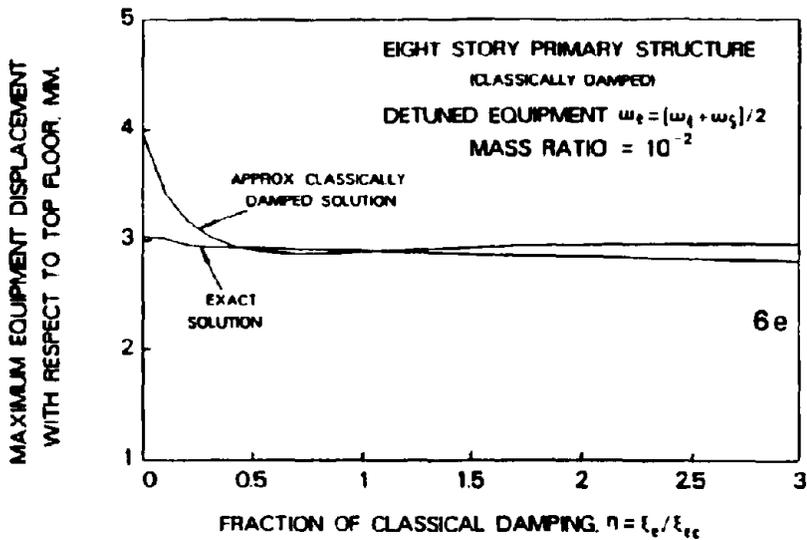
(e) EQUIPMENT FREQUENCY $\omega_e = (\omega_4 + \omega_5)/2$, MASS RATIO $\gamma - 10^{-2}$, $\xi_{ec} - 18\%$.
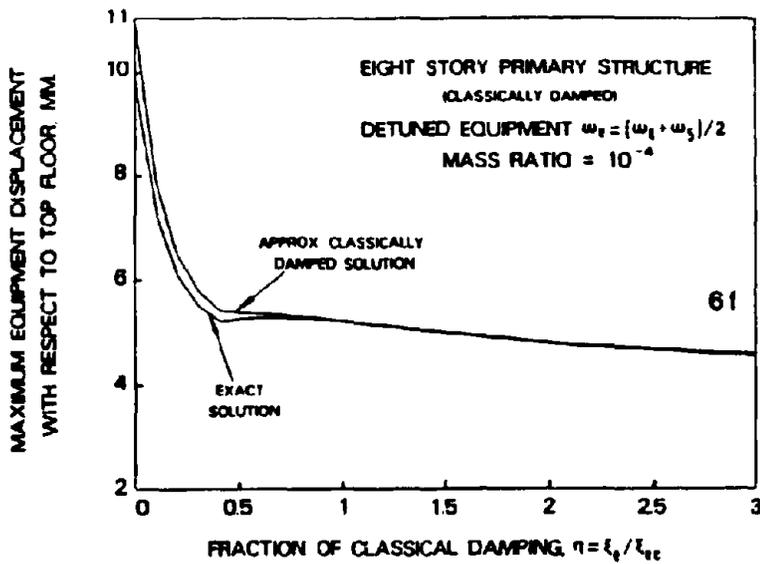
(f) EQUIPMENT FREQUENCY $\omega_e - (\omega_4 + \omega_5)/2$, MASS RATIO $\gamma - 10^{-4}$, $\xi_{ec} - 3.2\%$.

FIGURE 6   Maximum Relative Displacement of Detuned Equipment Attached to Eight Story Primary Structure as Function of Equipment Damping, $\eta = \xi_e/\xi_{ec}$ (Cont'd).
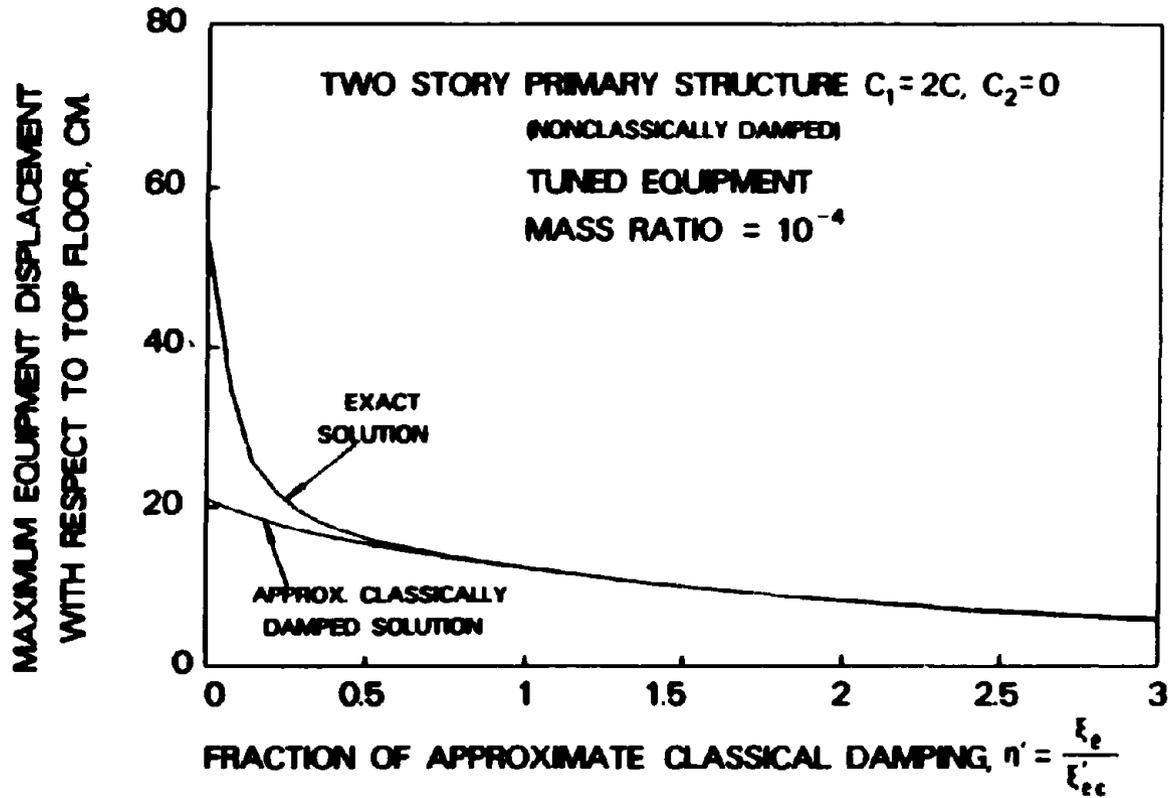
FIGURE 7   Maximum Relative Displacement of Tuned Equipment Attached to Nonclas-
sically Damped Two Story Primary Structure as Function of Equipment
Damping, $\eta' = \xi_e/\xi'_{ec}$.
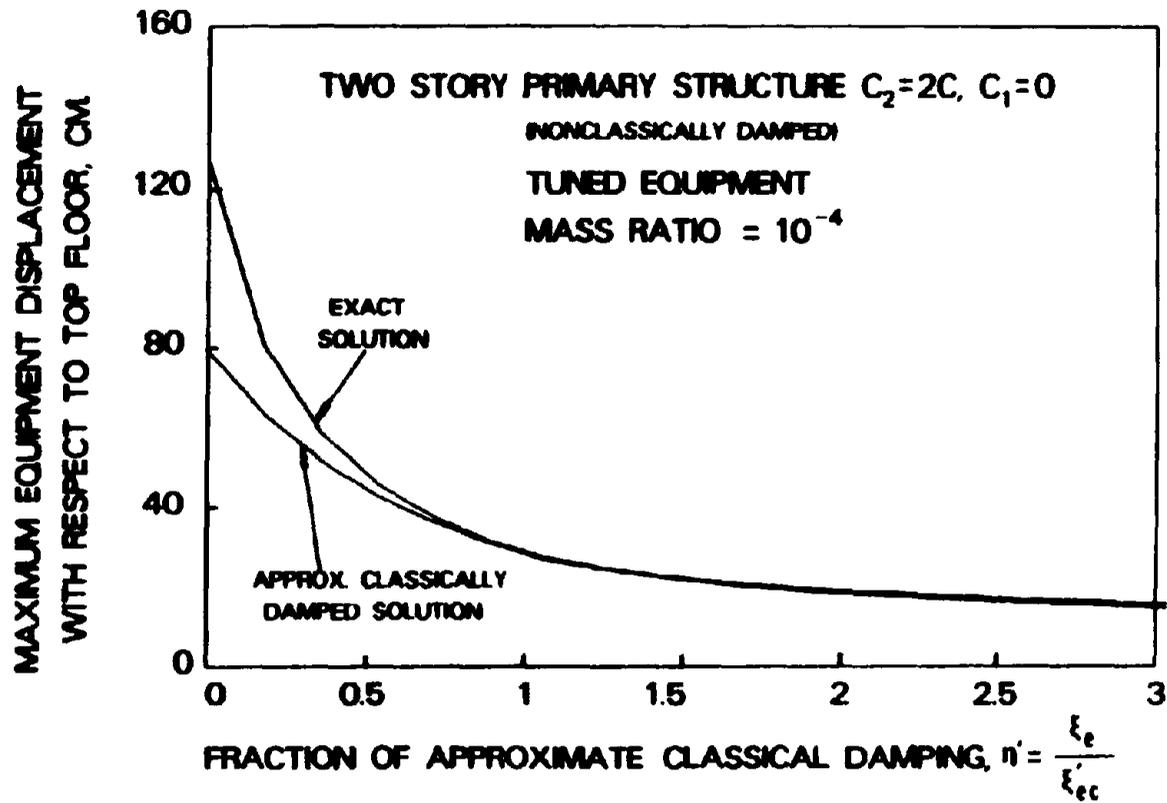
FIGURE 8   Maximum Relative Displacement of Tuned Equipment Attached to Nonclassically Damped Two Story Primary Structure as Function of Equipment Damping, $\eta' = \xi_e/\xi'_{ec}$.
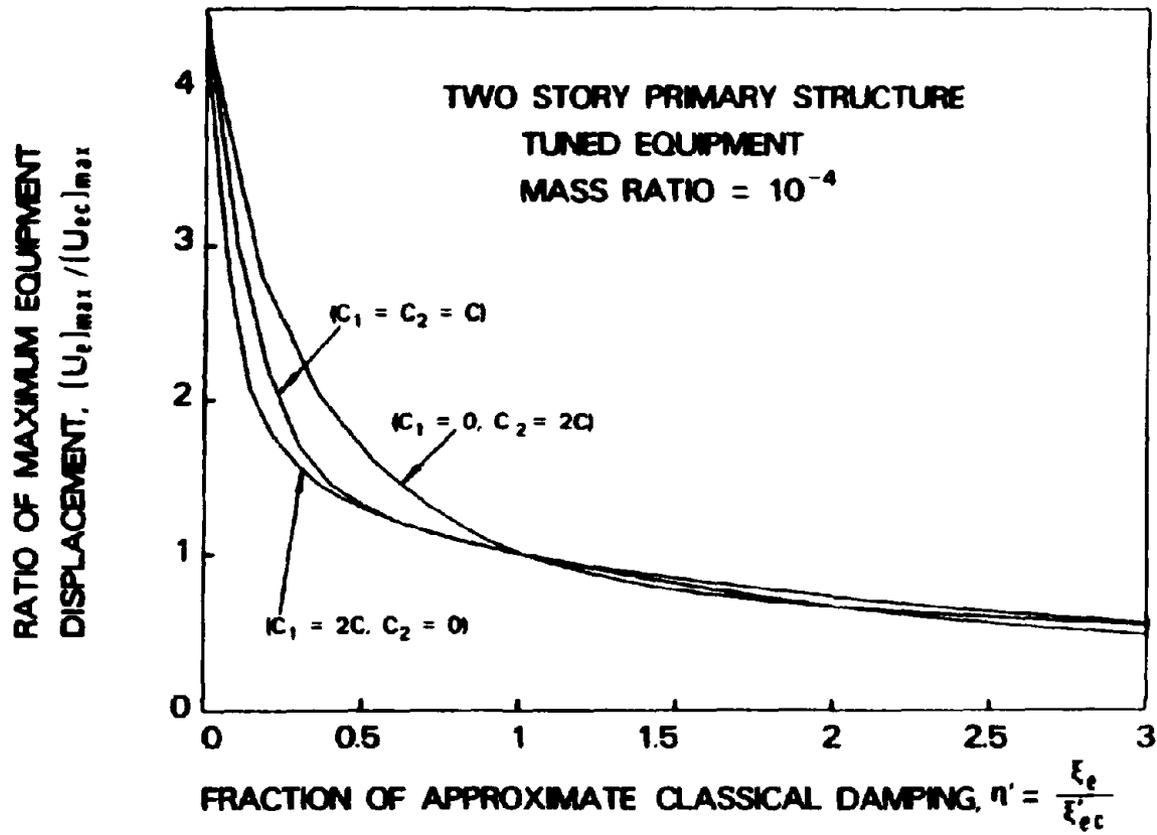
FIGURE 9 Maximum Relative Displacement of Tuned Equipment Attached to Nonclassically Damped Two Story Primary Structure as Function of Equipment Damping, $\eta' = \xi_e/\xi'_{ec}$.
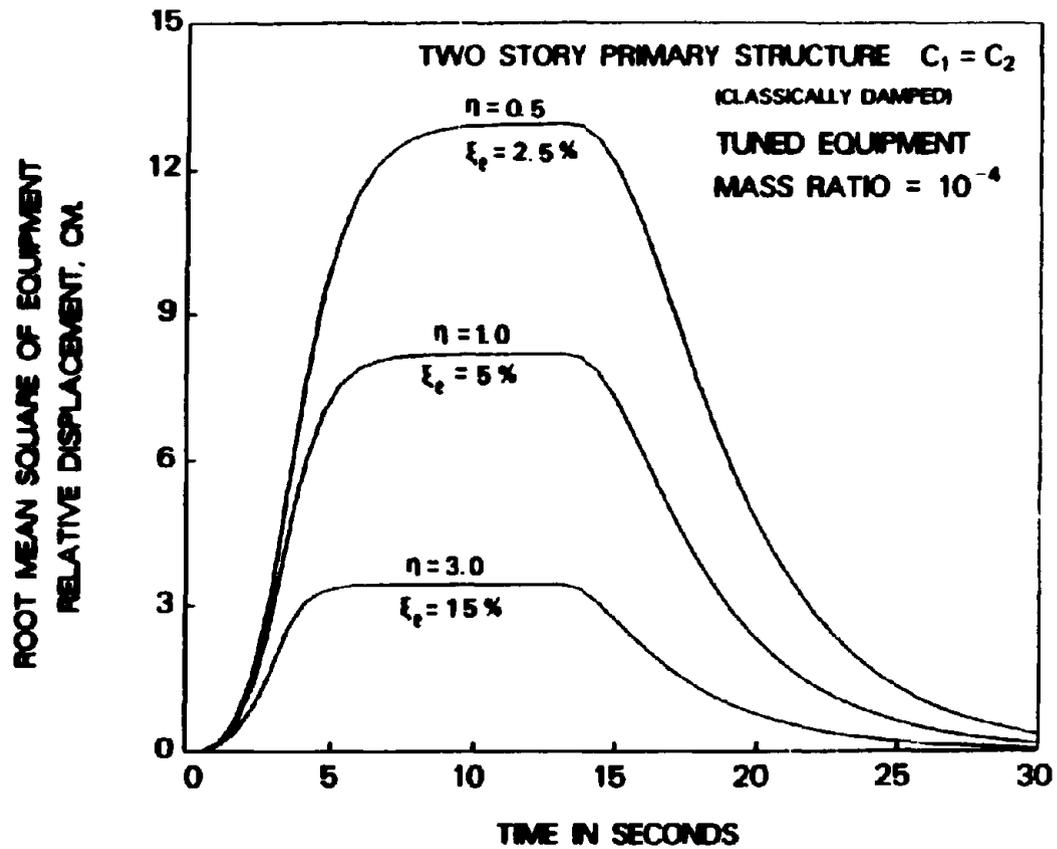
FIGURE 10   Root Mean Square of Response of Tuned Equipment Attached to Classically Damped Two Story Primary Structure as a Function of Time, t.

6-29

0.75

TWO STORY PRIMARY STRUCTURE $C_1 = C_2$

(CLASSICALLY DAMPED)

0.6

$\eta = 0.5,\ \xi_e = 4.5\%$

DETUNED EQUIPMENT

MASS RATIO $= 10^{-4}$

$\eta = 1.0,\ \xi_e = 9\%$

$\eta = 3.0,\ \xi_e = 18\%$

0.45

ROOT MEAN SQUARE OF EQUIPMENT RELATIVE DISPLACEMENT, CM

0.3

0.15

0

0   5   10   15   20   25   30

TIME IN SECONDS
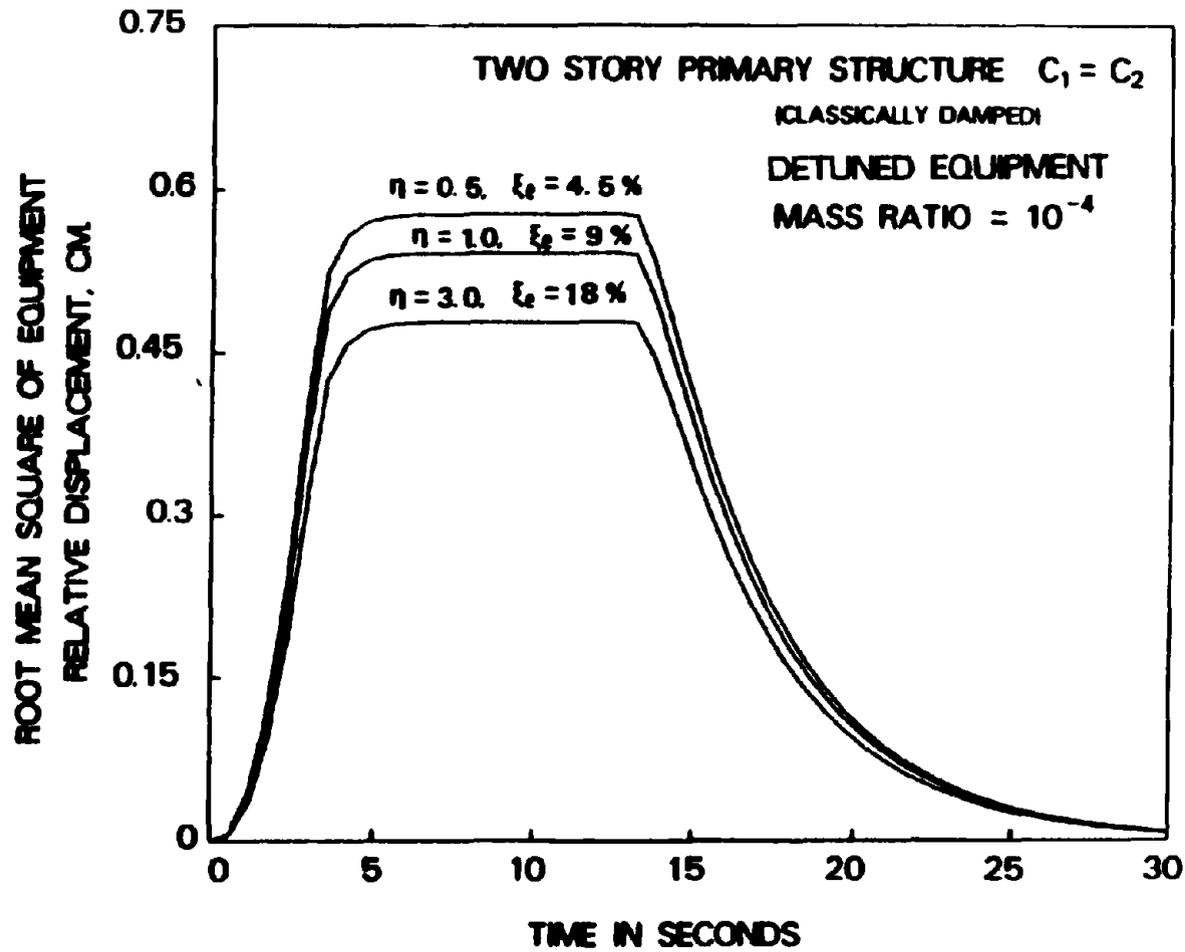
FIGURE 11   Root Mean Square of Response of Detuned Equipment Attached to Classically Damped Two Story Primary Structure as a Function of Time, t.
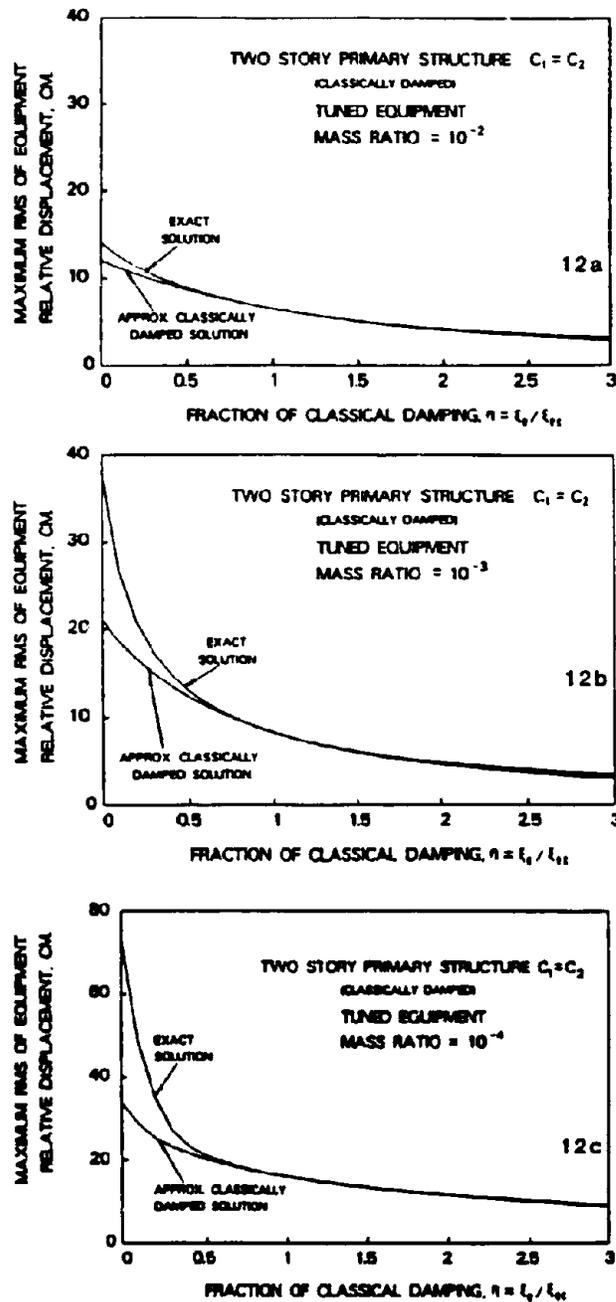
FIGURE 12   Maximum RMS Response of Tuned Equipment Attached to Two Story
Classically Damped Primary Structure as Function of Equipment Damping,
$\eta = \xi_e/\xi_{ec}$.

(a) Mass Ratio, $\gamma = 10^{-2}$, (b) Mass Ratio $\gamma = 10^{-3}$, (c) Mass Ratio $\gamma = 10^{-4}$.
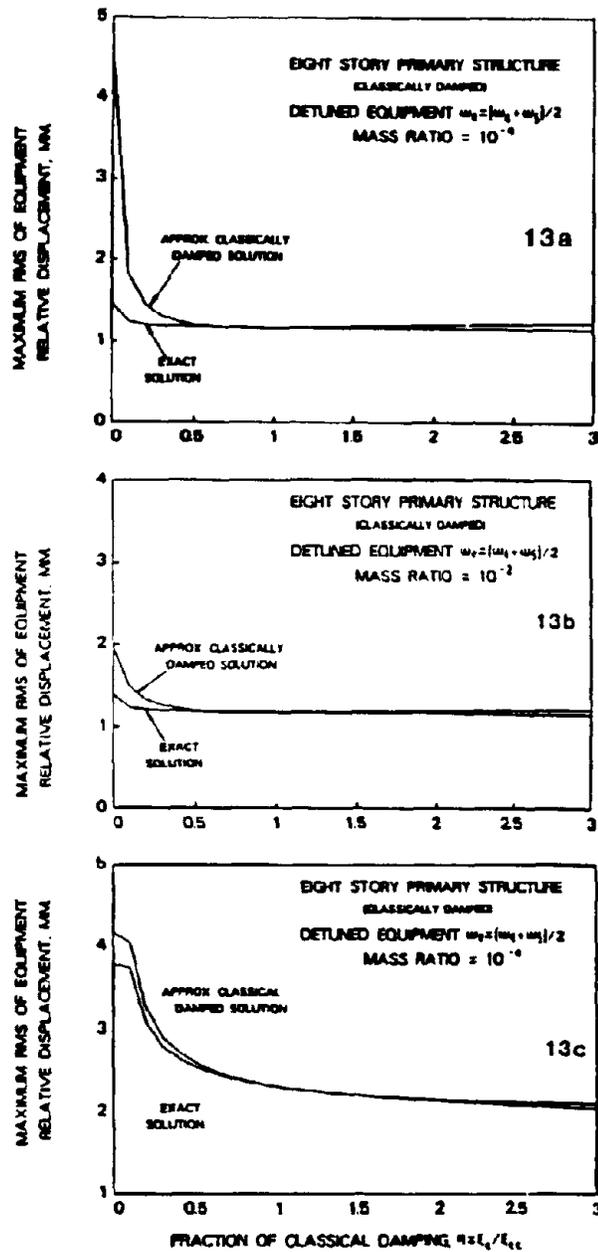
FIGURE 13 Maximum RMS Response of Detuned Equipment Attached to Eight Story Classically Damped Primary Structure as Function of Equipment Damping, $\eta = \xi_e/\xi_{ec}$.

(a) Mass Ratio $\gamma = 10^{-4}$, $\xi_{ec} = 18\%$, (b) Mass Ratio $\gamma = 10^{-2}$, $\xi_{ec} = 18\%$,

(c) Mass Ratio $\gamma = 10^{-4}$, $\xi_{ec} = 3.6\%$.
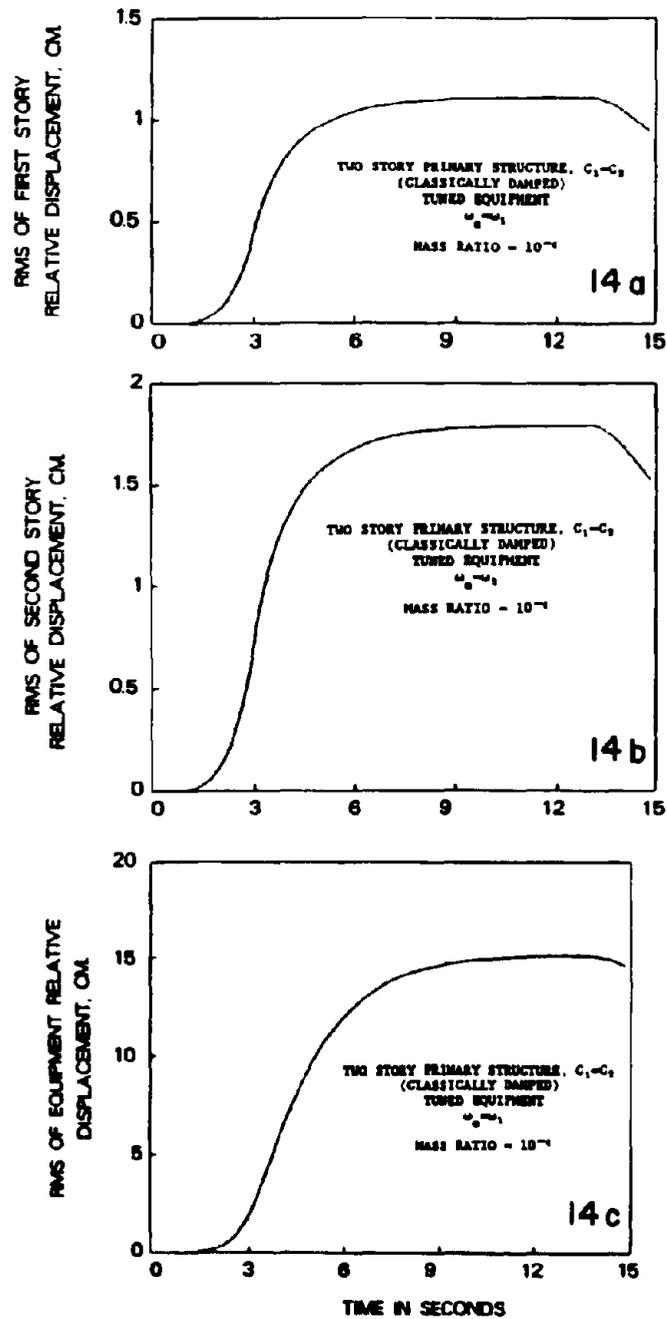
FIGURE 14   RMS as a Function of Time.

(a) First Story, (b) Second Story, (c) Equipment Displacement.

RMS OF FIRST STORY ACCELERATION   M/S.²

3

2

TWO STORY PRIMARY STRUCTURE, $C_1 = C_2$
(CLASSICALLY DAMPED)
TUNED EQUIPMENT
$\omega_e = \omega_1$

MASS RATIO $= 10^{-4}$

1

0

14d

0   3   6   9   12   15

RMS OF SECOND STORY ACCELERATION   M/S.²

4

3

TWO STORY PRIMARY STRUCTURE, $C_1 = C_2$
(CLASSICALLY DAMPED)
TUNED EQUIPMENT
$\omega_e = \omega_1$

MASS RATIO $= 10^{-4}$

2

1

0

14e

0   3   6   9   12   15

RMS OF EQUIPMENT ACCELERATION   M/S.²

40

30

TWO STORY PRIMARY STRUCTURE, $C_1 = C_2$
(CLASSICALLY DAMPED)
TUNED EQUIPMENT
$\omega_e = \omega_1$

MASS RATIO $= 10^{-4}$

20

10

0

14f

0   3   6   9   12   16

TIME IN SECONDS

FIGURE 14   RMS of Story Acceleration.

(d) First Story, (e) Second Story, (f) Equipment Displacement.

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SECTION 7

CONCLUSIONS

A modal analysis approach, referred to as the canonical modal decomposition procedure, for seismic analysis of nonclassically damped structural system is presented. The main advantage of this procedure is that the resulting decoupled equations of motion contain only real parameters. Procedures are outlined to solve the decoupled equations for deterministic ground excitations. Also presented is a procedure to solve these decoupled equations when the ground excitation is a nonstationary random process.

The canonical modal decomposition procedure is used to obtain the response of primary-secondary structural system and to perform parametric studies for the effect of nonclassical damping on the response of both primary and secondary structures. In parametric studies several examples were considered under both deterministtic and nonstationary stochastic ground accelerations. A single-degree-of-freedom equipment attached to a classically damped multi-degree-of-freedom primary structure was considered. Using the canonical modal decomposition procedures, the response of the equipment for various equipment dampings, mass ratios, and tunning and detuning have been calculated.

Based on both deterministric and stochastic earthquake ground motion inputs, the following conclusions are obtained from our sensitivity studies for the response of primary-secondary system, where the primary structure is classically damped. (1) The effect of nonclassical damping on the equipment response is significant when the following conditions are satisfied simultaneously, (i) the frequency of the equipment is tuned to that of any mode of the primary structure, (ii) the mass ratio is small, and (iii) the damping ratio ξ_e of the equipment is smaller than the damping ratio ξ_{ec} that

results in a classically damped primary-secondary system. Under this circumstance, the approximate classically damped solutions, i.e., the solutions obtained using the undamped modal matrix and disregarding the off-diagonal terms of the resulting damping matrix, are usually unconservative. (2) When the equipment is detuned at low frequency, the effect of nonclassical damping on the equipment response is negligible. Hence, the approximate classically damped approach can be used. However, under the following conditions, the effect of nonclassical damping on the detuned equipment response can be significant. (i) The equipment is detuned at high frequency, (ii) the mass ratio is small, (iii) the damping ratio of the equipment, ξ_{ec}, that results in a classically damped primary-secondary system is high, and (iv) the ratio of equipment damping ratio ξ_e to ξ_{ec} is smaller than unity. Under this circumstance, the approximate classically damped solutions for the equipment response are higher than the exact solutions.

Also studied were small equipment-structure systems in which the primary structure is nonclassically damped. Limited results indicate that for such primary-secondary systems, the effect of nonclassical damping on the equipment response can be estimated by approximating the primary structure as being classically damped. This is accomplished using the undamped modal matrix of the primary structure and disregarding the off-diagonal terms of the resulting damping matrix. Then the conclusions described above hold. Corresponding to ξ_{ec} for classically damped primary structure, a meaningful measure of equipment damping for nonclassically damped primary system, denoted by ξ'_{ec}, is determined using the approximate classically damped primary system.

With the consideration of nonstationary earthquake ground acceleration, extensive parametric studies indicate that the overshooting

phenomenon doesn't occur for the response of either primary or secondary system. In other words, the stationary response is always larger than the transient response. This may be attributed to the fact that both the earthquake excitation and the structural response are zero at time zero.

An eight story structure was considered and the distribution of damping in the structure was varied. Results indicate that the effect of nonclassical damping on the response is not significant if the damping distribution within the structure does not change drastically.

Finally, the equipment location may be an important factor for the behavior of the equipment response [5]. Such a problem is currently being investigated. In general, the conclusions obtained herein are consistent with those of Refs. 5 and 16 in which parametric studies for the frequency response function of equipment responses were conducted.

SECTION 9
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APPENDIX: CANONICAL TRANSFORMATION FOR EQUATION (2)

The equations of motion given by Eq. (2) is as follows

$$\underset{\sim}{A}_1 \; \dot{\underset{\sim}{Y}} + \underset{\sim}{B} \; \underset{\sim}{Y} = \underset{\sim}{P} \; \ddot{x}_g \qquad (I-1)$$

in which $\underset{\sim}{A}_1$ and $\underset{\sim}{B}$ are (2nx2n) symmetric matrices. The jth pair of eigen-
vectors, $\underset{\sim}{\phi}_{2j-1}$ and $\underset{\sim}{\phi}_{2j}$, are expressed as

$$\underset{\sim}{\phi}_{2j-1} = \underset{\sim}{a}_j + i \; \underset{\sim}{b}_j$$

$$\qquad (I-2)$$

$$\underset{\sim}{\phi}_{2j} = \underset{\sim}{a}_j - i \; \underset{\sim}{b}_j \qquad ; \quad j = 1, 2, \ldots, n$$

in which $\underset{\sim}{a}_j$ and $\underset{\sim}{b}_j$ are 2n real vectors.

Let

$$\underset{\sim}{Y} = \underset{\sim}{T} \; \underset{\sim}{\nu} \qquad (I-3)$$

in which $\underset{\sim}{T}$ is a (2nx2n) real matrix constructed in the following

$$\underset{\sim}{T} = \left[\underset{\sim}{a}_1, \; \underset{\sim}{b}_1, \; \underset{\sim}{a}_2, \; \underset{\sim}{b}_2, \ldots, \; \underset{\sim}{a}_n, \; \underset{\sim}{b}_n \right] \qquad (I-4)$$

Substituting Eq. (I-3) into Eq. (I-1) and premultiplying it by the
transpose of $\underset{\sim}{T}$ matrix, $\underset{\sim}{T}'$, one obtains

$$\underset{\sim}{\Gamma} \; \dot{\underset{\sim}{\nu}} + \underset{\sim}{\Delta} \; \underset{\sim}{\nu} = \underset{\sim}{T}' \; \underset{\sim}{P} \; \ddot{x}_g \qquad (I-5)$$

A-1

in which $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Delta}$ are of canonical form

$$\underset{\sim}{\Gamma} = \underset{\sim}{T}' \underset{\sim}{A}_1 \underset{\sim}{T} \quad , \quad \underset{\sim}{\Delta} = \underset{\sim}{T}' \underset{\sim}{B} \underset{\sim}{T} \tag{I-6}$$

where

$$\underset{\sim}{\Gamma} = \begin{bmatrix} \underset{\sim}{\Gamma}_1 & & & \\ & \underset{\sim}{\Gamma}_2 & & 0 \\ & & \ddots & \\ 0 & & & \underset{\sim}{\Gamma}_n \end{bmatrix} \quad , \quad \underset{\sim}{\Delta} = \begin{bmatrix} \underset{\sim}{\Delta}_1 & & & \\ & \underset{\sim}{\Delta}_2 & & 0 \\ & & \ddots & \\ 0 & & & \underset{\sim}{\Delta}_n \end{bmatrix} \tag{I-7}$$

and $\underset{\sim}{\Gamma}_j$ and $\underset{\sim}{\Delta}_j$ are (2x2) matrices given by

$$\underset{\sim}{\Gamma}_j = \left[\begin{array}{c|c} \underset{\sim}{a}'_j \underset{\sim}{A}_1 \underset{\sim}{a}_j & \underset{\sim}{a}'_j \underset{\sim}{A}_1 \underset{\sim}{b}_j \\ \hline \underset{\sim}{b}'_j \underset{\sim}{A}_j \underset{\sim}{a}_j & \underset{\sim}{b}'_j \underset{\sim}{A}_1 \underset{\sim}{b}_j \end{array} \right] \quad , \quad \underset{\sim}{\Delta}_j = \left[\begin{array}{c|c} \underset{\sim}{a}'_j \underset{\sim}{B} \underset{\sim}{a}_j & \underset{\sim}{a}'_j \underset{\sim}{B} \underset{\sim}{b}_j \\ \hline \underset{\sim}{b}'_j \underset{\sim}{B} \underset{\sim}{a}_j & \underset{\sim}{b}'_j \underset{\sim}{B} \underset{\sim}{b}_j \end{array} \right] \tag{I-8}$$

Equation (I-5) consists of n pairs of equations in which each pair represents one vibrational mode, and it is uncoupled with other pairs. The jth pair of coupled equations in Eq. (I-5) is given as follows

$$\underset{\sim}{a}'_j \underset{\sim}{A}_1 \underset{\sim}{a}_j \; \dot{\nu}_{2j-1} + \underset{\sim}{a}'_j \underset{\sim}{A}_1 \underset{\sim}{b}_j \; \dot{\nu}_{2j} + \underset{\sim}{a}'_j \underset{\sim}{B} \underset{\sim}{a}_j \; \nu_{2j-1} + \underset{\sim}{a}'_j \underset{\sim}{B} \underset{\sim}{b}_j \; \nu_{2j} = \underset{\sim}{a}'_j \underset{\sim}{P} \ddot{x}_g \tag{I-9a}$$

$$\underset{\sim}{b}'_j \underset{\sim}{A}_1 \underset{\sim}{a}_j \; \dot{\nu}_{2j-1} + \qquad\qquad\qquad\qquad\qquad j \quad -1 \; -j \quad \dot{2j} \quad -j \; - \; -j \quad 2j-1 \quad -j$$

Since $\underset{\sim}{A}_1$ and $\underset{\sim}{B}$ are symmetric, the orthogonality conditions are given by

$$\underset{\sim}{\ell}'_m \underset{\sim}{A}_1 \underset{\sim}{\ell}_k = 0$$

$$m \neq k \qquad \text{(I-10)}$$

$$\boldsymbol{\ell}'_m \, \underline{B} \, \boldsymbol{\ell}_k = 0$$

Substituting Eq. (I-2) into Eq. (I-10) with $m = 2j-1$ and $k = 2j$, one obtains

$$(\underline{a}'_j + \underline{i} \, \underline{b}'_j) \, \underline{A}_1 \, (\underline{a}_j - \underline{i} \, \underline{b}_j) = 0$$

$$\text{(I-11)}$$

$$(\underline{a}'_j + \underline{i} \, \underline{b}'_j) \, \underline{B} \, (\underline{a}_j - \underline{i} \, \underline{b}_j) = 0$$

Hence the following orthogonality conditions are obtained from Eq. (I-11),

$$\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j = -\underline{b}'_j \, \underline{A}_1 \, \underline{b}_j \quad , \quad \underline{a}'_j \, \underline{A}_1 \, \underline{b}_j = -\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j$$

$$\text{(I-12)}$$

$$\underline{a}'_j \, \underline{B} \, \underline{a}_j = -\underline{b}'_j \, \underline{B} \, \underline{b}_j \quad , \quad \underline{a}'_j \, \underline{B} \, \underline{b}_j = -\underline{b}'_j \, \underline{B} \, \underline{a}_j$$

Eliminating $\overset{\bullet}{\nu}_{2j}$ and $\overset{\bullet}{\nu}_{2j-1}$ respectively, using Eq. (I-9) and the orthogonality properties of Eq. (I-12), one obtains the following two equations for the jth vibrational mode

$$\overset{\bullet}{\nu}_{2j-1} + \frac{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)(\underline{a}'_j \, \underline{B} \, \underline{a}_j) + (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)(\underline{b}'_j \, \underline{B} \, \underline{b}_j)}{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)^2} \, \nu_{2j-1}$$

$$+ \frac{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)(\underline{b}'_j \, \underline{B} \, \underline{a}_j) - (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)(\underline{a}'_j \, \underline{B} \, \underline{a}_j)}{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)^2} \, \nu_{2j}$$

$$- \left[\frac{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)\underline{a}'_j + (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)\underline{b}'_j}{(\underline{a}'_j \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}'_j \, \underline{A}_1 \, \underline{a}_j)^2} \right] \underline{P} \, \ddot{x}_g \qquad \text{(I-13a)}$$

$$\dot{v}_{2j} + \frac{(\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{a}_j' \, \underline{B} \, \underline{a}_j) + (\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{b}_j' \, \underline{B} \, \underline{a}_j)}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \, v_{2j-1}$$

$$+ \frac{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{a}_j' \, \underline{B} \, \underline{a}_j) - (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)(\underline{b}_j' \, \underline{B} \, \underline{b}_j)}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \, v_{2j}$$

$$- \left[\frac{(\underline{a}_j' \, \underline{A}_1 \, \underline{b}_j)\underline{a}_j' - (\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)\underline{b}_j'}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \right] \underline{P} \, \ddot{x}_g \qquad\qquad \text{(I-13b)}$$

A comparison between Eqs. (20) and (I-13) leads to the following alternate expressions for the elements of the $\underline{F}$ vector.

$$F_{2j-1} = \left[\frac{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)\underline{a}_j' + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)\underline{b}_j'}{\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \right] \underline{P}$$

$$\text{(I-14)}$$

$$F_{2j} = \left[\frac{(\underline{a}_j' \, \underline{A}_1 \, \underline{b}_j)\underline{a}_j' + (\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)\underline{b}_j'}{(\underline{a}_j' \, \underline{A}_1 \, \underline{a}_j)^2 + (\underline{b}_j' \, \underline{A}_1 \, \underline{a}_j)^2} \right] \underline{P}$$

REDUCTION TO SOLUTION FOR CLASSICALLY DAMPED SYSTEM

If the structural system is classically damped, then the $\underline{T}$ matrix is greatly simplified. For such a system, the displacement part of the eigenvector is real. Since the velocity part of the eigenvector is the eigenvalue multiplied by the displacement part of the eigenvector, Eq. (6), $\underline{a}_j$ and $\underline{b}_j$ can be expressed as:

$$\underline{a}_j = \left\{ \begin{array}{c} -\xi_j \, \omega_j \, \underline{\phi}_j \\ \cdots\cdots \\ \underline{\phi}_j \end{array} \right\} \quad , \quad \underline{b}_j = \left\{ \begin{array}{c} \omega_{Dj} \, \underline{\phi}_j \\ \cdots\cdots \\ \underline{0} \end{array} \right\} \qquad\qquad \text{(I-15)}$$

in which $\underline{\Phi}_j$ is the lower half of the $\underline{a}_j$ vector representing the displacement eigenvector. Note that for classically damped structural systems, $\underline{\Phi}_j$, is simply the jth eigenvector of the undamped system. Hence, for a classically damped system, the following simplifications can be made

$$
\underline{a}_j{}' \; \underline{A}_1 \; \underline{a}_j \;=\; \left\{ \begin{matrix} -\xi_j \; \omega_j \; \underline{\Phi}_j \\ \dots\dots \\ \underline{\Phi}_j \end{matrix} \right\}' \; \left[\begin{array}{c|c} \underline{0} & \underline{M} \\ \hline \underline{M} & \underline{C} \end{array} \right] \; \left\{ \begin{matrix} -\xi_j \; \omega_j \; \underline{\Phi}_j \\ \dots\dots \\ \underline{\Phi}_j \end{matrix} \right\}
$$

$$
=\; \left\{ \begin{matrix} -\xi_j \; \omega_j \; \underline{\Phi}_j \\ \dots\dots \\ \underline{\Phi}_j \end{matrix} \right\}' \; \left\{ \begin{matrix} \underline{M} \; \underline{\Phi}_j \\ \dots\dots\dots\dots\dots \\ -\xi_j \; \omega_j \; \underline{M} \; \underline{\Phi}_j + \underline{C} \; \underline{\Phi}_j \end{matrix} \right\}
$$

$$
=\; - \; \xi_j \; \omega_j \; \underline{\Phi}_j' \; \underline{M} \; \underline{\Phi}_j \; - \; \xi_j \; \omega_j \; \underline{\Phi}_j{}' \; \underline{M} \; \underline{\Phi}_j + \underline{\Phi}_j{}' \; \underline{C} \; \underline{\Phi}_j
$$

$$
=\; -2\xi_j \; \omega_j \; \underline{M}^* + \underline{C}^*
$$

$$
=\; 0. \tag{I-16}
$$

Furthermore, expressions for F_{2j-1} and F_{2j} can be simplified using the orthogonality condition, $\underline{a}_j' \; \underline{A}_1 \; \underline{a}_j = \underline{b}_j' \; \underline{A}_1 \; \underline{b}_j$,

$$
F_{2j-1} \;=\; \left[\frac{0 \cdot (\underline{a}_j' \; \underline{A} \; \underline{b}_j)\underline{b}_j'}{(\underline{b}_j' \; \underline{A}_1 \; \underline{a}_j)^2} \right] \underline{P}
$$

$$
=\; \frac{\underline{a}_j' \; \underline{A}_1 \; \underline{b}_j}{(\underline{b}_j' \; \underline{A}_1 \; \underline{a}_j)^2} \; \left\{ \begin{matrix} \omega_{Dj} \; \underline{\Phi}_j \\ \dots\dots \\ \underline{0} \end{matrix} \right\} \; \left\{ \begin{matrix} \underline{0} \\ \dots\dots \\ -\underline{M} \; \underline{r} \end{matrix} \right\}
$$

$$
=\; 0. \tag{I-17}
$$

A-5

and

$$F_{2j} = \left[\frac{(\underline{a}'_j \; \underline{A}_1 \; \underline{b}_j) \underline{a}'_j}{(\underline{b}'_j \; \underline{A}_1 \; \underline{a}_j)^2} \right] \; \underline{P}$$

$$= \frac{\underline{a}'_j \; \underline{P}}{\underline{b}'_j \; \underline{A}_1 \; \underline{a}_j}$$

$$= \frac{-\underline{\Phi}'_j \; \underline{M} \; \underline{r}}{\left\{ \begin{matrix} \omega_{Dj} \; \underline{\Phi}_j \\ \cdots \\ \underline{0} \end{matrix} \right\}' \; \left[\begin{array}{c|c} \underline{0} & \underline{M} \\ \hline \underline{M} & \underline{C} \end{array} \right] \; \left\{ \begin{matrix} \xi_j \; \omega_j \; \underline{\Phi}_j \\ \cdots \\ \underline{\Phi} \end{matrix} \right\}}$$

$$= \frac{-\underline{\Phi}'_j \; \underline{M} \; \underline{r}}{\omega_{Dj} \; \underline{\Phi}'_j \; \underline{M} \; \underline{\Phi}_j} \qquad\qquad (I\text{-}18)$$