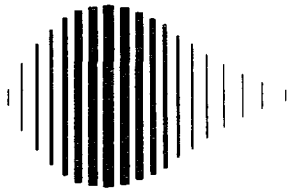

NATIONAL CENTER FOR EARTHQUAKE
ENGINEERING RESEARCH
State University of New York at Buffalo

Damping of Structures: Part 1 - Theory of Complex Damping

by

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Technical Report NCEER-91-0004

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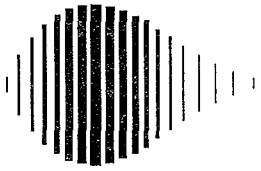

Damping of Structures: Part 1 - Theory of Complex Damping

by

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PREFACE

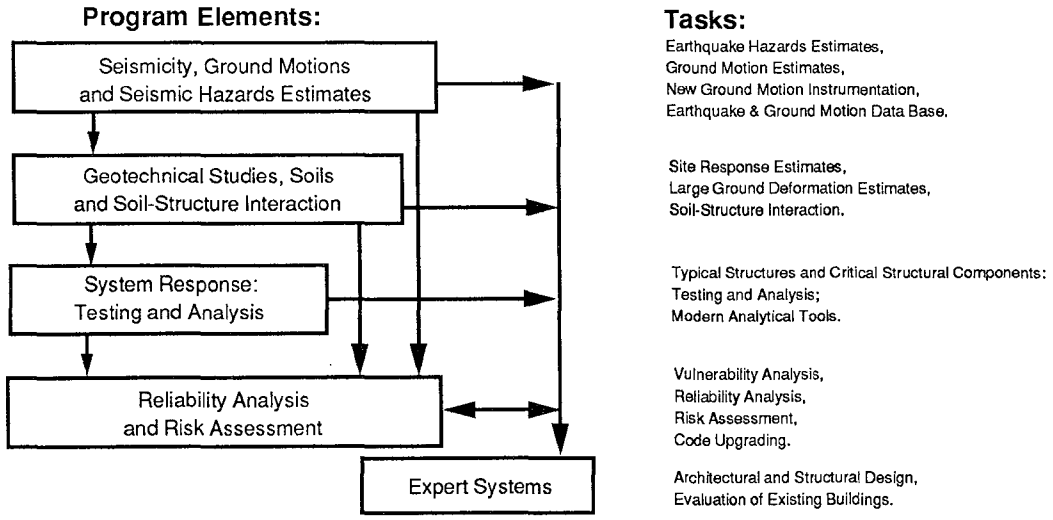
The National Center for Earthquake Engineering Research (NCEER) is devoted to the expansion and dissemination of knowledge about earthquakes, the improvement of earthquake-resistant design, and the implementation of seismic hazard mitigation procedures to minimize loss of lives and property. The emphasis is on structures and lifelines that are found in zones of moderate to high seismicity throughout the United States.

NCEER's research is being carried out in an integrated and coordinated manner following a structured program. The current research program comprises four main areas:

- Existing and New Structures
- Secondary and Protective Systems
- Lifeline Systems
- Disaster Research and Planning

This technical report pertains to Program 1, Existing and New Structures, and more specifically to system response investigations.

The long term goal of research in Existing and New Structures is to develop seismic hazard mitigation procedures through rational probabilistic risk assessment for damage or collapse of structures, mainly existing buildings, in regions of moderate to high seismicity. The work relies on improved definitions of seismicity and site response, experimental and analytical evaluations of systems response, and more accurate assessment of risk factors. This technology will be incorporated in expert systems tools and improved code formats for existing and new structures. Methods of retrofit will also be developed. When this work is completed, it should be possible to characterize and quantify societal impact of seismic risk in various geographical regions and large municipalities. Toward this goal, the program has been divided into five components, as shown in the figure below:



System response investigations constitute one of the important areas of research in Existing and New Structures. Current research activities include the following:

1. Testing and analysis of lightly reinforced concrete structures, and other structural components common in the eastern United States such as semi-rigid connections and flexible diaphragms.
2. Development of modern, dynamic analysis tools.
3. Investigation of innovative computing techniques that include the use of interactive computer graphics, advanced engineering workstations and supercomputing.

The ultimate goal of projects in this area is to provide an estimate of the seismic hazard of existing buildings which were not designed for earthquakes and to provide information on typical weak structural systems, such as lightly reinforced concrete elements and steel frames with semi-rigid connections. An additional goal of these projects is the development of modern analytical tools for the nonlinear dynamic analysis of complex structures.

In structural design against dynamic loads, damping is perhaps the least understood aspect. Because damping in typical structures is small, approximations have been made to facilitate the design process with negligible effects. One typical example is the assumption that the structures are proportionally damped. Today, much interest is given to passive and active control of structural vibrations. In most cases, the damping of structures is increased so that the effects of the approximations concerning damping may not necessarily be negligible.

This report presents a complex energy-based damping theory which can better describe the dynamic responses of generally (non-proportionally) damped structures. It is formulated by considering both dissipative and conservative energy components of vibrating systems simultaneously by using a complex-valued quantity. A second report, Part II, is being developed, which will provide examples to illustrate the scope of potential applications of the theory. One special emphasis is given to the determination of the damping matrix coefficients of non-proportionally damped structures. Other topics, such as damper design and passive control, are discussed in Part II.

ABSTRACT

Damping is perhaps the least understood aspect of structural analysis and design against dynamic loads. Because damping is small in typical civil engineering structures, approximations have been made to facilitate the design process with negligible effects. One example is the assumption that the structures are proportionally damped. Briefly speaking, the development of the theory of structures damping has been an area of study for quite some time - beginning with traditional Rayleigh damping, to general proportional damping, and finally to "non-proportional (non-classic)" damping. In recent years, dampers have been used successfully in the World Trade Center in New York City, for example, and in other tall structures around the world to reduce vibrations from the wind, and/or seismic ground motions. From shaking table and other methodological studies, various dampers have been shown to provide good seismic control. Today, much interest is given to vibration control in structural engineering research and practice. In most cases, the damping of structures is increased so that it is no longer a property that can be treated lightly. For example, the assumption of proportional damping for non-proportionally damped systems can no longer be taken for granted.

This report, consisting of two parts, presents a complex energy-based damping theory and its applications. The theory is formulated by considering the dissipative and the conservative energy components of damped vibrating systems simultaneously by complex-valued quantities. It provides a theoretical foundation for the analyses of generally (non-proportionally) damped systems. At the same time, the complex damping theory offers new approaches to model the dynamic responses of multiple-degree-of-freedom systems (a relatively underdeveloped area in Newtonian mechanics) and to deal with vibration control problems in structural engineering.

The mathematical tools used in this report are briefly summarized in Appendix A. Readers may omit this appendix if they are not interested in the mathematical development of the theory.

Chapter 1 selectively reviews the basic concepts of structural dynamics and damping, as they will be needed in subsequent presentations in the report.

Chapter 2 presents the theory of complex damping. This theory gives a unified representation of energy dissipation and energy transfer by means of one complex quantity. The real part of the complex quantity represents the traditional damping ratio, a ratio of the energy dissipation during a given period to the total energy at the beginning of the given period; the imaginary part is the ratio of energy transfer in the same period to the total energy at the beginning of this period.

Chapter 3 is concerned with lightly damped systems. In such systems, the real part of the eigenvalue of the state matrix and the damping ratio possess a special linearity. This property is directly deduced from the complex damping theory. To facilitate the understanding of the use of the lightly damping approach in engineering applications, several typical examples are given.

Chapter 4 deals with evaluation methods for the damping matrices. Based on the theories of complex damping and a theory of matrix representation, a simplified damping model is presented. An attempt is made to introduce a general method for evaluating damping matrices. Also in Chapter 4, the complex damping in structurally (hysteretically) damped structures is discussed.

Chapter 5 first presents the reasons that the quantitative value of the damping matrix depends not only on damping configurations, but also on the mass and stiffness matrices of the structure. This explains why conventional finite element and/or other methods cannot model a damping matrix directly. Then, based on the complex damping theory and by using the lightly damped approach, a method of directly assembling general damping matrices is presented.

The aforementioned topics are design problems and thus are "forward" problems. Damping and/or dynamic parameter identifications are inverse problems, which are discussed in Part II (Application of Complex Damping Theory will be available in the near future). In Part II, certain direct applications based on the theoretical considerations described in Chapter 2 through Chapter 5, such as damper design and passive control, will also be discussed.

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NOMENCLATURE

$\mathbf{A}$	Matrix
$\mathbf{A}^{(.)}$	Product of Matrices
$\mathbf{B}$, $\mathbb{B}$	Matrix, Operator, Loss β Matrix
B	Constant Scalar
$\mathbf{C}$	Matrix, Damping Matrix
$\mathbb{C}^{nxn}$	Set of All nxn Complex Matrices
$\mathbf{E}$	Temporal Matrix
$\mathbf{F}$, F	Driving Force Matrix, Driving Force Vector
G	Driving Force Vector
G_1	Loss Modulus of VE material
$\mathbf{H}$, $\hat{\mathbf{H}}$	State Matrices
$\mathbf{I}$	Identity Matrix
$\mathbf{K}$	Matrix, Stiffness Matrix
$\mathbf{L}$	Differential Operator, Triangular Matrix
$\mathbf{M}$	Matrix, Mass Matrix
$\mathbf{O}$, 0	Null Matrix, Zero Vector
$\mathbf{P}$, P	Eigenvector Matrix, Eigenvector
$\mathbf{P}_{(.)}$	Eigenvector Matrix of Matrix (.)
$\mathbf{Q}$, Q	Orthogonal Matrix, Vector
$\mathbf{Q}_p$	P^{th} Order Quasi-Vandermonde Matrix
$\mathbf{R}$	Matrix
$\mathbf{S}$, S	Input Matrix, Vector
$\mathbb{S}$	$\mathbb{S} = \text{diag}(\mathbf{S}^{-1})$
$\mathbf{T}$, T	Output Matrix, Period
$\mathbb{T}$	$\mathbb{T} = \text{diag}(\mathbf{T})$
$\mathbf{V}$	Matrix
$\mathbf{U}$	Matrix
W	Energy, Work
X,X',X"	Displacement, Velocity, Acceleration Vectors
Y,Y',Y"	Displacement, Velocity, Acceleration Vectors
Z,Z',Z"	Displacement, Velocity, Acceleration Vectors
a	Amplitude
a_i, b_i	i^{th} Real Damping Coefficient, Imaginary Damping Coefficient

c, $c_{(.)}$	Entry of Damping Matrix
d_i	i^{th} Complex Damping Coefficient
$\mathbf{e}_s$	s^{th} Unit Vector
$f_{(.)}$,	$(.)^{th}$ Frequency
f_{dij}	Damping force with frequency ω_i applying on j^{th} mass
k_i	i^{th} Eigenvalue of Matrix **K**
m_i	i^{th} Eigenvalue of Matrix **M**
n	Order of a Lumped Mass System
p_{ij}	ij^{th} Entry of Matrix **P**
i,j,k,l,n,p	Integers
j	$(-1)^{1/2}$
$r_{(.)}$	Scalar, Inner Product
${}_0r_{ij}$	j^{th} displacement amplitude under driving frequency ω_i
s	Laplace Variable
$s_{(.)}$	Scalar, Inner Product
t	Time
u,u',u"	Scalar: Displacement, Velocity, Acceleration
x,x',x"	Scalar: Displacement, Velocity, Acceleration
y,y',y"	Scalar: Displacement, Velocity, Acceleration
z,z',z"	Scalar: Displacement, Velocity, Acceleration
α, β, $\bar{\beta}$, γ,	Scalars
$\hat{\beta}$	Vector
σ	Stress Tensor
σ_{max}, σ_{min}	maximum and minimum eigenvalues
ε	Strain Tensor
η	Complex Scalar, Eigenfunction
$\Lambda_{(.)}$	Eigenvalue Matrix of Matrix (.)
$\lambda_{(.)}$	$(.)^{th}$ Eigenvalue
μ	Eigenvalue
ν	Complex Scalar, Eigenfunction, Friction Coefficient
Φ	State Transfer Matrix, Eigenfunction
Ψ, ψ	Eigenfunction
ϕ	Phase Angle, Eigenfunction
θ	Phase Angle
Ω	Bounded Region

$\omega_{(.)}$	$(.)^{th}$ Undamped Natural Frequency
$\vartheta_{(.)}$	$(.)^{th}$ Complex Damping Ratio
$\xi_{(.)}$	$(.)^{th}$ Damping Ratio
$\zeta_{(.)}$	$(.)^{th}$ Imaginary Damping Ratio
τ	Time
$\mathfrak{L}$, L, L	Laplace Operator, Linear Operator
Ξ	Eigenfunction
$\mathscr{R}$	Generalized Rayleigh Quotient
$\mathcal{S}$	$\mathcal{S}= \mathfrak{L}(S)$
$\mathscr{X}$	$\mathscr{X}= \mathfrak{L}(X)$
■	End of Proof

Superscript

(c), (s)	Function Resulted by Cosine or Sine Excitations
+	Pseudo-inverse of Matrix
*	Complex Conjugate
H	Hermitian Transpose
T	Regular Matrix Transpose

1 DAMPING MECHANISM AND DYNAMIC MODELING

1.1 INTRODUCTION

In structural dynamics the equilibrium of a vibrating system is usually described by a set of ordinary differential equations in the following form (see Clough, 1979, for example)

$$\mathbf{M}\, X''(t) + \mathbf{C}\, X'(t) + \mathbf{K}\, X(t) = F(t) \qquad (1.1a)$$

where, **M**, **C** and **K** are constant coefficient matrices of mass, damping and stiffness, respectively. All three matrices are symmetric. **M** and **K** are particularly restricted to be positive definite; **C** is required to be positive definite or positive semi-definite. In the above equation, X''(t), X'(t) and X(t) are respectively the acceleration, velocity and displacement vectors, the superscript ' and '' denote the first and second derivative with respect to time variable t; F(t) is the forcing function vector. The corresponding homogeneous equation of (1.1a) is

$$\mathbf{I}\, X'' + \mathbf{M}^{-1}\mathbf{C}\, X' + \mathbf{M}^{-1}\mathbf{K}\, X = 0 \qquad (1.1b)$$

where **I** and 0 are the identity matrix and the null vector.

A system described by equation (1.1a) or (1.1b) is referred to as an *M-C-K system*, which is a linear, damped, lumped mass system. In this report, we are only concerned with *simple systems* in which all the eigenvectors are linearly independent (see Meirovitch, 1969). If the spring, inertial and damping forces are respectively denoted by F_s, F_i and F_d, we have

$$\mathbf{K}X = F_s \qquad (1.2a)$$

$$\mathbf{M}X'' = F_i \qquad (1.2b)$$

$$\mathbf{C}X' = F_d \qquad (1.2c)$$

In above equations, the spring force (1.2a) observes Hooke's Law, and the inertial force (1.2b) observes the generalized Newton's Second Law. The two forces have clear theoretical bases and are conservative in nature. Relatively speaking, in many M-C-K systems, the spring and the inertial forces are much stronger than the damping force.

For most engineering applications, viscous-damping is assumed when seeking the constant coefficient matrices of an M-C-K system. It follows that damping force CX' is then a viscously dissipative force. When equation (1.1a) is transformed into the frequency domain by the Fourier transform, the damping force is seen to have a near 90 degree phase difference with respect to both the spring and inertial forces. This characteristic can be seen in most damping forces. If an M-C-K system is used to model an engineering structure, such as a high-rise building or a steel bridge, then the damping force is very small in amplitude such that the force can be omitted, as long as the design range of the dynamic behaviors of the structure is outside the resonant region. For these reasons, the damping force and the coefficient matrix C are more difficult to model than their spring and inertial counterparts. In this report, the modeling of damping coefficient matrix C for multiple-degree-of-freedom systems is a primary objective. To accomplish this objective, a complex energy-based theory is first established and then utilized. In this report, SDOF and MDOF are used for single and multiple degree of freedom systems.

1.2 TRADITIONAL APPROACHES IN DYNAMIC MODELING OF DAMPED SYSTEMS

Traditional treatment of damped systems include: 1) modeling (representation) damping force and coefficients, 2) the eigen-decomposition or modal analysis of damped systems, 3) calculation of free and forced responses or response bounds, and 4) system identification and vibration control. Some of the basic ideas of these approaches are briefly outlined as follows, starting with SDOF systems and then going to MDOF systems.

1.2.1 DAMPING FORCE AND DISSIPATED ENERGY

A SDOF linear system is described by a scalar equation

$$mx'' + cx' + kx = 0 \qquad (1.3a)$$

which is a simplified equation (1.1b) written in one-dimensional space. The term cx' implies that the damping force is proportional to the velocity (*viscous damping*).

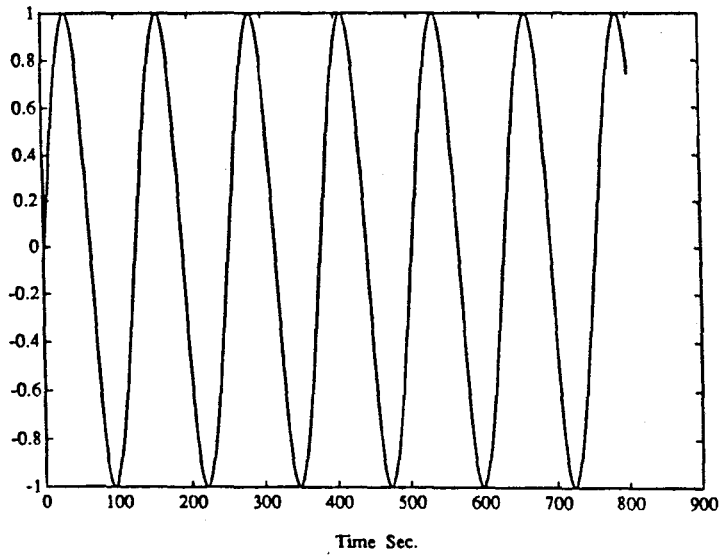


Figure 1.1 (a) Undamped Vibration

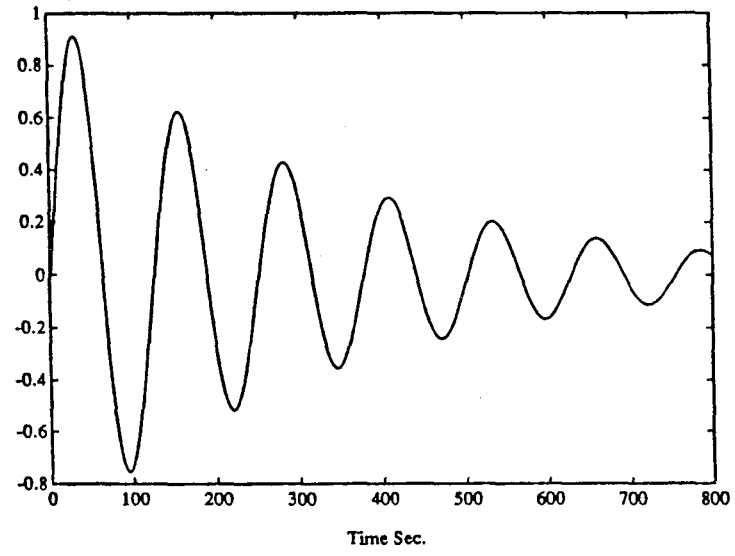


Figure 1.1 (b) Damped Vibration

If the term cx' is absent, the free vibration is said to be *undamped* (see Figure 1.1a). In reality, however, free vibration is always damped (Figure 1.1b), and the damping effect appears in the form of energy dissipation which is seen as the cause of diminishing free vibration responses (Figure 1.1b). Equation (1.3a) is then rewritten as

$$x'' + 2\xi\omega x' + \omega^2 x = 0 \qquad (1.3b)$$

where,

$$\omega = \sqrt{\frac{k}{m}}, \qquad (1.4)$$

is called the *undamped natural frequency* or simply the *natural frequency*; and

$$\xi = \frac{c}{2\sqrt{km}}, \qquad (1.5)$$

is called the *damping ratio*. Natural frequency and damping ratio are the two basic parameters that describe the dynamic behaviors of a SDOF system.

When the damping ratio is sufficiently small, it is approximately equal to the ratio of the energy dissipated during a cycle of motion, W_c (the period of the cycle $T = 2\pi/\omega$), and the vibrational energy contained in the system before the cycle, W . That is,

$$\xi \approx \frac{W_c}{4\pi W} \qquad (1.6)$$

In equation (1.6), the subscript c is used to specifically indicate that the quantity of energy loss is caused by viscous damping. (In later discussions, a subscript d will be used to indicate the energy loss related to general damping.) The above definitions of dynamic parameters for SDOF systems can be extended to MDOF systems.

1.2.2 DAMPING REPRESENTATION OF SDOF SYSTEMS

To model the damping of MDOF systems, one of the difficulties is the determination of the damping coefficient matrix. Often, the entries of the

matrix cannot be directly determined because of the lack of a material law and the unknown relationships between damping and mass/stiffness in the systems. For a SDOF system, however, the damping coefficient can be determined from equation (1.5) by the following relationship,

$$c = 2\,\xi\,(mk)^{1/2} = 2\xi\omega\,m = \frac{2\xi}{\omega}\,k \qquad (1.7a)$$

This equation expresses c in terms of the mass m and stiffness k which are relatively easy to obtain through direct measurement. Equation (1.7a) may be rewritten as

$$c = \alpha\,m = \beta\,k = \gamma\,(mk)^{1/2} \qquad (1.7b)$$

where α, β and γ are coefficients proportional to damping ratio ξ. ξ can also be measured from experiments. Thus, equations (1.7a) and (1.7b) show that, instead of modeling damping directly, we can use mass/stiffness to represent the damping coefficient.

In MDOF systems, this general approach is still useful, but not easy to accomplish. Instead of solving for an exact representation, some approximations are usually employed. These approximations may introduce errors. Alternative approaches to assemble the damping matrix will be introduced in Chapter 5 of this report.

1.2.3 PROPORTIONAL AND NON-PROPORTIONAL DAMPING

Perhaps the earliest notion of proportional damping came from Rayleigh's assumption that damping is proportional to the mass and/or stiffness, i.e.

$$\mathbf{C} = \alpha\,\mathbf{M} + \beta\,\mathbf{K} \qquad (1.8)$$

Modeling the damping matrix $\mathbf{C}$ by equation (1.8) is essentially equivalent to assuming a form of matrix representation for MDOF systems. That is, *Rayleigh damping* may be regarded as an extension of equations (1.7a) and (1.7b) for SDOF systems. Because it is easy to use, the Rayleigh damping remains a popular assumption today. However, equation (1.8) does not cover all possible cases of damping. Clough and Penzien (1977) had subsequently suggested a more

comprehensive form:

$$\mathbf{C} = \sum_{b} \mathbf{C}_b = \mathbf{M} \sum_{b} a_b (\mathbf{M}^{-1}\mathbf{K})^b \quad , \quad -\infty < b < \infty \tag{1.9}$$

where $\mathbf{C}_b = a_b \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^b$ and a_b's are the scalar coefficients. Equation (1.9) will be discussed in more detail in Chapter 4. At present we may regard equation (1.9) as a form of proportional damping with proportional coefficients a_b's.

The exact concept of proportional damping was first given by Caughey and O'Kelley (1965). According to their definition, a system is proportionally damped if and only if it satisfies the following criterion

$$\mathbf{C}\,\mathbf{M}^{-1}\,\mathbf{K} = \mathbf{K}\,\mathbf{M}^{-1}\,\mathbf{C} \tag{1.10}$$

Equation (1.10) is known as the *Caughey criterion* and is very convenient to apply. If a system does not satisfy the criterion, it is defined as *non-proportionally damped.*

In equations (1.8), (1.9) and (1.10), the terms $\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{M}^{-1}\mathbf{K}$ have the same eigenvector matrix. Hence equations (1.1a) and (1.1b) can be simultaneously diagonalized. This "*decoupling* " helps to simplify the system so that modal analysis can be performed on MDOF systems in the n-dimensional space, and the normal modes can be determined.

In the analysis of proportionally damped systems, the eigenvectors are all real-valued. (Rigorously speaking, this statement is not always true. In Appendix A and Chapter 2, we will discuss this in detail.) This makes the computations easier. However, proportional damping in conjunction with finite element modeling can only yield the normal mode, and in many cases such treatment introduces errors since certain complex modes of non-proportionally damped structures are not well represented (see section 1.3.2). Therefore, in engineering design against dynamic loading, complex modal model may be required (see Hall (1970), Thoren (1972), Caravani (1984), Hanugud (1984), Debelauwe (1989)).

Because of this, much effort has been devoted to the study of non-proportional damping. Since the 1970's, several authors have suggested different methods to approximate non-proportional damping by proportional damping. Among them, Caravani and Thomson (1974), Berman and Nagy (1983), Golla and Hughes (1985), Mau (1988) and Buhariwala and Hanson (1989) have proposed various methods of solution.

Although Clough damping (1.9) cannot represent general non-proportional damping, it does exactly represent all the damping ratios (equation (1.5)) of any damped system (Penzien, 1984). Therefore, by means of (1.9), a general damping matrix can be decomposed into two parts, $\mathbf{C}_P$ and $\mathbf{C}_N$:

$$\mathbf{C} = \mathbf{C}_P + \mathbf{C}_N \tag{1.11a}$$

where $\mathbf{C}_P$ is the *Clough damping* with the damping ratios same as those of matrix $\mathbf{C}$; and $\mathbf{C}_N = \mathbf{C} - \mathbf{C}_P$ consists of the other damping components of $\mathbf{C}$ not included in $\mathbf{C}_p$. Equation (1.11a) is referred to as the *Clough damping decomposition*. In general, a damping matrix may alternatively be decomposed into *pure proportional/non-proportional* matrices:

$$\mathbf{C} = \mathbf{C}_d + \mathbf{C}_O \tag{1.11b}$$

with

$$\mathbf{C}_d = \mathbf{Q}\,\mathbf{D}_d\,\mathbf{Q}^{-1},\quad \mathbf{C}_O = \mathbf{Q}\,\mathbf{D}_O\mathbf{Q}^{-1}$$

In this decomposition, matrices $\mathbf{D}_d$ and $\mathbf{D}_O$ are respectively the diagonal and the off-diagonal parts of matrix

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{C}\,\mathbf{Q} \tag{1.12}$$

In equation (1.12) matrix $\mathbf{D}$ is called the *modal damping matrix*, and $\mathbf{Q}$ satisfies the following relationships:

$$\mathbf{Q}^{-1}\mathbf{M}\,\mathbf{Q} = \mathbf{I},\quad \mathbf{Q}^{-1}\mathbf{K}\,\mathbf{Q} = \text{diagonal matrix}$$

The decomposition of damping matrices will be discussed further in Chapter 4.

1.3 EIGEN-DECOMPOSITION AND MODAL ANALYSIS

1.3.1 NORMAL MODES

The method used to deal with SDOF systems can be applied to MDOF systems if the MDOF system can be "decoupled" into a set of SDOF subsystems. In the case of proportionally damped system, the decoupling can be achieved by pre-multiplying the eigenvector matrix $\mathbf{Q}^{-1}$ of $\mathbf{M}^{-1}\mathbf{K}$ with equation (1.1b). This treatment will result in diagonalizing $\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{M}^{-1}\mathbf{K}$ simultaneously. Then the resulting matrix equation becomes a set of n scalar equations, spatially separated from each other. We can thus find the damping ratio and natural frequency of each scalar equation. This procedure is known as *modal analysis*. Each reduced SDOF subsystem coincides with an individual mode and these subsystems span the modal domain of the original MDOF system.

In the modal domain, the scalar equations can be solved for their forced responses. The actual response of the MDOF system is obtained by combining all the individual responses. This reformation is known as the *modal superposition*.

The above discussion is now explained mathematically. Multiplying equation (1.1b) by $\mathbf{Q}^{-1}$ and inserting identity ($\mathbf{Q}\ \mathbf{Q}^{-1}$) in between $\mathbf{M}^{-1}\mathbf{C}$ and X', and also in between $\mathbf{M}^{-1}\mathbf{K}$ and X, we have

$$\mathbf{Q}^{-1}\mathbf{X}'' + \mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{C})\ \mathbf{Q}\ \mathbf{Q}^{-1}\mathbf{X}' + \mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{K})\ \mathbf{Q}\ \mathbf{Q}^{-1}\mathbf{X} = \mathbf{0}$$

Then, matrices $\mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{C})\mathbf{Q}$ and $\mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{K})\mathbf{Q}$ are diagonalized (see Inman (1989)):

$$\mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{C})\mathbf{Q} = \mathrm{diag}(2\xi_i\omega_i) \qquad (1.13a)$$

and

$$\mathbf{Q}^{-1}(\mathbf{M}^{-1}\mathbf{K})\mathbf{Q} = \mathrm{diag}(\omega_i^2) \qquad (1.13b)$$

where ξ_i and ω_i are the i^{th} damping and undamped natural frequencies respectively. If we denote

$$\mathbf{U} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = \mathbf{Q}^{-1}\mathbf{X} \qquad (1.14)$$

we obtain n scalar equations:

$$u_i'' + 2\xi_i\omega_i u_i' + \omega_i^2 u_i = 0 \ , \ i = 1,...,n \qquad (1.15a)$$

The characteristic equation of (1.15a) is

$$\lambda_i^2 + 2\xi_i\omega_i\lambda_i + \omega_i^2 = 0 \ , \ i = 1,...,n \qquad (1.15b)$$

If $\xi_i < 1$, then

$$\lambda_i = -\xi_i\,\omega_i \pm j\sqrt{1-\xi_i^2}\;\omega_i \qquad (1.16)$$

The solved λ_i is said to be the i^{th} *eigenvalue* of the system. And

$$u_i = a_i \exp(\lambda_i t) \qquad (1.17)$$

is a solution of equation (1.15a).

For the SDOF system, the initial conditions can be also transformed into the modal domain to determine the amplitude a_i's.

Likewise, equation (1.15a) can be extended to the non-homogeneous case by means of eigen-decomposition, such as equation (1.13). Since equation (1.15a) (and therefore equation (1.16)) contains the unique damping ratio ξ_i and the natural frequency ω_i, i.e. without other ξ_j's and ω_j's ($j \neq i$) involved, this equation describes the i^{th} *mode* of the system, and both ξ_i and ω_i are *modal parameters* of this mode (assume that the system has no repeated roots).

Another modal parameter is the i^{th} column of the eigenvector matrix **Q**, denoted by Q_i. That is

$$Q_i = \begin{Bmatrix} q_{i1} \\ q_{i2} \\ \vdots \\ q_{in} \end{Bmatrix}$$

where q_{ij} is the j^{th} amplitude of i^{th} lumped mass. Q_i is, in fact, the

spatial shape function of the i^{th} mode of the system. Therefore Q_i is the i^{th} *mode shape* .

It can be shown that all the mode shape Q_i's are real-valued. Modal analysis of the proportionally damped system therefore yields *real modes*, or *normal modes.*

1.3.2 COMPLEX MODES

The above mentioned modal analysis for proportionally damped systems is rather straightforward. However, if damping of the system is non-proportional, then the terms $\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{M}^{-1}\mathbf{K}$ in equation (1.1b) cannot be simultaneously diagonalized. This makes the direct modal analysis impossible. In order to transform the system described by (1.1a) into a set of decoupled "SDOF" equations, a different approach must be employed. For example, we may use

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix} \begin{Bmatrix} X'' \\ X' \end{Bmatrix} = \begin{bmatrix} -\mathbf{C} & -\mathbf{K} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} X' \\ X \end{Bmatrix} \tag{1.18a}$$

Or, alternatively, we may denote the system as

$$\begin{Bmatrix} X'' \\ X' \end{Bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} X' \\ X \end{Bmatrix} \tag{1.18b}$$

As a third approach, we may transform the homogeneous equation (1.1b) into the following equations by pre-multiplying $\mathbf{M}^{-1/2}$ to (1.1b)

$$\mathbf{M}^{-1/2}\mathbf{M}\,\mathbf{M}^{-1/2}\mathbf{M}^{1/2}X'' + \mathbf{M}^{-1/2}\mathbf{C}\,\mathbf{M}^{-1/2}\mathbf{M}^{1/2}X' + \mathbf{M}^{-1/2}\mathbf{K}\,\mathbf{M}^{-1/2}\mathbf{M}^{1/2}X = \mathbf{0}$$

This treatment will enable us to define

$$\begin{aligned} \tilde{\mathbf{C}} &= \mathbf{M}^{-1/2}\mathbf{C}\,\mathbf{M}^{-1/2} \\ \tilde{\mathbf{K}} &= \mathbf{M}^{-1/2}\mathbf{K}\,\mathbf{M}^{-1/2} \end{aligned} \tag{1.19a}$$

and

$$Y = \mathbf{M}^{1/2}\,X \tag{1.19b}$$

Thus, we have

$$Y'' + \tilde{C}\, Y' + \tilde{K}\, Y = 0 \tag{1.20a}$$

and then

$$\begin{Bmatrix} Y'' \\ Y' \end{Bmatrix} = \begin{bmatrix} -\tilde{C} & -\tilde{K} \\ I & 0 \end{bmatrix} \begin{Bmatrix} Y' \\ Y \end{Bmatrix} \tag{1.20b}$$

Equations (1.18a), (1.18b) and (1.20b) are often referred to as the *state equations* in the literature. Denote the 2n x 2n matrix in the state equations by **H**, i.e.

$$\mathbf{H} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{1.21a}$$

Or

$$\mathbf{H} = \begin{bmatrix} -\tilde{C} & -\tilde{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{1.21b}$$

The matrices **H**'s in equations (1.21a) and (1.21b) are said to be the *state matrices* of the systems described by (1.19b) and (1.20b), respectively.

Equations (1.18a), (1.18b) and (1.20b) are the eigen-problems or generalized eigen-problems. Let us consider only the approach of equation (1.20b). It can be shown that the state matrix **H** (1.21b) in equation (1.20b) has the following eigen-decomposition (see Liang (1987) for instance)

$$\mathbf{H} = \begin{bmatrix} -\tilde{C} & -\tilde{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} = \mathbf{P}\, \Lambda\, \mathbf{P}^{-1} \tag{1.22}$$

with

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \Lambda_1 & \mathbf{P}_1^* & \Lambda_1^* \\ \mathbf{P}_1 & & \mathbf{P}_1^* & \end{bmatrix} \tag{1.23}$$

and

$$\Lambda = \begin{bmatrix} \Lambda_1 & & \\ & & \Lambda_1^* \end{bmatrix} \tag{1.24}$$

Since the **H** matrix is not symmetric, its 2n x 2n eigenvector matrix **P** is complex-valued in general, and so is the n x n matrix $\mathbf{P}_1$. $\mathbf{P}_1$ is called the *complex mode shape* matrix. In equation (1.23) the superscript * refers to the complex conjugate operation.

In equation (1.24),

$$\Lambda_1 = \text{diag}\ (\lambda_i) \qquad i = 1,...,N \tag{1.25}$$

When the system is underdamped (see Inman (1989) for instance), it always has complex-valued λ_i's; therefore, we can write

$$\lambda_i = -\xi_i\ \omega_i + j\sqrt{1-\xi_i^2}\ \ \omega_i \tag{1.26}$$

which is the i^{th} *eigenvalue* of the generally damped system, or it is called the *complex frequency*. Further,

$$\lambda_i = \lambda_{i+n}^* \tag{1.27}$$

Note that in equation (1.26), both ξ_i and ω_i are positive scalars.

Comparing with equation (1.16) and using the same treatment for proportionally damped MDOF systems, ξ_i and ω_i are defined as the i^{th} damping ratio and the i^{th} natural frequency, respectively. Because the mode shape matrix is complex, modal analysis yields the *complex mode* instead of the normal mode. The complex mode shape causes different lumped masses to reach their maximum amplitudes at different times. Physically, there are phase-shifts among different lumped masses in each mode. This is the classical approach to explain the phenomenon of the complex mode. If we denote

$$U=\begin{Bmatrix} u_1 \\ u_2 \\ : \\ u_{2n} \end{Bmatrix} = \mathbf{P}^{-1}\begin{Bmatrix} Y' \\ Y \end{Bmatrix} \tag{1.28}$$

then using eigen-decomposition (1.22), we can obtain 2n scalar equations in the state-space such that

$$u_i' + \lambda_i u_i = 0 , \qquad i = 1,...,2n \tag{1.29a}$$

Using the complex frequency λ_i defined by equation (1.26), we can see that

$$u_i = a_i \exp(\lambda_i t) \tag{1.29b}$$

is a solution of equation (1.29a), an analogy to the solution of equation (1.17).

It should also be noted that, although the state matrices are not unique, such as in (1.21a) and (1.21b), the eigenvalues of state matrices are all identical (if one does not count the order of those eigenvalues). This means that all the state matrices are similar, and we may use the state matrix **H** to represent the system. For example, we may use $\mathbf{H}_i$ to represent system i, and use $\mathbf{H}_{ci}$ to represent the system with damping $\mathbf{C}_i$. Also, we may use the eigenvalue matrix $\Lambda^{(i)}$ to represent system i, etc.

As soon as we obtain the damping ratio ξ_i, by means of either normal or complex modal analysis, equation (1.6) can be used to describe the energy status for the i^{th} mode.

1.4 RESPONSES OF FREE AND FORCED VIBRATION OF MDOF SYSTEMS

The above discussion is concerned with the system's (eigen) parameters, i.e., the damping ratios, natural frequencies and mode shapes. These parameters can be obtained from the state matrices or strictly speaking, from the eigen matrix (see Appendix A).

In the following, we will briefly review the response of a system excited by initial conditions and/or input forces. Obviously, a forced response not only

depends on the characteristics of the system, but also depends on the form of the forcing function, the right hand side of equation (1.1a).

1.4.1 CLASSIFICATION OF DYNAMIC RESPONSES

There are several ways to describe the solutions of equation (1.1a). In this report, we use the following two approaches.

First, the solution of equation (1.1a) is considered as the sum of a free response X_f and a (pure) forced response X_p. That is,

$$X = X_f + X_p \tag{1.30}$$

The free response can be determined by solving the homogeneous equation of (1.1a) with initial conditions. Recall equations (1.15a) and (1.29a). They are homogeneous modal equations decoupled from proportionally damped and non-proportionally damped systems, respectively. Equations (1.17a) and (1.29b) are respectively the corresponding free responses.

The forced response or the particular solution X_p is obtained from the non-homogeneous equation (1.1a) without initial conditions. Many time domain modal analyses use the free response X_f.

As a second approach, the solution of equation (1.1a) can be considered as the sum of a transient response X_t and a steady state response X_s, i.e.

$$X = X_t + X_s \tag{1.31}$$

The transient response X_t typically has a decaying envelope of a negative exponential function because of the damping effect. The steady state response X_s generally represents the fact that the system is forced into a state resembling the form of input excitation. Most frequency domain modal analyses employ the steady state response X_s .

1.4.2 METHODS FOR SOLVING SDOF SYSTEMS

For a SDOF system with given initial conditions, there are two basic approaches to obtain a solution: the time domain approach, which relies on the well-known Duhamel integration method, and the complex-frequency domain

approach, which uses Laplace and inverse-Laplace transformations.

In this report, we obtain the solutions of SDOF systems from the solutions of modal equations decoupled from proportionally damped MDOF systems.

1.4.3 METHODS FOR SOLVING MDOF SYSTEMS

The time domain and the complex-frequency domain methods can also be used directly to obtain solutions of MDOF systems. In addition, we can use the idea of modal analysis by first transforming an MDOF system into n or 2n individual "SDOF" equations, and then obtaining the SDOF solutions by using a SDOF-method. These results are then transformed back to give the solutions of the MDOF system. Let us examine such a method in detail. Rewrite the equation (1.1a) into a state space equation:

$$\begin{aligned} Z' &= \mathbf{H}\, Z + \mathbf{S}\, S \\ V &= \mathbf{V}\, Z \end{aligned} \qquad (1.32)$$

where, for example we let

$$Z = \begin{Bmatrix} Y' \\ Y \end{Bmatrix}, \quad \mathbf{H} = \begin{bmatrix} -\tilde{C} & -\tilde{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad S = \begin{Bmatrix} \mathbf{M}^{-1/2}\mathbf{F} \\ 0 \end{Bmatrix}$$

S is the input matrix (in this report **S** is set to be identity), **V** is an output matrix (in this report **V** is also set to be identity), and V is an output variable (Note that V = Z.)

As was done for the homogeneous case (1.29a), we can transform equation (1.32) and the associated initial conditions into the modal domain. In other words, we now extend equation (1.29a) to the non-homogeneous case, as

$$U' = \Lambda\, U + R \qquad (1.33a)$$

or

$$u_i' + \lambda_i u_i = r_i \qquad , i = 1,...,N \qquad (1.33b)$$

$$\text{and } u_i(o) = u_{i0} \qquad (1.33c)$$

where

$$R = [r_1\ r_2\ \ldots r_{2n}\ ]^T = \mathbf{P}^{-1}S \tag{1.34}$$

$$U_0 = [u_{10}\ u_{20}\ \ u_{2n,0}\ ]^T = \mathbf{P}^{-1}Z_0 \tag{1.35}$$

and

$$Z_0 = \begin{Bmatrix} \mathbf{M}^{1/2}\ X_0, \\ \mathbf{M}^{1/2}\ X_0 \end{Bmatrix}$$

which contains the initial velocity $X_{0'}$ and initial displacement X_0. After we solve these 2n (n) individual equations (1.33), we can obtain the solution X from

$$\begin{Bmatrix} X' \\ X \end{Bmatrix} = \begin{bmatrix} \mathbf{M}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1/2} \end{bmatrix} (\mathbf{P}\ \mathbf{U}) \tag{1.36}$$

This procedure is referred to as the *modal superposition* method, which is quite common in solving problems of MDOF systems.

Next, let us review the method of *state transfer matrix* for MDOF systems. The state transfer matrix is defined as

$$\Phi(t, t_0) = e^{\int_{to}^{t} H\ d\tau} = e^{(t\text{-}to)H} \tag{1.37}$$

We can simply represent $\Phi(t, t_0)$ by $\Phi(t)$.

For a free vibration with initial condition Z_0 the solution is given as

$$Z(t) = \Phi(t)Z_0 = e^{(t\text{-}to)H}\ Z_0 \tag{1.38}$$

The meaning of equation (1.38) is that the state after time $(t\text{-}t_0)$ is transferred from the initial state Z_0 by the state-transfer-matrix $e^{(t\text{-}to)H}$.

Now consider the general response included from both the initial condition and the forcing functions.

$$Z(t) = \Phi(t)[Z_0 + \int_{to}^{t} \Phi(\tau)^{-1}\ \mathbf{S}\ S(\tau)d\tau\ ] \qquad (1.39)$$

In equation (1.39), the term $\Phi(t)Z_0$ is the free response Z_f, and the term

$\Phi(t)\int_{to}^{t} \Phi(\tau)^{-1}\ \mathbf{S}\ S(\tau)d\tau$ is the pure forced response Z_p .

The following equations about the matrix $\Phi(t)$ are useful

$$e^{Ht} = \mathbf{I} + \mathbf{H}t + \mathbf{H}^2t^2/\ 2! + \mathbf{H}^3t^3/\ 3! + \ .... \qquad (1.40)$$

and

$$\Phi(t) = \mathbf{Z}(t)\ \mathbf{Z}(t_0)^{+} \qquad (1.41)$$

where

$$\mathbf{Z}\ (t_i) = [\ Z(t_i)\ Z(t_{i+1})\ .....\ Z(t_{i+m-1}\ )] \qquad m \geq n \qquad (1.42)$$

and the superscript + denotes the pseudo-inverse of a matrix. (Note that equation (1.41) is the base of a major time-domain-modal-testing method, the ITD method (Ibrahim (1977).)

Note that the product

$$\Phi(t,\tau)\ \Phi(\tau,s) = \Phi(t,s) \qquad (1.43)$$

$$\Phi(t,\tau) = \Phi(\tau,t)^{-1} \qquad (1.44)$$

With the help of equations (1.43) and (1.44), the term Z in equation (1.39) can be written as

$$Z_p = \int_{to}^{t} e^{H(t-\tau)}\ \mathbf{S}\ S(\tau)\ d\tau \qquad (1.45)$$

Equation (1.45) is called the *Duhamel Integration.* Methods using state transfer matrix and Duhamel integration are also regarded as techniques of *time domain analysis.*

Next, consider the method of Laplace transformation. Taking the Laplace transform of equation (1.32) we have

$s\mathcal{X}(s)-\mathcal{X}(0) = \mathbf{H}\,\mathcal{X}(s) + \mathbf{S}\,\mathcal{J}(s)$

where $\mathcal{X}(s)$ and $\mathcal{J}(s)$ are Laplace transforms of Z(t) and S(t), respectively.

Alternatively we can have $(s\mathbf{I} - \mathbf{H})\,\mathcal{X}(s) = \mathcal{X}(0) + \mathbf{S}\mathcal{J}(s)$. It can be shown that $(s\mathbf{I} - \mathbf{H})^{-1}$ exists and

$$(s\mathbf{I} - \mathbf{H})^{-1} = \mathbf{I}/s + \mathbf{H}/s^2 + \mathbf{H}^2/s^3 + \ldots\ldots \qquad (1.46)$$

So

$$\mathcal{X}(s) = (s\mathbf{I} - \mathbf{H})^{-1}\mathcal{X}(o) + (s\mathbf{I} - \mathbf{H})^{-1}\,\mathbf{S}\mathcal{J}(s)$$

The inverse Laplace transform of $(s\mathbf{I} - \mathbf{H})^{-1}$ is

$$\mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{H})^{-1}] = \mathbf{I} + \mathbf{H}t + \mathbf{H}^2t^2/2! + \mathbf{H}^3t^3/3! + \ldots = \Phi(t)$$

Thus, we can obtain a similar result as expressed by equation (1.39)

$$Z(t) = e^{(t-to)}Z(0) + \int_{to}^{t} e^{H(t-\tau)}\,SS(\tau)\,d\tau \qquad (1.47)$$

This is the *complex-frequency domain analysis.*

1.5 EXAMPLES OF SOLVING AN MDOF SYSTEM

Consider a two-degree-of-freedom system shown in figure 1.2.

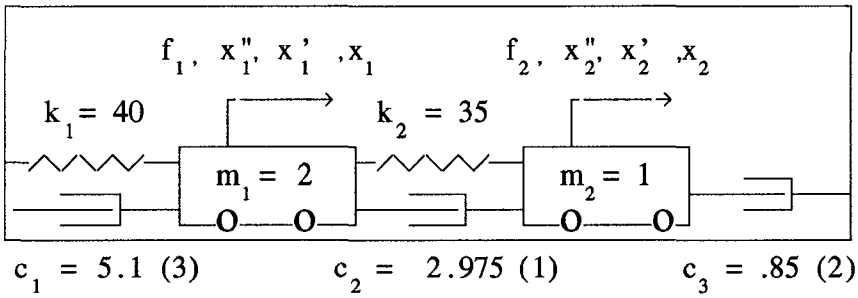


Figure 1.2 A 2-DOF M-C-K Vibrating System

Equilibrium of mass m_1 requires the inertial force m_1x_1'', the damping force $c_1x_1' + c_2(x_1' - x_2')$ and the spring force $k_1x_1 + k_2(x_1 - x_2)$ be balanced by the external force f_1, i.e.

$$m_1x_1'' + c_1x_1' + c_2(x_1' - x_2') + k_1x_1 + k_2(x_1 - x_2) = f_1$$

Similarly, for mass m_2, we can write

$$m_2 x_2'' + c_3 x_2' + c_2(x_2' - x_1') + k_2(x_2 - x_1) = f_2$$

Therefore, in the form of a matrix equation, we can write

$$\mathbf{M}\, X'' + \mathbf{C}\, X' + \mathbf{K}\, X = F \qquad (1.48)$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 75 & -35 \\ -35 & 35 \end{bmatrix}.$$

By choosing a system: $c_1 = 5.1$, $c_2 = 2.975$, $c_3 = .85$, we have

$$\mathbf{C}_1 = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 8.075 & -2.975 \\ -2.975 & 3.825 \end{bmatrix};$$

Next choosing a second system $c_1 = 3$, $c_2 = 1$, $c_3 = 2$, we have

$$\mathbf{C}_2 = \begin{bmatrix} c_1 + c_2 & c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}.$$

Since $\mathbf{C}_1\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}_1$, system 1 is proportionally damped. However, since $\mathbf{C}_2\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}_2$, system 2 is non-proportionally damped. As mentioned above, we can find a matrix **P**, which is an eigenvector matrix of both $\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{M}^{-1}\mathbf{K}$, to decouple system 1 into 2 normal modes, and thus find the damping ratios and natural frequencies of the 2-DOF systems. The results are as follows

$$\Lambda(\mathbf{M}^{-1}\mathbf{C}_1) = \begin{bmatrix} 1.8247 & \\ & 6.037 \end{bmatrix}, \quad \Lambda(\mathbf{M}^{-1}\mathbf{K}) = \begin{bmatrix} 11.469 & \\ & 61.0255 \end{bmatrix}$$

Therefore, we have

$\xi_1 = 0.2694$, $\xi_2 = 0.23864$ $\omega_1 = 0.5390 \times 2\pi$ and $\omega_2 = 1.2433 \times 2\pi$.

The mode shape matrix is given by

$$\mathbf{P} = \begin{bmatrix} -0.0024 - 0.1580j & -0.0413 + 0.0635j \\ -0.0035 - 0.2350j & 0.0556 - 0.0854j \end{bmatrix}$$

Matrix **P** can be normalized to be real-valued, that is,

$$\mathbf{P} = \begin{bmatrix} -0.5579 & 0.5968 \\ -0.8299 & -0.8024 \end{bmatrix}$$

Because the damping of system 2 is non-proportional, we can not decouple system 2 into n independent scalar equations. Instead, we introduce the state matrix

$$\mathbf{H} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C}_2 & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -2 & 0.5 & -37.5 & 17.5 \\ 1 & -3 & 35 & -35 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathbf{P}\,\Lambda\,\mathbf{P}^{-1}$$

and the state space equation

$$\dot{\bar{Z}} = \mathbf{H}\,\bar{Z} + S \qquad (1.49)$$

where $\bar{Z} = \begin{Bmatrix} X' \\ X \end{Bmatrix}$, and $S = \begin{Bmatrix} \mathbf{M}^{-1}F \\ O \end{Bmatrix}$

The eigenvalue matrix of the state matrix **H** is

$$\Lambda = \begin{bmatrix} -0.9141 + 3.2717j & & & \\ & -1.5859 + 7.6254j & & \\ & & -0.9141 - 3.2717j & \\ & & & -1.5859 - 7.6254j \end{bmatrix}$$

Correspondingly, we have $\xi_{1,3} = 0.2691$, $\xi_{2,4} = 0.2036$, $\omega_{1,3} = 0.5406 \times 2\pi$ and $\omega_{2,4} = 1.2396 \times 2\pi$. The mode shape matrix is

$$\mathbf{P}_1 = \begin{bmatrix} -0.0385 - 0.1914j & 0.0351 + 0.0885j \\ -0.0792 - 0.0283j & -0.0261 - 0.1257j \end{bmatrix}$$

which is a complex matrix. The eigenvector matrix **P** is

$$\mathbf{P} = \begin{bmatrix} 0.6616 + 0.0490j & -0.7307 + 0.1275j & 0.6616 + 0.0490j & -0.7307 + 0.1275j \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.0385 - 0.1914j & 0.0351 + 0.088\text{-}j & 0.0385 - 0.1914j & 0.0351 + 0.0885j \\ -0.0792 - 0.0283j & -0.0261 - 0.125\text{-}j & 0.0792 - 0.0283j & -0.0261 - 0.1257j \end{bmatrix}$$

Now, let us consider the free vibration of both systems 1 and 2. If we let $f_2 = 0$, and $f_1 =$ unit impulse, the corresponding response is shown in Figure 1.3

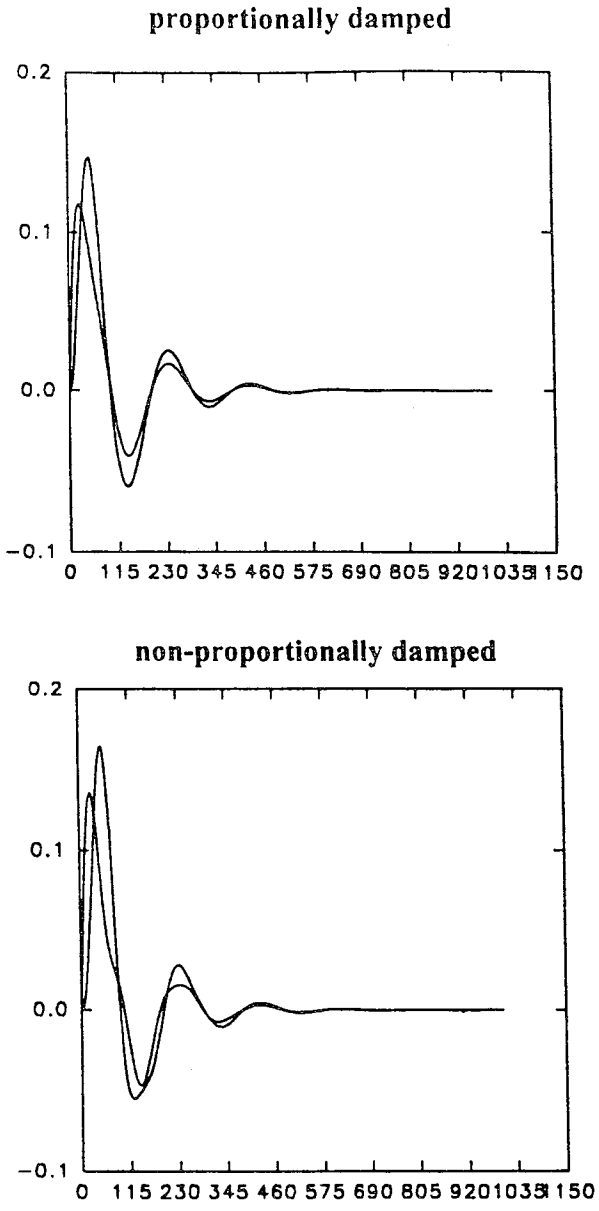


Figure 1.3 Response Time Histories

In Tables 1.1 and 1.2 we list the i^{th} peak values of both x_1 and x_2, denoted by $x_{i,1}$ and $x_{i,2}$, for both systems 1 and 2 (i = 1, 2, 3, 4, 5). Here, the time interval used is 0.01 sec. We also calculated the values

$\hat{\delta}_{ij} = \ln(x_{i,j} / x_{i+1,j})$ and $\hat{\xi}_{ij} = \hat{\delta} / (4\pi^2 + \hat{\delta}^2)^{1/2}$. These quantities are useful in determining damping ratios from measured data.

Table 1.1 Results from Response of System 1 (proportional damping)

mass 1				mass 2			
time	x_{i1}	$\hat{\delta}$	$\hat{\xi}$(%)	time	x_{i2}	$\hat{\delta}$	$\hat{\xi}$(%)
27 th	.11785559	7.375	29.65	49 th	.14690656	5.858	27.08
233 th	.01675239	5.780	26.89	233 th	.02507695	5.817	26.98
426 th	.00289819	5.800	26.94	426 th	.00431130	5.801	26.94
619 th	.00049970	5.802	26.94	619 th	.00074327	5.801	26.94
811 th	.00008615			811 th	.00012814		

Table 1.2 Results from Response of System 2 (non-proportional damping)

mass 1				mass 2			
time	x_{i1}	$\hat{\delta}$	$\hat{\xi}$(%)	time	x_{i2}	$\hat{\delta}$	$\hat{\xi}$(%)
27 th	.13563800	8.845	32.78	52 th	.16440750	5.967	27.34
234 th	.01532689	5.106	25.12	230 th	.02755899	6.448	28.43
424 th	.00300176	5.858	27.08	424 th	.00427405	5.617	26.48
614 th	.00051247	5.824	27.00	617 th	.00075287	5.770	26.86
807 th	.00008800			809 th	.00013049		

It is interesting to see that the proportionally damped system becomes sufficiently stable after the second peak. Thus, the time interval is about 1.93 (sec) which is very close to the period of the first mode (1.926 (sec)). Further, the calculated damping ratio of $\hat{\xi}$, 26.94 %, is very close to the damping ratio of the first mode (26.94 %). Because both masses gave nearly identical results, we may use the free response of any mass of a proportionally damped system to estimate the damping ratio and natural frequency of its first mode. In other words, if we use damping to describe the energy relationship, we may directly use the free response of any mass to estimate the main portion of the energy dissipation during one cycle of motion. This is generally true for proportionally damped systems.

On the other hand, the non-proportionally damped system 2 behaves differently. First, each mass yields different result. Secondly, the value of $\hat{\xi}$ is not equal to any of the damping ratios. Thirdly, the time interval varies between mass 1 and 2. Therefore, it is difficult to use the free response of any mass of a non-proportionally damped system to estimate the damping ratio and natural frequency of its first mode. In a non-proportionally damped system, the energy relationship is not as explicit as that of a proportionally damped system. To illustrate the modal superposition method, let us suppose that the force $f_1 = 2 \exp(j\ 1.0\ t)$, and $f_2 = 0$. The amplitudes of x_1 and x_2, denoted by x_{1o} and x_{2o} respectively, will be solved. Pre-multiplying the matrix $\mathbf{P}^{-1}$ to both sides of equation (1.49) yields

$$\mathbf{P}^{-1}\bar{Z}' - \mathbf{P}^{-1}\mathbf{H}\,\mathbf{P}\,\mathbf{P}^{-1}\,\bar{Z} = \mathbf{P}^{-1}\mathbf{S}$$

Denoting $\bar{U} = \mathbf{P}^{-1}\bar{Z} = [\bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_3 \quad \bar{u}_4\ ]^T$ and $R = \mathbf{P}^{-1}\mathbf{S}$

$= [r_1 \quad r_2 \quad r_3 \quad r_4]^T \exp\ (j\ 1.0\ t) =$

$[0.3556 + 0.1009j\ ,\ \ -0.3556 - 0.0775j,\ 0.3556 - 0.1009j,\ -0.3556 + 0.775j\ ]^T$

$\exp(j\ 1.0\ t\ )$, we have the following individual equations,

$$\bar{u}_i' - \lambda_i\,\bar{u}_i = r_i \exp\ (j\ 1.0\ t) \quad (\ i = 1,\ 2,\ 3,\ 4)$$

The steady state solution can be obtained as

$$\bar{u}_i = \bar{u}_{i0} \exp(j\ 1.0\ t)$$

So, we have

$$\bar{u}_{i0} = \ r_i\ /\ (\ j\ 1.0 - \lambda_i\ )$$

That is,

$\bar{u}_{10} = 0.0160 + 0.1501j\ ,\ \bar{u}_{20} = -0.0011 - 0.0534j\ ,$

$\bar{u}_{30} = -0.0560 - 0.0844j\ ,\ \bar{u}_{40} = 0.0014 + 0.0415j\ .$

Finally, we obtain the response-amplitude by modal superposition:

$$\bar{Z} = \mathbf{P}\,\bar{U} = [\ 0.0073 + 0.0529j,\ 0.01107 + 0.0537j;$$

$$0.0529 - 0.0073j,\ 0.0537 - 0.0107j]^T = [X'^T\ X^T]^T. \qquad (1.50a)$$

It is noted that we can also calculate the response amplitude from the following equation:

$$X = (-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K})\ \mathbf{F} = [0.0529 - 0.0073j,\ 0.0537 - 0.0107j]^T \qquad (1.50b)$$

The solution by modal superposition (1.50a) agrees with the solution obtained by using equation (1.50b).

1.6 OTHER TYPES OF DAMPING

1.6.1 ENERGY DISSIPATION AND EQUIVALENT DAMPING

Besides viscous damping, there are various other types of dissipative forces. Energy in a vibration system is, in most cases, dissipated into heat. Such dissipation is usually determined under conditions of cyclic oscillations. In any rate, if damping exists, the force-displacement curve will form a loop which is referred to as the *hysteresis loop*. The area enclosed in the loop is proportional to the energy dissipated in per cycle of motion, denoted by W_d,

$$W_d = \oint f_d\, dx \qquad (1.51a)$$

where f_d is the damping force of the SDOF system (recall equation (1.2c) which is the definition of damping force of a viscously damped MDOF system). For viscous damping, we have

$$W_d = \oint f_d\, dx = \oint cx'\, dx = \oint cx'^2 dt = c\omega^2 x_0^2 \int_0^{2\pi/\omega} \cos^2(\omega t - \phi)dt$$

$$= \pi\, c\, \omega\, x_0^2 \qquad (1.51b)$$

With the help of equations (1.4) and (1.5), we can have

$$W_d = 2\xi\pi k x_o^2 \tag{1.52}$$

It can be shown that the following is true (Thomson, 1981),

$$\left[\frac{f_d}{c\omega x_0}\right]^2 + \left[\frac{x}{x_0}\right]^2 = 1 \tag{1.53}$$

Thus, an ellipse with f_d and x plotted along the major and minor axes, as shown in Figure 1.4, encloses an area which is the energy dissipated per cycle of motion.

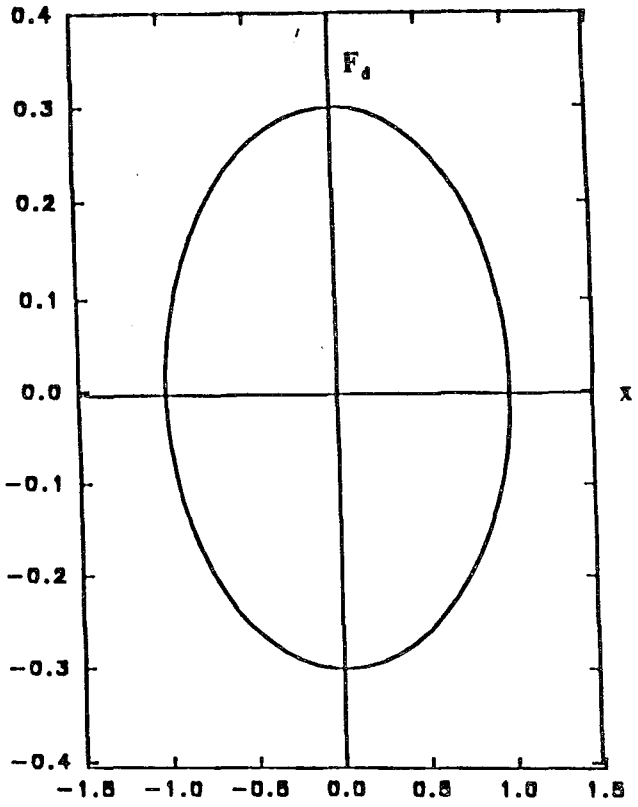


Figure 1.4 Damping Ellipse

For other types of damping, we can introduce the concept of *equivalent damping*, which is established by equating the energy dissipated by viscous

damping to that of the nonviscous damping force with assumed harmonic motion. Referring to equation (1.3b), we can define an equivalent damping c_{eq} as follows:

$$c_{eq} = \frac{W_d}{\pi \omega x_0^2} \tag{1.54}$$

1.6.2 STRUCTURAL DAMPING

Another commonly used damping model is *structural damping* or *hysteretic damping*. In certain cases, damping appears to be frequency-dependent. Then, the behavior of dissipation cannot be described properly by viscous damping. We need a different model where the damping coefficient varies inversely with frequency, i.e.

$$c = \alpha/\omega. \tag{1.55}$$

where α is a proportionality coefficient.

Structural damping satisfies equation (1.55) and provides a much simpler analysis for an MDOF system. However, in such modeling there is difficulty in carrying out rigorous free vibration analysis. This is the main reason that structural damping is less useful than viscous damping.

Generally, we can write a SDOF structurally damped system as

$$mx'' + \left[\frac{\alpha}{\pi \omega} \right] x' + kx = f \tag{1.56a}$$

Alternatively, using the concept of *complex stiffness* (see below), we have

$$mx'' + (k + j\frac{\alpha}{\pi})x = f \tag{1.56b}$$

where $j = \sqrt{-1}$; m, k, f, x, x" are as defined in equation (1.3a), and c is the structural damping coefficient. Note that the damping force is no longer proportional to the velocity x'. Instead, it is proportional to the displacement x. However, the damping force is still perpendicular to the spring and inertial forces. The term

$$k^* = k + j\frac{\alpha}{\pi} \tag{1.57}$$

is called the *complex stiffness.*

Solving the characteristic equation of (1.56b) gives the natural frequency of the structurally damped SDOF system:

$$\omega_s^2 = -\omega^2(1 + j\eta) \tag{1.58}$$

where, $\omega = (k/m)$, ω_s is the natural frequency and η is defined as the *loss factor*. If the structure is lightly damped (see Chapter 3 for details),

$\omega_s \approx (k/m)^{1/2}$, and

$$\eta \approx \frac{\alpha}{\pi\ \omega^2\ m} .$$

The equivalent damping is given by

$$c_{eq} = \frac{\alpha}{\pi\ \omega} \tag{1.59}$$

The structurally damped MDOF systems will be discussed in Chapter 4.

1.6.3 COULOMB DAMPING

Coulomb damping results from the sliding of two dry surfaces under normal pressure. From the coulomb law of friction force,

$$f_d = \upsilon\ N \tag{1.60}$$

where υ is the coefficient of friction, and N is the normal force. The damping force is independent of the velocity. Figure 1.5 shows the free vibration with Coulomb damping.

It is worth pointing out that, in many engineering applications, the dry-friction effect often provides a large amount of damping, and in such cases equation (1.60) is no longer a good model. The Coulomb damping force (1.61) is given by

$$f_d = \frac{1}{2} k (x_0 - x_{-0}) \quad (1.61)$$

where f_d is the Coulomb damping force, x_{-0} is the amplitude after the half

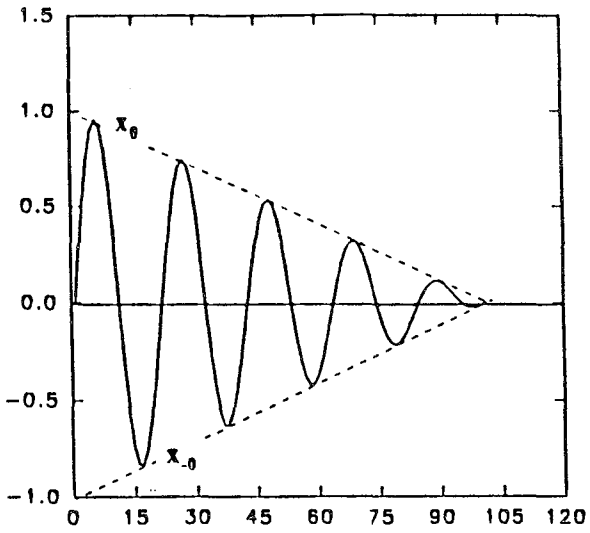


Figure 1.5 Coulomb Damped Vibration

cycle of motion as shown in Figure 1.5. From Figure 1.5, we can see that the decay in amplitude in per cycle of motion is a constant and the amplitude is equal to

$$x_1 - x_2 = 4f_d/k \quad (1.62)$$

With the help of (1.62), we can write the equivalent damping as

$$c_{eq} = \frac{4 f_d}{\pi \omega x_o} \quad (1.63)$$

1.6.4 AERODYNAMIC DAMPING

For air and acoustic damping mechanisms, the *aerodynamic damping* is defined by

$$f_d = \alpha x'^2 \quad (1.64)$$

where α is a proportional coefficient.

It can be shown that the equivalent damping corresponding to equation (1.64)

is given by

$$c_{eq} = \frac{8}{3\pi} \alpha\omega x_o \tag{1.65}$$

(See Timoshenko, 1974, for instance.)

1.6.5 VISCOELASTIC DAMPING

Viscoelastic damping results from the use of certain types of damping materials which, as the name implies, act partly as a viscous material (as an energy absorber), and partly as an elastic material (as an energy restorer). According to the theory of linear viscoelasticity: if the strain of the material is $\gamma = \gamma_0 \sin \omega t$, then the stress will be $\sigma = \sin(\omega t - \phi)$, where ϕ is the *loss angle*. The stress may be separated into two parts, one in phase with the strain and the other leading the phase by a quarter cycle

$$\sigma = \gamma_0 [\, G_s(\omega)\sin\omega t + G_l(\omega)\cos\omega t \,] \tag{1.66}$$

where $G_s(\omega)$ is known as the storage modulus, and $G_l(\omega)$ is the loss modulus. Both are functions of frequency and temperature. In unit volume of viscoelastic damping material, the dissipated energy during a cycle of motion is

$$W_d^{(u)} = \int_0^{2\pi/\omega} \sigma \, d\gamma = \int_0^{2\pi/\omega} \omega \gamma_0^2 \, [G_s(\omega)\sin\omega t + G_l(\omega)\cos\omega t] \cos\omega t dt$$

$$= \pi \gamma_0^2 G_l(\omega) \tag{1.67}$$

where the superscript (u) is used to indicate the lost energy in a unit volume of the damping material. The total loss of energy is then

$$W_d = W_d^{(u)} V \tag{1.68}$$

where V is the total volume of the viscoelastic material.

It is noted that viscoelastic damping is a property of the material used in the dampers. The damping of a system is the results of combined effects of dampers and the structure. It is thus difficult to determine the "equivalent damping (coefficient)" without a given structure. Therefore, instead of

dealing with the value c_{eq}, it is more convenient to employ the concept of an equivalent damping ratio, ξ_{eq}. With the assumption of a SDOF system and the help of equation (1.6), we can define ξ_{eq} for a SDOF system with viscoelastic damper(s) as

$$\xi_{eq} = \frac{W_d}{4\pi W} = \frac{\pi \gamma_0^2 G_1(\omega) V}{4 \pi W} = \frac{\gamma_0^2 G_1 V}{2 k x_0^2} = \frac{\gamma_0^2 G_1 V}{2 m \omega^2 x_0^2} \quad (1.69)$$

The damping effect from a viscoelastic damper is usually much greater than that from the material of the structure only (usually more than ten times, see Lin and Liang (1988) for example). Therefore, for a structure that can be approximated by the SDOF assumption, all damping may be regarded as the contribution of the viscoelastic damper(s). Under this assumption, we have the following equivalent damping coefficient:

$$c_{eq} = 2\xi_{eq}\sqrt{k\ m} \quad (1.70)$$

where m is the mass of the original system, since the mass of viscoelastic damper(s) is often much less than the mass m. On the other hand, in equation (1.70), k is the stiffness contributed by both the structure and the damper.

1.6.6 ACTIVE DAMPING AND DESIGN CONSIDERATIONS

If the aforementioned damping comes from the structure itself and/or added dampers, then the value of the damping coefficient is fixed once the structure is designed or constructed. Although the damping may change slightly due to other environmental factors, we often omit such variations. This is often referred to as *passive damping*. On the other hand, we may use actuators or other means to apply external forces to a structure to control its vibration. If the applied force is proportional to the velocity, the applied force is acting as a damping force. This is called *active damping*. Passive damping absorbs energy from the structure and active damping may add energy to the structure.

A structure with controlled vibration is a *stable* system (or *marginally stable* in some literature) which has a bounded response, i.e.

$$||X(t)|| \le B \quad (1.71)$$

where B is a positive constant.

There are many criteria to judge whether or not a system is stable. In this report, to be consistent with our discussion on the eigen-problem, we use the theorem that a stable system should have all the eigenvalues without positive real parts. For an M-C-K system, before and after the controlled force is added, the mass matrix should be positive, the damping and stiffness matrices should at least be semi-positive.

Another design consideration is whether a structure or a system is 1) underdamped, 2) critically damped or 3) overdamped. In this report, we adopt the following definitions and the related theories.

For a SDOF system, or a mode of a MDOF system, we say that this system or this mode is *underdamped*, if the corresponding damping ratio is less than 1, i.e.

$$0 < \xi < 1 \tag{1.72}$$

In this case, the response of this system or this mode exhibit decaying oscillation with frequency $\sqrt{1-\xi^2}\ \omega$.

A system or a mode is said to be *overdamped*, if the corresponding damping is greater than 1, i.e.

$$\xi > 1 \tag{1.73}$$

In this case, the response does not oscillate.

A system is a *critically damped* system if it has a mode with the damping ratio equal to 1, i.e.,

$$\xi = 1 \tag{1.74}$$

A critically damped system can be thought of as one that has the minimum value of damping corresponding to a nonoscillating response. It can also be thought of as the case separating nonoscillation from oscillation.

For a general MDOF system, we first define a *critical damping matrix*, $\mathbf{C}_{cr}$(see

Inman and Andry (1980)) if the system is given by equation (1.1b), that is

$$\mathbb{C}_{cr} = 2\ \mathbb{K}^{1/2} \tag{1.75}$$

Then, the following classifications can be defined:
1) If $\mathbb{C}_{cr} - \mathbb{C}$ is positive definite, then the system (1.1b) is said to be an underdamped system. Each mode then is underdamped and all the eigenvalues are in the form of equation (1.26). The eigenvectors are, in general, complex (unless $\mathbb{C}\mathbb{K} = \mathbb{K}\mathbb{C}$, see Chapter 2 for detail). In this report, systems are assumed to be underdamped system unless otherwise noted.
2) If $\mathbb{C} - \mathbb{C}_{cr}$ is positive definite, then the system (1.1b) is said to be an overdamped system. Each mode then is overdamped and all the eigenvalues are negative real number. All eigenvectors are real.
3) If $\mathbb{C} = \mathbb{C}_{cr}$, then the system (1.1b) is said to be a critically damped system. Each mode then is critically damped. Each eigenvalue is a repeated negative real number. All eigenvectors are real.

1.7 VISCOUSLY DAMPED MDOF SYSTEMS

In the above section, we reviewed traditional treatment and certain basic theories of damping. They include simple energy considerations, the representation of the damping matrix, and the modal analysis of proportionally and non-proportionally damped systems. In the following, viscously damped MDOF systems will be briefly reviewed.

1.7.1 MODAL SUPERPOSITION

Modal superposition is a useful concept in structural dynamics. The most common version of modal superposition is used in response calculations. It is known as the time history method. This method first obtains modal responses (1.33b) which are then transformed into the actual time response, (1.36), of the system. The modal superpositions can be classified into two categories: 1) the normal mode which yields the real valued method (Caughey, 1965); and 2) the complex mode which yields the complex valued or generalized method (Foss, 1958).

The normal mode method is more convenient from the computational viewpoint. It maintains the physical nature of the parameters after modal decomposition.

Furthermore, the normal mode method can handle viscoelastic materials with frequency-dependent damping.

However, in many practical engineering problems, we do have complex modes in which the physical nature of the parameters is lost after modal decomposition.

The complex-valued method has been given increased attention by researchers in recent years. Foss (1966), Singh (1980), Traill-Nash (1981) and Veletsos and Ventura (1986) have made efforts to explain the physical meaning by using the complex mode shape and complex frequency.

In 1976, Clough and Mojtahedi compared different approaches of modal superposition methods, including the methods of direct integration and weighted damping ratios. They favored the direct integration approach.

In 1984, Nicholson and Bergman examined a non-proportional damping configuration in their development of a theory for combined damped systems. In 1986, Sigh and Ghafory introduced a recursive step approach in the time domain which utilized information from the complex modal parameters.

1.7.2 COMPARISONS BETWEEN NORMAL AND COMPLEX-VALUED METHODS

Theoretically speaking, the real-valued method is not applicable to non-proportionally damped systems. In practice, one rarely finds ideal proportional damping. But the real-valued method can be satisfactorily used in cases where accuracy is not important. However, in cases such as structures with added energy absorbers, use of the proportional damping approach may introduce large errors. The approximation of a complete set of equations by an uncoupled set is also often used. Duncan and Taylor (1979) have pointed out that the extent of errors generated by these kind of approximations is difficult to assess in large MDOF systems.

Methods to reduce the modal damping matrix into a diagonal matrix have also been suggested. The most simple case is to omit the off-diagonal elements of the transformed damping matrix. This method has been used in earthquake

response analysis (Young, 1960 and Crede, 1963). But even for lightly damped systems, such damping can be highly non-proportional. Therefore, this approach has received strong criticism (Van Loon, 1974; Clough and Mojtahedi, 1976; Hasselman, 1976). Several papers have analyzed the conditions under which the errors of a decoupling approximation may be small (Warburton and Soni, 1977; Duncan and Taylor, 1979; Ozguven and Semercigil, 1982; Prater and Singh, 1986, Nair and Singh, 1986; Ozguven, 1986; Bellos and Inman, 1988). Hasselman (1976), also Warburton (1977), have shown for systems with close natural frequencies, if normal modes were used to approximate those modes, then the error would be unacceptably large. Duncan and Taylor found that significant errors may occur if modal truncation is employed. These conclusions strongly suggest the need to have a better understanding of the non-proportionally damped system.

1.7.3 APPROXIMATIONS AND ERROR ESTIMATES OF DECOUPLING

Knowing that normal mode methods inevitably involve errors and that complex mode methods require intensive computations, many authors have tried to develop approximate approaches. Hammill and Andrew (1971) suggested a substructure technique to calculate the harmonic response of an undamped structure loaded with discrete dampers. This is a non-proportionally damped system with the dampers considered as a secondary system. This method, used in many subsequent publications, is based on a principle of receptance coupling introduced by Bishop and Johnson in 1968. Following this method, Abhary (1975) and Sainsbury (1976) identified the receptances of a structure at certain selected coordinates. Wang, Clark and Chu (1985) discussed the effects of the local modifications on the damping characteristics. Thomson, Calkins and Garavani (1974) used an optimization technique to establish a diagonal damping matrix. Cronic (1976) developed a perturbation method to approach the damping matrix. Cronic's method was recently used by Chung and Lee (1986) to obtain better eigen-decompositions for non-proportional damping.

It should be mentioned that Warburton et al, (1977) had established a criterion to keep the errors in the responses within accepted limits when the uncoupled-modes are assumed in the analysis. Ozguven (1981) suggested a single mode method to decouple equation (1.1) at a particular undamped natural frequency. The accuracy of approximation for the total response by one mode is

discussed. The accuracy depends on whether it is negligible when the excitation frequency is close to an undamped natural frequency. Ozguven and Cowley (1981) also used receptance as the basis to determine the frequency response of non-proportionally damped and continuously damped plates, by first representing internal damping by equivalent external dampers. Later, in 1984, they developed a matrix inversion method to calculate non-proportionally damped structures. Instead of using equivalent external dampers, the internal damping is treated as a substructure with damping parameter only. In 1987, Ozquven further developed a different method to calculate the receptances, avoiding the matrix inversion. The method is computationally intensive and sometimes unstable. In 1988, Bollos and Inman suggested an approximate time domain technique, which, by taking into account the modal coupling, gives the system response. It was stated that the results were improved.

1.7.4 RESPONSE BOUNDS

In many practical situations, the exact or approximate solutions of equation (1.1a) are not required. Instead, information about the response bounds is needed. Estimates of response bounds are especially necessary in design and in calculations of large or complex systems (Yae and Inman, 1986). Although it is easy to establish the response bounds for proportionally damped systems because the system can be decoupled, it is a rather involved process for non-proportionally damped systems. Ahmadi (1986) derived the bounds on the earthquake response of MDOF structures. Nicholson (1987) suggested a time-decaying bounds under impulse and step loads. Shahruz and Ma (1988) developed error bounds to evaluate the approximations of replacing the modal damping matrix by selecting diagonal matrices. Liang and Soong (1989) have suggested a tighter bounds calculation under consideration of work done by internal and external forces.

1.7.5 NON-PROPORTIONALITY INDICES

In the open literature, information regarding the physical nature of non-proportional damping and complex modes is very limited. One advancement in recent years is the use of various non-proportionality indices. Prater and Singh (1985) suggested several numerical indices and gave a good comparison among them. These indices are based on

the damping ratio are continuous functions of the perturbation but the mode shape may not be. On the other hand, if the mode shape can be precisely measured and then used to determine the other parameters, one should obtain more accurate and stable results. Spatial domain methods are time consuming. But the methods have the advantage of high accuracy. In physical parameter identifications, the methods are also developed in three domains. Since physical parameters are more difficult to determine, to conduct a successful modal testing the requirements of high signal-to-noise ratio and participation of all modes (if it goes through modal testing) are essential. It is noted that, for modal parameter identification, most algorithms are linear (the calculations of identified items are linear functions of signals in the frequency or the time domain). However, in most cases, such requirements for physical parameter identification can not be satisfied. Therefore, some nonlinear iteration algorithms have been developed to overcome these difficulties. Because such algorithms require much more computation, physical parameter identification is not popular. Most published works in the literature are only concerned with undamped or proportionally damped systems. An example is Thoren's method (1972), which depends on the orthogonal mode shape vectors for determining mass, damping, and stiffness. However, most systems have complex modes, and the corresponding mode shape vectors are not orthogonal. Hall et al (1979) offered a method to identify a distributed parameter system. In their approach,the damping factor is completely omitted. More recently, Caravani and Thomson (1976), Hanagud (1982) and Inman (1984) have offered improvements on these shortcomings for non-proportionally damped systems. All of these developments assume that some or all the entries in the mass and stiffness matrices are known. Therefore, their identification is mostly concerned with the damping matrix. Johnkins et al have developed a time domain method to identify all the mass, damping and stiffness matrices (1984). This method is only valid when the displacement, velocity and acceleration in all dimensions can be measured simultaneously. This requirement encounters a technical difficulty in that we can not obtain the displacement X and velocity X' by integrating the acceleration signals X" without the phase-shift. At any rate, fewer nonlinear algorithms have been published than linear algorithms. Fuh (1981), Beliveau (1986) and Hollowell (1989) have all given good examples. These methods require more computations if higher accuracy is required (Link (1990)).

2 A THEORY OF COMPLEX DAMPING

2.1 INTRODUCTION

In Chapter 1, we selectively reviewed traditional considerations of damping and dynamic systems. We have pointed out that proportionally damped systems are essentially different from non-proportionally damped systems, and that their treatments are also different. It was also pointed out that, in many cases, using the proportional damping to approximate non-proportional damping may introduce errors. In this chapter an energy based damping theory is presented that explains the difference between these two systems by simultaneously describing energy dissipation and energy transmission of each vibration mode by a complex-valued quantity. For this reason, this newly developed approach is termed complex damping theory or complex energy-based damping theory.

2.2 CONCEPT OF COMPLEX DAMPING RATIOS, MATHEMATICAL TREATMENT

2.2.1 INTRODUCTION OF COMPLEX DAMPING COEFFICIENT

In this section we will introduce the complex damping coefficient. First, recall some definitions from Appendix A:

A *true-complex eigenvector* or *strongly-complex eigenvector* of a simple dynamic system is a non-zero eigenvector P_i associated with the eigenvalue λ_i such that P_i satisfies

$$\lambda_i^2 P_i + \lambda_i \mathbf{C} P_i + \mathbf{K} P_i = 0$$

and P_i can not be written as a complex linear combination of real-valued eigenvectors of λ_i, Q_j's, $0 \le j < p < n$. Here p is the number of the multiple of the corresponding eigenvalue λ_i, $0 < p < n$. On the other hand, if

$$P_i = \alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_p Q_p$$

where the α_i's are real or complex-valued scalars, $i = 1,...p < n$, and at least one $\alpha_i \neq 0$, then P_i is said to be a *pseudo-complex eigenvector* or a *weakly-complex eigenvector* of the system.

There are only two distinct events: either P_i is an eigenvector of both $\tilde{C}$ and $\tilde{K}$ and P_i is weakly-complex, or P_i is not an eigenvector of $\tilde{C}$ or $\tilde{K}$ and is strongly-complex (Theorem A.8).

If a matrix $\mathbb{P}$ is an eigenvector matrix of a system, then two distinct cases can result: $\mathbb{P}$ either contains true-complex eigenvectors of the system, or it does not. In the first case, $\mathbb{P}$ is called a *true-complex eigenvector matrix*. If $\mathbb{P}$ does not contain true-complex eigenvectors, it is called a *pseudo-complex eigenvector matrix* of the system. Note that a true complex eigenvector matrix can not be changed into a pseudo-complex eigenvector matrix of the same system by linear transformation and vice versa.

For any M-C-K system, there exists a corresponding C-Λ_k system which has the identical eigenvalues of the M-C-K system. Further, if the i^{th} eigenvector of the M-C-K system is a weakly-complex eigenvector, so is the corresponding eigenvector of the C-Λ_k system (Lemma 6). This fact tells us that the M-C-K system and its C-Λ_k system have the same eigen-structure or eigen-problem. The C-Λ_k system is referred to as the canonical vibration system (CVS) .

Now, let us consider a CVS, for the purpose of conceptually visualizing the energy transfers. Figure 2.1 shows a three DOF CVS.

To study the formation of such a CVS, rewrite equations (A.134), (A.135) and (A.136) as follows:

(A.134) $\quad \mathbb{P}_i \Lambda^2 + \tilde{C}\, \mathbb{P}_i \Lambda + \tilde{K}\, \mathbb{P}_i = \mathbb{0}$

(A.135) $\quad \tilde{K} = \mathbb{Q}\, \Lambda_k\, \mathbb{Q}^T$

(A.136) $\quad \Lambda_k = \mathrm{diag}(\, \omega_{ni}^2 \,)$

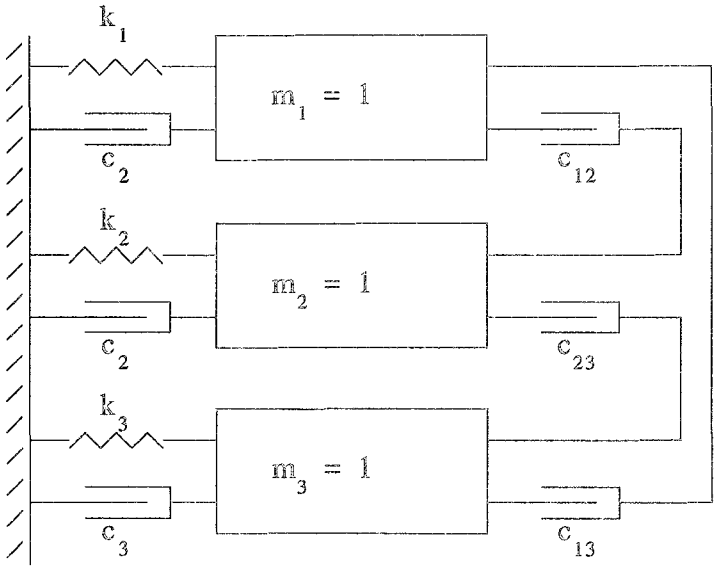


Figure 2.1 3-DOF Canonical Vibration System

Equation (A.134) describes the eigen-problem of a monic system (with identity mass). $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{K}}$ here are generalized damping and stiffness matrices, both are symmetric. Such a monic system can be further reduced into a canonical vibration system (CVS), by pre-multiplying $\mathbb{Q}^T$. $\mathbb{Q}^T$ is the transpose of eigenvector matrix of $\tilde{\mathbb{K}}$. Using the linear transformation of $\mathbb{Q}^T$ and equation (A.135), we can diagonlize the generalized stiffness matrix $\tilde{\mathbb{K}}$ into Λ_k .That is, $\mathbb{Q}^T \tilde{\mathbb{K}} \mathbb{Q} = \Lambda_k$. In Figure 2.1, we use k_i to denote the ii^{th} entry of matrix Λ_k . Note that k_i is the i^{th} eigenvalue of $\tilde{\mathbb{K}}$ ($k_i = \omega_{ni}^2$) and

$$k_i = Q_i^T \tilde{\mathbb{K}} Q_i \Big|_{Q_i^T Q_i = 1} \qquad (2.1a)$$

If the system is proportionally damped, and $\mathbb{Q}$ is also the eigenvector matrix of $\tilde{\mathbb{C}}$, then $\tilde{\mathbb{C}}$ will be also diagonalized. Denote the diagonal entries by c_i. We have $\mathbb{Q}^T \tilde{\mathbb{C}} \mathbb{Q} = \text{diag}\ (c_i)$. In this case, all the corresponding off-diagonal entries are zero. Then, in Figure 2.1 we have three isolated vibrators, each stands for a specific vibration mode. There is no relation between these isolated modes. Note that

$$c_i = Q_i^T \tilde{\mathbb{C}} Q_i \Big|_{Q_i^T Q_i = 1} \qquad (2.1b)$$

Equations (2.1a) and (2.1b) are regular Rayleigh quotients. Since k_i and c_i are eigenvalues of $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{C}}$, they are at standing points of the corresponding Rayleigh quotients, respectively. Since the eigenvector are identically Q_i, these standing points are reached simultaneously. At this moment there is no energy transfer between the modes. These three isolated vibrators represent the physical essence of normal modes of a proportionally damped system.

However, when the system is non-proportionally damped, certain off-diagonal entries of matrix product $\mathbf{Q}^T \tilde{\mathbf{C}} \mathbf{Q}$ are non-zero. Denote such a non-zero entry at the i^{th} row and j^{th} column c_{ij}. We can see, in Figure 2.1, that this quantity couples the generalized masses i and j. Or, certain energy is transferred between mass i and j. In general, c_i in equation (2.1b) is no longer at the standing point of the Rayleigh quotient, because Q_i is not the eigenvector of non-proportional damping and c_i is not the corresponding eigenvalue. The energy transfer is responsible for the coupled motions among the vibrators. In Figure 2.1, suppose that mass 1 vibrates with a frequency f_1. Due to energy transfer, mass 2 and 3 will also vibrate with the frequency close to f_1. However, these vibrations will have certain phase-shifts, then complex-valued vectors are needed to describe both the magnitudes and phase angles. This is the physical essence of complex modes of a non-proportionally damped system.

To see the above explanation mathematically, let us consider the following derivation. Pre-multiplying equation (A.134) by $\mathbf{Q}^T$ results in

$$\mathbf{R}\,\Lambda^2 + \mathbf{Q}^T\tilde{\mathbf{C}}\,\mathbf{P}_1\,\Lambda + \Lambda_k\,\mathbf{R} = 0 \qquad (2.2a)$$

where

$$\mathbf{R} = \mathbf{Q}^T\mathbf{P}_1 = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \\ r_{n1} & r_{n2} & & r_{nn} \end{bmatrix} \qquad (2.2b)$$

By using the notation (A.136) together with equation (A.135), we can have the following expression of equations for the ii^{th} entry, that is,

$$\lambda_i^2 r_{ii} + Q_i^T \tilde{C}\ P_{1i}\ \lambda_i + \omega_{ni}^2\ r_{ii} = 0 \qquad (2.3)$$

where P_{1i} is the i^{th} column of matrix $\mathbf{P}_1$. Note that,

$$r_{ii} = Q_i^T P_{1i} \tag{2.4}$$

If the system is proportionally damped, r_{ii} must be non-zero.
With

$$\omega_{ni}^2 Q_i = \tilde{K} Q_i \quad \text{or} \quad Q_i^T = \frac{1}{\omega_{ni}^2} Q_i^T \tilde{K}$$

we have

$$r_{ii} = \frac{1}{\omega_{ni}^2} Q_i^T \tilde{K} P_{1i} \tag{2.5}$$

In the case of a non-proportionally damped system, it can be shown that at least two of the terms r_{ij}, i = 1,...,n, are not equal to zero for at least one P_{1j}. Therefore, we suppose $r_{ii} \neq 0$. Dividing equation (2.4) by r_{ii} yields

$$\lambda_i^2 + \alpha_i \lambda + \omega_{ni}^2 = 0 \tag{2.6}$$

where

$$\alpha_i = \frac{Q_i^T \tilde{C} Q_i}{r_{ii}}$$

The term

$$(Q_i^T \tilde{C} P_{1j}) / Q_i^T P_{1j} = \Re_{ij} \tag{2.7}$$

is defined as a *generalized Rayleigh quotient* .

Since P_{1i} is generally a complex vector, it follows that the product of ω_{ni}^2 and the generalized Rayleigh quotient

$$\{\omega_{ni}^2\} \{ \Re_{ij} \} = \omega_{ni}^2 \{ (Q_i^T \tilde{C} P_{1i}) / (Q_i^T \tilde{K} P_{1i}) \}$$

is also complex. Also, the quantity α_i can be denoted by $\alpha_i = a_i + jb_i$. That is,

$$\alpha_i = a_i + jb_i = \omega_{ni}^2 (Q_i^T \tilde{C} P_{1i}) / (Q_i^T \tilde{K} P_{1i}) \tag{2.8}$$

Equation (2.7) and its corresponding differential equation

$$u_i'' + (a_i + j\, b_i)\, u_i' + u_i = 0 \qquad (2.9)$$

are the characteristic equation and the differential equation of the i^{th} virtual mode, respectively. Therefore, in Figure 2.1, regardless whether the system is proportionally damped or not, the i^{th} vibrator always refers to the i^{th} mode. We call it the i^{th} *virtual mode*. It is an extension of the definition of the normal mode. When the system is non-proportionally damped, the corresponding damping coefficient is complex-valued in nature.

Let us examine the above point in more detail. Consider again the physical meaning of pre-multiplying $\mathbb{Q}^T$ of equation (A.134). We refer to the terms $\mathbb{P}_1\Lambda^2$, $\tilde{\mathbb{C}}\mathbb{P}_1\Lambda$ and $\tilde{\mathbb{K}}\mathbb{P}_1$ as the inertial force, damping force and spring force, respectively (see Clough,1985, for example). Therefore $\mathbb{Q}^T\mathbb{P}_1\Lambda^2$, $\mathbb{Q}^T\tilde{\mathbb{C}}\mathbb{P}_1\Lambda$ and $\mathbb{Q}^T\tilde{\mathbb{K}}\mathbb{P}_1$ are the virtual work done along the virtual displacement $\mathbb{Q}$. Furthermore, we see that the quantity α_i is a ratio of virtual work ($Q_i^T\tilde{\mathbb{C}}P_1$) done by the damping force to the virtual work ($Q_i^T P_{1i}$) done by the inertial force (or virtual work ($\mathbb{Q}_i^T\tilde{\mathbb{K}}\,\mathbb{P}_{1i}$) done by the spring force) . The quantity α_i plays an important role in vibration analyses. α_i is defined as the i^{th} *complex damping coefficient.*

Suppose we take

$$\lambda_i = - \xi_i \omega_i + j\sqrt{1 - \xi_i^2}\ \omega_i$$

and then

$$\lambda_i^2 = (2\,\xi_i^2 - 1)\,\omega_i^2 - 2\,j\,\xi_i\sqrt{1 - \xi_i^2}\ \omega_i^2$$

Substituting these above notations into equation (2.6) and rearranging the result in two equations, for the real part we have

$$(2\,\xi_i^2 - 1)\,\omega_i^2 + \omega_{ni}^2 = \xi_i\omega_i a_i + \sqrt{1 - \xi_i^2}\ \omega_i\, b_i \qquad (2.10a)$$

and for the imaginary part we have

$$-2\,\xi_i\sqrt{1-\xi_i^2}\;\;\omega_i^2 = -\sqrt{1-\xi_i^2}\;\;\omega_i\;a_i + \xi_i\,\omega_i\,b_i \qquad (2.10b)$$

Combining the above two equations, we have

$$a_i = \frac{\xi_i}{\omega_i}\,(\,\omega_{ni}^2 + \omega_i^2\,) \qquad (2.11a)$$

$$b_i = \frac{\sqrt{1-\xi_i^2}}{\omega_i}\,(\,\omega_{ni}^2 - \omega_i^2\,) \qquad (2.11b)$$

Similarly, if we take

$$\lambda_i = -\xi_i\,\omega_i - j\sqrt{1-\xi_i^2}\;\;\omega_i$$

we can get the same result as (2.11a) for a_i, but an expression (from (2.11b)) with a changing sign for b_i

$$b_i = \frac{\sqrt{1-\xi_i^2}}{\omega_i}\,(-\,\omega_{ni}^2 + \omega_i^2\,) \qquad (2.11c)$$

From equation (2.11a), a_i is always greater than zero, if ξ_i is non-zero. However, the sign of b_i of (2.11b) or (2.11c) appears undefined. In fact for a given M-C-K system, the sign of b_i is uniquely determined.

Taking the complex conjugate of equation (2.9), we have

$$\alpha_i^* = a_i - jb_i = \omega_{ni}^2\,(Q_i^T\tilde{C}\,P_{1i}^*)/(\,Q_i^T\,\tilde{K}\,P_{1i}^*\,)$$

The sign of b_i is completely determined by the i^{th} eigenvector P_{1i}. The eigenvalues and the associated eigenvectors are in complex conjugate pairs. When we calculate the complex damping coefficients, these eigenvalues and eigenvectors should be chosen correspondingly. Then, the sign of b_i is uniquely determined.

Recall equation (A.134), which describes only one-half of the eigen-structure of the system (a half of the entire complex conjugate pairs.) The other half portion is given by

$$\mathbf{P}_1^* \Lambda^{*2} + \tilde{\mathbf{C}} \mathbf{P}_1^* \Lambda^* + \tilde{\mathbf{K}} \mathbf{P}_1^* = \mathbf{0}$$

The reason to choose only (A.134) is that this one-half actually covers the entire eigen-information.

2.2.2 CRITERIA FOR PROPORTIONAL DAMPING

The above discussion suggests that b_i can be used as an index to calculate the difference between the undamped natural frequency of the system, ω_i, and that of the corresponding non-proportionally damped system, ω_{ni}. In this section, we shall show that, in any proportionally damped system, there is no difference between ω_i and ω_{ni} for all modes. The b_i's are unanimously zero. If we can find a single non-zero b_i, the system must be non-proportionally damped.

Mathematically, if the system is proportionally damped, i.e.,

$$\omega_{ni} = \omega_i \tag{2.12}$$

then, $b_i = 0$. Note that equation (2.11b) or (2.11c) is not satisfied unless b_i is equal to zero. Thus, if the system is non-proportionally damped, it has a complex mode. Then the natural frequencies ω_{ni} and ω_i are not equal and the imaginary part of complex damping coefficient is not zero. The above discussion is summarized in Theorem 2.1:

Theorem 2.1: The following five statements are equivalent for the description of a proportionally damped system:

1) The system is proportionally damped, when

$$\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C} \quad \text{or} \quad \tilde{\mathbf{C}}\,\tilde{\mathbf{K}} = \tilde{\mathbf{K}}\,\tilde{\mathbf{C}}$$

2) The eigen-matrix **A** (defined in Appendix A, equation (A.58a)) of the system has the following properties:

$$\mathrm{Re}(\mathbf{A})\,\mathrm{Im}(\mathbf{A}) = \mathrm{Im}(\mathbf{A})\,\mathrm{Re}(\mathbf{A}) \quad \text{and} \quad \mathbf{A}\,\mathrm{Re}(\mathbf{A}) = \mathrm{Re}(\mathbf{A})\,\mathbf{A}$$

3) The system has only normal modes (all eigenvectors of system (2.65) are weakly-complex eigenvectors), and $\mathbf{P} = \mathbf{Q}\,\mathbf{A}$.

4) The undamped natural frequencies of the system and the corresponding eigenvalues of the generalized stiffness $\tilde{\mathbf{K}}$ are all equal. Namely,

$\omega_i = \omega_{ni}$, $i = 1, ..., n$

5) All the imaginary parts of the generalized Rayleigh quotient $\Re_{ij}$ are zero, or, $b_i = 0$, $i = 1, ..., n$.

Since statements 1), 2) and 3) are well established (see Appendix A, Theorem A.6 and Corollary A.6), we give proofs only for 4) and 5).

PROOF:

For sufficiency: If a system has no complex mode, then b_i's must be zero. From this condition, $\mathbf{P}_1$ is weakly-complex. With certain linear transformation, we can have

$Q_i = P_{1i}$, $i = 1,...n$

Then, from equation (2.8), α_i is a real scalar, or, $b_i = 0$; Also, from the argument $Q_i = P_i$ we know that other generalized Rayleigh Quotient $\Re_{ij}$'s, $i \neq j$, are zero.

For necessity: If all b_i's = 0, $\mathbf{P}_1$ is weakly-complex. It is clear from equation (2.10b), that this condition is equivalent to equation (2.12):

$\omega_i = \omega_{ni}$, $i = 1,...,n$

We can rewrite this relationship as

$$\Lambda \Lambda^* = \mathbf{Q}^T \tilde{\mathbf{K}} \mathbf{Q} = \Lambda_k^2 \qquad (2.13)$$

In Appendix A (Theorem A.8) we have shown that for an M-C-K system, if equation (2.12) holds, it can have a weakly-complex eigenvector matrix. Therefore, we know the necessary condition is also true. ■

Theorem 2.1 is the basis of the complex damping theory. Its usefulness and importance are pursued further in the following.

When $\tilde{\mathbf{K}}\ \tilde{\mathbf{C}} \neq \tilde{\mathbf{C}}\ \tilde{\mathbf{K}}$, for any strongly-complex eigenvector P_i and its associate generalized Rayleigh quotient, we have

$$\frac{Q_i^T \tilde{C} P_j}{Q_i^T P_j} = \Re_{ij}$$

where at least one of the $\mathrm{Im}(\Re_{ij})$'s does not equal zero.

Consider the eigen-equation associated with P_i in a way that

$$(I \lambda_i^2 + \tilde{C} \lambda_i + \tilde{K}) P_i = 0 \tag{2.14}$$

where λ_i is the corresponding eigenvalue. Pre-multiplying equation (2.28) by Q^T, we have

$$Q^T(I \lambda_i^2 + \tilde{C} \lambda_i + \tilde{K}) Q Q^T P_i = 0$$

Denote

$$R = Q^T P_{1i} \tag{2.15}$$

According to Lemma A.5, P_i is a strongly-complex eigenvector of the system, so is vector R. Then we can write

$$(I \lambda_i^2 + Q^T \tilde{C} Q \lambda_i + \Lambda_k) R = 0 \tag{2.16}$$

Without loss of generality, we can always arrange

$$\Lambda_k = \begin{bmatrix} {}_1\omega_{n1}^2 & & & & & & & \\ & {}_2\omega_{n1}^2 & & & & & & \\ & & \ddots & & & & & \\ & & & {}_{r1}\omega_{n1}^2 & & & & \\ & & & & {}_1\omega_{n2}^2 & & & \\ & & & & & \ddots & & \\ & & & & & & {}_{r2}\omega_{n2}^2 & \\ & & & & & & & \ddots \\ & & & & & & & {}_{rp}\omega_{np}^2 \end{bmatrix} \tag{2.17}$$

where the pre-subscript ri is the multiple number of the i^{th} repeated eigenvalue ω_{ni}^2, $1 \le ri < n$. Here

$$\omega_{ni}^2 \neq \omega_{nj}^2 \text{ , if } i \neq j. \tag{2.18}$$

The condition (2.17) implies that Λ_k has at least two distinct eigenvalues, since ri = n (Λ_k has n repeated eigenvalues) is contradictory to the condition $\tilde{\mathbf{K}}\ \tilde{\mathbf{C}} \neq \tilde{\mathbf{C}}\ \tilde{\mathbf{K}}$.

Corresponding to the arrangement (2.16), we can write vector R as

$$\mathbf{R} = \left\{ \begin{array}{c} r_{11} \\ r_{12} \\ .. \\ r_{1r1} \\ r_{21} \\ .. \\ r_{2r2} \\ .. \\ r_{prp} \end{array} \right\} \tag{2.19}$$

Then we know that R is a strongly-complex eigenvector if and only if R is not an eigenvector of $\tilde{\mathbf{K}}$ or Λ_k (see Theorem A.6). In other words, there exists at least one r_s and one r_t which are non-zero. We choose r_{11} and r_{21} in the following discussion.

Next, one of the following inequalities must be true, either

$$\lambda_i\ \lambda_i^* \neq \omega_{ns}^2 \tag{2.20a}$$

or

$$\lambda_i\ \lambda_i^* \neq \omega_{nt}^2 \tag{2.20b}$$

Suppose it is the case of (2.20a) (it can be seen that the following arguments are also applicable for the case of (2.20b)). Then, we can have

$\mathbf{e}_s = [\ 0\ 0\ ..1...\ 0\ ]$ to make
$\qquad\qquad\qquad\quad s$

$$\Lambda_k\ \mathbf{e}_s = \omega_{ns}^2\ \mathbf{e}_s \tag{2.21}$$

Note that since both r_s and r_t are non-zero, $\mathbf{e}_s^T\ \mathbf{R} \neq 0$. Pre-multiplying

equation (2.30) by $\mathbf{e}_s^T$ yields

$$\mathbf{e}_S^T (I \lambda_i^2 + \mathbf{Q}^T \tilde{C} \mathbf{Q} \lambda_i + \Lambda_k) R = 0$$

or

$$r_s \lambda_i^2 + \mathbf{e}_1^T \mathbf{Q}^T \tilde{C} \mathbf{Q} R \lambda + \omega_{ns}^2 r_s = 0$$

Because $r_s \neq 0$, we have

$$\lambda_i^2 + (\mathbf{e}_1^T \mathbf{Q}^T \tilde{C} \mathbf{Q} R)/ r_s \lambda_i + \omega_{ns}^2 = 0 \qquad (2.22)$$

Since (2.20) holds, the term $(\mathbf{e}_1^T \mathbf{Q}^T \tilde{C} \mathbf{Q} R)/ r_s$ must be complex-valued, i.e.,

$$(\mathbf{e}_1^T \mathbf{Q}^T \tilde{C} \mathbf{Q} R)/ r_s = a + jb \quad \text{and} \quad b \neq 0 \qquad (2.23)$$

Note that

$$\mathbf{Q} R = P_i$$

Denoting

$$\mathbf{e}_s^T \mathbf{Q} = Q_s$$

we have

$$s_s = \mathbf{e}_s^T R = \mathbf{e}_s^T \mathbf{Q}^T \mathbf{Q} R$$

then, from equation (2.23) we have

$$\text{Im} \left(\frac{Q_s^T \tilde{C} P_{1i}}{Q_s^T P_{1i}} \right) \neq 0 . \qquad (2.24)$$

Theorem 2.1 provides two new criteria for the determination of whether or not a system is proportionally damped. First, if all the imaginary parts of $\Re_{ij} = 0$, and secondly if $\omega_{ni} = \omega_i$, i = 1, ...n. In addition, Theorem 2.1 also reassures us the sufficient and necessary relationship between the complex damping coefficient and the damping property of the system.

From equation (2.10b), we have a simple but important corollary:

Corollary 2.1: For a damped system, if its i^{th} complex damping coefficient is real-valued, then its i^{th} undamped natural frequency is equal to that of the corresponding proportionally damped system.

In summary, when a system has non-zero imaginary damping, the system must be non-proportionally damped. If a system is non-proportionally damped, then it must have non-zero imaginary damping. There is no single exception. Analyzing a generally damped system, we can use the concept of CVS to transform the system into a SDOF virtual mode in order to examine its energy relationship. To do that, complex damping is a necessary tool.

2.3 ENERGY CONSIDERATIONS OF COMPLEX DAMPING

In the above section, we have defined a mathematical quantity, complex damping, to describe the damping properties of a system. The physical meaning of complex damping can be explored through energy considerations.

2.3.1 SYSTEM INVARIANTS

In energy-related formulations, equations are established containing certain parameters that are invariants. In this section, we first consider the invariants of a proportionally damped systems. Then, they are extended to non-proportionally damped systems.

Traditionally, the "undamped natural frequency" is defined by the quantity ω_i in equation (1.26). Corollary 2.1 supports this definition for a proportionally damped system, since no matter how the damping matrix changes, the quantity ω_i remains unchanged. However, if the system is non-proportionally damped, the quantity ω_i may be affected by damping, according to Theorem 2.1. Thus the natural frequency may no longer be "undamped". For the sake of coherence, we continue to use the traditional definition for non-proportionally damped systems.

Recall from Chapter 1, a general damping matrix can be written as the summation of a proportional component and a non-proportional component

$$\mathbf{C} = \mathbf{C}_p + \mathbf{C}_n \tag{2.25}$$

With the help of the proportional portion (C_p), we can obtain all the exact damping ratios of the system, ξ_i's, by means of the relationship:

$$\xi_i = \frac{d_{ii}}{2|\lambda_i|} = \frac{d_{ii}}{2\,\omega_i} \tag{2.26}$$

where d_{ii} is the iith entry of matrix $\mathbf{Q}^T\mathbf{C}_p\mathbf{Q}$, which is a real number.

Now, consider the ith eigenvalue of the system with damping $\tilde{\mathbf{C}}$, denoted by $\lambda_i(\tilde{\mathbf{C}})$, and that of the system with damping $\mathbf{C}_p$, denoted by $\lambda_i(\mathbf{C}_p)$. Both systems have the same generalized stiffness matrices $\tilde{\mathbf{K}}$. The second system is referred to as the *corresponding proportionally damped system* or simply the *corresponding system*, of the first system. We denote the first system by $\mathbf{H}(\tilde{\mathbf{C}})$ and the corresponding system by $\mathbf{H}(\mathbf{C}_p)$. Since both systems have the same damping ratio and undamped natural frequency, we know that $\lambda_i(\tilde{\mathbf{C}}) = \lambda_i(\mathbf{C}_p)$. This is true for all the corresponding eigenvalues. If $b_i = 0$, or $\omega_{ni} = \omega_i$, for all i = 1,..,n, then the system $\mathbf{H}(\tilde{\mathbf{C}})$ and the corresponding proportionally damped system $\mathbf{H}(\mathbf{C}_p)$ will have the same eigenvalue matrices.

Next let us consider some invariants of non-proportionally damped system $\mathbf{H}(\tilde{\mathbf{C}})$ by comparing it with the corresponding system $\mathbf{H}(\mathbf{C}_p)$. We know that all the three different formulas of state matrices of the system are similar matrices. Namely,

$$\mathbf{H} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & 0 \end{bmatrix}$$

and

$$\mathbf{H} = \begin{bmatrix} -\tilde{\mathbf{C}} & -\tilde{\mathbf{K}} \\ \mathbf{I} & 0 \end{bmatrix} \tag{2.27}$$

and

$$\mathbf{H} = \begin{bmatrix} -\tilde{\mathbf{C}} & j\,\mathbf{K}_s \\ j\,\mathbf{K}_s & 0 \end{bmatrix}$$

are all similar matrices and therefore have an identical determinant. In the following, we use the term " determinant of the state matrix" but do not

specify which one of the above three state matrices.

First, consider the proportionally damped system. We have the following corollary.

Corollary 2.2: For a M-C-K system with proportional damping, no matter how the damping matrix **C** changes, as long as the system is proportionally damped, all undamped natural frequencies remain unchanged. That is,

$$\omega_i = \text{constant} \quad i = 1,...,n \tag{2.28}$$

This corollary is a direct deduction from Theorem 2.1, condition (4). Thus, for any proportionally damped system, if the mass and stiffness matrices remain unchanged while only the damping matrix varies, that system will have the invariant undamped natural frequencies. Also from Theorem 2.1, we know that equation (2.24) is invalid if the system is non-proportionally damped.

Lemma 2.1: The determinant of the state matrix and that of the corresponding generalized stiffness matrix are identical. That is,

$$\det(\mathbf{H}) = \det(\tilde{\mathbf{K}}) \tag{2.29}$$

This follows the second expression of **H** matrix of equation (2.27).

Corollary 2.3 For a M-C-K system, if only the damping matrix **C** changes while both the **M** and **K** matrices remain unchanged, the product of all undamped natural frequencies also remains unchanged. That is,

$$\prod_{i=1}^{2N} \omega_i = \text{constant} \tag{2.30}$$

With the help of Lemma 2.1, this result is clear, since

$$\prod_{i=1}^{2N} \omega_i = \det(\mathbf{H}) = \det(\tilde{\mathbf{K}})$$

The invariants are useful in the energy analyses of generally damped systems.

2.3.2 ENERGY DISSIPATION

The concept of virtual mode enables us to examine the dynamic response of MDOF systems in an n dimensional space utilizing SDOF systems. Because there may exist certain phase shifts among these "SDOF systems". It is convenient to use complex-valued solutions to describe the dynamic behaviors of such systems. In the following, first, a SDOF system with real-valued damping is considered to examine energy dissipation. Then an imaginarily damped SDOF system is examined for energy transfer. These two cases are then combined to formulate the complex-valued damping coefficient of a SDOF system.

Consider a SDOF system with free decay vibration:

$$m\,x'' + c\,x' + k\,x = 0 \qquad (2.31)$$

where m, c, and k are considered to be modal mass, damping and stiffness. We rewrite it in the form of (2.32):

$$x'' + 2\xi\omega\,x' + \omega^2\,x = 0 \qquad (2.32)$$

where ω and ξ are defined as in (1.4) and (1.5) of Chapter 1. The solution of equation (2.32) can be written as (see Inman, 1989 for instance)

$$x_s = a\,\exp(-\xi\omega t)\,\sin(\sqrt{1-\xi^2}\,\omega\,t + \theta)$$

where

$$a = \frac{\sqrt{(v_o + \xi\omega\,x_o)^2 + (x_o\sqrt{1-\xi^2}\,\omega)^2}}{\sqrt{1-\xi^2}\,\omega} \quad \text{and} \quad \theta = \tan^{-1}\left[\frac{x_o\sqrt{1-\xi^2}\,\omega}{v_o + \xi\omega\,x_o}\right]$$

where x_o and v_o are the initial displacement and the initial velocity respectively, and the subscript s denotes the solution is in sine form. We may also choose another set of initial conditions to have the cosine form of solution with the same amplitude a and phase θ, that is,

$$x_c = a\,\exp(-\xi\omega t)\,\cos(\sqrt{1-\xi^2}\,\omega\,t + \theta)$$

Now, to form a complex-valued response, let the solution be the combination of x_c and $j\,x_s$, that is, $x = x_c + j\,x_s$. We can prescribe the third set of initial

condition so that the amplitude equals one and the phase angle equals zero. Then

$$x = \exp(\lambda t) \tag{2.33}$$

where $\lambda = -\xi\omega \pm j \sqrt{1 - \xi^2}\,\omega$. Let us first take $\lambda = -\xi\omega + j \sqrt{1 - \xi^2}\,\omega$.

Consider the work done by the inertial force, W_m, the work done by damping force, W_c, and the work done by spring force, W_k. Note that since the quantity x in (2.33) is in general complex-valued, the work will be expressed in complex value as well. For a SDOF system in physical domain, we often do not use the complex-valued solution. However, using the SDOF system to describe a vibration mode of a generally damped MDOF system, we employ the complex-valued response. The work done can be considered as virtual work because the quantity x can be treated as a virtual displacement. For simplicity, we call it the *complex virtual work* or simply the *complex work*. Consider the work done in one cycle of motion, with period of $T = 2\pi/\omega$. We have

$$\begin{aligned} W_m &= \int_o^{2\pi/\omega} x'' \, x \, dt = \frac{\lambda}{2} \eta \\ W_c &= \int_o^{2\pi/\omega} 2\xi\omega \, x' \, x \, dt = 2 \xi\omega \eta \\ W_k &= \int_o^{2\pi/\omega} k \, x \, x \, dt = \frac{\omega^2}{2 \lambda} \eta \end{aligned} \tag{2.34}$$

where η is a complex number

$$\eta = (\exp(2\lambda T) - 1) \tag{2.35}$$

Now rewrite η in the following form

$$\eta = \eta_o \, e^{j\phi} \tag{2.36}$$

in a regular complex plane. We denote the regular plane by $\mathbb{C}_p^{(n)}$
To simplify the representation in the regular complex plane, we adopt another

modified complex plane $\mathbb{C}_p^{(m)}$ in which the unit of measurement is η_o (instead of unity) and the line with angle ϕ in plane $\mathbb{C}_p^{(n)}$ is set to be the x-axis (see Figure. 2.2.). Suppose that a complex-valued number $\alpha = \alpha_0 e^{j\Psi_o}$ in the regular complex plane is transformed into the $\mathbb{C}_p^{(m)}$ plane. Its module α_0 will be multiplied by factor η_o and its phase will be shifted by ϕ_o in a conuterclock-wise direction, i.e.,

$$\alpha_{\text{transfermed}} = (\alpha_0\ \eta_0)\ \ e^{j(\Psi_o + \phi_o)}$$

Now, in the modified plane, we have

$$\left.\begin{aligned} W_m &= \frac{\lambda}{2} = \frac{1}{2}\left(-\xi\omega + j\sqrt{1-\xi^2}\,\omega\right) \\ W_c &= \xi\omega \\ W_k &= \frac{\omega}{2\lambda} = \frac{1}{2}\left(-\xi\omega - j\sqrt{1-\xi^2}\,\omega\right) \end{aligned}\right\} \qquad (2.37)$$

The energy equation (2.37) satisfies the law of conservation of energy with dissipative forces, or the law of virtual work. That is,

$$W_m + W_c + W_k = 0 \qquad (2.38)$$

Furthermore, the dissipated energy W_c is real-valued. It can be seen on the x-axis of the $\mathbb{C}_p^{(m)}$ plane. However, both W_m and W_k are complex-valued. The sum of their real parts $\xi\omega$ is the energy lost from the system in the specific cycle of motion (energy dissipation). And, both W_m and W_k will have an angle γ with respect to the x-axis, which is defined as the *loss angle*. If the damping ratio ξ is sufficiently small, we can have the following relationship:

$$\gamma = \xi = \tan\left(\frac{Im(\lambda)}{Re(\lambda)}\right) = \tan\left[\frac{Im(W_m)}{Re(W_m)}\right] = \tan\left[\frac{Im(W_k)}{Re(W_k)}\right] \qquad (2.39)$$

The three tangent forms are called *loss tangents*.

Theorem 2.2 For a SDOF system with real valued damping coefficient, denoted by (2.32), the damping ratio equals to the ratio of work done by the damping force and the geometric sum of the work done by the inertia and the spring

forces during a cycle of motion. That is,

$$\xi = \frac{W_c}{2\sqrt{W_m W_k}} \tag{2.40}$$

Equations (2.39) and (2.40) can also be obtained from Figure 2.2.

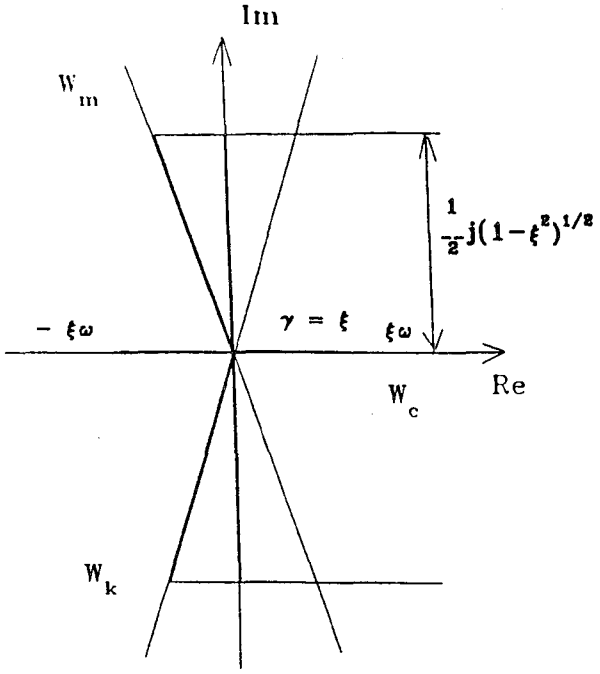


Figure 2.2 Complex $\mathbb{C}_p^{(m)}$ plane

The module of the complex work W_m (or W_k) is equal to ω. If we return to the $\mathbb{C}_p^{(n)}$ plane, and suppose at the beginning of the cycle the amplitude of the displacement is unity, then the quantity ω represents the amount of kinetic energy at this moment. For convenience, we also call the undamped natural frequency *virtual energy*, and denote it by $\mathcal{E}_v$. We can write the following corollary:

Corollary 2.4 For a SDOF system with a real-valued damping coefficient, denoted by (2.32), its virtual energy equals the square root of the generalized stiffness,

$$\mathcal{E}_v = \sqrt{m^{-1}k} = \sqrt{\omega^2} = \omega \qquad (2.41)$$

Comparing Corollary 2.2 with this corollary, we see that the virtual energy of system (2.32) is an invariant. For a system (2.32), no matter how damping coefficient c changes, the virtual energy remains unchanged. Also, an MDOF proportionally damped system can be decoupled into n real modes of n scalar equations, like equation (2.32), as stated by the following corollary:

Corollary 2.5 If an MDOF system with proportional damping is decoupled into n real modes, then each mode has invariant virtual energy, whether or not the damping matrix changes.

Finally, let us consider the imaginary parts of the complex work W_m and W_k. From the above analysis, the work done by the inertial force and the spring force contains the real part (the energy that has been dissipated), and the imaginary part (the energy that has been transferred). This imaginary part stands for conservative energy. We therefore have the energy equation with the virtual energy equal to ω.

Since this amount of imaginary energy is the conservative portion of the energy during this cycle, the conservative portion of the energy is equal to ω, if the damping is equal to zero. However, as the damping ratio ξ becomes larger, this portion of energy will become smaller by the factor $\sqrt{1 - \xi^2}$, as the dissipated energy is increased through damping. The imaginary part of the energy, or work done, considered on the $\mathbb{C}_p^{(m)}$ plane, is perpendicular to W_c . This phenomena is also true in MDOF systems. It will be discussed in Section 2.3.4 and Chapter 3.

2.3.3 ENERGY TRANSMISSION

Now, let us consider the imaginary coefficient of the velocity term in equation of motion

$$m\,x'' + j\,c\,x' + k=0$$

which is written in the familiar form,

$$x'' + 2j\,\zeta\omega\, x' + \omega^2 x = 0 \qquad (2.42)$$

where m, c and k are also real scalars.

Equation (2.42) can not be used to describe a real SDOF system. However, this equation takes on a clear physical meaning if it is used to express a virtual mode of a MDOF system.

Suppose the solution to equation (2.42) is $x = \exp(\nu t)$, then

$x' = \nu \exp(\nu t)$ and $x'' = \nu^2 \exp(\nu t)$

Therefore we have the characteristic equation as follows

$$\nu^2 + 2j\zeta\omega\,\nu + \omega^2 = 0 \qquad (2.43)$$

The solution to equation (2.42) is given by

$$\nu = j\,\omega\,(-\zeta \pm \sqrt{1 + \zeta^2}\,)$$

For simplicity, we choose

$$\nu = j\,\omega\,(-\zeta + \sqrt{1 + \zeta^2}\,)$$

Note that, the solution does not have a real part.

Again we use the concept of a modified $\mathbb{C}_p^{(m)}$ plane. With the complex number, $\eta = (\exp(2\nu T) - 1)$, we obtain the work done in one cycle with period $T = 2\pi/\omega$:

$$\left.\begin{aligned} W_m &= \frac{\nu}{2} = \frac{1}{2}\, j\,\omega\,(-\zeta + \sqrt{1 + \zeta^2}\,) \\ W_c &= j\;\zeta\omega \\ W_k &= \frac{\omega^2}{2\,\nu} = \frac{1}{2}\, j\,\omega\,(-\zeta - \sqrt{1 + \zeta^2}\,) \end{aligned}\right\} \qquad (2.44)$$

The above quantities expressed by equations (2.44) satisfy the law of conservation of energy or the law of virtual work. That is,

$$W_m + W_c + W_k = 0 \qquad (2.45)$$

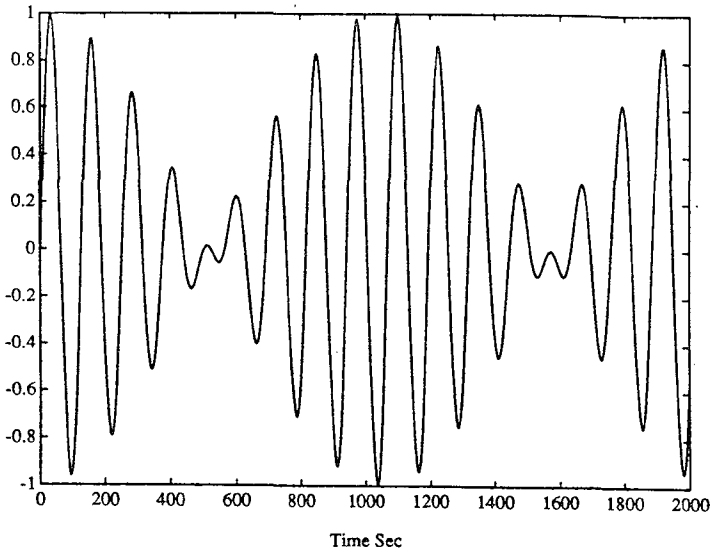


Figure 2.3 Imaginarily Damped Vibration

In addition, these quantities of work are all imaginary. Thus, during a cycle of motion, no energy is dissipated. Figure 2.3 gives the typical response time history of system (2.42) subjected to an impulse. It is seen that without real part of the damping coefficient, the amplitude of vibration will not decrease. No energy is dissipated. The above described impulse response does not behave like a SDOF system. Rather, it behaves like an MDOF system with many natural frequencies (see Figure 2.3). Such a response coincides with the changing amplitude of vibration, increased in certain cycles and decreased in other cycles of motion. It is the result of energy transfer in the system.

Energy also can be transferred within a system with real-valued damping. It is understandable that such an energy-transfer is essentially different from the case where the damping is only imaginary-valued.

The portion of energy transfer between kinetic energy and potential energy is included in both the system with real-valued damping described in (2.37) and the system with imaginary-valued damping described in (2.44). In the first case the energy transfer is $\pm\frac{1}{2}j\omega\sqrt{1 - \xi^2}$; and in the second case, it is $\pm\frac{1}{2}j\omega\sqrt{1 + \zeta^2}$.

In the first case, the energy transfer or dissipation to the the damper is represented by the part $\xi\omega$. The energy transferred to an "imaginary" device in the second case is represented by the part $j\zeta\omega$. The major difference is that the energy quantity $\xi\omega$ is changed from mechanical work to another type of

energy, in most cases thermal energy, while the quantity $j\zeta\omega$ remains in the form of mechanical work. One may think of this energy as having been transferred somewhere and stored there for a period of time and then transferred back to the mass-spring system at a later time. Based on this concept we may call the quantity $j2\zeta\omega$ the *imaginary damping coefficient* and the quantity ζ the *imaginary damping ratio*. The energy decreases in real-valued and imaginary-valued systems are also different. This is stated in the following theorem:

Theorem 2.3 For an imaginarily-damped system, denoted by (2.42), the imaginary damping ratio equals the absolute ratio of work done by the imaginary damping force and the geometric sum of work done by the inertia and spring force in a cycle. That is,

$$\zeta = \left| \frac{W_c}{2\sqrt{W_m W_k}} \right| \tag{2.46}$$

We refer to the change of energy *of* $\xi\omega$ as the *energy dissipation*, and the change of $j\zeta\omega$ as the *energy transmission* (or *energy transfer*).

It is interesting to note that with a given amount of energy transferred, the virtual energy, $\mathscr{E}_v$, i.e. the "undamped" natural frequency $|\nu|$, is no longer equal to ω. It is modified by the factor $(-\zeta \pm \sqrt{1 + \zeta^2})$. This tells us that with the energy transfer, the total energy during a cycle changes. It also suggests that for a SDOF system, equation (2.31) does not have real meaning, just like the virtual energy of SDOF system does not have real meaning. We will show in a later section that a " SDOF system" is a virtual mode for an MDOF system. For any given MDOF system, its n undamped natural frequencies are uniquely determined. Therefore, we have the following corollary:

Corollary 2.6: A given n dimensional MDOF system has one and only one set of n virtual modes with a corresponding set of n virtual energy quantities. That is

$$\mathscr{E}_{vi} = \omega_i \quad \text{for } i = 1,...,n$$

Typically, "virtual energy" is not used in energy considerations of a SDOF

system. However, for convenience, we state the following corollary using the virtual energy of a SDOF system to actually express the energy relationship between different virtual modes of an MDOF system.

Corollary 2.7: For a SDOF system (2.42) with an imaginary damping coefficient jc, if the value of jc changes, the virtual energy of the system will also be changed.

2.3.4 COMPLEX WORK

Consider now the case of a SDOF system that has both real and imaginary damping coefficients,

$$x'' + 2\,\vartheta\,\omega\,x' + \omega^2 x = 0 \qquad (2.47a)$$

where

$$\vartheta = \xi + j\zeta \qquad (2.47b)$$

and ϑ is referred to as the complex damping ratio. With the help of sections 2.3.2 and 2.3.3, let us discuss the results of complex work. Complex work consists of both the energy dissipation and the energy transmission. Moreover, its virtual energy or undamped natural frequency will not be an invariant. Also, we know that an MDOF system can be "decoupled" into n virtual modes, each with the form (2.47a). Therefore, each virtual mode (2.42) will no longer have invariant virtual energy in general. On the other hand, from Theorem 2.3, we know that the product of total virtual energy is still an invariant.

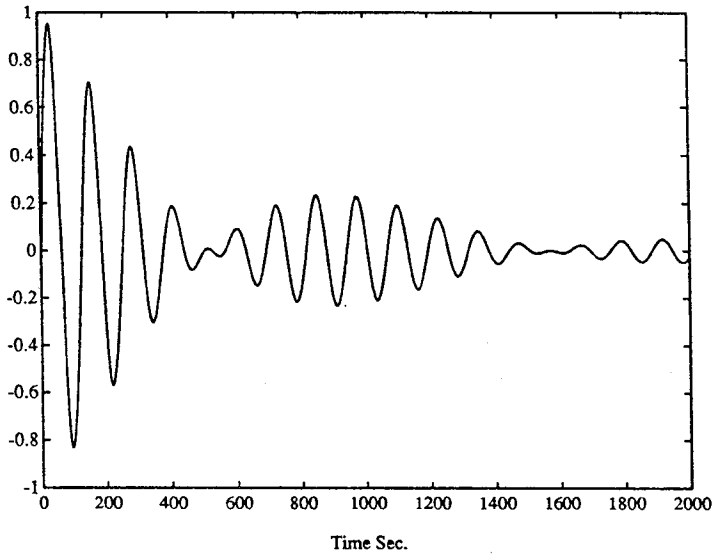


Figure 2.4 Vibration with Complex Damping

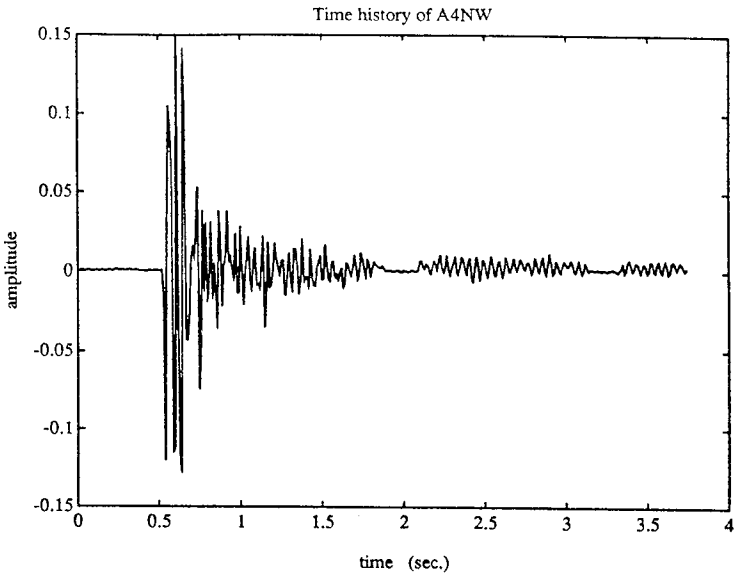


Figure 2.5 Impulse Response of a Real Structure

It is a complicated task to quantitatively describe the complex work and all its implications. Certain cases of practical significance are more straightforward. For example, complex damping of lightly damped structures will be considered in a later section.

In Figure 2.4 we show the time history of an impulse response of the complex damped system described by (2.47a). For comparison, figure 2.5 shows an impulse response from a real structure (see Figure 2.6), which is non-

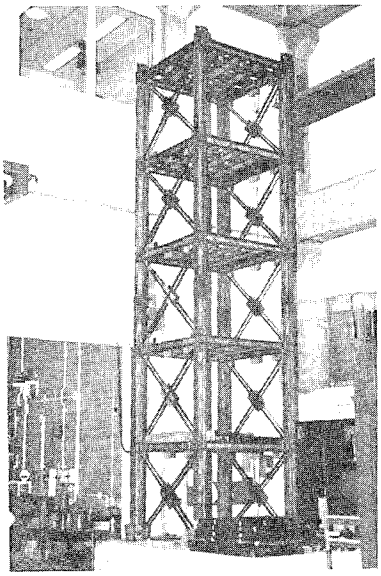


Figure 2.6 A Generally Damped Structure

proportionally damped. It can be seen that the time histories are certain combinations of the processes of energy dissipation and energy transmission.

2.4 RESPONSE ANALYSES FOR GENERALLY DAMPED SYSTEMS

Before this section, we concentrated mainly on the eigen-structure of MDOF systems.

In this section, we continue to discuss the physical nature of complex damping by examining various responses. First, based on the concept of virtual mode, we explain the physical sense of the "imaginary device." Then, we discuss the methods of solving the equation of motion of an MDOF system in n dimensional domains. Because of the aforementioned energy transfer among the virtual modes in n dimensional modal and/or physical domains, we can not directly obtain the exact responses of non-proportionally damped systems. Instead, we can either use iterative algorithms to achieve the convergent solutions (which will be discussed briefly in Section 2.4.2); or use virtual mode method to approximate the solution (which will be introduced in Part II).

2.4.1 PHYSICAL SENSE OF AN "IMAGINARY DEVICE"

The analytical relationships discussed above can be followed mathematically. Yet, the physical meaning of the "imaginary device" for a SDOF system is not clear. We now offer the explanation by using the concept of the virtual mode in MDOF systems.

Consider first a proportionally damped system which can be decoupled with the i^{th} real mode denoted by

$$u_i'' + a_i u_i' + \omega_i^2 u_i = 0 \qquad (2.48)$$

where $u_i = Q_i^T Y_i$, a_i , ω_i , Q_i and Y_i are standard notations defined before. Since the damping coefficient a_i is real-valued, there is no energy transmission but only energy dissipation. In the case of proportionally damped systems, response analyses can be made in n dimensions of both the modal and the physical domains.

In the mode domain, the system is decoupled: every normal mode is separated

from other modes. Therefore the virtual mode domain and the regular mode domain are identical. In the physical domain of a CVS, each lumped parameter subsystem contains and dissipates its own energy, but no energy is exchanged between such subsystems. Therefore, the mode shapes can all be reduced into the forms of $\mathbf{e}_i$'s. The fact that no energy is transferred between lumped subsystems defines proportional damping, the real damping coefficient and normal (real) modes in a CVS.

The above statement can be seen more clearly by using the 3-DOF CVS (Figure 2.1) as shown in Table 2.1 In Table 2.1 and in what follows, we use the term *subsystem* to denote dynamic behavior of the individual mass for the purpose of convenience. From Table 2.1, it is clear that no "imaginary device" can be found in non-proportionally damped system.

Now, we have established that, in the case of proportional damping, the CVS of this system is decoupled not only in the modal coordinate, but also in the spatial coordinate simultaneously. In other words, a proportionally damped system can be decoupled not only modally in the frequency domain, but also in the spatial domain. This is summarized as follows:

Table 2.1 Decoupling in both Frequency and Spatial Domains of a CVS

		Spatial Domain		
Frequency domain		Mass 1	Mass 2	Mass 3
	λ_1	$h_{11}[\tilde{u}_{11}'' + c_{11}\tilde{u}_{11}' + \omega_1^2\tilde{u}_{11}]=0$	$h_{21}[\tilde{u}_{21}'' + c_{11}\tilde{u}_{21}' + \omega_1^2\tilde{u}_{21}]=0$	$h_{31}[\tilde{u}_{31}'' + c_{11}\tilde{u}_{31}' + \omega_1^2\tilde{u}_{31}]=0$
	λ_2	$h_{12}[\tilde{u}_{12}'' + c_{22}\tilde{u}_{12}' + \omega_2^2\tilde{u}_{12}]=0$	$h_{22}[\tilde{u}_{22}'' + c_{22}\tilde{u}_{22}' + \omega_2^2\tilde{u}_{22}]=0$	$h_{32}[\tilde{u}_{32}'' + c_{22}\tilde{u}_{32}' + \omega_2^2\tilde{u}_{32}]=0$
	λ_3	$h_{13}[\tilde{u}_{13}'' + c_{33}\tilde{u}_{13}' + \omega_3^2\tilde{u}_{13}]=0$	$h_{23}[\tilde{u}_{23}'' + c_{33}\tilde{u}_{23}' + \omega_3^2\tilde{u}_{23}]=0$	$h_{33}[\tilde{u}_{33}'' + c_{33}\tilde{u}_{33}' + \omega_3^2\tilde{u}_{33}]=0$
		Subsystem 1	Subsystem 2	Subsystem 3

Corollary 2.8: A proportionally damped system can be decoupled in both modal

coordinates (frequency domain) and physical coordinates (spatial domain) simultaneously. Therefore in free vibrations, no energy is transferred between modes and individual mass-subsystems.

For the case of non-proportional damping, the system can not be decoupled. There can be a certain amount of energy transferred between modes or between subsystems. The "imaginary device" of a virtual mode is actually an energy reservoir formed by other virtual modes which exchange the conservative energy with this virtual mode. To see that, let us consider the following equation again:

$$X'' + \tilde{C}X' + \tilde{K}X = 0$$

Upon pre-multiplying the transpose of the eigenvector matrix of $\tilde{K}$, the $\tilde{K}$ matrix will be diagonalized, whereas the $\tilde{C}$ matrix will not be. Denote

$$\hat{C} = Q^T \tilde{C} Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots\cdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \tag{2.49}$$

and consider the equation (2.16) again

$$\lambda_i^2 r_{ii} + Q_i^T \tilde{C} P_{1i} \lambda_i + \omega_{ni}^2 r_{ii} = 0 \tag{2.50}$$

which can be rewritten as

$$\lambda_i^2 r_{ii} + (c_{i1} r_{i1} + \ldots + c_{in} r_{in}) \lambda_i + \omega_{ni}^2 r_{ii} = 0$$

or

$$\lambda_i^2 + (c_{ii} + \frac{1}{r_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij} r_{ij}) \lambda_i + \omega_{ni}^2 = 0 \tag{2.51}$$

Comparing the above equation with equation (2.21), we obtain

$$d_i = c_{ii} + \frac{1}{r_{ii}} \sum_{\substack{j=1 \\ j \neq i}}^{N} c_{ij} r_{ij} \tag{2.52}$$

However, from (2.22), d_i is a complex number, but the quantity c_{ii} is real, therefore we can denote that

$$\frac{1}{r_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij} r_{ij} = a_i^o + j\, b_i \qquad (2.53)$$

It is quite clear that the imaginary part of the complex damping coefficient comes from the term $\frac{1}{r_{ii}} \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij} r_{ij}$, which is the contribution of the other modes. Therefore, the "imaginary device" is formed for other modes. From equation (2.53), it can be seen that, these modes not only contribute the imaginary portion of the complex coefficient, but also determine the real part a_i^o . This is understandable because when energy is transferred to other modes, a certain amount of dissipation occurs. This portion of energy is dissipative and is therefore described by the real-valued quantity a_i^o .

We also conclude that these other modes generally have different natural frequencies from the i^{th} natural frequency, which result in the time history shown in Figures 2.4 and 2.5.

The same approach can be used to study non-proportionally damped systems. In equation (2.55), both terms $Q_i^T P_j$ and c_{ij} are in general not necessarily zero, for some i,j =1,..., n. This means that we can not obtain the equivalence of equation (2.59) or Table 2.1. This is stated in the following corollary:

Corollary 2.9: A non-proportionally damped system can not be decoupled in the n dimensional virtual modal domain nor in the physical domain. In the virtual modal domain, at least two modes influence each other. In the spatial domain, at least two subsystems will interact with each other.

Corollaries 2.8 and 2.9 could be introduced after the concept of complex damping is presented. However, we introduced them in this section to explain the physical meaning of why a system processes the complex mode or complex damping when energy transfer takes place between subsystems. Corollaries 2.8 and 2.9 also explain the results of the example in Section 1.5 of Chapter 1.

2.4.2 ITERATIVE SOLUTIONS OF MDOF SYSTEMS IN N DIMENSIONAL SPACE

Because of the energy transfer we cannot solve a non-proportionally damped system in n dimensional space directly. The 2n state space method is computationally intensive. Furthermore, approximating a non-proportionally damped system using a proportionally damped system may introduce large errors.

Udwadia et al (1990), Tong et al (1991) and Tsai et al (1991) have suggested certain iterative methods for the solution of MDOF systems in n dimensional space. Udwadia et al are the first to suggest the iterative approaches. Tong et al published the first unconditional convergent algorithm and provided rigorous mathematical proofs. The work is based on the complex damping theory by choosing proper ratio of energy transfer and dissipation (TD ratio). The TD ratio is adjusted by selecting certain iterative parameters to achieve an optimal convergent speed. In fact, the suggested algorithm yields a faster convergent speed than other published methods (see Tong et al 1991). The iterative methods proposed by Tong et al and the concept of TD ratio are briefly described in the following.

By using the definition of (2.49), the governing equation of a CVS can be written as:

$$Y'' + \hat{C}\, Y' + \Lambda_k\, Y = F \tag{2.54}$$

where F is the generalized force.
Denoting a diagonal matrix

$$\mathbf{D} = \begin{bmatrix} c_{11} & & & \\ & c_{22} & & \\ & & \cdots & \\ & & & c_{nn} \end{bmatrix} \tag{2.55}$$

we can have

$$Y'' + \mathbf{D}\, Y' + \Lambda_k\, Y = F + (\mathbf{D} - \hat{C})\, Y \tag{2.56}$$

Equation (2.56) can be further transformed into an iterative formula, namely

$$Y''^{(k)} + \mathbf{D}\, Y'^{(k)} + \Lambda_k\, Y^{(k)} = F + (\mathbf{D} - \hat{C})\, Y^{(k-1)} \tag{2.57}$$

where $Y^{(k)}$ denotes the k^{th} iterative result. Initially Y^0 is set to be null to obtain the first result Y^1 by (2.57). Consequently, (2.57) is used iteratively to find more accurate result (see Udwadia et al 1990).

The first step is equivalent to a proportional approximation method by omitting the off-diagonal entries in the generally damping matrix $\hat{C}$, which may introduce unacceptable errors. However, if the iteration converges, it yields correct answer of the response by the force F and initial conditions. The convergence then is the key issue of this method. Unfortunately, equation (2.57) does not guarantees unconditional convergence. In fact, it has a large chance to diverge (see Tong et al, 1991).

Intuitively, it can be understood that if the off-diagonal terms in matrix $\hat{C}$ is small, the term $(\mathbf{D} - \hat{C})\ Y^{(k-1)}$ in equation (2.57) will also be small so that the iteration can get more chance to converge. In other words, if the energy dissipation of each virtual mode is sufficiently larger in quantity than that of the energy transfer, we can make sure that equation (2.57) converges.

Define a TD ration

$$\gamma_i = \zeta_i / \xi_i \tag{2.58}$$

which is referred to as the ratio of energy transfer over the energy dissipation in the i^{th} virtual mode. The goal is to decrease the TD ratio so that the energy dissipation suppress the energy transfer. The value of the TD ration depends on the damping matrix and the external force. However, the TD ratio can be lowered by the following approach. Rewrite (2.57) as

$$Y''^{(k)} + \eta\ \mathbf{D}\ Y'^{(k)} + \Lambda_k\ Y^{(k)} = F + (\eta\mathbf{D} - \hat{C})\ Y^{(k-1)} \tag{2.59}$$

where η is a scalar. By increased proper values of η, the TD ratio can be increased. However, η cannot be too large. Otherwise,the term $\eta\ \mathbf{D}\ Y'^{(k)}$ will be much larger than $(Y''^{(k)} + \Lambda_k\ Y^{(k)})$ and we will have

$$\eta\ \mathbf{D}\ Y'^{(k)} = \eta\ \mathbf{D}\ Y'^{(k-1)} + \varepsilon \quad \text{and} \quad Y'^{(k)} = Y'^{(k-1)} + \mathbf{D}^{-1}\varepsilon/\eta$$

where ε is a small quantity. From the above equation, we can see that the convergent speed will be very slow.

It can be shown that the proper amount of energy transfer is equal to the arithmetic average of the

$$\eta = (\sigma_{max} + \sigma_{min})/2 \qquad (2.60)$$

where σ_{max} and σ_{min} are the largest and smallest eigenvalues of matrix $(\mathbf{D}^{-1}\hat{\mathbf{C}})$, respectively. η corresponds to the optimal TD ratio and guarantees convergence of the iterative scheme unconditionally.

2.4.3 COMPARISON BETWEEN VIRTUAL MODES AND TRADITIONALLY DEFINED MODES

The emphasis of this chapter is to introduce the theory of complex damping of virtual modes. To further clarify the concept of virtual mode, we now compare the newly defined virtual mode with the traditionally defined normal and complex modes.

Generally, a vibration mode is defined by certain energy standing points. There are three basic parameters to describe a mode: natural frequencies, damping ratios (generally real valued for traditionally defined mode and complex valued for virtual mode) and mode shapes. The first two parameters are unique if a system is given. The third parameter is not unique in form, but can be reduced to a uniquely defined quantity. Virtual modes of a system are obtained from virtual work and generalized Rayleigh quotients, by assigning the corresponding monic system a virtual displacement $\mathbf{Q}^T$. The same procedure can be used to locate the normal modes, if the system is proportionally damped. The complex modes are determined from state space methods, by eigen-decompositions of various state matrices. In this chapter we only use the natural frequencies and damping ratios to describe a virtual mode, but not the mode shapes.

When a system is proportionally damped, the set of virtual modes and the set of normal modes are identical. Therefore, we say that the concept of virtual

modes is a direct extension of the normal mode. Any proportionally damped system can only have n normal modes.

We define the virtual modes in n dimensional space, because it is not necessary to express them in 2n space. The half portion of an eigenstructure in complex conjugate pairs gives the whole eigen-information and the entire massage of energy transfer, which are all we need. However, the complex modes must be in 2n space.

Both virtual and complex modes yield the identical eigen-information of a non-proportionally damped system. The former can give more information about energy transfer whereas the latter do not provide these messages. From conventional modal testing, we cannot obtain the complex damping ratios.

In response calculation, the 2n state space method is well developed. Both time domain direct integration methods and frequency modal superposition methods result in exact solutions of the responses for non-proportionally damped systems. The virtual mode methods do not yield exact solutions. It can be shown that when the imaginary part b_i increases, using virtual modes to approximate the solutions introduces certain errors, though they are generally smaller than those associated with using the proportional damping assumption for non-proportionally systems. On the other hand, the state space methods are quite computationally intensive. An optimal iterative method in n dimensional space based on the complex damping theory can save the computation cost and is unconditionally convergent.

Table 3.2 Damping Ratios and the Approximations

Approximation of ϑ_o		Approximation of $\sqrt{1-\vartheta_o^2}$		
ϑ_o	$sh(\vartheta_o)$ & error	$\sqrt{1-\vartheta_o^2}$	1 & error	$1-\vartheta_o^2/2$ & error
.001	1.0000005 0.%	0.9999995	1 5e-5 %	.9999995 0.%
.01	.0100001 1.6e-5%	0.99995	1 5e-3 %	.999995 0.%
.05	.0500208 .042 %	0.9987492	1 0.125 %	.99875 -8e-5 %
.1	.1001667 .167 %	0.9949874	1 0.501 %	.995 .00126 %
.2	.201336 .668 %	0.9797959	1 2.02 %	.98 .0204 %
.25	.2526123 1.04 %	0.9682458	1 3.175 %	.96875 .0504 %
.3	.3045202 1.51 %	0.9539392	1 3.606 %	.955 .106 %

In Table 3.2, the largest error occurs when we use unity for $\sqrt{1-\vartheta_o^2}$. However, if the value of ϑ_o is less than 10 % , the error in using unity for $\sqrt{1-\vartheta_o^2}$ is no more than 0.5 %. If the value of ϑ_o is less than 30 % , approximating $\sqrt{1-\vartheta_o^2}$ by using unity only introduces an error of less than 4 %. If we approximate $(1-\vartheta_o^2/2)$ by unity the error is no more than 0.106 % . These are acceptable errors in most engineering computations. Therefore, *light damping* is when the absolute value of the damping ratio is less than 30 %. From Table 3.1, we can see that for most structures the damping ratio is normally in the range of several percent and is rarely over 10% .

The following definition is used for light damping:

$$\sqrt{1-\vartheta_o^2} \approx 1 \qquad (3.1a)$$

As a result, all the complex functions (analytical functions) of ξ: $\sqrt{1-\vartheta^2}$, $\exp(\vartheta)$, $sh(\vartheta)$ and $ch(\vartheta)$ etc, are approximated accordingly as shown in (3.1b) when they are treated as real functions.

$$\sqrt{1-\vartheta^2} \approx 1 - \vartheta^2/2$$

$$\exp(\vartheta) \approx 1 + \vartheta + \vartheta^2/2 \qquad (3.1b)$$

$$ch(\vartheta) \approx 1$$

$$sh(\vartheta) \approx \vartheta$$

Equations (3.1a) and (3.1b) are the assumptions of *light damping* used in this report.

3.2 COMPLEX DAMPING OF LIGHTLY DAMPED STRUCTURES

3.2.1 THEOREM OF COMPLEX DAMPING RATIO

With the definition of light damping we may proceed to establish the formula for the complex damping ratio for lightly damped structures. Equation (2.47a) may be rewritten in the following form:

$$u'' + (a + jb)\, u' + \omega_n^2\, u = 0 \qquad (3.2)$$

which may be thought of as representing a virtual mode in an MDOF system. For convenience, we omit the subscript i. The characteristic equation of (3.2) is given by

$$\lambda^2 + (a + jb)\,\lambda + \omega_n^2 = 0$$

From Chapter 2, this equation is known as the equation of motion for virtual modes. Upon examining this equation, a useful theorem will be introduced which gives the relationship between the real part of the complex damping ratio and the conventionally defined damping ratio, and the relationship between the imaginary part of the complex damping ratio and the natural frequency.

Reconsidering the value of λ with the light damping assumption, we have

$$\lambda = \frac{1}{2}\left[-(a+jb) \pm \sqrt{(a+jb)^2 - 4\,\omega_n^2} \right]$$

$$= \omega_n \left[-\left(\frac{a+jb}{2\omega_n} \right) \pm \sqrt{\left(\frac{a+jb}{2\omega_n} \right)^2 - 1} \right]$$

$$= j\,\omega_n \left[\left(\frac{a+jb}{2\omega_n} j \right) \pm \sqrt{1 + \left(\frac{a+jb}{2\omega_n} \right)^2} \right] \tag{3.3}$$

Introducing the assumption of light damping gives us

$$\left| \left(\frac{a+jb}{2\omega_n} j \right) \right| \ll 1 \tag{3.4}$$

and

$$\sqrt{1 + \left(\frac{a+jb}{2\omega_n} j \right)^2} \approx 1 + \frac{1}{2} \left(\frac{a+jb}{2\omega_n} j \right)^2 \tag{3.5}$$

We first take the positive part of

$$\sqrt{1 + \left(\frac{a+jb}{2\omega_n} j \right)^2}$$

Then we obtain

$$\lambda = j\,\omega_n \left[\left(\frac{a+jb}{2\omega_n} j \right) + 1 + \frac{1}{2} \left(\frac{a+jb}{2\omega_n} j \right)^2 \right] \tag{3.6}$$

Furthermore, from the light damping assumption,

$$\left(\frac{a+jb}{2\omega_n} j \right) + 1 + \frac{1}{2} \left(\frac{a+jb}{2\omega_n} j \right)^2 \approx \exp \left(\frac{ja-b}{2\omega_n} \right)$$

$$= \exp \left(j \frac{a}{2\omega_n} \right) \exp \left(\frac{-b}{2\omega_n} \right)$$

$$\approx \left[1 + j \frac{a}{2\omega_n} - \frac{1}{2} \left(\frac{a}{2\omega_n} \right)^2 \right] \exp \left(\frac{-b}{2\omega_n} \right)$$

$$\approx \left[j \frac{a}{2\omega_n} + \sqrt{1 - \left(\frac{a}{2\omega_n} \right)^2} \right] \exp \left(\frac{-b}{2\omega_n} \right)$$

we then have

$$\lambda = j\,\omega_n \left[j \frac{a}{2\omega_n} + \sqrt{1 - \left(\frac{a}{2\omega_n} \right)^2} \right] \exp \left(\frac{-b}{2\omega_n} \right)$$

$$= -\frac{a}{2\omega_n} \exp \left(\frac{-b}{2\omega_n} \right) \omega_n + j \sqrt{1 - \left(\frac{a}{2\omega_n} \right)^2} \exp \left(\frac{-b}{2\omega_n} \right) \omega_n \tag{3.7}$$

Now taking the negative part of

$$\sqrt{1 + \left[\frac{a+jb}{2\omega_n} j \right]^2}$$

and introducing it into equation (3.3), we acquire

$$\lambda = -\frac{a}{2\omega_n} \exp\left[\frac{b}{2\omega_n} \right] \omega_n - j\sqrt{1 - \left(\frac{a}{2\omega_n}\right)^2} \exp\left[\frac{b}{2\omega_n} \right] \omega_n$$

Combining with equation (2.44), we get

$$\lambda = -\frac{a}{2\omega_n} \exp\left[\frac{\mp b}{2\omega_n} \right] \omega_n \pm j\sqrt{1-\left(\frac{a}{2\omega_n}\right)^2} \exp\left[\frac{\mp b}{2\omega_n} \right] \omega_n \qquad (3.8)$$

By comparing equation (2.44) with the standard form of λ, i.e.

$$\lambda = -\xi\omega \pm j\sqrt{1-\xi^2}\,\omega$$

we have

$$\frac{a}{2\omega_n} = \xi \quad \text{or} \quad a = 2\,\xi\,\omega$$

and

$$\exp\left[\frac{\mp b}{2\omega_n} \right] \omega_n = \omega \qquad (3.9)$$

Using assumption of light damping once more,

$$\exp\left[\frac{\mp b}{2\omega_n} \right] \approx 1 \pm \sqrt{1 \mp \left(\frac{b}{2\omega_n}\right)^2}$$

yields

$$\frac{b}{2\omega_n} = \zeta \quad \text{or} \quad b = 2\,\zeta\,\omega_n$$

We can see that the term b is the same as the one used in equation (2.47).

Extending the above approximation for MDOF systems in more detail, we define that an MDOF system is a *lightly damped system*, if all of its complex damping ratios satisfy the aforementioned definition of light damping. In this case,

we may think of the equation (3.2) as the equation of motion of the i^{th} virtual mode of the system. Hence the subscript i is added to represent a designated MDOF system. That is,

$$\begin{aligned} a_i &= 2\ \xi_i \omega_{ni} \\ b_i &= 2\ \zeta_i \omega_{ni} \end{aligned} \tag{3.10}$$

and

$$\lambda_i = -\ \xi_i \exp(\mp\zeta_i)\ \omega_{ni} \pm j\sqrt{1-\xi^2}\ \exp(\mp\zeta_i)\ \omega_{ni} \tag{3.11a}$$

Sometimes, it may be more convenient to approximate λ by:

$$\lambda = j\ \omega_n \exp(\ \xi \pm j\ \zeta\ ) \tag{3.11b}$$

If we define the i^{th} *complex damping ratio* of a lightly damped system, ϑ_i, by

$$\vartheta_i = \frac{d_i}{2\omega_{ni}} = \xi_i \pm j\ \zeta_i \tag{3.12}$$

we can state the following theorem.

Theorem 3.1 For a lightly damped MDOF system, if its complex damping coefficient of the i^{th} virtual mode is given by

$$d_i = 2\ (\ \xi_i + j\ \zeta_i)\ \omega_{ni}$$

then the real part of the corresponding complex damping ratio ξ_i is just the conventionally defined damping ratio (here we use ξ_{icon} to denote the conventionally defined damping ratio), i.e.

$$\xi_i = \frac{\text{real}\ (d_i)}{2\ \omega_{ni}} = \xi_{icon} \tag{3.13}$$

The imaginary part of the complex damping ratio, ζ_i, only contributes to the change in the undamped natural frequency ω_i

$$\omega_i = \exp(\zeta_i)\ \omega_{ni}\ . \tag{3.14}$$

Theorem 3.1 states that for a lightly damped system, we can treat the real and

imaginary parts of its complex damping ratio (or complex damping coefficient) separately. It also implies that, for a lightly damped M-C-K system, the value of ω_i is determined and the sign of ζ_i is determined (so is the sign of b_i). With the definitions of the complex damping coefficient d_i (see equation (2.18)) and the concept of an eigen-matrix, we can deal with both energy dissipation and energy transmission simultaneously. Theorem 3.1 is useful because we can treat pure real damping and pure imaginary damping independently with sufficient accuracy.

3.2.2 COMPLEX WORK AND ENERGY EQUATION

Theorem 3.2 : For a lightly damped MDOF system, if its complex damping coefficient of the i^{th} virtual mode is given by

$$d_i = 2 (\xi_i + j \zeta_i) \omega_{ni}$$

then the complex work done by inertia, damping and spring forces during one cycle of motion in the $\mathbb{C}_p^{(m)}$ plane is given by equation (3.15) (note that $\eta_i = (\exp(2\lambda_i T) -1)$):

$$\left.\begin{aligned} W_{mi} &= \frac{\lambda_i}{2} = \frac{1}{2}\omega_{ni} \exp(-\zeta_i) (-\xi_i + j \sqrt{1 - \xi_i^2}) \\ W_{ci} &= (\xi_i + j \zeta_i) \omega_{ni} \\ W_{ki} &= \frac{\omega^2}{2} = \frac{1}{2}\omega_{ni} \exp(\zeta_i) (-\xi_i - j \sqrt{1 - \xi_i^2}) \end{aligned}\right\} \qquad (3.15)$$

Theorem 3.2 can be proven by using equations (3.11) and (3.12).

With the help of this theorem, the connection between the two parts of the complex work quantities can be analyzed. First, consider the real part of the complex work, i.e.

$$\begin{aligned} \mathrm{Re}(W_{mi}) &= - \frac{1}{2} \xi_i \omega_{ni} \exp(-\zeta_i) \\ \mathrm{Re}(W_{ki}) &= - \frac{1}{2} \xi_i \omega_{ni} \exp(\zeta_i) \end{aligned} \qquad (3.16)$$

and

$$\mathrm{Re}(W_{ci}) = \xi_i \omega_{ni}$$

The sum of the first two terms is the work done by conservative forces

$$\mathrm{Re}(W_{mi}) + \mathrm{Re}(W_{ki}) = -\frac{1}{2}\xi_i \omega_{ni} \exp(-\zeta_i) - \frac{1}{2}\xi_i \omega_{ni} \exp(\zeta_i)$$

$$= -\xi_i \omega_{ni} \mathrm{ch}(\zeta_i) \qquad (3.17)$$

With the light damping assumption, $\mathrm{ch}(\zeta_i) \approx 1$, equation (3.17) reduces to

$$\mathrm{Re}(W_{mi}) + \mathrm{Re}(W_{ki}) = -\xi_i \omega_{ni} \qquad (3.18)$$

Hence, the total work done is

$$\mathrm{Re}(W_{mi}) + \mathrm{Re}(W_{ki}) + \mathrm{Re}(W_{ci}) = 0 \qquad (3.19)$$

Equation (3.19) verifies the first part of Theorem 3.1, by showing the energy equality of the real parts of virtual complex work.

Next, consider the imaginary parts of the work done, that is,

$$\mathrm{Im}(W_{mi}) = \frac{1}{2}\omega_{ni}\exp(-\zeta_i)\sqrt{1-\xi_i^2}$$

$$\mathrm{Im}(W_{ki}) = -\frac{1}{2}\omega_{ni}\exp(\zeta_i)\sqrt{1-\xi_i^2}$$

and (3.20)

$$\mathrm{Im}(W_{ci}) = \zeta_i \omega_{ni}$$

Again, using the light damping assumption and letting $\exp(\pm \zeta_i) \approx 1 \pm \zeta_i$ (from Table 3.1 this treatment is acceptable) and $\sqrt{1-\xi_i^2} \approx 1$ we have

$$\mathrm{Im}(W_{mi}) + \mathrm{Im}(W_{ki}) = \frac{1}{2}\omega_{ni}\exp(-\zeta_i)\sqrt{1-\xi_i^2} - \frac{1}{2}\omega_{ni}\exp(\zeta_i)\sqrt{1-\xi_i^2} =$$

$$\left[-\frac{1}{2}\omega_{ni}\zeta_i + \frac{1}{2}\omega_{ni}(-\zeta_i)\right] + \left[\frac{1}{2}\omega_{ni}\sqrt{1-\xi_i^2} - \frac{1}{2}\omega_{ni}\sqrt{1-\xi_i^2}\right] \qquad (3.21)$$

The term contained in the first bracket equals $-\zeta_i\omega_{ni}$, which is in balance

with $\zeta_i \omega_{ni}$, the imaginary part of W_c. We can see that the term contained in the first bracket is the energy transmission referred to as the "imaginary device" in Chapter 2. The second bracketed term, which equals zero, is the energy transmission between potential energy and kinetic energy, which causes the oscillation. In summary, for the imaginary part of the virtual work, we can write

$$\mathrm{Im}(W_{mi}) + \mathrm{Im}(W_{ki}) + \mathrm{Im}(W_{ci}) = 0 \tag{3.22}$$

Equation (3.22) verifies the second part of Theorem 3.1 by showing the equality of the energy conservation.

Equations (3.21) and (3.22) deal with the sum of the real part and the imaginary part of the virtual work, respectively. The second energy equation may be obtained by using equation (3.11b)

$$W_{mi} + W_{ki} + W_{ci} = \frac{\lambda_i}{2} + \frac{\omega_{ni}^2}{2\lambda_i} + (\xi_i + j\zeta_i)\omega_{ni}$$

$$= \omega_{ni}\left[\frac{\exp(-\xi_i - j\zeta_i) - \exp(\xi_i + j\zeta_i)}{2}\right] + (\xi_i + j\zeta_i)\omega_{ni}$$

$$= \omega_{ni}\,\mathrm{sh}(-\xi_i - j\zeta_i) + (\xi_i + j\zeta_i)\omega_{ni}$$

$$= -(\xi_i + j\zeta_i)\omega_{ni} + (\xi_i + j\zeta_i)\omega_{ni} = 0$$

With Theorem 3.2, we now consider the relationship between the complex damping ratio and the complex work done.

Theorem 3.3: For a lightly damped MDOF system, if its complex damping ratio of the i^{th} virtual mode is given by

$$\vartheta_i = \xi_i + j\zeta_i$$

then ξ_i is the ratio of the real part of the virtual work done by the damping force and the geometric sum of the virtual work done by the spring and inertial forces in one cycle of motion. Further, ζ_i is the ratio of the imaginary part of the virtual work done by the damping force and the geometric

sum of the virtual work done by the spring and inertial forces. These ratios are mathematically given by equations (3.23) and (3.24):

$$\xi_i = \frac{\mathrm{Re}\,(\,W_{c\,i}\,)}{2\sqrt{W_{m\,i}\,W_{k\,i}}} \tag{3.23}$$

$$\zeta_i = \frac{\mathrm{Im}\,(\,W_{c\,i}\,)}{2\sqrt{W_{m\,i}\,W_{k\,i}}} \tag{3.24}$$

PROOF

Since $W_{ci} = (\xi_i + j\zeta_i)\,\omega_{ni}$, we have

$\mathrm{Re}(W_{ci}) = \xi_i\,\omega_{ni}$

and

$\mathrm{Im}(W_{ci}) = \zeta_i\,\omega_{ni}$

Now, consider the term of geometric sum of W_{mi} and W_{ki}, namely

$$2\sqrt{W_{m\,i}\,W_{k\,i}} = 2\left[\,\{\,\frac{1}{2}\,\omega_{ni}\exp(\,\zeta_i)\,(\,-\xi_i - j\sqrt{1-\xi_i^{\,2}}\,)\,\}\right.$$

$$\left.\{\frac{1}{2}\;\omega_{ni}\,\exp(-\zeta_i)\,(\,-\xi_i + j\,\sqrt{1-\xi_i^{\,2}}\,)\,\}\,\right]^{1/2}$$

$$= \omega_{ni}\,\mathrm{ch}(\,\xi_i\,)^{1/2} \tag{3.25}$$

With the light damping assumption, $\mathrm{ch}(\,\xi_i\,)^{1/2} \approx 1$ we have

$$2\sqrt{W_{mi}\,W_{ki}} = \omega_{ni}$$

Alternatively we can directly use equation (3.15), i.e.

$$W_{mi} = \frac{\lambda_i}{2} \quad \text{and} \quad W_{ki} = \frac{\omega_{n\,i}^{\,2}}{2\lambda_i}$$

to arrive at

$$2\sqrt{W_{mi}\,W_{ki}} = 2\left(\frac{\lambda_i}{2}\,\frac{\omega_{ni}^2}{2\lambda_i}\right)^{1/2} = \omega_{ni}$$

Hence, we have

$$\xi_i = \frac{\mathrm{Re}(W_{ci})}{2\sqrt{W_{mi}\,W_{ki}}}$$

and

$$\zeta_i = \frac{\mathrm{Im}(W_{ci})}{2\sqrt{W_{mi}\,W_{ki}}}$$

■

This theorem states that the real part of the complex damping ratio is the energy dissipative ratio in one cycle of motion and that the imaginary part is the energy transfer ratio in the same cycle.

3.3 EXAMPLE OF LIGHT DAMPING ASSUMPTION

In this section we consider a numerical example of a lightly damped system. We will compare the eigenvalues, damping ratios and natural frequencies of the original system and those quantities calculated with the light damping assumption.

Suppose we have the following M-C-K system:

$$\mathbf{M} = \begin{bmatrix} .3333 & .1667 \\ .1667 & .3333 \end{bmatrix},\quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} 31.68 & -31.68 \\ -31.68 & 89.76 \end{bmatrix} \qquad (3.26)$$

Since $\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}$, the system is non-proportionally damped. The eigenvalues of the system are as follows

$-1.1049 \pm 6.2147j,\ -6.8951 \pm 22.5082j$

The undamped natural frequencies and damping ratios are

$f_1 = 1.0046$ (Hz), $f_2 = 3.7466$ (Hz) and $\xi_1 = .1750$, $\xi_2 = .2929$

The system is lightly damped (at the upper limit) according to our assumption.

From equation (2.2), we have

$$\tilde{C} = \begin{bmatrix} 9.7321 & -7.0000 \\ -7.0000 & 6.2679 \end{bmatrix} \quad \tilde{K} = \begin{bmatrix} 205.6425 & -248.1600 \\ -248.1600 & 406.8753 \end{bmatrix} \qquad (3.27)$$

From equation (3.27), we can find the eigenvalue and eigenvector matrix of $\tilde{K}$ as follows:

$$\Lambda_k = \begin{bmatrix} 38.4654 & \\ & 573.0146 \end{bmatrix}$$

$$Q = \begin{bmatrix} .8294 & -.5587 \\ .5587 & .8294 \end{bmatrix}$$

The mode shape matrix of the system can be calculated

$$P_1 = \begin{bmatrix} .6027 + .0310j & -.4113 + .1774j \\ .4005 + .0650j & .5864 - .0101j \end{bmatrix}$$

Then, the complex damping ratios can be calculated as follows:

$$\vartheta_1 = .1751 \pm .0173j \qquad \vartheta_2 = .2929 \pm .0168j$$

The real parts of ϑ_1 and ϑ_2 are very close to the damping ratios ξ_1 and ξ_2.

By using the imaginary parts, the undamped natural frequencies are given by $\omega_1 = (38.4654)^{1/2}\exp(.0173)=6.3105$ and $\omega_2=(573.0146)^{1/2}\exp(-.0168) = 23.5588$ Comparing with the exact values, 6.3122 and 23.5406, the errors are .000269 and -.000773 respectively, which are small. Note that the system has a damping ratio of 0.2929 which is close to the upper limit 0.3 of the light damping assumption. Even at the upper limit, the error is still acceptable. For systems with smaller damping ratios, we can expect much smaller errors.

The above example shows clearly that the imaginary parts of the damping ratios affect the undamped natural frequencies. For any non-proportionally damped MDOF system, if the lowest (highest) mode is complex, the energy transfer will always increase (decrease) the value of the natural frequency

(frequencies) from ω_{n1} (ω_{nn}). However, the natural frequencies in between the lowest and highest values may either increase or decrease. This is shown in Figure 3.1.

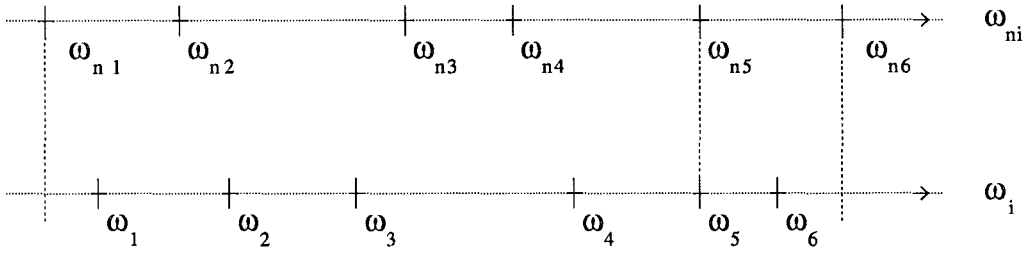


Figure 3.1 Distribution of ω_{ni} and ω_i

Now we calculate the products of all the natural frequencies. It is shown that the values are invariant, namely,

$\lambda_{k1}\ \lambda_{k2} = \det(\tilde{\mathbf{K}}) = 22079.692$

and

$\omega_1^2\ (\ 1-\xi_1^2)\ \ \omega_2^2\ (1-\xi_2^2) = \det(\mathbf{H}) = 22079.692$

If we use the light damping approach, the product of the natural frequency is 22090.738, with an error of 0.0005, which is quite acceptable.

The light damping assumption allows us to treat the real-valued energy dissipation and the imaginary-valued energy transfer separately. We then have the *energy equation* (3.15) for virtual modes. This treatment is useful not only in theoretical analysis, but also in many engineering applications. The linear property of the damping ratios discussed in the following section can be used directly for damper design in vibration control.

3.4 DAMPING RATIO CALCULATIONS

3.4.1 LINEARITY APPROACH

For lightly damped systems the damped natural frequencies are approximately equal to the undamped natural frequencies. That is if the value ζ_i is sufficiently small (from Table 3.1, this is true for most structures), i.e.,

$$\exp(\zeta_i) \approx 1 \tag{3.28}$$

then

$$\omega_i = \exp(\zeta_i)\ \omega_{ni} \approx \omega_{ni} \tag{3.29}$$

With the help of equation (3.29), we can see that if two lightly damped systems, $\mathbf{H}_1$ and $\mathbf{H}_2$, have the same mass and stiffness, then,

$$\Lambda_{Im}^{(1)} \approx \Lambda_{Im}^{(2)} \tag{3.30}$$

Further, if we can trace the change of ω_i's from ω_{ni}'s one by one, or if the system has no close space natural frequencies (see Warburton, 1967), we have

$$\lambda_{iIm}^{(1)} \approx \lambda_{iIm}^{(2)}\ ,\ i = 1, \ldots 2n \tag{3.31}$$

Without sufficient modal information, we cannot always trace the change. However, in the following, it is assumed that when (3.30) holds, we can always obtain (3.31).

Here, we arrange all the eigenvalues in the following order

$$\lambda_1,\quad \lambda_2,\quad \ldots\ldots\quad \lambda_n$$

if (3.32)

$$\omega_1 \le\ \omega_2 \le \ldots\ldots \le \omega_n$$

For proportional systems, we have the following lemma.

Lemma 3.1 For a lightly damped system **H** with the proportional damping matrix $\mathbf{C}_C$, which can be shown as the sum of two proportional damping matrices $\mathbf{C}_{1C}$ and $\mathbf{C}_{2C}$, i.e.

$$\mathbf{C}_C = \mathbf{C}_{1C} + \mathbf{C}_{2C} \tag{3.33}$$

then, we have

$$\Lambda_{Re} \quad = \quad \Lambda_{Re}^{(1C)} + \Lambda_{Re}^{(2C)} \tag{3.34}$$

or

$$\lambda_{iRe} = \lambda_{iRe}^{(1C)} + \lambda_{iRe}^{(2C)}, \quad i = 1,...2n \tag{3.35}$$

and,

$$\Lambda_{Im} \approx \Lambda_{Im}^{(1C)} \approx \Lambda_{Im}^{(2C)} \tag{3.36a}$$

or

$$\lambda_{iIm} \approx \lambda_{1Im}^{(1C)} \approx \lambda_{1Im}^{(2C)}, \quad i = 1,...2n \tag{3.36b}$$

for the subsystem $\mathbf{H}_{1C}$ and $\mathbf{H}_{2C}$.

This lemma states that for a system with proportional damping, if it consists of two subsystems whose damping are also proportional, the real part of its eigenvalues can be seen as a sum of the corresponding real parts of the two subsystems. In other words, for the damping ratios, we can write

$$\xi_i^{(C)} = \xi_i^{(1C)} + \xi_i^{(2C)}, \quad i = 1,.....2n \tag{3.37}$$

where the superscript (.) stands for the system (.).

This lemma further states that if a certain amount of damping is to be added to a structure, equation (3.37) can be used to determine the required damping in a simple fashion. In Section 3.4.2, some examples will be provided to illustrate this point. Lemma 3.1 can be expressed for systems with general non-proportional damping.

Lemma 3.2: For any lightly damped system with damping matrix **C** and state matrix **H**, separating the **C** matrix by means of equation (1.11a), we can generate two subsystems $\mathbf{H}_P$ and $\mathbf{H}_N$. From these, we can obtain

$$\Lambda_{Im} \approx \Lambda_{Im}^{(P)} \tag{3.38a}$$

or

$$\lambda_{iIm} \approx \lambda_{1Im}^{(P)}, \quad i = 1,...2n \tag{3.38b}$$

and,

$$\Lambda_{Re} \approx \Lambda_{Re}^{(P)} + \Lambda_{Re}^{(N)}$$

or

$$\lambda_{iRe} \approx \lambda_{iRe}^{(P)} + \lambda_{iRe}^{(N)}, \quad i = 1,...2n \tag{3.39}$$

This lemma is easily understood by noting that systems **H** and system $\mathbf{H}_P$ have the identical damping ratios for each mode, and that **H** and $\mathbf{H}_P$ have almost the same natural frequencies per each mode.

Theorem: 3.4 Given a lightly damped system **H**, the damping matrix **C** can be seen as a sum of two matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, i.e.

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$$

then, for the subsystems $\mathbf{H}_1$ and $\mathbf{H}_2$, we have

$$\Lambda_{Im} \approx \Lambda_{Im}^{(1)} \approx \Lambda_{Im}^{(2)} \tag{3.40a}$$

or

$$\lambda_{iIm} \approx \lambda_{iIm}^{(1)} \approx \lambda_{iIm}^{(2)}, \quad i = 1,...2n \tag{3.40b}$$

and,

$$\Lambda_{Re} \approx \Lambda_{Re}^{(1)} + \Lambda_{Re}^{(2)} \tag{3.41a}$$

or

$$\lambda_{iRe} \approx \lambda_{iRe}^{(1)} + \lambda_{iRe}^{(2)}, \quad i = 1,...2n \tag{3.41b}$$

PROOF

The first part of the theorem described by (3.40a) and (3.40b) is obvious.

Consider the second part, namely equations (3.41a) and (3.41b). Let

$$\mathbf{C}_1 = \mathbf{C}_{1P} + \mathbf{C}_{1N}$$

and

$$\mathbf{C}_2 = \mathbf{C}_{2P} + \mathbf{C}_{2N}$$

then,

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 = (\mathbf{C}_{1P} + \mathbf{C}_{2P}) + (\mathbf{C}_{1N} + \mathbf{C}_{2N})$$

According to Lemma 3.2, we have

$$\Lambda_{Re} \approx \Lambda_{Re}^{(1P + 2P)} = \Lambda_{Re}^{(1P)} + \Lambda_{Re}^{(2P)} \approx \Lambda_{Re}^{(1)} + \Lambda_{Re}^{(2)}$$

■

Corollary 3.1 Suppose that two lightly damped systems $\mathbf{H}_1$ and $\mathbf{H}_2$ have the same mass and stiffness matrices and the first subsystem has a damping matrix $\mathbf{C}_1$, and the damping matrix $\mathbf{C}_2$ of the second subsystem is given by

$$\mathbf{C}_2 = \beta\, \mathbf{C}_1$$

where β is a constant.

Then, for the subsystem $\mathbf{H}_1$ and $\mathbf{H}_2$, we have

$$\Lambda_{Im}^{(1)} \approx \Lambda_{Im}^{(2)} \tag{3.42a}$$

or

$$\lambda_{iIm}^{(1)} \approx \lambda_{iIm}^{(2)} \quad , \quad i = 1,...2n \tag{3.42b}$$

and

$$\Lambda_{Re}^{(2)} \approx \beta\, \Lambda_{Re}^{(1)} \tag{3.43a}$$

or

$$\lambda_{iRe}^{(2)} \approx \beta\, \lambda_{iRe}^{(1)} \quad , \quad i = 1,...2n \tag{3.43b}$$

3.4.2 APPLICATIONS AND EXAMPLES

In this section, we present some application examples of Theorem 3.4. A simplified damping matrix representation for the theory of lightly damping. is given in Chapter 4.

Example I

Figure 3.2 shows a structure with added dampers. Figure 3.3 shows the damper.

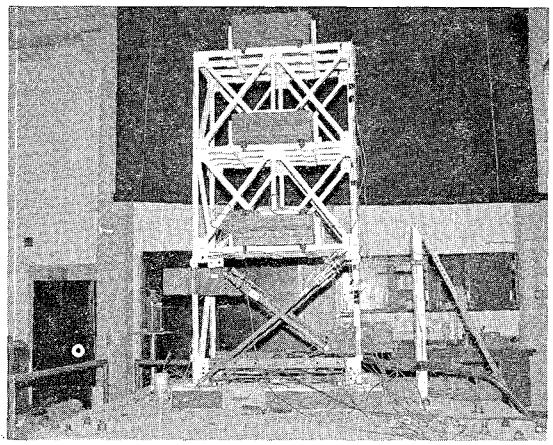


Figure 3.2 Structure with Dampers

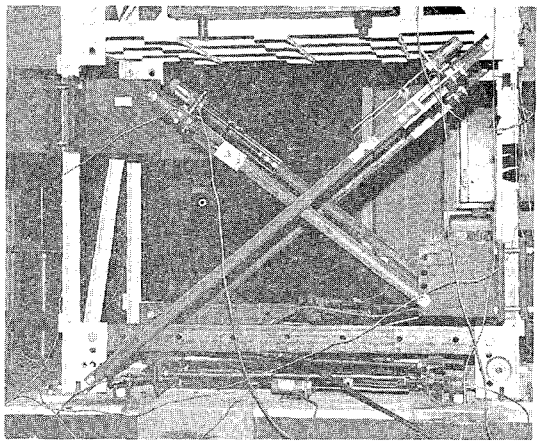


Figure 3.3 Dampers

The purpose of adding damping to a structure is to increase the damping ratio and thereby to control the responses due to seismic excitations. For this test structure, we are given the maximum allowable displacements of the structure and the finite element model of the structure, which is treated as a 3-DOF system without any dampers. The damping ratios of the base structure are listed in Table 3.3 (interested readers can find this example in NCEER Tech. Report #88-0018).

Table 3.3 Damping Ratios of the Base Structure

Mode	I	II	III
damping ratio	.0102	.0087	.0079

We are also given certain properties of the dampers; most importantly, the energy dissipation due to its relative displacement are given by

$$W_c = \alpha x_r^2 \tag{3.44}$$

where W_c stands for the energy dissipated during a cycle of motion, α is a proportional coefficient, which is a function of temperature and frequency and is usually supplied by the manufacturer; x_r is the relative displacement

between the two ends of the damper.

With the above information, it is not difficult to calculate the damping ratios of the structure with added dampers. However, we cannot yet obtain the damping coefficients of the system from equation (3.44) or from other approaches available in the literature. We therefore do not have the damping matrix (even though we knew the dissipated energy W_c's). Lacking information of the damping matrix, we cannot solve the equation of motion (1.1a) nor can we use a conventional finite element model. The relative displacement x_d's are not available, but they are essential in computing treatment of dissipative energy W_c. All these lead to difficulties in constructing equation (1.1a). This is a typical problem in engineering design that often requires trial-and-error approaches, which costs intensive computation and often does not provide satisfactory results.

In earthquake engineering design, we are rarely given a damping matrix. Hence the equation of motion (1.1a) cannot be solved and trial-and-error approaches must be used. The assumption of light damping provides us the convenience needed to perform the damping calculations. In Chapter 5, a method is presented to assemble the general damping matrix, so that we can avoid a trial-and-error procedure.

Now, let us use Theorems 3.3 and 3.4 to calculate the required damping ratios. Suppose we have obtained the complex work done of the base system, W_m and W_k, with the help of a finite element model (note that it is not difficult in practice to obtain this kind of model), these quantities of work done are listed in Table 3.4, row 2. The work done W_c calculated by equation (3.44) is listed in row 3. We now use equation (3.44) for a lightly damped system (this structure satisfies the condition of light damping), in which the undamped natural frequencies do not change too much. We also know that an undamped natural frequency is in fact the virtual work. Therefore, we can use the work done for the base structure to approximate the damper-added-system. The fourth row of Table 3.4 shows the calculated damping ratios contributed by dampers. The fifth row of Table 3.4 are the damping ratios of the base structure taken from Table 3.3. With the help of Theorem 3.3, these damping ratios are added and are given in the sixth row. The last row in Table 3.4 is the experimental

data. They agree well with the calculated results.

Table 3.4 Calculation of Damping Ratio

mode	I	II	III
$2\sqrt{W_m W_k}$	83.70	9.01	0.712
W_c	23.02	.910	0.053
ξ_c	.275	.101	.0744
ξ_b	.0102	.0087	.0079
calculated ξ	.2852	.1097	.0752
tested ξ	.2970	.0877	.0620

In fact, this example expresses a simplified design procedure. Suppose we are given a structure with insignificantly small damping which can be omitted. By adding dampers to this structure, we increase the damping ratio to 5% . If we desire up to a 10% damping ratio, we can just double the number of dampers with the same configuration. Comparing this with the trial-and-error method, we see the advantage of this design procedure.

Example II

This second example concerns the damping matrix decompositions. Thus far we have already discussed three kinds of damping matrix decompositions, namely

1) the Clough-Penzien decomposition (1.11a)

$$\mathbf{C} = \mathbf{C}_P + \mathbf{C}_N$$

gives a proportional damping matrix $\mathbf{C}_p$ and hence all the damping ratios of the system can be calculated;

2) the pure proportional and non-proportional decomposition (1.11b)

$$\mathbf{C} = \mathbf{C}_d + \mathbf{C}_o$$

gives the matrix $\mathbf{C}_o$, the pure non-proportional damping; and

3) the real-imaginary decomposition

$$\mathbf{C} = \mathbf{C}_r + \mathbf{C}_i \tag{3.45}$$

gives the matrix $\mathbf{C}_r$ to yield the real part of complex damping ratio and the matrix $\mathbf{C}_i$ to provide the imaginary part of the complex damping ratio.

In practice, all the above decompositions are useful. However, the direct calculation required for equation (3.45) is more difficult to accomplish than the decompositions in (1.11a) and (1.11b). Alternatively, with the help of Theorem 3.3. we can use the formula

$$\mathbf{C} = \mathbf{C}_d + \mathbf{C}_o$$

to approximate

$$\mathbf{C} = \mathbf{C}_r + \mathbf{C}_i$$

It can be shown that the approximation is simple in calculation.

Suppose we have the following M-C-K system,

$$\mathbf{M} = \mathbf{I}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 4 & -2 \\ 0 & -1 & -2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 180 & -48 & 0 & 0 \\ -48 & 136 & -88 & 0 \\ 0 & -88 & 180 & -92 \\ 0 & 0 & -92 & 92 \end{bmatrix}$$

Since

$$\mathbf{C}\,\mathbf{K} = \begin{bmatrix} 408 & -144 & -92 & 92 \\ -324 & 544 & -352 & 0 \\ -132 & -440 & 992 & -552 \\ 48 & 40 & -732 & 644 \end{bmatrix}$$

we know that the system has non-proportional damping. With the Clough-Penzien decomposition (also called Clough decomposition), we have

$$\mathbf{C}_p = \begin{bmatrix} 3.8488 & -.2799 & .0338 & -.3457 \\ -.2799 & 3.6543 & -1.1758 & -.5689 \\ .0338 & -1.1758 & 3.6289 & -1.3251 \\ -.3457 & -.5689 & -1.3251 & 2.9056 \end{bmatrix}$$

and

$$C_n = \begin{bmatrix} -1.8488 & -.7201 & -1.0338 & .3457 \\ -.7201 & -.6543 & .1758 & -.4311 \\ -1.0338 & .1758 & .3711 & -.6749 \\ .3457 & -.4311 & -.6749 & 2.0944 \end{bmatrix}$$

With the pure proportional decomposition, we have

$$C_d = \begin{bmatrix} 3.8507 & -.2994 & .0301 & -.3307 \\ -.2944 & 3.6314 & -1.1827 & -.5486 \\ .0301 & -1.1827 & 3.6328 & -1.3303 \\ -.3307 & -.5486 & -1.3303 & 2.8851 \end{bmatrix}$$

and

$$C_o = \begin{bmatrix} -1.8507 & -.7006 & -1.0301 & .3307 \\ -.7006 & -.6314 & .1827 & -.4514 \\ -1.0301 & .1827 & .3672 & -.6697 \\ .3307 & -.4514 & -.6697 & 2.1149 \end{bmatrix}$$

With the real-imaginary decomposition, we have

$$C_r = \begin{bmatrix} 3.8314 & -.2606 & .0334 & -.3529 \\ -.2606 & 3.6538 & -1.1541 & -.5829 \\ .0334 & -1.1541 & 3.6033 & -1.3055 \\ -.3529 & -.5829 & -1.3055 & 2.9120 \end{bmatrix}$$

and

$$C_i = \begin{bmatrix} -1.8314 & -.7394 & -1.0334 & .3529 \\ -.7394 & -.6538 & .1541 & -.4171 \\ -1.0334 & .1541 & .3967 & -.6945 \\ .3529 & -.4171 & -.6945 & 2.0880 \end{bmatrix}$$

Comparing these results, we can visualize that the errors of using C_d - C_o to approximate C_r - C_i and even C_p - C_n are acceptably small.

The eigenvalues and complex damping ratios of C_p - C_n , C_d - C_o and C_r - C_i are shown in Tables 3.5, 3.6 and 3.7, respectively. For comparison, column 2 in each table lists the eigenvalues and complex damping ratios of the original system. In column 3 of each table, the parameters of C_p, C_d and C_r are given, respectively. The last column of each table shows the parameters of C_n, C_o and C_i. It can be seen that the C_r - C_i decomposition gives the best approximation of the real part of eigenvalues. The C_p - C_n decomposition shows the closest complex damping ratio to the original ones. However, all the three decompositions give a good approximation and agree well with Theorem 3.4. Therefore, we

can use the simplest decomposition $C_d - C_o$ as a further approximation with sufficient accuracy. Note that, the maximum damping ratio of the original system is about 21 %, which approaches the upper limit of light damping. If the damping ratio is smaller, we can expect an even better approximation. To see that, let us consider another system with damping matrix equal to C/10. In Tables 3.8, 3.9 and 3.10 we list the eigenvalues and complex damping ratios of $C_p - C_n$, $C_d - C_o$ and $C_r - C_i$ decompositions, respectively. Also for comparison, column 2 in each table gives the eigenvalues and complex damping ratios of the system with damping C/10. In column 3 of each table, the corresponding data of C_p, C_d and C_r are shown, respectively. The last column of each table presents the parameters of C_n, C_o and C_i.

It can be seen again that the $C_r - C_i$ decomposition gives the closest real part of the eigenvalues to the system with damping C/10. The $C_p - C_n$ decomposition provides the best approximation. Again, all three substantiate the linearity Theorem 3.4 well. Further the results are better than those of the case with a heavier damping ratio, since in this case, the damping ratio is ten times smaller.

Comparing Tables 3.5 and 3.8, the imaginary part of those data in Table 3.5 is about ten times larger than those given in Table 3.8. This agrees with the conclusion of Theorem 3.4. The same phenomena can be observed by comparing Table 3.6 with Table 3.9, also by comparing Table 3.7 with Table 3.10.

Table 3.5 $\lambda(H_{cp})$ and $\vartheta(H_{cp})$, $\lambda(H_{cn})$ and $\vartheta(H_{cn})$

	Original System	System with C_p	System with C_n
λ	-2.5182 ± 16.5207j	-2.5434 ± 16.6860j	0.0031 ± 16.7148j
	-1.8893 ± 13.6937j	-1.9019 ± 13.7850j	-0.0414 ± 13.8245j
	-1.9617 ± 9.8809j	-1.9491 ± 9.8175j	0.0434 ± 10.0734j
	-0.6307 ± 3.0123j	-.62444 ± 2.9819j	0.0037 ± 3.0769j
ϑ	1 X	1 X	1 X
	.1507 + .0098 j	.1507 + .0000j	-0.0008 + .0098j
	.1367 + .0066 j	.1367 + .0000j	0.0030 + .0066j
	.1947 - .0063 j	.1947 + .0000j	-0.0043 - .0064j
	.2050 - .0100 j	.2049 + .0000j	-0.0012 - .0099j

Table 3.6 $\lambda(\mathbf{H}_{cd})$ and $\vartheta(\mathbf{H}_{cd})$, $\lambda(\mathbf{H}_{co})$ and $\vartheta(\mathbf{H}_{co})$

	System with C	System with C_d	System with C_o
λ	-2.5182 ± 16.5207j	-2.5484 ± 16.6852j	-.0179 ± 16.7148j
	-1.8893 ± 13.6937j	-1.9087 ± 13.7841j	.0323 ± 13.8242j
	-1.9617 ± 9.8809j	-1.9226 ± 9.8228j	-.0148 ± 10.0737j
	-0.6307 ± 3.0123j	-0.6203 ± 2.9827j	.0004 ± 3.0769j
ϑ	1 X	1 X	1 X
	.1507 + .0098j	.1492 + .0000j	-.0011 + .0098j
	.1367 + .0066j	.1358 + .0000j	.0023 + .0066j
	.1947 - .0063j	.1960 - .0000j	-.0015 - .0064j
	.2050 - .0100j	.2070 - .0000j	.0001 - .0099j

Table 3.7 $\lambda(\mathbf{H}_{cr})$ and $\vartheta(\mathbf{H}_{cr})$, $\lambda(\mathbf{H}_{ci})$ and $\vartheta(\mathbf{H}_{ci})$

	System with C	S tem with C_r	System with C_i
λ	-2.5182 ± 16.5207j	-2.5184 ± 16.6898j	-.0122 ± 16.7146j
	-1.8893 ± 13.6937j	-1.8893 ± 13.7867j	-.0550 ± 13.8248j
	-1.9617 ± 9.8809j	-1.9618 ± 9.8150j	.0572 ± 10.0731j
	-0.6307 ± 3.0123j	-0.6308 ± 2.9805j	.0103 ± 3.0769j
ϑ	1 X	1 X	1 X
	.1507 + .0098j	.1491 + .0000j	.0007 + .0098j
	.1367 + .0066j	.1358 + .0000j	.0040 + .0065j
	.1947 - .0063j	.1960 - .0000j	-.0057 - .0064j
	.2050 - .0100j	.2070 - .0000j	-.0033 - .0099j

Table 3.8 $\lambda(\mathbf{H}_{cp})$ and $\vartheta(\mathbf{H}_{cp})$, $\lambda(\mathbf{H}_{cn})$ and $\vartheta(\mathbf{H}_{cn})$

	System with C/10	System with C_p	System with C_n
λ	-0.2548 ± 16.8752j	-0.2548 ± 16.8768j	0.0000 ± 16.8771j
	-0.1909 ± 13.9143j	-0.1909 ± 13.9143j	-0.0000 ± 13.9147j
	-0.1923 ± 10.0079j	-0.1923 ± 10.0073j	0.0000 ± 10.0098j
	-0.0620 ± 3.0462j	-0.0620 ± 3.0459j	0.0000 ± 3.0468j
ϑ	.1 X	.1 X	0.0001 X
	.1510 + .0010j	.1510 + .0000j	-0.0076 + .9811j
	.1372 + .0006j	.1372 + .0000j	0.0256 + .6170j
	.1922 - .0006j	.1922 + .0000j	-0.0380 - .6156j
	.2036 - .0010j	.2036 + .0000j	-0.0117 - .9824j

Table 3.9 $\lambda(H_{cd})$ and $\vartheta(H_{cd})$, $\lambda(H_{co})$ and $\vartheta(H_{co})$

	System with C/10	System with C_d	System with C_o
λ	-0.2548 ± 16.8752j	-0.2458 ± 16.8768j	.0000 ± 16.8771j
	-0.1909 ± 13.9143j	-0.1909 ± 13.9143j	-.0000 ± 13.9147j
	-0.1923 ± 10.0079j	-0.1923 ± 10.0073	-.0000 ± 10.0098j
	-0.0620 ± 3.0462j	-0.0620 ± 3.0459j	.0000 ± 3.0468j
ϑ	.1 X	.1 X	0.0001 X
	.1510 + .0010j	.1510 + .0001j	-.0100 + .9811j
	.1372 + .0006j	.1372 + .0001j	.0216 + .6176j
	.1922 - .0006j	.1922 + .0001j	-.0136 - .6156j
	.2036 - .0010j	.2036 + .0001j	.0012 - .9824j

Table 3.10 $\lambda(H_{cr})$ and $\vartheta(H_{cr})$, $\lambda(H_{ci})$ and $\vartheta(H_{ci})$

	System with C/10	System with C_r	System with C_i
λ	-0.2548 ± 16.8752j	-0.2548 ± 16.8768j	-.0000 ± 16.8771j
	-0.1909 ± 13.9143j	-0.1909 ± 13.9143j	-.0000 ± 13.9147j
	-0.1923 ± 10.0079j	-0.1923 ± 10.0073j	.0000 ± 10.0098j
	-0.0620 ± 3.0462j	-0.0620 ± 3.0459j	.0000 ± 3.0468j
ϑ	.1 X	1 X	0.0001 X
	.1510 + .0010j	.0151 + .0001j	.0072 + .9811j
	.1372 + .0006j	.0137 + .0001j	.0341 + .6170j
	.1922 - .0006j	.0192 + .0001j	-.0499 - .6156j
	.2036 - .0010j	.0204 + .0001j	-.0318 - .9824j

4 REPRESENTATION OF DAMPING MATRIX

4.1 INTRODUCTION

In the previous chapters we have established the fundamentals of the complex energy-based damping theory. Complex energy comes from non-proportional damping. From Chapter 4 on, this report will mainly be concerned with discussing the modeling of non-proportional damping matrices.

We consider modeling of a damping matrix in two steps. First, in this chapter, we introduce a simplified representation of a damping matrix in terms of its mass and stiffness matrices. This proposed relationship is based on matrix representation, Theorem A.3, which states that, under certain circumstances, a matrix C can be represented accurately by a polynomial of two given matrices **M** and **K**. The result is simplified with acceptable accuracy for engineering applications by taking only the first few terms of the polynomial. This approach can be adopted in finite element analyses of structures and is equally simple to apply in system identification. In the second step, we formulate general damping coefficient matrices (Chapter 5).

It is noted here that the theory of complex damping is not only useful for viscous damping, but also for many other types of damping, such as the visco-elastic damping, friction damping and structural damping. The last section of this chapter will focus on structural damping.

4.2 DAMPING MATRIX REPRESENTATION

The representation of a damping matrix in terms of the mass and stiffness matrices is considered to be a convenient approach to indirectly obtain the damping matrix. Generally speaking, a damping matrix does depend on specific mass and stiffness matrices. This dependence will be discussed in Chapter 5.

In recent years, significant progress has been made in the methods of analysis

and design of structures and mechanical systems under dynamic loading conditions. As mentioned above, one of the difficulties is the lack of accurate modeling to deal with the damping coefficient matrix. Traditionally, *Rayleigh damping* and/or general *proportional damping* has been assumed in most dynamic structural analyses primarily for simplicity.

Rayleigh damping can be written as (see equation (1.8))

$$\mathbf{C} = \alpha\, \mathbf{M} + \beta\, \mathbf{K}$$

Because this equation does not cover all cases of proportional damping, an improved version was suggested by Clough and Penzien (see equation (1.9))

$$C = \sum_b C_b = M \sum_b a_b \, (M^{-1}K)^b \quad , \quad -\infty < b < \infty$$

However, these above formulas can only yield the normal mode and are not very accurate in many engineering applications.

Many studies have been devoted to the subject of non-proportional damping in dynamically restrained structures. Among the more recent efforts, Caravani and Thomson (1974), Berman and Nagy (1983), Golla and Hughes (1985), Buhariwala and Hanson (1988) and Mau et al (1988) have tried to approximate non-proportional damping by using of additional (dissipation) coordinates, integro-differential equation and complex modal damping factors as well as other ideas.

However, no explicit expression for general non-proportional damping has been established. The lack of an explicit formula for damping representation remains a major problem area in analysis and design of structures and mechanical systems against dynamic load (see Clough (1979) and Liang (1987)).

4.3 SIMPLIFIED MODEL OF NON-PROPORTIONAL DAMPING

In typical structural engineering applications, the finite element method is often used for numerical solutions of equations of motion in which the damping matrix **C** is generally represented by Rayleigh's method. As mentioned in Chapter 1, Rayleigh's idea is one type of a damping representation. Clough damping is another example of damping representation. In this section, we

consider a simple model to represent the general case, non- proportional damping.

4.3.1 SIMPLIFIED FORMULA

Theorem A.3 in Appendix A shows that the damping matrix **C** can be treated as an exact function of the mass and stiffness matrices. The relationship may be expressed by a Taylor series at a certain "point," for example at 0 we have

$$\mathbf{C} = f(\mathbf{M}, \mathbf{K}) = f\mathbf{A}^{(0)1} + \frac{1}{2}\left(f_{\mathrm{M}}\,\mathbf{A}^{(1)1} + f_{\mathrm{K}}\,\mathbf{A}^{(1)2}\right) + \frac{1}{6} f_{\mathrm{MK}}\,\mathbf{A}^{(2)1} +$$

$$+ \ldots. = \alpha\,\mathbf{I} + \beta\,\mathbf{M} + \gamma\,\mathbf{K} + \delta\,\mathbf{M}\,\mathbf{K} + \ldots. \qquad (4.1)$$

where

$$f_{\mathrm{a}} = \frac{\partial f(0)}{\partial a}, \qquad f_{\mathrm{ab}} = \frac{\partial^2 f(0)}{\partial a\,\partial b}, \ldots..$$

and $\mathbf{A}^{(.)\cdot}$ is a matrix product defined in Appendix A.

If only the first three terms of equation (4.1) are considered, the damping matrix **C** can be described approximately by

$$\mathbf{C} \approx \alpha\,\mathbf{I} + \beta\,\mathbf{M} + \gamma\,\mathbf{K}$$

where the matrix **M** is chosen to be the consistent mass matrix (see Zienkiewicz, 1982 for example). Rewrite the above equation as follows

$$\mathbf{C} \approx \alpha\,\mathbf{M} + \beta\,\mathbf{K} + \gamma\,\mathbf{I} \qquad (4.2)$$

We see that the first two terms of equation (4.2) represent Rayleigh damping. However, equation (4.2) is essentially different from Rayleigh's formula (1.8), because the extra term provides a basic change over Rayleigh damping. The difference is that equation (4.1) can yield the complex mode. In the following, we can show this point rigorously.

Theorem 4.1 In equation (1.1), if

$$\mathbf{M}\,\mathbf{K} \neq \mathbf{K}\,\mathbf{M}, \qquad (4.3)$$

and $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I}$, then the M-C-K system yields complex modes.

PROOF:

The necessary and sufficient condition for an M-C-K system to have complex modes is

$$\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}$$

Introducing equation (4.3) into the left hand side of the above inequality yields

$$\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} = (\alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I})\,\mathbf{M}^{-1}\mathbf{K} = \alpha\mathbf{K} + \beta\mathbf{K}\mathbf{M}^{-1}\mathbf{K} + \gamma\mathbf{M}^{-1}\mathbf{K}$$

Similarly,

$$\mathbf{K}\,\mathbf{M}^{-1}\,\mathbf{C} = \alpha\,\mathbf{M} + \beta\,\mathbf{K}\,\mathbf{M}^{-1}\mathbf{K} + \gamma\,\mathbf{K}\,\mathbf{M}^{-1}$$

Because $\mathbf{M}\,\mathbf{K} \neq \mathbf{K}\,\mathbf{M}$, then $\mathbf{M}^{-1}\,\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-1}$. Therefore, $\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} \neq \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}$ and then the M-C-K system yields complex mode. ∎

In most engineering problems, the condition of equation (4.3) is satisfied. Therefore, equation (4.2) will provide complex modes.

The polynomial approximation of the damping matrix **C** by Clough and Penzien is based on the consideration of simultaneously decoupling of the **M**, **C** and **K** matrices. Therefore, it can only yield normal mode information and can not handle non-proportional damping. The advantage of Clough's approach is its ability to represent any proportional damping, if the term $\mathbf{C}_b$'s are adequate. For the real mode, we can equivalently let each term $\mathbf{C}_b$ in Clough's summation

$$\mathbf{C} = \sum_b \mathbf{C}_b$$

be $(\mathbf{K}\,\mathbf{M}^{-1})\mathbf{K}$, $\mathbf{K}(\mathbf{M}^{-1}\mathbf{K})$ or $\mathbf{M}^{1/2}(\mathbf{M}^{-1/2}\mathbf{K}\,\mathbf{M}^{-1/2})\mathbf{M}^{1/2}$, etc.

Alternatively, we may let $\mathbf{C}_b$ be $\mathbf{K}(\mathbf{K}^{-1}\mathbf{M})$, etc. It should be pointed out that in Clough's damping $\mathbf{C} = \mathbf{M}\sum_b a_b\,(\mathbf{M}^{-1}\mathbf{K})^b$ the exponent b does not necessarily range from $-\infty$ to ∞. In fact, according to the Hamilton-Cayley Theorem, the maximum number of linearly independent terms in the summation must not exceed

n, where n is the dimension of **M**, **C** and **K** matrices.

Note that the polynomial approximation of Clough and Penzien is a special case of equation (4.1), when the α_{ij} 's are taken to be

$$|\alpha_{i1}| + |\alpha_{i2}| + |\alpha_{i3}| + \ldots = k \, .$$

Comparing Rayielgh's and Clough's damping with formula (4.1), we can see the advantages of the proposed formula. Formula (4.1) not only covers the traditional treatments but also provides more information. And (4.1) can be used in any truncated form under certain accuracy requirement.

4.3.2 PHYSICAL MEANING OF DAMPING MATRIX PARAMETERS

The physical meaning of the three terms, **M**, **K** and **I** in equation (4.2) may be briefly explained as follows. If the *energy dissipated* by damping is denoted by δW_c (see Zienkiewicz, 1982), then

$$\delta W_c = \delta \left\{ \int_V \mathbf{u}^T f_c \, dv \right\} \tag{4.4}$$

where **u** is the displacement vector and f_c is the damping or dissipation force vector per unit volume.

Case 1 In viscous damping, the dissipation force is proportional to the velocity, thus we have

$$f_{c1} = \beta_1 \frac{d}{dt} (\mathbf{u}) = \beta_1 \, \mathbf{u}' \tag{4.5}$$

where β_1 is the proportional coefficient.

Dividing a structure into small finite elements gives the approximate displacement field **u** in terms of the nodal displacement **x** and interpolation function N,

$$\mathbf{u} = N \, \mathbf{x} \tag{4.6}$$

The dissipative energy can then be expressed by

$$\delta W_c = \delta \left\{ \int_V \mathbf{u}^T f_c \, dv \right\} = \delta \left\{ \beta_1 \int_V \mathbf{x}^T N^T N \mathbf{x}' \, dv \right\}$$

$$= \delta \mathbf{x}^T \left\{ \beta_1 \int_V N^T N \, dv \right\} \mathbf{x}'$$

Therefore the damping matrix under the assumption of viscous damping can be written as

$$\mathbf{C}_1 = \alpha_1 \left\{ \int_V N^T N \, dv \right\} \tag{4.7}$$

where the subscript 1 of the damping **C** stands for the damping mechanism of this first case. It is noted that the *consistent mass* matrix is given by

$$\mathbf{M} = \rho \left\{ \int_V N^T N \, dv \right\} \tag{4.8}$$

where ρ is the mass density. Comparing this with the expression of $\mathbf{C}_1$, we now can write

$$\mathbf{C}_1 = \frac{\alpha_1}{\rho} \left[\rho \left\{ \int_V N^T N \, dv \right\} \right] = \alpha \mathbf{M} \tag{4.9}$$

where $\alpha = \dfrac{\alpha_1}{\rho}$

Case 2 By introducing a viscous damping stress tensor σ_d, which is proportional to the time derivative of the normal stress tensor σ, we may write

$$\sigma_d = \beta \frac{d}{dt} (E \varepsilon) = \beta E \varepsilon' \tag{4.10}$$

where ε is the strain tensor of the body, and $\sigma = E \varepsilon$; β is a proportional coefficient.

For small strain, the strain field ε can also be expressed by the displacement field,

$$\varepsilon = L\,\mathbf{u} = L\,N\,\mathbf{x} = B\,\mathbf{x} \tag{4.11}$$

where L is a suitable differential operator developed by geometric considerations, and $B = L\,N$. Now, the dissipated energy can be written as

$$W_d = \int_V \delta\,\varepsilon^T \sigma_d\,dv = \int_V \delta\,\mathbf{x}^T\,B^T\,[\beta\,E\,B]\,\mathbf{x}'dv = \delta\mathbf{x}^T \left\{ \beta \int_V B^T\,E\,B\,dv \right\}\mathbf{x}'$$

In the second case, the damping matrix is given by

$$\mathbf{C}_2 = \beta \left\{ \int_V B^T\,E\,B\,dv \right\} = \beta\,\mathbf{K} \tag{4.12}$$

where

$$\mathbf{K} = \left\{ \int_V B^T\,E\,B\,dv \right\} \tag{4.13}$$

is the stiffness matrix generated by the finite element method.

If one considers only the above two cases of damping, i.e. $\mathbf{C}_1$ and $\mathbf{C}_2$, the generalized damping matrix **C**, may be written as:

$$\mathbf{C} = \alpha\,\mathbf{M} + \beta\,\mathbf{K}$$

which is Rayleigh damping. The above derivation also shows that Rayleigh damping is *consistent* with the mass and stiffness matrices.

Case 3 Returning to the damping matrix of equation (4.2), the term $\gamma\mathbf{I}$ may be considered a " *lumped damping* ." In other words, damping may also be lumped at nodal points. Such a concept can be readily understood in many structural engineering applications, where the damping force occurs not only in damping materials, but also at structural connections. In fact, the latter is often much greater than the former. In such a case, the lumped damping assumption is a better approach, because the term

$$\int_V N^T\,N\,dv$$

reduces to a diagonal matrix. To simplify the procedure, if we use an identity matrix **I** to approximate the lumped damping with a proper coefficient α, the

damping matrix can be simply expressed as

$$\mathbf{C}_3 = \gamma\, \mathbf{I} \tag{4.14}$$

Combining equation (4.9) with equations (4.12) and (4.14), equation (4.2) will result.

In addition to the above, there are other special cases worth noting. One such case is the damping representation of

$$\tilde{\mathbf{C}} = 2\, \xi\, \tilde{\mathbf{K}}^{1/2} \tag{4.15a}$$

in which $\tilde{\mathbf{K}}^{1/2}$ is the the matrix product of

$$\tilde{\mathbf{K}}^{1/2} = (\mathbf{M}^{-1/2}\, \mathbf{K}\, \mathbf{M}^{-1/2})^{1/2} = \mathbf{M}^{-1/4}\, \mathbf{K}^{1/2}\, \mathbf{M}^{-1/4}.$$

Or

$$\tilde{\mathbf{C}} = \delta\, \tilde{\mathbf{K}}^{-1/2} \tag{4.15b}$$

where

$$\tilde{\mathbf{K}}^{-1/2} = (\mathbf{M}^{-1/2}\, \mathbf{K}\, \mathbf{M}^{-1/2})^{-1/2} = \mathbf{M}^{1/4}\, \mathbf{K}^{-1/2}\mathbf{M}^{1/4}.$$

The following cases are good examples of (4.15a) and (4.15b). In traditional viscoelastic (VE) damper design, the equivalent damping ratio, defined in equation (1.69), is employed. Recently, the SDOF formula (1.69) is suggested for proportionally damped MDOF structures. That is, the i^{th} damping ratio of such a structure is assumed to be (see Zhang, 1986)

$$\xi_i = \frac{W_{d\,i}}{4\pi\, W_i} = \frac{\sum_j \gamma_{ij}^2\, G_1\, V_j}{2\, Q_i^T \tilde{\mathbf{K}}\, Q_i}$$

where the subscript i stands for the i^{th} mode; j denotes the j^{th} VE damper; all other symbols are as previously defined.

Within the proportional assumption, once all the damping ratios are determined, the damping coefficient matrix $\tilde{\mathbf{C}}$ can be obtained by

$$\tilde{\mathbf{C}} = \mathbf{Q}\ \mathrm{diag}(\, 2\, \xi_i \omega_i)\ \mathbf{Q}^T$$

Now, suppose the damping ratio ξ_i equals constant for i = 1, ... ,n, then,

$$\tilde{C} = Q \; \text{diag}(2 \; \xi_i \omega_i) \; Q^T = 2 \; \xi \; Q \; \text{diag}(\omega_i) \; Q^T = 2 \; \xi \; \tilde{K}^{1/2} \qquad (4.16)$$

Thus, the damping matrix is proportional to the square root of the stiffness matrix $\tilde{K}$ and equation (4.15a) results.

Next, suppose the damping matrix is constant. In this case,

$$\text{diag}(2 \; \xi_i \omega_i) = Q^T \; \tilde{C} \; Q = \text{constant matrix (of } \omega_i \text{)}$$

Therefore, $\xi_i \propto 1/\omega_i$. Note that the proportional coefficient may vary. Under the condition $2 \; \xi_i \omega_i = \gamma$ = constant for all i's, we have

$$\tilde{C} = Q \; \text{diag}(\gamma) \; Q^T = \gamma \; Q \; Q^T = \gamma \; I \qquad (4.17)$$

which is the case of equation (4.14).

As a third case, we suppose the loss modulus G_l is constant. Furthermore, assume

$$W_{di} = (G_l \; A / t \;)\pi \; a \; Q_i^T \; Q_i$$

then

$$\xi_i = \frac{W_{di}}{4\pi \; W_i} = \frac{(G_l \; A / t \;)\pi \; a \; Q_i^T \; Q_i}{4 \; \pi \; Q_i^T \; \tilde{K} \; Q_i / 2} = \frac{a \; G_l \; A}{2 \; t \; \omega_i^2} \propto 1/\omega_i^2$$

where A is the area of the VE material, t is the corresponding thickness, V = At; and a is a proportional coefficient. Thus, we obtain equation (4.15b) by

$$\tilde{C} = Q^T \; \text{diag} \; (\; 2 \; \frac{a \; G_l \; A}{2 \; t \; \omega_i} \;) \; Q = \frac{a \; G_l \; A}{t} \; Q^T \text{diag} \; (1/\omega_i) \; Q$$

$$= \delta \; \tilde{K}^{-1/2} \qquad (4.18)$$

where the proportional coefficient $\delta = \dfrac{a \; G_l \; A}{t}$ = constant

Other aspects of the above damping representations will be discussed in

4.4 DAMPING CONTRIBUTIONS OF M, K AND I

We first examine the contributions of the terms **M**, **K** and **I** in equation (4.2). Here the damping contributions refer to the contributions to pure energy dissipation. They are known to be related only with the real parts of the complex damping ratio. Therefore, we omit the imaginary part and/or complex mode in subsequent discussions of this section.

With the help of the eigen-matrix **A**, let us examine the eigenvalue of the M-C-K system, where $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I}$, i.e

$$\mathbf{M}\,\mathbf{A}^2 + \mathbf{C}\,\mathbf{A} + \mathbf{K} = \mathbf{M}\,\mathbf{A}^2 + (\alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I})\,\mathbf{A} + \mathbf{K} = \mathbf{0} \qquad (4.19)$$

According to Theorem 3.4, if the system is lightly damped, we can separately consider

$$\mathbf{M}\,\mathbf{A}_M^2 + \alpha\,\mathbf{M}\,\mathbf{A}_M + \mathbf{K} = \mathbf{0}$$

$$\mathbf{M}\,\mathbf{A}_K^2 + \beta\,\mathbf{K}\,\mathbf{A}_K + \mathbf{K} = \mathbf{0}$$

$$\mathbf{M}\,\mathbf{A}_I^2 + \gamma\,\mathbf{I}\,\mathbf{A}_I + \mathbf{K} = \mathbf{0}$$

and subsequently add the real parts of $\lambda_i(\mathbf{A}_M)$, $\lambda_i(\mathbf{A}_K)$ and $\lambda_i(\mathbf{A}_I)$ altogether to approximate the real part of $\lambda_i(\mathbf{A})$ for i = 1,...,n. Here the subscripts M, K and I correspond to the terms **M**, **K** and **I**, respectively.

First, consider the equation

$$\mathbf{M}\,\mathbf{A}^2 + \alpha\,\mathbf{M}\,\mathbf{A} + \mathbf{K} = \mathbf{0}$$

or

$$\mathbf{A}^2 + \alpha\,\mathbf{A} + \mathbf{M}^{-1}\,\mathbf{K} = \mathbf{0} \qquad (4.20)$$

According to Corollary A.5, we know that

$$\mathbf{A} = \left(-\alpha\,\mathbf{I} + \sqrt{\alpha^2\,\mathbf{I} - 4\,\mathbf{M}^{-1}\mathbf{K}} \right) / 2$$

Since the system is underdamped, we have

$$\mathbf{A} = (- \alpha \mathbf{I} + j \sqrt{4 \mathbf{M}^{-1}\mathbf{K} - \alpha^2 \mathbf{I}}) / 2 \qquad (4.21)$$

From equation (4.21), we know that, $\mathrm{Re}(\lambda_i(\mathbf{A})) = - \alpha/2$.
However, $\mathrm{Re}(\lambda_i(\mathbf{A})) = - \xi_i \omega_i$ therefore, in the case of equation (4.20), we have

$$\xi_i = \frac{\alpha}{2 \omega_i} \qquad (4.22)$$

From equation (4.22), we can see that, the value of ω_i is inversely proportional to the value of ξ_i. If we approximate the damping ratio of the lower modes, we will not be able to obtain good approximations for the higher modes, since ξ_i's of the higher modes will become very small in value. In fact, with a properly chosen coefficient α, we can approximate the damping ratio of the lower modes only.

Secondly, consider the case

$$\mathbf{M A}^2 + \beta \mathbf{K A} + \mathbf{K} = \mathbf{0}$$

or

$$\mathbf{A}^2 + \beta \mathbf{M}^{-1}\mathbf{K A} + \mathbf{M}^{-1}\mathbf{K} = \mathbf{0} \qquad (4.23)$$

Similarly, we have

$$\mathbf{A} = (\beta \mathbf{M}^{-1}\mathbf{K} + j \sqrt{4 \mathbf{M}^{-1} \mathbf{K} - \beta^2 (\mathbf{M}^{-1}\mathbf{K})^2}) / 2 \qquad (4.24)$$

Then, $\mathrm{Re}(\lambda_i(\mathbf{A})) = - \frac{\beta}{2} \omega_i^2 = - \xi_i \omega_i$, where $\omega_i = \lambda_i [\mathbf{M}^{-1}\mathbf{K}]^{1/2}$. So, in the case of equation (4.23), we have

$$\xi_i = \frac{\beta}{2} \omega_i \qquad (4.25)$$

From equation (4.25), we can conclude that in case (4.23), the value of ω_i is directly proportional to the value of ξ_i. Opposite to case (4.20), equation (4.23) does not yield good results for lower modes and small ω_i's. With a properly chosen coefficient β, we can approximate the damping ratio of higher

modes.

Combining (4.22) and (4.25), we may choose proper coefficients α and β to fit damping ratios for lower and higher modes. Therefore, we can use Rayleigh damping $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ to effectively approach the damping ratio of the first few and the last few modes. However, it may not fit the middle modes. This then is a severe drawback when approximating for an entire spectrum of modes.

The above conclusions from equations (4.22) and (4.25) can be extended to the case of Clough damping, where the term $\mathbf{M}\,(\mathbf{M}^{-1}\mathbf{K})^b$ is involved. In this case, for the b^{th} term with the coefficient a_b , we can have

$$\xi_i = -\frac{\alpha_b}{4}\,\omega_i^{2b-1} \tag{4.26}$$

Equation (4.26) says that we can choose proper value of b's from negative to positive numbers to fit these ξ_i's from lower to higher modes. Thus when we use n terms of $\mathbf{M}\,(\mathbf{M}^{-1}\mathbf{K})^b$, we can exactly fit all damping ratios. However, this is computationally intensive.

Now, let us consider

$$\mathbf{M}\,\mathbf{A}^2 + \gamma\,\mathbf{A} + \mathbf{K} = \mathbf{0} \tag{4.27}$$

For the time being, we are not interested in the complex mode but only the contribution of the terms $\alpha\mathbf{M}$, $\beta\mathbf{K}$ and $\gamma\mathbf{I}$. We may use the Clough damping term $\mathbf{M}\,(\mathbf{M}^{-1}\mathbf{K})^b$ to approach $\gamma\mathbf{I}$, that is, let

$$\gamma\mathbf{I} \approx \sum_i \alpha_i\,\mathbf{M}\,(\mathbf{M}^{-1}\mathbf{K})^i \tag{4.28}$$

and i is chosen from negative integers to positive integers symmetrically. For example, if $\mathbf{I}$ is 3x3, i = -4, -3, -2, -1, 0, 1, 2, 3, 4.

The introduction of equation (4.28) is not for the convenience of calculation but rather for the purpose of analyzing the influence of modal damping ratios by the term $\gamma\mathbf{I}$.

From equation (4.28), we can see that the portion of the term $\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^i$ with negative i's contribute to lower modes. The portion of $\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^i$ with positive i's works for higher modes. And the portion of $\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^i$ with i's close to the origin benefits middle modes. In the following section, we will show this by a numerical example.

Thus, a properly chosen coefficient γ may let us fit the damping ratios of middle modes. Combining the cases of (4.20), (4.21) and (4.27), we can choose a group of coefficients α, β and γ to fit the entire spectrum of modes. That is, equation $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I}$ will give a better approach than Rayleigh damping. And, from equation (4.1), when more terms of $\mathbf{A}^{(i)}$'s are used, better results for an approximation of the **C** matrix will be obtained.

Now, let us examine the contributions of the approximation (4.2) to the imaginary damping ratio. First of all, the terms $\alpha\mathbf{M}$ and $\beta\mathbf{K}$ have no influence on the imaginary damping ratio. The entire contribution is only provided by the term $\gamma\mathbf{I}$.

Unlike equations (4.22), (4.26) and (4.28), we do not yet have enough information to locate the contribution of $\gamma\mathbf{I}$. However, for the lowest (highest) modes, since ζ_1 (ζ_n) is always positively-(negatively-)valued, the contribution of $\gamma\mathbf{I}$ helps to describe the trend of energy transfer correctly. On the other hand, for certain middle modes, $\gamma\mathbf{I}$ may provide wrong signs for the ζ_i's.

4.5 EXAMPLES FOR SIMPLIFIED MODELS

4.5.1 NUMERICAL EXAMPLES:

The following are numerical examples, which show the application of the matrix representation theorem and equation (4.2).

(1) Let a system be given such that

$$\mathbf{M} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 200 & -200 \\ -200 & 500 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1.7 & -1.7 \\ -1.7 & 2.9 \end{bmatrix}$$

Since

$$\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 4080 & -7140 \\ -5520 & 10020 \end{bmatrix}$$

is not symmetric, the system will have complex modes. In fact, the M-C-K system has eigenvalues of

$$-12.0264 \pm 56.9165j$$

and $\quad -.5736 \pm 14.5749j$

The eigenvector matrix of the system is given by

$$\begin{bmatrix} -.8293 \pm .0314j & 1 \\ 1 & .5496 \pm .0173j \\ .0024 \pm .0141j & -.0027 \pm .0685j \\ -.0036 \pm .0168j & -.0003 \pm .0377j \end{bmatrix}$$

Next let $\mathbf{A}_0 = \mathbf{I}$, $\mathbf{A}_1 = \mathbf{M}$, $\mathbf{A}_2 = \mathbf{M}$, then we can have

$$\hat{\beta} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 200 \\ 1 & 1/6 & -200 \\ 1 & 1/3 & 200 \end{bmatrix}^{-1} \begin{bmatrix} 1.7 \\ -1.7 \\ 2.9 \end{bmatrix} = \begin{bmatrix} 2.7 \\ -4.4 \\ .004 \end{bmatrix}$$

that is

$$\mathbf{C} = 2.7\,\mathbf{I} - 4.4\,\mathbf{M} + .004\,\mathbf{K} = \begin{bmatrix} 1.700 & -1.700 \\ -1.700 & 2.900 \end{bmatrix}$$

Next, if $\mathbf{A}_0$ and $\mathbf{A}_1$ are kept unchanged but $\mathbf{A}_3 = \mathbf{K}^2$, we can similarly obtain

$$\hat{\beta} = [\ 3.0429 \quad -4.4 \quad 4.71428\text{x}10^{-6}\ ]^T$$

Then,

$$\mathbf{C} = 3.0429\,\mathbf{I} - 4.4\,\mathbf{M} + 4.71428\text{x}10^{-6}\,\mathbf{K}^2 = \begin{bmatrix} 1.700 & -1.700 \\ -1.700 & 2.900 \end{bmatrix}$$

Therefore, for a system and/or subsystem with order 2, the use of three terms

in equation (4.2) yields the exact answer.

(2) As a second example, let a system be defined by the following properties:

$$\mathbf{M} = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 30 & -20 & 0 \\ -20 & 50 & 0 \\ 0 & 0 & 60 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Again, let $\mathbf{A}_0 = \mathbf{I}$, $\mathbf{A}_1 = \mathbf{M}$, $\mathbf{A}_2 = \mathbf{K}$, we are then able to obtain

$$\hat{\beta} = \begin{bmatrix} 1 & 6 & 50 \\ 0 & 2 & -20 \\ 1 & 4 & 50 \\ 1 & 5 & 60 \end{bmatrix}^{+} \begin{bmatrix} 4 \\ -1 \\ 5 \\ 5 \end{bmatrix} = [4.722 \quad -.3889 \quad 0.0167]^{T}$$

where the superscript + stands for the pseudo-inverse (see Golub, 1985).

Generally speaking, we should use a constrained least squares method here to calculate $\hat{\beta}$, because a heavily damped system using a pseudo-inverse may result in the calculated matrix **C** being non-positive. However, the pseudo-inverse method can often be used to simplify the computation. For this case, the calculated **C** matrix is given by

$$\mathbf{C} = \begin{bmatrix} 3.8889 & -1.1111 & 0 \\ -1.1111 & 4.0000 & 0 \\ 0 & 0 & 4.7778 \end{bmatrix}$$

The eigenvalues of the original system are

$$-.2337 \pm 1.5437j$$

$$-.5000 \pm 3.4278j$$

and $\quad -1.0163 \pm 4.6400j$

The damping ratio of the original system are 0.1497 (1st), 0.1443 (2nd) and 0.2140 (3rd).

The undamped natural frequencies of the original system are

0.2485 Hz, 0.5513 Hz and 0.7560 Hz

Whereas, the eigenvalues of the calculated system are

$-.2213 \pm 1.5456j$

$-.4778 \pm 3.4310j$

and $-1.0287 \pm 4.6372j$

The damping ratios of the calculated system are 0.1417, 0.1379 and 0.2166. The undamped natural frequencies of the calculated system are 0.2485 Hz, 0.5513Hz and 0.7560 Hz.

In this example, the maximum error for the calculated **C** matrix is about 0.1111, the maximum error for the damping ratio is 0.0532 and the maximum error for the undamped natural frequency is 1.005×10^{-15}

When we use 0.467 **M** to approximate the damping matrix, we have the damping ratios 0.1497, 0.0674 and 0.0482. It can be seen that α **M** mainly contributes to the first modes. When we use 0.09 **K** to approximate the damping matrix, the calculated damping ratios are 0.070, 0.1560 and 0.2140. So, the term β **K** mainly affects the third mode. If we use 5 **I** to approximate the damping matrix, we have damping ratios 0.2213, 0.1443 and 0.1904. Thus, the contribution is evenly distributed.

If we let $\mathbf{A}_2 = \mathbf{M}\,\mathbf{K}$, or use four terms, the exact answer can be obtained.

(3) In order to check the accuracy of using different numbers of terms in equation (4.1), let us consider the following example, where

$$\mathbf{M} = \begin{bmatrix} .25 & .125 & .125 & 0 & 0 & 0 \\ & .25 & .125 & .125 & 0 & 0 \\ & & .25 & .125 & .125 & 0 \\ & & & .25 & .125 & 0 \\ & & & & .25 & .125 \\ & & & & & .5 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 900 & -240 & 0 & 0 & 0 & 0 \\ & 680 & -440 & 0 & 0 & 0 \\ & & 900 & -460 & 0 & 0 \\ & & & 510 & -50 & 0 \\ & & & & 1000 & -30 \\ & & & & & 180 \end{bmatrix}$$

and the damping matrix is specially designed to be

$$\mathbf{C} = 4.\text{x}10^{-13}\ \mathbf{M}^5\mathbf{K}^5\mathbf{M}^5$$

Since the order of the system is 6, and the system is symmetric, theoretically 21 terms (n(n+1)/2) in equation (4.1) need to be taken in order to exactly represent the damping matrix C. We compare the accuracy of using 1 to 21 terms. Figures 4.1 (a-d) give these numerical results. Figure 4.1 (a) shows the errors in using various number of terms in terms of the damping ratio of the last mode. Figure 4.1 (b) illustrates the errors related to various number of terms in terms of the damping ratio of the first mode. Figure 4.1 (c) gives the errors obtained when using various number of terms in terms of the imaginary damping ratio of the last mode. Figure 4.1 (d) shows the errors in using various number of terms in terms of the imaginary damping ratio of the first mode.

In these calculations, no term of $\mathbf{M}^5\mathbf{K}^5\mathbf{M}^5$ is used. These results show that in this special case, three terms of $\alpha\mathbf{M} + \beta\mathbf{K} + \gamma\mathbf{I}$ is a good choice.

(4) In addition to the above three simple examples, we have also tested several larger matrices, such as a system with 8x8 matrices used by Mau (1988) and Itoh (1973). Comparisons were also made by using the formula of Clough and Penzien with the one proposed by the authors. All comparisons show that the proposed approach is far more accurate than those results reported by others using Rayleigh and Clough damping.

4.5.2 SUMMARY OF DAMPING MATRIX REPRESENTATION

A matrix can be represented by a polynomial of two given matrices under certain conditions (the conditions can be easily checked). Most engineering problems satisfy these conditions. The damping matrix of a system can be expressed approximately by its mass and stiffness matrices, which are normally given or can be easily obtained. Representing damping by mass and stiffness matrices, instead of using Rayleigh damping, will generally yield complex modes, because of using non-proportional damping.

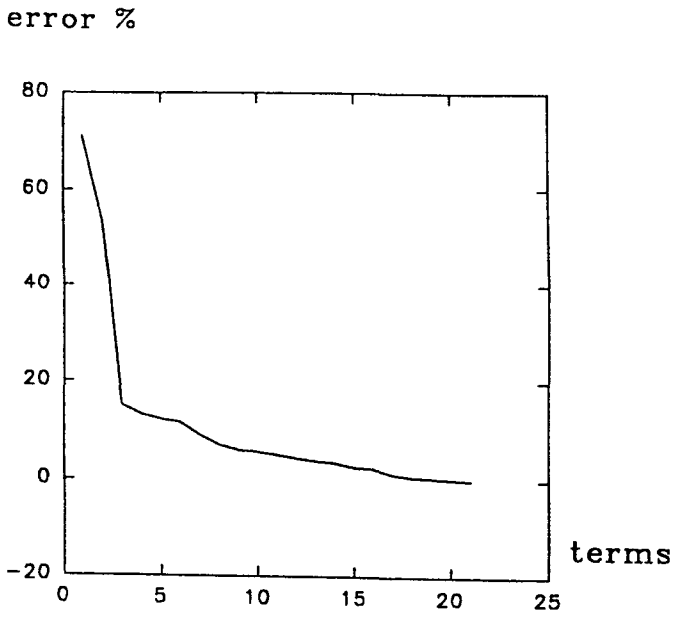


Figure 4.1 (a) Errors of Damping Ratios of the Last Mode

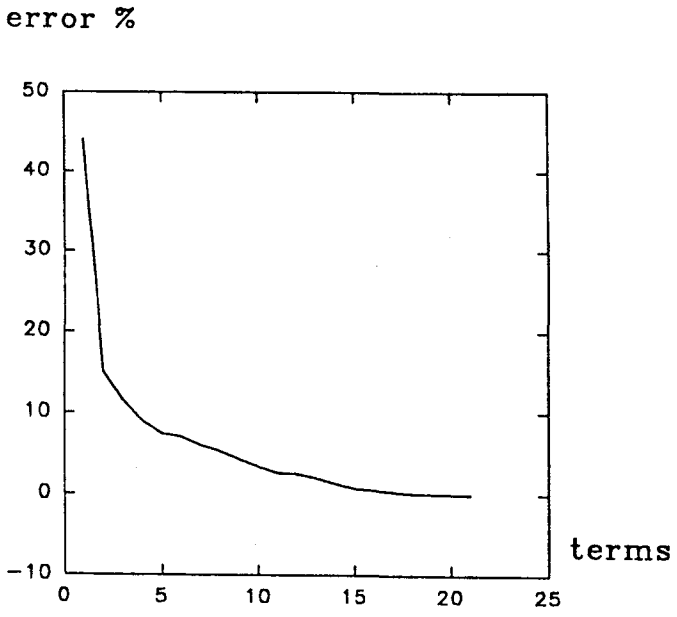


Figure 4.1 (b) Errors of Damping Ratios of the 1st Mode

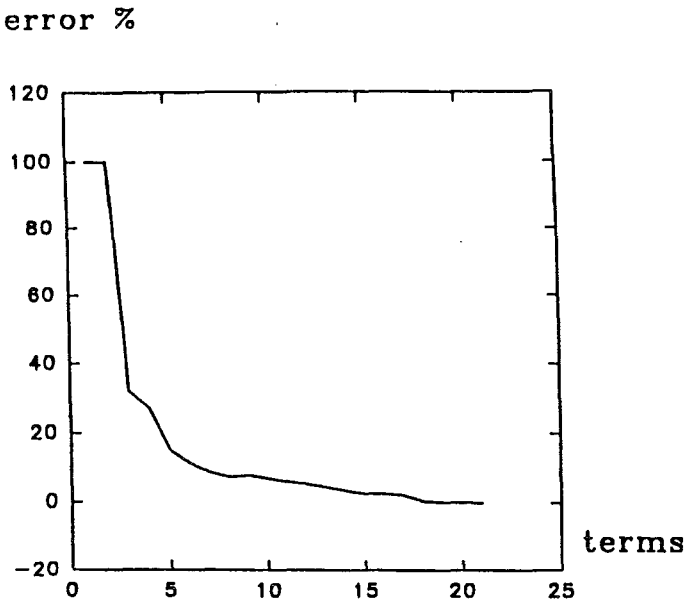


Figure 4.1 (c) Errors of Imaginary Damping Ratios of the Last Mode

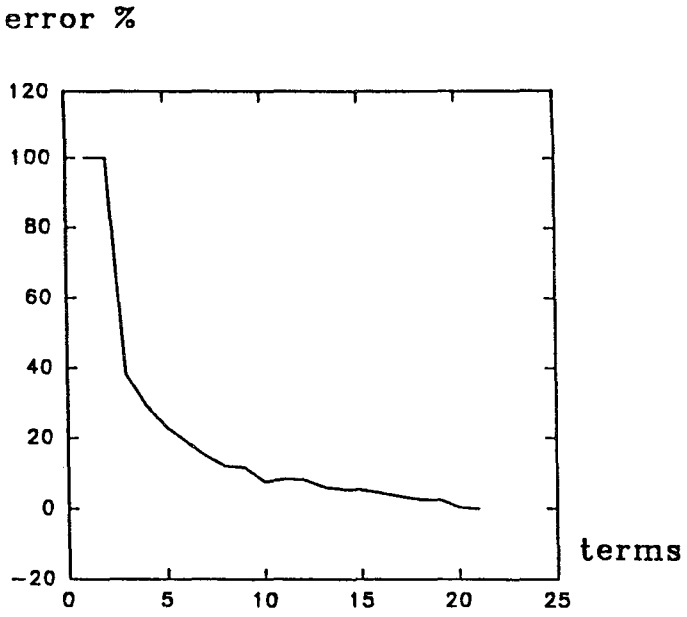


Figure 4.1 (d) Errors of Imaginary Damping Ratios of the 1st Mode

A simplified formula of damping representation is presented based on the matrix representation theorem. The simplified representation contains only the first few terms of the matrix-product of mass and stiffness matrices (one more term than Rayleigh damping). The simplest case consists of only three terms, the mass, stiffness and identity matrices. This three term equation can be easily adopted to finite element algorithms or to system identification formulations. The simplified approach can yield the complex mode and is shown to be more accurate than using Rayleigh damping in dynamic structural analysis. Furthermore, this approach is simpler than other known methods for handling non-proportional damping and can result in significant savings of computing time.

However, it should be noted that the approximation of the damping matrix by equation (4.2) does not necessarily provide the best result in all cases. Care must be exercised in the application of equation (4.2).

4.6 COMPLEX DAMPING IN STRUCTURALLY DAMPED SYSTEMS AND DAMPING REPRESENTATION

4.6.1 EIGEN-MATRIX OF STRUCTURALLY DAMPED SYSTEMS

Proportional and non-proportional damping also exist in structurally damped systems. If the damping is non-proportional, both energy dissipation and energy transmission occur in a manner similar to that of a system with viscous damping.

Whatever the nature of its damping, a structurally damped system can always be decoupled into n individual modes. This is one of the major differences between viscously and structurally damped systems. To show this, we rewrite the homogeneous equation of a structurally damped system in the following form:

$$\mathbf{M}\, X'' + (j\, \mathbf{C} + \mathbf{K})\, X = 0$$

Or,

$$X'' + \mathbf{H}\, X = 0 \qquad (4.29)$$

where

$$\mathbf{H} = j\,\mathbf{M}^{-1}\,\mathbf{C} + \mathbf{M}^{-1}\,\mathbf{K} \tag{4.30}$$

An nxn matrix **P** is available to decouple the system (4.29) into n individual equations or modes, if **P** is the eigenvector matrix of **H**. To further examine this difference between viscously and structurally damped systems, let

$$X = \mathbf{P}\,E \tag{4.31}$$

where, **P** is an nxn matrix which is different from the case of a viscously damped system, and

$$E = \begin{Bmatrix} \exp(\lambda_1 t) \\ \cdots \\ \exp(\lambda_n t) \end{Bmatrix}$$

Substituting (4.31) into equation (4.29) and considering the spatial part only, we have the following eigen-problem:

$$\mathbf{P}\,\Lambda^2 + \mathbf{H}\,\mathbf{P} = \mathbf{0} \tag{4.32}$$

where

$$\Lambda^2 = \mathrm{diag}(\lambda_i) \tag{4.33}$$

Here, we call matrix **H** the *state matrix* and define matrix **A** the *eigen-matrix* of a structurally damped system if **A** satisfies the following relationship:

$$\mathbf{A}^2 + \mathbf{H} = \mathbf{0} \tag{4.34}$$

One important property of system (4.29) is that its state matrix and eigen-matrix commute, i.e.,

$$\mathbf{H}\,\mathbf{A} = \mathbf{A}\,\mathbf{H} \tag{4.35}$$

By using the definition of mode shapes in viscously damped systems, it can be seen that the mode shape **P** is real-valued if and only if

$$\mathrm{Re}(\mathbf{H})\,\mathrm{Im}(\mathbf{H}) = \mathrm{Im}(\mathbf{H})\,\mathrm{Re}(\mathbf{H}) \tag{4.36}$$

Equation (4.31) is a different version of the Caughey criterion

$$\mathbf{C}\,\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\,\mathbf{M}^{-1}\mathbf{C}$$

It can also be seen that

$$\mathrm{Re}(\mathbf{A})\ \mathrm{Im}(\mathbf{A}) = \mathrm{Im}(\mathbf{A})\ \mathrm{Re}(\mathbf{A}) \tag{4.37}$$

if and only if (4.36) is true or **P** is weakly complex. So, Theorem A.5 also holds for structurally damped systems.

4.6.2 LIGHTLY DAMPED STRUCTURES

In the following, it will be shown that Theorem 2.1 also applies to lightly damped structures. For convenience, equation (4.29) can be rewritten as

$$\mathbf{I}\ Y'' + (j\tilde{\mathbf{C}} + \tilde{\mathbf{K}})Y = \mathbf{0} \tag{4.38}$$

where **C** and **K** are defined in equation (2.2).

Then, in equation (4.38), $\mathbf{H} = j\tilde{\mathbf{C}} + \tilde{\mathbf{K}}$. As in the case of viscous damping, we multiply the transpose of eigenvector matrix, $\mathbf{Q}^T$, on the left side of equation (4.38)

$$\mathbf{Q}^T\ \mathbf{P}\ \Lambda^2 + \mathbf{Q}^T\ (j\tilde{\mathbf{C}} + \tilde{\mathbf{K}})\ \mathbf{P} = \mathbf{0}$$

Considering the i^{th} equation, slightly different from equation (2.18), we will have

$$\lambda_i^2 + (j\ \mathbf{Q}_i^T\ \tilde{\mathbf{C}}\ \mathbf{P}_i)/r_{ii} + \omega_{ni}^2 = 0 \tag{4.39}$$

where r_{ii} is the ii^{th} entry of matrix $\mathbf{R} = \mathbf{Q}^T\mathbf{P}$. Then we have

$$r_{ii} = \frac{1}{\omega_{ni}^2}\ \mathbf{Q}_i^T\ \tilde{\mathbf{K}}\ \mathbf{P}_i \tag{4.40}$$

Substituting (4.40) into (4.39) yields

$$\lambda_i^2 = -\ \omega_{ni}^2\ \left(\frac{j\ \mathbf{Q}_i^T\ \tilde{\mathbf{C}}\ \mathbf{P}_i}{\mathbf{Q}_i^T\ \tilde{\mathbf{K}}\ \mathbf{P}_i} + 1\right) \tag{4.41}$$

We then define the i^{th} *complex damping coefficient* $\not{d}_i$ as follows:

$$d_i = a_i - jb_i = \omega_{ni}^2 \frac{Q_i^T \tilde{C} P_i}{Q_i^T \tilde{K} P_i} \tag{4.42}$$

From the above discussion and equation (4.36), we know that if the structure is proportionally damped,

$$b_i = 0 , \qquad i = 1,...,n \tag{4.43}$$

Since the system is assumed to be lightly damped, in parallel with the case of viscous damping, we may define a quantity υ_i by

$$\upsilon_i = \frac{d_i}{\omega_{ni}^2} = \frac{Q_i^T \tilde{C} P_i}{Q_i^T \tilde{K} P_i} = \alpha_i - j\,\beta_i \tag{4.44}$$

Substitution of (4.44) into (4.41) yields

$$\lambda_i^2 = - \omega_{ni}^2(1 + \beta_i + j\alpha_i) = - [\omega_{ni}^2(1+ \beta_i)] \left[(1 + \frac{j\alpha_i}{1+ \beta_i})\right] \tag{4.45a}$$

Equation (4.45a) is obtained without a light damping assumption. Recall that the natural frequency λ_i in viscously damped systems has cross terms of real and imaginary parts of complex damping ratios. However, in structurally damped systems, we have clear separation of $[\omega_{ni}^2(1+ \beta_i)]$ and $\left[(1 + \frac{j\alpha_i}{1+ \beta_i})\right]$ in equation (4.45a).

With a light damping assumption, equation (4.45a) can be further simplified as

$$\lambda_i^2 \approx -\omega_{ni}^2 \exp(\beta_i) \left(1 + j\frac{\alpha_i}{1+ \beta_i} \right) \tag{4.45b}$$

Then, we have

$$\lambda_i = \pm j\omega_{ni}(1 + \beta_i + j\alpha_i)^{1/2} = \pm j\omega_{ni} \left(1 + \frac{j\alpha_i}{1+ \beta_i}\right)(1+ \beta_i)]^{1/2}$$

$$\approx \pm j\omega_{ni} \exp(\beta_i/ 2) \left(1 + j\frac{\alpha_i/ 2}{1+ \beta_i} \right) \tag{4.46}$$

Comparing equation (4.46) with equation (1.58), we can write

$$\lambda_i = \omega_{si} = j\omega_{ni} \exp(\iota_i)(1+ j\eta_i/2) \tag{4.47}$$

First, we see that the natural frequency is changed by a factor of

$$\iota_i = \beta_i / 2 \tag{4.48}$$

This result is similar to the case of viscous damping. If the system is proportionally damped, β_i's are equal to zero and the natural frequencies remain unchanged from those of the corresponding undamped system $\mathbf{MX''}+ \mathbf{KX} = \mathbf{0}$. However, if the system is non-proportionally damped, β_i's are not equal to zero, and the frequencies are changed by ι_i's, a result of the existence of damping matrix **C**. In other words, if the damping is not proportional, energy is transferred among modes.

Secondly, we see that

$$\frac{\alpha_i}{1+ \beta_i} = \eta_i \tag{4.49}$$

is the *loss factor* of the i^{th} mode (virtual) of the structurally damped system (1.1). η_i is not only affected by the imaginary part of the complex damping coefficient, but also affected by its real part. This is one of the major differences between viscous damping and structural damping.

A second difference can be seen by comparing equation (4.44) with equation (3.10). For viscous damping,

$$a_i = 2\, \xi_i\, \omega_{ni} \,,\; b_i = 2\, \zeta_i\, \omega_{ni}$$

but in the case of structural damping (here suppose β_i is sufficiently small)

$$a_i \approx \eta_i\, \omega_{ni}^2 \,, \qquad b_i = 2\, \iota_i\, \omega_{ni}^2 \tag{4.50}$$

They differ by a factor ω_{ni} .

If the structure is proportionally damped, then b_i's = 0. Therefore,

$$\lambda_i = \omega_{si} = j\omega_{ni}(1+ ja_i/2) \tag{4.51}$$

In this case, if two lightly damped systems $\mathbf{H}(\mathbf{C}_1)$ and $\mathbf{H}(\mathbf{C}_2)$ are given, then for the third system $\mathbf{H}(\alpha\mathbf{C}_1+\beta\mathbf{C}_2) = \mathbf{H}(\mathbf{C}_3)$, we have

$$\Lambda_{re}^{(3)} = \Lambda_{re}^{(1)} = \Lambda_{re}^{(2)} \tag{4.52}$$

$$\Lambda_{im}^{(3)} = \alpha\, \Lambda_{im}^{(1)} + \beta\, \Lambda_{im}^{(2)} \tag{4.53}$$

The equation can be proven by noting that $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ have identical eigenvectors. Comparing the above equations with equations (3.36a) and (3.35), we have strict equality for structurally damped systems (with proportional damping).

For non-proportionally damped systems, we can write

$$\Lambda_{re}^{(3)} \approx \Lambda_{re}^{(1)} \approx \Lambda_{re}^{(2)} \tag{4.54}$$

$$\Lambda_{im}^{(3)} \approx \alpha\, \Lambda_{im}^{(1)} + \beta\, \Lambda_{im}^{(2)} \tag{4.55}$$

First, equation (4.54) can be understood by recalling equation (1.59a). Then, following the same procedure in the proof of Theorem (3.4), we can prove (4.55).

4.6.3 DAMPING MATRIX REPRESENTATION

Similar to viscously damped systems, the structural damping matrix can also be represented by the polynomial of the mass and stiffness matrices, namely,

$$\mathbf{C} = \alpha\, \mathbf{M} + \beta\, \mathbf{K} + \gamma\, \mathbf{I} + \ldots\ldots \tag{4.56}$$

The conclusions of the contributions by matrices **M**, **K** and **I** to the real part of the complex damping still hold, although each formula will be divided by the factor ω_{ni} according to equation (4.50). The only difference is, in this case we use the loss factor η_i instead of damping ratio ξ_i. Therefore, equation (4.56) has the same advantages as the viscously-damped-system over the Rayleigh damping.

However, comparing equations (4.50) and (3.10) results in the difference of contributions to the imaginary part of the complex damping between the two cases. Since the ζ_i and ι_i differ by the factor ω_{ni}, we can see that the identity matrix will contribute more to the imaginary part for structurally damped systems.

5 DAMPING COEFFICIENT MATRICES

5.1 INTRODUCTION

Of the three coefficient matrices, mass **M**, damping **C** and stiffness **K** in a dynamic system described by (1.1a), **K** and **M** are relatively easy to determine. For example, we can use the finite element approach (4.8) and (4.13) to construct the mass and stiffness matrices, respectively. However, we do not in general have an explicit approach to construct the damping matrices. This is the reason we introduced damping matrix representation in the previous chapter. It provides an approximation of the damping matrix for design purposes. The coefficients in such a representation can be obtained from experimental data. Therefore, detailed information about damping materials used in the structure is not needed. This is an advantage of the matrix representation method. However, in many cases, we do need exact damping coefficient matrices (also called damping matrices in the literature). This chapter is concerned with methods of assembling damping coefficient matrices.

In vibration control or damping design, damping calculations may be carried out by two approaches. The first is to take the damping ratios for the design parameters (see Keel et al (1986) for example). This is the easier approach because several well established formulas are available. However, in many cases, the effects of damping control need to be evaluated by responses under different forcing functions. Knowing only the parameters of damping ratios, it is not possible to perform response calculations.

The second method of damping design and/or vibration control, which takes into consideration the damping matrices as design parameters, is more attractive in damping calculations. General damping matrices are often non-proportional. However, in most engineering practice, a simple design procedure is used in which the proportional damping matrices are employed. They are obtained from the following formula:

$$\tilde{C} = Q \; \mathrm{diag}(2\xi_i\omega_i) \; Q^T$$

In the literature, there are various methods for calculating the damping ratios and the corresponding damping matrices, such as the modal-damping-ratio methods suggested by Zhang (1989) and the modal-strain-energy methods suggested by Johnson et al (1982) and by Soong et al (1991). These approaches are essentially identical. They can be unified by a proportional damping formula, which will be briefly discussed in Section 5.3.1.

The proportional damping assumption is often unsatisfactory (not sufficiently precise) for damping designs and vibration controls and will be discussed in this chapter. It will be shown that, although a universal approach is not available for damping determination, it is possible to establish criteria for special cases of damping design, such as VE damper design. To accomplish this, we must first identify an important property of the damping coefficient - i.e. there exist certain invariant quantities in a damper-added system when the dampers are given. We can model the damping invariants by using the force method. These invariant quantities enable us to extend a SDOF system approach to models of MDOF systems without the proportional damping assumption. This relationship between the invariant quantities and damping matrices enables us to determine the general damping matrix, if mass and stiffness matrices and loss factors or loss moduli of the damping materials are known. Application to VE-type damper design is also described in this chapter.

5.2 A SPECIAL PROPERTY OF THE DAMPING COEFFICIENT

Within the scope of Newtonian mechanics, both the mass and the stiffness of a structure are invariants of time and space. For example, a wide-flange steel structural member of certain length has certain mass and stiffness regardless where and how this structural member is supported and loaded.

However, the damping coefficient is often not invariant with respect to time and space. Nor does a given commercially available damper always contribute the same damping coefficient to a structure regardless of where the damper is attached. This is a characteristic of the damping coefficient which is useful in damper design.

To illustrate this viewpoint, we first use the energy method (see Timoshenko, 1982) to analyze an isolated damper (Figure 5.1).

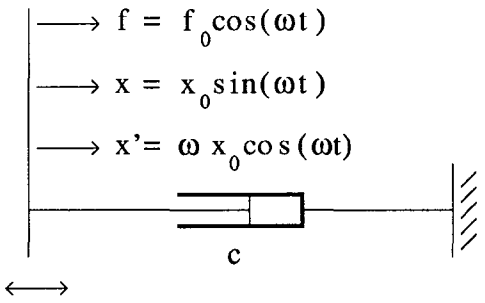


Figure 5.1 A Damper with One End Fixed

The forced displacement x in Figure 5.1 is 90° out of phase with respect to the force but its velocity x' is in phase with the force. Now consider the work done by the external force $f = f_0\cos(\omega t)$, where the subscript 0 denotes amplitudes and ω is a driving frequency. The work done W_f by the external force in one cycle of motion from 0 to $2\pi/\omega$ is given by

$$W_f = \int_0^{2\pi/\omega} f\,dx = \int_0^{2\pi/\omega} f_0\cos(\omega t)\;\omega\;x_0\cos(\omega t)\,dt$$

$$= \int_0^{2\pi/\omega} f_0\;x_0\cos^2(\omega t)\,d\omega t = f_0\;x_0/2\;(\omega t + \sin(\omega t)\cos(\omega t))\Big|_0^{2\pi/\omega}$$

$$= \pi\;f_0\;x_0$$

The corresponding work W_d done by the damper is

$$W_d = \int_0^{2\pi/\omega} cx'dx = \int_0^{2\pi/\omega} cx_0\omega\cos(\omega t)\;\omega\;x_0\cos(\omega t)\,dt$$

$$= \int_0^{2\pi/\omega} c\;\omega\;x_0^2\cos(\omega t)\,d\omega t = c\omega\;x_0^2/2\;(\omega t + \sin(\omega t)\cos(\omega t))\Big|_0^{2\pi/\omega}$$

$$= \pi\;c\omega\;x_0^2$$

Since $W_f = W_d$ we have

$$\pi f_0 x_0 = \pi c\omega x_0^2 \tag{5.1}$$

Then, we can let

$$\beta = \frac{f_0}{x_0} = c\ \omega \tag{5.2}$$

Equation (5.2) can be directly obtained by equating the damping and the external forces. The reason we use the energy equation is for convenience in later discussions. From (5.2), if the value of c is fixed, β will be proportional to ω. However, in many cases, β is measured to be non-proportional to ω. In this case, c will vary due to different value of ω's. We will see that the ratio β plays a key role in establishing the damping matrix.

We now consider a damped SDOF system where the damping can be expressed as a function of mass and stiffness. For a SDOF system, the equation of motion may be written as

$$m\ x'' + c\ x' + k\ x = \bar{f}$$

Alternatively, we can write

$$x'' + 2\xi\omega\ x' + \omega^2\ x = f$$

where c/m is defined to be $2\xi\omega$, k/m is defined to be ω^2 and

$f = \bar{f}/m$.

Therefore,

$$\xi \propto \frac{c}{\omega} \tag{5.3a}$$

or

$$c \propto \xi\ \omega \tag{5.3b}$$

Suppose the forcing function can be expressed by $f = f_0 \sin(\omega t)$. We have the displacement $x = x_0 \cos(\omega t)$. Then the energy dissipated W_d can be calculated by

$$W_d = \int_0^{2\pi/\omega} \frac{c}{m} x' dx = \frac{c}{m} x_o^2 \omega\pi \tag{5.4}$$

Let us consider a damper that is made of viscoelastic material. The energy dissipated by the damper is given by (see Lin (1989))

$$W_d = [\int_0^{2\pi/\omega} [\sigma \, d\gamma]] V = \frac{G_1 A}{t} x_0^2 \pi \tag{5.5}$$

where σ is stress, γ is strain, V is the volume of the damping material, G_1 is the loss modulus, A and t are the effective area and the thickness of the damping material, respectively.

Equating the dissipated works by the SDOF system and by the damper, we have

$$\frac{c}{m} x_o^2 \omega \pi = \frac{G_1 A}{t} x_0^2 \pi$$

Therefore,

$$c \propto \frac{G_1}{\omega} \tag{5.6}$$

or

$$G_1 \propto c \omega \tag{5.7}$$

Combining equations (5.2) and (5.6), we have

$$\xi \propto \frac{c}{\omega} \propto \frac{G_1}{\omega^2} \tag{5.8}$$

or we have

$$G_1 \propto c \omega \propto \xi \omega^2 \tag{5.9}$$

In the case of a SDOF, the relationships of G_1, c and ξ are given in Table 5.1

Table 5.1 Relationship among ξ, c and G_I

Loss modulas G_I	damping coefficient c	damping ratio
$\propto 1/\omega$	$\propto 1/\omega^2$	$\propto 1/\omega^3$
= constant	$\propto 1/\omega$	$\propto 1/\omega^2$
$\propto \omega$	= constant	$\propto 1/\omega$
$\propto \omega^2$	$\propto \omega$	= constant
$\propto \omega^3$	$\propto \omega^2$	$\propto \omega$

We see that in a SDOF system, if m and k are given, the value of the natural frequency ω is fixed, and hence according to Table 5.1, G_I , c and ξ are all fixed. However, when two identical dampers are mounted on two different structures of different m and k, different values of G_I or c and/or ξ are likely to result.

In the case of a viscoelastic damper, the loss modulus is almost constant in a wide frequency range (see Lin (1989), also see Mahmoodi (1985)). Therefore, both the damping coefficient and the damping ratio will change more or less by the factor $1/\omega$ and the factor $1/\omega^2$, respectively.

Similar to equation (5.4), we can calculate the work done by the driving force, that is,

$$W_f = \int_0^{2\pi/\omega} f\, dx = f_0\, x_0\, \pi \qquad (5.10)$$

Since the work done W_f must be equal to the energy dissipated by the damper W_d, from equations (5.1) and (5.10), we have

$$f_0\, x_0\, \pi = \frac{G_I\, A}{t}\, x_0^2\, \pi$$

Or

$$\frac{f_0}{x_0} = \beta \qquad (5.11)$$

Here

$$\beta = \frac{G_1 A}{t} \tag{5.12}$$

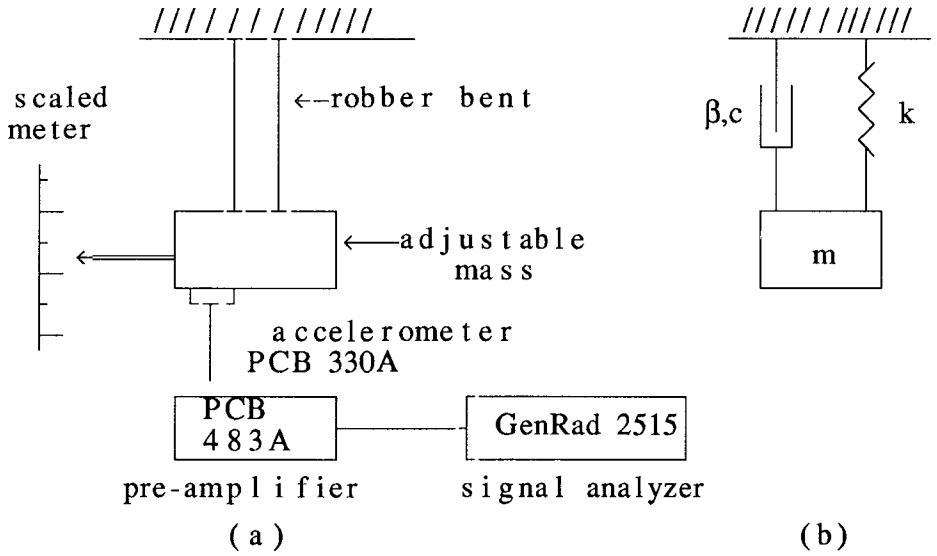
Then, we have

$$c = \frac{\beta}{\omega} \tag{5.13}$$

Equation (5.13) is of the same form as equation (5.2).

We call this factor defined by equation (5.13) the *loss* β. Now, suppose G_1 is not proportional to the frequency ω. (In practice, we rarely have the loss modulus proportional to ω. Rather, it is likely to be constant for most VE material in certain frequency ranges.) Further, ω is only determined by the mass and stiffness coefficients of the system. In conclusion, we have the following:

Two identical dampers, with their loss β not proportional to ω, are mounted on two separate m-k SDOF systems. If the natural frequencies of the two m-k systems are different, the corresponding damping coefficients are different. Especially, if the loss β of the identical dampers is constant, the corresponding coefficients are proportional to $1/\omega$.



Figures 5.2 (a and b) An Experiment with a SDOF System

In order to explain the above mathematical statement physically, we have carried out the following SDOF system.experiment.

As shown in Figures 5.2 (a), a robber bent is fixed to the ceiling at one end and connected to an adjustable mass at other end. An accelerometer is placed at the bottom face of the adjustable mass. The signal, through a pre-amplifier, is analyzed by a digital signal analyzer that can display the natural frequency and damping ratio of the system. The static displacement is recorded by a ruler. Figure 5.2 (b) shows the corresponding analytical model.

First, we increase the mass and record the corresponding displacements. The static force vs. displacement relationship (a hysteristic loop) is shown in Figure. 5.3. We then conclude that the robber bent is made of viscoelastic material (Lazan, 1966.)

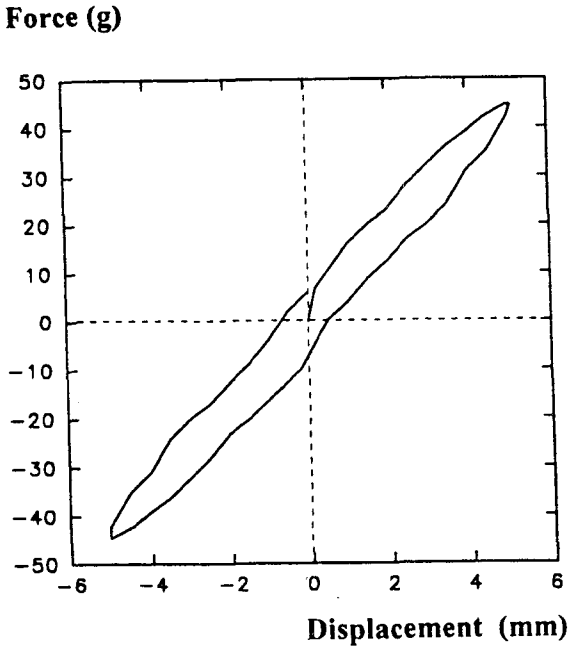


Figure 5.3 Experimental Damping Loop

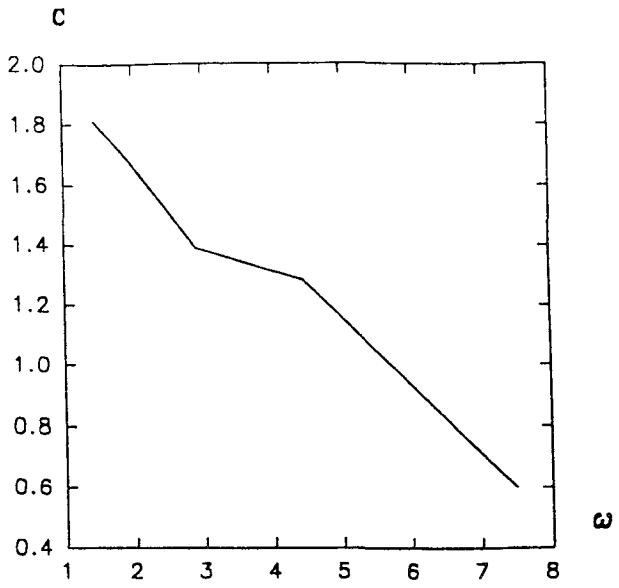


Figure 5.4 (a) Experimental Damping Coefficient vs. Frequency

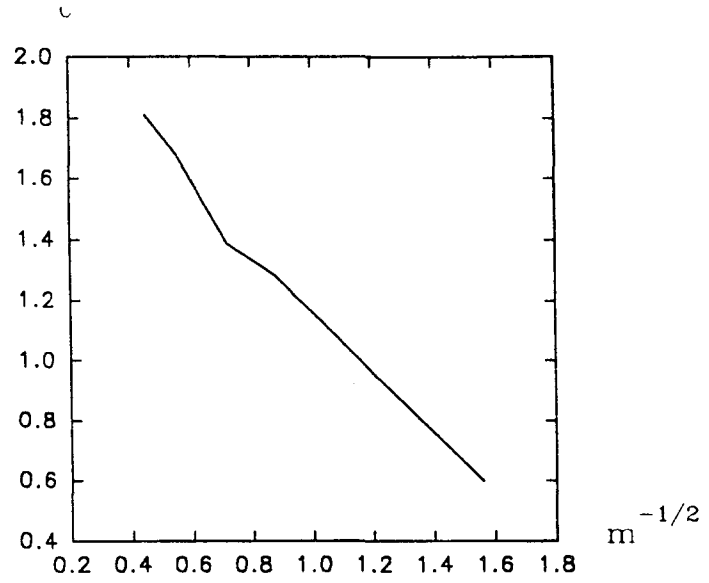


Figure 5.4 (b) Experimental Damping Coefficient vs. $m^{-1/2}$

Next, we perform dynamic tests. Under each fixed mass, the corresponding natural frequency and damping ratio are measured by 30 averages. The corresponding damping coefficient is calculated by using

$$c = 2 \xi \omega m$$

The tested results are shown in Figures 5.4 (a) and 5.4 (b). They verify the mathematical theorem.

The above experiment shows that there is an explicit relationship between the loss β and the damping coefficient of a SDOF system.

5.3 A GENERAL METHOD FOR THE EVALUATION OF DAMPING MATRICES

In this section, we discuss damper design for MDOF systems, this chapter major emphasis.

5.3.1 PROPORTIONALLY DAMPED SYSTEMS

Let us first consider the situation of a proportionally damped system and start from the monic equation,

$$\mathbf{I}\,\tilde{X}'' + \tilde{C}\,\tilde{X}' + \tilde{K}\,\tilde{X} = \tilde{F}$$

and

$$\tilde{C}\,\tilde{K} = \tilde{K}\,\tilde{C}$$

From Appendix A, we know that for such a system both damping and stiffness matrices have the identical eigenvector matrix and it is also the eigenvector matrix of the system itself. If we know a priori the mass and the stiffness matrices, we will also know the eigenvector matrix **Q**. Therefore, if we can obtain the damping ratio, we can solve for the damping matrix from

$$\tilde{C} = \mathbf{Q}\ \mathrm{diag}(\ 2\ \xi_i \omega_i)\ \mathbf{Q} \qquad (5.13)$$

As mentioned before, Zhang et al (1990) suggested a formula to determine the damping ratios by

$$\xi_i = \frac{W_{d\,i}}{4\pi\ W_{ki}} = \frac{\sum_j G_{l\,i}\,\gamma^2_{i\,j}\,V_j}{1/2\ Q_i^T\ \tilde{\mathbf{K}}\ Q_i} \tag{5.14}$$

Alternatively, the modal strain energy method for viscoelastic damper design can be calculated (see Johnson (1982), also see Soong and Lai (1991)). The i^{th} damping ratio is calculated by

$$\xi_i = \frac{\eta_{v\,i}}{2}\,\frac{E_{v\,i}}{E_{si}} \tag{5.15}$$

where $\eta_{v\,i}$ is the loss factor of the viscoelastic material, E_{vi} and E_{si} are the modal strain energy of the system and the energy stored in the damping material respectively, and

$$E_{vi} = Q_i^T\ \mathbf{K}_v\ Q_i/2 \quad \text{and} \quad E_{si} = Q_i^T\ \tilde{\mathbf{K}}\ Q_i/2.$$

Here $\mathbf{K}_s$ is the stiffness matrix attributed to the added dampers. Note that $\eta_{vi} = G_{li}/\ G_{si}$, where G_{si} is the shear modulus of the viscoelastic material for the i^{th} mode, which is directly related to the stiffness $\mathbf{K}_{si}$. It can be shown theoretically that the strain energy method is identical to the modal energy formula (5.14).

In equation (5.14), the shear strain is given by

$$\gamma_j = \bar{q}_{ij} / t_j$$

where $\bar{q}_{ij}$ is the relative displacement between the two ends of the j^{th} damper. Regardless of whether or not dampers exist between masses j and k, we always have

$$\bar{q}_{ij} = q_{ij} - q_{ik}$$

Note that

$$V_j = A_j\ t_j$$

The numerator in equation (5.14) becomes

$$\sum_j \frac{G_{1i} A_j}{t_j} \overline{q}^2_{ij} = Q_j^T \mathbf{B}_i Q_j$$

Here, the matrix $\mathbf{B}_i$ is determined by both the damper configuration for specific structures and loss β of the dampers. The following 3-DOF structure illustrates the approach for the determination of the $\mathbf{B}_i$ matrix.

Example:

The force-method is used to determine the $\mathbf{B}_i$ matrix of the structure shown in Figure 5.5. We first find the loss β coefficients by using the force-method with corresponding frequency ω_i; then, the coefficient matrix $\mathbf{B}_i$ will be assembled with respect to the global coordinates.

For the first mode, the natural frequency is ω_1, the structure is assumed to be supported laterally at the 2nd and 3rd floor levels (see Figure 5.6).

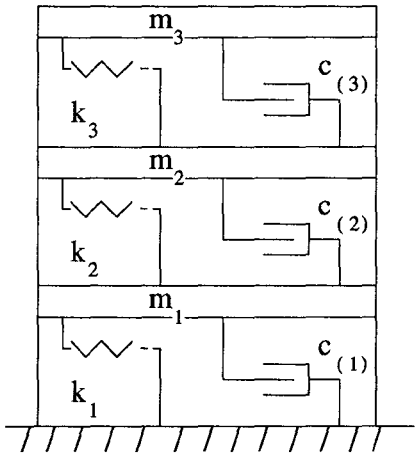


Figure 5.5 A 3-DOF System with Springs and Dampers

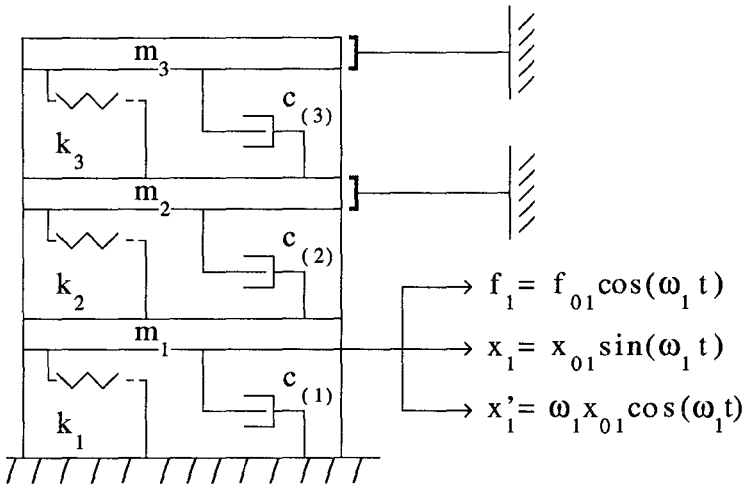


Figure 5.6 A 3-DOF System with 2nd and 3rd Floors Fixed to Grounds

For the first floor, the governing equation of motion is

$$m_1 x_1'' + (c_1 + c_2) x_1' + (k_1 + k_2) x_1 = f_1 = f_{01}\cos(\omega_1 t)$$

Then we see that,

$$x_1 = x_{01}\sin(\omega_1 t)$$

$$x_1' = \omega_1 x_{01}\cos(\omega_1 t)$$

Similar to the case mentioned in Figure. 5.2, for a SDOF systems, the work done by the external force f_1 during one cycle is equal to the energy dissipated by the damper itself. That is, $W_f = W_d$. Therefore, equation (5.2) or equation (5.11) still holds. The same results are also true for all other dampers. We thus can use the loss β to mark all the dampers shown in Figure. 5.7.

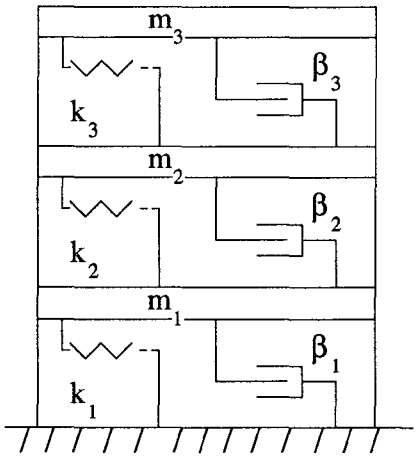


Figure 5.7 A 3-DOF System with Dampers Marked by Loss β

Consider the procedure to determine the loss β for the specific case of the VE damper design (the loss modulus G_1 and total volume V of the damping material are given). The energy dissipated during a cycle can be denoted by

$$W = \pi \bar{\beta} \gamma^2$$

where γ is the strain caused by deformation of the damping material, and $\bar{\beta}$ is a proportional coefficient. If the damper is a shear-damper, in the above case

$$\gamma = x_0 / t$$

where t is the thickness of the damping material.

Thus, we have

$$f_0 x_0 \pi = W_f = W_d = \pi \bar{\beta} \gamma^2 = \pi \bar{\beta}/t^2 x_0^2$$

or,

$$f_0 = \beta x_0 \qquad (5.16)$$

where the loss $\beta = \bar{\beta}/t^2$.

We now denote the coefficient β in the first case by β_1. Then

$$f_{01} = (\beta_1 + \beta_2)x_{01} \tag{5.17}$$

Following the same procedure, we have

$$f_{02} = (\beta_2 + \beta_3)x_{02} \tag{5.18}$$

and

$$f_{03} = \beta_3 x_{03} \tag{5.19}$$

Figures 5.8 and 5.9 explain the procedure for determining equations (5.18) and (5.19).

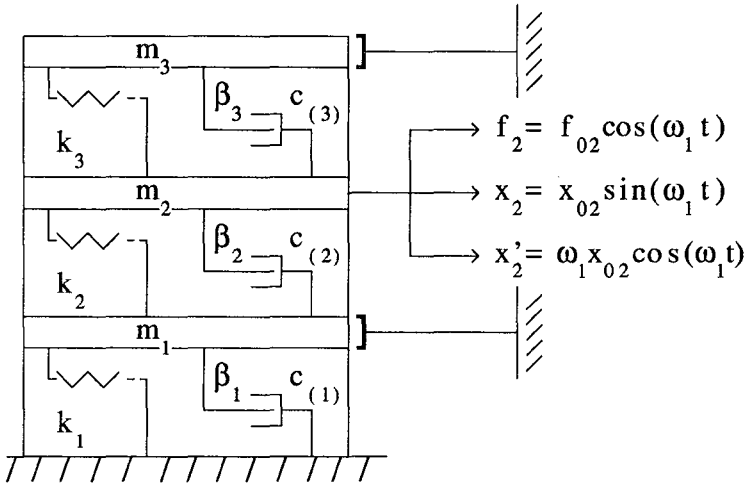


Figure 5.8 A 3-DOF System with 1st and 3rd Floors Fixed

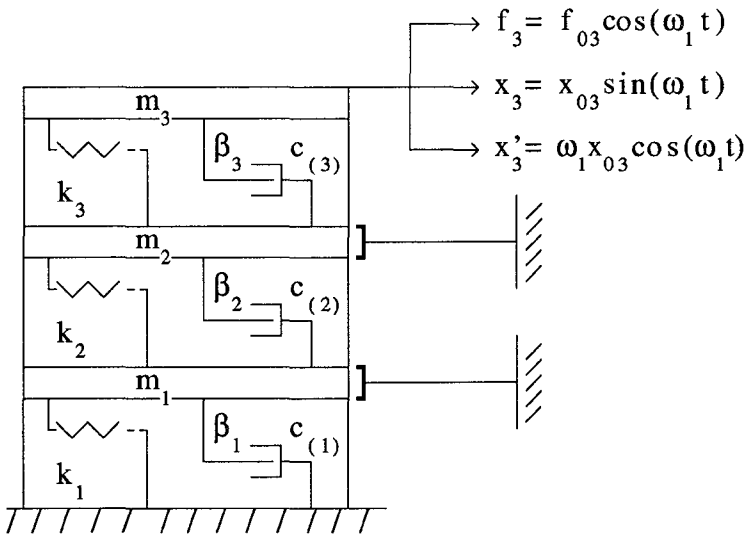


Figure 5.9 A 3-DOF System with 1st and 2nd Floors Fixed

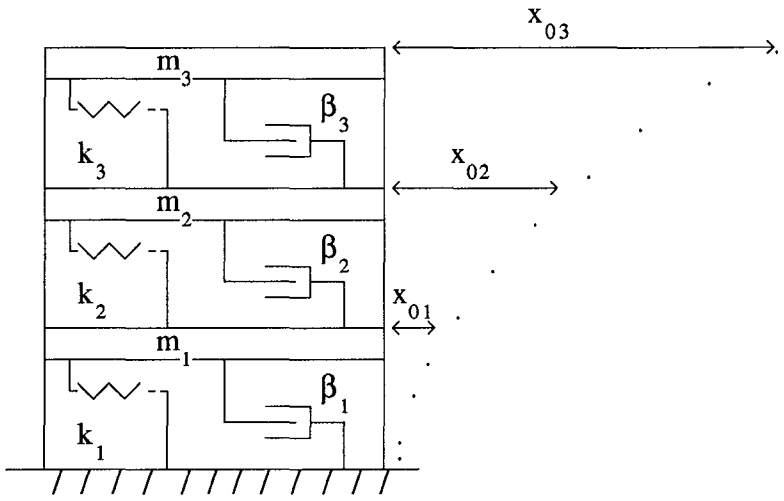


Figure 5.10 First Mode and its Mode Shape

Now, let us consider the first vibration mode and its mode shape shown in Figure 5.10. In this case, we see that

$$f_{01} = \beta_1 x_{01} + \beta_2 (x_{01} - x_{02})$$

$$f_{02} = \beta_2 x_{02} + \beta_3 (x_{02} - x_{03})$$

$$f_{03} = \beta_3 x_{03}$$

Note that the displacement x_{0i} is just the i^{th} element of the first mode shape, denoted by

$$Q_1 = \begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{Bmatrix} = \begin{Bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{Bmatrix}$$

In the matrix form, we can rewrite the damping force F_1 of the first mode by

$$F_1 = \begin{Bmatrix} f_{01} \\ f_{02} \\ f_{03} \end{Bmatrix} = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \\ 0 & -\beta_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{Bmatrix} \tag{5.20}$$

Then, similar to using the force method to assemble the stiffness matrix the $\mathbf{B}_1$ matrix can be assembled by

$$\mathbf{B}_1 = \begin{bmatrix} \beta_1 + \beta_2 & -\beta_2 & 0 \\ -\beta_2 & \beta_2 + \beta_3 & -\beta_3 \\ 0 & -\beta_3 & \beta_3 \end{bmatrix} \tag{5.21}$$

and, $F_1 = \mathbf{B}_1 Q_1$ (5.22)

We now extend the procedure of determining the damping force and the $\mathbf{B}_{(.)}$ matrices of the first mode to higher modes. That is, we have

$$F_i = \mathbf{B}_i Q_i , \quad i = 1, ..., n \tag{5.23}$$

In equation (5.23) the $\mathbf{B}_i$ matrix will have the same configuration as the $\mathbf{B}_1$ matrix. However, because the loss β will contain a different value of loss modulus which is typically a function of frequency, all the $\mathbf{B}_i$ matrices will be different by a certain constant coefficient. That is

$$\mathbf{B}_i = \mathbf{B}_i(\omega_i) \tag{5.24}$$

Therefore, we can in general represent the $\mathbf{B}_i$ matrix by a configuration matrix

$\mathbb{B}$ and a scalar β_i which equals to $G_1(\omega_i)$ A/t. So,

$$\mathbf{B}_i = \beta_i \, \mathbb{B} \tag{5.25}$$

Note that, the matrix $\mathbb{B}$ is only a function of the damper configuration and it is independent of the frequency. For example, suppose in the structure of Figure 5.5, we mount seven identical dampers, four between the ground and the first floor, one between first and the second floors, and two between the second and the third floors; then, the configuration matrix $\mathbb{B}$ should be

$$\mathbb{B} = \begin{bmatrix} 5+1 & -1 & 0 \\ -1 & 1+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Now, we can unify the formula suggested by Zhang and Johnson (5.14). With the help of the **B** matrix and equation (5.25), we rewrite the numerator of Zhang's formula, denoted by N, as follows,

$$N = \sum_j \frac{G_{1i} A_j}{t_j} \overline{q}^2_{ij} = Q_i^T \mathbf{B}_i Q_i = \beta_i Q_i^T \mathbb{B} Q_i = \frac{G_{1i} A}{t} Q_i^T \mathbb{B} Q_i$$

By using the factor $\eta_i = \dfrac{G_{1i}}{G_{si}}$ in Johnson's formula, we have

$$N = \eta_i \frac{G_{si} A}{t} Q_i^T \mathbb{B} Q_i = \eta_i Q_i^T \mathbf{K}_v Q_i$$

where the quantity $\dfrac{G_{si} A \, \mathbb{B}}{t}$ is actually the stiffness matrix of the damping material under frequency ω_i. This matrix can be denoted by $\mathbf{K}_{vi}$. If the shear modulus G_{si} is nearly constant within a certain frequency range, we can use one matrix $\mathbf{K}_v$ to approximate this quantity. Substitution of the expression of the numerator, N, into Zhang's equation (5.14) yields

$$\xi_i = \frac{\eta_i}{2} \frac{Q_i^T \mathbf{K}_{v\,i} Q_i}{Q_i^T \tilde{\mathbf{K}} Q_i}$$

which is exactly Johnson's formula.

Generally, the damping material together with the values of the loss modulus $G_l(\omega)$ and loss factor $\eta(\omega)$ will be provided by the manufactures. Therefore, both Zhang's and Johnson's formulas are applicable. Since Johnson's formula does not in general contain the variety of damper's stiffness, Zhang's formula may yield more precise results.

As soon as all the damping ratios are determined, the damping matrix $\tilde{\mathbf{C}}$ or $\mathbf{C}$ can be obtained from (5.13):

$$\tilde{\mathbf{C}} = \mathbf{Q}^T \text{ diag } (2\ \xi_i \omega_i)\ \mathbf{Q}$$

Note that, in equation (5.14) or (5.15), $\mathbf{Q}^T\ \tilde{\mathbf{K}}\ \mathbf{Q} = \omega^2$. Thus, in equation (5.13), we have

$$\tilde{\mathbf{C}} = \mathbf{Q}^T \text{ diag } (2\ \xi_i \omega_i)\ \mathbf{Q} = \mathbf{Q}^T \text{ diag } \left(2\ \frac{\beta_i\ \mathbf{Q}_i^T \mathbb{B}\ \mathbf{Q}_i}{2\ \omega_i^2}\ \omega_i\right) \mathbf{Q}$$

$$= \mathbb{B}\ \mathbf{Q}^T \frac{\beta_i}{\omega_i}\ \mathbf{Q} \qquad (5.26)$$

For the case of VE material, the loss β is nearly constant. Therefore, we may write

$$\tilde{\mathbf{C}} = \beta\ \mathbb{B}\ \tilde{\mathbf{K}}^{-1/2} \qquad (5.27)$$

According to Maxwell's Law, the damping matrix must be symmetric. Further, the configuration matrix $\mathbb{B}$ must also be symmetric. From equation (5.27), we have

$$\mathbb{B}\ \tilde{\mathbf{K}}^{-1/2} = (\ \mathbb{B}\ \tilde{\mathbf{K}}^{-1/2}\ ) = \tilde{\mathbf{K}}^{-1/2}\ \mathbb{B} \qquad (5.28)$$

We know that the two matrices commute, as shown in equation (5.28). Therefore they must have the same eigenvectors. Moreover, the damping matrix $\tilde{\mathbf{C}}$ must have the same eigenvectors as $\mathbb{B}$. According to Appendix A, we can verify again that the system is proportionally damped.

It should be noted that the $\mathbb{B}$ matrix is established from the configuration of dampers only. Then, we can see that the requirement of identical eigenvector matrices for both $\mathbb{B}$ and $\mathbf{K}$ is seldom satisfied. In fact, different damper

configurations may result in quite different damping matrices, which will affect greatly the damping design and control.

5.3.2 NON-PROPORTIONALLY DAMPED SYSTEMS - GENERAL APPROACH

From the previous chapters, we know that a non-proportional formulation for damper design is necessary if complex modes of the structure prevail. Consider again the equation of motion:

$$I\,\tilde{X}'' + \tilde{C}\,\tilde{X}' + \tilde{K}\,\tilde{X} = \tilde{F}$$

Assume that the system vibrates at a certain driving frequency ω_i with an amplitude vector R_i, (for a linear system, this assumption is always true). That is

$$X = R_i \sin(\omega_i t) = \begin{Bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{in} \end{Bmatrix} \sin(\omega_i t) \qquad (5.29)$$

where r_{ij} is the j^{th} real-valued amplitude of the displacement and no phase shift exist between those r_{ij}'s.

Substitution of equation (5.29) into the equation of motion yields

$$[-\omega_i^2 R_i \sin(\omega_i t) + \tilde{K} R_i \sin(\omega_i t)] + [\tilde{C} R_i \omega_i \cos(\omega_i t)] = F_i$$

Then we have

$$F_i = [(-\omega_i^2 I + \tilde{K}) R_i \sin(\omega_i t)] + [\tilde{C} R_i \cos(\omega_i t)] \qquad (5.30)$$

In equation (5.30), the first square bracket on the right hand side represents the conservative force. It is balanced by a component of the driving force F_i.

$$F_{ic} = {}_c F_i \sin(\omega_i t) = (-\omega_i^2 I + \tilde{K}) R_i \sin(\omega_i t) \qquad (5.31)$$

Similarly, to balance the term in the second square bracket of equation (5.30) (the dissipative force) the component of F_i is given by

$$F_{id} = {}_dF_i \cos(\omega_i t) = \tilde{C} R_i \cos(\omega_i t) \tag{5.32}$$

and

$$F_i = F_{ic} + F_{id} \tag{5.33}$$

In the above the real-valued quantities ${}_cF_i$ and ${}_dF_i$ are the amplitudes of conservative forces F_{ic} and damping F_{id} respectively.

When the system is proportionally damped and if ω_i and R_i are taken to be the i^{th} natural frequency and mode shape respectively, then

$${}_cF_i = (-\omega_i^2 I + \tilde{K}) R_i = 0$$

and

$$F_i = F_{id} = {}_dF_i \cos(\omega_i t)$$

In this case, it can be seen that the analysis by the force-method in the previous section (Figures 5.6, 5.8 and 5.9) is valid.

When the system is non-proportionally damped, both forces F_{ic} and F_{id} are non-zero in general. We only consider the force F_{id} and still have the same relationship between displacement X and force F_{id} as the case of proportional damping, that is,

$$X = R_i \sin(\omega_i t) \text{ and } F_{id} = {}_dF_i (\cos\omega_i t) = \begin{Bmatrix} {}_df_{i1} \\ {}_df_{i2} \\ \vdots \\ {}_df_{in} \end{Bmatrix} \cos(\omega_i t)$$

where ${}_df_{ij}$ is the damping force amplitude at the j^{th} point. It can be seen that both X and F_{id} have real-valued amplitude and X has the phase 90^o leading to F_{id}. Note that there is no phase shift between those ${}_df_{ij}$'s. Therefore, the force-method (Figures 5.6, 5.8 and 5.9) still holds for non-proportionally damped system, that is

$$_{d}F_{i} = \mathbf{B}_{i} R_{i}$$

or

$$_{d}F_{i} = \beta_{i} \mathbb{B} R_{i} \tag{5.34}$$

On the other hand, we have

$$_{d}F_{i} = \tilde{\mathbf{C}} R_{i}\omega_{i} \tag{5.35}$$

Now, substituting equation (5.35) into (5.34) yields

$$\tilde{\mathbf{C}} R_{i} = (\beta_{i}/\omega_{i}) \mathbb{B} R_{i} \qquad i = 1,...n \tag{5.36}$$

There are an infinite number of equations (5.36), but n equations are mathematically sufficient for the determination of the damping matrix $\tilde{\mathbf{C}}$. Therefore, we let i = 1,...n. Equation (5.36) becomes a generalized eigen-problem. We then can determine the matrix $\tilde{\mathbf{C}}$ if matrix $\mathbb{B}$ and scalars β_{i}/ω_{i}'s are given (see Ortega, 1988).

Note that the generalized eigen-problem requires only n pairs of eigenvalues and eigenvectors. In equation (5.29), the frequency ω_{i} and amplitude R_{i} cannot be chosen arbitrarily, In a latter discussion, we will see that the ω_{i}'s are in fact the natural frequencies of the system. In this way, the ω_{i}'s and R_{i}'s are uniquely determined.

If both $\mathbb{B}$ and $\tilde{\mathbf{C}}$ are symmetric (which are guaranteed by Maxwell's Law), we have

$$\tilde{\mathbf{C}} = \mathbf{R}^{T} \text{ diag } (\tilde{\beta}_{i}/\omega_{i}) \mathbf{R} \tag{5.37}$$

where $\tilde{\beta}_{i}$ is the i^{th} eigenvalue of matrix $(\beta_{i}\mathbb{B})$, and $\mathbf{R}$ is the eigenvector matrix of matrix $\mathbb{B}$.

From equations (5.36) and (5.37), if matrix $\mathbb{B}$ has the same eigenvector matrix as that of matrix $\tilde{\mathbf{K}}$, the generally damped system becomes proportionally damped. In this case, equation (5.26) as well as Johnson's and Zhang's

formulas can be obtained from this approach also. However, the condition of proportional damping is seldom satisfied. In practical damper design, it is difficult to install dampers such that $\mathbb{B}$ and $\tilde{\mathbf{K}}$ have the same eigenvectors.

The relationship described in equations (5.24) and (5.34) can also be obtained by the energy-method, which states that at any particular point j, the work done by external damping force (denoted by ${}_{d}F_{j}$), ${}_{ed}W_{j}$, equals the work done by internal damping force, ${}_{id}W_{j}$. That is,

$$ {}_{ed}W_{j} = {}_{id}W_{j} \tag{5.38} $$

For convenience, we omit the subscript denoting the i^{th} driving frequency and use the subscript i to denote the internal force only. A simple example (see Figure 5.11) is given below to show the energy-method. At mass j, we have

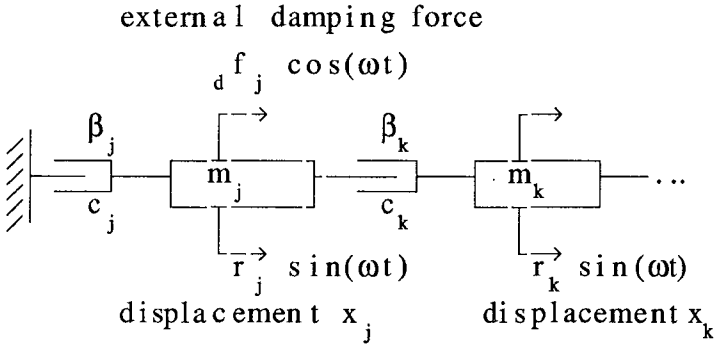


Figure 5.11 Energy-Method

$$ {}_{ed}W_{j} = \int_{0}^{2\omega/\pi} {}_{d}f_{j} \cos(\omega t)\ r_{j} \cos(\omega t)\ d(\omega t) = \pi\ {}_{d}f_{j}\ r_{j} \tag{5.39} $$

and

$$ {}_{id}W_{j} = \int_{0}^{2\omega/\pi} (c_{j} + c_{k})\ r_{j} \cos(\omega t)\ r_{j} \cos(\omega t)\ d(\omega t) - $$

$$ \int_{0}^{2\omega/\pi} c_{k}\ r_{k} \cos(\omega t)\ r_{j} \cos(\omega t)\ d(\omega t) = \pi\ \omega\ (c_{j}\ r_{j}^{2} - c_{k}\ r_{k}\ r_{j}) \tag{5.40} $$

Then, with the help of equation (5.38), we have

$$\pi \, {}_{d}f_{j} \, r_{j} = \pi \, \omega \, (c_{j} + c_{k}) \, r_{j}^{2} - \pi \, \omega \, c_{k} \, r_{k} \, r_{j}$$

or

$${}_{d}f_{j} \, r_{j} = \omega \, (c_{j} + c_{k}) \, r_{j}^{2} - \omega \, c_{k} \, r_{k} \, r_{j} \tag{5.41}$$

Similar to the case of proportional damping, we extend the idea to more general cases with j, k = 1,...n and $j \neq k$. Then equation (5.41) can be rewritten as

$${}_{d}f_{j} = (\beta_{j} + \beta_{k}) \, r_{j} - \beta_{k} \, r_{j} \, , \; j, k = 1,...n, \; j \neq k \tag{5.42}$$

Writing equation (5.42) in the matrix form, we can arrive at the same result as that obtained by using the force-method, that is

$$\begin{Bmatrix} {}_{d}f_{i1} \\ {}_{d}f_{i2} \\ \vdots \\ {}_{d}f_{in} \end{Bmatrix} = \begin{bmatrix} \beta_{i1} + \beta_{i2} & -\beta_{i2} & 0 \\ -\beta_{i2} & \beta_{i2} + \beta_{i3} & \vdots \\ & \cdots & \\ 0 & \cdots & \beta_{in} \end{bmatrix} \begin{Bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{in} \end{Bmatrix} \tag{5.43}$$

In equation (5.44), we re-add the subscript i again to denote that the system vibrates under with frequency ω_{i}. In this way, equation (5.34) results.

Thus far, we still have not determined all the frequency ω_{i}'s. From equation (5.36) with i = 1,...n, we have the damping matrix that fits the n frequency points. At any other frequency point, the damping matrix may be only an approximation. Since the damping has most significant effect on resonant frequency points, it is reasonable to choose these frequencies to be the n natural frequencies. Under this assumption, if we can solve for all the natural frequency ω_{i}'s, we can then obtain the generalized damping matrix.

From Chapter 2, if a system is non-proportionally damped, its natural frequencies will be affected by the damping matrix. That is, before the damping matrix is known, we cannot exactly locate these natural frequencies.

However, we have shown in Chapter 3 that, these natural frequencies are quite close to ω_{ni}, which are the square roots of the eigenvalues of the generalized stiffness matrix $\tilde{K}$. Therefore in typical engineering applications, we may simply use these ω_{ni}'s to approximate the ω_i's.

The above discussion about the establishment of damping matrices depends on damping materials or dampers such as VE dampers. However, it should be pointed out that, the approach, and in particular equation (5.36), is generally applicable to most damping design and damping matrix evaluation*, when the quantity G_I, or its equivalent, is known.

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* The formulation presented in this chapter to construct general damping matrices is an approximate approach based on the theoretical formulation. It is also based on an assumption that at resonant point the damping coefficient matrix has its most significant affect on the structure. Further experimental validation and theoretical refinement are currently being pursued by the authors. As an application in structural engineering, the general principle of determining damping matrix will be introduced in detail in VE and other type damper design in passive vibration control, in Part II, to appear.

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APPENDIX A
SOME MATHEMATICAL CONSIDERATIONS

A.1 INTRODUCTION

In this appendix we first briefly describe some necessary mathematical tools used in this report. Then, some new developments by the authors for the consideration of complex damping are presented. The latter have not yet appeared in the literature (Sections A.3 through A.8). Readers not interested in the mathematical background of the complex damping theory may go directly from Chapter 1 to Chapter 2.

A.2 SOME BASIC CONCEPTS OF MATRIX ALGEBRA

Linear algebra, especially **matrix theory**, is a major mathematical tool in structural dynamics (modal analyses, identification, design and control of structures, etc), particularly from the viewpoint of computational convenience.

The following notations will be used in this report. $\mathbb{R}$ denotes the real field, $\mathbb{C}$ denotes the complex field, $\mathbb{R}^{mxn}$ denotes the set of all real mxn matrices, $\mathbb{S}^{nxn}$ denotes the set of all real symmetric matrices and $\mathbb{C}^{mxn}$ denotes all the general mxn matrices.

A.2.1 MATRIX DECOMPOSITIONS

Singular Value Decomposition (SVD) :
If $\mathbf{A} \in \mathbb{R}^{mxn}$, then there exist orthogonal matrices

$$\mathbf{U} = [U_1\ U_2 \quad U_m] \in \mathbb{R}^{mxm}$$

and

$$\mathbf{V} = [V_1\ V_2 \quad V_n] \in \mathbb{R}^{nxn}$$

such that

$$\mathbf{U}^T\ \mathbf{A}\ \mathbf{V} = \text{diag}(\ \sigma_1\ \sigma_2 \ldots \sigma_p\ )\ , \qquad p = \min(m,n) \tag{A.1}$$

where,

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$

and U, V denote the the i^{th} column of matrix **U** and **V**, respectively.

The following is true for SVD:

If,

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq \sigma_{r+1} = \ldots = \sigma_p = 0$

then,

$\text{rank}(A) = r$ (A.2)

$N(A) = \text{span}(V_{r+1}, \ldots, V_n)$ (A.3)

$R(A) = \text{span}(U_1, \ldots, U_r)$ (A.4)

$A = \sum \sigma_i U_i V_i^T = U_r \Sigma_r V_r^T$, where $U_r = [U_1 \ldots U_r]$,

$V_r = [V_1 \ldots V_r]$ and $\Sigma_r = [\sigma_1 \ldots \sigma_r]$ (A.5)

$\| A \|_f = (\sigma_1^2 + \ldots + \sigma_p^2)^{1/2}$ (Frobenius norm) (A.6)

$\| A \|_2 = \sigma_1$ (2-norm) (A.7)

$A V_i = \sigma_i U_i$ (A.8a)

$A^T U_i = \sigma_i V_i$ (A.8b)

Note that, if **A** is square and positive semi definite, **U** = **V** is the eigenvector matrix of **A**, and $\Sigma = \text{diag}(\sigma_1 \sigma_2 \ldots \sigma_p)$ which is identical to the eigenvalue matrix of **A**.

Rank-Decomposition

Any matrix $A \in \mathbb{C}^{mxn}$ with rank r can be decomposed into two matrices $M \in \mathbb{C}^{mxr}$ and $N \in \mathbb{C}^{rxn}$ both with rank r, such that

$A = M N$ (A.9)

L-R Decomposition

If $\mathbf{A} \in \mathbb{C}^{nxn}$ with rank n, then

$$\mathbf{A} = \mathbf{L}\,\mathbf{R} \tag{A.10}$$

where **L** is the unit lower triangular matrix, and **R** is the upper triangular matrix.

Q-R Decomposition

If $\mathbf{A} \in \mathbb{C}^{nxn}$ with rank n, then

$$\mathbf{A} = \mathbf{Q}\,\mathbf{R} \tag{A.11}$$

where **Q** is the orthogonal matrix, and **R** is the upper triangular matrix. If **A** is nonsingular, and if **R** has all non-negative diagonal entries, the decomposition is unique. If **A** is rank deficient, then at least one diagonal entry of **R** is zero.

Choleski Decomposition

If $\mathbf{A} \in S^{nxn}$ is positive definite, then, there exists a lower triangular matrix $\mathbf{G} \in \mathbb{R}^{nxn}$ with positive diagonal entries such that,

$$\mathbf{A} = \mathbf{G}\,\mathbf{G}^T \tag{A.12}$$

Note that **G** is unique and is called the *Choleski triangular* matrix.

A.2.2 PSEUDO-INVERSE OF MATRICES

The following matrix equations are used to define generalized inverse of matrices, where $\mathbf{A} \in \mathbb{C}^{mxn}$

$$\text{i) } \mathbf{A}\,\mathbf{X}\,\mathbf{A} = \mathbf{A}$$

$$\text{ii) } \mathbf{X}\,\mathbf{A}\,\mathbf{X} = \mathbf{X}$$

$$\text{iii) } (\mathbf{X}\,\mathbf{A})^H = \mathbf{X}\,\mathbf{A} \tag{A.13}$$

$$\text{iv) } (\mathbf{A}\,\mathbf{X})^H = \mathbf{A}\,\mathbf{X}$$

Definition :

a) A *generalized inverse* of a matrix **A** is a matrix $\mathbf{X} = \mathbf{A}^g$ which satisfies i);

b) A *reflexive generalized inverse* of a matrix **A** is a matrix $\mathbf{X} = \mathbf{A}^r$ which

satisfies i) and ii);

c) A *left weak generalized inverse* of a matrix **A** is a matrix $\mathbf{X} = \mathbf{A}^{w}$ which satisfies i), ii) and iii);

d) A *right weak generalized inverse* of a matrix **A** is a matrix $\mathbf{X} = \mathbf{A}^{n}$ which satisfies i), ii) and iv);

e) A *pseudo-inverse* of a matrix **A** is a matrix $\mathbf{X} = \mathbf{A}^{+}$ which satisfies i), ii), iii) and iv).

The following is true for pseudo-inverse matrices:

$$\mathbf{A}^{+} \subset \mathbf{A}^{w} \subset \mathbf{A}^{r} \subset \mathbf{A}^{g}$$

$$\mathbf{A}^{+} \subset \mathbf{A}^{n} \subset \mathbf{A}^{r} \tag{A.14}$$

The equation

$$\mathbf{A\,X\,B} = \mathbf{C} \tag{A.15}$$

has a solution, if and only if

$$\mathbf{A\,A}^{+}\,\mathbf{C\,B}^{+}\,\mathbf{B} = \mathbf{C} \tag{A.16}$$

In this case, the general solution is

$$\mathbf{X} = \mathbf{A}^{+}\,\mathbf{C\,B}^{+} + \mathbf{Y} - \mathbf{A}^{+}\,\mathbf{A\,Y\,B\,B}^{+} \tag{A.17}$$

$$(\mathbf{A}^{+})^{+} = \mathbf{A} \tag{A.18}$$

$$(\mathbf{A}^{+}\,\mathbf{A})^{2} = \mathbf{A}^{+}\,\mathbf{A},\quad (\mathbf{A\,A}^{+})^{2} = \mathbf{A\,A}^{+}, \tag{A.19}$$

$$(\mathbf{A}^{H})^{+} = (\mathbf{A}^{+})^{H} \tag{A.20}$$

$$(\mathbf{A\,A}^{H})^{+} = (\mathbf{A}^{+})^{H}\mathbf{A}^{+},\quad (\mathbf{A}^{H}\,\mathbf{A})^{+} = \mathbf{A}^{+}(\mathbf{A}^{H})^{+}, \tag{A.21}$$

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^{+}) = \text{rank}(\mathbf{A}^{+}\mathbf{A}) = \text{rank}(\mathbf{A\,A}^{+}) \tag{A.22}$$

$$(\mathbf{U\,A\,V})^{+} = \mathbf{V}^{H}\,\mathbf{A}^{+}\,\mathbf{U}^{H}\text{, if } \mathbf{U}, \mathbf{V} \text{ are unitary} \tag{A.23}$$

$$(\alpha\,\mathbf{A})^{+} = \alpha^{-1}\,\mathbf{A}^{+} \quad \text{provided } \alpha \neq 0,\ 0^{+} = 0^{T} \tag{A.24}$$

$$\mathbf{A}^{+} = (\mathbf{A}^{H}\,\mathbf{A})^{+}\,\mathbf{A}^{H} \tag{A.25}$$

$$\mathbf{A}^{+} = \mathbf{A}^{H}\,(\mathbf{A\,A}^{H})^{n} = (\mathbf{A}^{H}\,\mathbf{A})^{w}\,\mathbf{A}^{H} \tag{A.26}$$

$$\mathbf{A}^{+} = \mathbf{A}^{w}, \text{ if } \text{rank}(\mathbf{A}) = m = \min(m,n) \tag{A.27}$$

$$\mathbf{A}^{+} = \mathbf{A}^{n}, \text{ if } \text{rank}(\mathbf{A}) = n = \min(m,n) \tag{A.28}$$

$$\mathbf{A}^{+} = \mathbf{A}^{n} = \mathbf{A}^{w} = \mathbf{A}^{r} = \mathbf{A}^{g} = \mathbf{A}^{-1}\text{, if } \mathbf{A} \text{ is nonsingular and square} \tag{A.29}$$

$A^+ A$, $A A^+$, $I - A^+ A$, $I - A A^+$ are hermitian and idempotent (A.30)

$(A A^H)^+ A A^H = A A^+$ (A.31)

$A A^+ = A^+ A$, if A is normal (A.32)

$(A^k)^+ = (A^+)^k$, if A is normal and k is a positive integer (A.33)

$A B = 0$ if and only if $B^+ A = 0$ (A.34)

$A^+ = (A^H A)^{-1} A^H$, if rank(A) = n (A.35a)

$A^+ = A^H (A A^H)^{-1}$, if rank(A) = m (A.35b)

$(A B)^{-1} = B_1^+ A_1^+$, where $B_1 = A^+ A B$ and $A_1 = A B_1 B_1^+$ (A.36)

$(A B)^+ = B^+ A^+$, if and only if both equations

$A^+ A B (A B)^H = B (A A^H)$

and (A.37)

$B B^+ A^H A B = A^H A B$

are satisfied. Namely, if and only if M(B) and $M(A)^H$ are invariant subspaces of $A^H A$ and $B B^H$, respectively.

If $A = U \Sigma V^H$, the singular value decomposition, and

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

then

$$A^+ = V \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^H \quad \text{(A.38)}$$

If $A = M N$, by rank decomposition, then

$$A^+ = N^+ M^+ = N^H (N N^H)^{-1} (M^H M)^{-1} M^H \quad \text{(A.39)}$$

With the above notations and definitions, the following theorem can be established:

Theorem A.1: A vector $B \in \mathbb{C}^{n \times 1}$ is linearly dependent to all the column vectors

A_1, A_2, ... and $A_t \in \mathbb{C}^{nx1}$ of matrix A, i.e.

$A = [A_1, A_2, \ldots A_t] \in \mathbb{C}^{nxt}$

In other words, there exists a non-zero vector $\alpha \in \mathbb{C}^{nx1}$, such that

$$A\,\alpha = B \tag{A.40}$$

if and only if B is an eigenvector of matrix ($A\,A^g$) associated with a unit eigenvalue. That is,

$$(A\,A^g)\,B = B \tag{A.41}$$

PROOF

If $t \geq n$, the proof is trivial. Therefore we only need to consider the case of $t < n$. First, consider the necessity condition. Suppose we have t scalars α_1 , α_2 and α_t such that

$$A \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_t \end{Bmatrix} = B$$

Pre-multiplying the above equation by the matrix product $A\,A^g$ results in

$$A\,A^g A \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_t \end{Bmatrix} = A \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_t \end{Bmatrix} = A\,A^g\,B \tag{A.42}$$

Comparing equations (A.42) with (A.40), we know that

$(A\,A^g)\,B = B$

Next, consider the sufficient condition. Suppose equation (A.40) is true. We can always have a non-zero vector

$\alpha = [\alpha_1 \quad \alpha_2 \quad \alpha_t]^T$ with

$$\alpha = A^g\,B \tag{A.43}$$

Since, if $\alpha = 0$, then $A\,A^gB$ will be a zero vector which contradicts (A.41). Pre-multiplying equation (A.43) by matrix A, we have

$$A\ \alpha = A\ A^g\ B = B \qquad \blacksquare$$

The above theorem leads directly to the following corollary.

Corollary A.1: If a vector $B \in \mathbb{C}^{n\times 1}$ is an eigenvector of matrix $A\ A^+$, associate a unit eigenvalue, where $A \in \mathbb{C}^{n\times t}$, that is

$$(A\ A^+)\ B = B \tag{A.44}$$

then, B is linearly dependent with all the column vectors of A. That is, there exists a non-zero vector $\alpha \in \mathbb{C}^{t\times 1}$, such that

$$A\ \alpha = B \tag{A.45}$$

In practice, it is not necessary to use the generalized inverse A^g. Instead, we use the pseudo-inverse of Corollary A.1. Theorem A.1 will be helpful when we discuss the method of matrices representation for a damping matrix.

A.3 THEOREM OF PSEUDO-SIMILARITY OF MATRICES

In this section, some materials concerning pseudo-similarity of matrices are presented.

From equation (1.20) or equation (1.21), we know that state matrices play an important role in modal analysis. Although they may be expressed in different forms, they are all similar matrices. Also, we know from linear algebra that similar matrices will have identical eigenvalues.

First, let us briefly recall some properties regarding similarity transformations: If matrix **H** is similar to matrix **L**, denoted by

$$\mathbf{H} \simeq \mathbf{L} \tag{A.46}$$

then the following is true

$$\mathbf{H} = \mathbf{P}\ \mathbf{L}\ \mathbf{P}^{-1} \tag{A.47}$$

and

$$\Lambda_H = \Lambda_L \tag{A.48}$$

where, $\mathbf{H} \in \mathbb{C}^{r \times r}$, $\mathbf{L} \in \mathbb{C}^{r \times r}$, and $\mathbf{P} \in \mathbb{C}^{r \times r}$ with full rank; also

$\Lambda_H \in \mathbb{C}^{r \times r}$, ${}_L\Lambda \in \mathbb{C}^{r \times r}$

We often have to deal with a system with all of its non-zero eigenvalues equal to those of another system, except for the sizes that may vary from system to system. For example, in time domain damping measurement and/or modal analysis, we often can not avoid such larger-sized "state" matrices. In these cases, equations (A.47) and (A.48) are no longer valid. To establish a rigorous theoretical base to deal with such problems, we introduce the concept of pseudo-similarity matrices.

Definition: For any full rank matrices $\mathbf{W} \in \mathbb{C}^{r \times r}$, and $\mathbf{B} \in \mathbb{C}^{q \times r}$, with $q \geq r$, the matrix $\mathbf{S} = \mathbf{B}\,\mathbf{W}\,\mathbf{B}^{+} \in \mathbb{C}^{q \times q}$, is said to be *pseudo-similar* to matrix $\mathbf{W}$, denoted by

$$\mathbf{S} \underset{\sim}{:} \mathbf{W} \tag{A.49}$$

In the above, $\mathbf{B}$ is the *pseudo-similarity transformation* matrix.

The following theorem tells us that a pseudo-similarity transformation $\mathbf{S}$ contains the same spectrum of $\mathbf{W}$ except that $\mathbf{S}$ has (q-r) more zeros.

Theorem A.2: For any full rank matrix $\mathbf{W} \in \mathbb{C}^{r \times r}$, with eigenvalue matrix

$\Lambda_W = \text{diag}(\lambda_i(\mathbf{W}))$,

we can define

$$\begin{aligned} \Lambda_S = \text{diag}(\lambda_i(\mathbf{S})) &= \text{diag}(\lambda_1(\mathbf{S}), \lambda_2(\mathbf{S}), \ldots \lambda_r(\mathbf{S}), 0, 0, \ldots 0) \\ &= \text{diag}(\lambda_1(\mathbf{W}), \lambda_2(\mathbf{W}), \ldots \lambda_r(\mathbf{W}), 0, 0, \ldots 0) \end{aligned} \tag{A.50}$$

where $\Lambda_W \in \mathbb{C}^{r \times r}$, $\Lambda_S \in \mathbb{C}^{q \times q}$, $\mathbf{S} \in \mathbb{C}^{q \times q}$, and $\mathbf{S} = \mathbf{B}\,\mathbf{W}\,\mathbf{B}^{+}$, $\mathbf{B} \in \mathbb{C}^{q \times r}$,

with $q \geq r$;

if and only if

$$\text{rank}(\mathbf{B}) = r. \tag{A.51}$$

PROOF:

To show that equation (A.51) is the necessary and sufficient condition for the pseudo similarity transformation

$$\mathbf{S} = \mathbf{B}\ \mathbf{W}\ \mathbf{B}^{+} \underset{\sim}{:}\ \mathbf{W}$$

let

$$\mathbf{B} = \mathbf{U}\ \Sigma\ \mathbf{V}^{+}$$

the singular value decomposition, where $\mathbf{U} \in \mathbb{C}^{qxq}$ and $\mathbf{V} \in \mathbb{C}^{rxr}$ are unitary matrices

$$\Sigma = \begin{bmatrix} \tilde{\Sigma} \\ \cdots \\ \mathbf{0} \end{bmatrix}$$

and

$$\tilde{\Sigma} = \text{diag}(\ \sigma_1\ \sigma_2 \quad \sigma_r\ )$$

Since rank($\mathbf{B}$) = r, then $\sigma_i \neq 0$, i = 1,...,r, and we have

$$\mathbf{B}^{+} = \mathbf{V}\ \Sigma^{+}\ \mathbf{U}^{H}$$

where

$$\Sigma^{+} = [\ \tilde{\Sigma}^{-1} : \mathbf{0}\ ]$$

and

$$\tilde{\Sigma}^{-1} = \text{diag}\ (\ \sigma_1^{-1}\ \ \sigma_2^{-1} \ldots\ \ \sigma_r^{-1}\ )$$

With the above notations, we have

$$\mathbf{S} = \mathbf{B}\ \mathbf{W}\ \mathbf{B}^{+} = \mathbf{U}\ \Sigma\ \mathbf{V}^{H}\ \mathbf{W}\ \mathbf{V}\ \Sigma^{+}\ \mathbf{U}^{H} = \mathbf{U}\ \Sigma\ \mathbf{W}_1\ \Sigma^{+}\ \mathbf{U}^{H}$$

where

$$\mathbf{W}_1 = \mathbf{V}^{H}\ \mathbf{W}\ \mathbf{V}.$$

Since $\mathbf{V}$ is unitary, $\mathbf{W} \simeq \mathbf{W}$.

Next, le t

$$\mathbf{W}_2 = \Sigma\ \mathbf{W}_1\ \Sigma^{+}$$

then

$$\mathbf{W}_2 = \begin{bmatrix} \Sigma\, \mathbf{W}_1\, \Sigma^{-1} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{0}_{22} \end{bmatrix} \tag{A.52}$$

where $\mathbf{0}_{ij}$'s are null matrices with proper dimensions. Then we see that $\lambda(\mathbf{W}_2)$ consists of two parts. We can always arrange these non-zero eigenvalues as the first part, from $\lambda_1(\mathbf{W}_2)$ to $\lambda_r(\mathbf{W}_2)$, denoted by $\Lambda_1(\mathbf{W}_2)$. Therefore, the second part, arranging from $\lambda_{r+1}(\mathbf{W}_2)$ to $\lambda_q(\mathbf{W}_2)$, should be identically zero. Thus, $\Lambda_1(\mathbf{W}_2)$ consists of the same values of $\Lambda(\mathbf{W}_1)$, and thus $\Lambda(\mathbf{W})$. The above proves the sufficiency of equation (A.51).

The necessity of the condition rank(**B**) = r is obvious. Since rank(**B**) < r, at least one of the σ_i's will become zero. Then in equation (A.52), this zero σ_i will delete a non-zero eigenvalue of $\mathbf{W}_1$ or **W**. Thus, $\Lambda(\mathbf{W}_2)$ will no longer contain the same non-zero spectrum of **W**. Note that, the condition rank(**B**) = r implies $q \geq r$. If $q < r$, the statement does not hold. ■

Comparing this statement with the similarity transformation (A.47) and (A.48), we can see the difference between the similarity transformation **P** and the pseudo-similarity transformation **B**. The former is a non singular square matrix while the later is a full rank rectangular matrix with more rows than columns. The results of a pseudo-similarity transformation **S** will contain all the eigenvalues of **A**, but also contain a number of zeros, n_s. The number n_s is just equal to the difference (q-r), i.e.

$$n_s = q - r \tag{A.53}$$

A.4 THEOREM OF MATRIX REPRESENTATION

In this section, we introduce a matrix representation theory. Before doing so, some definitions and lemmas are presented.

Definition: A matrix $\mathbf{Q} \in \mathbb{C}^{nxn}$ of the following form is defined to be a *Quasi-Vandermonde* matrix:

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{V}_y & x_1\mathbf{V}_y & \dots & x_1^{n-1}\mathbf{V}_y \\ \mathbf{V}_y & x_2\mathbf{V}_y & \dots & x_2^{n-1}\mathbf{V}_y \\ & & \dots & \\ \mathbf{V}_y & x_n\mathbf{V}_y & \dots & x_n^{n-1}\mathbf{V}_y \end{bmatrix} \qquad \text{(A.54)}$$

where the matrix $\mathbf{V}_y$ is given by

$$\mathbf{V}_y = \begin{bmatrix} 1 & y_1 & \dots & y_1^{n-1} \\ 1 & y_2 & \dots & y_2^{n-1} \\ & & \dots & \\ 1 & y_n & \dots & y_n^{n-1} \end{bmatrix}$$

and x_i's and y_i's are scalars.

Lemma A.1: The value of the determinant of the Quasi-Vandermonde matrix is

$$|\mathbf{Q}_v| = |\mathbf{V}_x|^n |\mathbf{V}_y|^n \qquad \text{(A.55)}$$

where $|\ \ |$ denotes the determinant of corresponding matrices.

PROOF:

Equation (A.54) can rewritten as

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{I} & x_1\mathbf{I} & \dots & x_1^{n-1}\mathbf{I} \\ \mathbf{I} & x_2\mathbf{I} & \dots & x_2^{n-1}\mathbf{I} \\ & \dots\dots & & \\ \mathbf{I} & x_n\mathbf{I} & \dots & x_n^{n-1}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V}_y & & & \\ & \ddots & & \\ & & & \\ & & & \mathbf{V}_y \end{bmatrix}$$

where $\mathbf{I}$ is an identity matrix with proper dimensions.

The above equation may be written as

$$\mathbf{Q}_v = \mathbf{A}\ \mathbf{B}$$

Then $|\mathbf{Q}_v| = |\mathbf{A}||\mathbf{B}|$. We see that $|\mathbf{B}| = |\mathbf{V}_y|^n$. In order to obtain the value of $|\mathbf{A}|$, we interchange the i^{th} row and the $\{(i-1)xn+1\}^{th}$ row of **A**, for i = 2,...n, and so on, then

$$|\mathbf{A}| = (-1)^{n-1}\begin{bmatrix} 1\ 0..0\ x_1\ 0\ ..0\ ....x_1^{n-1}\ 0\ ..\ 0 \\ \cdots\cdots \\ 1\ 0..0\ x_n\ 0\ ..0\ ....x_n^{n-1}\ 0\ ..\ 0 \\ 0\ 1..0\ 0\ \ x_1..0\ ....\ 0\ x_1^{n-1}..\ 0 \\ \cdots\cdots \\ 0\ 1..0\ 0\ \ x_n..0\ ....\ 0\ x_n^{n-1}..\ 0 \\ \cdots\cdots \\ \cdots\cdots \\ 0\ 0..1\ 0\ \ 0\ ..x_1....\ 0\ \ 0\ ..x_1^{n-1} \\ \cdots\cdots \\ 0\ 0..1\ 0\ \ 0\ ..x_n....\ 0\ \ 0\ ..x_n^{n-1} \end{bmatrix}$$

Similarly, interchanging the i^{th} column and the $\{(i-1)xn+1\}^{th}$ column of **A**, for i= 2,...n, and so on, we have

$$|\mathbf{A}| = (-1)^{2x(n-1)}\begin{bmatrix} \mathbf{V}_x & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & \mathbf{V}_x \end{bmatrix} = |\mathbf{V}_x|^n$$

Therefore, $|\mathbf{Q}_v| = |\mathbf{A}||\mathbf{B}| = |\mathbf{V}_x|^n|\mathbf{V}_y|^n$. ■

From the value of the quasi-Vandermonde determinant, the following lemmas can be obtained.

Lemma A.2: If $x_i \neq x_j$ and $y_i \neq y_j$, for all $i \neq j$, the Quasi-Vandermonde matrix is non-singular.

Lemma A.3: For any given matrices M, K $\in \mathbb{C}^{nxn}$, let m_i and k_i (i = 1,2,..,n) be the eigenvalues of M and **K**, respectively. If $m_i \neq m_j$ and $k_i \neq k_i$ for all $i \neq j$ and $\mathbf{P} = \mathbf{P}_m^{-1}\mathbf{P}_k$ with no zero entries where $\mathbf{P}_m$ and $\mathbf{P}_k$ are eigenvector matrices

of $\mathbf{M}$ and $\mathbf{K}$ (in other words, none of rows of $\mathbf{P}_m^{-1}$ and columns of $\mathbf{P}_k$ are orthogonal), then the following n^2 matrices

$$\mathbf{I},\ \mathbf{K},\ldots,\mathbf{K}^{n-1};\ \mathbf{M},\ \mathbf{MK},\ldots,\mathbf{MK}^{n-1};\ \ \ldots,\ \ \mathbf{M}^{n-1},\ \ \mathbf{M}^{n-1}\mathbf{K},\ldots,\mathbf{M}^{n-1}\mathbf{K}^{n-1}$$

are linearly independent.

PROOF:

Suppose that there exist n^2 scalars α_{ij} $i,j = 1,..n$, such that

$$\alpha_{11}\mathbf{I} + \alpha_{12}\mathbf{K} + \ldots + \alpha_{1n}^{n-1}\mathbf{K}$$

$$+ \ \alpha_{21}\mathbf{M} + \alpha_{22}\mathbf{MK} + \ldots + \alpha_{2n}^{n-1}\mathbf{MK}^{n-1}$$

$$+ \ \ldots .$$

$$+ \ \alpha_{n1}\mathbf{M}^{n-1} + \alpha_{n2}\mathbf{M}^{n-1}\mathbf{K} + \ldots \alpha_{nn}^{n-1}\mathbf{M}^{n-1}\mathbf{K}^{n-1} = \mathbf{O}_{nxn}$$

Furthermore, let $\Lambda_k = \text{diag}(k_1,\ldots,k_n)$, then $\mathbf{K} = \mathbf{P}_k \Lambda_k \mathbf{P}_k^{-1}$ and

$$\mathbf{P}_k(\alpha_{11}\mathbf{I} + \alpha_{12}\Lambda_k + \ldots + \alpha_{1n}\Lambda_k^{n-1})\mathbf{P}_k^{-1}$$

$$+ \ \mathbf{MP}_k(\alpha_{21}\mathbf{I} + \alpha_{22}\Lambda_k + \ldots + \alpha_{2n}\Lambda_k^{n-1})\mathbf{P}_k^{-1}$$

$$+ \ \ldots$$

$$+ \ \mathbf{M}^{n-1}\mathbf{P}_k(\alpha_{n1}\mathbf{I} + \alpha_{n2}\Lambda_k + \ldots + \alpha_{nn}\Lambda_k^{n-1})\mathbf{P}_k^{-1} = \mathbf{O}_{nxn}$$

Let $\Lambda_m = \text{diag}(m_1,\ldots,m_n)$, then $\mathbf{M} = \mathbf{P}_m \Lambda_m \mathbf{P}_m^{-1}$

and

$$\mathbf{P}_k(\alpha_{11}\mathbf{I} + \alpha_{12}\Lambda_k + \ldots + \alpha_{1n}\Lambda_k^{n-1})\mathbf{P}_k^{-1}$$

$$+ \ \mathbf{P}_m\Lambda_m\mathbf{P}_m^{-1}\mathbf{P}_k(\alpha_{21}\mathbf{I} + \alpha_{22}\Lambda_k + \ldots + \alpha_{2n}\Lambda_k^{n-1})\mathbf{P}_k^{-1}$$

$$+ \ \ldots$$

$$+ \ \mathbf{P}_m\Lambda_m^{n-1}\mathbf{P}_m^{-1}\mathbf{P}_k(\alpha_{n1}\mathbf{I} + \alpha_{n2}\Lambda_k + \ldots + \alpha_{nn}\Lambda_k^{n-1})\mathbf{P}_k^{-1} = \mathbf{O}_{nxn}$$

Multiplying the left hand side of the above equation by $\mathbf{P}_m^{-1}$ and the right hand side by $\mathbf{P}_k$, we have

$$\mathbf{P}\ (\ \alpha_{11}\mathbf{I} + \alpha_{12}\Lambda_k + \ \ldots + \alpha_{1n}\Lambda_k^{n-1})$$

$$+ \qquad \Lambda_m\mathbf{P}\ (\ \alpha_{21}\mathbf{I} + \alpha_{22}\Lambda_k + \ \ldots + \alpha_{2n}\Lambda_k^{n-1})$$

$$+ \ \ldots$$

$$+ \qquad \Lambda_m^{n-1}\mathbf{P}(\ \alpha_{n1}\mathbf{I} + \alpha_{n2}\Lambda_k + \ \ldots + \alpha_{nn}\Lambda_k^{n-1}) \ = \ \mathbf{O}_{nxn}$$

where

$$\mathbf{P} = \mathbf{P}_m^{-1}\,\mathbf{P}_k \ = \ \begin{bmatrix} p_{11} \cdots p_{1n} \\ \cdots\cdots \\ p_{n1} \cdots p_{nn} \end{bmatrix}$$

Writing the equation in matrix form, we have

$$\begin{bmatrix} \alpha_{11}p_{11} & \ldots \alpha_{11}p_{1n} \\ \ldots & \\ \alpha_{11}p_{n1} & \ldots \alpha_{11}p_{nn} \end{bmatrix} + \begin{bmatrix} \alpha_{12}p_{11}k_1 & \ldots & \alpha_{12}p_{1n}k_n \\ & \ldots & \\ \alpha_{12}p_{n1}k_1 & \ldots & \alpha_{12}p_{nn}k_n \end{bmatrix} + \ \ldots$$

$$\begin{bmatrix} \alpha_{1n}p_{11}k_1^{n-1} \ldots & \alpha_{1n}p_{1n}k_n^{n-1} \\ \ldots & \\ \alpha_{1n}p_{n1}k_1^{n-1} \ldots & \alpha_{1n}p_{nn}k_n^{n-1} \end{bmatrix} +$$

$$\begin{bmatrix} \alpha_{21}m_1p_{11} \ldots & \alpha_{21}m_1p_{1n} \\ \ldots & \\ \alpha_{21}m_np_{n1} \ldots & \alpha_{21}m_np_{nn} \end{bmatrix} + \ \ldots \ + \begin{bmatrix} \alpha_{22}m_1p_{11}k_1 \ldots & \alpha_{22}m_1p_{1n}k_n \\ \ldots & \\ \alpha_{22}m_np_{n1}k_1 \ldots & \alpha_{22}m_np_{nn}k_n \end{bmatrix}$$

$$+ \ \ldots$$

$$\begin{bmatrix} \alpha_{nn}m_1^{n-1}p_{11}k_1^{n-1} \ldots & \alpha_{nn}m_1^{n-1}p_{1n}k_n^{n-1} \\ \ldots & \\ \alpha_{nn}m_n^{n-1}p_{n1}k_1^{n-1} \ldots & \alpha_{nn}m_n^{n-1}p_{nn}k_n^{n-1} \end{bmatrix} = \mathbf{O}_{nxn}$$

Since each entry of the above matrix is equal to zero, we now have a homogeneous linear system containing n^2 equations. Without loss of generality, let us consider the first equation, that is

$$\alpha_{11}p_{11} + \alpha_{12}p_{11}k_1 + \ldots + \alpha_{1n}p_{11}k_1^{n-1} + \alpha_{21}m_1p_{11} + \ldots + \alpha_{2n}m_1p_{11}k_1^{n-1}$$

$$+ \alpha_{n1}m_1^{n-1}p_{11} + \ldots + \alpha_{nn}\,m_1^{n-1}p_{11}k_n^{n-1} = 0$$

Since p_{11} is not zero, it can be divided from the above equation. Similarly, p_{ij} can be divided from other equations correspondingly. Therefore,

$$\begin{bmatrix} 1 & k_1 .. k_1^{n-1} & m_1 m_1 k_1 .. m_1 k_1^{n-1} ... m_1^{n-1} k_1 .. m_1^{n-1} k_1^{n-1} \\ 1 & k_1 .. k_1^{n-1} & m_2 m_2 k_1 .. m_2 k_1^{n-1} ... m_2^{n-1} k_1 .. m_2^{n-1} k_1^{n-1} \\ & ..... & \\ 1 & k_1 .. k_1^{n-1} & m_n m_n k_1 .. m_n k_1^{n-1} ... m_n^{n-1} k_1 .. m_n^{n-1} k_1^{n-1} \\ & ..... & \\ 1 & k_n .. k_n^{n-1} & m_1 m_1 k_n .. m_1 k_n^{n-1} ... m_1^{n-1} k_n .. m_1^{n-1} k_n^{n-1} \\ & ..... & \\ 1 & k_n .. k_n^{n-1} & m_n m_n k_n .. m_n k_n^{n-1} ... m_n^{n-1} k_n .. m_n^{n-1} k_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \\ \alpha_{21} \\ \alpha_{22} \\ \vdots \\ \alpha_{2n} \\ \vdots \\ \alpha_{n1} \\ \vdots \\ \alpha_{nn} \end{bmatrix}$$

$$= \mathbf{O}_{n^2 \times 1}$$

The coefficient matrix of the above equation can be written as a Quasi-Vandermonde matrix

$$\mathbf{Q}_v = \begin{bmatrix} \mathbf{V}_K & m_1 \mathbf{V}_K & ... & m_1^{n-1} \mathbf{V}_K \\ \mathbf{V}_K & m_2 \mathbf{V}_K & ... & m_2^{n-1} \mathbf{V}_K \\ & .. . . & & \\ \mathbf{V}_K & m_n \mathbf{V}_K & ... & m_n^{n-1} \mathbf{V}_K \end{bmatrix}$$

where matrix $\mathbf{V}_k$ is defined to be:

$$\mathbf{V}_k = \begin{bmatrix} 1 & k_1 & ... & k_1^{n-1} \\ 1 & k_2 & ... & k_2^{n-1} \\ & ..... & & \\ 1 & k_n & ... & k_n^{n-1} \end{bmatrix}$$

Therefore, $\mathbf{Q}_v$ will be nonsingular since $m_i \neq m_j$ and $k_i \neq k_j$ for all $i \neq j$.

According to Lemma A.2, all these scalar α_{ij}'s must be zero. So, we have

$\mathbf{I}, \mathbf{K},...., \mathbf{K}^{n-1} \quad \mathbf{M}, \mathbf{MK},...\mathbf{MK}^{n-1},.....\mathbf{M}^{n-1},\mathbf{M}^{n-1}\mathbf{K},...\mathbf{M}^{n-1}\mathbf{K}^{n-1}$

are linearly independent. ■

Definition: Given matrices $\mathbf{M}, \mathbf{K} \in \mathbb{C}^{nxn}$, a matrix $\mathbf{A}^{(k)i}$ is defined as

$$\mathbf{A}^{(k)i} = \mathbf{M}^{\alpha i1}\mathbf{K}^{\alpha i2}\mathbf{M}^{\alpha i3}...$$

and $\alpha_{i1} + \alpha_{i2} + \alpha_{i3} + ... = k$

where the superscript i of (k) denotes the distinct permutation of α_{il}, $i = 1,....,m \le 2^k$. That is, if $i \neq j$, and

$$\mathbf{A}^{(k)i} = \mathbf{M}^{\alpha i1}\mathbf{K}^{\alpha i2}\mathbf{M}^{\alpha i3}...$$

$$\mathbf{A}^{(k)j} = \mathbf{M}^{\alpha j1}\mathbf{K}^{\alpha j2}\mathbf{M}^{\alpha j3}...$$

then at least one pair

$\alpha_{i1} \neq \alpha_{jl}, \quad l = 1,...,m \le 2^k$

Theorem 2.3: Given matrices $\mathbf{M}, \mathbf{K} \in \mathbb{C}^{nxn}$, if there exist two of the products $\mathbf{A}^{(q)s}$ and $\mathbf{A}^{(r)t}$, and each has no repeated eigenvalues, and $\mathbf{P} = \mathbf{P}_q^{-1}\mathbf{P}_r$ has no zero entries, where $\mathbf{P}_q$ and $\mathbf{P}_r$ are eigenvector matrices of $\mathbf{A}^{(q)s}$ and $\mathbf{A}^{(r)t}$, respectively, then any matrix $\mathbf{C} \in \mathbb{C}^{nxn}$ can be expressed as a polynomial of $\mathbf{M}$ and $\mathbf{K}$,

$$\mathbf{C} = \sum_{k=1}^{K}\sum_{i=1}^{I}\beta_{ki}\mathbf{A}^{(k)i} \qquad (A.56)$$

where K and I are finite integers.

PROOF:

Let $\mathbf{M}_1 = \mathbf{A}^{(q)s}$ and $\mathbf{K}_1 = \mathbf{A}^{(r)t}$.

By Lemma A.3, we know that the following matrices

$$\mathbf{I}, \mathbf{K}_1... \mathbf{K}_1^{n-1}, \quad \mathbf{M}_1, \mathbf{M}_1\mathbf{K}_1,... \mathbf{M}_1\mathbf{K}_1^{n-1},... \quad \mathbf{M}_1^{n-1}, \mathbf{M}_1^{n-1}\mathbf{K}_1,... \mathbf{M}_1^{n-1}\mathbf{K}_1^{n-1}$$

are linearly independent. Therefore, they form a complete linear basis for

$\mathbb{C}^{nxn}$, since $\mathbb{C}^{nxn}$ is an n^2- dimensional space. Therefore any matrix $\mathbf{C} \in \mathbb{C}^{nxn}$ is a linear combination of the basis. It follows that **C** can be expressed as a polynomial of **M** and **K** . ■

Corollary A.2: Under the same conditions as the above theorem except **P** has some zero entries, the theorem is still true provided that matrix **C** is known to have $c_{ij} = 0$ whenever $P_{ij} = 0$.

PROOF:

Consider the matrix equation

$$\begin{bmatrix} p_{11} & & & & & & & \\ & p_{12} & & & & & & \\ & & \ddots & & & & & \\ & & & p_{1n} & & & & \\ & & & & p_{21} & & & \\ & & & & & \ddots & & \\ & & & & & & p_{2n} & \\ & & & & & & & \ddots \\ & & & & & & & & p_{n1} \\ & & & & & & & & & \ddots \\ & & & & & & & & & & p_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \\ y_{n^2} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \\ c_{nn} \end{bmatrix}$$

Since $p_{ij} = 0$ implies $c_{ij} = 0$, thus the above linear system is consistent. We choose one of its solution denoted by $\hat{Y}_o$, then the following equation

$$
\begin{bmatrix}
1 & k_1 \ldots & k_1^{n-1} & m_1 & m_1k_1 & m_1k_1^{n-1} & m_1^{n-1}k_1 \ldots & m_1^{n-1}k_1^{n-1} \\
1 & k_1 \ldots & k_1^{n-1} & m_2 & m_2k_1 & m_2k_1^{n-1} & m_2^{n-1}k_1 \ldots & m_2^{n-1}k_1^{n-1} \\
 & \ldots\ldots & & & & & & \\
1 & k_1 \ldots & k_1^{n-1} & m_n & m_nk_1 & m_nk_1^{n-1} & m_n^{n-1}k_1 \ldots & m_n^{n-1}k_1^{n-1} \\
 & \ldots\ldots & & & & & & \\
1 & k_n \ldots & k_n^{n-1} & m_1 & m_1k_n & m_1k_n^{n-1} & m_1^{n-1}k_n \ldots & m_1^{n-1}k_n^{n-1} \\
 & \ldots\ldots & & & & & & \\
1 & k_n \ldots & k_n^{n-1} & m_n & m_nk_n & m_nk_n^{n-1} & m_n^{n-1}k_n \ldots & m_n^{n-1}k_n^{n-1}
\end{bmatrix}
\begin{bmatrix}
\beta_{11} \\
\beta_{12} \\
\vdots \\
\beta_{1n} \\
\beta_{21} \\
\beta_{22} \\
\vdots \\
\beta_{2n} \\
\vdots \\
\beta_{n1} \\
\vdots \\
\beta_{nn}
\end{bmatrix}
= \hat{Y}_o
$$

has a unique solution denoted as

$$\hat{\beta} = [\beta_{11}\ \beta_{12} \ldots\ \beta_{nn}]^T$$

It follows immediately that

$$[I\ \ K_1 \ldots K_1^{n-1}\ \ M_1\ \ M_1K_1 \ldots M_1K_1^{n-1} \ldots M_1^{n-1}\ \ M_1^{n-1}K_1 \ldots M_1^{n-1}K_1^{n-1}]\ \hat{\beta} = C \qquad \blacksquare$$

The above matrices in the square brackets are no longer linearly independent.

A.5 THEORY OF EIGEN-MATRIX

As mentioned in Chapter 1, the eigen-problem of a damped system is an important subject in vibration analysis. One of the approaches to address eigen-problems is by using the concept of *matrix polynomials* or the λ-*matrix* (see Gohberg et al (1982) or Lancaster (1966)). The λ-matrix for a damped system (1.1a) is defined by

$$\mathbf{L}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}$$

In the following, a new concept, somewhat different from the λ-matrix, will be discussed, together with the description of an eigen-matrix for the system described by (1.1a).

Theorem A.4: If an M-C-K system described by equation (1.1a) has eigenvalue and eigenvector matrices denoted respectively by

$$\begin{bmatrix} \Lambda & \\ & \Lambda^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{P}\Lambda & \mathbf{P}^*\Lambda^* \\ \mathbf{P} & \mathbf{P}^* \end{bmatrix} \tag{A.57}$$

then there exists a simple matrix **A**, which has no complex conjugate pairs of eigenvectors and whose eigen-decomposition is given by

$$\mathbf{A} = \mathbf{P}\,\Lambda\,\mathbf{P}^{-1}\,, \tag{A.58a}$$

if and only if **A** satisfies the following matrix equation

$$\mathbf{M}\,\mathbf{A}^2 + \mathbf{C}\,\mathbf{A} + \mathbf{K} = \mathbf{0} \tag{A.59}$$

PROOF:

Since (A.57) is the eigen-problem of the M-C-K system, we know that

$$\mathbf{M}\,\mathbf{P}\,\Lambda^2 + \mathbf{C}\,\mathbf{P}\,\Lambda + \mathbf{K}\,\mathbf{P} = \mathbf{0} \tag{A.60a}$$

and

$$\mathbf{M}\,\mathbf{P}^*\Lambda^{*2} + \mathbf{C}\,\mathbf{P}^*\Lambda^* + \mathbf{K}\,\mathbf{P}^* = \mathbf{0} \tag{A.60b}$$

Suppose matrix **A** satisfies equation (A.59), and **A** has the following eigen-

decomposition

$$\mathbf{A} = \tilde{\mathbf{P}}\,\tilde{\Lambda}\,\tilde{\mathbf{P}}^{-1} \tag{A.61}$$

(Note that, since A is simple, this assumption is acceptable.) We then have

$$\mathbf{M}\,\tilde{\mathbf{P}}\,\tilde{\Lambda}^{2}\,\tilde{\mathbf{P}}^{-1} + \mathbf{C}\,\tilde{\mathbf{P}}\,\tilde{\Lambda}\,\tilde{\mathbf{P}}^{-1} + \mathbf{K} = \mathbf{0}$$

from which we can write

$$\begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda}^{2} \\ \tilde{\mathbf{P}}\,\tilde{\Lambda} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda} \\ \tilde{\mathbf{P}} \end{bmatrix} \tag{A.62a}$$

Taking the complex conjugate of equation (A.62a) yields

$$\begin{bmatrix} \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*2} \\ \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \\ \tilde{\mathbf{P}}^{*} \end{bmatrix} \tag{A.62b}$$

Combining equations (A.62a) and (A.62b), we have

$$\begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda}^{2} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*2} \\ \tilde{\mathbf{P}}\,\tilde{\Lambda} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \\ \tilde{\mathbf{P}} & \tilde{\mathbf{P}}^{*} \end{bmatrix}$$

or

$$\begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \\ \tilde{\mathbf{P}} & \tilde{\mathbf{P}}^{*} \end{bmatrix} \begin{bmatrix} \tilde{\Lambda} & \\ & \tilde{\Lambda}^{*} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \\ \tilde{\mathbf{P}} & \tilde{\mathbf{P}}^{*} \end{bmatrix}$$

Note that there exists no complex conjugate pair of eigenvectors inside matrix $\tilde{\mathbf{P}}$, and thus the matrix

$$\begin{bmatrix} \tilde{\mathbf{P}}\,\tilde{\Lambda} & \tilde{\mathbf{P}}^{*}\,\tilde{\Lambda}^{*} \\ \tilde{\mathbf{P}} & \tilde{\mathbf{P}}^{*} \end{bmatrix}$$

will not be singular. Therefore, the eigen-decomposition of (A.58a) is

satisfied regardless of the order of the eigenvalues and the constant factor of each column of the eigenvector matrix, since $\Lambda = \tilde{\Lambda}$, and $\mathbf{P} = \tilde{\mathbf{P}}$. Thus we have shown the sufficient condition of Theorem A.5.

Next, consider the value of $\mathbf{B} = \mathbf{M}\,\mathbf{A}^2 + \mathbf{C}\,\mathbf{A} + \mathbf{K}$. Suppose $\mathbf{A}$ has the eigen-decomposition (A.58a). We have

$$\mathbf{M}\,\mathbf{P}\,\Lambda^2\,\mathbf{P}^{-1} + \mathbf{C}\,\mathbf{P}\,\Lambda\,\mathbf{P}^{-1} + \mathbf{K} = \mathbf{B}$$

Since $\mathbf{P}$ is not singular, we know

$$\mathbf{M}\,\mathbf{P}\,\Lambda^2 + \mathbf{C}\,\mathbf{P}\,\Lambda + \mathbf{K}\,\mathbf{P} = \mathbf{B}\,\mathbf{P}$$

From equation (A.60a), we know that $\mathbf{B}$ is identical to the null matrix. Therefore $\mathbf{A}$ satisfies equation (A.59). This is the necessary condition of Theorem A.5. ∎

Definition: Equation (A.59) is defined to be an *eigen-equation* of the corresponding system (1.1).

Definition: A matrix $\mathbf{A}$ is defined to be an *eigen-matrix* of system (1.1) if it satisfies the eigen-equation (A.59).

In the following and from Chapter 2 through Chapter 5, unless otherwise noted, we regularize the eigenvalue matrix Λ in equation (A.58a) so that

$$\Lambda = \mathrm{diag}\,(-\xi_i\omega_i + j\sqrt{1-\xi_i^2}\;\omega_i) \qquad \text{(A.58b)}$$

In equation (A.58b), all the imaginary parts of eigenvalues in equation (A.58a) are taken to be positive.

Corollary A.3: If a matrix $\mathbf{A}$ is the eigen-matrix of system (1.1), then $\mathbf{A}^*$ is also an eigen-matrix of the same system.

The proof of this corollary is straight forward from the complex conjugate of equation (A.59).

Corollary A.4: A system (1.1) is proportionally damped if and only if its

eigen-matrix **A** has the following equivalent properties:

1) Re(**A**), Im(**A**), **A** and $\mathbf{A}^*$ commute with each other.

2) $\mathbf{P}(\mathbf{A}) = \mathbf{P}(\mathbf{A}^*)$ and **P** can be real-valued.

3) $\mathbf{P}(\mathrm{Re}(\mathbf{A})) = \mathbf{P}(\mathrm{Im}(\mathbf{A})) = \mathbf{P}(\mathbf{A})$

4) $\mathrm{Re}(\Lambda(\mathbf{A})) = \Lambda(\mathrm{Re}(\mathbf{A})) = \mathrm{diag}(-2\xi_i\omega_i)$

$\mathrm{Im}(\Lambda(\mathbf{A})) = \Lambda(\mathrm{Im}(\mathbf{A})) = \mathrm{diag}\left((\sqrt{1-\xi_i^2})\,\omega_i\right)$

where $\mathbf{P}(.)$ is the eigenvector matrix of matrix (.); $\Lambda(.)$ is the eigenvalue matrix of matrix (.).

Corollary A.5: If and only if a system (1.1) is proportionally damped, the eigen-matrix **A** can be obtained by a generalized quadratic formula:

$$\mathbf{A} = \frac{-\mathbf{M}^{-1}\mathbf{C} \pm \sqrt{(\mathbf{M}^{-1}\mathbf{C})^2 - 4\,\mathbf{M}^{-1}\mathbf{K}}}{2} \qquad \text{(A.63)}$$

Equation (A.59) can be rewritten as a *standard eigen-equation* of the form

$$\tilde{\mathbf{A}}^2 + \tilde{\mathbf{C}}\,\tilde{\mathbf{A}} + \tilde{\mathbf{K}} = \mathbf{0} \qquad \text{(A.64a)}$$

where $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ are defined in equation (1.17); however, for convenience we rewrite them together with the definition of $\tilde{\mathbf{A}}$:

$$\begin{aligned} \tilde{\mathbf{C}} &= \mathbf{M}^{-1/2}\mathbf{C}\,\mathbf{M}^{-1/2} \\ \tilde{\mathbf{K}} &= \mathbf{M}^{-1/2}\mathbf{K}\,\mathbf{M}^{-1/2} \\ \tilde{\mathbf{A}} &= \mathbf{M}^{1/2}\mathbf{A}\,\mathbf{M}^{-1/2} \end{aligned} \qquad \text{(A.64b)}$$

The eigen equations (A.59) and (A.64) have the same eigenproblem, furthermore both $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{K}}$ are symmetric. The matrix $\tilde{\mathbf{A}}$ is called a *standard eigen-matrix*. For a standard eigen-matrix, the following results hold.

Theorem A.5: A system (A.64a) is proportionally damped if and only if its standard eigen-matrix is symmetric. That is,

$$\tilde{A} = \tilde{A}^T \tag{A.65}$$

PROOF:

Sufficiency: if equation (A.65) holds, the eigenvector matrix $\tilde{P}$ of matrix $\tilde{A}$ can be real-valued. Note that $\tilde{P}$ is also the eigenvector of system (A.65). A system with a real eigenvector matrix is proportionally damped.

Necessity: If system (A.64a) is proportionally damped, its eigenvector $\tilde{P}$, which is also the eigenvector matrix of $\tilde{A}$, can be real-valued and orthomonal. A matrix with such a eigenvector matrix should be symmetric.

Theorem A.6 (*generalized root theorem of standard eigen-equation*): A matrix $\tilde{A}$ satisfies equation (A.64a), if and only if

$$\tilde{A} + \tilde{A}^H = -\tilde{C}$$

and (A.66)

$$\tilde{A}^H \tilde{A} = \tilde{K}$$

PROOF:

Necessity: Suppose equation (A.66) holds. Substituting equations (A.66) into equation (A.64a) will show that the matrix $\tilde{A}$ does satisfy equation (A.64a).

Sufficiency: Suppose that the matrix $\tilde{A}$ satisfies equation (A.64), and

$$\tilde{A} + \tilde{A}^H = -\tilde{C} + \mathbf{X}$$

$$\tilde{A}^H \tilde{A} = \tilde{K} + \mathbf{Y} \quad . \tag{A.67}$$

Since

$$[\tilde{A} + \tilde{A}^H]^H = \tilde{A} + \tilde{A}^H$$

and

$$[\tilde{A}^H \tilde{A}]^H = \tilde{A}^H \tilde{A}$$

we know that $\mathbf{X} = \mathbf{X}^H$ and $\mathbf{Y} = \mathbf{Y}^H$.

From the proof of necessity, conditions (A.67) yield

$$\tilde{\mathbf{A}}^2 + (\tilde{\mathbf{C}} + \mathbf{X})\,\tilde{\mathbf{A}} + (\tilde{\mathbf{K}} + \mathbf{Y}) = \mathbf{0}$$

Thus,

$$\mathbf{X}\,\tilde{\mathbf{A}} = \mathbf{Y}$$

and

$$\mathbf{X}\,\tilde{\mathbf{A}} = (\mathbf{X}\,\tilde{\mathbf{A}})^H = \tilde{\mathbf{A}}^H\,\mathbf{X}$$

Note that a Lyapnove equation

$$\mathbf{X}\,\tilde{\mathbf{A}} - \tilde{\mathbf{A}}^H\,\mathbf{X} = \mathbf{0} \qquad \text{(A.68)}$$

has the non-zero solution $\mathbf{X}$, if and only if matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^H$ possess at least one identical eigenvalue. Since $\tilde{\mathbf{A}}$ satisfies equation (A.64), $\tilde{\mathbf{A}}^T$ must satisfy

$$(\tilde{\mathbf{A}}^T)^2 + \tilde{\mathbf{A}}^T\,\tilde{\mathbf{C}} + \tilde{\mathbf{K}} = \mathbf{0} \qquad \text{(A.69)}$$

Therefore $\tilde{\mathbf{A}}^T$ possesses the same set eigenvalues, Λ, as A does (see equation (A.58a)).

On the other hand, let us take the complex conjugate of equation (A.69). Now, all the eigenvalues of matrix $\tilde{\mathbf{A}}^H$ are just the complex conjugates of Λ. Therefore, matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^H$ have no eigenvalues in common. This implies that equation (A.68) has only the zero solution $\mathbf{X} = \mathbf{0}$. The same is true for $\mathbf{Y}$. Thus equation (A.66) holds. ∎

Corollary A.6: A matrix $\tilde{\mathbf{A}}$ satisfies equation (A.64a), then

$$\begin{aligned} \mathrm{Im}(\tilde{\mathbf{A}}) &= [\,\mathrm{Im}(\tilde{\mathbf{A}})]^T \\ \mathrm{Im}(\tilde{\mathbf{A}})\mathrm{Re}(\tilde{\mathbf{A}}) &= [\mathrm{Im}(\tilde{\mathbf{A}})\mathrm{Re}(\tilde{\mathbf{A}})]^T \end{aligned} \qquad \text{(A.70)}$$

Equation (A.70) can be obtained directly from equation (A.66).

Corollary A.7: The singular values of the standard eigen-matrix $\tilde{\mathbf{A}}$ is invariant of the arbitrary damping matrix $\tilde{\mathbf{C}}$, as long as the stiffness matrix $\tilde{\mathbf{K}}$

is fixed. The singular values of matrix $\tilde{\mathbf{A}}$, denoted by $\Sigma(\tilde{\mathbf{A}})$, are just the eigenvalues of $\tilde{\mathbf{K}}$ matrix, that is

$$\Sigma(\tilde{\mathbf{A}}) = \lambda(\tilde{\mathbf{A}}^H \tilde{\mathbf{A}}) = \lambda(\tilde{\mathbf{K}}) \tag{A.71}$$

Therefore, matrix $\tilde{\mathbf{C}}$ does not affect $\Sigma(\tilde{\mathbf{A}})$.

To introduce and prove the next corollary, we need the following definition and lemma.

Definition: For any positive definite matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$, if a lower triangular matrix $\mathbf{L} \in \mathbb{C}^{n\times n}$ exists such that

$$\mathbf{K} = \mathbf{L}\,\mathbf{L}^H \tag{A.72}$$

then $\mathbf{L}$ is called a *quasi-Choleski triangular* of $\mathbf{K}$, and equation (A.72) is called a *quasi-Choleski decomposition* of $\mathbf{K}$.

Lemma A.4: A lower triangular matrix $\mathbf{L} \in \mathbb{C}^{n\times n}$ is a quasi-Choleski triangular of a positive definite matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$ if and only if

$$\mathbf{L} = \mathbf{G}\,\mathrm{diag}(\exp(j\,\theta_i)) \tag{A.73}$$

Here $\mathbf{G}$ is regular Choleski triangular; θ_i's are scalars. (Note that $\mathbf{G}$ is unique, but the i^{th} angle θ_i is arbitrary; see equation (A.12)).

PROOF

Sufficiency:

$$\mathbf{L}\,\mathbf{L}^H = \mathbf{G}\,\mathrm{diag}(\exp(j\,\theta_i))\,\mathrm{diag}(\exp(-j\,\theta_i))\,\mathbf{G}^T = \mathbf{G}\,\mathbf{G}^T = \mathbf{K}$$

Necessity: Suppose a lower triangular matrix $\mathbf{L}$

$$\mathbf{L} = \begin{bmatrix} l_{11}\exp(j\theta_1) & & & \\ l_{21}\exp(j\theta_{21}) & l_{22}\exp(j\theta_2) & & \\ l_{31}\exp(j\theta_{31}) & l_{32}\exp(j\theta_{32}) & l_{33}\exp(j\theta_3) & \cdots \\ \cdots & & & \\ l_{n1}\exp(j\theta_{n1}) & l_{n2}\exp(j\theta_{n2}) & l_{n3}\exp(j\theta_{n3}) & l_{n3}\exp(j\theta_{n3}) \end{bmatrix}$$

satisfies equation (A.72). Considering the first column of the product $\mathbf{L}\,\mathbf{L}^H$, we have

$$l_{11}\exp(j\theta_1)\;\; l_{11}\exp(-j\theta_1) = k_{11}$$

$$l_{21}\exp(j\theta_{21})\; l_{11}\exp(-j\theta_1) = k_{21}$$

$$l_{31}\exp(j\theta_{31})\; l_{11}\exp(-j\theta_1) = k_{31}$$

$$\cdots\cdots$$

$$l_{n1}\exp(j\theta_{n1})\, l_{11}\exp(-j\theta_1) = k_{n1}$$

Since $\mathbf{K}$ is positive definite, $k_{11} > 0$. Therefore, $l_{11} = \pm(k_{11})^{1/2}$. Considering the requirement of Choleski triangularity, by choosing

$l_{11} = (\,k_{11}\,)^{1/2}$, one can see that

$\theta_{i1} = \theta_1 \quad i = 2,\ldots n$

and

$l_{ni} = k_{ni}\,(k_{11})^{-1/2}$

Similarly, considering the second column of product $\mathbf{L}\,\mathbf{L}^H$, we have

$$l_{11}\exp(j\theta_1)\;\; l_{21}\exp(-j\theta_1) = k_{12}$$

$$l_{21}\exp(j\theta_1)\; l_{21}\exp(-j\theta_1) + l_{22}\exp(j\theta_2)\; l_{22}\exp(-j\theta_2) = k_{22}$$

$$l_{31}\exp(j\theta_1)\; l_{21}\exp(-j\theta_1) + l_{32}\exp(j\theta_{32})\; l_{22}\exp(-j\theta_2) = k_{32}$$

$$\cdots\cdots$$

$$l_{n1}\exp(j\theta_1)\; l_{21}\exp(-j\theta_1) + l_{n2}\exp(j\theta_{n2})\; l_{22}\exp(-j\theta_2) = k_{n2}$$

First, we know $l_{22} = (k_{22} - l_{21}^2)^{1/2} = (k_{22} - k_{12}^2 / k_{11})^{1/2}$.

Since **K** is positive definite,

$k_{22} - k_{12}^2 / k_{11} > 0$, and $l_{22} > 0$.
Also, we know that $\theta_{i2} = \theta_2$, i = 3,...n, and

$$l_{i2} = (k_{i2} k_{11} - k_{i1} k_{12})/(k_{22} k_{11} - k_{12}^2)$$

Re-iterating the same procedure, we have

$\theta_{pq} = \theta_q$, q = 1,..n, p = q+1,..n.

and $l_{qq} = (K_q / K_{q-1})^{1/2}$

where K_q is the determinant of the q^{th} principal submatrix of **K**.

Hence, $l_{qq} = (K_q / K_{q-1})^{1/2} > 0.$ (A.74)

Note that θ_q can be chosen arbitrarily. Furthermore, all l_{pq}'s are real-valued, not affected by the θ_q's. Therefore, we write

$$\mathbf{L} = \mathbf{L}_1 \text{diag}(\exp(j\theta_i)) \quad (A.75)$$

The ij^{th} entry of $\mathbf{L}_1$ is l_{ij} .

Substituting equation (A.75) into equation (A.72) yields

$$\mathbf{K} = \mathbf{L}_1 \text{diag}(\exp(j\theta_i))\ \text{diag}(\exp(-j\theta_i))\mathbf{L}_1^T = \mathbf{L}_1\ \mathbf{L}_1^T$$

Since inequality (A.74) holds, and because of the uniqueness of the Choleski decomposition of a positive definite matrix, we know

$$\mathbf{L}_1 = \mathbf{G}$$

■

We now can introduce the following corollary, which is useful for *order-determination* of an existing structure. In Part II, to appear, we will discuss the applications of this corollary in detail.

Corollary A.8: If a standard eigen-matrix $\tilde{A}$ has a Q-R decomposition (see equation (A.11)), that is,

$$\tilde{A} = Q\ R$$

then a matrix $|R|$, whose ij^{th} entry is the module of the ij^{th} entry of R, is invariant of the arbitrary damping matrix $\tilde{C}$, as long as the stiffness matrix $\tilde{K}$ is fixed.

PROOF:

From the second equation of (A.66), we have

$$\tilde{A}^H\ \tilde{A} = \tilde{K} = (Q\ R)^H\ (\ Q\ R\ ) = R^H\ Q^H\ Q\ R = R^H\ R$$

Comparing with the quasi-Cholesky decomposition (A.72), we know

$L^H = R$. Therefore $R = \text{diag}(\exp(-j\theta_i))\ G^T$, and $|R| = |G^T|$. Then

$|R|$ is not affected by the damping matrix $\tilde{C}$. ∎

Corollary A.9: A matrix A satisfies equation (A.59), if and only if

$$A\ M^{-1} + M^{-1}\ A^H = -\ C \tag{A.76}$$

and

$$A^H\ M\ A = K$$

Equation (A.76) can be obtained from the generalized root Theorem A.6 by substituting the notations (A.64b) into (A.66). This corollary may be called a *generalized root theorem* for regular eigen-equation.

A.6 SOME PROPERTIES OF EIGENSYSTEMS

In this section, we discuss some major properties of the eigensystems of a given structural dynamic system (1.1). From Chapter 1, equation (1.19), the following two eigen equations have the same eigenvalue problems:

$$M\ A^2 + C\ A + K = 0$$

$$\tilde{\mathbb{A}}^2 + \tilde{\mathbb{C}}\,\tilde{\mathbb{A}} + \tilde{\mathbb{K}} = \mathbb{0}$$

where $\tilde{\mathbb{C}}$ and $\tilde{\mathbb{K}}$ are defined in equation (1.17) or (A.64b); $\tilde{\mathbb{K}}$ is positive definite and $\tilde{\mathbb{C}}$ is positive semi-definite. We now present some important properties about eigen-problems of system (1.1) with the help of the concept of eigen-equation. For convenience, we only consider the above standard eigen-equation and omit the overhead bar " ~ ," that is,

$$\mathbb{A}^2 + \mathbb{C}\,\mathbb{A} + \mathbb{K} = \mathbb{0} \tag{A.77}$$

Again using the notation of eigen decomposition (A.58a), we have

$$\mathbb{P}\,\Lambda^2 + \mathbb{C}\,\mathbb{P}\,\Lambda + \mathbb{K}\,\mathbb{P} = \mathbb{0} \tag{A.78}$$

Definition: In equation (A.78), matrix $\mathbb{P}$ and Λ are said to be the *eigenvector* and *eigenvalue matrices* of system (A.78) respectively.

Definition : A non-zero vector P_i in equation (A.78) is said to be a *true-complex eigenvector* or *strongly-complex* for eigenvalue λ_i and P_i which make the following equation hold:

$$\lambda_i^2\,P_i + \lambda_i\,\mathbb{C}\,P_i + \mathbb{K}\,P_i = 0 \tag{A.79}$$

if P_i can not be written as a complex linear combination of real-valued eigenvectors of λ_i, Q_j's, $0 \le j < p < n$. Here p is the number of the multiple of the corresponding eigenvalue λ_i, $0 < p < n$. On the other hand, if

$$P_i = \alpha_1\,Q_1 + \alpha_2\,Q_2 + \ldots + \alpha_p\,Q_p \tag{A.80}$$

where the α_i's are real or complex-valued scalars, $i = 1,\ldots p < n$, and at least one $\alpha_i \neq 0$, then P_i is said to be a *pseudo-complex eigenvector* or a *weakly-complex* of equation (A.78).

Lemma A.5: All the complex-valued eigenvectors , $P_i \in \mathbb{C}^n$, in the eigenvector matrix $\mathbb{P} \in \mathbb{C}^{n\times n}$ are not strongly-complex eigenvectors of equation (A.78) (all are weakly-complex eigenvectors of (A.78)), if and only if the following

equation holds:

$$\mathbf{P} = \mathbf{Q}\,\mathbf{A} \qquad \text{(A.}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is a real-valued eigenvector matrix of (A.78) and $\mathbf{A} \in \mathbb{C}$ full-rank matrix.

The necessity of the lemma can be established from the definition of a w complex valued eigenvector. For the sufficiency of this lemma, let us ext the subscript p to n. Then, for any P_i, we have

$$P_i = \alpha_{1i}Q_1 + \alpha_{2i}Q_2 + \ldots + \alpha_{ni}Q_n \qquad \text{(A.}$$

Thus,

$$\mathbf{P} = [\,P_1\ P_2\ \ldots P_n\,] = [\,Q_1\ Q_2\ \ldots Q_n\,]\begin{bmatrix} \alpha_{11} & \alpha_{12} \ldots & \alpha_{1n} \\ \alpha_{11} & \alpha_{12} \ldots & \alpha_{1n} \\ & \ldots & \\ \alpha_{11} & \alpha_{12} \ldots & \alpha_{1n} \end{bmatrix}$$

Let

$$\mathbf{Q} = [\,Q_1\ Q_2\ \ldots Q_n\,]$$

and

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \ldots & \alpha_{1n} \end{bmatrix}$$

Lemma A.5 also implies that a vector $P_i \in \mathbb{C}^n$, in the eigenvector matrix $\mathbb{C}^{n\times n}$ is a strongly-complex eigenvector of equation (A.78), if and only if equation (A.82) does not hold. Here the matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ may be any g full-rank matrix.

Therefore, Lemma A.6 tells us that if a matrix **P** is an eigenvector matri system (A.78), there exist two possibilities for **P**: **P** either contains strongly-complex eigenvectors of a system (A.78), or it does not. In the

first case, **P** can be called a *strongly-complex eigenvector matrix*. If **P** does not contain a strongly-complex eigenvector, it may be called a *weakly-complex eigenvector matrix* of system (A.78). It is important to realize that a strongly-complex eigenvector matrix can not be changed into a weakly-complex eigenvector matrix of the same system (A.78) by equation (A.81) and vice versa.

Lemma A.6: For any M-C-K system, there exists a corresponding C-Λ_k system which has the identical eigenvalues of the **M-C-K** system. Further, if the i^{th} eigenvector of the M-C-K system is a weakly-complex eigenvector, so is the corresponding eigenvector of the C-Λ_k system.

PROOF:

Consider the eigen-equation of a M-C-K system (A.78):

$$\mathbf{M A^2 + C A + K = 0}$$

where $\mathbf{A = P \Lambda P^{-1}}$.

Pre- and post- multiplying matrices **S** and $\mathbf{S^{-1}}$ on both sides of the above equation respectively, we have

$$\mathbf{S M S^{-1} S A^2 S^{-1} + S C S^{-1} S A S^{-1} + S K S^{-1} = 0}$$

where

$$\mathbf{S = Q^T M^{-1/2}} \qquad \text{(A.83)}$$

and **Q** is the eigenvector matrix of matrix $\mathbf{M^{-1/2} K M^{1/2}}$. Then, we know that $\mathbf{S M S^{-1} = I}$, $\mathbf{S K S^{-1}} = \Lambda_k$, where Λ_k is the eigenvalue matrix of **K**. Denoting

$$\bar{\mathbf{A}} = \mathbf{S A S^{-1}} \qquad \text{(A.84)}$$

and

$$\bar{\mathbf{C}} = \mathbf{S C S^{-1}}$$

we have

$$\bar{\mathbf{A}}^2 + \bar{\mathbf{C}}\,\bar{\mathbf{A}} + \Lambda_k = \mathbf{0} \qquad \text{(A.85)}$$

The system (A.85) has the same eigenvalues as system (A.78), in view of the relationships with (A.84). Because both Q and $M^{-1/2}$ are real-valued, the second part of this lemma can be clearly shown from equation (A.83).

Note that the system (A.85) has some advantages over its original system (A.78) since Λ_k is diagonal. If we are only concerned with the eigenvalues of a system, or if we are only interested in knowing whether or not an eigenvector is strongly-complex, then, we will often use equation (A.85), rather than (A.78). We call the C-Λ_k system described by equation (A.85) the *canonical vibration system* (CVS) corresponding to its original system (A.64a).

Lemma A.7: For a CVS, if one of its eigenvector P_i is a weakly-complex eigenvector, then

$$\omega_i^2 = k_i \tag{A.86}$$

where ω_i is the corresponding i^{th} undamped natural frequency, k_i is the i^{th} eigenvalue of matrix Λ_k.

PROOF:

Denote

$$P_i = [\, p_1 \ldots p_n \,]^T \tag{A.87}$$

Consider the i^{th} eigen-equation of the system (A.85)

$$(I \lambda_i^2 + C \lambda_i + \Lambda_k) P_i = 0 \tag{A.88}$$

where for convenience, we omit the overhead bar -.
Among the equations from (A.88),

$$\lambda_i^2 p_j + \Big(\sum_{m=1}^{n} c_{jm} p_m \Big) \lambda_i + k_i p_j = 0 \qquad j = 1,\ldots n \tag{A.89}$$

where c_{jm} is the jm^{th} entry of matrix C, there exists at least one equation in which $p_j \neq 0$, such that

$$\lambda_i^2 + [(\sum_{m=1}^{n} c_{jm}\, p_m)/p_j]\, \lambda_i + k_i = 0 \qquad (A.90)$$

Otherwise P becomes a zero vector.

In equation (A.90), since P_i is real-valued, the term $[(\sum_{m=1}^{n} c_{jm} p_m)/p_j]$ will be real-valued as well. Therefore, from (A.90), we see that

$$\lambda_i\, \lambda_i^* = \omega_i^2 = k_j$$

Denoting k_j here by k_i proves this lemma. ∎

Corollary A.10: For a CVS, if $\omega_i^2 \neq k_i$, then the corresponding eigenvector P_i is strongly-complex eigenvector.

With the help of the above lemmas and corollaries, we have the following theorems:

Theorem A.7: In equation (A.78), if $4\mathbb{K} - \mathbb{C}^2 > 0$ and $\mathbb{K}\,\mathbb{C} \neq \mathbb{C}\,\mathbb{K}$, then at least two strongly-complex eigenvectors P_i and P_j exist such that equation (A.79) holds.

PROOF :

Suppose there exists no weakly-complex eigenvector, then according to Lemma A.5, equation (A.81) holds. So, we have

$$Q\,\Lambda^2 + C\,Q\,\Lambda + \mathbb{K}\,Q = 0 \qquad (A.91)$$

Multiplying Q^{-1} on the left of equation (A.91) yields

$$\Lambda^2 + Q^{-1}\,C\,Q\,\Lambda + Q^{-1}\,\mathbb{K}\,Q = 0 \qquad (A.92)$$

Denote

$$Q^{-1}\,C\,Q = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and

$$\mathbf{Q}^{-1}\,\mathbf{K}\,\mathbf{Q} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Equation (A.92) can be rewritten as

$$\begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & \cdots & & \\ & & & \lambda_n^2 \end{bmatrix} + \begin{bmatrix} a_{11}\lambda_1 & a_{12}\lambda_2 & \cdots & a_{1n}\lambda_n \\ a_{21}\lambda_1 & a_{22}\lambda_2 & \cdots & a_{2n}\lambda_n \\ & \cdots & & \\ a_{n1}\lambda_1 & a_{n2}\lambda_2 & \cdots & a_{nn}\lambda_n \end{bmatrix} +$$

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & \cdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \mathbf{0} \qquad \text{(A.93)}$$

because $\mathbf{K}\,\mathbf{C} \neq \mathbf{C}\,\mathbf{K}$, $\mathbf{Q}$ cannot be the eigenvector matrix of both $\mathbf{C}$ and $\mathbf{K}$. Therefore, there exists at least one pair of a_{ij} and b_{ij}, which are not zero simultaneously, when $i \neq j$. From equation (A.93) we may write

$$\lambda_j\, a_{ij} = b_{ij} \qquad \text{(A.94)}$$

where $\lambda_j \neq 0$. Now suppose $a_{ij} \neq 0$. Then, $b_{ij} \neq 0$. Similarly, if $b_{ij} \neq 0$, then, $a_{ij} \neq 0$. However, both a_{ij} and b_{ij} must be real-valued, since $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{Q}$ are all real. Therefore λ_j must be real. But this conclusion contradicts the condition that $4\mathbf{K} - \mathbf{C}^2 > \mathbf{0}$, which implies that all λ_j's are complex-valued. Thus, we conclude that at least one of the eigenvectors of equation (A.78) must be a strongly-complex eigenvector.

Suppose there exists only one strongly-complex eigenvector P_i and others are all weakly-complex eigenvectors. We can see that $\det(\mathbf{H}) = \det(\mathbf{K})$.

$$\mathbf{H} = \begin{bmatrix} -\mathbf{C} & -\Lambda_k \\ \mathbf{I} & \mathbf{O} \end{bmatrix}$$

is the state matrix of the CVS. Note that,

$$\det(\mathbf{H}) = \prod_{j=1}^{2n} \lambda_j = \prod_{j=1}^{n} \omega_j^2 = \omega_i^2 \prod_{\substack{j=1 \\ j\neq i}}^{n-1} \omega_j^2$$

$$\det(\mathbf{K}) = \prod_{j=1}^{n} k_j = k_i \prod_{\substack{j=1 \\ j\neq i}}^{n-1} k_j$$

where k_j is the j^{th} eigenvalue of **K**.

Since $\omega_i^2 \neq k_i$,

$$\prod_{\substack{j=1 \\ j\neq i}}^{n-1} \omega_j^2 \neq \prod_{\substack{j=1 \\ j\neq i}}^{n-1} k_j \qquad \text{(A.95)}$$

However, according to Lemma A.8 all the corresponding undamped natural frequencies

$$\omega_j = k_j^{1/2} \quad , j = 1,..n,\ j \neq i \qquad \text{(A.96)}$$

because we assumed there is only one $\omega_i \neq k_i$. The contradiction of (A.95) and (A.96) implies the incorrect assumption. Therefore, we conclude that in this case, there exists at least two strongly-complex eigenvectors. ∎

Theorem A.8: Suppose vector P_i is an eigenvector of the eigen-problem (A.78). P_i is an eigenvector of **C** (or **K**), and P_i is not a strongly-complex eigenvector of (A.78), if and only if P_i is also an eigenvector of **K** (or **C**).

PROOF :

Sufficiency: Since P_i is also an eigenvector of **K**, we have $\mathbf{K}\, P_i = \omega_{ni}^2 P_i$, where ω_{ni}^2 is the corresponding eigenvalue of **K** (recall that **K** is positive definite). However, suppose P_i is not an eigenvector of **C**. Denoting $P_i = [\, p_{1i}\ p_{2i} .. p_{ni}]^T$, we write

$$\mathbf{C}\ P_i = \begin{Bmatrix} c_1 p_{1i} \\ c_2 p_{2i} \\ .. \\ c_n p_{ni} \end{Bmatrix} \tag{A.97}$$

and at least one pair of $c_s \neq c_t$, and $c_s\ p_{si} \neq 0$, $c_t\ p_{ti} \neq 0$. Consider the following equations corresponding to the pair c_s and c_t, i.e.

$$\lambda_i^2\ p_{si} + c_s\ p_{si}\ \lambda_i + \omega_{ni}^2\ p_{si} = 0$$

$$\lambda_i^2\ p_{ti} + c_t\ p_{ti}\ \lambda_i + \omega_{ni}^2\ p_{ti} = 0$$

In the above equations, neither p_{si} nor p_{ti} equals zero and since $c_s \neq c_t$, the above two equations are mutually contradicting. Therefore, the assumption that P_i is not an eigenvector of **C** is false. That is, P_i must be an eigenvector of both **K** and **C**. By using this same reasoning, we can first suppose P_i as an eigenvector of **C** and then prove that P_i is also an eigenvector of **K**. Now, we can have $\mathbf{C}\ P_i = 2\ \xi_i \omega_i\ P_i$, where we can always denote the corresponding eigenvalue by $2\ \xi_i \omega_i$.

It remains to show that in this case P_i is not a strongly-complex eigenvector of the system (A.78). Suppose λ_i is a v multiple repeated eigenvalue of system (A.78), since

$$\lambda_i = -\ \xi_i\ \omega_{ni} + j(\ 1 - \xi_i^2\ )^{1/2}\ \omega_{ni}$$

where for convenience, we take the sign of the imaginary part of λ_i to be positive, both ξ_i and ω_{ni} are uniquely determined by λ_i. Therefore $2\xi_i \omega_i$ and ω_{ni}^2 must be v multiple repeated eigenvalues of **C** and **K**, respectively. However, since **K** is positive definite, there exist v linearly independent eigenvectors of **K** associate with ω_{ni}^2. This implies that, any other real-valued or complex-valued vector P_i must be only weakly-complex eigenvector of **K** or **C** of the system (A.78).

The above argument establishes the sufficient condition for Theorem A.8.

Necessity: Suppose P_i is not a strongly-complex eigenvector of system (A.78). According to Lemma A.7, we can always change P_i into a real-valued vector. Since equation (A.79) holds, we can take the complex conjugate of (A.79) and the following should also hold

$$\lambda_i^{*2} P_i + \lambda_i^* \mathbf{C} P_i + \mathbf{K} P_i = 0 \qquad \text{(A.98)}$$

Suppose P_i is not an eigenvector of $\mathbf{K}$ or $\mathbf{C}$. Similar to equation (A.97), we can always write

$$\mathbf{K} P_i = \begin{Bmatrix} k_1 p_{1i} \\ k_2 p_{2i} \\ \cdots \\ k_n p_{ni} \end{Bmatrix} \qquad \text{(A.99)}$$

and at least one pair of $k_s \neq k_t$ with $k_s p_{si} \neq 0$ and $k_t p_{ti} \neq 0$ exists.

The corresponding equations (A.79) and (A.98) of this pair may be written as

$$\lambda_i^2 p_{si} + c_s p_{si} \lambda_i + k_s p_{si} = 0$$

$$\lambda_i^2 p_{ti} + c_t p_{ti} \lambda_i + k_t p_{ti} = 0$$

and

$$\lambda_i^{*2} p_{si} + c_s p_{si} \lambda_i^* + k_s p_{si} = 0$$

$$\lambda_i^{*2} p_{ti} + c_t p_{ti} \lambda_i^* + k_t p_{ti} = 0$$

Note that, c_s c_t k_s and k_t are all real-valued and $k_s \neq k_t$, and that p_{si} and p_{ti} must not be zero, The above four equations are mutually contradicting. So, the assumption that P_i is not an eigenvector of $\mathbf{K}$ or $\mathbf{C}$ is not true. Therefore, if P_i is a weakly-complex eigenvector of system (A.78), it must also be an eigenvector of both $\mathbf{C}$ and $\mathbf{K}$. ∎

Lemma A.8: (Ortega, 1986) For any symmetric matrix $\mathbf{A} \in \mathbb{R}^{nxn}$, denoting its first (smallest) and last (largest) eigenvalues by λ_1 and λ_n, respectively, if

and only if its Rayleigh quotient equals either λ_1 or λ_n, i.e

$$\frac{X^H A X}{X^H X} = \lambda_1 \qquad \text{(A.100)}$$

or

$$\frac{X^H A X}{X^H X} = \lambda_n \qquad \text{(A.101)}$$

then, the corresponding non-zero vector $X \in \mathbb{C}^n$ is the first or last eigenvector of **A**.

Corollary A.11: For a given Rayleigh quotient

$$\frac{X^H A X}{X^H X} = \lambda_i$$

where $A \in S^{nxn}$, if λ_i is the i^{th} eigenvalue of **A**, and λ_i is greater than the first eigenvalue of **A** and less than the last eigenvalue of **A**, vector **X** is not necessarily the corresponding i^{th} eigenvector of **A**.

Lemma A.8 can play an important role in modal analysis. Now, with the help of Lemma A.8, let us consider Theorem A.9, which is the basis of the discussions in Chapters 2, 3 and 4. First let us introduce a definition:

Definition: For a CVS, if the damping matrix is in the following form, the system is said to be *self-decoupled* into two subsystems.

$$C = \left[\begin{array}{ccc:ccc} c_{11} & \cdots & c_{1t} & & & \\ & \cdots & & & \mathbf{0} & \\ c_{1t} & \cdots & c_{tt} & & & \\ \hdashline & & & c_{t+1,t+1} & \cdots & c_{t+1,n} \\ & \mathbf{0} & & & \cdots & \\ & & & c_{t+1,n} & \cdots & c_{nn} \end{array}\right]$$

A diagonal **C** makes the system self-decoupled into n subsystems. We can see

that each subsystem has its own CVS vibrators, which is spatially separated from other subsystems.

Theorem A.9: For an M-C-K system, if the following is true:

$\omega_i^2 = k_i \quad , i = 1,...n$

then, it has a real-valued eigenvector matrix.

PROOF:

From Lemma A.6, we examine the corresponding CVS of the M-C-K system where the overhead bar "-" can be omitted for simplicity.

First, for the case where all eigenvalues of Λ_k are distinct, consider the lowest mode, ω_1, the first undamped natural frequency and P_1, the first eigenvector. According to Ewins (1986), we have

$$\frac{P_1^H \; C \; P_1}{P^H \; P} = 2\xi_1\omega_1 \tag{A.102}$$

and

$$\frac{P_1^H \; \mathbf{K} \; P_1}{P^H \; P} = \omega_1^2 \tag{A.103}$$

Since $\omega_1^2 = k_1$, according Theorem A.7, P_1 must be the first eigenvector of Λ_k. It can be $\mathbf{e}_1$. This result is concluded directly from Lemma A.8. Here the i^{th} element of vector $\mathbf{e}_i$ equals unity and others are zero. Then, in the eigen-equation of the system:

$$[\; \mathbf{I} \; \lambda_1^2 + \mathbf{C} \; \lambda_1 + \Lambda_k \;] \; P_1 = 0$$

we have

$$\lambda_1^2 p_1 + [\sum_{m=1}^{p} c_{1m} p_m] \lambda_1 + k_1 p_1 = 0$$

$$0 + [\sum_{m=1}^{p} c_{2m} p_m] \lambda_1 + 0 = 0$$

$$\cdots\cdots \qquad (A.104)$$

$$0 + [\sum_{m=1}^{p} c_{nm} p_m] \lambda_1 + 0 = 0$$

Equations (A.104) implies that

c_{1m}'s = 0 , m = 1, ... p

Note that, C is symmetric. We have the damping matrix C in the following form:

$$C = \begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22} & \cdots & c_{2n} \\ & & \cdots & \\ 0 & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

In this case, the CVS becomes self-decoupled : the first vibrator has no relationship with the rest.

As we proceed to consider the second lowest ω_2 and its corresponding eigenvector P_2 of the CVS, we see that ω_2 becomes the lowest undamped natural frequency of the remaining subsystem. Therefore, the eigenvector P_2 must be the real-valued vector $\mathbf{e}_2$. The corresponding damping matrix is

$$C = \left[\begin{array}{cc:ccc} c_{11} & 0 & & & \\ 0 & c_{22} & & \mathbf{0} & \\ \hdashline & & c_{33} & \cdots & c_{3n} \\ & \mathbf{0} & & \cdots & \\ & & c_{3n} & \cdots & c_{nn} \end{array}\right]$$

Following this same procedure, we finally can obtain a diagonal C matrix; n subsystems and all real-valued eigenvector $\mathbf{e}_i$'s, i = 1,...,n.

Secondly consider that Λ_k has r repeated eigenvalues k_r. We can always assume that C has t repeated eigenvalues $2\xi_r\omega_r$, $t \leq s$. First we only consider the case t = s; other possibilities will be discussed as a third case. Let us now consider the lowest mode. In this situation, the first eigenvector can be chosen as $\mathbf{e}_1$, and so on, until the t^{th} eigenvector is chosen to be $\mathbf{e}_t$. According to Theorem A.7, if both C and Λ_k have an identical eigenvector, the vector must also be an eigenvector of the system. In this case, $\mathbf{e}_i$ can be an eigenvector of both C and Λ_k. These eigenvectors make t subsystems which are separated from each other and are separated from the rest of the subsystems. Following this same procedure, we obtain the same results as the first case. That is, we can obtain a diagonal matrix C, n subsystems and all real eigenvectors, $\mathbf{e}_i$'s, i = 1,...,n.

The third case to be considered is when Λ_k has t repeated eigenvalues. Accordingly, C only has s repeated eigenvalues. Here, $1 \leq s < t$. Following the discussion of the second case, we can concentrate on the situation s = 1, for the third case. Now suppose that the lowest mode of Λ_k has t repeated eigenvalue k_1's. We know that, for a symmetric matrix A, we can always choose an orthogonal matrix T such that $\mathbf{T}^T\mathbf{A}\,\mathbf{T}$ = diagonal matrix. Let us choose

$$\mathbf{T} = \left[\begin{array}{c|c} \begin{matrix} t_{11} \cdots t_{1t} \\ \cdots \\ t_{t1} \cdots t_{tt} \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]$$

with

$$\mathbf{T}^T \Lambda_k \mathbf{T} = \Lambda_k$$

and

$$T^T C T = \hat{C} = \left[\begin{array}{c|c} \begin{matrix} \hat{c}_{11} & & \\ & \ddots & \\ & & \hat{c}_{tt} \end{matrix} & \begin{matrix} \hat{c}_{1,t+1} & \cdots & \hat{c}_{1n} \\ & \cdots & \\ \hat{c}_{t,t+1} & \cdots & \hat{c}_{tn} \end{matrix} \\ \hline \text{symmetric} & \begin{matrix} c_{t+1,t+1} & \cdots & c_{t+1,n} \\ & \cdots & \\ c_{t+1,n} & \cdots & c_{nn} \end{matrix} \end{array}\right]$$

As in case one, we can show that the left lower and the right upper block matrices in the above equation are null, i.e.

$$\left[\begin{matrix} \hat{c}_{1,t+1} & \cdots & \hat{c}_{1n} \\ & \cdots & \\ \hat{c}_{t,t+1} & \cdots & \hat{c}_{tn} \end{matrix}\right]^T = \left[\begin{matrix} \hat{c}_{1,t+1} & \cdots & \hat{c}_{1n} \\ & \cdots & \\ \hat{c}_{t,t+1} & \cdots & \hat{c}_{tn} \end{matrix}\right] = \mathbf{0}$$

Therefore, we have

$$\hat{C} = \left[\begin{array}{c|c} \begin{matrix} \hat{c}_{11} & & \\ & \ddots & \\ & & \hat{c}_{tt} \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} c_{t+1,t+1} & \cdots & c_{t+1,n} \\ & \cdots & \\ c_{t+1,n} & \cdots & c_{nn} \end{matrix} \end{array}\right]$$

Pre- and post-multiplying T and T^T on $\hat{C}$ we have,

$$C = T \hat{C} T^T = \left[\begin{array}{c|c} \begin{matrix} c_{11} & \cdots & c_{1t} \\ & \ddots & \\ c_{1t} & & c_{tt} \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \begin{matrix} c_{t+1,t+1} & \cdots & c_{t+1,n} \\ & \cdots & \\ c_{t+1,n} & \cdots & c_{nn} \end{matrix} \end{array}\right]$$

Continuing with the same manipulation, we finally can obtain all self-decoupled subsystems, each one has its own stiffness matrix with repeated diagonal elements. Such stiffness matrices can commute with any matrices with the same order. Therefore, according to Chaughey's criteria, these subsystems are all proportionally damped systems. In other words, all the eigenvalues can be real-valued. Now, consider a subsystem with t repeated k_r's, $t \geq 1$, that is

$$[\, \mathbf{I}\, \lambda_r^2 + \mathbf{C}_{sr} \lambda_r + \mathrm{diag}(\, k_r\,)\,]\, \mathbf{P}_r = \mathbf{0} \qquad \text{(A.105)}$$

where $\mathbf{C}_r$ stands for the r^{th} diagonal block damping matrix.

Since the matrix diag(k_r) will commute with any txt matrices,

$$\mathbf{C}_{sr}\ \mathrm{diag}\ (\mathbf{K}_r) = \mathrm{diag}(\mathbf{K}_r)\ \mathbf{C}_{sr}$$

then, we have all these subsystems with real-valued eigenvectors. In other words, the CVS has a real-valued eigenvector matrix. A C matrix may contain mixed cases. That is, all the above three cases may occur together. However, there is no essential technical difficulties in applying the same analysis mentioned above because we can always obtain the C matrix and, thus, the system is self-decoupled. Each self-decoupled subsystem only has real or weakly-complex eigenvectors. Therefore Theorem 2.9 is proven. ∎

Note that this theorem not only works with positive or semi-positive matrices Λ_k and C, but also works with certain other kind of symmetric matrices. However, we are only interested in the M-C-K system in this report.

A.7 EIGEN-PROPERTIES OF DISTRIBUTED MASS SYSTEMS

For linear distributed mass systems, Lemmas A.8 and A.9, Corollaries A.10 and A.11 and Theorems A.7, A.8 and A.9 all hold true. The vibration of a distributed mass system complies with the partial differential equation

$$U_{tt}(p,t) + \mathbf{L}_1[U_t(p,t)] + \mathbf{L}_2[U(p,t)] = f(p,t) \quad \text{for } p \in \Omega \tag{A.106a}$$

$$\mathbf{B}[U(p,t)] = 0 \quad \text{for } p \in \partial\Omega \tag{A.106b}$$

where U_t denotes the partial derivative of u with respect to the time variable t, and Ω is a bounded open region in $\mathbb{R}^n$, n = 1, 2, or 3 with boundary $\partial\Omega$. The two spatial linear differential operators, $\mathbf{L}_1$ and $\mathbf{L}_2$, are of order n_1 and n_2 (n_1 and n_2 are even numbers) respectively. These differential operators are self-adjoint, positive definite, and in particular $\mathbf{L}_2$ has a compact inverse. The domain of $\mathbf{L}_1$ and $\mathbf{L}_2$ is

$$\mathcal{D} = \{\, u(p,t) : \frac{\partial^k u}{\partial p^k} \in L^2(\Omega),\ k = 0, 1, \ldots, n;\ \ n = \mathrm{Max}(2n_1, n_2) \,\}$$

The boundary condition equation (A.106b) is under a restriction that operator **B** is linear differential with an order up to $\mathrm{Max}(n_1, n_2) - 1$.

Caughey and O'Kelly (1965) pointed out that the system (A.106a) is proportionally damped when $\mathbf{L}_1$ and $\mathbf{L}_2$ commute. In this case, equation (A.106a) has a solution of the following form

$$U(p,t) = \sum_{i=1}^{\infty} a_i(t)\Phi_i(p) \tag{A.107}$$

where $\{\Phi_i(p)\}$ consists of a complete set of orthogonal spatial functions. Technically, this criterion is related with another property of the distributed mass system, namely being underdamped. Inman (1989) proved that with $\mathbf{L}_1\mathbf{L}_2 = \mathbf{L}_2\mathbf{L}_1$ a system described by equation (A.106a) is underdamped provided that $4\mathbf{L}_2 - \mathbf{L}_1^2$ is positive definite. Although the given proof uses equation (A.107), this result is likely valid without this commutative condition. At any rate, since we are only interested in underdamped systems, it is expedient to assume in the following that all the systems discussed are

underdamped.

In the following discussion, we assume that the spatial domain under consideration is $L^2(\mathbb{R}^n)$, n = 1, 2, or 3. In the context of distributed mass systems, the interested spatial domain is actually restricted on the intersection of $\mathcal{D}$ and $L^2(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n)$ is much more generalized than $\mathcal{D} \cap L^2(\mathbb{R}^n)$, the equality in the following should be understood as equal in the sense of L^2-norm.

Definition: For given operators $\mathbf{L}_1$ and $\mathbf{L}_2$ as described in equation (A.106a), an *eigenvalue and eigenfunction* of the corresponding system of equation (A.106a) is a scalar λ and a non-zero spatial function $\Phi(p)$ such that the function is in $L^2(\mathbb{R}^n) \cap \mathcal{D}$ and satisfies

$$(\lambda^2 + \lambda \mathbf{L}_1 + \mathbf{L}_2)\, \Phi(p) = 0 \qquad \text{(A.108)}$$

and the boundary conditions associated with equation (A.106a). We shall refer to equation (A.108) as the spatial operator system of equation (A.106a), or simply the operator system.

Note that the above defined eigen parameters differ from their counter-parts of a self-adjoint compact operator in that the eigenfunctions of different eigenvalues may no longer be orthogonal to each other.

Definition: An eigenfunction $\Phi(p)$ associated with eigen value λ is said to be *strongly-complex* if it can not be expressed as a finite linear combination of the real-valued eigen-functions of λ. $\Phi(p)$ is said to be *weakly-complex* if it is not strongly complex. For simplicity, we may use Φ for $\Phi(p)$.

The following preliminary results show some basic properties of the vibrational systems connected with the above concepts. The results can also be established for lumped mass systems.

Lemma A.9: For a given pair of system eigen parameters λ and $\Phi(p)$ the following conditions are equivalent:

a. $\Phi(p)$ is an eigenfunction of the damping operator $\mathbf{L}_1$.

b. $\Phi(p)$ is an eigenfunction of the stiffness operator $\mathbf{L}_2$.

c. $\Phi(p)$ is weakly-complex.

PROOF:

First, consider the conditions *a.* and *b.*
Suppose

$$\mathbf{L}_1\Phi(p) = \alpha\,\Phi(p)\ . \tag{A.109}$$

Substituting equation (A.109) into equation (A.108) results in

$$\lambda^2\Phi(p) + \lambda(\alpha\,\Phi(p)) + \mathbf{L}_2\Phi(p) = 0.$$

So

$$\mathbf{L}_2[\Phi(p)] = -(\lambda^2 + \lambda\alpha)\Phi(p).$$

Since $-(\lambda^2 + \lambda\alpha)$ is a scalar, $\Phi(p)$ is an eigenfunction of $\mathbf{L}_2$. Also, the boundary conditions should be satisfied. So the condition from *a.* to *b.* is proved. Conversely, suppose

$$\mathbf{L}_2[\Phi(p)] = \beta\Phi(p). \tag{A.110}$$

Following the same argument, we can similarly show the validity from *b.* to *a.*

Secondly, consider the conditions *b.* and *c.*

Suppose $\Phi(p)$ is complex-valued, i.e. $\Phi(p) = \psi(p) + j\eta(p)$, where ψ and η are real-valued functions. Then by linearity

$$\mathbf{L}_2[\Phi(p)] = \mathbf{L}_2[\psi(p)] + j\,\mathbf{L}_2[\eta p)].$$

It follows that $\mathbf{L}\,[\psi(p)]$ is real and $j\mathbf{L}\,[\eta(p)]$ is imaginary. On the other hand, by assuming $\Phi(p)$ to be an eigenfunction of $\mathbf{L}_2$, then

$$\mathbf{L}_2[\Phi(p)] = \alpha\Phi(p) = \alpha\psi(p) + j\alpha\eta(p).$$

Since $\mathbf{L}_2$ is self-adjoint, positive definite and has a compact inverse, all of its eigenvalues must be real. Therefore, $\alpha\psi(p)$ is real and $j\alpha\eta(p)$ is imaginary. Comparing the above two equations, we conclude

$$\mathbf{L}_2[\psi(p)] = \alpha\psi(p)\ ,$$

$$\mathbf{L}_2[\eta(p)] = \alpha\eta(p) \ .$$

Thus, both ψ and η are real-valued eigenfunctions of $\mathbf{L}_2$. Since $\Phi(p)$ is a linear combination of real-valued eigenfunctions $\psi(p)$ and $\eta(p)$, $\Phi(p)$ is weakly-complex.

Next, we prove the converse. It is known that for a linear self-adjoint compact operator $\mathbf{L}$ of $L^2(\mathbb{R}^n)$, n = 1, 2, or 3, there is a complete set of orthogonal eigenfunctions $\{ Y_j(p) \}$ such that for $V(p) \in L^2(\mathbb{R}^n)$

$$V(p) = \sum_{j=1}^{\infty} b_j Y_j(p)$$

and

$$\mathbf{L}[V(p)] = \sum_{i=1}^{\infty} \mu_i b_i Y_i(p)$$

where μ_i is the associated eigenvalue of Y_i. Since $\mathbf{L}_2$ is self- adjoint and has a compact inverse, the above spectral theorem is still valid except that

$$\mathbf{L}_2[V(p)] = \sum_{i=1}^{\infty} \frac{1}{\mu_i} b_i Y_i(p)$$

By assumption, $\Phi(p)$ is a weakly-complex eigenfunction of the operator system equation (A.108) , i.e. there is a system eigenvalue λ such that

$$\Phi(p) = l_1 \Xi_1(p) + \ldots + l_m \Xi_m(p) \qquad \text{(A.111)}$$

where l_i are complex coefficients, and

$$\mathbf{L}_2[\Xi_j(p)] = \lambda \ \Xi_j(p) \ , \qquad j = 1, 2, \ldots m.$$

We show that Ξ_j are eigenfunctions of $\mathbf{L}_2$, and they have the same eigenvalue $\lambda \ \lambda^*$.

Choose an arbitrary $\Xi_j(p)$ in equation (A.111). Rewrite $\Xi_j(p)$ as

$$\Xi_j(p) = \sum_{i=1}^{\infty} d_i Y_i(p)$$

For the following equation

$$(\lambda^2 + \lambda \mathbf{L}_1 + \mathbf{L}_2)\, \Xi_j(p) = 0$$

there exists a unique eigenfunction of $\mathbf{L}_2$, denoted as $Y_i(p)$, such that

$\langle Y_i, \Xi_j \rangle \neq 0$ and

$$\lambda^2 \langle Y_i, \Xi_j \rangle + \lambda \langle Y_i, \mathbf{L}_1[\Xi_j] \rangle + \langle Y_i, \mathbf{L}_2[\Xi_j] \rangle = 0$$

We know that $\mathbf{L}_2$ is self-adjoint, so

$$\langle Y_i, \mathbf{L}_2[\Xi_j] \rangle / \langle Y_i, \Xi_j \rangle = \lambda\lambda^* = \mu_i$$

Thus, $\Xi_j(p) = d_i Y_i(p)$ is an eigenfunction of $\mathbf{L}_2$. It follows that $\Phi(p)$ is an eigenfunction of $\mathbf{L}_2$.

Lemma A.10: Suppose $\Phi(p)$ is an eigenfunction of the operators $\mathbf{L}_1$ and $\mathbf{L}_2$ in equation (A.108), then it is also a *system eigenfunction*. Moreover, $\Phi(p)$ must be weakly-complex.

PROOF:

According to the assumption, we have

$$\mathbf{L}_1[\Phi(p)] = \alpha\Phi(p)$$

$$\mathbf{L}_2[\Phi(p)] = \beta\Phi(p)$$

It follows that

$$\lambda^2\Phi(p) + \lambda\mathbf{L}_1\Phi(p) + \mathbf{L}_2\Phi(p) = (\lambda^2 + \alpha \lambda + \beta)\Phi(p)$$

Choosing a particular λ such that $(\lambda^2 + \alpha \lambda + \beta) = 0$, then we have

$$(\lambda^2 + \alpha \lambda + \beta)\Phi(p) = 0$$

So $\Phi(p)$ is a system eigenfunction. Then, $\Phi(p)$ is also weakly-complex.

Lemma A.11: For a self-adjoint compact operator $\mathbf{L}$, denote its greatest eigenvalue by λ_1. If the Rayleigh quotient equals to λ_1, i.e.

$$\frac{< \Psi , L\Psi >}{< \Psi , \Psi >} = \lambda_1 , \qquad \text{(A.112)}$$

then the non-zero function Ψ is an eigenfunction of L, and its associated eigenvalue is λ_1.

PROOF:

Since operator **L** is self-adjoint and compact, it has a complete set of orthogonal eigenfunctions, i.e. for any function f(p) in $L^2(\mathbb{R}^n)$, n = 1, 2 or 3

$$f(p) = \sum_{i=1}^{\infty} <f, \Phi_i> \Phi_i(p) \quad \text{where } <\Phi_i, \Phi_j> = \delta_{ij} \text{ (the delta function)}$$

and

$$\mathbf{L}[f(p)] = \sum_{i=1}^{\infty} \mathbf{L}[<f, \Phi_i>\Phi_i(p)] = \sum_{=1}^{\infty} <f, \Phi_i>\lambda_i \Phi_i(p)$$

where λ_i is the associated eigenvalue for Φ_i. Moreover, among the eigenvalues of **L**, there exists one which is the greatest, λ_1. Consider the given function $\Psi(p)$, it has the representation

$$\Psi(p) = \sum_{i=1}^{\infty} <\Psi, \Phi_i> \Phi_i(p)$$

Applying **L** to Ψ, we have

$$\mathbf{L}[\Psi(p)] = \sum_{i=1}^{\infty} < \Psi, \Phi_i> \lambda_i \Phi_i(p)$$

By assuming $<\Psi, \mathbf{L}\Psi> = \lambda_1 <\Psi,\Psi>$, we get

$$< \Psi, \mathbf{L}\Psi > = \sum_{i=1}^{\infty} < \Psi, \Phi_i>^* < \Psi, \Phi_i>\lambda_i = \lambda_1 \sum_{i=1}^{\infty} < \Psi, \Phi_i>^* <\Psi, \Phi_i>$$

which implies

$$\sum_{i=1}^{\infty} <\Psi, \Phi_i>^* <\Psi, \Phi_i> (\lambda_i - \lambda_1) = 0$$

where $<\Psi, \Phi_i>^* <\Psi, \Phi_i> \geq 0$ for all i. Also, λ_1 is the greatest eigenvalue, so $(\lambda_i - \lambda_1) \leq 0$. By now, the above equation forces $<\Psi,\Phi_i> = 0$, for all i such that $\lambda_i \neq \lambda_1$ and $< \Psi, \Phi_i> \neq 0$ for some i such that $\lambda_i = \lambda_1$.

Thus, we conclude that

$$\Psi = \sum_{\lambda_i = \lambda_1} <\Psi, \Phi_i>\Phi_i .$$

This shows that Ψ is an eigenfunction of $\mathbf{L}$, and is associated with the greatest eigenvalue λ_1.

One of the essential conditions in the above lemma is the compactness of operator $\mathbf{L}$. This condition requires that operator $\mathbf{L}$ has a maximal eigenvalue, but no limit point beside zero among its eigenvalues. By analogy, if $\mathbf{L}$ is a linear self-adjoint operator with compact inverse, a corresponding lemma is true for the smallest eigenvalue of $\mathbf{L}$.

Theorem A.10: For a distributed mass system described by equation (A.106a), suppose we know all of its system eigenvalues λ_i. If the following relationships (repetition counted) hold valid

$$\beta_i = \lambda_i \lambda_i^* , \qquad i = 1, 2, ...$$

where β_i are the eigenvalues of stiffness operator $\mathbf{L}_2$, then the corresponding system eigenfunctions consist of a complete orthogonal set and are weakly-complex. The converse of this statement is also true.

PROOF.

We carry out the proof for the first part of the criterion by induction on the system eigenvalues λ_i. Since operator $\mathbf{L}_2$ is self-adjoint, positive definite and has compact inverse, its eigenvalues contain a minimal element and have no limit point. As a result, there is a complete basis of orthogonal eigenfunctions of $\mathbf{L}_2$, $\{ \Psi_i \}_{i=1}^{\infty}$, and the corresponding eigen-values β_i are linearly ordered by $\beta_1 < \beta_2 < ...$. Utilizing this ordering and the given assumption, $\lambda_i^* \lambda_i = \beta_i$, we suppose for the proof to follow, without loss of generality, that λ_i are ordered in an increasing sequence by their norms.

Consider the smallest $\lambda\lambda^*$, namely $\lambda_1 \lambda_1^*$. The corresponding eigenfunction is $\Phi_1(p)$ and

$$\lambda_1^2 \Phi_1(p) + \lambda_1 \mathbf{L}_1 \Phi_1(p) + \mathbf{L}_2 \Phi_1(p) = 0$$

Taking the inner product of $\Phi_1(p)$ with the above equation, we get

$$<\Phi_1, \lambda_1^2\Phi_1> + <\Phi_1, \lambda_1 L_1 \Phi_1> + <\Phi_1, L_2\Phi_1> = 0$$

Then

$$\lambda_1^2 + \lambda_1 \left\{\frac{<\Phi_1, L_1 \Phi_1>}{<\Phi_1, \Phi_1>}\right\} + \left\{\frac{<\Phi_1, L_2 \Phi_1>}{<\Phi_1, \Phi_1>}\right\} = 0$$

Note that the terms in { } are real scalars. Since both L_1 and L_2 are self-adjoint, so

$$\left\{\frac{<\Phi_1, L_2 \Phi_1>}{<\Phi_1, \Phi_1>}\right\} = \lambda_1\lambda_1^* = \beta_1$$

where β_1 is the smallest eigenvalue of L_2. Thus, $\Phi_1(p)$ is an eigenfunction of L_2. By the auxiliary lemma, we see that $\Phi_1(p)$ is weakly complex. It is noted that the same argument would be applicable to Φ_2, should it be the case that $\lambda_2\lambda_2^* = \lambda_1\lambda_1^*$. In such an event, we should have no problem in adjusting Φ_1 and Φ_2 to make them orthogonal.

For the inductive step, assume for all subscripts i with $\lambda_i\lambda_i^* \le \beta_{n-1}$ their corresponding system eigenfunctions Φ_i are weakly complex and orthogonal. We show that the operator system can be separated on the subspace spanned by $\{\Psi_i\}_{i=1}^{n-1}$ and its orthogonal complement space spanned by $\{\Psi_i\}_{i=n}^{\infty}$. Then by reiterating the foregoing argument on the latter space, Φ_n is shown to be weakly-complex and orthogonal to Φ_i, for $i < n$. As a consequence, we have for i where $\lambda_i\lambda_i^* \le \beta_n$, Φ_i are weakly complex and orthogonal.

Before proceeding to a case-by-case study, we form a useful representation for Φ_n. As aforementioned, operator L_2 has a complete set of orthogonal eigenfunctions $\{\Psi_i\}_{i=1}^{\infty}$. By the induction hypothesis, the system eigenfunctions Φ_i, $i < n$, are now eigenfunctions of L_2 as well. Thus, we may assume that Φ_i, $i < n$, are identical to Ψ_i, $i < n$ (strictly in order to validate this assumption, we need to mention that the system is simple). By now Φ_n has a representation of

$$\Phi_n(p) = \sum_{i=1}^{n-1} d_i \Phi_i(p) + \sum_{i=n}^{\infty} d_i \Psi_i(p) \qquad \text{(A.113)}$$

With the help of equation (A.113), we begin to examine the operator system

$$[\lambda_n^2 + \lambda_n L_1 + L_2]\ \Phi_n(p) = 0 \qquad \text{(A.114)}$$

Case 1: Suppose $\beta_n > \beta_{n-1}$. Replacing $\Phi_n(p)$ in equation (A.114) by equation (A.113) , and taking the inner product of Φ_i with equation (A.114), we have

$$\lambda_n^2 \langle \Phi_i, d_i \Phi_i \rangle + \lambda_n \langle \Phi_i, L_1 d_i \Phi_i \rangle + \langle \Phi_i, L_2 d_i \Phi_i \rangle = 0 \qquad \text{for } i < n$$

Note that the cross-terms are eliminated because Φ_i , $i < n$ are orthogonal to each other. If $d_i \neq 0$, $i < n$ in equation (A.113), then we have

$$\lambda_n^2 + \lambda_n \alpha_i + \beta_i = 0 \qquad \text{for } i < n \qquad \text{(A.115)}$$

where α and β are the eigenvalues of L_1 and L_2 associated with eigenfunction Φ_i . The above equation contradicts the fact

$\lambda_n \lambda_n^* = \beta_n > \beta_i$, for $i < n$

Therefore, we conclude that $d_i = 0$ for all $i < n$ in equation (A.113) .

Case 2: Suppose $\beta_n = \beta_{n-1}$. By resorting to the induction hypothesis, we already can conclude that Φ_n is weakly-complex and orthogonal to Φ_i , $i < n$. Therefore, $d_i = 0$ for $i < n$ in equation (A.113) .

The above study shows that Φ_n is orthogonal to Φ_i , $i < n$. By analogy, the representation of equation (A.113) can be directly generalized to any Φ_i, $i > n$. Using the same argument given for Φ_n , we can easily establish that Φ_i, $i \geq n$ are orthogonal to Φ_i, $i < n$. Hence for the analysis of Φ_i, $i \geq n$, the operator system can be treated as if being defined only on the space spanned by $\{ \Psi_i \}_{i=n}^{\infty}$. With this separation of the operator system, the remaining proof of the inductive step is straightforward.

For the converse statement, the proof is relatively easy. Consider an eigenfunction $\Phi_i(p)$ associated with eigenvalue λ_i. Suppose $\Phi_i(p)$ is weakly-complex. From

$$\lambda_i^2\ \Phi_i(p) + \lambda_i L_1 \Phi_i(p) + L_2 \Phi_i(p) = 0$$

we have

$$(\lambda_i^2 + \lambda_i \alpha_i + \beta_i) \Phi_i(p) = 0$$

Since $\Phi_i(p) \neq 0$, so

$$(\lambda_i^2 + \lambda_i \alpha_i + \beta_i) = 0.$$

Note that both α_i and β_i are real, so $\lambda_i \lambda_i^* = \beta_i$ holds.

A.8 EXISTENCE OF DAMPING MATRIX AND MODES

If two systems have exactly the same **M**, **C** and **K** matrices, they will have the same eigenvalue matrices. However, if two systems have the same **M** and **K** matrices and the same eigenvalue matrix Λ, they may not have identical damping matrices. In this section, more details will be given (the topic will be further discussed in Part II, to appear).

In practice, any identified value can involve certain errors. Therefore our concerns are the convergence of the values of the parameters themselves and whether they converge to the "real value." These concerns are difficult to deal with in physical parameter identification (mass, damping and stiffness). According to Liang and Lee (1990), to identify **M**, **C** and **K** accurately, we have to use a 3n-dimensional space.

A.8.1 SYMMETRIC STATE MATRIX

Consider the homogeneous form of the fundamental equations of motion (1.1a):

$$\mathbf{M}\, X'' + \mathbf{C}\, X' + \mathbf{K}\, X = 0$$

This system has n linearly independent modes (the system is simple). As discussed in Chapter 1, two approaches can be used to transform equation (1.1a) into the state space form. Alternatively, we introduce another approach here.

As in equation (1.18), since **M** is non-singular, we can let

$$Y = \mathbf{M}^{1/2} X$$

Multiplying $M^{-1/2}$ on the left hand side of equation (1.1a), we have

$$M^{-1/2} M M^{-1/2} M^{1/2} X'' + M^{-1/2} C M^{-1/2} M^{1/2} X' + M^{-1/2} K M^{-1/2} M^{1/2} X = 0$$

or

$$I Y'' + \tilde{C} Y' + \tilde{K} Y = 0 \qquad \text{(A.115a)}$$

where

$$\tilde{C} = M^{-1/2} C M^{-1/2}, \qquad \tilde{K} = M^{-1/2} K M^{-1/2} \qquad \text{(A.115b)}$$

Both $\tilde{C}$ and $\tilde{K}$ remain symmetric. For convenience, we call the matrix $\tilde{K}$ the generalized stiffness matrix, and $\tilde{C}$ the generalized damping matrix. It can be shown that systems (1.1a), (1.1b) and (A.115a) have the same eigenvalues (see Inman, 1989). We therefore denote

$$Y(t) = \tilde{P} E(t) \qquad \text{(A.116)}$$

where

$$\tilde{P} = \begin{bmatrix} P_1 & \Lambda & P_2 & \Lambda^* \\ & & & \\ P_1 & & P_2 & \end{bmatrix} \qquad \text{(A.117)}$$

is the eigenvector matrix (or spatial function) of system (A.115a) and

$$E = \begin{Bmatrix} \exp(\lambda_1 t) \\ \exp(\lambda_2 t) \\ \vdots \\ \exp(\lambda_{2n} t) \end{Bmatrix} \qquad \text{(A.118)}$$

is the vector of time exponentials (temporal functions).

Next, let

$$\mathbf{K}_s = \tilde{\mathbf{K}}^{1/2} \tag{A.119}$$

We can alternatively have 2n equations in the state space form

$$\begin{Bmatrix} Y'' \\ j\mathbf{K}_s\, Y' \end{Bmatrix} = \begin{bmatrix} -\mathbf{C} & j\mathbf{K}_s \\ j\mathbf{K}_s & 0 \end{bmatrix} \begin{Bmatrix} Y' \\ j\mathbf{K}_s\, Y \end{Bmatrix} \tag{A.120}$$

Substituting notation (A.115b), (A.117) and (A.118) into equation (A.120) and also deleting the temporal functions (A.116) from the resulting equation, we obtain an eigenproblem (A.121). This eigenproblem is similar to eigenproblems (1.15), (1.16) and/or (1.19). Furthermore equation (A.121) can be diagonalized and it has the same eigenvalues as (1.15), (1.16) and/or (1.19) (see Lancaster, 1960)

$$\begin{bmatrix} \mathbf{P}_1 & \Lambda & \mathbf{P}_2 & \Lambda^* \\ j\mathbf{K}_s\mathbf{P}_1 & & j\mathbf{K}_s\mathbf{P}_2 & \end{bmatrix} \begin{bmatrix} \Lambda & \\ & \Lambda^* \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{C} & j\mathbf{K}_s \\ j\mathbf{K}_s & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 & \Lambda & \mathbf{P}_2 & \Lambda^* \\ j\mathbf{K}_s\mathbf{P}_1 & & j\mathbf{K}_s\mathbf{P}_2 & \end{bmatrix} \tag{A.121}$$

Because the state matrix

$$\mathbf{H}_s = \begin{bmatrix} -\mathbf{C} & j\mathbf{K}_s \\ j\mathbf{K}_s & 0 \end{bmatrix} \tag{A.122}$$

is also symmetric, the eigen-decomposition of $\mathbf{H}_s$

$$\mathbf{H}_s = \mathbf{P}\,\Lambda_s\,\mathbf{P}^{-1} \tag{A.123}$$

where the eigenvalue matrix

$$\Lambda_s = \begin{bmatrix} \Lambda & \\ & \Lambda^* \end{bmatrix}, \tag{A.124}$$

defined in Section 2.5, must have the following eigenvector matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 & \Lambda & \mathbf{P}_2 & \Lambda^* \\ j\mathbf{K}_s\mathbf{P}_1 & & j\mathbf{K}_s\mathbf{P}_2 & \end{bmatrix} \tag{A.125}$$

Considering the i^{th} and j^{th} modes of equation (A.123), we have

$$\mathbf{H}_s\mathbf{P}_i = \lambda_i\mathbf{P}_i \tag{A.126}$$

and

$$\mathbf{H}_s\mathbf{P}_j = \lambda_j\mathbf{P}_j \tag{A.127}$$

Taking the transpose of (A.126) results in

$$\mathbf{P}_i^T\,\mathbf{H}_s = \lambda_i\,\mathbf{P}_i^T$$

so,

$$\mathbf{P}_i^T\,\mathbf{H}_s\,\mathbf{P}_j = \lambda_i\,\mathbf{P}_i^T\,\mathbf{P}_j \tag{A.128}$$

Similarly, we have

$$\mathbf{P}_j^T\,\mathbf{H}_s\,\mathbf{P}_i = \lambda_j\,\mathbf{P}_j^T\,\mathbf{P}_i \tag{A.129}$$

Subtracting the transpose of equation (A.129) from (A.128) yields

$$(\lambda_i - \lambda_j)\,\mathbf{P}_i^T\,\mathbf{P}_j = 0 \tag{A.130}$$

Therefore,

$$\mathbf{P}_i^T\,\mathbf{P}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{A.131}$$

(The above equation shows the orthogonal properties of eigen-vectors of the state matrix $\mathbf{H}_s$.) Thus, we have

$$\mathbf{P}\,\mathbf{P}^T = \mathbf{P}^T\,\mathbf{P} = \mathbf{I} \qquad \text{(A.132a)}$$

also

$$\mathbf{P}^T\,\mathbf{H}_s\,\mathbf{P} = \Lambda_s \qquad \text{(A.132b)}$$

Equations (A.132a) and (A.132b) show that eigenvector matrix $\mathbf{P}$ of the state matrix $\mathbf{H}_s$ can be orthonomal. Although the state matrix $\mathbf{H}_s$ is symmetric, the eigenvector matrix will generally be complex, since $\mathbf{H}_s$ is a complex matrix. In the discussion above, matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are mode shape matrices. It can also be shown that $\mathbf{P}_2$ can be chosen as $\mathbf{P}_1^*$. From the above equations, we can obtain the following relationships:

$$\begin{aligned}
&\mathbf{P}_1\,\Lambda^3\,\mathbf{P}_1^H + \mathbf{P}_2\,(\Lambda^*)^3\,\mathbf{P}_2^H = -\tilde{\mathbf{C}} \\
&\mathbf{P}_1\,\Lambda^2\,\mathbf{P}_1^H + \mathbf{P}_2\,(\Lambda^*)^2\,\mathbf{P}_2^H = \mathbf{I} \\
&\mathbf{P}_1\,\Lambda\,\mathbf{P}_1^H + \mathbf{P}_2\,(\Lambda^*)\,\mathbf{P}_2^H = 0 \\
&\mathbf{P}_1\,\mathbf{P}_1^H + \mathbf{P}_2\,\mathbf{P}_2^H = -\tilde{\mathbf{K}}^{-1}
\end{aligned} \qquad \text{(A.133a)}$$

or

$$\begin{aligned}
&\mathbf{P}_1\,\Lambda^3\,\mathbf{P}_1^H + \mathbf{P}_1^*\,(\Lambda^*)^3\,\mathbf{P}_1^T = -\tilde{\mathbf{C}} \\
&\mathbf{P}_1\,\Lambda^2\,\mathbf{P}_1^H + \mathbf{P}_1^*\,(\Lambda^*)^2\,\mathbf{P}_1^T = \mathbf{I} \\
&\mathbf{P}_1\,\Lambda\,\mathbf{P}_1^H + \mathbf{P}_1^*\,(\Lambda^*)\,\mathbf{P}_1^T = 0 \\
&\mathbf{P}_1\,\mathbf{P}_1^H + \mathbf{P}_1^*\,\mathbf{P}_1^T = -\tilde{\mathbf{K}}^{-1}
\end{aligned} \qquad \text{(A.133b)}$$

With the notation of (A.125), we see that

$$\mathbf{P}_1 \Lambda^2 + \tilde{\mathbf{C}} \mathbf{P}_1 \Lambda + \tilde{\mathbf{K}} \mathbf{P}_1 = 0 \qquad \text{(A.134)}$$

Next, since $\tilde{\mathbf{K}}$ is symmetric, it has the eigen-decomposition

$$\tilde{\mathbf{K}} = \mathbf{Q} \Lambda_k \mathbf{Q}^T \qquad \text{(A.135)}$$

where the eigenvector matrix $\mathbf{Q}$ must be real-valued and have a standard normalization $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$. And,

$$\Lambda_k = \text{diag}(\omega_{ni}^2) \qquad \text{(A.136)}$$

If a system is proportionally damped, the square roots of all the entries ω_{ni}^2 's , or all the ω_{ni}'s, are undamped natural frequencies of the system. However, if the damping is not proportional, these ω_{ni}'s may not be the natural frequencies.

If the system is proportionally damped, we may have the eigenvector matrix in the form,

$$\mathbf{P}=\mathbf{P}^T=\mathbf{P}^{-1}=\begin{bmatrix} \text{diag}(\exp(j\psi_i)\cdot(2\cos^{-1/2}(\psi_i)) & \text{diag}(\exp(-j\psi_i)\cdot(2\cos^{-1/2}(\psi_i)) \\ \text{diag}(-\exp(j\psi_i)\cdot(2\cos^{-1/2}(\psi_i)) & -\text{diag}(\exp(j\psi_i)\cdot(2\cos^{-1/2}(\psi_i)) \end{bmatrix} \qquad \text{(A.137)}$$

where $\psi_i = \sin^{-1}(\xi_i)$.

A.8.2 EXISTENCE OF MODES

Theorem A.11: The modes of an M-C-K system will be determined, both sufficiently and necessarily, by the stationary points of the following Rayleigh quotient:

$$\Re(X) = \frac{X^T \mathbf{H}_s X}{X^T X} \qquad \text{(A.138)}$$

PROOF:

Taking the partial derivative of $\Re(X)$ with respect to each element of $X = [x_1 \ldots x_n]^T$ yields

$$\frac{\partial \Re(X)}{\partial x_i} = \frac{(X^T X)(2\,\mathbf{H}_i . X) - (X^T \mathbf{H}_s X)\,2x_i}{(X^T X)^2} \quad i = 1,..,n \qquad \text{(A.139)}$$

where $\mathbf{H}_i. = [h_{i1} \ldots\ldots h_{in}]$ and h_{ij} is the ij^{th} entry of matrix $\mathbf{H}_s$.

Putting the above n equations together in matrix form and letting the corresponding matrix equal 0 results in

$$\mathbf{H}_s X_i - \Re(X) X_i = 0$$

or

$$\mathbf{H}_s X_i = \Re(X) X_i \qquad \text{(A.140)}$$

Equation (A.140) is an eigenproblem, if and only if

$X_i = P_i$, and $R(X_i) = \lambda_i$; i = 1,..., n. ∎

A.8.3 EXISTENCE AND UNIQUENESS OF DAMPING MATRICES

Corollary A.12: If two systems described by equation (1.1) have the same mass and stiffness matrices **M** and **K**, and thus, identical eigenvalue matrices, and if one system is given to be proportionally damped, then the another is also proportionally damped. Furthermore, if all eigenvalues of **K** matrix are distinct, their damping matrices are identical.

From Theorem A.9, this first statement is clear. If the eigenvalues of **K** yield $k_i \neq k_j$ $i \neq j$, then the matrix C_{cr} defined in equation (A.97) becomes a scalar. The scalar is equal to c_{rr} in the damping matrix of the given proportionally damped system. Therefore $\mathbf{C}_1 = \mathbf{C}_2$.

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