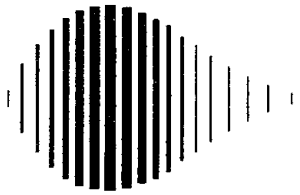

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STOCHASTIC FINITE ELEMENT EXPANSION FOR RANDOM MEDIA

by

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March 14, 1988

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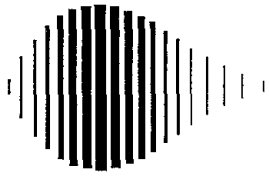

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ABSTRACT

A new method for the solution of problems involving random media is proposed. The medium property is represented by a stochastic process. The method makes use of an orthogonal expansion of the process with a finite set of random variables and leads to a formulation compatible with the finite element method. The usefulness of the method, in terms of accuracy and efficiency, is exemplified by considering a cantilever beam with random rigidity.

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SECTION 1

INTRODUCTION

The reliability of many engineering structures in the presence of uncertainty has been a crucial factor in their analysis and design. Primary and secondary systems related to structures such as nuclear power containments, space vehicles, and offshore platforms may be quite sensitive to small imperfections of pertinent design variables. Several of these variables are inherently random and can be most appropriately modeled as random processes. They may include quantities such as modulus of elasticity, poisson ratio, shear strength, ocean wave height, and a variety of other physical and mathematical parameters (Vanmarcke, 1977 ; Burnside, 1985). Clearly, the complexity of these modern structures requires the use of versatile numerical algorithms, such as the finite element method, to obtain accurate mathematical approximations to their physical behavior. Thus, a challenging task to the analyst is to accurately account for the randomness in a given problem while using some proven numerical algorithm. The result from such an analysis can be in the form of statistical quantities describing the response.

A number of researchers have attempted to solve problems with random media introducing assumptions of various degrees of severity. Collins and Thompson (1969) treated the eigenvalue problem for random systems using first order perturbation. Hart and Collins (1970) dealt with randomness in finite element modeling, using again first order perturbation. Nakagiri and Hisada (1982) initiated a series of investigations related to perturbation analysis in stochastic finite elements and concluded that the second order perturbation is impractical due to the scale of the requisite computational effort. The first order perturbation, however, gives rather crude approximations to the solutions. Therefore, it is of limited value. In all the approaches mentioned above, the medium randomness is accounted for by means of random variables. Shinozuka and various coworkers investigated probabilistic models for spatial distribution of material properties (1987). He used simulation methods (Shinozuka and Lenoe, 1976) and the Neuman expansion method (Yamazaki, Shinozuka and Dasgupta, 1986) to obtain the statistical properties of the response. Also, Shinozuka (1987) obtained analytical solutions for a class of statically determined structural members. The Neumann expansion was also used by Adomian (1984) and by Benaroya and Rehak (1987).

The present paper suggests a new method for solving problems involving random media. The random process describing the medium is expanded in an orthogonal decomposition (Loeve, 1977) which is then incorporated in a finite element formulation of the problem. The method is appealing in that simulation is not required for the solution, and only the average stiffness matrix needs to be inverted. Further, since the orthogonal decomposition is derived from the spectral theorem for positive definite operators (Mercer, 1909), it possesses some desirable convergence properties. Finally, the computational efficiency and the accuracy of its results for a wide range of problems may be the basis of a substantial improvement over available methods.

In the next section, the orthogonal decomposition for a one-parameter random process is derived. Next, a finite element formulation that incorporates the expansion is developed. Finally, a numerical example is discussed, and pertinent results are compared with those obtained from a Monte-Carlo simulation of the problem.

SECTION 2

ORTHOGONAL EXPANSION

Let S(x) denote a random process, a function of the position x defined over the domain L. Let $\bar{S}(x)$ denote the expected value of S(x) over all possible realizations of the process, and C(x,ξ) denote its covariance function associated with locations x and ξ. By definition, the covariance function is bounded, symmetric, and positive definite. Thus, it has the spectral decomposition

$$C(x,\xi) = \sum_{n=0}^{\infty} \lambda_n \phi_n(x) \phi_n(\xi) . \tag{1}$$

where λ_n and ϕ_n are the eigenvalue and the eigenvector of the covariance kernel, respectively. That is, they are the solution to the integral equation

$$\int_L C(x,\xi) \phi_n(x) d\xi = \lambda_n \phi_n(x) . \tag{2}$$

Due to the symmetry and the positive-definiteness of the covariance kernel (Loeve, 1977), its eigenfunctions form a complete set and they are orthogonal satisfying the equation

$$\int_L \phi_n(x) \phi_m(x) dx = \delta_{nm} , \tag{3}$$

where δ_{nm} is the Kronecker delta. The process S(x) can be written as

$$S(x) = \bar{S}(x) + \Delta S(x) , \tag{4}$$

where ΔS(x) as defined by equation (4) denotes a process with zero mean and covariance function C(x,ξ). The process ΔS(x) can be expanded in terms of ϕ_n as

$$\Delta S(x) = \sum_{n=0}^{\infty} b_n \sqrt{\lambda_n} \phi_n(x) , \tag{5}$$

where b_n is a random coefficient independent of x. In order to determine b_n, multiply both sides of equation (4) by ΔS(ξ) and take the expectation on both sides. Then,

$$\begin{aligned} C(x,\xi) &= E\left[\Delta S(x) \Delta S(\xi) \right] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} E\left[b_n b_m \right] \sqrt{\lambda_n \lambda_m} \phi_n(x) \phi_m(\xi) , \end{aligned} \tag{6}$$

where E[.] denotes the operator of mathematical expectation. Multiplying equation (6) by $\phi_k(\xi)$, integrating over the domain L, and making use of the orthogonality of $\{ \phi_n \}$ yield

$$\begin{aligned} \int_L C(x,\xi) \phi_k(\xi) d\xi &= \lambda_k \phi_k(x) \\ &= \sum_{n=0}^{\infty} E\left[b_n b_k \right] \sqrt{\lambda_n \lambda_k} \phi_n(x). \end{aligned} \tag{7}$$

Again, multiplying equation (7) by $\phi_l(x)$ and integrating gives

$$\lambda_k \int_L \phi_k(x)\,\phi_l(x)\,dx = \sum_{n=0}^{\infty} E\left[b_n\, b_k\right] \sqrt{\lambda_n\,\lambda_k}\,\delta_{nl}\,. \tag{8}$$

Then, using equation (3) leads to

$$\lambda_k\,\delta_{kl} = \sqrt{\lambda_k\,\lambda_l}\,E\left[b_k\, b_l\right]. \tag{9}$$

Equation (9) can be rearranged to give

$$E\left[b_k\, b_l\right] = \delta_{kl}\,. \tag{10}$$

Thus, the random process S(x) can be written as

$$S(x) = \bar{S}(x) + \sum_{n=0}^{\infty} b_n \sqrt{\lambda_n}\,\phi_n(x)\,, \tag{11}$$

where

$$E\left[b_n\right] = 0 \qquad , \qquad E\left[b_n\, b_m\right] = \delta_{nm}\,, \tag{12}$$

and λ_n, ϕ_n satisfy equation (2). The series in equation (11) is known to converge in the mean square. Further, if S(x) is a gaussian process, the series can be shown to also converge almost surely (Loeve, 1977). Truncating the series in equation (11) at the r^{th} term gives

$$S(x) = \bar{S}(x) + \sum_{n=0}^{r} b_n \sqrt{\lambda_n}\,\phi_n(x)\,. \tag{13}$$

Note that the above expansion is optimal in the Fourier expansion sense. That is, it minimizes the mean squared approximation error resulting from truncating the series at a finite number of terms. Further, equation (13) is an expression for the projection of the random process S(x), viewed as a curve in a Hilbert space, onto an r dimensional subspace. The expansion is used extensively in the field of pattern recognition and image processing as an efficient tool to store random processes (Devijver and Kittler, 1982). Of special interest in earthquake engineering, is the potential of the expansion in generating realizations of multidimensional and nonstationary random processes associated with earthquakes such as ground motion and material variability. Note that since the random process S(x) is defined over a finite domain, it is not ergodic. This fact does not affect the solvability of the class of problems being investigated herein; it eliminates an assumption necessary to other approaches.

In the special case where the random process S(x) possesses a rational spectrum, the integral eigenvalue problem can be replaced by an equivalent differential equation which is more tractable mathematically (Van Trees, 1968).

SECTION 3

EXAMPLES OF KERNEL EXPANSION

As an example of the orthogonal expansion discussed above, two quite common covariance kernels are discussed in this section.

3.1 Exponential Covariance

Consider the covariance kernel defined by the equation

$$C(x,\xi) = \sigma_S^2 \, e^{-c|x-\xi|} \,, \tag{14}$$

where σ_S denotes the standard deviation of the random process under consideration. Clearly, $C(x,\xi)$ can be made rapidly attenuating versus $|x-\xi|$ by selecting a suitable value of the parameter c. This kernel is related to a 1st order markovian process (Vanmarcke, 1983) and is used extensively in geophysics and in earthquake engineering. Realizations of this process are considered on the interval [-a,+a]. The eigenfunctions and eigenvalues of the covariance function given by equation (14) are the solutions to the following integral equation

$$\sigma_s^2 \int_{-a}^{+a} e^{-c|x-\xi|} \, \phi(\xi) \, d\xi = \lambda \, \phi(x) \,. \tag{15}$$

Equation (15) can be written as

$$\sigma_s^2 \int_{-a}^{x} e^{-c(x-\xi)} \, \phi(\xi) \, d\xi \; + \; \sigma_s^2 \int_{x}^{a} e^{c(x-\xi)} \, \phi(\xi) \, d\xi \; = \; \lambda \, \phi(x) \,. \tag{16}$$

Differentiating equation (16) with respect to x and rearranging gives

$$\lambda \, \phi'(x) \; = \; -c \, \sigma_s^2 \int_{-a}^{x} e^{-c(x-\xi)} \, \phi(\xi) \, d\xi \; + \; c \, \sigma_s^2 \int_{x}^{+a} e^{c(x-\xi)} \, \phi(\xi) \, d\xi \,. \tag{17}$$

Differentiating once more with respect to x, the following equation is obtained

$$\lambda \, \phi''(x) \; = \; (-2 \, c \, \sigma_s^2 + c^2 \, \lambda \,) \, \phi(x) \,. \tag{18}$$

Introducing the new variable

$$\omega^2 = \frac{2 \, c \, \sigma_s^2 - c^2 \, \lambda}{\lambda} \,, \tag{19}$$

equation (18) becomes

$$\phi''(x) + \omega^2 \, \phi(x) = 0 \qquad\qquad -a < x < +a \,. \tag{20}$$

To find the boundary conditions associated with the differential equation (20), equations (16) and (17) are evaluated at $x = -a$ and $x = +a$. After rearrangement, the boundary conditions are

$$c \, \phi \, (a) + \phi'(a) = 0 \tag{21}$$

$$c\,\phi(-a) - \phi'(-a) = 0\,. \tag{22}$$

Thus, the integral equation given by equation (15) is transformed into the ordinary differential equation (20) with appended boundary conditions given by equations (21) and (22). It can be shown that $\omega^2 > 0$ is the only range for ω where equation (20) admits of solutions. In this case the solution is

$$\phi(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x)\,. \tag{23}$$

Further, applying the boundary conditions, equations (21) and (22), gives

$$\begin{cases} a_1 (c - \omega \tan(\omega a) + a_2 (\omega + c \tan(\omega a) = 0 \\ a_1 (c - \omega \tan(\omega a) - a_2 (\omega + c \tan(\omega a) = 0 \end{cases} \tag{24}$$

Nontrivial solutions exist only if the determinant of the homogeneous system in equation (24) is equal to zero. Setting this determinant equal to zero gives

$$\begin{cases} c - \omega \tan(\omega a) = 0 \\ \text{and} \\ \omega^* + c \tan(\omega^* a) = 0\,. \end{cases} \tag{25}$$

The resulting eigenfunctions are

$$\phi_n(x) = \frac{\cos(\omega_n x)}{\sqrt{a + \dfrac{\sin(2\omega_n a)}{2\,\omega_n}}} \tag{26}$$

$$\phi_n^*(x) = \frac{\sin(\omega_n^* x)}{\sqrt{a - \dfrac{\sin(2\omega_n^* a)}{2\,\omega_n^*}}}\,. \tag{27}$$

The corresponding eigenvalues are

$$\lambda_n = \frac{2\,c\,\sigma_s^2}{\omega_n^2 + c^2}\,. \tag{28}$$

and

$$\lambda_n^* = \frac{2\,c\,\sigma_s^2}{\omega_n^{*2} + c^2}\,, \tag{29}$$

where ω_n and ω_n^* are defined by equation (25). Thus, a process S(x) with covariance function given by equation(14) can be expanded as

$$S(x) = \sum_{n=0}^{\infty} \left[b_n \sqrt{\lambda_n}\,\phi_n(x) + b_n^* \sqrt{\lambda_n^*}\,\phi_n(x)^* \right]. \tag{30}$$

3.2 Triangular Covariance

The second kernel to be considered is given by the equation

$$C(x,\xi) = \sigma_S^2 (1-d|x-\xi|) . \tag{31}$$

This kernel represents the triangular covariance function. It provides for linear decrease in correlation, which may be useful for applications in quality control problems. Here, σ_S represents the variance of the process and d is a parameter which can be used to adjust the distance $|x-\xi|$ of null correlation between S(x) and S(ξ). Consider realizations of this process on the interval [0,a]. The eigenfunctions and eigenvalues of C(x,ξ) are obtained as the solution to the integral equation

$$\sigma_s^2 \int_0^a \left[1-c\,|x-\xi| \right] \phi(\xi)\, d\xi = \lambda\, \phi(x) . \tag{32}$$

Differentiating equation (32) twice with respect to x, the following equivalent differential equation is obtained

$$\phi''(x) + \omega^2\, \phi(x) = 0 \qquad\qquad 0 < x < +a . \tag{33}$$

The associated boundary conditions are given by the equations

$$\phi'(a) = -\phi'(0) \text{ and} \tag{34}$$

$$\phi'(0) = \frac{\phi(0) + \phi(a)}{\frac{2}{d} - a} , \tag{35}$$

where

$$\omega = \sqrt{\frac{2d}{\lambda}} . \tag{36}$$

The solution of equation (33) subjected to the boundary conditions described by equations (34) and (35) is, for even n,

$$\phi_n(x) = \frac{\cos(\omega_n x) + \tan(\frac{\omega_n a}{2})\sin(\omega_n x)}{\sqrt{a + \frac{1}{2}\,(\tan^2(\frac{\omega_n a}{2})\,(a - \frac{\sin(2\omega_n a)}{2\omega_n}) + \frac{1}{\omega}\,\sin^2(\omega_n a)\,\tan(\frac{\omega_n a}{2})}} \tag{37}$$

and, for odd n,

$$\phi_n(x) = \frac{\cos(\omega_n x)}{\sqrt{\frac{a}{2} + \frac{\sin(2\omega_n a)}{4}}} . \tag{38}$$

In equation (37), ω_n is the solution to the transcendental equation

$$\tan(\frac{\omega_n a}{2}) = \frac{2}{\omega_n\,(\frac{2}{d} - a)} . \tag{39}$$

In equation (38), ω_n is defined as,

$$\omega_n = m\frac{\pi}{a} \qquad m = 1,3,5... \tag{40}$$

Figures 3-1 and 3-2 show, for a typical value of the parameter c, plots of the exponentially decaying covariance function and of the 4-term approximation, respectively. Figures 3-3 and 3-4 show corresponding plots for the triangular covariance function, for a typical value of the parameter d.

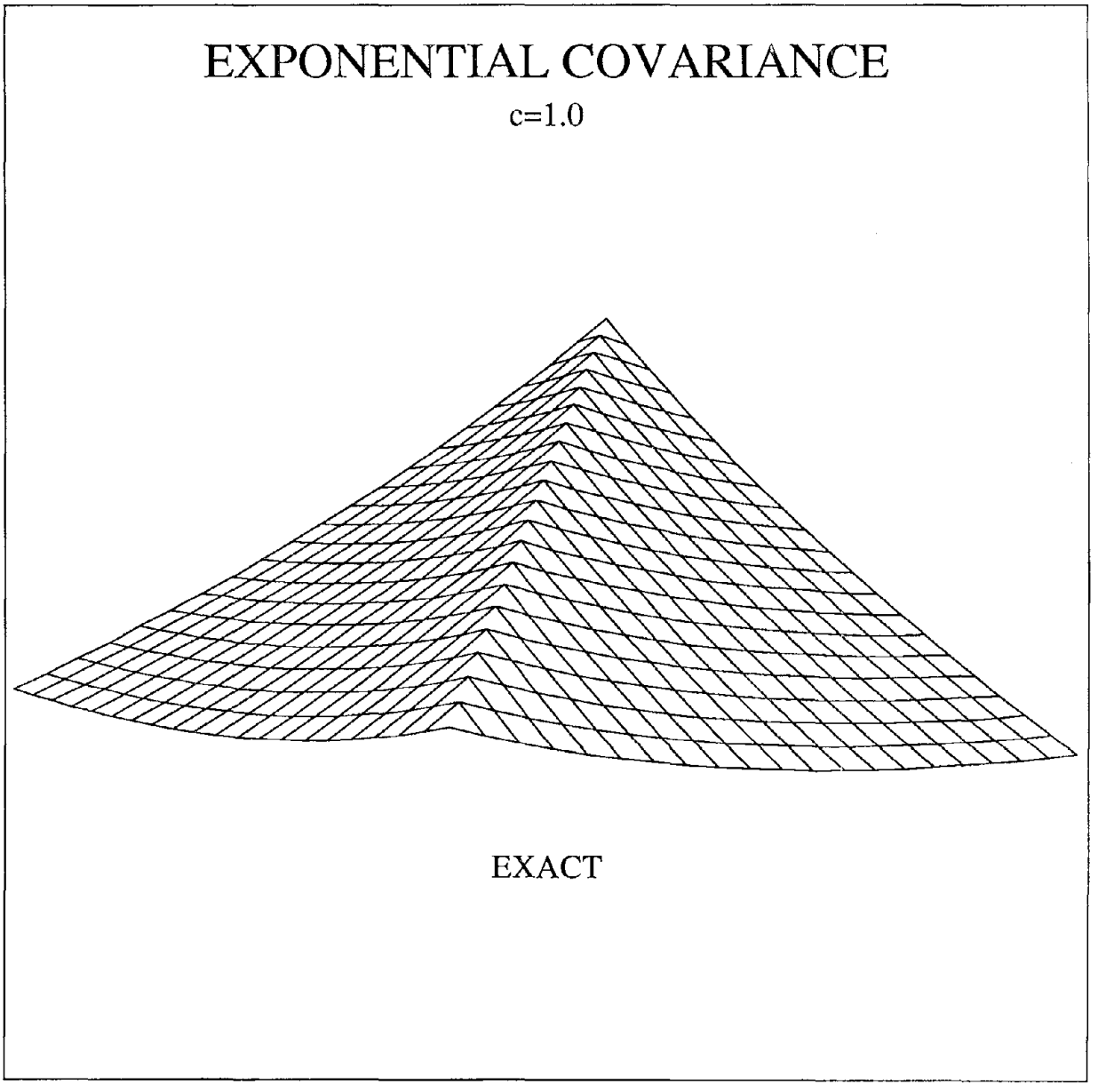

Fig. 3-1 Exact Exponential Covariance; c=1.0.

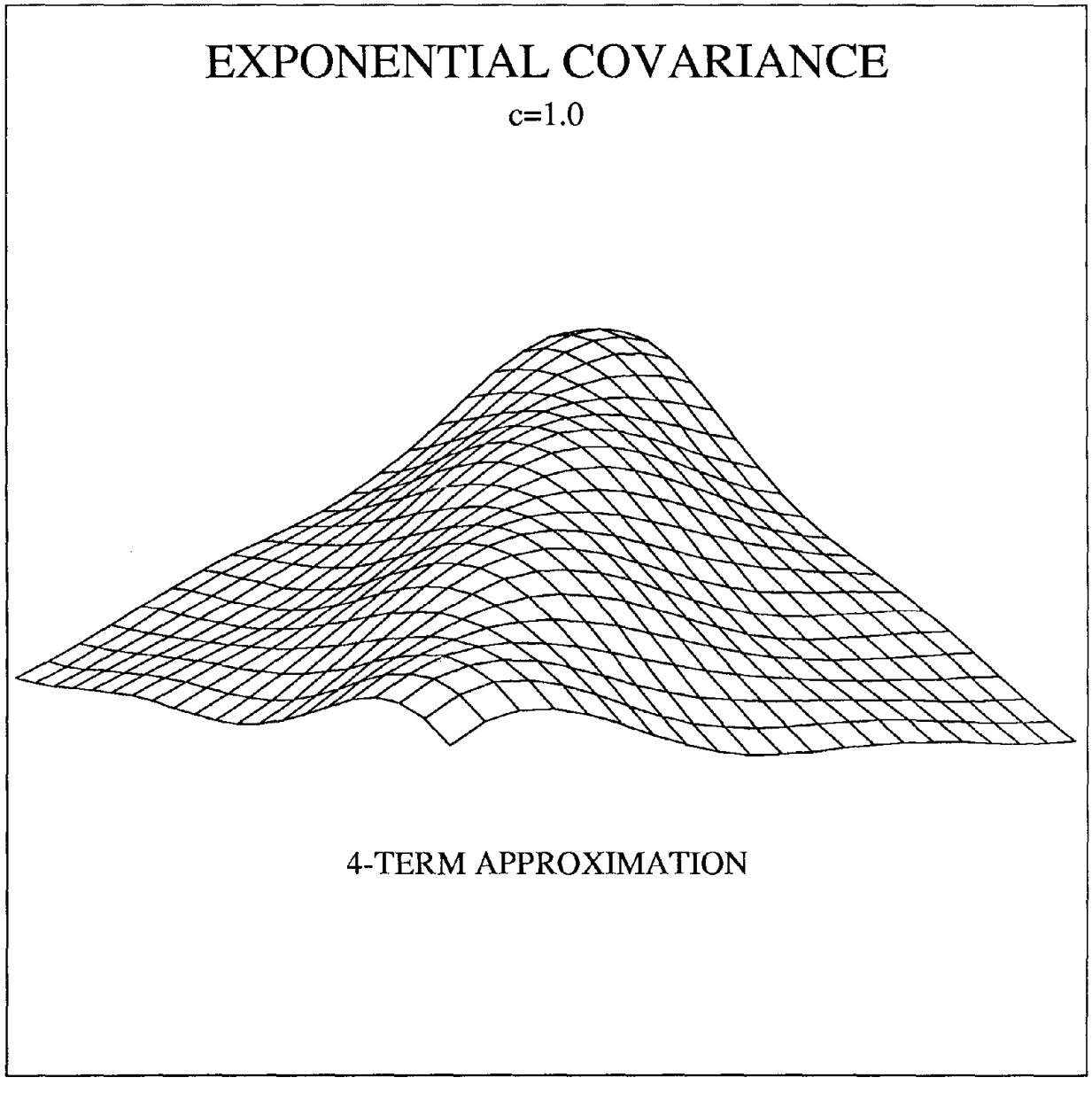

Fig. 3-2 Approximate Exponential Covariance; c=1.0, r=4.

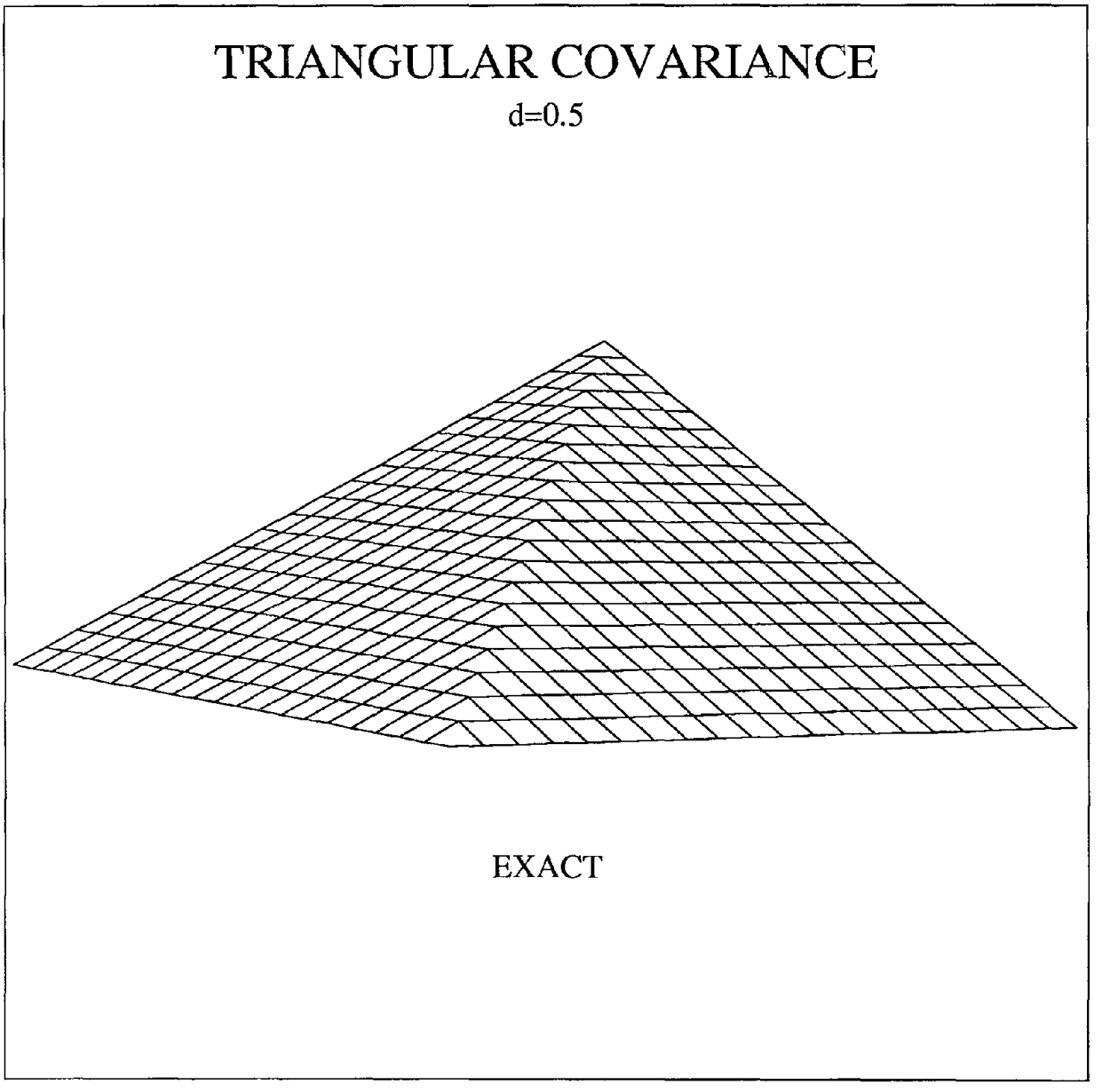

Fig. 3-3 Exact Triangular Covariance; d=0.5.

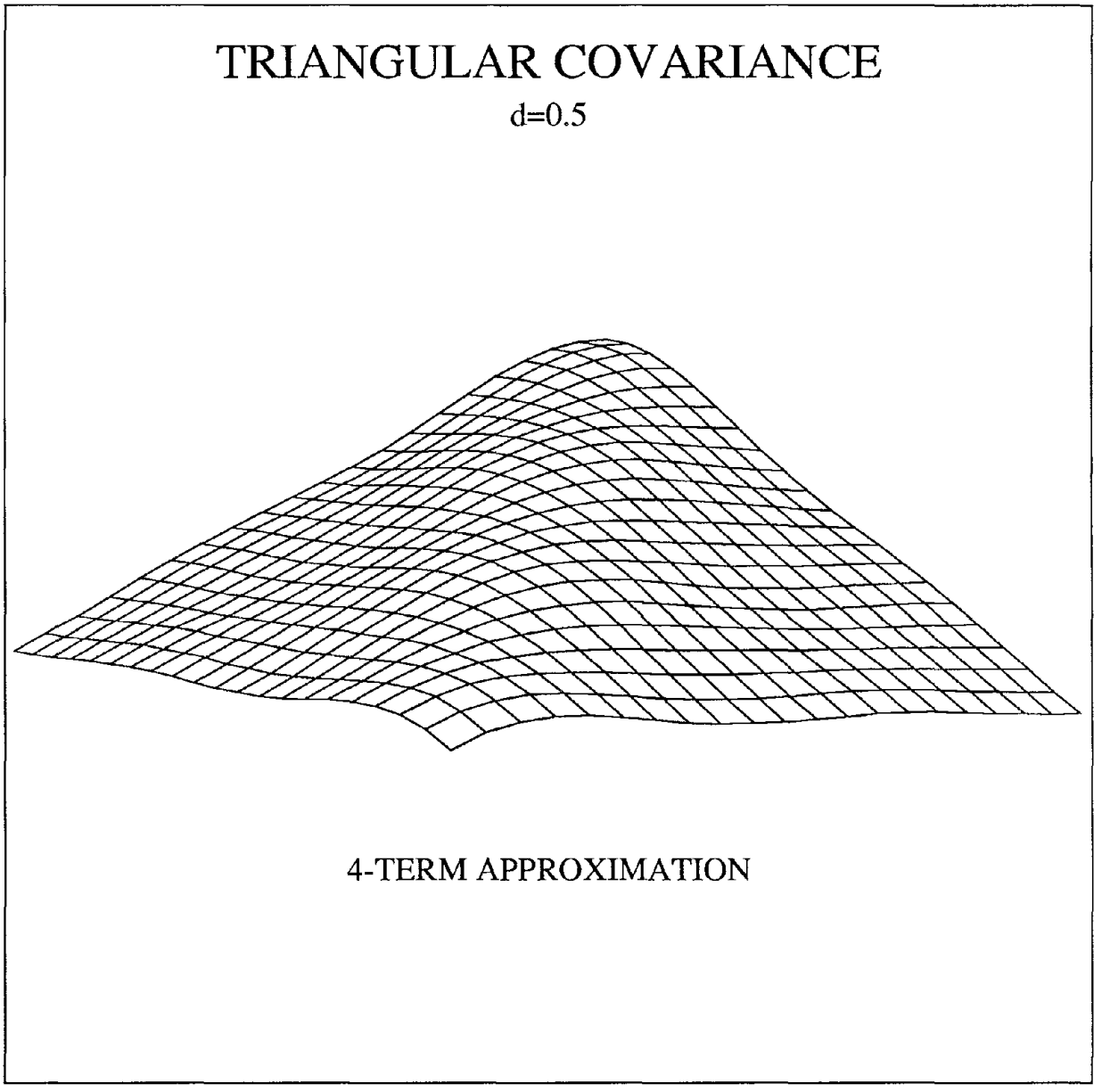

Fig. 3-4 Approximate Triangular Covariance; d=0.5, r=4.

SECTION 4

FINITE ELEMENT FORMULATION

Let L be a domain in R, and let S(x), $x \varepsilon L$, be a property of this domain which will be considered to be a random process with mean $\bar{S}(x)$ and covariance function $C(x,\xi)$. Let the domain L be subjected to a set of p external forces $\{ P_i \}$, i=1,...,p. Following the standard energy formulation (Zienckiewicz, 1977), subdivided L into m finite elements each of length l^e. The strain energy stored in each of these elements can be represented in terms of the strain ε^e and the stress σ^e of the element

$$V^e = \frac{1}{2} \int_{l^e} \sigma^e \varepsilon^e \, dD \tag{41}$$

$$= \frac{1}{2} \left\{ u^e \right\}^T \int_{l^e} \left[B^e(x) \right]^T \left[D^e(x) \right] \left[B^e(x) \right] dx \left\{ u^e \right\},$$

where $\left\{ u^e \right\}$ is the vector of nodal displacements, $\left[B^e(x) \right]$ is the strain-displacement matrix, and $\left[D^e(x) \right]$ is the random matrix of material properties. It is assumed that $\left[D^e(x) \right]$ can be represented as $S(x) \left[P^e \right]$ where $\left[P^e \right]$ is a deterministic matrix.

Summing up the contributions from all elements, the total strain energy stored in the domain L becomes

$$V = \sum_{e=1}^{m} V^e = \frac{1}{2} \sum_{e=1}^{m} \left\{ u^e \right\}^T \int_{l^e} \left[B^e(x) \right]^T \left[S^e(x) \right] \left[B^e(x) \right] dx \left\{ u^e \right\}. \tag{42}$$

Setting

$$\left[K^e \right] = \int_{l^e} \left[B^e(x) \right]^T \left[S^e(x) \right] \left[B^e(x) \right] dx \tag{43}$$

and expanding S(x) as in equation (13) gives

$$\left[K^e \right] = \left[\bar{K}^e \right] + \sum_{n=0}^{r} b_n \left[K_n^e \right]. \tag{44}$$

In this equation,

$$\left[K_n^e \right] = \lambda_n \int_{l^e} \phi_n(x) \left[B^e(x) \right]^T \left[P^e(x) \right] \left[B^e(x) \right] dx \tag{45}$$

is the n^{th} consistent component of the random element stiffness and ,

$$\left[\bar{K}^e \right] = \int_{l^e} \left[B^e(x) \right]^T \left[\bar{S}(x) \right] \left[B^e(x) \right] dx \tag{46}$$

is the consistent mean element stiffness. Substituting equations (43-46) into equation (42), the expression for the total strain energy becomes

$$V = \frac{1}{2} \sum_{n=0}^{r} b_n \sum_{e=1}^{m} \left\{ u^e \right\}^T \left[K_n{}^e \right] \left\{ u^e \right\} + \frac{1}{2} \sum_{e=1}^{m} \left\{ u^e \right\}^T \left[\bar{K}^e \right] \left\{ u^e \right\}. \tag{47}$$

The local displacements $\left\{ u^e \right\}$ for each element are related to the global displacements $\{ U \}$ of the whole system through the " bookkeeping " transformation

$$\left\{ u^e \right\} = \left[C^e \right] \{ U \}, \tag{48}$$

where $\left[C^e \right]$ is a permutation matrix. Combining equation (48) and equation (47) yields

$$V = \frac{1}{2} \sum_{n=0}^{r} b_n \{ U \}^T \left[K_n \right] \{ U \} + \frac{1}{2} \{ U \}^T \left[\bar{K} \right] \{ U \}, \tag{49}$$

where

$$\left[K_n \right] = \sum_{e=1}^{m} \left[C^e \right]^T \left[K_n{}^e \right] \left[C^e \right] \tag{50}$$

is the n^{th} consistent global random stiffness, and

$$\left[\bar{K} \right] = \sum_{e=1}^{m} \left[C^e \right]^T \left[\bar{K} \right] \left[C^e \right]. \tag{51}$$

is the consistent global mean stiffness.

The p external forces applied to the domain can be grouped in a vector $\{ P \}$. The work performed by $\{ P \}$ during the deformation of the domain is equal to

$$\Omega = \{ U \}^T \{ P \}. \tag{52}$$

Minimizing the total potential energy leads to

$$\frac{\partial \left(V - \Omega \right)}{\partial \{U\}} = 0. \tag{53}$$

Inserting in the above equation the expressions for V and Ω from equations (49) and (52), respectively, results in

$$\left[\bar{K} + \sum_{n=0}^{r} b_n \left[K_n \right] \right] \{U\} = \{P\}. \tag{54}$$

At this stage, the boundary conditions can be imposed on each $\left[K_n \right]$ separately. The spatial variation of the randomness has been incorporated in the deterministic $\left[K_n \right]$ matrices. In equation (54), the random coefficient matrix has to be inverted to determine the response vector. That is,

$$\{ U \} = \left[I + \sum_{n=0}^{r} b_n \left[\overline{K} \right]^{-1} \left[K_n \right] \right]^{-1} \left[\overline{K} \right]^{-1} \{ P \} . \tag{55}$$

The first term on the right hand side of equation (55) can be expanded in a Neumann series leading to

$$\{ U \} = \sum_{k=0}^{\infty} \left[- \sum_{n=0}^{r} b_n \left[Q_n \right] \right]^k \left[\overline{K} \right]^{-1} \{ P \} , \tag{56}$$

where

$$\left[Q_n \right] = \left[\overline{K} \right]^{-1} \left[K_n \right] . \tag{57}$$

For gaussian material properties, $\left\{ b_n \right\}$ forms a gaussian vector with independent, uncorrelated and jointly gaussian random variables. An important implication of the gaussian property of $\{ b_n \}$ can be expressed by the equations (Loeve, 1977)

$$E \left[b_1 \cdots b_{2m+1} \right] = 0 \tag{58}$$

$$E \left[b_1 \cdots b_{2m} \right] = \sum \prod E \left[b_i b_j \right] \tag{59}$$

In equation (59) the summation involves $\frac{(2m)!}{(2^m m!)}$ terms corresponding to the different ways by which 2m elements can be broken up into m pairs. This fact greatly simplifies the analysis. Assuming gaussian material property, for simplicity, and averaging both sides of equation (56) yields

$$E \left[U \right] = \sum_{k=\text{even}} \left[\sum_{n=0}^{\infty} b_n \left[Q_n \right] \right]^k \left[K \right]^{-1} \{ P \} \tag{60}$$

To compute the correlation matrix of the response, multiply equation (56) by its transpose and average both sides. This procedure leads to

$$E \left[UU^T \right] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} E \left[\left[\sum_{n=0}^{r} b_n \left[Q_n \right] \right]^l \left[\overline{K} \right]^{-1} \{ P \} \{ P \}^T \left[\overline{K} \right]^{-T} \left[\sum_{m=0}^{r} b_m \left[Q_m \right]^T \right]^k \right] \tag{61}$$

Clearly, this equation for the correlation matrix can be greatly simplified, if equations (58) and (59) are applicable.

SECTION 5

NUMERICAL EXAMPLE

The formulation presented in the previous sections is quite general in that it is applicable to any physical domain by simply using the corresponding stress-strain relations as indicated by the matrices $\left[B^e\right]$ and $\left[D^e\right]$. Also, random loadings can be readily accommodated in the analysis as can be seen from equations (60) and (61). The method described in the preceding sections is exemplified by considering a one-dimensional problem. The problem involves a cantilever Euler-Bernoulli beam subjected to a deterministic uniform transverse static load, as shown in figure 5-1. It is assumed that the bending rigidity $\bar{E}I$ of the beam, which involves the modulus of elasticity $\bar{E}$ and the cross-sectional mass moment of inertia I, is a random process of the spatial variable x, as shown in fig. 5-1. It is assumed that the process $\bar{E}I$ has a known mean value $\langle\bar{E}I\rangle$ and a known covariance function $C(x,\xi)$ reflecting two beam locations, at x and ξ. Two models for $C(x,\xi)$ are considered. The first involves the exponential form which lends itself to the expansion defined by equations (26) through (30). The second involves the triangular form which lends itself to the expansion specified by the equations (37) through (40). Further, the beam under consideration is assumed, without loss of generality, to have unit length and unit mean bending rigidity; it is subjected to a unit uniform load. In implementing the preceding stochastic finite element method, the matrix $\left[B^e(x)\right]$ must be determined. In doing this task, linear interpolation of the strains is used over each element. The resulting equation is

$$\left[B^e\right] = \left[\begin{array}{cccc} 12\eta-6 & l^e(6\eta-4) & 6-12\eta & l^e(6\eta-2) \end{array}\right] \frac{1}{l^{e^2}}, \tag{63}$$

where η represents the local coordinate over the element as shown in figure 5-1. This expression is then substituted into equation (45) to compute the matrices $\left[K_n^e\right]$. Finally, equations (60) and (61), truncated at an appropriate number of terms, are used to calculate the mean and the standard deviation of the response of the beam at any given nodal point.

To assess the reliability of the proposed stochastic finite element method, a Monte Carlo simulation of the beam response is undertaken. Specifically, an auto-regressive (AR) digital filter of order 20 (twenty) is used to synthesise realizations of the bending rigidity of the beam along its span. Upon generating a bending rigidity profile of the beam, its response to the uniform load is determined by relying on a standard numerical quadrature algorithm. This procedure is repeated several times to produce an ensemble of beam deflections along its span. Then, statistical algorithms are utilized to extract from the ensemble the mean value and the standard deviation of the deflection at selected nodal points.

Figures 5-2 through 5-5 show the results for the standard deviation σ_T of the deflection of the tip of the beam versus various values of the standard deviation $\sigma_{\bar{E}I}$ of the bending rigidity $\bar{E}I$. Also shown in these figures are the corresponding values of σ_T produced by a Monte Carlo study

involving 5000 (five thousand) realizations of the bending rigidity profile of the beam. Observe the excellent agreement between the theoretical and the simulated results as the number of terms in the orthogonal expansion reaches four (r=4) and combined terms of order up to eight (k+l=8) are maintained in the Neumann expansion. The same trend is observed in the results shown in figures 5-6 through 5-9 which pertain to the triangular covariance function. For both cases, the reliability of the proposed method is quite remarkable even for a beam with large $\sigma_{\overline{EI}}$.

To reflect the computational efficiency of the proposed method, define the time required to solve the deterministic problem as a unit of "computing effort". Table 1 shows the computing effort required by the proposed method for the cases shown in the plots. Examining this table, it is seen that even for small scale problems like those described in the previous section, the proposed method is notably more efficient than the Monte-Carlo method. Note that in few cases of interest, analytical expressions for the eigenfunctions and eigenvalues of the covariance function, which are involved in the expansion of the process $\Delta S(x)$, are possible. In general, however, it is necessary to numerically find the eigenvalues and eigenfunctions of the covariance matrix associated with a particular problem. Once computed, the eigenfunctions can be stored either numerically or by using interpolation functions (Masri at al. 1982, 1986).

Table 1 Comparison of the Computation Time Required by the Proposed Method to that Required by the Monte Carlo Simulation Method.

Computational Units				
	k+l=2	k+l=4	k+l=6	k+l=8
r*=2	1.2	1.2	3.0	13.0
r=4	1.2	3.0	36.0	737.0
MCS**	5000			

(*) r = number of terms in the orthogonal expansion

(**) 5000 samples in the Monte-Carlo simulation.

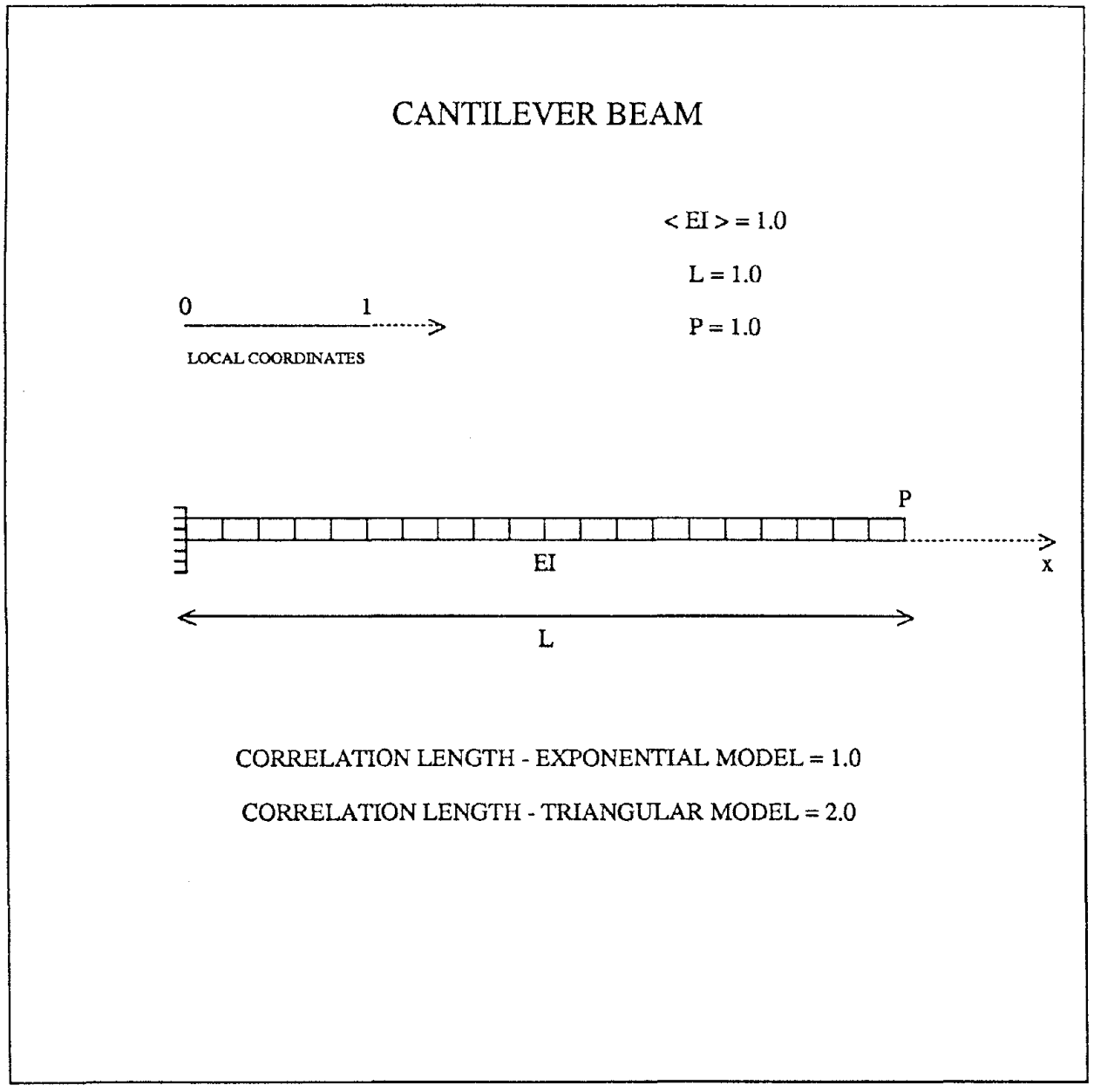

Fig. 5-1 Beam with Random Bending Rigidity under Uniform Load; Exponential and Triangular Covariance Models.

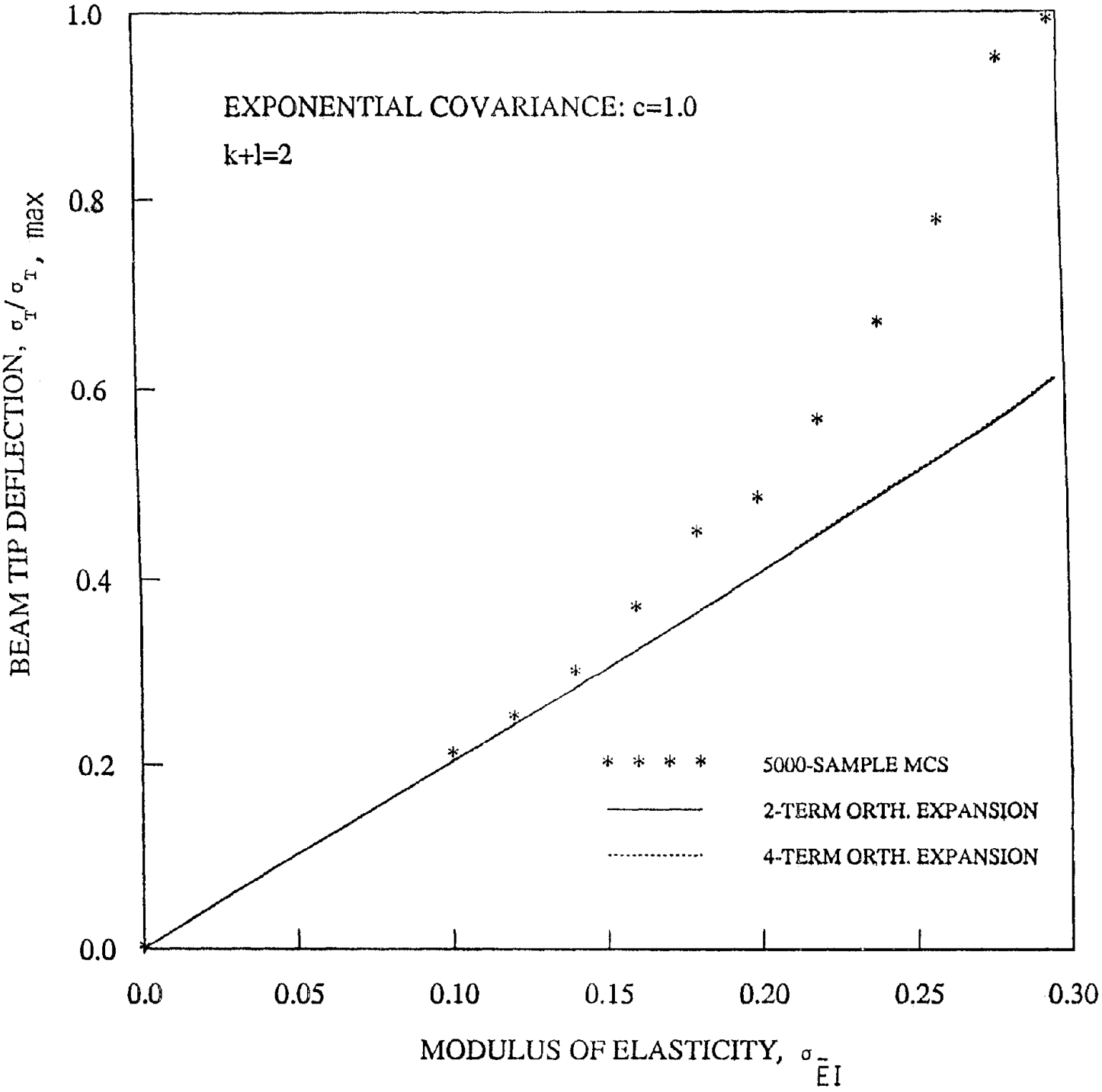

FIGURE 5-2 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Exponential Covariance, k+1=2

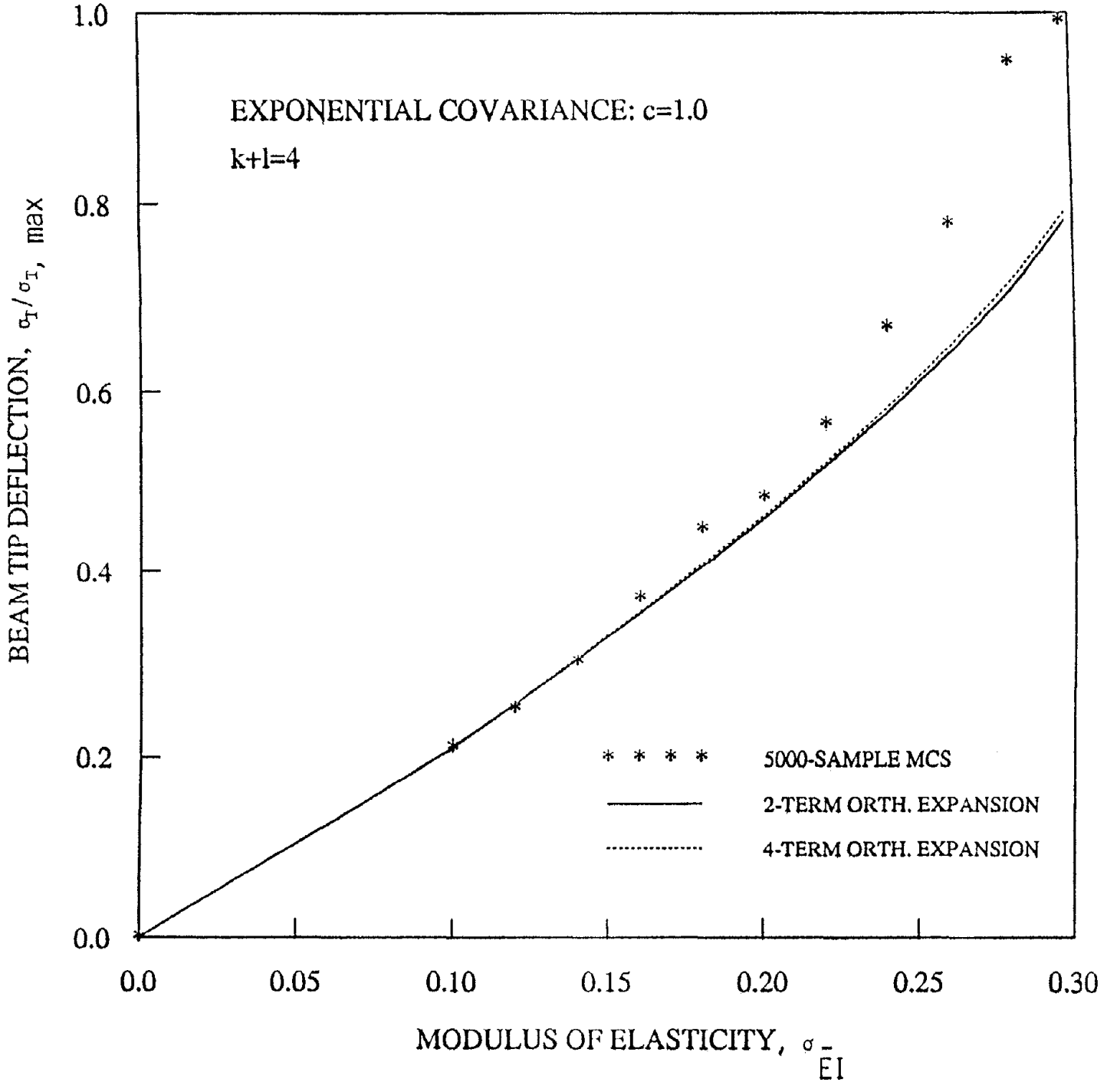

FIGURE 5-3 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Exponential Covariance, k+1=4

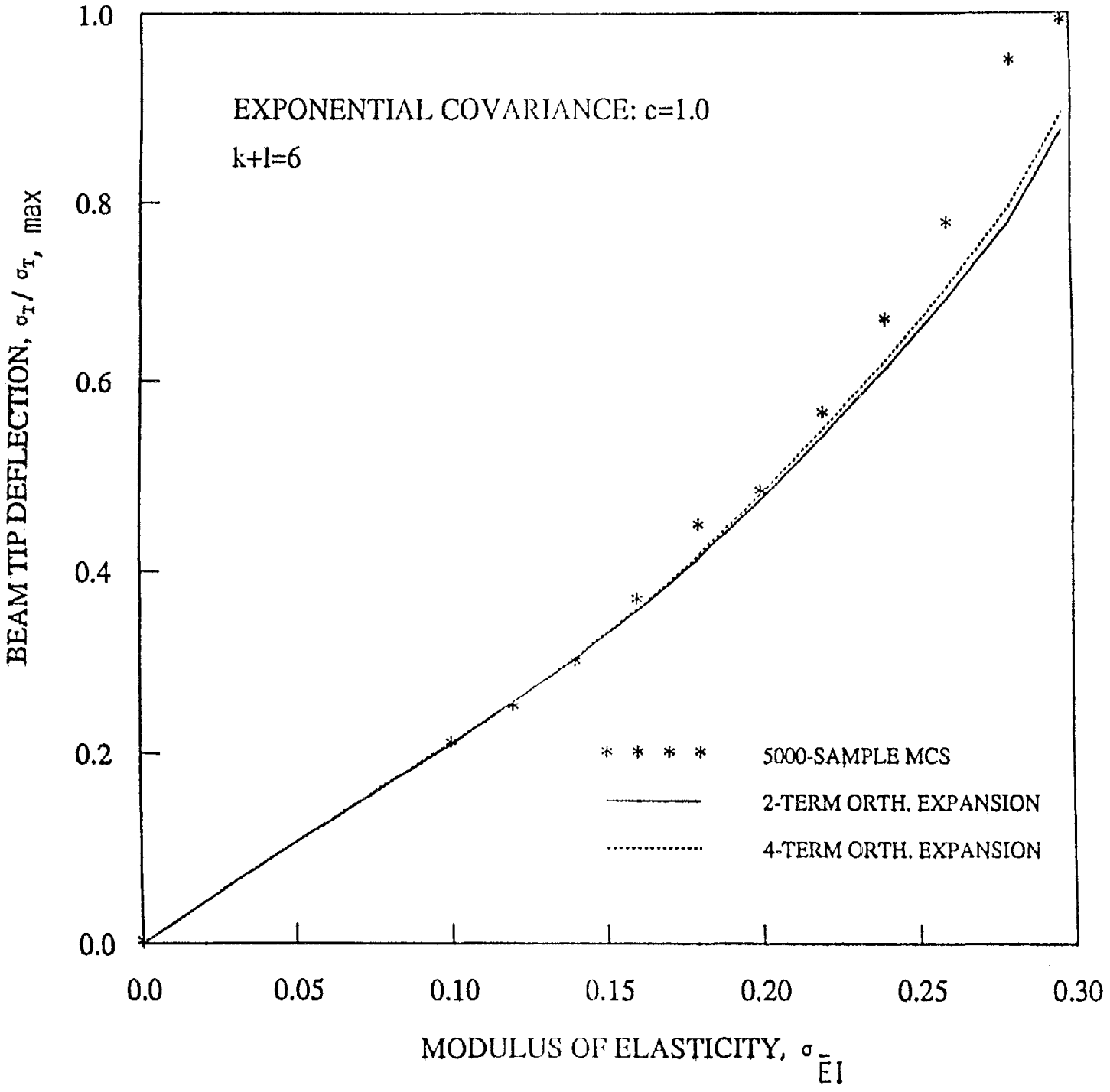

FIGURE 5-4 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Exponential Covariance, k+1=6

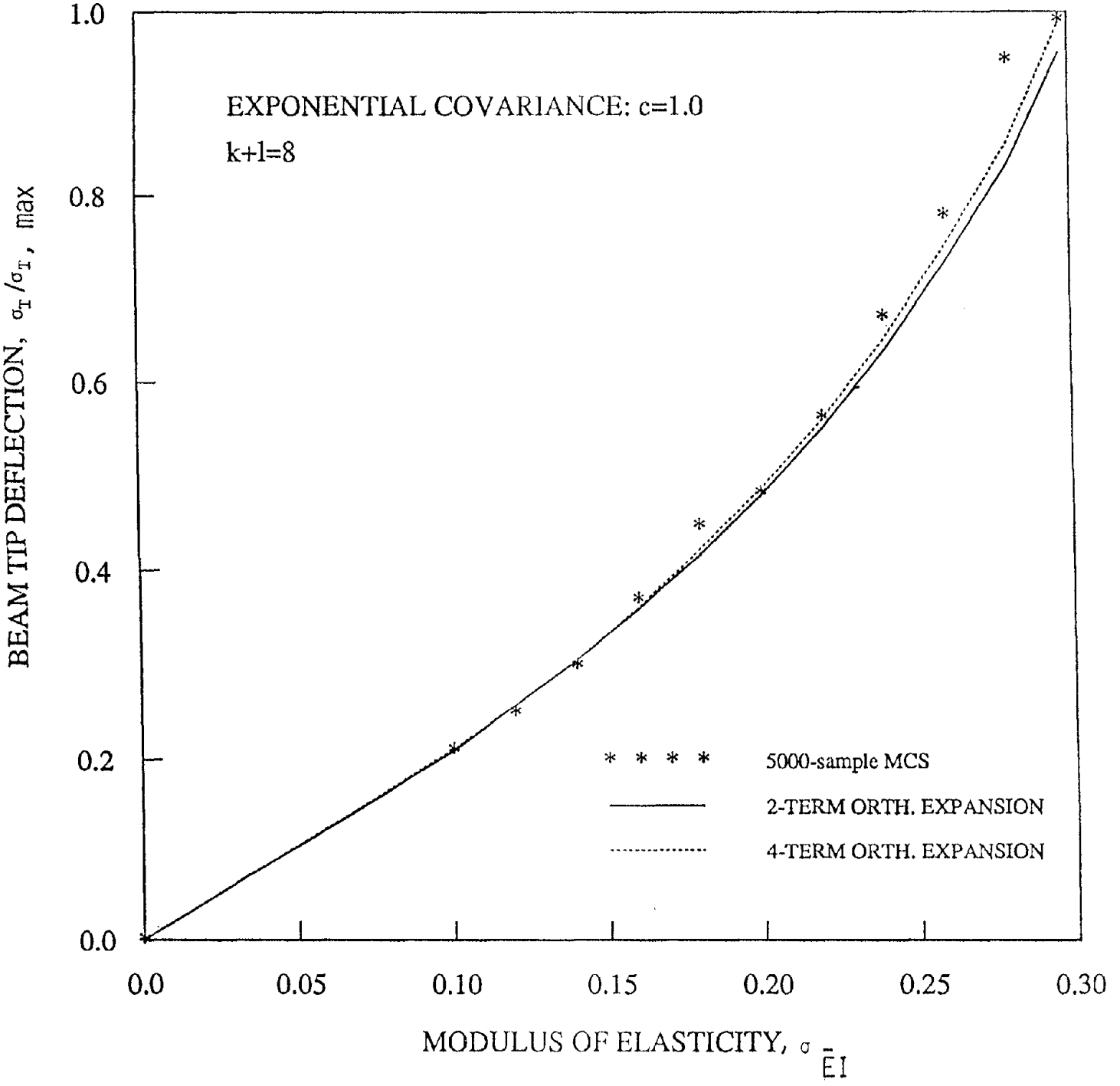

FIGURE 5-5 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Exponential Covariance, k+1=8

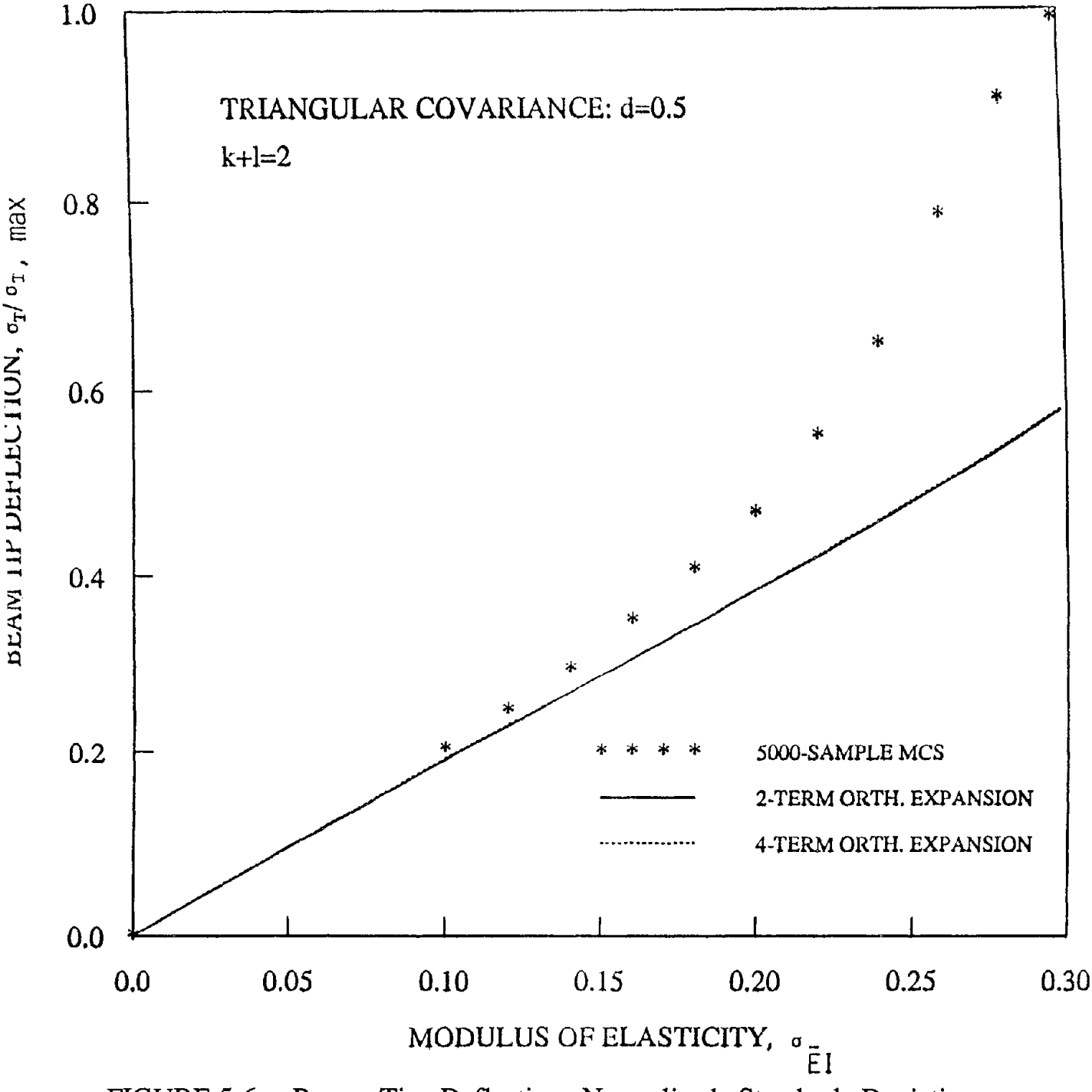

FIGURE 5-6 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Triangular Covariance, k+1=2

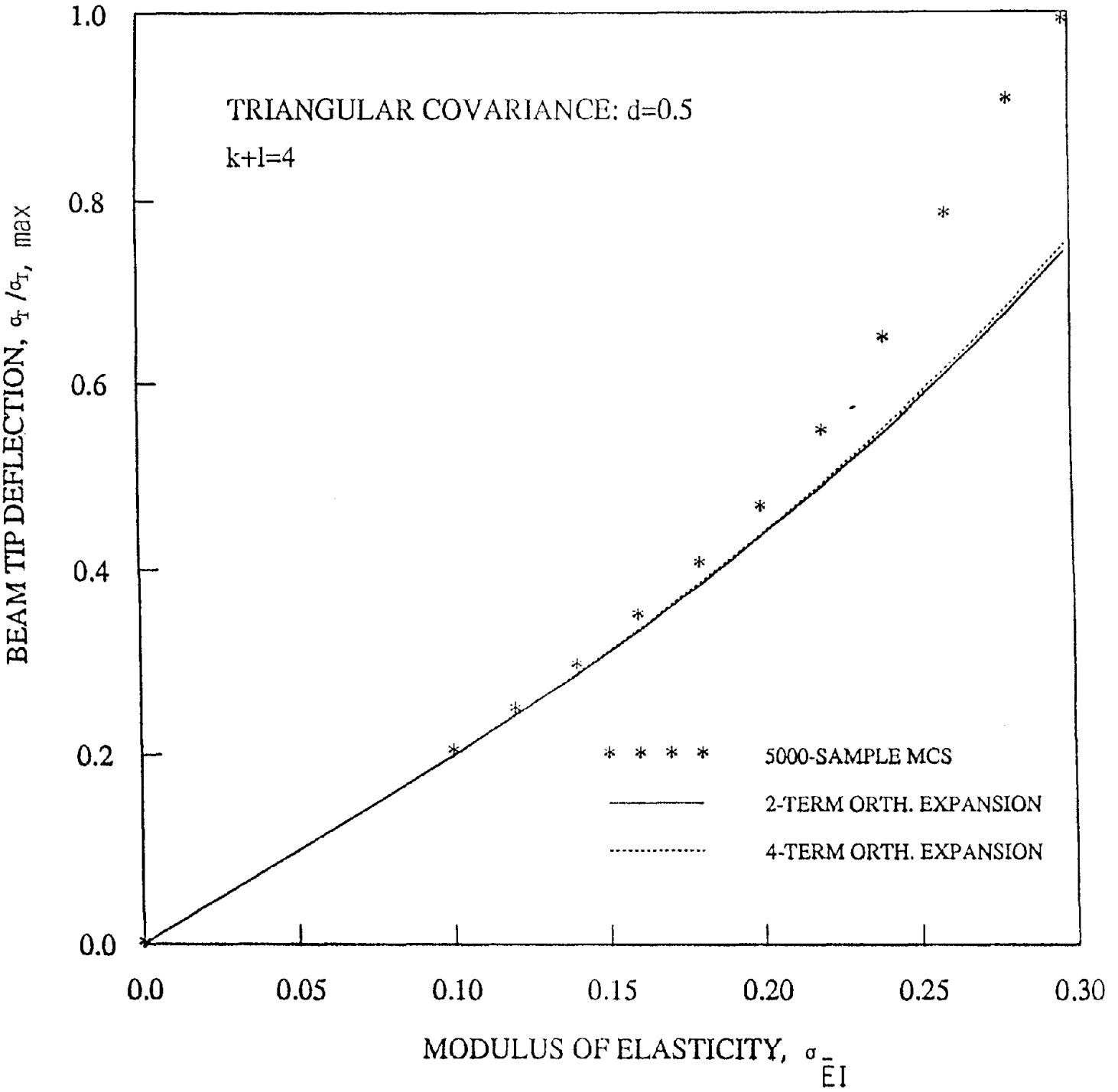

FIGURE 5-7 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Triangular Covariance, k+1=4

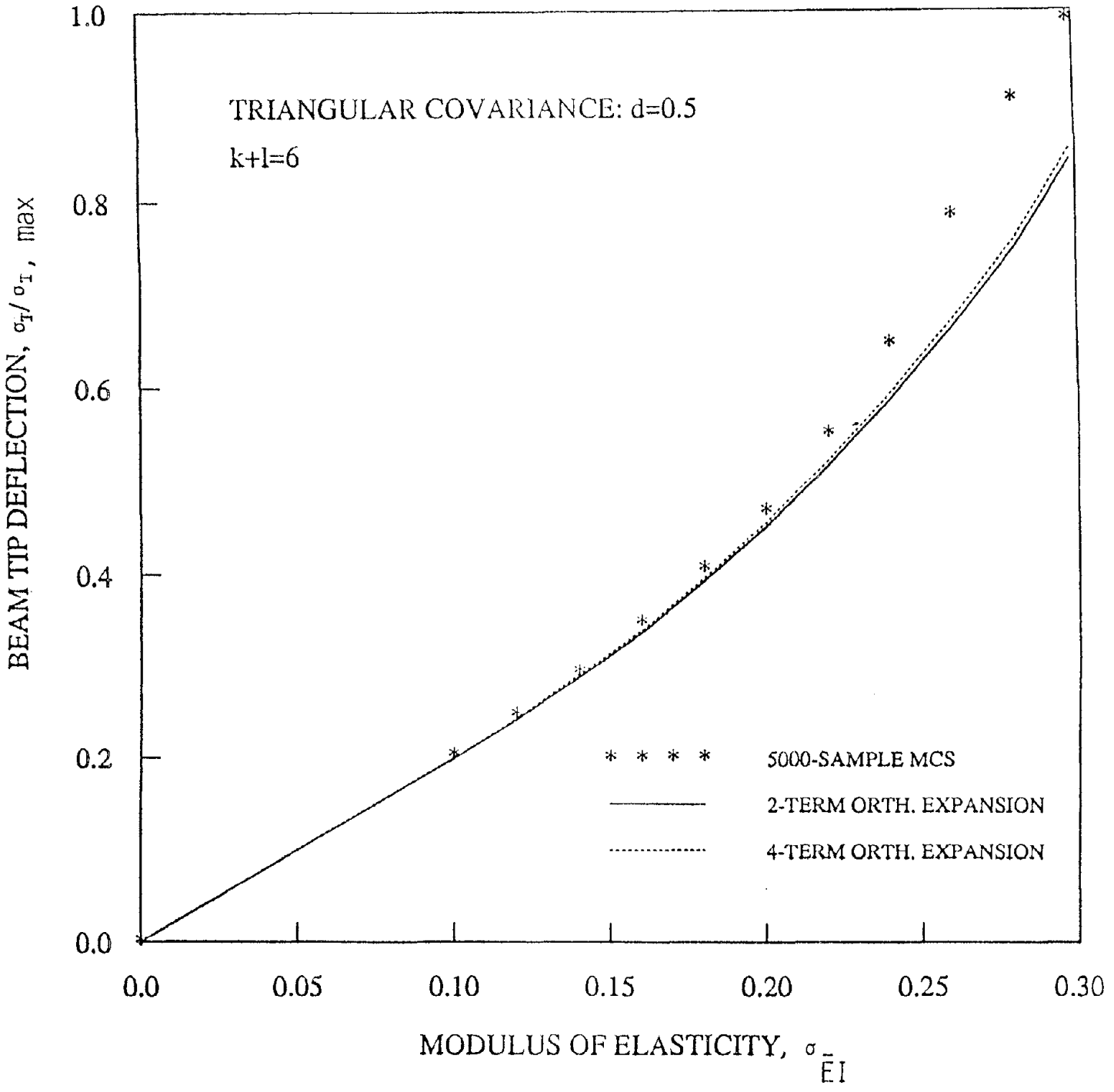

FIGURE 5-8 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Triangular Covariance, k+1=6

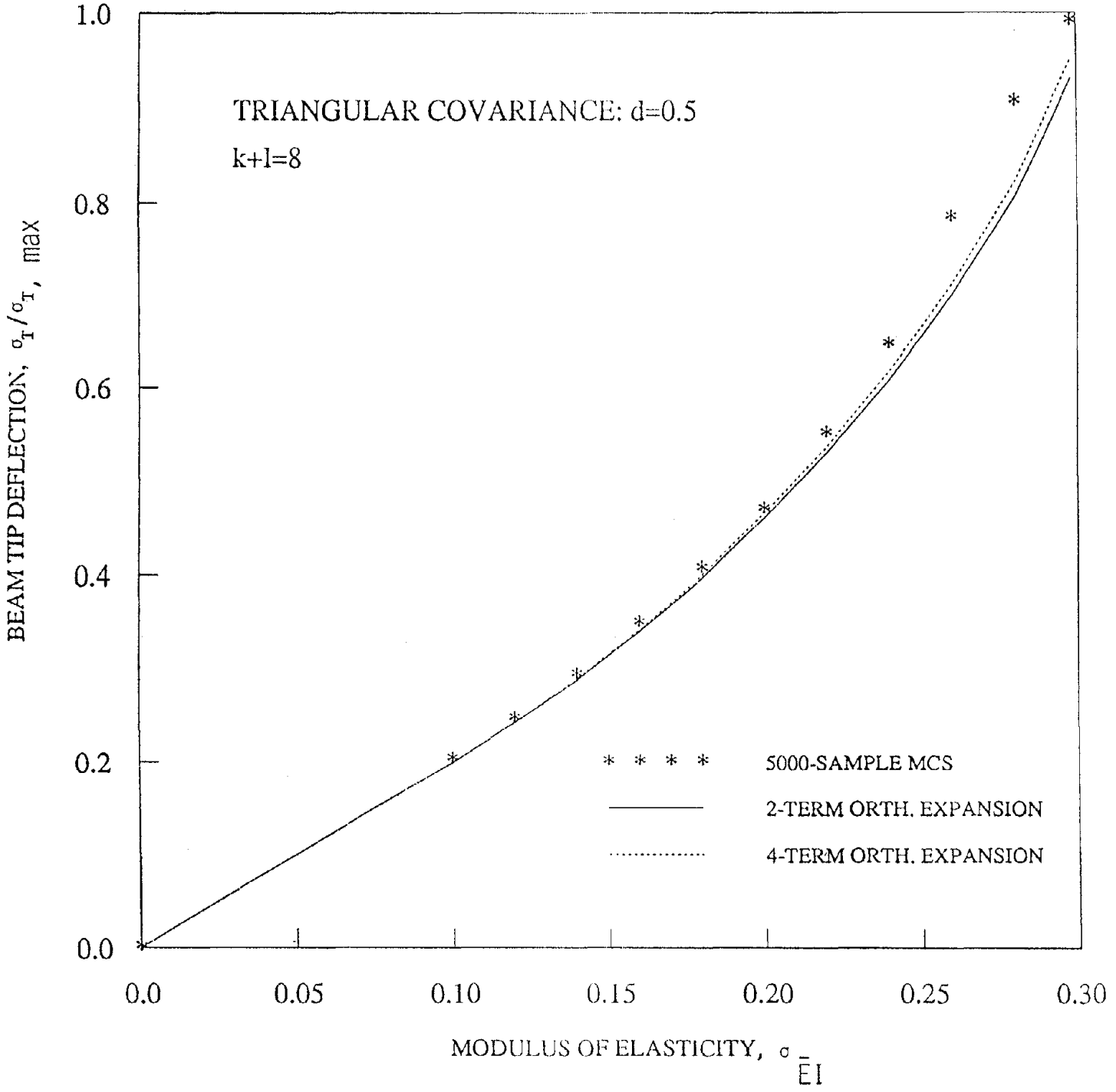

FIGURE 5-9 Beam Tip Deflection Normalized Standard Deviation versus Bending Rigidity Standard Deviation; Triangular Covariance, k+1=8

SECTION 6

SUMMARY

A new method is proposed for a numerical treatment of problems involving random media The method is based on the Karhunen-Loeve orthogonal expansion of a random process. The expansion consists of the projection of the process onto a space of orthogonal random variables. The method is incorporated into a consistent variational finite element formulation. It can be mechanised for computational efficiency. Further, it can be readily combined with any deterministic finite element code. To apply the method to a specific problem it is first required to determine, analytically or numerically, the eigenvalues and eigenvectors of the covariance function, which are then used in equation (45) to compute $\left[K_n^e\right]$. The indicated integrations can be performed analytically for some special cases. For arbitrary problems, however, resorting to numerical quadrature is necessary. The elemental random stiffnesses are then assembled into the global random stiffnesses as indicated by equation (50). The mean elemental and global matrices given in equations (46) and (51) can be assembled using a standard finite element code. Equations (60) and (61) can be conveniently automated to compute the average and the covariance matrix of the response to any desired accuracy. An application to the problem of a random cantilever beam was investigated. The results were found to be in good agreement with a Monte-Carlo data bank simulation. Substantial superiority, in terms of the requisite computational time, of the new method over the Monte Carlo approach was noted. It is believed that the proposed method has a great potential for dealing with problems encountered in earthquake engineering where both the excitation and the properties of the medium can be modeled as random fields.

SECTION 7

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SECTION 8

NOTATION

The following symbols are used in this report.

b_n = orthogonal random variables.

c = reciprocal of the correlation length of the random process -exponential model-.

d = reciprocal of the correlation length of the random process -triangular model-.

l^e = length of element e.

m = number of finite element in the mesh.

p = number of external loads applied to the beam.

r = number of terms used in the orthogonal expansion.

u^e = local displacement vector of element e.

$[B^e (x)]$ = strain-displacement matrix.

$[C^e]$ = permutation matrix for element e.

$[D^e (x)]$ = stress-strain matrix.

$\bar{E}$ = modulus of elasticity of the beam.

$E[.]$ = is the mathematical expectation operator.

$[K^e]$ = element stiffness matrix.

$[K]$ = global stiffness matrix.

$[K_n]$ = n^{th} consistent component of the random stiffness matrix.

$[\bar{K}^e]$ = element mean stiffness matrix.

$[\bar{K}]$ = global mean stiffness matrix.

I = cross-section mass moment of inertia of the beam.

L = Domain of definition of the process.

$[Q_n] = [\bar{K}]^{-1} [K_n]$.

S (x) = stochastic process.

$\bar{S}$ (x) = mathematical expectation of the process S.

U = global displacement vector.

V^e = strain energy in element e.

V = total strain energy in the beam.

δ_{mn} = Kronecker delta.

ε^e = strain over element e.

η = local coordinate over a finite element.

λ_n = n^{th} eigenvalue of $C(x,\xi)$.

$\phi_n(x)$ = n^{th} eigenvector of $C(x,\xi)$.

σ^e = stress over element e.

σ_S = standard deviation of the modulus of elasticity of the beam.

$\Delta S(x) = S(x) - \overline{S}(x)$.

Ω = external work on the beam.

$[\,.\,]^T$ = indicates matrix transposition.

$[\,.\,]^{-1}$ = indicates matrix inversion.

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